Waves and Solitons in Quantum Fluids

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In this report we describe theoretically the propagation of waves in a Bose-Einstein condensate (BEC). In order to do so, we first linearize the Gross-Pitaevskii equation (LGPE) up to first order ($O(\epsilon^2)$) and then, proposing a sinusoidal transversal wave as solution we find that it satisfies the Bogoliubov dispersion relation. This opens the possibility of defining a speed of sound of the phonons. Furthermore, we find the dark and bright soliton solutions of the GPE and calculate the corresponding conserved quantities: the number of particles N, the energy E and the linear momentum P. Finally we explain in depth that the relations established by these quantities suggest that the solitons behaves as classical particles.

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I. LINEARIZED GROSS-PITAEVSKII EQUATION

Of particular importance are the perturbations of the condensate that propagates as sound waves. We derive the behaviour of these perturbations for a homogeneous condensate that satisfies

$$\psi_0 = \sqrt{\frac{\mu}{g}}, \qquad \rho_0 = \frac{\mu}{g} \tag{1}$$

where μ is the chemical potential and g the nonlinear coupling constant. Thus, We expand the wavefunction as a homogeneous background with different orders of perturbation

$$\psi(x,t) = \psi_0 + \epsilon\psi_1(x,t) + \epsilon^2\psi_2(x,t) + \ ..., \qquad \epsilon << 1. \tag{2}$$

We expand the GPE up to first order noting that temporal and spatial derivatives of the steady uniform background are zero

$$\begin{aligned} i\hbar\partial_t(\psi_0 + \epsilon\psi_1(x,t)) &= -\frac{\hbar^2}{2m}\partial_x^2(\psi_0 + \epsilon\psi_1(x,t)) + g|\psi_0 + \epsilon\psi_1(x,t)|^2(\psi_0 + \epsilon\psi_1(x,t)) - \mu(\psi_0 + \epsilon\psi_1(x,t)), \\ i\hbar\partial_t\psi_1(x,t) &= -\frac{\hbar^2}{2m}\partial_x^2\psi_1(x,t) + g\psi_0^2(2\psi_1(x,t) + \psi_1^*(x,t)) - \mu\psi_1(x,t), \end{aligned}$$

$$\boxed{i\hbar\partial_t\psi_1(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\psi_1(x,t) + \mu\psi_1(x,t) + \mu\psi_1^*(x,t).} \tag{3}$$

Eq.(3) is the so-called Linear Gross-Pitaevskii Equation (LGPE) because the nonlinearity has disappeared from the original equation. As we explain in the next sections, it describes the propagation of sound waves in a homogeneous condensate.

II. SOUND WAVES IN BECS

We look for travelling wave solutions of the general form

$$\psi_1(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx-\omega t)}, \tag{4}$$

whit k being the wavenumber, ω the frequency of the wave and A and B complex coefficients to be determined by the initial conditions. Substituting the last ansatz into Eq.(3) yields the Bogoliubov dispersion relation. This solution is the only non-trivial for A and B

$$\boxed{\omega(k) = \sqrt{\left(\frac{\hbar^2k^2}{2m}\right)^2 + \frac{\rho_0 g}{m}k^2}.} \tag{5}$$

As long as ω depends on k the medium is, in general, dispersive. Let us explore the propagation possibilities depending on the value of the nonlinear coupling g.

- In the absence of interactions ($g = 0$), the wave behaves as a free particle: $\hbar\omega = \frac{\hbar^2k^2}{2m}$.
- For repulsive interactions ($g > 0$) the free particle behaviour is recovered in the limit of large k, i.e. Short waves. Nevertheless, for low k, i.e. Long waves, a linear dispersion relation dominates, characteristic of sound waves: $\hbar\omega \propto |k|$.
- For attractive interactions ($g < 0$), the situation is fundamentally different. In the regime of long waves ω becomes complex. Then, the exponential terms increase over time increasing the dynamical instability of the condensate. For short waves, it still behaves as a particle. Here the condensate prefers to collapse rather than stay as a uniform density profile.

