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The symmetry group of the harmonic oscillator and its reduction

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The symmetry of the harmonic oscillator is dealt with in the Hamiltonian formalism. Unitary operators representing the symmetry are studied from this point of view. Of additional interest is reduction of the symmetry group SU(4) for the four-dimensional harmonic oscillator. Subspaces are determined from the representation spaces for SU(4) so as to give those for SO(4).

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I. INTRODUCTION

This article deals with a quantum system as a Hamiltonian dynamical system which was formulated by Marsden.[1–3] A purpose of this paper is to discuss the symmetry group of the quantum harmonic oscillator in the Hamiltonian formalism. The harmonic oscillator is a rather simple system whose symmetry is well known. Of particular interest is then the symplectic point of view of symmetry transformations, which has received little attention. Another purpose is to set up a quantum version of a previous paper,[4] in which a reduction of the four-dimensional classical harmonic oscillator was dealt with. The results to be obtained will be utilized in the next paper.[5]

Section II contains Hamiltonian formalism of the quantum harmonic oscillator. Section III is concerned with the symmetry group dealt with in the Hamiltonian formalism. For the harmonic oscillator the usual technique of correspondence for constructing quantum operators from classical first integrals offers no problem, as far as the Cartesian coordinates are concerned. Of central interest is to integrate those operators to give unitary operators which describe the symmetry of the quantum system. Section IV is devoted to the four-dimensional harmonic oscillator. Reduction of eigenspaces of the Hamiltonian operator is discussed together with accompanying reduction of the symmetry group.

II. HAMILTONIAN FORMALISM OF THE HARMONIC OSCILLATOR

The classical harmonic oscillator is described on the space $\mathbb{R}^n \times \mathbb{R}^n$. Let (x_j, p_j) be the Cartesian coordinates. Introducing the coordinates

$$z_j = \lambda x_j + i p_j \quad [i = (-1)^{1/2}], \tag{2.1}$$

where λ is a positive constant, one can equip $\mathbb{R}^n \times \mathbb{R}^n$ with the structure of an n-dimensional complex vector space $\mathbb{C}^n$. The Hamiltonian function is then written in the form

$$H = \tfrac{1}{2} \Sigma p_j^2 + (\lambda^2/2) \Sigma x_j^2 = \tfrac{1}{2} \Sigma \bar{z}_j z_j, \tag{2.2}$$

where $\bar{z}_j$ is the complex conjugate to z_j. Further properties of the classical harmonic oscillator will be recalled when required.

To formulate the quantum harmonic oscillator as a Hamiltonian system, we have first to designate an infinite dimensional symplectic manifold. According to Marsden,[2] the suitable symplectic manifold and the symplectic form ω are, respectively, $L^2(\mathbb{R}^n)$, the Hilbert space of square integrable complex functions on $\mathbb{R}^n$, and

$$\omega(X, Y) = -\operatorname{Im}\langle X, Y\rangle, \tag{2.3}$$

where $X, Y \in L^2(\mathbb{R}^n)$ and $\operatorname{Im}\langle X, Y\rangle$ denotes the imaginary part of the inner product

$$\langle X, Y\rangle = \int_{\mathbb{R}^n} \bar{X} Y \, dx \tag{2.4}$$

with dx the standard volume element on $\mathbb{R}^n$.

The Hamiltonian function is defined as a quadratic form in $\varphi \in L^2(\mathbb{R}^n)$ with the Hamiltonian operator $\hat{H}$ by

$$H(\varphi) = \tfrac{1}{2}\langle \varphi, \hat{H}\varphi\rangle. \tag{2.5}$$

Here $\hat{H}$ is the self-adjoint extension of the operator determined from (2.2) by the Schrödinger procedure: $p_j = -i\partial/\partial x_j$. For simplicity we use the same letter for the operators formed from (2.1) as in the classical system. Then one has

$$\hat{H} = \tfrac{1}{2}\sum \bar{z}_j z_j + n\lambda/2 = \tfrac{1}{2}\sum z_j \bar{z}_j - n\lambda/2, \tag{2.6}$$

where we do not distinguish the essentially self-adjoint operator, the right-hand sides (2.6), from its self-adjoint extension $\hat{H}$.

