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Normalization theorems for bilateral natural deduction systems

Dennis Akdasli

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PD Dr. Luca Tranchini
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Abstract

Rumfitt presented a bilateral natural deduction system in [Rum00]. We study the relationships between this system and two of its fragments that are considered in programming language literature. The first fragment we consider is from Abe and Kimura [AK22], the second is from Lovas and Crary [LC06]. Both develop lambda-calculi that correspond to these fragments and prove their normalization.

In this thesis, we first study the relationship between Rumfitt's system and two of its fragments by giving embeddings and prove that these embeddings preserve the reduction relation. Second, we prove the normalization of all three systems in a uniform manner. Our proof strategy differs from those given in [AK22, LC06]. We prove by induction over ranks that order the different kinds of redexes. The uniformity of our approach illustrates how the proofs of the fragments contribute to the proof of the full system. Lastly, we prove the subformula property for the full system.

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Chapter 1

Introduction

Gentzen introduced his system for natural deduction NK in [Gen35]. With this he aimed to construct a formalism that is as close as possible to actual deduction [Gen35, p.176]. NK is a system for classical logic to prove assertions. It employs two main types of rules: introduction rules, which state the premises to prove assertions of formulas with a logical connective as main connective, and elimination rules, which restrict what can be inferred from a proof of an assertion with a certain logical connective as main connective. There is one more rule called *Reductio* that is neither an introduction nor an elimination rule.

Dummett proposes a constraint on every theory of assertion, called *proof-theoretic harmony*, expressing the idea that both the premises for asserting a sentence and the consequences that can be inferred by such an assertion need to be *balanced* [Dum91]. Thus, introduction and elimination rules of connectives in natural deduction should fulfill this as well. However, it is a well-known defect of natural deduction that the rules for negation do not fulfill this constraint.[1] This leads intuitionists to conclude "that classical rules such as double negation elimination are not logical (or that they are in some other sense defective), and that the logical rules we should adopt are those of intuitionistic logic" [Mur18, p.398].

Rumfitt proposes to overcome this difficulty by developing a natural deduction system for classical logic that represents proofs of assertions and rejections equally. Because of this equality such systems are called *bilateral*. To represent proofs of assertions and refutations directly, bilateral logic extends the language by two signs for formulas $+$ and $-$.[2] Then $+\,A$ is an abbreviation for "Is it the case that A? Yes" and $-\,A$ is an abbreviation for "Is it the case

[1] To be precise, there are many different formulations of *harmony* [Ste13]; all standard formulations of it entail the disharmony of negation.

[2] For arbitrary signs, we will use s as a metavariable and $\overline{s}$ as the inverse of s.

that A? No" [Rum00, p.800].[3] Derivations ending with $+\ A$ give a proof of the assertion of A. Derivations ending with $-\ A$ give a proof of the rejection of A [AK22, p.7]. Thus, following Rumfitt, every connective $\circ$ has four kinds of inference rules: positive introduction and elimination rules and negative introduction and elimination rules [Rum00]. These rules determine when we can infer a proof or rejection of a formula with $\circ$ as the main connective, and what we can infer from a given proof of an assertion or proof of a rejection of a formula with $\circ$ as the main connective.[4] The *harmonious* rules for negation in his system are as follows:[5]

$$\frac{-\ A}{+\ \neg A}\,(\neg I_+) \qquad \frac{+\ \neg A}{-\ A}\,(\neg E_+)$$

$$\frac{+\ A}{-\ \neg A}\,(\neg I_-) \qquad \frac{-\ \neg A}{+\ A}\,(\neg E_-)$$

Moreover, Rumfitt has three rules called *co-ordination principles.* They coordinate the relationship between the proof of an assertion and proof of a rejection of a formula [Rum00]. These rules involve another kind of deductive entity, next to signed formulas, called contradictions $\perp$.

Another argument in favour of bilateral natural deduction systems is the following: proofs and rules in natural deduction for classical logic should emphasize the duality of conjunction and disjunction. Looking at the inference rules for conjunction and disjunction in classical natural deduction [Pra65, p.20], the rules do not make this duality apparent:

$$\frac{A \qquad B}{A \wedge B}\,(\wedge I) \qquad \frac{A \wedge B}{A}\,(\wedge E) \qquad \frac{A \wedge B}{B}\,(\wedge E)$$

$$\frac{A}{A \vee B}\,(\vee I) \qquad \frac{A}{A \vee B}\,(\vee I) \qquad \frac{A \vee B \qquad \begin{matrix}[\mathrm{A}]\\ C\end{matrix} \qquad \begin{matrix}[\mathrm{B}]\\ C\end{matrix}}{C}\,(\vee E)$$

D'Agostino et. al. suggest the possible culprit to be in the "external" adding of the *law of excluded middle* to the intuitionistic calculus [DGM20, p.296f.]. This leads to a natural deduction system for classical logic, where the introduction and elimination rules for the logical connectives are "faithful to the intuitionistic meaning, but not to their classical meaning" [DGM20, p.293]. Hence, the classical rules cannot represent the classical sense of these operators and their inner duality.

[3] In this context, a formula without signs is a declarative sentence.

[4] The structure of the rules and the dualities between the connectives are motivated in the remark on Definition 1.

[5] While the introduction and elimination rules for negation are harmonious in the above sense, other difficulties arise in Rumfitt's natural deduction system in regard to proof-theoretic harmony, see [Fer08, Rum08, Kü21a, dVIS22].

Bilateral natural deduction systems on the other hand may provide this inner symmetry and might be closer to the classical meaning of the logical operators. For this, compare the dualities between the connectives in the inference rules in Fig. 3.2. These dualities also reflect in dual proofs of the DeMorgan's laws below:

$$\dfrac{\dfrac{\overset{x}{+\,\neg A \vee \neg B} \qquad \dfrac{\dfrac{\overset{y}{+\,\neg A}}{-\,A}\,(\neg E_+)}{-\,A \wedge B}\,(\wedge I_-) \qquad \dfrac{\dfrac{\overset{z}{+\,\neg B}}{-\,B}\,(\neg E_+)}{-\,A \wedge B}\,(\wedge I_-)}{-\,A \wedge B}\,(\vee E_+)^{y,z}}{+\,\neg(A \wedge B)}\,(\neg I_+)$$

$$\dfrac{\dfrac{\overset{x}{+\,\neg(A \wedge B)}}{-\,A \wedge B}\,(\neg E_+) \qquad \dfrac{\dfrac{\overset{\dot{y}}{-\,A}}{+\,\neg A}\,(\neg I_+)}{+\,\neg A \vee \neg B}\,(\vee I_+) \qquad \dfrac{\dfrac{\overset{\dot{z}}{-\,b}}{+\,\neg B}\,(\neg I_+)}{+\,\neg A \vee \neg B}\,(\vee I_+)}{+\,\neg A \vee \neg B}\,(\wedge E_-)^{\dot{y},\dot{z}}$$

From the *Curry-Howard isomorphism* [CFC$^+$58, How80], we know that proofs of assertions can be understood as programs. In this context Gentzen's *natural deduction* is isomorphic to the typed λ-calculus. Going further in the Curry-Howard isomorphism one might ask, what could be the computational content of refutations and bilateral natural deduction systems. Refutations can be understood as continuations or contexts that consume expressions [Gri89, Mur91, Gir91]. Hence, we can distinguish between *expressions* that correspond to proofs of assertions and *continuations* that correspond to refutations. We call languages based on bilateral natural deduction systems symmetric, due to the symmetric representations of expressions and continuations. In the literature on programming languages there are two such symmetric λ-calculi based on bilateralism. The first was developed by Lovas and Crary [LC06] and the second by Abe and Kimura [AK22]. Both correspond to fragments of Rumfitt's system, except that Abe and Kimura extend their system by the *but-not type*, which is dual to the *function type*.

In this thesis we aim to connect the literature in the field of programming language based on bilateralism with the literature on bilateralism in philosophical logic. To fulfill this goal we set these fragments in relation to the full system by giving embeddings and proving that the reductions are preserved by embedding. Moreover we provide proofs of the normalization of all three systems in a uniform manner. We use a general proof strategy—different from those given in [AK22, LC06]—by induction over ranks that order the different kinds of redexes. This uniformity of our approach will illustrate how the proofs of the fragments contribute to the proof of the full system. At last, we prove the subformula property of the full system.

Overview

The rest of this thesis is structured as follows:

- In Chapter 2, we develop a symmetric λ-calculus with typing and reduction rules. This is an extension of Abe and Kimura's λ-calculus by sum, product and negation types, including their respective terms, typing and reduction rules [AK22].

- In Chapter 3, we develop Rumfitt's bilateral natural deduction system extended by a *but-not* connective. We show that this system can be embedded into a fragment that is isomorphic to the symmetric λ-calculus presented in Chapter 2 and into a fragment that corresponds to Lovas and Crary's symmetric λ-calculus extended by implication and but-not connectives. Thereafter, we show that this embedding is reduction preserving.

- In Chapter 4, we characterize the normal forms and prove the normalization of these two fragments.

- In Chapter 5, we prove the normalization and the subformula property of the full bilateral natural deduction system.

- We conclude our results in Chapter 6.

Chapter 2

Symmetric λ-calculus

In this chapter, we develop a symmetric language with first-class configurations.[1] The language is called *symmetric*, because expressions and continuations are both handled as terms. We begin by giving a definition of symmetric λ-calculus terms (λTERMS). Following that, we proceed with the typing rules and a reduction relation.

2.1 Syntax

A summary of λTERMS is given in Fig. 2.1. We start with some useful definitions of variable names and metavariables, and thereafter use them to define λTERMS.

2.1.1 Names

$x, y, z, x_0, x_1, \ldots$ are variable names for expressions. Continuation variables share the same namespace; however, they are distinguished by dots above the variables.

Further, we define **ExpVar** as the set of all producer variables and **ConVar** as the set of all continuation variables.

$$x, y, z, x_0, x_1, \ldots \in \mathbf{ExpVar}$$
$$\dot{x}, \dot{y}, \dot{z}, \dot{x}_0, \dot{x}_1, \ldots \in \mathbf{ConVar}$$

2.1.2 Terms

As mentioned in the introduction, continuations and configurations are handled as first-class citizens. Therefore, we get three kinds of terms:

[1] As mentioned in the overview, this is an extension of Abe and Kimura's λ-calculus by sum, product and negation types [AK22].

expressions, continuations, and configurations.

$$B ::= E \mid C \mid D \text{ terms}$$

We write B, E, C, D (possibly with subscripts and primes) as metavariables for terms, expressions, continuations and configurations, respectively.

