Kinematics, dynamics and control of robots

Tutorial 3 – Kinematics of parallel manipulators

Mobility of a robot - Grübler–Kutzbach criterion

$$m = d(n-1-p) + \sum_{i=1}^{p} f_i$$

$d = \begin{cases} 3, & \text{Planar robot} \\ 6, & \text{Spatial robot} \end{cases}$

$n -$ Number of links (including ground!)

$p -$ Number of joints

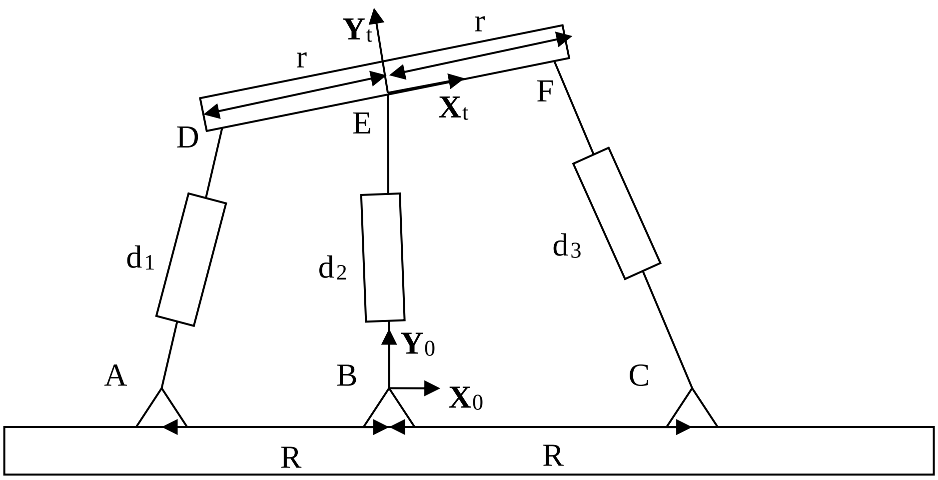
$f_i -$ Number of DOFs of joint i

When $m \le 0$ the robot is "stuck".

Comparison to serial manipulator:

	Serial manipulator	**Parallel manipulator**
DOFs	$m = p = n - 1$	$m = d(n - p - 1) + \sum f_i$
Actuators (motors)	All joint actuated	Some joints are passive
Precision	Low	High
Load/Weight ratio	Low	High
workspace	Large	Small
Forward kinematics	Very easy, single analytic solution	Very complicated, multiple solutions, numeric
Inverse kinematics	Easy, multiple solutions, analytic	Easy, analytic

Exercise 1:
Figure 1 shows planar parallel manipulator.



a. Calculate the mobility of the robot.
   For (d_1, d_2, d_3) actuated joints:
b. Solve the inverse kinematics problem.
c. Solve the forward kinematics problem.
d. Calculate the Jacobian matrices and find singularities.

__**Solution:**__

a. Mobility:

$$d = 3, \qquad n = 8, \qquad p = 9, \qquad f_i = 1 \Rightarrow M = 3(8 - 9 - 1) + 9 = 3$$

We have three degrees of freedom (x_t, y_t, θ).

b. Inverse kinematics:

Given (x_t, y_t, θ), calculate the values of (d_1, d_2, d_3).

The positions of points A-F:

$$A = \begin{pmatrix} -R \\ 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$$D = \begin{bmatrix} c_\theta & -s_\theta & x_t \\ s_\theta & c_\theta & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -r \\ 0 \\ 1 \end{pmatrix}, \qquad E = \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \qquad F = \begin{bmatrix} c_\theta & -s_\theta & x_t \\ s_\theta & c_\theta & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} r \\ 0 \\ 1 \end{pmatrix}$$

The joint values can be found from the magnitudes:

$$d_1^2 = (D_x - A_x)^2 + \left(D_y - A_y\right)^2 = (x_t - r\cos\theta + R)^2 + (y_t - r\sin\theta)^2$$

$$d_2^2 = (E_x - B_x)^2 + \left(E_y - B_y\right)^2 = x_t^2 + y_t^2$$

$$d_3^2 = (F_x - C_x)^2 + \left(F_y - C_y\right)^2 = (x_t + r\cos\theta - R)^2 + (y_t + r\sin\theta)^2$$

c. Forward kinematics:

For given values of (d_1, d_2, d_3) find (x_t, y_t, θ).

Using the equations we found above:

$$\begin{aligned} d_1^2 + d_3^2 = (x_t - r\,cos\,\theta + R)^2 + (y_t - r\,sin\,\theta)^2 + (x_t + r\,cos\,\theta - R)^2 \\ + (y_t + r\,sin\,\theta)^2 = 2(x_t^2 + y_t^2 + R^2 + r^2 - 2rRc_\theta) \\ = 2(d_2^2 + R^2 + r^2 - 2rRc_\theta) \end{aligned}$$

$$\cos\theta = \frac{2(d_2^2 + R^2 + r^2) - d_1^2 - d_2^2}{4rR}$$

$$\sin\theta = \pm\sqrt{1 - \cos\theta}$$

$$\theta = \text{atan2}(\sin\theta, \cos\theta)$$

We have 2 possible solutions for θ!