The time evolution is governed by Hamilton's equation

$$\frac{d\varphi}{dt} = X_H(\varphi), \tag{2.7}$$

where X_H is the Hamiltonian (or canonical) vector field determined by

$$i(X_H)\omega = -dH, \tag{2.8}$$

$i(\ )$ denoting the interior product.[2] Of course, X_H is defined in the domain of $\hat{H}$, dense in $L^2(\mathbb{R}^n)$. Applying (2.8) to (2.5), we have $X_H(\varphi) = -i\hat{H}\varphi$, so that Eq. (2.7) becomes the Schrödinger equation $d\varphi/dt = -i\hat{H}\varphi$.

We here make a mention of operators z_j and $\bar{z}_j$. The usual commutation relations are given by

$$[z_j, \bar{z}_k] = 2\lambda\delta_{jk}, \quad \text{and the others vanishing.} \tag{2.9}$$

In what follows, instead of dwelling upon the domain of z_j and $\bar{z}_j$, we understand that operators which are polynomials in z_j and $\bar{z}_j$ are defined at least in the linear subspace, dense in $L^2(\mathbb{R}^n)$,

$$\left\{(\text{polynomials in } x_j)\exp\left(-\tfrac{1}{2}\lambda\sum x_j^2\right)\right\}. \tag{2.10}$$

The operators z_j and $\bar{z}_j$ are then conjugate to each other.

In the remainder of this section we review the eigenspaces of $\hat{H}$. Let

$$\varphi_0(x_1,\ldots,x_n) = (\lambda/\pi)^{n/4}\exp(-\tfrac{1}{2}\lambda\sum x_j^2) \tag{2.11}$$

be the normalized ground state. All of normalized eigenfunctions are expressed in the form

$$\varphi_{k_1\cdots k_n} = \lambda^{-N/2}C_{k_1\cdots k_n}\bar{z}_1^{k_1}\cdots\bar{z}_n^{k_n}\varphi_0 \tag{2.12a}$$

$$= C_{k_1\cdots k_n}H_{k_1}(\lambda^{1/2}x_1)\cdots H_{k_n}(\lambda^{1/2}x_n)\varphi_0, \tag{2.12b}$$

where $C_{k_1\cdots k_n} = (2^N k_1!\cdots k_n!)^{-1/2}$ and $N = k_1 + \cdots + k_n$ with k_j a nonnegative integer and the H_{k_j} are Hermite's polynomials defined by

$$H_k(t) = (-1)^k\exp(t^2)(d/dt)^k\exp(-t^2). \tag{2.13}$$

The functions (2.12) form a complete orthonormal system in $L^2(\mathbb{R}^n)$. Every eigenspace is assigned by the nonnegative integer $N = k_1 + \cdots + k_n$:

$$\hat{H}\varphi_{k_1\cdots k_n} = \lambda(N + n/2)\varphi_{k_1\cdots k_n}. \tag{2.14}$$

The dimension of the eigenspace, or the degeneracy of the energy level $\lambda(N + n/2)$ is $\binom{N+n-1}{n-1}$, a binomial coefficient.

III. THE SYMMETRY GROUP

We first make a brief review of the symmetry of the classical harmonic oscillator (see Ref. 4). A general first integral is written in the form

$$F = (\lambda/2i)\sum C_{jk}z_k\bar{z}_j, \quad \mathrm{tr}(C_{jk}) = 0, \tag{3.1}$$

where $C = (C_{jk})$ is an anti-Hermitian matrix with vanishing trace. The Hamiltonian vector field X_F associated with F and the symplectic (or canonical) transformations exp (tX_F/λ) generated by X_F are, respectively, given by

$$X_F = -\lambda\sum C_{jk}z_k\frac{\partial}{\partial z_j} - \lambda\sum\overline{C_{jk}z_k}\frac{\partial}{\partial\bar{z}_j}, \tag{3.2a}$$

$$\exp(tX_F/\lambda): \quad z\to\exp(-tC)z, \quad \bar{z}\to\overline{\exp(-tC)z}, \tag{3.2b}$$