For expressions and continuations, two kinds of variables are necessary.

$$E ::= x \mid \ldots$$
$$C ::= \dot{x} \mid \ldots$$

Now that we established variables in our syntax, we can write down λ-abstractions for variables within a term. Thus, we obtain $\lambda x.E$ as an expression that abstracts the expression variable x from E. Similarly $\lambda \dot{x}.C$ is a continuation, abstracting over the continuation variable $\dot{x}$. Applications of two expressions or continuations are written EE or CC, respectively.

$$E ::= \ldots \mid \lambda x.E \mid EE$$
$$C ::= \ldots \mid \lambda \dot{x}.C \mid CC$$

Furthermore, we want to extend our language with pairs and projections. Again, separated into expressions and continuations such that pairs of expressions are denoted in angular brackets $\langle E, E\rangle$ and pairs of continuations in square brackets $[C, C]$. Projections of pairs are marked by $\wedge$ or $\vee$, respectively. The conjunction symbol is used for projections on expressions and disjunction symbol for projections on continuations.

$$E ::= \ldots \mid \langle E, E\rangle \mid E.1^{\wedge} \mid E.2^{\wedge}$$
$$C ::= \ldots \mid [C, C] \mid C.1^{\vee} \mid C.2^{\vee}$$

A continuation C can be reified as an expression by marking it with "not (C)" [LC06, p.9]. This step should also be reversible. Thus, the reverse operation "unpack (E)" is a continuation.

$$E ::= \ldots \mid \text{not}\,(C)$$
$$C ::= \ldots \mid \text{unpack}\,(E)$$

Configurations build a new category of terms next to expressions and continuations. They can be interpreted as "the state of a computation" [LC06, p.2]. Configurations appear in the form $\langle E \mid C\rangle$, where E is an expressions of a type and C a continuation of the same type.

$$D ::= \langle E \mid C\rangle$$

Besides expressions and continuations, we can also abstract over configurations by expression variables x or continuation variables $\dot{x}$. Then, $\mu\dot{x}.D$ becomes an expression waiting for its continuation. This also explains why it is considered as an expression. Dually $\mu x.D$ is a continuation waiting for an expression to further evaluate D.

$$E ::= \ldots \mid \mu\dot{x}.D$$
$$C ::= \ldots \mid \mu x.D$$

With this we have all terms of our language. The notion of free and bound variables as well as capture-avoiding substitution are defined as usual. We indicate the result of substituting B' for x in B as $B[x/B']$.

$x, y \in \mathbf{ExpVar}$ $\dot{x}, \dot{y} \in \mathbf{ConVar}$			variable names
E	::=	$x \mid \lambda x.E \mid EE \mid \mu\dot{x}.D$	expressions
		$\mid \langle E, E\rangle \mid E.1^{\wedge} \mid E.2^{\wedge} \mid \mathrm{not}\,(C)$	
C	::=	$\dot{x} \mid \lambda\dot{x}.C \mid CC \mid \mu x.D$	continuations
		$\mid [C, C] \mid C.1^{\vee} \mid C.2^{\vee} \mid \mathrm{unpack}\,(E)$	
D	::=	$\langle E \mid C\rangle$	configuration
B	::=	$E \mid C \mid D$	terms

Figure 2.1: Syntax specifications for λTERMS.

2.2 Types

In this section we introduce types, signed *typing judgments*, and thereafter define the *typing rules*.

2.2.1 Types in λterms

Type variables are denoted with p. As types we want to have a symmetric *function type*, a *product type*, a *sum type* and a *negation type* noted with their corresponding connectives from the *Curry-Howard isomorphism*. The *negation type* and terms with negation type are inspired from Lovas and Crary's primitive implementation of negation into their language [LC06].

$$A ::= p \mid A \to A \mid A \leftarrow A \mid A \wedge A \mid A \vee A \mid \neg A \qquad \text{types}$$

The relation between these types and the introduced terms will be defined by our type system in Section 2.2.3.

2.2.2 Typing judgments

A typing context is a sequence of expression and continuation variables and their types, in which every variable occurs at most once.

$$\Gamma ::= \emptyset \mid \Gamma, x : A \mid \Gamma, \dot{x} : A \qquad \text{Typing context}$$

As mentioned in the introduction, the symmetric language is based on a fragment of Rumfitt's bilateral natural deduction system [Rum00]. In bilateral natural deduction we can assert propositions $+\ A$ or reject them $-\ A$. By the Curry-Howard isomorphism these correspond to typing judgments of expressions $\Gamma \vdash_+ E : A$ and continuations $\Gamma \vdash_- C : A$. Configurations receive the type $\bot$ and typing judgments of the form $\Gamma \vdash_o \langle E \mid C \rangle : \bot$, if E and C have the same type. Consequently, the Curry-Howard interpretation of a configuration can be read as a contradiction ($\bot$), since E is a proof of some assertion A and C is a proof of the refutation of A [Zei08]. To summarize, we end up with the following kinds of typing judgments:

- for expressions: $\Gamma \vdash_+ E : A$,
- for continuations: $\Gamma \vdash_- C : A$,
- for configurations: $\Gamma \vdash_o \langle E \mid C \rangle : \bot$, and
- for a uniform notation of the above: $\Gamma \vdash B : A$.

2.2.3 Typing rules

A summary of the typing rules is in Fig. 2.2.[2] We require typing rules for the configurations in our language.

A configuration consists of two halves: the first halve is an expression, the second is a continuation. A continuation can be interpreted as *consumer* of type A, i.e. it "receives information of type A as its input" [DA18, p.19]. A expressions can be interpreted as *producer* of type A, i.e. it "sends information of type A as its output" [DA18, p.19]. Thus, for a well-typed configuration both expression and continuation need to have the same type.

$$\frac{\Gamma \vdash_+ E : A \qquad \Delta \vdash_- C : A}{\Gamma, \Delta \vdash_o \langle E \mid C \rangle : \bot}\ (\text{Nc.})$$

In an evaluation step of $\langle E \mid \mu x.D \rangle$ the continuation $\mu x.D$ "receives information of type A as its input" [DA18, p.19], if E has type A. Thus, the type of $\mu x.D$ is A, if the configuration D we abstract over is well-typed and x is in the typing context of D and has type A. The "R" in the rule name is an abbreviation for *Reductio*.

$$\frac{\Gamma, x : A \vdash_o D : \bot}{\Gamma \vdash_- \mu x.D : A}\ (\text{R}_-)$$

[2] In the following we use the variable i ranging over $\{0, 1\}$.

The typing rule for $\mu\dot{x}.D$ is dual to this.

The typing rules for the other expressions are defined in a standard manner. From the duality of the connectives, we can infer the typing rules for the continuations. To compare just two of them, given two expressions E_0 and E_1 with types A_0 and A_1, respectively, we can infer that the expression $\langle E_0, E_1\rangle$ can be typed with the product type $A_0 \wedge A_1$.

$$\frac{\Gamma \vdash_+ E_0 : A_0 \qquad \Delta \vdash_+ E_1 : A_1}{\Gamma, \Delta \vdash_+ \langle E_0, E_1\rangle : A_0 \wedge A_1}\,(\wedge I_+)$$

Dually, we can also construct a well-typed pair of continuations $[C_0, C_1]$ from two well-typed continuations C_0 and C_1.

$$\frac{\Gamma \vdash_- C_0 : A_0 \qquad \Delta \vdash_- C_1 : A_1}{\Gamma, \Delta \vdash_- [C_0, C_1] : A_0 \vee A_1}\,(\vee I_-)$$

This way, the symmetry of the language is once again emphasized, as the rules for $\to$ and $\wedge$ are dual to those of $\leftarrow$ and $\vee$, respectively.

The *substitution lemma* holds [AK22, p.5].

Lemma 1 (Substitution λTERMS)**.** *For all $E, C, B \in \lambda$TERMS:*

1. *if $\Gamma \vdash_+ E : A$ and $\Delta, x : A \vdash B : A'$ are derivable, then so is $\Gamma, \Delta \vdash B[x/E] : A'$.*
2. *if $\Gamma \vdash_- C : A$ and $\Delta, \dot{x} : A \vdash B : A'$ are derivable, then so is $\Gamma, \Delta \vdash B[\dot{x}/C] : A'$.*

Proof. The proof is as usual by induction on the derivations of $\Delta, x : A \vdash B : A'$ and $\Delta, \dot{x} : A \vdash B : A'$, respectively. □

Example 1. For every well-typed expression E, continuation C_0 and C_1 of type A_0, A_0 and A_1, respectively, it holds that
$\Gamma \vdash_o \langle \mu\dot{x}.\langle E \mid \dot{x}.1^{\vee}\rangle \mid [C_0, C_1]\rangle : \bot$.

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash_+ E : A_0 \qquad \dfrac{\dfrac{}{\dot{x} : A_0 \vee A_1 \vdash_- \dot{x} : A_0 \vee A_1}\,(\mathrm{Var}_-)}{\dot{x} : A_0 \vee A_1 \vdash_- \dot{x}.1^{\vee} : A_0}\,(\vee E_-)}{\Gamma, \dot{x} : A_0 \vee A_1 \vdash_o \langle E \mid \dot{x}.1^{\vee}\rangle : \bot}\,(\mathrm{Nc.})}{\Gamma \vdash_+ \mu\dot{x}.\langle E \mid \dot{x}.1^{\vee}\rangle : A_0 \vee A_1}\,(\mathrm{R}_+)^1 \qquad \dfrac{\Gamma \vdash_- C_0 : A_0 \qquad \Gamma \vdash_- C_1 : A_1}{\Gamma \vdash_- [C_0, C_1] : A_0 \vee A_1}\,(\vee I_-)}{\Gamma \vdash_o \langle \mu\dot{x}.\langle E \mid \dot{x}.1^{\vee}\rangle \mid [C_0, C_1]\rangle : \bot}\,(\mathrm{Nc.})$$

$$\frac{\Gamma \vdash_+ E : A \qquad \Delta \vdash_- C : A}{\Gamma, \Delta \vdash_o \langle E \mid C \rangle : \bot}\;(\mathrm{Nc.})$$

$$\frac{\Gamma, \dot{x} : A \vdash_o D : \bot}{\Gamma \vdash_+ \mu\dot{x}.D : A}\;(\mathrm{R}_+) \qquad \frac{\Gamma, x : A \vdash_o D : \bot}{\Gamma \vdash_- \mu x.D : A}\;(\mathrm{R}_-)$$

$$\frac{}{x : A \vdash_+ x : A}\;(\mathrm{Var}_+) \qquad \frac{}{\dot{x} : A \vdash_- \dot{x} : A}\;(\mathrm{Var}_-)$$

$$\frac{\Gamma, x : A_0 \vdash_+ E : A_1}{\Gamma \vdash_+ \lambda x.E : A_0 \to A_1}\;(\to I_+) \qquad \frac{\Gamma, \dot{x} : A_1 \vdash_- C : A_0}{\Gamma \vdash_- \lambda\dot{x}.C : A_0 \leftarrow A_1}\;(\leftarrow I_-)$$

