Control Pre-Lab 4

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Question 1:

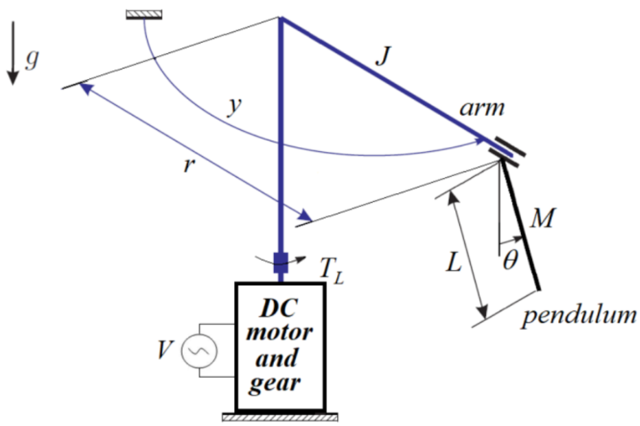


Figure 1: Experimental setup - schematic diagram

Consider the experimental setup depicted in Figure 1. The rotation of the arm is forced by the torque from the DC motor and gear, and the rotation of the pendulum is free. It is assumed that the pendulum is an uniform rigid rod with negligible thickness. The physical parameters are:

• J – moment of inertia of the arm (about the spin axis that passes through the arm mass center),

• f – arm viscous friction constant,

• M – pendulum mass, • c – pendulum viscous friction constant,

• L – pendulum length,

• r – distance between arm axis of rotation and pendulum plane of rotation,

• g – gravitational acceleration,

• Km - motor torque constant,

• Kb - back electromotive force constant (Kb = Km),

• Ra, La - electric resistance and inductance of the armature, respectively (La Ra),

• Nr - gear ratio,

• Ku - DAC + amplifier.

Tasks:

1. Write the nonlinear equations of motion for the dynamic system (see the Figure 1) assuming that the torque TL from the DC motor and gear is the system input, and the system outputs are the arm y and pendulum θ angles in the units of radian. It is assumed that the pendulum angle equals zero if the free pendulum coincides with an imaginary plumb line that is fastened to the pendulum pivot. Do not imply small deviations in the system input and outputs.

Ans)

We are going to solve these equations of motion using the Lagrange equation, which evaluates the equations using the Kinetic energy and Potential energy of the system.

We will start by taking the generalized coordinates for the system

$$q = [\theta_{arm}, \theta_{pendulum}] = [\theta_a, \theta_b]$$

Here we take K as the kinetic energy of the system and V is potential energy of the system. We assume F to be the generalized forces

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = F_q$$

For finding the Kinetic energy we will first find the centre of mass of the system

$$r_c = (r cos\theta_a \hat{e_1} + r sin\theta_a \hat{e_2}) + \left(\frac{L}{2} sin\theta_p \widehat{e_2}' - \frac{L}{2} cos\theta_p \widehat{e_3}'\right)$$

$\hat{e}_i$ and $\hat{e}_i'$ is:

$$e_1' = \cos\theta_a * e_1 + \sin\theta_a * e_2$$
$$e_2' = -\sin\theta_a * e_1 + \cos\theta_a * e_2$$
$$e_3' = e_3$$

Now for the equation for the position for the centre of mass

$$r_c = \left(r cos\theta_a - \frac{L}{2} sin\theta_p sin\theta_a\right)\hat{e_1} + \left(r sin\theta_a + \frac{L}{2} sin\theta_p cos\theta_a\right)\hat{e_2} - \frac{L}{2} cos\theta_p \hat{e_3}$$

Now we express the KE as:

$$K = K_{arm} + K_{pendulum} = \frac{1}{2} J\dot{\theta_a} + \frac{1}{2} M(\dot{r_c} * \dot{r_c}) + \frac{1}{2}\omega^T I_c \omega$$

$$\dot{\boldsymbol{r}}_c = \begin{bmatrix} -r\dot{\theta}_a \sin\theta_a - \frac{L}{2}\dot{\theta}_a \sin\theta_p \cos\theta_a - \frac{L}{2}\dot{\theta}_p \cos\theta_p \sin\theta_a \\ r\dot{\theta}_a \cos\theta_a - \frac{L}{2}\dot{\theta}_a \sin\theta_p \sin\theta_a + \frac{L}{2}\dot{\theta}_p \cos\theta_p \cos\theta_a \\ \frac{L}{2}\dot{\theta}_p \sin\theta_p \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\boldsymbol{\omega} = \begin{bmatrix} \dot{\theta}_p \\ \dot{\theta}_a \sin\theta_p \\ \dot{\theta}_a \cos\theta_p \end{bmatrix} \qquad I_c = \begin{bmatrix} \frac{ML^2}{12} & 0 & 0 \\ 0 & \frac{ML^2}{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The Potential Energy V can be expressed as:

$$V = V_{arm} + V_{pendulum} = 0 - Mg\left(\frac{L}{2}\right)cos\theta_p = -Mg\left(\frac{L}{2}\right)cos\theta_p$$