where $z = (z_j)$ and $\bar{z} = (\bar{z}_j)$ are column vectors and exp $(-tC)\in\mathrm{SU}(n)$. To quantize the classical observable F, we follow the Schrödinger procedure by writing out F in terms of x_j and p_j and substituting $-i\partial/\partial x_j$ for p_j. We use the symbols z_j and $\bar{z}_j$ as operators in the same way as in (2.6). Let (A_{jk}) and (B_{jk}) be the real and imaginary parts of (C_{jk}), respectively:

$$C_{jk} = A_{jk} + iB_{jk}. \tag{3.3}$$

Then, regardless of the condition $\mathrm{tr}(C_{jk}) = 0$, the quantized operator $\hat{F}$ takes the form

$$\hat{F} = \frac{1}{2}\sum B_{jk}\left(\lambda^2x_jx_k - \frac{\partial^2}{\partial x_j\partial x_k}\right) - \frac{\lambda}{i}\sum A_{jk}x_k\frac{\partial}{\partial x_j} \tag{3.4a}$$

$$= \frac{1}{2i}\sum C_{jk}z_k\bar{z}_j + \frac{i\lambda}{2}\mathrm{tr}(C_{jk}) \tag{3.4b}$$

$$= \frac{1}{2i}\sum C_{jk}\bar{z}_jz_k - \frac{i\lambda}{2}\mathrm{tr}(C_{jk}). \tag{3.4c}$$

It is clear that $\hat{F}$ is a symmetric operator on the domain (2.10). Commutation relations are calculated to give

$$\left[\frac{i}{\lambda}\frac{1}{2i}\sum C_{jk}z_k\bar{z}_j, \frac{i}{\lambda}\frac{1}{2i}\sum D_{jk}z_k\bar{z}_j\right] = \frac{i}{\lambda}\frac{1}{2i}\sum[C, D]_{jk}z_k\bar{z}_j, \tag{3.5}$$

where $[C, D]_{jk}$ denote the (j,k) components of the matrix $[C, D]$. Here the conditions tr $(C_{jk}) = \mathrm{tr}(D_{jk}) = 0$ have not been required, so that Eq. (3.5) is applicable to the operator $\hat{H} + n\lambda/2$ (see (2.6)). Since $\hat{H} + n\lambda/2$ has the unit matrix as the coefficient matrix, Eq. (3.5) ensures that $\hat{H}$ and $\hat{F}$ commute. Moreover, Eq. (3.5) shows that the mapping

$$(C_{jk})\to(i/\lambda)(1/2i)\sum C_{jk}z_k\bar{z}_j \tag{3.6}$$

is a Lie algebra homomorphism. From now on, we assume that $\mathrm{tr}(C_{jk}) = 0$, so that the ordering of z_j and $\bar{z}_j$ in expressing $\hat{F}$ offers no problem, as is seen from (3.4).

We turn to symplectic transformations associated with $\hat{F}$. Like (2.5), to the operator $\hat{F}$ there corresponds a quadratic form in φ:$F(\varphi) = \frac{1}{2}\langle\varphi,\hat{F}\varphi\rangle$. The Hamiltonian vector field X_F[6] is determined by the same condition as (2.8) to be $X_F(\varphi) = -i\hat{F}\varphi$. The symplectic transformations generated by X_F will be expressible by $\exp(tX_F/\lambda) = \exp(-it\hat{F}/\lambda)$ as in (3.2b). Our problem is then to show the existence and property of the operator exp $(-it/\hat{F}/\lambda)$.

Before approaching the problem, we touch upon a quantum version of the classical symplectic transformation (3.2b). Let us regard z_j and $\bar{z}_j$ in (3.2b) as operators. Then the linear transformation

$$z_j' = \sum(e^{-tC})_{jk}z_k, \quad \bar{z}_j' = \sum\overline{(e^{-tC})}_{jk}\bar{z}_k \tag{3.7}$$

makes no change in the canonical commutation relations (2.9), as $\exp(-tC)$ belongs to $\mathrm{SU}(n)$. In this sense the transformation (3.7) could be called a canonical transformation. However, in the Hamiltonian formalism we are not accurate in calling it so because canonical (or symplectic) transformations must be defined on $L^2(\mathbb{R}^n)$ so as to leave the symplectic structure (2.3) invariant.