$$\frac{\Gamma \vdash_+ E_0 : A_0 \to A_1 \qquad \Delta \vdash_+ E_1 : A_0}{\Gamma, \Delta \vdash_+ E_0 E_1 : A_1}\;(\to E_+)$$

$$\frac{\Gamma \vdash_- C_0 : A_0 \leftarrow A_1 \qquad \Delta \vdash_- C_1 : A_1}{\Gamma, \Delta \vdash_- C_0 C_1 : A_0}\;(\leftarrow E_-)$$

$$\frac{\Gamma \vdash_+ E_0 : A_0 \qquad \Delta \vdash_+ E_1 : A_1}{\Gamma, \Delta \vdash_+ \langle E_0, E_1 \rangle : A_0 \wedge A_1}\;(\wedge I_+) \qquad \frac{\Gamma \vdash_- C_0 : A_0 \qquad \Delta \vdash_- C_1 : A_1}{\Gamma, \Delta \vdash_- [C_0, C_1] : A_0 \vee A_1}\;(\vee I_-)$$

$$\frac{\Gamma \vdash_+ E : A_1 \wedge A_2}{\Gamma \vdash_+ E.i^{\wedge} : A_i}\;(\wedge E_+) \qquad \frac{\Gamma \vdash_- C : A_1 \vee A_2}{\Gamma \vdash_- C.i^{\vee} : A_i}\;(\vee E_-)$$

$$\frac{\Gamma \vdash_- C : A}{\Gamma \vdash_+ \mathrm{not}\,(C) : \neg A}\;(\neg I_+) \qquad \frac{\Gamma \vdash_+ E : \neg A}{\Gamma \vdash_- \mathrm{unpack}\,(E) : A}\;(\neg E_+)$$

Figure 2.2: Typing rules for λTERMS.

2.3 Reductions

We now define a *reduction relation* $\rightsquigarrow$ for our language as the least compatible relation satisfying the reductions in Fig. 2.3.[3]

Our first two rules are β-reductions of λ-abstractions $(\beta_\to), (\beta_\leftarrow)$. Next, we introduce our reduction rules for projections, which are also β-reductions. These reductions are defined as usual. The reduction of $\mathrm{unpack}\,(\mathrm{not}\,(C))$ unpacks the continuation C as expected.

The rules marked with ξ might seem odd at first. These are also called *permutative reductions* where we do not eliminate constructors immediately as with the β-reductions. Instead, we rearrange terms so that further reductions are possible. In Definition 5, we elaborate why we need to substitute with exactly those terms in the contractum. In brief, consider the rule $(\to_\xi)$. We

[3] Following Church, we call the terms on the left-hand side of $\rightsquigarrow$ in Fig. 2.3 *redex*, as an abbreviation for *reducible expressions* and the right-hand side *contractum*.

send the expression E further into the configuration D by substitution, since the μ-abstraction $(\mu\dot{x}.D)$ expects a continuation as an argument. Similarly, the other rules send the outer applications and projections on μ-abstractions further into the configuration.

At last, we define reduction steps for μ-abstractions. As previously stated, $\mu x.D$ can be interpreted as a continuation that receives an expression E for x. Thus, configurations of the form $\langle E \mid \mu x.D\rangle$ can be reduced by substituting the expression E into D for x. Dually, the reduction of $\langle \mu\dot{x}.D \mid C\rangle$ substitutes the continuation C into D for $\dot{x}$.

$$(\mu_L)\ \langle \mu\dot{x}.D \mid C\rangle \rightsquigarrow D[\dot{x}/C]$$
$$(\mu_R)\ \langle E \mid \mu x.D\rangle \rightsquigarrow C[x/D]$$

Configurations consisting of two μ-abstractions, might reduce with either (μ_L) or (μ_R) and this gives rise to critical pairs. Consider the following two configurations:[4]

- $\langle \mu\dot{x}.\, \langle x \mid \dot{x}\rangle \mid \mu y.\, \langle y \mid \dot{y}\rangle\rangle$, both possible reduction sequences reduce to the same term.

$$\langle \mu\dot{x}.\, \langle x \mid \dot{x}\rangle \mid \mu y.\, \langle y \mid \dot{y}\rangle\rangle \overset{(\mu_L)}{\rightsquigarrow} \langle x \mid \mu y.\, \langle y \mid \dot{y}\rangle\rangle \overset{(\mu_R)}{\rightsquigarrow} \langle x \mid \dot{y}\rangle$$
$$\langle \mu\dot{x}.\, \langle x \mid \dot{x}\rangle \mid \mu y.\, \langle y \mid \dot{y}\rangle\rangle \overset{(\mu_R)}{\rightsquigarrow} \langle \mu\dot{x}.\, \langle x \mid \dot{x}\rangle \mid \dot{y}\rangle \overset{(\mu_L)}{\rightsquigarrow} \langle x \mid \dot{y}\rangle$$

- $\langle \mu\dot{x}.\, \langle y \mid \dot{y}\rangle \mid \mu x.\, \langle y \mid \dot{y}\rangle\rangle$, both possible reduction sequences reduce to different terms.

$$\langle \mu\dot{z}.\, \langle x \mid \dot{x}\rangle \mid \mu z.\, \langle y \mid \dot{y}\rangle\rangle \overset{(\mu_L)}{\rightsquigarrow} \langle x \mid \dot{x}\rangle$$
$$\langle \mu\dot{z}.\, \langle x \mid \dot{x}\rangle \mid \mu z.\, \langle y \mid \dot{y}\rangle\rangle \overset{(\mu_R)}{\rightsquigarrow} \langle y \mid \dot{y}\rangle$$

  This example shows that the reduction system is not confluent, because different normal forms might arise depending on the choice of the reduction sequence.

Consider the general form of a critical pair consisting of two μ-abstractions $\langle \mu\dot{x}.D_e \mid \mu x.D_c\rangle$. An evaluation strategy might restore confluence. For this, any evaluation strategy needs to determine how—among other restrictions—critical pairs like above need to be evaluated. In *call-by-value* evaluations we would expect expressions to evaluate before passing them into continuations. Therefore, we substitute the continuation $\mu x.D_c$ into D_e by reducing with (μ_L). Then, D_e can be evaluated and use the continuation $\mu x.D_c$ later. In

[4]Both examples have two possible reduction sequences. In more complex critical pairs, the number of possible reductions might increase.

comparison, μ-abstractions over expression variables x stand for let-bindings and in *call-by-name* we substitute evaluations into the continuations as far as possible, before evaluating. Thus, we apply (μ_R) for *call-by-name* [AK22, Wad03].

$$\langle \mu\dot{x}.D_e \mid \mu x.D_c\rangle \rightsquigarrow \begin{cases} D_e[\dot{x}/\mu x.D_c], & \text{if the eval. order is } \textit{call-by-value} \\ D_c[x/\mu\dot{x}.D_e], & \text{if the eval. order is } \textit{call-by-name} \end{cases}$$

Configurations of these forms are not only blocking the proof of confluence, but they also represent problematic cases in the proofs of normalizations, to be discussed later on. We refer to configurations of this kind as *critical configurations*.

The *multi-step reduction* relation $\rightsquigarrow^*$, is the smallest congruence relation closed under reflexive, transitive closure of single-step reductions $\rightsquigarrow$ defined in Fig. 2.3 [Pie02, p.39].

Example 2. Considering the term from the previous example it holds $\langle \mu\dot{x}.\, \langle E \mid \dot{x}.1^{\vee}\rangle \mid [C_0, C_1]\rangle \rightsquigarrow^* \langle E \mid C_0\rangle$:[5]

$$\begin{aligned} \langle \mu\dot{x}.\, \langle E \mid \dot{x}.1^{\vee}\rangle \mid [C_0, C_1]\rangle &\overset{(\mu_L)}{\rightsquigarrow} \langle E \mid [C_0, C_1]\,.1^{\vee}\rangle \\ &\overset{(\beta_{1^\vee})}{\rightsquigarrow} \langle E \mid C_0\rangle \end{aligned}$$

Example 3. For $(\mu\dot{x}_1.\, \langle \mu\dot{x}_2.\, \langle y_1 \mid \dot{x}_2\rangle \mid \dot{x}_1\rangle)\,((\lambda x.x)y)$ the resulting reduction depends on the evaluation strategy:

$$\begin{aligned} (\mu\dot{x}_1.\, \langle \mu\dot{x}_2.\, \langle y_1 \mid \dot{x}_2\rangle \mid \dot{x}_1\rangle)\,((\lambda x.x)y) &\overset{(\beta_\to)}{\rightsquigarrow} (\mu\dot{x}_1.\, \langle \mu\dot{x}_2.\, \langle y_1 \mid \dot{x}_2\rangle \mid \dot{x}_1\rangle)\,y \\ &\overset{(\xi_\to)}{\rightsquigarrow} \mu\dot{x}_3.\langle \mu\dot{x}_2.\, \langle y_1 \mid \dot{x}_2\rangle \mid \mu x.\, \langle xy \mid \dot{x}_3\rangle\rangle \end{aligned}$$

with *call-by-value*:

$$\overset{(\mu_L)}{\rightsquigarrow} \mu\dot{x}_3.\, \langle y_1 \mid \mu x.\, \langle xy \mid \dot{x}_3\rangle\rangle$$

with *call-by-name*:

$$\begin{aligned} &\overset{(\mu_R)}{\rightsquigarrow} \mu\dot{x}_3.\, \langle (\mu\dot{x}_2.\, \langle y_1 \mid \dot{x}_2\rangle)\,y \mid \dot{x}_3\rangle \\ &\overset{(\xi_\to)}{\rightsquigarrow} \mu\dot{x}_3.\, \langle \mu\dot{x}_4.\, \langle y_1 \mid \mu x.\, \langle xy \mid \dot{x}_n\rangle\rangle \mid \dot{x}_3\rangle \end{aligned}$$

To establish a connection between the typing rules from Fig. 2.2 and the reduction relation of untyped terms defined here, we prove the *subject-reduction theorem* for λTERMS.