For repulsive interactions the phase velocity is approximately constant for all wavelengths, so, the speed of sound of the phonons can be defined. Phonons are propagating perturbations of the quantum fluid often referred to as a quasiparticle. Similar to think in photons as quantized light waves, phonons can be thought as quantized sound waves. Its phase velocity is defined as follows

$$\boxed{v_{ph} = \frac{\omega}{k} = \sqrt{\frac{\rho_0 g}{m}} \equiv c.} \tag{6}$$

This is the speed of sound that appears in the Linearized Superfluid equations after applying the Madelung transform to the GPE. The density perturbations of the superfluid satisfies the wave equation

$$\partial_t^2 \rho_1(x,t) = c^2 \partial_x^2 \rho_1(x,t). \tag{7}$$

The wave solution which we have found in one-dimensional spaces can be easily generalized to two and three dimensions. In a trapped condensate ($V(x) \neq 0$) the speed of sound will vary with the position due to the spatial dependence of the density.

Now, what follows to this report is the study of unchanging form and localized perturbations, the solitons.

III. SOLITONS

The 1D-GPE is well-studied not only in condensed matter but also in the context of nonlinear optics. It has the special property of being integrable such that its solutions possess an infinite set of conserved quantities. The most relevant for this study are the number of particles N, the momentum P and the energy E, that must be renormalized to represent finite values as follows

$$N = \int_{-\infty}^{+\infty} dx \; (\rho_0 - |\psi(x,t)|^2), \tag{8}$$

$$P = \frac{i\hbar}{2}\int_{-\infty}^{+\infty} dx \; (\psi(x,t)\partial_x \psi^*(x,t) - \psi^*(x,t)\partial_x\psi(x,t))\left(1 - \frac{\rho_0}{|\psi(x,t)|^2}\right), \tag{9}$$

$$E = \int_{-\infty}^{+\infty} dx \; \left(\frac{\hbar^2}{2m}|\partial_x \psi(x,t)|^2 + \frac{g}{2}(|\psi(x,t)|^2 - \rho_0)^2\right). \tag{10}$$

Before entering into details we need to solve the GPE. The adimensional time-independent GPE we want to solve is

$$-\frac{1}{2}\frac{d^2\psi(x)}{dx^2} = \psi^3 + \psi = -\frac{d}{d\psi}(\psi^2 + \frac{1}{2}\psi^4), \; (\mu < 0, \; g > 0). \tag{11}$$

This is mathematically analog to a mechanical potential (if you let me to assume the position coordinate as a time coordinate) with potential energy

$$W(\psi) = \psi^2 + \frac{1}{2}\psi^4. \tag{12}$$

Thus, we can solve it applying the "energy conservation" i.e.

$$\underbrace{\frac{1}{2}\left(\frac{d\psi}{dx}\right)^2}_{Kinetic} + \underbrace{W(\psi)}_{Potential} = K, \tag{13}$$

with K a constant associated with the total energy of the system. The asymptotic behaviour fix the boundary conditions

$$\lim_{x\to+\infty} \psi(x) = 0 \implies K = 0,$$

$$x = 0 \implies W(\psi_0) = 0 \implies \psi_0 = \sqrt{2}\sqrt{\frac{\mu}{g}}.$$

We can reduce the problem to the first integral

$$x = \int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{-2W(\psi)}} = \int_{\psi_0}^{\psi} \frac{d\psi}{\psi\sqrt{2-\psi}}, \tag{14}$$

which solution is known as the dark soliton

$$\psi(x) = tanh(x/\xi). \tag{15}$$

For $g < 0$ the solution obtained is called bright soliton and has the next form

$$\psi(x) = sech(x/\xi). \tag{16}$$

Even though the physical differences of the two types of solitons they share three basic properties:

1. They have permanent unchanging form due to the nonlinear term, which can be understood as an infinite energy storage that the soliton uses to not disperse.
2. They are localized in space.
3. They emerge unscathed from collisions with other solitons.

Now we devote the rest of the report to dig into the properties of these two types of solitons.

A. Dark Solitons

Dark solitons are supported for repulsive interactions that creates propagating density dips along the fluid. The general spatio-temporal solution is given by the next equation

$$\psi(x,t) = \sqrt{\rho_0}\left[\gamma tanh\left(\frac{(x-vt)\gamma}{\xi}\right) + i\nu\right] e^{-i\mu t/\hbar}, \tag{17}$$

with $\gamma = \sqrt{1-\nu^2}$ and $\nu = \frac{v}{c}$, is the velocity of the soliton in terms of the speed of sound. As can be seen in Fig. 1, the amplitude of the soliton is inversely proportional to its velocity.