We are now going to give a definite meaning to the operator exp $(it\hat{F}/\lambda)$. In view of (3.7) we first define a one-parameter family of functions $\Phi_{k_1\cdots k_n}(t)$ on $\mathbb{R}^n$ by

$$\Phi_{k_1\cdots k_n}(t) = \left(\sum(e^{tC})_{j_1 1}\bar{z}_{j_1}\right)^{k_1}\cdots\left(\sum(e^{tC})_{j_n n}\bar{z}_{j_n}\right)^{k_n}\varphi_0. \tag{3.8}$$

It is clear that $\Phi_{k_1\cdots k_n}(0) = \bar{z}_1^{k_1}\cdots\bar{z}_n^{k_n}\varphi_0$ and that for all $t\in\mathbb{R}$ the function (3.8) still remains in the eigenspace for $\hat{H}$ assigned by $N = k_1 + \cdots + k_n$. If we look on $\bar{z}_j$ as formal variables, each eigenspaces of $\hat{H}$ is regarded as the space of homogeneous polynomials in $\bar{z}_j$ of degree $N = k_1 + \cdots + k_n$, in which the unitary group $\mathrm{U}(n)$ is represented unitarily and irreducibly.[7] By T we mean the representation.[8] Then we have

$$\Phi_{k_1\cdots k_n}(t) = T(e^{tC})\bar{z}_1^{k_1}\cdots\bar{z}_n^{k_n}\varphi_0. \tag{3.9}$$

Let us consider the representations T's for all N at the same time. These define a one-parameter family of invertible linear mappings T_t from the dense subspace (2.10) of $L^2(\mathbb{R}^n)$ onto itself. We recall that the inner product of functions $\bar{z}_1^{k_1}\cdots\bar{z}_n^{k_n}\varphi_0$ is calculated by using the commutation relations (2.9). Since the transformation (3.7) makes no change in the commutation relations (2.9), the mappings T_t defined above take the complete orthonormal system (2.12a) into another. Therefore, T_t extends uniquely to a unitary operator U_t on $L^2(\mathbb{R}^n)$. Moreover, as T is a representation, the operators U_t, $t\in\mathbb{R}$, form a one-parameter group. Thus we have in particular

$$\Phi_{k_1\cdots k_n}(t) = U_t \bar{z}_1^{k_1}\cdots\bar{z}_n^{k_n}\varphi_0. \tag{3.10}$$

In this respect the domain of $\hat{H}$ is worth noticing.

Remark 3.1: The action of the unitary operator U_t decomposes into unitary transformations of the eigenspaces for $\hat{H}$, so that U_t leaves the domain of $\hat{H}$ invariant, as is readily verified by the spectral decomposition of $\hat{H}$ (see also Ref. 9).

Our next task is to find the infinitesimal generator of U_t. To this end, we calculate the derivative of $\Phi_{k_1\cdots k_n}(t)$ with respect to t to obtain, after a long calculation,

$$\frac{d}{dt}\Phi_{k_1\cdots k_n}(t) = \frac{i}{\lambda}\hat{F}\Phi_{k_1\cdots k_n}(t). \tag{3.11}$$

We notice here only that use has been made of the commutation relations (2.9) together with their immediate consequences and of the fact that $z_j\varphi_0 = 0$.

From Eqs. (3.10) and (3.11) it follows that

$$\frac{d}{dt}U_t f = \frac{i}{\lambda}\hat{F}U_t f \tag{3.12}$$

for any function f in the domain (2.10). Thus we may conclude[10]

Theorem 3.2: The symmetric operator $\hat{F}$ extends to a self-adjoint operator to generate the unitary operator

$$U_t = \exp(it\hat{F}/\lambda), \tag{3.13}$$

where $\hat{F}$ and its extension are not distinguished in notation.

Furthermore, since U_t is a unitary operator and the symplectic form (2.3) is defined through the inner product, one obtains

Theorem 3.3: The operator U_t is a symplectic transformation on $L^2(\mathbb{R}^n)$.