[5] We mark occurring redexes in blue.

$$
\begin{array}{lrcl}
(\beta_{\rightarrow}) & (\lambda x.E)E' & \rightsquigarrow & E[x/E'] \\
(\beta_{\leftarrow}) & (\lambda \dot{x}.C)C' & \rightsquigarrow & C[\dot{x}/C'] \\
(\beta_{i\wedge}) & \langle E_1, E_2 \rangle .i^{\wedge} & \rightsquigarrow & E_i \\
(\beta_{i\vee}) & [C_1, C_2] .i^{\vee} & \rightsquigarrow & C_i \\
(\beta_{\neg}) & \text{unpack}\,(\text{not}\,(C)) & \rightsquigarrow & C \\
(\xi_{\rightarrow}) & (\mu \dot{x}.D)E & \rightsquigarrow & \mu \dot{x}_n.D[\dot{x}/\mu x.\, \langle xE \mid \dot{x}_n \rangle] \\
(\xi_{\leftarrow}) & (\mu x.D)C & \rightsquigarrow & \mu x_n.D[x/\mu \dot{x}.\, \langle x_n \mid \dot{x}C \rangle] \\
(\xi_{i\wedge}) & (\mu \dot{x}.D).i^{\wedge} & \rightsquigarrow & \mu \dot{x}_n.D[\dot{x}/\mu x.\, \langle x.i^{\wedge} \mid \dot{x}_n \rangle] \\
(\xi_{i\vee}) & (\mu x.D).i^{\vee} & \rightsquigarrow & \mu x_n.D[x/\mu \dot{x}.\, \langle x_n \mid \dot{x}.i^{\vee} \rangle] \\
(\xi_{\neg}) & \text{unpack}\,(\mu \dot{x}.D) & \rightsquigarrow & \mu x_n.D[\dot{x}/\mu x_m.\, \langle x_n \mid \text{unpack}\,(x_m) \rangle] \\
(\mu_L) & \langle \mu \dot{x}.D \mid C \rangle & \rightsquigarrow & D[\dot{x}/C] \\
(\mu_R) & \langle E \mid \mu x.D \rangle & \rightsquigarrow & C[x/D]
\end{array}
$$

Figure 2.3: Reduction rules for untyped λTERMS. Assume x_n and x_m are fresh on the right-hand side.

Theorem 1 (Subject-reduction theorem)**.** *For all $B, B' \in \lambda$TERMS and every type A: if $\Gamma \vdash B : A$ and $B \rightsquigarrow^* B'$, then $\Gamma \vdash B' : A$.*

Proof. By induction on the structure of derivations $\mathfrak{D}$ of $\Gamma \vdash B : A$. We need to consider all rules which are applicable in the last step.
Induction base:

- $\mathfrak{D}$ ends with (Var$_+$). Then, $B = x$ and $x : A \vdash x : A$. Therefore, the theorem is vacuously true with $B' = x$.

- $\mathfrak{D}$ ends with (Var$_-$). This is analogue to (Var$_+$).

Induction hypothesis: The assertion holds for the derivations of the premises of the last rule application in $\mathfrak{D}$.
Induction step: We only consider the complex cases where $\mathfrak{D}$ ends with (Nc.), all other cases are omitted.

- $\mathfrak{D}$ ends with (Nc.).

$$
\dfrac{\begin{array}{c}\mathfrak{D}_1 \\ \Gamma \vdash_+ E : A\end{array} \qquad \begin{array}{c}\mathfrak{D}_2 \\ \Delta \vdash_- C : A\end{array}}{\Gamma, \Delta \vdash_o \langle E \mid C \rangle : \bot}\,(\text{Nc.})
$$

  There are four possible cases to consider:

  1.
$$\langle E \mid C \rangle \rightsquigarrow^* \langle E' \mid C \rangle .$$

     We know by induction hypothesis on the derivation $\mathfrak{D}_1$ of $\Gamma \vdash_+ E : A$ with $E \rightsquigarrow^* E'$, that $\Gamma \vdash_+ E' : A$ holds. Thus:

$$\frac{\Gamma \vdash_{+} E' : A \qquad \Delta \vdash_{-} C : A}{\Gamma, \Delta \vdash_{o} \langle E' \mid C \rangle : \bot}\ (\text{Nc.})$$

2. The case in which

$$\langle E \mid C \rangle \rightsquigarrow \langle E \mid C' \rangle\,.$$

   is similar to the previous case.

3. The most interesting case is:

$$\langle E \mid C \rangle = \langle \mu \dot{x}.D \mid C \rangle \rightsquigarrow D[\dot{x}/C].$$

   We can specify the typing derivation of $\langle \mu \dot{x}.D \mid C \rangle$:

$$\frac{\dfrac{\begin{array}{c}\mathfrak{D}_{1.1}\\ \Gamma, \dot{x} : A \vdash_{o} D : \bot\end{array}}{\Gamma \vdash_{+} \mu \dot{x}.D : A}\ (\text{R}_{+}) \qquad \begin{array}{c}\mathfrak{D}_{2}\\ \Delta \vdash_{-} C : A\end{array}}{\Gamma, \Delta \vdash_{o} \langle \mu \dot{x}.D \mid C \rangle : \bot}\ (\text{Nc.})$$

   Applying the *substitution lemma* to the derivation $\mathfrak{D}_{1.1}$ of $\Gamma, \dot{x} : A \vdash_{o} D : \bot$ and to the derivation $\mathfrak{D}_{2}$ of $\Delta \vdash_{-} C : A$, leads to $\Gamma \vdash_{o} D[\dot{x}/C] : \bot$.

4. The case in which

$$\langle E \mid C \rangle = \langle E \mid \mu x.D \rangle \rightsquigarrow D[x/E].$$

   is similar to the previous one.

□

Chapter 3

Bilateral natural deduction

In this chapter, we present a bilateral natural deduction calculus (Bi-ND$_{\text{FULL}}$). Subsequently, we consider a fragment (Bi-ND), which is isomorphic to the symmetric λ-calculus of the previous chapter, and a fragment (Bi-ND$_{\text{INT}}$), which is isomorphic to an extension of Lovas and Crary's λ-calculus [LC06]. We then embed Bi-ND$_{\text{FULL}}$ into the two fragments and prove that these embeddings preserve the reduction relation.

3.1 Full system Bi-ND$_{\text{FULL}}$

3.1.1 Formulas

Our formulas are exactly the types from above.

$$A ::= p \mid A \to A \mid A \leftarrow A \mid A \wedge A \mid A \vee A \mid \neg A$$

A brief explanation will be provided for the inclusion of $\leftarrow$ as a primitive connective.[1] Terms with the $A \leftarrow B$ type are functions between continuations, the logical interpretation of this type is a connective, where $A \leftarrow B$ is logically equivalent to
$A \wedge \neg B$ [AK22, p.8]. Therefore, it is called *but-not* connective. Another point is that $A \wedge \neg B$ is equivalent to $\neg(A \to B)$, which also explains the duality of the typing rules for function types and but-not types. Thus, by this duality and the duality between conjunction and disjunction we have a stronger symmetry between our rules in Bi-ND$_{\text{FULL}}$.

As stated in the introduction, in the Curry-Howard isomorphism expressions are interpreted as proofs of assertions, continuations as proofs of refutations, configurations as proofs of contradictions and variables as labels

[1] The $\leftarrow$ connective is usually not included as a primitive connective in bilateral natural deduction systems.

for assumptions [Wad03]. We call formulas $+\ A$ and $-\ B$ and their rules *positive* or *negative*, respectively.

$$F ::= +\ A \mid -\ A$$
$$F^{\perp} ::= F \mid \perp$$

F is the metavariable for signed formulas. $\perp$ is not a signed formula, wherefore $F^{\perp}$ means that either we have a signed formula or $\perp$.

3.1.2 Inference rules

First, we define the inference rules of Bi-ND$_{\text{FULL}}$. Then, we define how the rules can be combined to construct derivations and how assumptions are handled in this natural deduction system. Finally, the inference rules are motivated.

Definition 1.

1. The *inference rules* of Bi-ND$_{\text{FULL}}$ are listed in Figs. 3.2 and 3.3.

2. Next to the inference line is the *rule name*. To give two examples: $(\rightarrow I_{+})$ stands for the *introduction rule of an assertion of an implication* and $(\vee E_{-})$ stands for the *elimination rule of a refutation of a disjunction*.

3. Below the inference line is the *conclusion* of the rule. The formula(s) immediately above the inference line are the premise(s) of the rule.

4. In elimination rules, if there is more than one premise, the left-most premise is the *main premise*. The others are the *minor premises* of the rules.

5. For every signed formula $s\ A$ and every variable x,[2] $\overset{x}{s\ A}$ is an *assumption* of $s\ A$.

Remark. The inference rules were first stated by Rumfitt [Rum00], except for the rules of $\leftarrow$. Here, we shortly comment the inference rules and justify their structure. The rules (Nc.), (R_{+}) and (R_{-}) are the *co-ordination principles*, mentioned in the introduction. They coordinate the relationship between the proof of an assertion of a formula, the proof of a rejection of a formula and contradictions [Rum00].

The other rules are *operational rules* of the connectives. The rules governing the inference of positive formulas remain identical to those of standard natural deduction, as standard natural deduction only concerns itself

[2] In this definition x stand for continuation variables $\dot{x}$, if $s = -$.

with the proof of formula assertions. From the duality of the connectives—except for negation—the inference rules for their negative formulas can be derived. Thus, explaining these dualities will suffice for their justification.

The rules for disjunction have a dual structure to those of conjunction. The reason for this can be emphasized by the fact that a rejection of $A_0 \vee A_1$ only holds, if both A_0 *and* A_1 are rejected. This "and" makes this duality more obvious. Similarly, we can infer from a proof of a rejection of $A_0 \vee A_1$ that both A_0 and A_1 have to be rejected. This duality also goes the other way around, such that the positive disjunction rules are also dual to the negative conjunction rules.[3]

The duality of the rules for the *but-not* connective to the rules of *implication* can be explained by inspecting when $- \, A_0 \leftarrow A_1$ holds. Refuting $A_0 \leftarrow A_1$ is logically equivalent to refuting $A_0 \wedge \neg A_1$ in classical logic [AK22, p.8]. The latter is again logically equivalent to refuting $\neg(A_0 \rightarrow A_1)$ or proving $A_0 \rightarrow A_1$ in classical logic. This makes the relation to *implication* clear, but we can go a step further and apply contraposition to prove $\neg A_1 \rightarrow \neg A_0$. Thus, the negations become refutations, such that we can infer $- \, A_0 \leftarrow A_1$, if there is a derivation from the assumption $- \, A_1$ to $- \, A_0$. The elimination rule $(\leftarrow E_-)$ then, is an application of *modus ponendo ponens*, analogue to $(\rightarrow E_+)$. Again, this duality also holds for the positive but-not and negative implication rules.

The rules for negation have no direct counterpart in standard natural deduction. We can infer a proof of an assertion $\neg A$, if a proof of the rejection of A is given. Dually, we can infer a proof of a rejection $\neg A$, if a proof of the assertion of A is given. These rules are explicitly constructed to fulfil proof-theoretic harmony. Thus, the elimination rules are the mirrored introduction rules.

The inference rules of BI-ND$_{\text{FULL}}$ also correspond to typing rules in symmetric λ-calculi. Some of the inference rules correspond to the typing rules of λTERMS. Thus, the corresponding terms for those inference rules are evident. The rule corresponding to $(\vee I_+)$ introduces the expressions inl E and inr E. The positive elimination $(\vee E_+)$ introduces a *case* construct [Pie02]. Dually, the negative rules for conjunction introduce fst C, snd C and a *case*-construct for continuations. The negative rules for implications, and the dual positive rules of the *but-not* connective correspond to the creation and projection of pairs. These pairs are then combinations of expressions and continuations.[4] The negative introduction rule for negation reifies expressions as continuations.

