# Harmonic approximation of optical traps

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T. Sánchez-Pastor

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An optical lattice in one spatial direction can be formed by overlapping two counter-propagating laser beams. It can be shown that the potential in this kind of systems depends on:

- The polarizability $\alpha$.
- The wavelength of the laser $k = \frac{2\pi}{\lambda}$.
- The intensity of the laser in the j-direction $I_{o,j}$.

The potential depth in the j-direction is defined as follows:

$$V_{0,j} = \frac{-Re(\alpha)}{2c\epsilon_0} \sum_{j=x,y,z} I_{0,j} \sin^2(k_j j). \tag{1}$$

Therefore, the optical lattice potential for a system of two atoms is:

$$\boxed{V_{oL} = \sum_{i=1,2} \sum_{j=x,y,z} V_{0,j}^{(i)} \sin^2(k_j j_i)} \tag{2}$$

The harmonic trap potential is central in this studies and can be ovtained by expanding the optical lattice potencial in a Taylor Series up to the second order:

$$V_{oL} \approx \sum_{i=1,2} \sum_{j=x,y,z} V_{0,j}^{(i)} k_j^2 j_i^2 \tag{3}$$

In ultracold atoms systems is always required to separate the contributions of the relative motion, the center of mass and the coupling between them. The CM coordinates are:

$$\begin{cases} \vec{r} &= \vec{r_1} - \vec{r_2} \\ \vec{R} &= \mu_1 \vec{r_1} + \mu_2 \vec{r_2} \end{cases} \tag{4}$$

where, $\mu_{1,2} = \frac{m_{1,2}}{m_1+m_2}$. Taking advantage of the symmetry of the spatial variables, we are going to perform this calculation only in one dimension and generalize it to the other two dimensions:

$$V_{oL} = \sum_{i=1,2} V_{0,x}^{(i)} k_x^2 x_i^2 \tag{5}$$

$$\begin{cases} x &= x_1 - x_2 \\ X &= \mu_1 x_1 + \mu_2 x_2 \end{cases} \tag{6}$$

$$x_1 = x + x_2 \rightarrow X = \mu_1 x + x_2(\mu_1 + \mu_2)$$

Hence, using appendix A.1 and A.2:

$$\begin{aligned} x_2 &= X - \mu_1 x && (7) \\ x_1 &= X + x(1-\mu_1) = X + \mu_2 x && (8) \end{aligned}$$

We need to calculate an expression for $x_1^2$ and $x_2^2$:

$$\begin{aligned} x_1^2 &= (X+\mu_2 x)^2 = X^2 + \mu_2^2 x^2 + 2xX\mu_2 && (9) \\ x_2^2 &= (X-\mu_1 x)^2 = \underbrace{X^2}_{c.m} + \underbrace{\mu_1^2 x^2}_{rel} - \underbrace{2xX\mu_1}_{c.m-rel} && (10) \end{aligned}$$

Plugging the last equations in eq.5 we obtain the three contributions to the harmonic potential:

$$\begin{aligned} V_{cm} &= \sum_{i=1,2} V_{0,x}^{(i)} k_x^2 X^2 && (11) \\ V_{rel} &= \sum_{i=1,2} V_{0,x}^{(i)} k_x^2 \mu_\eta^2 x^2, \ \ \eta = 2, 1 && (12) \\ V_{c.m-rel} &= 2k_x^2 xX \left[ V_{0,x}^{(1)} \mu_2 - V_{0,x}^{(2)} \mu_1 \right] && (13) \end{aligned}$$

Finally, for the three spatial dimensions:

$$\begin{aligned} V_{cm} &= \sum_{i=1,2} \sum_{j=x,y,z} V_{0,j}^{(i)} k_j^2 R_j^2 && (14) \\ V_{rel} &= \sum_{i=1,2} \sum_{j=x,y,z} V_{0,j}^{(i)} k_j^2 \mu_\eta^2 r_j^2, \ \ \eta = 2, 1 && (15) \\ V_{c.m-rel} &= 2 \sum_{j=x,y,z} k_j^2 r_j R_j \left[ V_{0,j}^{(1)} \mu_2 - V_{0,j}^{(2)} \mu_1 \right] && (16) \end{aligned}$$

The discrepancy with Sala's thesis is that the c.m-rel coupling is multiplied by $\frac{1}{2}$ and not by 2.

# Appendix

## A.1: $\mu_1 + \mu_2$

$$\mu_1 + \mu_2 = \frac{m_1}{m_1 + m_2} + \frac{m_2}{m_1 + m_2} = 1$$

## A.2: $1 - \mu_1$

$$1 - \mu_1 = 1 - \frac{m_1}{m_1 + m_2} = \frac{m_1 + m_2 - m_1}{m_1 + m_2} = \mu_2$$