Mention should be here made of angular momentum operators. We discuss the unitary operator (3.13) with the restriction that the coefficient matrix (C_{jk}) of $\hat{F}$ is a real matrix (A_{jk}). Then $\hat{F}$ becomes, from (3.4a), an angular momentum operator

$$\hat{F} = -\frac{\lambda}{i}\sum A_{jk}x_k\frac{\partial}{\partial x_j}. \tag{3.14}$$

Consider the unitary mapping $f\to U_t f$. For simplicity we restrict f within the domain (2.10). From (3.12) and (3.14), U_t satisfies

$$\frac{d}{dt}U_t f = -\sum A_{jk}x_k\frac{\partial}{\partial x_j}U_t f. \tag{3.15}$$

On the other hand, a one-parameter group $\exp(tA)$ acting on $\mathbb{R}^n$ is represented by $f(\exp(-tA)x)$, $x\in\mathbb{R}^n$. Simple calculation yields

$$\frac{d}{dt}f(\exp(-tA)x) = -\sum A_{jk}x_k\frac{\partial}{\partial x_j}f(\exp(-tA)x). \tag{3.16}$$

From (3.15) and (3.16) we see that $U_t f(x)$ and $f(\exp(-tA)x)$ satisfy the same differential equation with the same initial condition at $t=0$. Therefore, we obtain

$$U_t f(x) = f(\exp(-tA)x), \tag{3.17}$$

which extends so as to hold for all f in $L^2(\mathbb{R}^n)$.

We return to (3.7). The operators z_j and $\bar{z}_j$ are considered as vector fields on the domain (2.10). The mapping U_t induces a mapping U_{t*}, the differential,[11] on these vector fields. Because of linearity, the differential U_{t*} is equal to U_t itself.[11] Then we have by definition

$$U_{t*}z_j(\varphi) = U_t z_j U_{-t}(\varphi), \quad U_{t*}\bar{z}_j(\varphi) = U_t\bar{z}_j U_{-t}(\varphi) \tag{3.18}$$

for φ in the domain (2.10). We note here that the domain (2.10) is invariant under U_t. We now look into the right-hand side of (3.18). The following holds on the domain (2.10):

$$U_t z_j U_{-t} = \sum(e^{-tC})_{jk}z_k, \tag{3.19a}$$

$$U_t\bar{z}_j U_{-t} = \sum\overline{(e^{-tC})}_{jk}\bar{z}_k. \tag{3.19b}$$

For both sides of Eqs. (3.19) satisfy as functions of t the same differential equation with the same initial value. Thus we see that (3.7) are transformations induced on vector fields z_j and $\bar{z}_j$ by U_t. From (3.18), U_{t*} preserves the commutation relations (2.9).

So far we have obtained U_t and U_{t*} whose properties are summed up in

Proposition 3.4: The operators U_t and U_{t*} leave invariant the symplectic structure (2.3) and the commutation relations (2.9), respectively.

We are now to discuss the property of U_t. In the Hamiltonian formalism, a symplectic transformation is called a symmetry transformation (or a symmetry for brevity) if it leaves the Hamiltonian function H invariant. We now show that U_t is indeed a symmetry. Let $\exp(-it\hat{H})$ be the unitary operator generated from the Hamiltonian operator $\hat{H}$, which may be formed in the same method as applied for $U_t = \exp(it\hat{F}/\lambda)$. It is clear that $\exp(-it\hat{H})$ and U_s commute when they are operated with on the dense subspace (2.10). Because of unitarity they commute also on $L^2(\mathbb{R}^n)$. On this account one has

$$\langle U_s\varphi, \exp(-it\hat{H})U_s\varphi\rangle = \langle\varphi, \exp(-it\hat{H})\varphi\rangle. \tag{3.20}$$

Differentiating (3.20) with respect to t, we get $H\circ U_s = H$ for φ in the domain of $\hat{H}$ [see (2.5) and Remark 3.1]. Thus we have

Theorem 3.5: The operator U_t is a symmetry of the harmonic oscillator, that is, it leaves invariant both the symplectic structure (2.3) and the Hamiltonian function (2.5).