For the sake of brevity, we give the next two definitions for the fragments

[3] This duality essentially encapsulate the *DeMorgan's laws*. See also the deliberations on this in the introduction.

[4] This new kind of pair combines expressions and continuations *in one term*, in contrast to $\langle E, E \rangle$ and $[C, C]$.

consisting of positive implication and coordination principles. The definitions can be easily extended to $\textsc{Bi-ND}_{\textsc{full}}$ and any fragment of the latter.

Definition 2. A *derivation* is an upward branched tree, inductively defined as follows:

1. Every assumption of $s\ A$ is a derivation of $s\ A$.

2. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are derivations,[5] and x and $\dot{x}$ are variables, then the following are derivations as well:

$$\dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2 \\ +\ A & -\ A\end{array}}{\bot}\ (\text{Nc.})$$

$$\dfrac{\begin{array}{c}\mathfrak{D}_1 \\ \bot\end{array}}{+\ A}\ (\text{R}_+)^{\dot{x}} \qquad \dfrac{\begin{array}{c}\mathfrak{D}_1 \\ \bot\end{array}}{-\ A}\ (\text{R}_-)^{x}$$

$$\dfrac{\begin{array}{c}\mathfrak{D}_1 \\ +\ A_1\end{array}}{+\ A_0 \to A_1}\ (\to I_+)^{x} \qquad \dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2 \\ +\ A_0 \to A_1 & +\ A_0\end{array}}{+\ A_1}\ (\to E_+)$$

Definition 3. We define the *set of open assumptions* $FV\,(\mathfrak{D})$ of a derivation $\mathfrak{D}$ recursively:[6]

1. $FV\left(\begin{array}{c} x \\ s\ A\end{array}\right) = \left\{\begin{array}{c} x \\ s\ A\end{array}\right\}$.

2. $FV\left(\dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2 \\ +\ A & -\ A\end{array}}{\bot}\ (\text{Nc.})\right) = FV\left(\begin{array}{c}\mathfrak{D}_1 \\ +\ A\end{array}\right) \cup FV\left(\begin{array}{c}\mathfrak{D}_2 \\ -\ A\end{array}\right)$.

3. $FV\left(\dfrac{\begin{array}{c}\mathfrak{D}_1 \\ \bot\end{array}}{+\ A}\ (\text{R}_+)^{\dot{x}}\right) = FV\left(\begin{array}{c}\mathfrak{D}_1 \\ \bot\end{array}\right) \setminus \left\{\begin{array}{c}\dot{x} \\ -\ A\end{array}\right\}$.
   Accordingly, $(\text{R}_-)^x$ is dual.

4. $FV\left(\dfrac{\begin{array}{c}\mathfrak{D}_1 \\ +\ A_1\end{array}}{+\ A_0 \to A_1}\ (\to I_+)^{x}\right) = FV\left(\begin{array}{c}\mathfrak{D}_1 \\ +\ A_1\end{array}\right) \setminus \left\{\begin{array}{c}x \\ +\ A_0\end{array}\right\}$.

[5] We write the conclusion of the derivation immediately below $\mathfrak{D}_i$.

[6] The assumptions of a derivation are equivalent to the variables in λTERMS. Hence, we use FV as an abbreviation for *free variables*.

5. $$FV\left(\dfrac{\begin{matrix}\mathfrak{D}_1 \\ +\ A_0 \to A_1\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2 \\ +\ A_0\end{matrix}}{+\ A_1}(\to E_+)\right) \quad = \quad FV\left(\begin{matrix}\mathfrak{D}_1 \\ +\ A_0 \to A_1\end{matrix}\right) \cup FV\left(\begin{matrix}\mathfrak{D}_2 \\ +\ A_0\end{matrix}\right).$$

Remark. All assumptions that are not open in $\mathfrak{D}$ are called *discharged*. It is noteworthy that $\perp$ is not an assumption.

Remark. We summarize the definition above by giving the inference rules in the format of Fig. 3.2. Later on, we will just introduce systems by stating the inference rules in this format.

Example 4. In BI-ND$_{\text{FULL}}$ *Pierce's law* can be derived:

$$\dfrac{\dfrac{\dfrac{\overset{x}{+\ (A \to B) \to A} \qquad \dfrac{\dfrac{\dfrac{\overset{y}{+\ A} \qquad \overset{\dot{x}}{-\ A}}{\perp}(\text{Nc.})}{+\ B}(\text{R}_+)}{+\ A \to B}(\to I_+)^y}{+\ A}(\to E_+) \qquad \overset{\dot{x}}{-\ A}}{\perp}(\text{Nc.})}{\dfrac{+\ A}{+\ ((A \to B) \to A) \to A}(\to I_+)^x}(\text{R}_+)^{\dot{x}}$$

3.1.3 Reduction steps

We begin by stating a few definitions.

Definition 4.
- A sequence of formula occurrences $F_1^\perp, \ldots, F_n^\perp$ with $n \geq 2$ is called *segment*, if
  1. $F_1^\perp$ is not the conclusion of $(\vee E_+)$ or $(\wedge E_-)$,
  2. $F_n^\perp$ is not a minor premise of $(\vee E_+)$ or $(\wedge E_-)$, and
  3. for every consecutive pair $F_i^\perp, F_{i+1}^\perp$ it holds: $F_i^\perp$ is a minor premise of $(\vee E_+)$ or $(\wedge E_-)$ and $F_{i+1}^\perp$ its conclusion.
- A formula occurrence is called *β-redex*, if it is the conclusion of an introduction rule and the main premise of an elimination rule.[7]
- A formula occurrence is called *ι-redex*, if it is the premise of (Nc.) and the conclusion of an introduction rule and the other premise of (Nc.) is the conclusion of an introduction rule.

[7] In the literature on logic redexes are also called *maximal* formula occurrences.

$$\dfrac{+\,A \qquad -\,A}{\bot}\,(\mathrm{Nc.})$$

$$\begin{array}{c}\dot{x}\\ -\,A\\ \dfrac{\bot}{+\,A}\end{array}(\mathrm{R}_{+})^{\dot{x}} \qquad \begin{array}{c}x\\ +\,A\\ \dfrac{\bot}{-\,A}\end{array}(\mathrm{R}_{-})^{x}$$

$$\begin{array}{c}x\\ +\,A_0\\ \dfrac{+\,A_1}{+\,A_0 \to A_1}\end{array}(\to I_{+})^{x} \qquad \dfrac{+\,A \qquad -\,B}{-\,A \to B}\,(\to I_{-}) \qquad \dfrac{+\,A \qquad -\,B}{+\,A \leftarrow B}\,(\leftarrow I_{+}) \qquad \begin{array}{c}\dot{x}\\ -\,A_1\\ \dfrac{-\,A_0}{-\,A_0 \leftarrow A_1}\end{array}(\leftarrow I_{-})^{\dot{x}}$$

$$\dfrac{+\,A_0 \to A_1 \qquad +\,A_0}{+\,A_1}\,(\to E_{+}) \qquad \dfrac{-\,A \to B}{+\,A}\,(\to E_{-}) \qquad \dfrac{+\,A \leftarrow B}{+\,A}\,(\leftarrow E_{+}) \qquad \dfrac{-\,A_0 \leftarrow A_1 \qquad -\,A_1}{-\,A_0}\,(\leftarrow E_{-})$$

$$\dfrac{-\,A \to B}{-\,B}\,(\to E_{-}) \qquad \dfrac{+\,A \leftarrow B}{-\,B}\,(\leftarrow E_{+})$$

$$\dfrac{+\,A_0 \qquad +\,A_1}{+\,A_0 \wedge A_1}\,(\wedge I_{+}) \qquad \dfrac{-\,A_i}{-\,A_1 \wedge A_2}\,(\wedge I_{-}) \qquad \dfrac{+\,A_i}{+\,A_1 \vee A_2}\,(\vee I_{+}) \qquad \dfrac{-\,A_0 \qquad -\,A_1}{-\,A_0 \vee A_1}\,(\vee I_{-})$$

$$\dfrac{+\,A_1 \wedge A_2}{+\,A_i}\,(\wedge E_{+}) \qquad \dfrac{-\,A_1 \wedge A_2 \qquad \begin{array}{c}\dot{x}\\ -\,A_1\\ F^{\bot}\end{array} \qquad \begin{array}{c}\dot{y}\\ -\,A_2\\ F^{\bot}\end{array}}{F^{\bot}}\,(\wedge E_{-})^{\dot{x},\dot{y}} \qquad \dfrac{+\,A_1 \vee A_2 \qquad \begin{array}{c}x\\ +\,A_1\\ F^{\bot}\end{array} \qquad \begin{array}{c}y\\ +\,A_2\\ F^{\bot}\end{array}}{F^{\bot}}\,(\vee E_{+})^{x,y} \qquad \dfrac{-\,A_1 \vee A_2}{-\,A_i}\,(\vee E_{-})$$

Figure 3.2: Inference rules for BI-ND$_{\mathrm{FULL}}$ without negation.

$$\frac{-\ A}{+\ \neg A}\,(\neg I_+) \qquad \frac{+\ \neg A}{-\ A}\,(\neg E_+)$$

$$\frac{+\ A}{-\ \neg A}\,(\neg I_-) \qquad \frac{-\ \neg A}{+\ A}\,(\neg E_-)$$

Figure 3.3: Inference rules for negation in BI-ND$_{\text{FULL}}$.

- A segment $A_1, \ldots, A_n$ is called *ξ-redex*, if A_n is the main premise of an elimination rule.
- A formula occurrence is called *ξ-redex*, if it is the conclusion of (R_s) and the main premise of an elimination rule.
- A segment $A_1, \ldots, A_n$ is called *μ-redex*, if A_n is premise of (Nc.).
- A formula occurrence is called *μ-redex*, if it is premise of (Nc.) and conclusion of (R_s).
- If both premises of (Nc.) are μ-redexes, we refer to both redexes as *critical*.
- We call all segments ending with the same formula occurrence $s\ A_n$ *siblings* of each other.
- As in the λ-calculus of Chapter 2, we define a *reduction* relation $\rightsquigarrow$ that allow us either to get rid of redexes or to shorten their length. Some of the reduction steps are listed in Figs. 3.4 and 3.5.[8]
- A derivation is called *normal*, if it does not contain a redex.
- If we can transform a derivation $\mathfrak{D}$ into a normal form by applying sequences of applications of reduction steps, we call that initial derivation *normalizable*.
- The *degree deg* of a (signed) formula is defined as follows:
  1. $deg(\bot) \coloneqq 0$.
  2. $deg(s\ A) \coloneqq 0$, if A is atomic.
  3. $deg(s\ \neg A) \coloneqq deg(s\ A) + 1$.
  4. $deg(s\ A \circ B) \coloneqq deg(s\ A) + deg(s\ B) + 1$, if $\circ$ is a binary connective $\wedge, \vee, \rightarrow$ or $\leftarrow$.[9]

[8] The variables x_n and $\dot{x}_n$, occurring in the permutative cases, are new variables not occurring in the derivations on the left-hand side.