The only generalized non conservative force is the torque applied by the DC motor

$$F = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$

Now we use the Rayleigh's Dissipation function

So from this dissipation function the unconservative forces due to viscous friction can be written as

$$D = \frac{1}{2}\left(f\dot{\theta_a}^2 + c\dot{\theta_p}^2\right)$$

Now taking the derivative with respect to $\dot{q}$

$$\frac{\partial D}{\partial \dot{q}} = \begin{bmatrix} f\dot{\theta}_{arm} \\ c\dot{\theta}_{pendulum} \end{bmatrix}$$

Now substituting the in the equation

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = F_q$$

Now in the Matrix form

$$\begin{bmatrix} J + \frac{M}{3}(L^2 + 3r^2 - L^2\cos^2\theta_p) & \frac{1}{2}MLr\cos\theta_p \\ \frac{1}{2}MLr\cos\theta_p & \frac{1}{3}ML^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_a \\ \ddot{\theta}_p \end{bmatrix} - \frac{1}{6}\begin{bmatrix} ML\dot{\theta}_p(3r\dot{\theta}_p \sin\theta_p - 2L\dot{\theta}_a \sin(2\theta_p)) \\ ML^2\dot{\theta}_a^2 \sin(2\theta_p) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ MLg\sin\theta_p \end{bmatrix} = \begin{bmatrix} \tau - f\dot{\theta}_a \\ -c\dot{\theta}_p \end{bmatrix}$$

We can write it as two equations, and after substituting the variables $[\theta_a, \theta_b] = [y, \theta]$, we get

$$\begin{cases} \left(J + Mr^2 + \frac{1}{3}ML^2\,sin^2(\theta)\right)\ddot{y} + \frac{1}{2}MLr\ddot{\theta}\,cos(\theta) + \frac{2}{3}ML^2\dot{y}\dot{\theta}\,sin(\theta)\,cos(\theta) - \frac{1}{2}MLr\dot{\theta}^2\,sin(\theta) + f\dot{y} = N_r\frac{K_m}{R_a}(K_u u - K_b N_r \dot{y}) \\ \frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}MLr\ddot{y}\,cos(\theta) + \frac{1}{2}MgLsin(\theta) - \frac{1}{3}ML^2\dot{y}^2\,sin(\theta)\,cos(\theta) + c\dot{\theta} = 0 \end{cases}$$

Qustion2

Section (a) Find all equilibrium points for the experimental setup model assuming ue = 0.

Equilibrium points are obtained in case, all acceleration and velocity of the arm and pendulum are zero, meaning $\ddot{\theta} = \dot{\theta} = \ddot{y} = \dot{y} = 0$. In that case, the given equations are reduced to

$$\begin{cases} N_r \dfrac{K_m}{R_a} K_u u_e = 0 \\ \dfrac{1}{2} MgLsin(\theta_e) = 0 \end{cases}$$

The first equation is automatically zero given that $u_e = 0$, for the second equation to hold, we require that

$$sin(\theta_e) = 0 \rightarrow \theta_e = k\pi, k\epsilon Z$$

Thus, the equilibrium points of the system are

$$(y_e, \theta_e) = (y, k\pi), y \in R, k\epsilon Z$$

Section (b) Implement in Simulink the dynamics of the experimental setup

Given,

$$\begin{cases} \left(J + Mr^2 + \frac{1}{3}ML^2 \, sin^2(\theta)\right)\ddot{y} + \frac{1}{2}MLr\ddot{\theta}\, cos(\theta) + \frac{2}{3}ML^2\dot{y}\dot{\theta}\, sin(\theta)\, cos(\theta) - \frac{1}{2}MLr\dot{\theta}^2\, sin(\theta) + f\dot{y} = N_r \frac{K_m}{R_a}(K_u u - K_b N_r \dot{y}) \\ \frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}MLr\ddot{y}\, cos(\theta) + \frac{1}{2}MgLsin(\theta) - \frac{1}{3}ML^2\dot{y}^2\, sin(\theta)\, cos(\theta) + c\dot{\theta} = 0 \end{cases}$$