The infinitesimal version of this theorem is well known. That is, one has $U_{t*}\hat{H} = U_t\hat{H}U_{-t} = \hat{H}$ and $[\hat{F},\hat{H}] = 0$, as long as the commutator makes sense.

We can easily extend Theorem 3.5 to a theorem for the group SU(n) which acts on the basis $\bar{z}_1^{k_1}\cdots\bar{z}_n^{k_n}\varphi_0$ of $L^2(\mathbb{R}^n)$, as in (3.8), in the form

$$\Big(\sum V_{j_1 1}\,\bar{z}_{j_1}\Big)^{k_1}\cdots\Big(\sum V_{j_n n}\,\bar{z}_{j_n}\Big)^{k_n}\varphi_0, \tag{3.21}$$

where $(V_{jm})\in\mathrm{SU}(n)$.

Theorem 3.6: The group $SU(n)$ is represented unitarily in $L^2(\mathbb{R}^n)$ as a symmetry group for the harmonic oscillator.

This theorem could be interpreted as a quantization of the symmetry group $SU(n)$ for the classical harmonic oscillator.

IV. THE FOUR-DIMENSIONAL HARMONIC OSCILLATOR

We have discussed in Ref. 4 a reduction of the energy surface of the four-dimensional classical harmonic oscillator together with the accompanying reduction of the symmetry group. We consider in this section a quantum analog to these reductions. An eigenspace of the Hamiltonian operator $\hat{H}$ is taken to be associated with an energy surface of the classical system, so that our problem amounts to reducing the eigenspace. The representation (3.21) fits our problem well.

Let N_3 be a 4×4 matrix[12] given by

$$N_3 = \begin{pmatrix} K & \\ & K \end{pmatrix} \quad \text{with } K = \begin{pmatrix} & -\frac{1}{2} \\ \frac{1}{2} & \end{pmatrix}. \tag{4.1}$$

By $SO(2)$ we mean the group $\exp(tN_3)$ acting on $\mathbb{R}^4$. We introduce the complex variables

$$\xi = x_1 + ix_2, \quad \eta = x_3 + ix_4 \tag{4.2}$$

to get a concise expression of $SO(2)$ action on $\mathbb{R}^4 = \mathbb{C}^2$ as

$$(\xi,\eta) \to (e^{it/2}\xi,\, e^{it/2}\eta), \quad t\in\mathbb{R}. \tag{4.3}$$

The action (4.3) yields the orbit space

$$\mathbb{R}^4/SO(2) = \mathbb{R}^3, \tag{4.4}$$

which has a close relation to the Hopf fibering $S^3/S^1 = S^2$.[13] Introduction of the local coordinates (r,θ,φ,ψ) by

$$\xi = r^{1/2}e^{i(\psi+\varphi)/2}\cos(\theta/2),\ \eta = r^{1/2}e^{i(\psi-\varphi)/2}\sin(\theta/2) \tag{4.5}$$

allows us to have a fair idea of the orbit space; the coordinates (r,θ,φ) becomes the polar spherical coordinates in $\mathbb{R}^3$.

Since N_3 is real and antisymmetric, we can apply (3.14) and (3.16) to N_3. By the help of (4.3), we find that the operator (3.14) with A replaced by N_3, denoted by $\hat{N}_3$, takes the form

$$\hat{N}_3 = \frac{\lambda}{2i}\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}\right) = -\frac{\lambda}{i}\frac{\partial}{\partial\psi}. \tag{4.6}$$

The reduction of the eigenspace of $\hat{H}$ should be carried out by means of the group $SO(2)$ so as to be compatible with (4.4).