[9] From here on, i use $\circ$ as an arbitrary unary or binary connective.

Remark. In cases where μ-redexes are segments of length two, their reduction can result in μ-redexes, ι-redexes, or no redex at all. Similarly, ξ-redexes may give rise to new ξ- or β-redexes or none at all.

The reduction steps that shorten the length of segments are called *γ-reductions*. The respective reductions for all other cases are called after the redexes, e.g., *β-reductions* reduce β-redexes. The β- and γ-reductions on positive redexes stem from Prawitz [Pra65]. ξ- and γ-reductions are also called *permutative reductions*.

The *multi-step reduction* relation $\rightsquigarrow^*$ is the smallest congruence relation closed under reflexive, transitive closure of single-step reductions $\rightsquigarrow$ defined in Definition 4 [Pie02, p.39].

Examples

We analyze three example derivations by applying the definitions from above.

Example 5. Below is an example of a reduction sequence in Bi-ND.[10] This shows that the initial derivation is normalizable, since the reduction sequence ends in normal form. In this example sequence, the degrees of the occurring redexes are one.[11] The derivations and their reductions here correspond through the Curry-Howard isomorphism to the terms and reductions in Example 2.

$$\dfrac{\dfrac{\dfrac{\overset{x}{+\,A} \qquad \dfrac{\overset{\dot{x}}{-\,A \vee B}}{-\,A}\,(\vee E_-)}{\bot}\,(\mathrm{Nc.})}{\color{blue}{+\,A \vee B}}\,(\mathrm{R}_+)^{\dot{x}} \qquad \dfrac{\overset{\dot{y}}{-\,A} \qquad \overset{\dot{z}}{-\,B}}{-\,A \vee B}\,(\vee I_-)}{\bot}\,(\mathrm{Nc.}) \qquad \overset{(\mu_L)}{\rightsquigarrow}$$

$$\dfrac{\overset{x}{+\,A} \qquad \dfrac{\dfrac{\overset{\dot{y}}{-\,A} \qquad \overset{\dot{z}}{-\,B}}{\color{blue}{-\,A \vee B}}\,(\vee I_-)}{-\,A}\,(\vee E_-)}{\bot}\,(\mathrm{Nc.}) \qquad \overset{(\beta_{1\vee})}{\rightsquigarrow}$$

$$\dfrac{\overset{x}{+\,A} \qquad \overset{\dot{y}}{-\,A}}{\bot}\,(\mathrm{Nc.})$$

Example 6. In the example below we can show that the initial derivation is normalizable. The initial derivation here has two redexes with degree one and two. Thus, the reduction is non-deterministic. Below we only give one of the two possible reduction sequences, the one that results by reducing the

[10] We continue to highlight redexes in blue.

[11] This is the case if A and B are atomic.

redex of higher degree on the left-hand side.[12] Observe moreover that although the application of (Nc.) followed by (R$_+$) could be omitted, $\bot$ is not a redex. Hence, we cannot reduce it and a derivation containing this pair of rules can be in normal form, and still have these kind of redundancies.[13] Another interesting point is that the degree of the new constructed redexes is never higher than the considered redex before we reduce. This behaviour is further investigated in the *normalization theorems* in Chapter 4.

$$\dfrac{\dfrac{\dfrac{\overset{x}{+\,A_0\to(A_1\vee B)} \qquad \overset{\dot{x}}{-\,A_0\to(A_1\vee B)}}{\bot}\,(\text{Nc.})}{+\,A_0\to(A_1\vee B)}\,(\text{R}_+)^{\dot{x}} \qquad \dfrac{\dfrac{\overset{y}{+\,A_0} \qquad \overset{y}{+\,A_0}}{+\,A_0\wedge A_0}\,(\wedge I_+)}{+\,A_0}\,(\wedge E_+)}{+\,A_1\vee B}\,(\to E_+)$$

$$\overset{(\xi_\to)}{\rightsquigarrow}$$

$$\dfrac{\dfrac{\overset{x}{+\,A_0\to(A_1\vee B)} \qquad \dfrac{\dfrac{\dfrac{\overset{\dot{y}}{-\,A_0\to(A_1\vee B)} \qquad \dfrac{\dfrac{\overset{y}{+\,A_0} \qquad \overset{y}{+\,A_0}}{+\,A_0\wedge A_0}\,(\wedge I_+)}{+\,A_0}\,(\wedge E_+)}{+\,A_1\vee B}\,(\to E_+) \qquad \overset{\dot{z}}{-\,A_1\vee B}}{\bot}\,(\text{Nc.})}{-\,A_0\to(A_1\vee B)}\,(\text{R}_-)^{\dot{y}}}{\bot}\,(\text{Nc.})}{+\,A_1\vee B}\,(\text{R}_+)^{\dot{z}}$$

$$\overset{(\mu_R)}{\rightsquigarrow}$$

$$\dfrac{\dfrac{\dfrac{\overset{x}{+\,A_0\to(A_1\vee B)} \qquad \dfrac{\dfrac{\overset{y}{+\,A_0} \qquad \overset{y}{+\,A_0}}{+\,A_0\wedge A_0}\,(\wedge I_+)}{+\,A_0}\,(\wedge E_+)}{+\,A_1\vee B}\,(\to E_+) \qquad \overset{\dot{z}}{-\,A_1\vee B}}{\bot}\,(\text{Nc.})}{+\,A_1\vee B}\,(\text{R}_+)^{\dot{z}}$$

$$\overset{(\beta_\wedge)}{\rightsquigarrow}$$

$$\dfrac{\dfrac{\dfrac{\overset{x}{+\,A_0\to(A_1\vee B)} \qquad \overset{y}{+\,A_0}}{+\,A_1\vee B}\,(\to E_+) \qquad \overset{\dot{z}}{-\,A_1\vee B}}{\bot}\,(\text{Nc.})}{+\,A_1\vee B}\,(\text{R}_+)^{\dot{z}}$$

[12] Here, the reduction order does not matter in the sense that every reduction sequence in this example leads to the same normal form. This is not always the case, as the next example shows.

[13] Reductions omitting these kinds of rule applications are called η-reductions or η-conversions.

Example 7. Next, we look at the logical counterpart of the critical configurations $\langle \mu\dot{x}.D_e \mid \mu x.D_c \rangle$ from λTERMS.

$$
\begin{array}{c}
x \\ +\,A \\ \mathfrak{D}_c \\ \dfrac{\perp}{-\,A}\,(\mathrm{R}_-)^x \\ \mathfrak{D}_e \\ \perp
\end{array}
\;\overset{(\mu_L)}{\leftsquigarrow}\;
\dfrac{
\begin{array}{c} \dot{x} \\ -\,A \\ \mathfrak{D}_e \\ \dfrac{\perp}{+\,A}\,(\mathrm{R}_+)^{\dot{x}} \end{array}
\qquad
\begin{array}{c} x \\ +\,A \\ \mathfrak{D}_c \\ \dfrac{\perp}{-\,A}\,(\mathrm{R}_-)^x \end{array}
}{\perp}\,(\mathrm{Nc.})
\;\overset{(\mu_R)}{\rightsquigarrow}\;
\begin{array}{c}
\dot{x} \\ -\,A \\ \mathfrak{D}_e \\ \dfrac{\perp}{+\,A}\,(\mathrm{R}_+)^{\dot{x}} \\ \mathfrak{D}_c \\ \perp
\end{array}
$$

If we choose to evaluate (μ_L) before (μ_R) in such cases, we end up with a call-by-value like evaluation strategy. The other way around leads to call-by-name, as explained in Section 2.3. As we see, we now have an example where unrestricted non-determinism possibly leads to different normal forms. We will not delve into this aspect any further, instead, we search for normal forms in general. It will not be of importance whether there are multiple normal forms or only one. As defined in Definition 4, we refer to μ-redexes in such pairs as *critical μ-redexes*.

$$\dfrac{\dfrac{\begin{array}{c}x\\+A_0\\\mathfrak{D}_0\\+A_1\end{array}}{+A_0\to A_1}(\to I_+)^x \quad \begin{array}{c}\mathfrak{D}_1\\+A_0\end{array}}{+A_1}(\to E_+) \overset{(\beta_\to)}{\rightsquigarrow} \begin{array}{c}\mathfrak{D}_1\\+A_0\\\mathfrak{D}_0\\+A_1\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\dot{x}\\-A_1\\\mathfrak{D}_0\\-A_0\end{array}}{-A_0\leftarrow A_1}(\leftarrow I_-)^{\dot{x}} \quad \begin{array}{c}\mathfrak{D}_1\\-A_1\end{array}}{-A_0}(\to E_-) \overset{(\beta_\leftarrow)}{\rightsquigarrow} \begin{array}{c}\mathfrak{D}_1\\-A_1\\\mathfrak{D}_0\\-A_0\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\+A_1\end{array}\quad\begin{array}{c}\mathfrak{D}_2\\+A_2\end{array}}{+A_1\wedge A_2}(\wedge I_+)}{+A_i}(\wedge E_+) \overset{(\beta_{i\wedge})}{\rightsquigarrow} \begin{array}{c}\mathfrak{D}_i\\+A_i\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\-A_1\end{array}\quad\begin{array}{c}\mathfrak{D}_2\\-A_2\end{array}}{-A_1\vee A_2}(\vee I_-)}{-A_i}(\vee E_-) \overset{(\beta_{i\vee})}{\rightsquigarrow} \begin{array}{c}\mathfrak{D}_i\\-A_i\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\dot{x}\\-A\\\mathfrak{D}_0\\\bot\end{array}}{+A}(\mathrm{R}_+)^{\dot{x}} \quad \begin{array}{c}\mathfrak{D}_1\\-A\end{array}}{\bot}(\mathrm{Nc.}) \overset{(\mu_L)}{\rightsquigarrow} \begin{array}{c}\mathfrak{D}_1\\-A\\\mathfrak{D}_0\\\bot\end{array}$$

