

# Madelung\_Representation

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[1]: __author__ = "@Tssp"
__date__ = "17/03/2021"
import sympy as sp
import numpy as np
```

This notebook is aimed to obtain the superfluid hydrodynamic equations of motions through the Madelung representation of the Gross Pitaevskii equation.

```
[10]: x, t, m, hbar = sp.symbols('x t m \hbar', real=True)
psi = sp.Function('\psi', real=False)(x, t) # wavefunction
rho = sp.Function('\rho', real=True)(x, t) # probability density
V = sp.Function('V', real=True)(x, t) # Trap Potential
S = sp.Function('S', real=True)(x, t) # Madelung Ansatz
v = sp.Function('v', real=True)(x, t) # fluid velocity
Pq = sp.Function('P_Q', real=True)(x, t) # Quantum Preassure
g = sp.symbols('g', real=True) # g constant
```

```
[11]: Schr_eq = sp.I * hbar * sp.diff(psi, t) + (hbar**2/(2*m) * sp.diff(psi, x, 2) -
↳ V * psi - g*abs(psi)**2 * psi)
Schr_eq
```

```
[11]: 
$$\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + i\hbar \frac{\partial}{\partial t} \psi(x, t) - g\psi(x, t) |\psi(x, t)|^2 - V(x, t)\psi(x, t)$$

```

Madelung transformation:

$$\psi(x, t) = \sqrt{\rho(x, t)} e^{imS(x, t)/\hbar}$$

```
[12]: Schr_madelung = sp.simplify(Schr_eq.subs({psi:sp.sqrt(rho)*sp.exp(sp.I*m*S/
↳ hbar)})).\
collect(sp.exp(sp.I*m*S/hbar))/sp.exp(sp.I*m*S/hbar)
Schr_madelung
```

```
[12]:
```

$$\frac{\hbar^2 \left( \frac{2 \frac{\partial^2 \rho(x, t)}{\partial x^2} \rho(x, t) - \left( \frac{\partial}{\partial x} \rho(x, t) \right)^2}{4 \sqrt{\rho(x, t)}} + \frac{m \left( i \frac{\partial^2 S(x, t)}{\partial x^2} - \frac{m \left( \frac{\partial S(x, t)}{\partial x} \right)^2}{\hbar} \right) \sqrt{\rho(x, t)}}{\hbar} + \frac{im \frac{\partial}{\partial x} S(x, t) \frac{\partial}{\partial x} \rho(x, t)}{\hbar \sqrt{\rho(x, t)}} \right)}{2m} + i\hbar \left( \frac{\frac{\partial}{\partial t} \rho(x, t)}{2 \sqrt{\rho(x, t)}} + \frac{im \sqrt{\rho(x, t)} \frac{\partial}{\partial t} S(x, t)}{\hbar} \right) - g \sqrt{\rho(x, t)} \left| \sqrt{\rho(x, t)} \right|^2 - V(x, t) \sqrt{\rho(x, t)}$$

+

```
[13]: Schr_madelung = Schr_madelung.expand()
      Schr_madelung
```

$$[13]: \frac{\hbar^2 \frac{\partial^2}{\partial x^2} \rho(x, t)}{4m \sqrt{\rho(x, t)}} - \frac{\hbar^2 \left( \frac{\partial}{\partial x} \rho(x, t) \right)^2}{8m \rho^{\frac{3}{2}}(x, t)} + \frac{i\hbar \sqrt{\rho(x, t)} \frac{\partial^2}{\partial x^2} S(x, t)}{2} + \frac{i\hbar \frac{\partial}{\partial x} S(x, t) \frac{\partial}{\partial x} \rho(x, t)}{2 \sqrt{\rho(x, t)}} + \frac{i\hbar \frac{\partial}{\partial t} \rho(x, t)}{2 \sqrt{\rho(x, t)}} - g \sqrt{\rho(x, t)} \left| \sqrt{\rho(x, t)} \right|^2 - m \sqrt{\rho(x, t)} \frac{\partial}{\partial t} S(x, t) - \frac{m \sqrt{\rho(x, t)} \left( \frac{\partial}{\partial x} S(x, t) \right)^2}{2} - V(x, t) \sqrt{\rho(x, t)}$$