The physical state vector is given by $x = [y \quad \dot{y} \quad \theta \quad \dot{\theta}]'$, we'll substitute the state variables in the dynamics. Equation

$$\begin{cases} \left(J + Mr^2 + \frac{1}{3}ML^2 \, sin^2(x_3)\right)\dot{x}_2 + \frac{1}{2}MLr\dot{x}_4 cos(x_3) + \frac{2}{3}ML^2 x_2 x_4 sin(x_3)\, cos(x_3) - \frac{1}{2}MLrx_4^2 sin(x_3) + fx_2 = N_r \frac{K_m}{R_a}(K_u u - K_b N_r \dot{y}) \ (*) \\ \frac{1}{3}ML^2\dot{x}_4 + \frac{1}{2}MLr\dot{x}_2 cos(x_3) + \frac{1}{2}MgLsin(x_3) - \frac{1}{3}ML^2 x_2^2 sin(x_3)\, cos(x_3) + cx_4 = 0 \quad (**) \end{cases}$$

In order to write the dynamic system in standard state space manner, we need to the following transformations $\frac{L}{3} \cdot (*) - \frac{rcos(x_2)}{2}(**)$, and we get

$$\left(\frac{L}{3}\left(J + Mr^2 + \frac{1}{3}ML^2\, sin^2(x_3)\right) - \frac{1}{4}MLr^2\, cos^2(x_3)\right)\dot{x}_2 =$$

$$\frac{rcos\, x_3}{2}\left(\frac{1}{2}MgLsin(x_3) - \frac{1}{3}ML^2x_2^2 sin(x_3)\, cos(x_3) + cx_4\right) + \frac{L}{3}(T_L - \frac{2}{3}ML^2x_2x_4 sin\,(x_3) cos\,(x_3) + \frac{1}{2}MLrx_4^2 sin\,(x_3) - fx_2)$$

Next, we will perform the following transformation $(*) \cdot \frac{1}{2}MLr\cos(x_3) - (**) \cdot (J + Mr^2 + \frac{1}{3}ML^2 \sin^2(x_3))$, and we get the following result:

$$\left(\frac{1}{4}M^2L^2r^2\cos^2(x_3)-\frac{1}{3}ML^2\left(J+Mr^2+\frac{1}{3}ML^2\sin^2(x_3)\right)\right)\dot{x}_4=\frac{1}{2}MLr\cos(x_3)\left(T_L-\frac{2}{3}ML^2x_2x_4\sin(x_3)\cos(x_3)+\frac{1}{2}MLrx_4^2\sin(x_3)-fx_2\right)+\left(J+Mr^2+\frac{1}{3}ML^2\sin^2(x_3)\right)\cdot\left(\frac{1}{2}MgL\sin(x_3)-\frac{1}{3}ML^2x_2^2\sin(x_3)\cos(x_3)+cx_4\right)$$

Now, we can write the dynamics equation in standard form

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=\dfrac{1}{M}\left[\dfrac{r\cos x_3}{2}\left(\dfrac{1}{2}MgL\sin(x_3)-\dfrac{1}{3}ML^2x_2^2\sin(x_3)\cos(x_3)+cx_4\right)+\dfrac{L}{3}\left(T_L-\dfrac{2}{3}ML^2x_2x_4\sin(x_3)\cos(x_3)+\dfrac{1}{2}MLrx_4^2\sin(x_3)-fx_2\right)\right]\\ \dot{x}_3=x_4\\ \dot{x}_4=\dfrac{1}{N}\left[\dfrac{3}{L^2}\left(-\dfrac{1}{2}MLr\dot{x}_2\cos(x_3)-\dfrac{1}{2}\mathrm{MgL}\sin(x_3)+\dfrac{1}{3}ML^2x_2^2\sin(x_3)\cos(x_3)-cx_4\right)+\left(J+Mr^2+\dfrac{1}{3}ML^2\sin^2(x_3)\right)\cdot\left(\dfrac{1}{2}MgL\sin(x_3)-\dfrac{1}{3}ML^2x_2^2\sin(x_3)\cos(x_3)+cx_4\right)\right]\end{cases}$$

Where,

$$M=\left(\frac{L}{3}\left(J+Mr^2+\frac{1}{3}ML^2\sin^2(x_3)\right)-\frac{1}{4}MLr^2\cos^2(x_3)\right)$$

$$N=\left(\frac{1}{4}M^2L^2r^2\cos^2(x_3)-\frac{1}{3}ML^2\left(J+Mr^2+\frac{1}{3}ML^2\sin^2(x_3)\right)\right)$$