Lemma 4.1: A smooth function f on $\mathbb{R}^4$ is $SO(2)$-invariant if and only if

$$\hat{N}_3 f = 0, \quad f(x) = f(-x). \tag{4.7}$$

Proof: If f is $SO(2)$-invariant, one has $f(\exp(tN_3)x) = f(x)$ for any $x\in\mathbb{R}^4$. Equation (3.16) with (4.6) then gives $\hat{N}_3 f = 0$. The remaining equation of (4.7) is evident from (4.3) with $t = 2\pi$. Conversely, if f satisfies (4.7), it turns out to be independent of ψ and invariant under the inversion $x\to -x$. The inversion invariance implies that f is periodic in φ with the period 2π, as is seen from (4.5). Therefore f can be looked on as a function on $\mathbb{R}^3$. This is a practical idea of the reduced function f^{red} on $\mathbb{R}^3$ determined for $SO(2)$-invariant functions by

$$f^{\text{red}}([x]) = f(x) \quad \text{for } x\in\mathbb{R}^4, \tag{4.8}$$

where $[x]$ denotes the equivalence class of x. Clearly any function on $\mathbb{R}^3$ is pulled back through the natural projection associated with (4.4) to an $SO(2)$-invariant function on $\mathbb{R}^4$. This completes the proof.

We are now to choose $SO(2)$-invariant subspace of the eigenspace for $\hat{H}$. All we need is to find eigenfunctions which satisfy (4.7). First we consider $\hat{N}_3 f = 0$. An easy access to this equation will be given by finding a basis of the eigenspace which diagonalizes the operator $\hat{N}_3$. The following linear transformation of operators z_j and $\bar{z}_j$, which is a quantum analog to that employed in Ref. 4 for the classical system, will be successful;

$$\begin{aligned} w_1 &= z_1 + iz_2, \quad w_2 = z_3 + iz_4, \\ w_3 &= z_1 - iz_2, \quad w_4 = z_3 - iz_4, \end{aligned} \tag{4.9}$$

together with the adjoint operators $\bar{w}_j$. In terms of w_j and $\bar{w}_j$ the operator $\hat{N}_3$ takes the form

$$\hat{N}_3 = \tfrac{1}{8}(w_1\bar{w}_1 + w_2\bar{w}_2 - w_3\bar{w}_3 - w_4\bar{w}_4). \tag{4.10}$$

Since the transformation (4.9) is nondegenerate, the totality of

$$\bar{w}_1^{k_1}\cdots\bar{w}_4^{k_4}\varphi_0 \quad \text{with } N = k_1 + \cdots + k_4, \tag{4.11}$$

where the k_j are nonnegative integers and N is fixed, form a basis of the eigenspace assigned by the nonnegative integer N. We operate with $\hat{N}_3$ on the basis (4.11) to obtain

$$\hat{N}_3\bar{w}_1^{k_1}\cdots\bar{w}_4^{k_4}\varphi_0 = (\lambda/2)(k_1 + k_2 - k_3 - k_4)\bar{w}_1^{k_1}\cdots\bar{w}_4^{k_4}\varphi_0, \tag{4.12}$$

which shows that $\hat{N}_3$ is diagonalized, as was expected.

The condition $f(x) = f(-x)$ is rather easy to treat. The inversion $x\to -x$ gives rise to the inversion of the operators $\bar{z}_j = x_j - \partial/\partial x_j$, and hence that of $\bar{w}_j$, i.e., $\bar{w}_j \to -\bar{w}_j$. The φ_0 is clearly inversion-invariant. Accordingly, we have under the inversion

$$\bar{w}_1^{k_1}\cdots\bar{w}_4^{k_4}\varphi_0 \to (-1)^N\bar{w}_1^{k_1}\cdots\bar{w}_4^{k_4}\varphi_0. \tag{4.13}$$

From (4.12) and (4.13) it follows that for the eigenfunctions (4.11) the conditions (4.7) read

$$\begin{aligned} &k_1 + k_2 - k_3 - k_4 = 0, \\ &k_1 + k_2 + k_3 + k_4 = N = \text{ an even number.} \end{aligned} \tag{4.14}$$

The $SO(2)$-invariant eigenfunctions belonging to (4.11) are assigned by the exponents $(k_1,\dots,k_4)$ satisfying (4.14). Therefore, we find from the number of solutions to (4.14) the following:

Theorem 4.2: In the eigenspace of the four-dimensional harmonic oscillator, there exists a space of $SO(2)$-invariant eigenfunctions, which is of dimension $(N/2 + 1)^2$, where N is a nonnegative integer assigning the energy level.