```
[14]: Im_eq = Schr_madelung.coeff(sp.I, n=1)
      Im_eq
```

$$[14]: \frac{\hbar \sqrt{\rho(x, t)} \frac{\partial^2}{\partial x^2} S(x, t)}{2} + \frac{\hbar \frac{\partial}{\partial x} S(x, t) \frac{\partial}{\partial x} \rho(x, t)}{2 \sqrt{\rho(x, t)}} + \frac{\hbar \frac{\partial}{\partial t} \rho(x, t)}{2 \sqrt{\rho(x, t)}}$$

## 0.1 Imaginary part

Defining:

$$\vec{v} = \frac{\partial S(x, t)}{\partial x}$$

```
[15]: Im_eq = (Im_eq * 2 * m/hbar* sp.sqrt(rho)).expand()
      Im_eq
```

$$[15]: m \rho(x, t) \frac{\partial^2}{\partial x^2} S(x, t) + m \frac{\partial}{\partial x} S(x, t) \frac{\partial}{\partial x} \rho(x, t) + m \frac{\partial}{\partial t} \rho(x, t)$$

```
[16]: sp.diff(rho*sp.diff(S, x), x)
```

$$[16]: \rho(x, t) \frac{\partial^2}{\partial x^2} S(x, t) + \frac{\partial}{\partial x} S(x, t) \frac{\partial}{\partial x} \rho(x, t)$$

```
[17]: Im_eq.subs({rho*sp.diff(S, x, 2): sp.diff(rho*sp.diff(S, x), x) - sp.diff(S, x)
      ↪ sp.diff(rho, x)})
```

$$[17]: m \rho(x, t) \frac{\partial^2}{\partial x^2} S(x, t) + m \frac{\partial}{\partial x} S(x, t) \frac{\partial}{\partial x} \rho(x, t) + m \frac{\partial}{\partial t} \rho(x, t)$$

$$m \frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x} \left( \left( \rho \frac{\partial}{\partial x} S \right) - S - \rho + S + \rho \right) = 0$$

Or:

$$m \frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x} \left( \left( \rho \frac{\partial}{\partial x} S \right) \right) = 0$$

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x} (\rho v) = 0$$

A continuity equation, in 3D

$$\frac{\partial \rho}{\partial t} - \nabla (\rho v) = 0$$

## 0.2 Real part

In this part we are going to treat  $\sqrt{\rho}$  as an unique variable in order to forbid Python to literally express the derivative and therefore simplify the equations

```
[18]: sqrt_rho = sp.Function('\sqrt{\rho}', real=True)(x, t)
```

```
[19]: Schr_madelung = sp.simplify(Schr_eq.subs({psi:sqrt_rho*sp.exp(sp.I*m*S/hbar)})).
    → \
    collect(sp.exp(sp.I*m*S/hbar))/sp.exp(sp.I*m*S/hbar)
    Schr_madelung
```

$$\begin{aligned} [19]: \quad & \hbar^2 \left( \frac{\partial^2}{\partial x^2} \sqrt{\rho}(x, t) + \frac{m \left( i \frac{\partial^2}{\partial x^2} S(x, t) - \frac{m \left( \frac{\partial}{\partial x} S(x, t) \right)^2}{\hbar} \right) \sqrt{\rho}(x, t)}{\hbar} + \frac{2im \frac{\partial}{\partial x} S(x, t) \frac{\partial}{\partial x} \sqrt{\rho}(x, t)}{\hbar} \right) \\ & + i\hbar \left( \frac{\partial}{\partial t} \sqrt{\rho}(x, t) + \frac{im \sqrt{\rho}(x, t) \frac{\partial}{\partial t} S(x, t)}{\hbar} \right) - g\sqrt{\rho}^3(x, t) - V(x, t)\sqrt{\rho}(x, t) \end{aligned}$$