Shown below is the Simulink we built in MATLAB

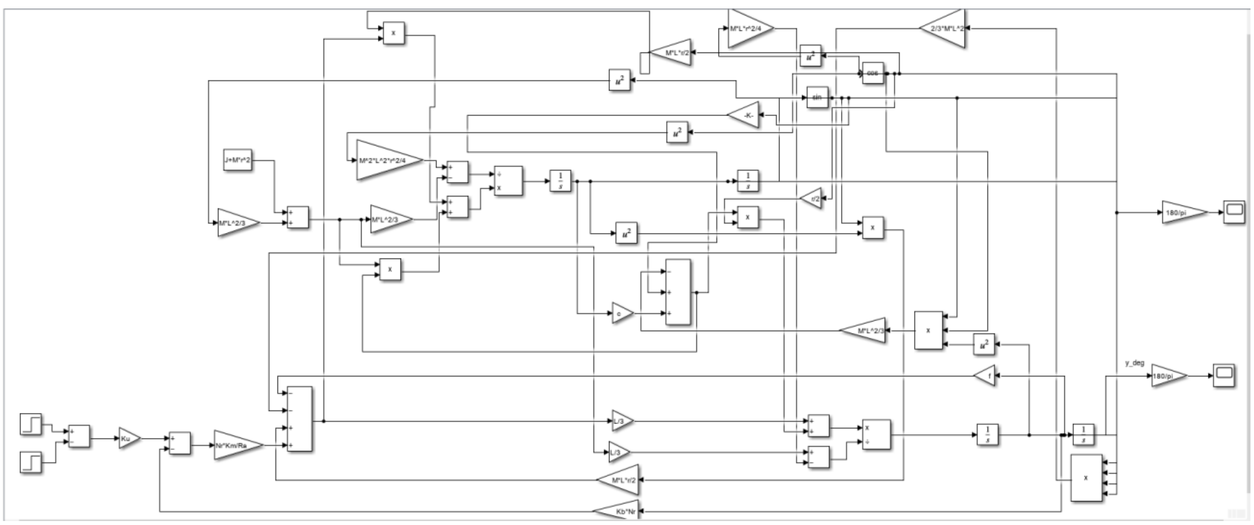


Figure:2 Simulink model of the dynamic system

We choose u to be a step input which last for $0.1s$, a simulation of an impulse, shown below is the simulated arm rotation in degree we get

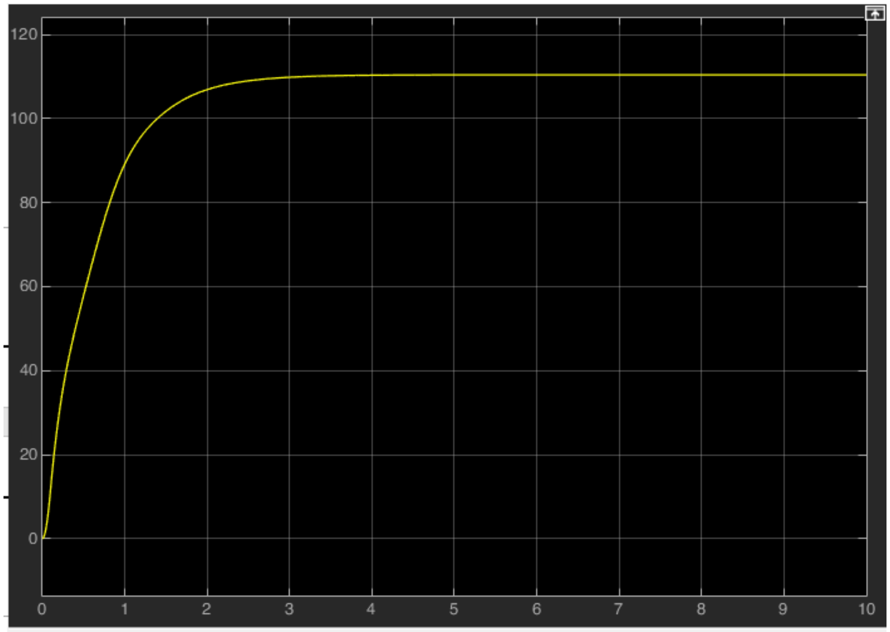


Figure 3: Simulated arm rotation degree in response to the 'impulse' input

c. Linearize the equations of the experimental setup dynamics about the equilibrium point with the downward pendulum (see Subsection (a)), and then build the corresponding block diagram.