Next, we find x_t, y_t:

$$\begin{aligned} d_1^2 - d_3^2 = (x_t - r\,cos\,\theta + R)^2 + (y_t - r\,sin\,\theta)^2 - (x_t + r\,cos\,\theta - R)^2 \\ - (y_t + r\,sin\,\theta)^2 = -4y_t r s_\theta + 4Rx_t \Rightarrow \end{aligned}$$

$$y_t = \frac{R - rc_\theta}{rs_\theta} x_t + \frac{d_3^2 - d_1^2}{rs_\theta} = ax_t + b$$

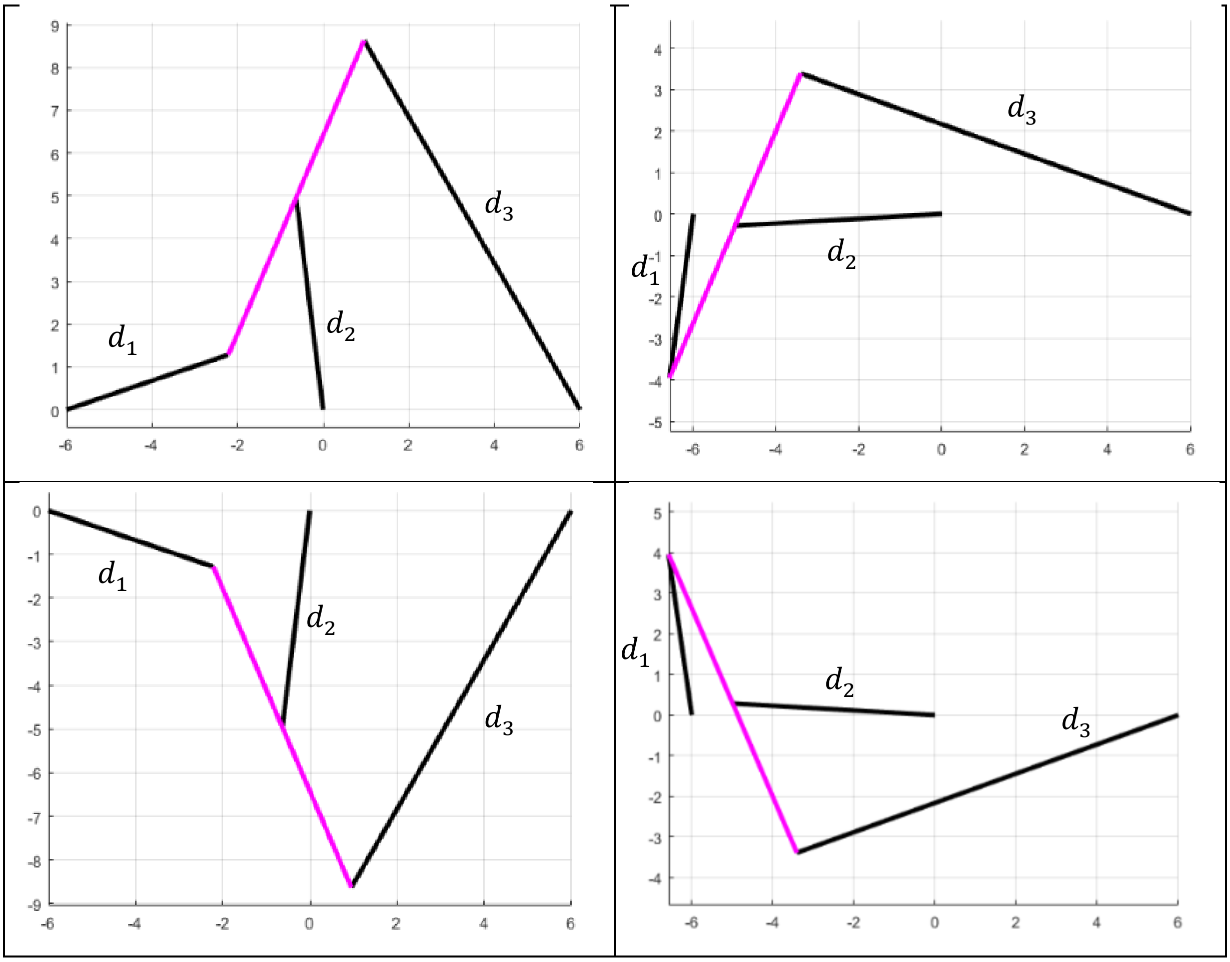
We have two equations:

$$\begin{matrix} y_t = ax_t + b \\ x_t^2 + y_t^2 = d_2^2 \end{matrix} \Rightarrow (ax_t + b)^2 + x_t^2 = d_2^2$$

$$x_t = \frac{-2ab \pm \sqrt{4a^2b^2 - 4(a^2+1)(b^2 - d_2^2}}{2(a^2+1)}$$

For each solution of θ we have two solutions for x_t that each leads to a single solution for y_t:

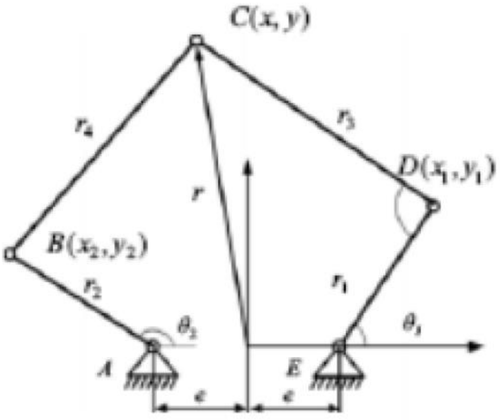
$$\begin{array}{ll} & x \rightarrow\rightarrow y \\ \theta & \begin{matrix}\nearrow \\ \searrow\end{matrix} \\ & x \rightarrow\rightarrow y \\ & x \rightarrow\rightarrow y \\ \theta & \begin{matrix}\nearrow \\ \searrow\end{matrix} \\ & x \rightarrow\rightarrow y \end{array}$$


d_1
d_2
d_3
d_1
d_2
d_3
d_1
d_2
d_3
d_1
d_2
d_3

Exercise 2 (self-practice)
Figure 2 shows a model for a planar parallel manipulator with 2 DOFs.

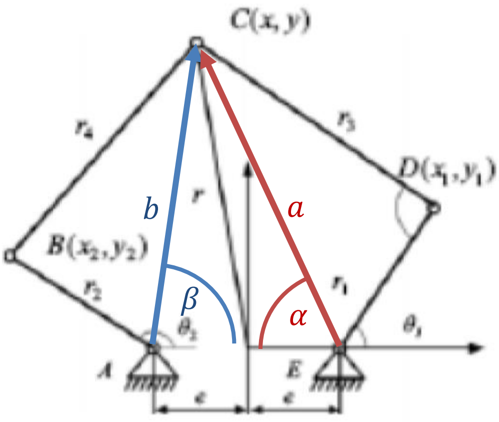
$$|\underline{\boldsymbol{r}}_1| = |\underline{\boldsymbol{r}}_2| = l_1, \qquad |\underline{\boldsymbol{r}}_3| = |\underline{\boldsymbol{r}}_4| = l_2$$

The end-effector is at the point $C(x, y)$ and the actuated joints are θ_1, θ_2.



Solve the inverse kinematics of for the task of tool position (x, y).
Solve the forward kinematics
Solve the above problems for the case where the actuated joints are at point B and D.

Solution:



The line a:

$$\underline{a} = \begin{pmatrix} x - e \\ y \end{pmatrix} \to |\underline{a}|^2 = a^2 = (x - e)^2 + y^2$$

Using the law of cosines:

$$|\underline{r}_3|^2 = |\underline{r}_1|^2 + |\underline{a}|^2 - 2|\underline{r}_1||\underline{a}|\cos(\pi - \theta_1 - \alpha)$$

Where $\alpha = \arccos\left(\frac{e-x}{a}\right)$

Substituting, we have:

$$l_2^2 = l_1^2 + a^2 + 2al_1\cos(\theta_1 + \alpha)$$

$$\cos(\theta_1 + \alpha) = \frac{l_2^2 - l_1^2 - a^2}{2l_1 a}, \sin(\theta_1 + \alpha) = \pm\sqrt{1 - \left(\frac{l_2^2 - l_1^2 - a^2}{2l_1 a}\right)^2}$$

$$\theta_1 = \operatorname{atan2}[\sin(\theta_1+\alpha), \cos(\theta_1+\alpha)] - \arccos\left(\frac{e-x}{a}\right)$$

Similarly, for b:

$$\underline{b} = \binom{x+e}{y} \rightarrow |\underline{b}|^2 = b^2 = (x+e)^2 + y^2$$

$$|\underline{r}_4|^2 = |\underline{r}_2|^2 + |\underline{b}|^2 - 2|\underline{r}_2||\underline{b}|\cos(\theta_2-\beta)$$

$$\beta = \arccos\left(\frac{e+x}{b}\right)$$

$$l_2^2 = l_1^2 + b^2 - 2bl_1\cos(\theta_2+\beta)$$

$$\cos(\theta_2-\beta) = \frac{-l_2^2+l_1^2+b^2}{2l_1b}, \sin(\theta_2-\beta) = \pm\sqrt{1-\left(\frac{-l_2^2+l_1^2+b^2}{2l_1b}\right)^2}$$

$$\theta_2 = \operatorname{atan2}[\sin(\theta_2-\beta), \cos(\theta_2-\beta)] + \arccos\left(\frac{e+x}{b}\right)$$