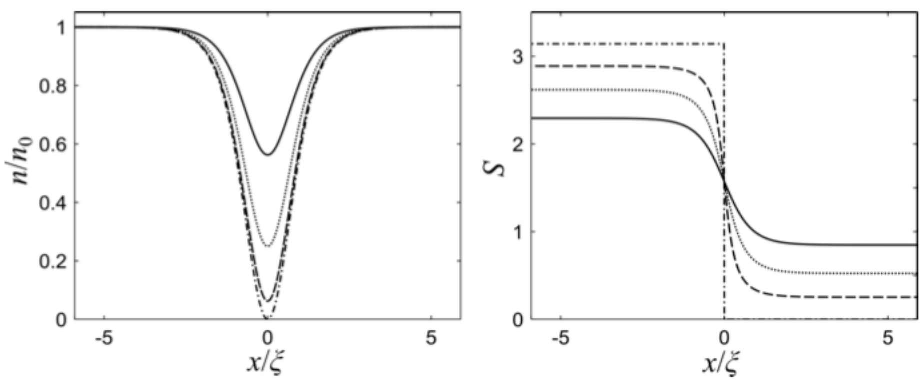

Figure 1. Density $n(x)$ and phase $S(x)$ profiles of dark solitons with various speeds: $\nu = 0$ (dot-dashed line), $\nu = 0.25$ (dashed line), $\nu = 0.5$ (dotted line) and $\nu = 0.75$ (solid line). Density is scaled in terms of the background density of the homogeneous system, n_0, and position in terms of the healing length, ξ.

Moreover, the phase jump is characteristic of a 1D phase deffect.

B. Bright Solitons

In contrast, bright solitons are a localized density accumulation originated by the attractive interactions. Its equation is quite similar to the dark solitons

$$\psi(x,t) = \sqrt{\rho_0}\left[\gamma sech\left(\frac{(x-vt)\gamma}{\xi}\right) + i\nu\right] e^{-i\mu t/\hbar}. \tag{18}$$

IV. PARTICLE-LIKE BEHAVIOUR

Using equations (8), (9) and (10) we obtain the following expressions for a dark soliton at a given time

$$N = 2\gamma\rho_0\xi, \tag{19}$$
$$P = 2\hbar\rho_0 \arctan(\gamma/\nu) - 2\hbar\rho_0 c\nu\gamma, \tag{20}$$
$$E = \frac{4}{3}\hbar\rho_0 c\gamma^3. \tag{21}$$

Differenciating the energy and momentum with respect to the velocity we find

$$\frac{dE}{dv} = -\frac{4\hbar\rho_0 v\gamma}{c}, \tag{22}$$
$$\frac{dP}{dv} = -\frac{4\hbar\rho_0\gamma}{c}. \tag{23}$$

Note that both the energy and momentum decrease as the soliton gets faster; the maximum energy corresponds to a static dark soliton. Now, we can compare the classical and the quantum results

$$\text{Classical: } \frac{dE}{dP} = \frac{dE}{dv}\frac{dv}{dP} = mv \cdot \frac{1}{m} = v \qquad \text{Quantum: } \frac{dE}{dP} = \frac{dE}{dv}\frac{dv}{dP} = -\frac{4\hbar\rho_0 v\gamma}{c} \cdot -\frac{c}{4\hbar\rho_0\gamma} = v \tag{24}$$

There is a correspondence between quantum solitons and classical particles. Particles are usually thought as localized wave-packets because their norm can be adjusted to 1. To highlight this analogy we can compute the energy in the non-relativistic limit.

First, we see that the effective mass of a dark soliton is negative

$$m_{eff} = \frac{dP}{dv} = -\frac{4\hbar\rho_0\gamma}{c} \tag{25}$$

This sign is responsible of the decreasing energy for increasing velocities. Furthermore, it is not surprising given that the dark soliton is an absence of atoms. Let us finally see the limit of slow solitons ($\gamma \approx 1$)

$$\boxed{E(v) = \frac{4}{3}\hbar\rho_0 c + \frac{1}{2}m_{eff}(v \to 0)v^2 + O(v^3),} \tag{26}$$

which is the form of a classical particle with rest energy $E_0 = \frac{4}{3}\hbar c\rho_0$.

For a bright soliton this approach is completely analogous with the key difference that they behave as classical particles of positive mass.

V. FINAL REMARKS

This report tries to show one of the exciting examples we can find in the BEC physics. As seen in the last section, with ultracold matter, it is possible to study the physics of negative mass particles. Nevertheless this is not new because solid states physics predicts electrons of arbitrary masses, it exemplifies the broadness and flexibility of this systems.