$$\dfrac{\begin{array}{c}\mathfrak{D}_1\\+A\end{array} \quad \dfrac{\begin{array}{c}x\\+A\\\mathfrak{D}_0\\\bot\end{array}}{-A}(\mathrm{R}_-)^x}{\bot}(\mathrm{Nc.}) \overset{(\mu_R)}{\rightsquigarrow} \begin{array}{c}\mathfrak{D}_1\\+A\\\mathfrak{D}_0\\\bot\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\dot{x}\\-A_0\to A_1\\\mathfrak{D}_0\\\bot\end{array}}{+A_0\to A_1}\dot{x} \quad \begin{array}{c}\mathfrak{D}_1\\+A_0\end{array}}{+A_1} \overset{(\xi_\to)}{\rightsquigarrow} \dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\mathfrak{D}_1\\+A_0\end{array}\quad\begin{array}{c}\dot{x}_n\\-A_1\end{array}}{-A_0\to A_1}\\\mathfrak{D}_0\\\bot\end{array}}{+A_1}\dot{x}_n$$

$$\dfrac{\dfrac{\begin{array}{c}x\\+A_0\leftarrow A_1\\\mathfrak{D}_0\\\bot\end{array}}{-A_0\leftarrow A_1}x \quad \begin{array}{c}\mathfrak{D}_1\\-A_1\end{array}}{-A_0} \overset{(\xi_\leftarrow)}{\rightsquigarrow} \dfrac{\begin{array}{c}\dfrac{\begin{array}{c}x_n\\+A_0\end{array}\quad\begin{array}{c}\mathfrak{D}_1\\-A_1\end{array}}{+A_0\leftarrow A_1}\\\mathfrak{D}_0\\\bot\end{array}}{-A_0}x_n$$

$$\dfrac{\dfrac{\begin{array}{c}\dot{x}\\-A_1\wedge A_2\\\mathfrak{D}_0\\\bot\end{array}}{+A_1\wedge A_2}\dot{x}}{+A_i} \overset{(\xi_{i\wedge})}{\rightsquigarrow} \dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\dot{x}_n\\-A_i\end{array}}{-A_1\wedge A_2}\\\mathfrak{D}_0\\\bot\end{array}}{+A_i}\dot{x}_n$$

$$\dfrac{\dfrac{\begin{array}{c}x\\+A_1\vee A_2\\\mathfrak{D}_0\\\bot\end{array}}{-A_1\vee A_2}x}{-A_i} \overset{(\xi_{i\vee})}{\rightsquigarrow} \dfrac{\begin{array}{c}\dfrac{\begin{array}{c}x_n\\+A_i\end{array}}{+A_1\vee A_2}\\\mathfrak{D}_0\\\bot\end{array}}{-A_i}x_n$$

$$\dfrac{\dfrac{\begin{array}{c}x\\+A\\\mathfrak{D}_0\\+B\end{array}}{+A\to B}(\to I_+)^x \quad \dfrac{\begin{array}{c}\mathfrak{D}_2\\+A\end{array}\quad\begin{array}{c}\mathfrak{D}_3\\-B\end{array}}{-A_0\to A_1}(\to I_-)}{\bot}(\mathrm{Nc.}) \overset{(\iota_\to)}{\rightsquigarrow} \dfrac{\begin{array}{c}\mathfrak{D}_2\\+A\\\mathfrak{D}_0\\+B\end{array}\quad\begin{array}{c}\mathfrak{D}_3\\-B\end{array}}{\bot}(\mathrm{Nc.})$$

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_i\\+A_i\end{array}}{+A_1\vee A_2}(\vee I_+) \quad \dfrac{\begin{array}{c}\mathfrak{D}_3\\-A_1\end{array}\quad\begin{array}{c}\mathfrak{D}_4\\-A_2\end{array}}{-A_1\vee A_2}(\vee I_-)}{\bot}(\mathrm{Nc.}) \overset{(\iota_\vee)}{\rightsquigarrow} \dfrac{\begin{array}{c}\mathfrak{D}_i\\+A_i\end{array}\quad\begin{array}{c}\mathfrak{D}_{3/4}\\-A_i\end{array}}{\bot}(\mathrm{Nc.})$$

Figure 3.4: Some reduction steps for BI-ND$_{\text{FULL}}$.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_0\\ +\,D\vee E\end{array}\quad \begin{array}{c}\mathfrak{D}_1\\ F^{\perp}_{n-1}\end{array}\quad \begin{array}{c}\mathfrak{D}_2\\ F^{\perp}_{n-1}\end{array}}{F^{\perp}_n}\,(\vee E_+)\quad \begin{array}{c}\mathfrak{D}_k\\ \{G^{\perp}\}\end{array}}{s\ C}\,(\circ E_s) \qquad \overset{\gamma}{\rightsquigarrow}$$
$$\mathfrak{D}_4$$

$$\dfrac{\begin{array}{c}\mathfrak{D}_0\\ +\,D\vee E\end{array}\quad \dfrac{\begin{array}{c}\mathfrak{D}_1\\ F^{\perp}_{n-1}\end{array}\quad \begin{array}{c}\mathfrak{D}_k\\ \{G^{\perp}\}\end{array}}{s\ C}\,(\circ E_s)\quad \dfrac{\begin{array}{c}\mathfrak{D}_2\\ F^{\perp}_{n-1}\end{array}\quad \begin{array}{c}\mathfrak{D}_k\\ \{G^{\perp}\}\end{array}}{s\ C}\,(\circ E_s)}{s\ C}$$
$$\mathfrak{D}_4$$

Figure 3.5: Summary of some of the γ-reduction step schemes. $\{G^{\perp}\}$ indicates further possible minor premises.

3.2 Embeddings

Bi-ND$_{\text{FULL}}$ is a rather dense system with a large number of inference rules and reductions. In the literature on programming languages, two fragments of Bi-ND$_{\text{FULL}}$ and their corresponding λ-calculus have been studied. An extension of the first one from Abe and Kimura was presented in Chapter 2. The definitions of the fragments, called Bi-ND and Bi-ND$_{\text{INT}}$, are given by their inference rules in Figs. 3.6 and 3.7. Following Abe and Kimura, we choose for each connective in Bi-ND one among the positive or negative sets of rules. Analogously, following Lovas and Crary we include in Bi-ND$_{\text{INT}}$ for each connective only the introduction rules, positive and negative [LC06].[14] However, we could also take a third path with solely primitive elimination rules or similar to Abe and Kimura, we choose for each connective either the positive or negative sets of rules.[15]

In the remaining part of this chapter we give two embeddings of Bi-ND$_{\text{FULL}}$ into Bi-ND and Bi-ND$_{\text{INT}}$, respectively. From this embeddings it already follows that those fragments are as expressive as Bi-ND$_{\text{FULL}}$. Furthermore, we prove that the embeddings are reduction preserving, if we extend the reduction relations of Bi-ND and Bi-ND$_{\text{INT}}$.

3.2.1 Embedding functions

We start by stating the embedding into Bi-ND. Following that, we state the embedding into Bi-ND$_{\text{INT}}$.

[14] The only difference to Lovas and Crary's system is: we implement implication and but-not primitively in Bi-ND$_{\text{INT}}$, whereas they represent implication as syntactic sugar by using negation and conjunction primitively.

[15] All mentioned fragments are reduction preserving in the sense of Section 3.2.2.

Definition 5. For every derivation $\mathfrak{D}$ in Bi-ND$_{\text{FULL}}$ we define a derivation $\mathfrak{D}^{AK}$ in Bi-ND, that is

$$\begin{aligned} {}^{AK} :& \text{Bi-ND}_{\text{FULL}} \to \text{Bi-ND} \\ & \mathfrak{D} \mapsto \mathfrak{D}^{AK} \end{aligned}$$

by recursion on $\mathfrak{D}$ as follows:
Recursion base:

1. $$\left(\overset{x}{+A} \right)^{AK} = \overset{x}{+A}$$

2. $$\left(\overset{\dot{x}}{-A} \right)^{AK} = \overset{x}{-A}$$

Recursive step:

3. $$\left(\dfrac{\overset{\mathfrak{D}_0}{+A} \qquad \overset{\mathfrak{D}_1}{-B}}{-A \to B}(\to I_-) \right)^{AK} \overset{\text{def}}{=} \dfrac{\dfrac{\dfrac{\overset{x_n}{+A \to B} \qquad \overset{\mathfrak{D}_0^{AK}}{+A}}{+B}(\to E_+) \qquad \overset{\mathfrak{D}_1^{AK}}{-B}}{\perp}(\text{Nc.})}{-A \to B}(\text{R}_-)^{x_n}$$

4. $$\left(\dfrac{\overset{\mathfrak{D}}{-A \to B}}{+A}(\to E_-) \right)^{AK} \overset{\text{def}}{=} \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{x_n}{+A} \qquad \overset{\dot{x}_n}{-A}}{\perp}(\text{Nc.})}{+B}(\text{R}_s)}{+A \to B}(\to I_+)^{x_n} \qquad \overset{\mathfrak{D}^{AK}}{-A \to B}}{\perp}(\text{Nc.})}{+A}(\text{R}_+)^{\dot{x}_n}$$

5. $$\left(\dfrac{\overset{\mathfrak{D}}{-A \to B}}{-B}(\to E_-) \right)^{AK} \overset{\text{def}}{=} \dfrac{\dfrac{\dfrac{\overset{x_n}{+B}}{+A \to B}(\to I_+) \qquad \overset{\mathfrak{D}^{AK}}{-A \to B}}{\perp}(\text{Nc.})}{-B}(\text{R}_-)^{x_n}$$

6. $$\left(\dfrac{\overset{\mathfrak{D}}{+A_i}}{+A_0 \lor A_1}(\lor I_+) \right)^{AK} \overset{\text{def}}{=} \dfrac{\dfrac{\overset{\mathfrak{D}^{AK}}{+A_i} \qquad \dfrac{\overset{\dot{x}_n}{-A_0 \lor A_1}}{-A_i}(\lor E_-)}{\perp}(\text{Nc.})}{+A_0 \lor A_1}(\text{R}_+)^{\dot{x}_n}$$

7.
$$\left(\dfrac{\begin{array}{c}\mathfrak{D}\\ +\,A_0\vee A_1\end{array}\qquad\begin{array}{c}x\\ +\,A_0\\ \mathfrak{D}_0\\ s\;C\end{array}\qquad\begin{array}{c}y\\ +\,A_1\\ \mathfrak{D}_1\\ s\;C\end{array}}{s\;C}\,(\vee E_+)^{x,y}\right)^{AK}\overset{\text{def}}{=}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}^{AK}\\ +\,A_0\vee A_1\end{array}\qquad\dfrac{\dfrac{\dfrac{\begin{array}{c}x\\ +\,A_0\\ \mathfrak{D}_0^{AK}\\ s\;C\end{array}\quad\begin{array}{c}x_n\\ \overline{s}\;C\end{array}}{\bot}\,(\text{Nc.})}{-\,A_0}\,(\text{R}_-)^{x}\qquad\dfrac{\dfrac{\begin{array}{c}y\\ +\,A_1\\ \mathfrak{D}_1^{AK}\\ s\;C\end{array}\quad\begin{array}{c}x_n\\ \overline{s}\;C\end{array}}{\bot}\,(\text{Nc.})}{-\,A_1}\,(\text{R}_-)^{y}}{-\,A_0\vee A_1}\,(\vee I_-)}{\bot}\,(\text{Nc.})}{s\;C}\,(\text{R}_s)^{x_n}$$

If $(\vee E_+)$ ends with $\bot$, the last application of $(\text{R}_s)^{x_n}$ and the applications of (Nc.) on C are omitted.