c) We have the equation

$$\begin{cases} \left(J + Mr^2 + \frac{1}{3}ML^2 \sin^2(\theta)\right)\ddot{y} + \frac{1}{2}MLr\ddot{\theta}\cos(\theta) + \frac{2}{3}ML^2\dot{y}\dot{\theta}\sin(\theta)\cos(\theta) - \frac{1}{2}MLr\dot{\theta}^2\sin(\theta) + f\dot{y} = N_r\frac{K_m}{R_a}(K_u u - K_b N_r \dot{y}) \\ \frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}MLr\ddot{y}\cos(\theta) + \frac{1}{2}MgLsin(\theta) - \frac{1}{3}ML^2\dot{y}^2\sin(\theta)\cos(\theta) + c\dot{\theta} = 0 \end{cases}$$

We can take $T_l = N_r\frac{K_m}{R_a}(K_u u - K_b N_r \dot{y})$

So the equation becomes:

$$\begin{cases} \left(J + Mr^2 + \frac{1}{3}ML^2 \sin^2(\theta)\right)\ddot{y} + \frac{1}{2}MLr\ddot{\theta}\cos(\theta) + \frac{2}{3}ML^2\dot{y}\dot{\theta}\sin(\theta)\cos(\theta) - \frac{1}{2}MLr\dot{\theta}^2\sin(\theta) + f\dot{y} = T_l \\ \frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}MLr\ddot{y}\cos(\theta) + \frac{1}{2}MgLsin(\theta) - \frac{1}{3}ML^2\dot{y}^2\sin(\theta)\cos(\theta) + c\dot{\theta} = 0 \end{cases}$$

Now in order to linearize this system we take small angle approximations where

$sin\theta \approx \theta \; and \; cos\theta \approx 1$, and the equilibrium values are $\theta_e \approx y_e \approx 0$

So upon taking the small angle approximations

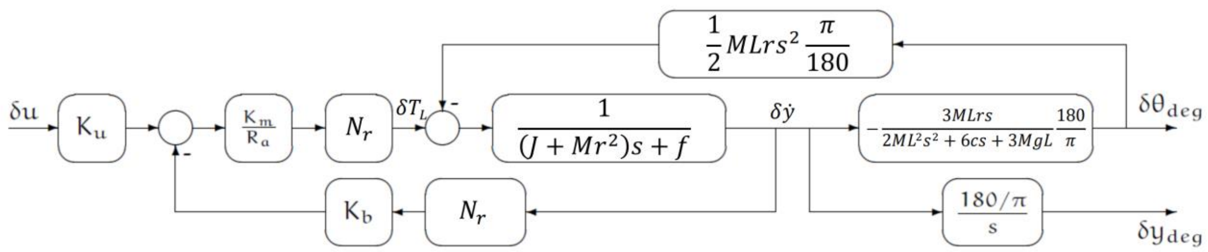
$$\begin{cases} (J + Mr^2)\ddot{y} + \frac{1}{2}MLr\ddot{\theta} + \frac{2}{3}ML^2\dot{y}\dot{\theta} - \frac{1}{2}MLr\dot{\theta}^2 + f\dot{y} = T_l \\ \frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}MLr\ddot{y} + \frac{1}{2}MgL\theta - \frac{1}{3}ML^2\dot{y}^2 + c\dot{\theta} = 0 \end{cases}$$

Applying Laplace transformation, we get

$$\begin{cases} (J+Mr^2)s^2y(s)+\frac{1}{2}MLr^2\theta(s)+fsy(s)=T_L \\ \frac{1}{3}ML^2s^2\theta(s)+\frac{1}{2}MLrs^2y(s)+\frac{1}{2}MgL\theta(s)+cs\theta(s)=0 \end{cases}$$

Rearranging the equation, we get

$$\begin{cases} \dot{y}(s)=\frac{180s}{\pi}y_{deg}(s)=\frac{T_L-\frac{1}{2}MLrs^2\frac{\pi}{180}\theta_{deg}(s)}{(J+Mr^2)s+f} \\ \theta_{deg}(s)=-\frac{180}{\pi}\frac{3MLrsy(s)}{2ML^2s^2+6cs+3MgL} \end{cases}$$



Appendix

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%%% prelab4
% % Numeric values we used for the Simulink
Km=39.6*0.001;
Kb=6.46e-3*60/(2*pi);
Ra=6.8;
La=620*10^-6
J=0.06;
f=0.0088;
Nr=4.5
J=0.0057;
M=51e-3;
r=0.26;
L=0.35;
% c=0.01;  not given
g=9.807;
Ku=12;
u=1;
```