We proceed to a reduction of the symmetry group $SU(4)$ acting on the eigenspace of $\hat{H}$. Recall that $\hat{N}_3$ is the operator $\hat{F}$ having the coefficient matrix N_3. We see from (3.13) and (3.17) that a function f is $SO(2)$-invariant if and only if $\exp(it\hat{N}_3/\lambda)f = f$ for all t. When the domain of $\exp(it\hat{N}_3/\lambda)$ is restricted to the eigenspace of $\hat{H}$, the operator $\exp(it\hat{N}_3/\lambda)$ becomes $T(\exp(tN_3))$ introduced in (3.9), so that an eigenfunction f is SO(2)-invariant if and only if

$$T(\exp(tN_3))f = f. \tag{4.15}$$

We have obtained in Ref. 4 the subgroup $G = SU(2) \times SU(2)$ of SU(4) which commute with U(1), where U(1) is the group $\exp(tN_3)$ acting on $\mathbb{C}^4$. Let g be an arbitrary element of G. Since G and U(1) commute, one can conclude from (4.15) that for an SO(2)-invariant eigenfunction f one has

$$T(\exp(tN_3))T(g)f = f. \tag{4.16}$$

This means that $T(g)f$ is also SO(2)-invariant. In other words, G acts on the space of SO(2)-invariant eigenfunctions. The action of G, describable by (3.21) with $(V_{jm}) \in G$, has a simpler form when the basis (4.11) is adopted because elements g of G then get the simpler matrix form[4]

$$g = \begin{pmatrix} u_{jk} & \\ & v_{jk} \end{pmatrix} \in SU(2) \times SU(2), \tag{4.17}$$

where (u_{jk}) and (v_{jk}) are 2×2 matrices. The operators $\bar{w}_j$ transform according to (4.17). Thus we have

$$T(g)\bar{w}_1^{k_1} \cdots \bar{w}_4^{k_4}\varphi_0 = \left(\sum_{j=1}^{2} u_{j1}\bar{w}_j\right)^{k_1} \cdots \left(\sum_{j=3}^{4} v_{j4}\bar{w}_j\right)^{k_4}\varphi_0. \tag{4.18}$$

In view of (4.18) we find that inversion $x \to -x$ induces the operator $T(-1)$, where 1 denote the 4×4 unit matrix. Clearly, -1 is an element of (4.17). From this, for an inversion-invariant eigenfunction f one has $T(-1)f = f$ and hence

$$T(-g)f = T(g)f \quad \text{for all } g \in G. \tag{4.19}$$

From (4.19) we conclude that the representation of G induces a representation of $G/\{1, -1\} = SO(4)$.

Theorem 4.3: The group SO(4) acts on the space of SO(2)-invariant eigenfunctions for the four-dimensional harmonic oscillator.

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[1] J. E. Marsden, Arch. Rat. Math. Anal. **28**, 362 (1968).

[2] J. Marsden, *Applications of Global Analysis in Mathematical Physics* (Publish or Perish, Boston, 1974).

[3] P. R. Chernoff and J. E. Marsden, *Properties of Infinite Dimensional Hamiltonian Systems*, Lecture Notes in Mathematics, Vol. 425 (Springer-Verlag, Berlin, 1974).

[4] T. Iwai, J. Math. Phys. **22**, 1628 (1981).

[5] T. Iwai, J. Math. Phys. **23**, 1093 (1982).

[6] The same notation as in the classical system will cause no confusion.

[7] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1950), pp. 139, 299.

[8] G. A. Baker, Jr., Phys. Rev. **103**, 1119 (1956); M. I. Petrashen and E. D. Trifonov, *Applications of Group Theory in Quantum Mechanics* (MIT Press, Cambridge, Mass., 1969), p. 195.

[9] M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis* (Amer. Math. Soc., New York, 1932), p. 262.

[10] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis* (Academic, New York, 1972), p. 269.

[11] S. Lang, *Differential Manifolds* (Addison-Wesley, Reading, Mass., 1972), pp. 8, 106.

[12] We keep the same notation as in Ref. 4. Missing matrix entries are all zero.

[13] J. Dieudonné, *Éléments d'analyse, Tome III*, 2nd ed. (Gauthier-Villars, Paris, 1974), Chap. 16.