```
[20]: Schr_madelung = Schr_madelung.expand()
    Schr_madelung
```

$$\begin{aligned} [20]: \quad & \frac{\hbar^2 \frac{\partial^2}{\partial x^2} \sqrt{\rho}(x, t)}{2m} + \frac{i\hbar \sqrt{\rho}(x, t) \frac{\partial^2}{\partial x^2} S(x, t)}{2} + i\hbar \frac{\partial}{\partial x} S(x, t) \frac{\partial}{\partial x} \sqrt{\rho}(x, t) + i\hbar \frac{\partial}{\partial t} \sqrt{\rho}(x, t) - g\sqrt{\rho}^3(x, t) - \\ & m\sqrt{\rho}(x, t) \frac{\partial}{\partial t} S(x, t) - \frac{m\sqrt{\rho}(x, t) \left( \frac{\partial}{\partial x} S(x, t) \right)^2}{2} - V(x, t)\sqrt{\rho}(x, t) \end{aligned}$$

```
[25]: Re_eq = Schr_madelung.coeff(sp.I, n=0)
    Re_eq = sp.simplify(-Re_eq/(sqrt_rho*m))
    Re_eq
```

$$[25]: \quad -\frac{\hbar^2 \frac{\partial^2}{\partial x^2} \sqrt{\rho}(x, t)}{2m^2 \sqrt{\rho}(x, t)} + \frac{g\sqrt{\rho}^2(x, t)}{m} + \frac{\partial}{\partial t} S(x, t) + \frac{\left( \frac{\partial}{\partial x} S(x, t) \right)^2}{2} + \frac{V(x, t)}{m}$$

The negative term corresponds with the quantum preassure contribution to the fluid. With some manipulation we get the superfluid hydrodynamic equations:

```
[26]: Re_eq = Re_eq.subs({-hbar**2*sp.diff(sqrt_rho, x, 2)/(2*m**2*sqrt_rho): Pq})
    Re_eq
```

$$[26]: \quad \frac{g\sqrt{\rho}^2(x, t)}{m} + P_Q(x, t) + \frac{\partial}{\partial t} S(x, t) + \frac{\left( \frac{\partial}{\partial x} S(x, t) \right)^2}{2} + \frac{V(x, t)}{m}$$

```
[27]: m*rho*sp.diff(Re_eq, x).subs({sp.diff(S, x): v})
```

$$[27]: \quad m \left( \frac{2g\sqrt{\rho}(x, t) \frac{\partial}{\partial x} \sqrt{\rho}(x, t)}{m} + v(x, t) \frac{\partial}{\partial x} v(x, t) + \frac{\partial}{\partial x} P_Q(x, t) + \frac{\partial}{\partial t} v(x, t) + \frac{\frac{\partial}{\partial x} V(x, t)}{m} \right) \rho(x, t)$$

or:

$$m\rho\left(\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right) = -m\rho\nabla P_Q - \rho\nabla V - g\nabla\rho$$

Notice that the pressure depends only on the density. This property makes the condensate a barotropic fluid; as a consequence, surfaces of constant pressure are also surfaces of constant density. The quantum pressure is a pure quantum effect, and vanishes if we set Planck's constant equal to zero. It has the same origin as the quantum kinetic energy, i.e. zero point motion, which creates a pressure that opposes any 'squashing' or 'bending' of the condensate. In a uniform condensate the quantum pressure is zero because  $n$  is constant.

This equation is very similar to the classical Euler equation for an inviscid fluid. If in addition, the trapping potential is absent ( $V = 0$ ) then the equation reduces to the classical Euler equation, which describes the motion of a classical fluid without viscosity.