8.
$$\left(\dfrac{\begin{array}{c}\mathfrak{D}\\ +\,A\end{array}}{-\,\neg A}\,(\neg I_-)\right)^{AK}\overset{\text{def}}{=}\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}^{AK}\\ +\,A\end{array}\qquad\dfrac{\begin{array}{c}x_n\\ +\,\neg A\end{array}}{-\,A}\,(\neg E_+)}{\bot}\,(\text{Nc.})}{-\,\neg A}\,(\text{R}_-)^{x_n}$$

9.
$$\left(\dfrac{\begin{array}{c}\mathfrak{D}\\ -\,\neg A\end{array}}{+\,A}\,(\neg E_-)\right)^{AK}\overset{\text{def}}{=}\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}^{AK}\\ -\,\neg A\end{array}\qquad\dfrac{\begin{array}{c}\dot{x}_n\\ -\,A\end{array}}{+\,\neg A}\,(\neg I_+)}{\bot}\,(\text{Nc.})}{+\,A}\,(\text{R}_+)^{\dot{x}_n}$$

10. The embeddings of $(\leftarrow I_+)$, $(\leftarrow E_+)$, $(\wedge I_-)$ and $(\wedge E_-)$ are dual to the above, respectively.

11. The inductive clause of the definition for all other rules is obvious, as they belong both to Bi-ND$_{\text{FULL}}$ and Bi-ND.

Definition 6. For every derivation $\mathfrak{D}$ in Bi-ND$_{\text{FULL}}$ we define a derivation $\mathfrak{D}^{LC}$ in Bi-ND$_{\text{INT}}$, that is

$$\begin{aligned}{}^{AK}:&\text{Bi-ND}_{\text{FULL}}\to\text{Bi-ND}_{\text{INT}}\\ &\mathfrak{D}\mapsto\mathfrak{D}^{LC}\end{aligned}$$

by recursion on $\mathfrak{D}$ as follows:
Recursion base:

1. $\left(\overset{x}{+\,A}\right)^{LC} = \overset{x}{+\,A}$

2. $\left(\overset{\dot{x}}{-\,A}\right)^{LC} = \overset{x}{-\,A}$

Recursive step:

3.

$$\left(\dfrac{\begin{matrix}\mathfrak{D}_1 \\ +\,A_0 \to A_1\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2 \\ +\,A_0\end{matrix}}{+\,A_1}\,(\to E_+) \right)^{LC} \stackrel{\text{def}}{=} \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1^{LC} \\ +\,A_0 \to A_1\end{matrix} \qquad \dfrac{\begin{matrix}\mathfrak{D}_2^{LC} \\ +\,A_0\end{matrix} \qquad \begin{matrix}\dot{x}_n \\ -\,A_1\end{matrix}}{-\,A_0 \to A_1}\,(\to I_-)}{\bot}\,(\mathrm{Nc.})}{+\,A_1}\,(\mathrm{R}_+)^{\dot{x}_n}$$

4.

$$\left(\dfrac{\begin{matrix}\mathfrak{D} \\ -\,A_0 \vee A_1\end{matrix}}{-\,A_i}\,(\vee E_-) \right)^{LC} \stackrel{\text{def}}{=} \dfrac{\dfrac{\dfrac{\begin{matrix}x_n \\ +\,A_i\end{matrix}}{+\,A_0 \vee A_1}\,(\vee I_+) \qquad \begin{matrix}\mathfrak{D}^{LC} \\ -\,A_0 \vee A_1\end{matrix}}{\bot}\,(\mathrm{Nc.})}{-\,A_i}\,(\mathrm{R}_-)^{x_n}$$

5.

$$\left(\dfrac{\begin{matrix}\mathfrak{D} \\ +\,\neg A\end{matrix}}{-\,A}\,(\neg E_+) \right)^{AK} \stackrel{\text{def}}{=} \dfrac{\dfrac{\dfrac{\begin{matrix}x_n \\ +\,A\end{matrix}}{-\,\neg A}\,(\neg I_-) \qquad \begin{matrix}\mathfrak{D}^{LC} \\ +\,\neg A\end{matrix}}{\bot}\,(\mathrm{Nc.})}{-\,A}\,(\mathrm{R}_-)^{x_n}$$

6. The embeddings of $(\leftarrow\ E_-)$ and $(\wedge E_+)$ are dual to the above, respectively.

7. The embeddings for $(\to\ E_-)$, $(\leftarrow\ E_+)$, $(\vee E_+), (\wedge E_-)$ and $(\neg E_-)$ are analogous to their embeddings in Definition 5.

8. The inductive clause of the definition for all other rules is obvious, as they belong both to Bi-ND$_{\text{FULL}}$ and Bi-ND$_{\text{INT}}$.

Remark. These embeddings need to apply the coordination principles on arbitrary formulas. There is a perfect trade-off between choosing a system with introduction and elimination rules for one of the two signs, or alternatively choosing a system with both introduction rules for both signs. This is the case as long as we are allowed to apply the coordination principles to arbitrary formula. It should however, be observed that if one restricts the coordination principles to atomic formula we get a weaker system, even if we keep all operational rules. This is due to the fact that we cannot derive the unrestricted

$$\frac{+\ A \qquad -\ A}{\bot}\ (\text{Nc.})$$

$$\frac{\begin{array}{c}\dot{x}\\ -\ A\\ \bot\end{array}}{+\ A}\ (\text{R}_+)^{\dot{x}} \qquad \frac{\begin{array}{c}x\\ +\ A\\ \bot\end{array}}{-\ A}\ (\text{R}_-)^{x}$$

$$\frac{\begin{array}{c}x\\ +\ A_0\\ +\ A_1\end{array}}{+\ A_0 \to A_1}\ (\to I_+)^{x} \qquad \frac{\begin{array}{c}\dot{x}\\ -\ A_1\\ -\ A_0\end{array}}{-\ A_0 \leftarrow A_1}\ (\leftarrow I_-)^{\dot{x}}$$

$$\frac{+\ A_0 \to A_1 \qquad +\ A_0}{+\ A_1}\ (\to E_+) \qquad \frac{-\ A_0 \leftarrow A_1 \qquad -\ A_1}{-\ A_0}\ (\leftarrow E_-)$$

$$\frac{+\ A_0 \qquad +\ A_1}{+\ A_0 \wedge A_1}\ (\wedge I_+) \qquad \frac{-\ A_0 \qquad -\ A_1}{-\ A_0 \vee A_1}\ (\vee I_-)$$

$$\frac{+\ A_1 \wedge A_2}{+\ A_i}\ (\wedge E_+) \qquad \frac{-\ A_1 \vee A_2}{-\ A_i}\ (\vee E_-)$$

$$\frac{-\ A}{+\ \neg A}\ (\neg I_+) \qquad \frac{+\ \neg A}{-\ A}\ (\neg E_+)$$

Figure 3.6: Inference rules for Bi-ND.

rule of (R_s): the cases where $-\ A \wedge B$ or $+\ A \vee B$ are the conclusions of (R_s) are not derivable [dVIS22, p.199ff]. This can be avoided, if we change the introduction and elimination rules of negative conjunction and positive disjunction, as del Valle-Inclan and Schlöder suggest [dVIS22, p.200f].[16]

3.2.2 Reduction preservation

The reduction relation for Bi-ND$_{\text{FULL}}$ transfers to the fragments. Thus, all redexes in combination with their contractums are in the fragments, if all occurring rules are in the fragments.

In Bi-ND$_{\text{INT}}$ all β-, γ- and ξ-reductions are omitted, but all ι-reductions are kept. In Bi-ND all $\xi-, \iota$- and γ-reductions, and half of the β-reductions are omitted.[17] Without γ-reductions it is clear that, if we talk in regard to the

[16] For an overview on the debate of whether coordination principles should be applied to arbitrary formulas see [Fer08, Rum08, dVIS22].

[17] Every reduction step in Bi-ND corresponds to the reduction rule in Fig. 2.3 with the same name. ξ-reductions take a special role, which is discussed later.

$$\frac{+\,A \qquad -\,A}{\bot}\,(\text{Nc.})$$

$$\frac{\begin{array}{c}\dot{x}\\ -\,A\\ \bot\end{array}}{+\,A}\,(\mathrm{R}_+)^{\dot{x}} \qquad\qquad \frac{\begin{array}{c}x\\ +\,A\\ \bot\end{array}}{-\,A}\,(\mathrm{R}_-)^{x}$$

$$\frac{\begin{array}{c}x\\ +\,A_0\\ +\,A_1\end{array}}{+\,A_0 \to A_1}\,(\to I_+)^{x} \qquad \frac{\begin{array}{c}\dot{x}\\ -\,A_1\\ -\,A_0\end{array}}{-\,A_0 \leftarrow A_1}\,(\leftarrow I_-)^{\dot{x}}$$

$$\frac{+\,A \qquad -\,B}{-\,A \to B}\,(\to I_-) \qquad \frac{+\,A \qquad -\,B}{+\,A \leftarrow B}\,(\leftarrow I_+)$$

$$\frac{+\,A_0 \qquad +\,A_1}{+\,A_0 \wedge A_1}\,(\wedge I_+) \qquad \frac{-\,A_0 \qquad -\,A_1}{-\,A_0 \vee A_1}\,(\vee I_-)$$

$$\frac{-\,A_i}{-\,A_1 \wedge A_2}\,(\wedge I_-) \qquad\qquad \frac{+\,A_i}{+\,A_1 \vee A_2}\,(\vee I_+)$$

$$\frac{-\,A}{+\,\neg A}\,(\neg I_+) \qquad\qquad \frac{+\,A}{-\,\neg A}\,(\neg I_-)$$

Figure 3.7: Inference rules for BI-ND$_{\text{INT}}$.

fragments of redexes, we always refer to formula occurrences.

In the previous section, it is evident from the provided embeddings that derivability is preserved. Is the relation of reducibility also preserved? The answer is "not without extended reduction relations for the fragments".

For the preservation of the β-reductions we need in both systems η-reductions that were already mentioned in Example 6:

$$\cfrac{\cfrac{\begin{array}{c}\mathfrak{D}\\ +\,A\end{array} \qquad \begin{array}{c}\dot{x}\\ -\,A\end{array}}{\bot}\,(\text{Nc.})}{+\,A}\,(\mathrm{R}_+)^{\dot{x}} \quad \overset{(\eta_+)}{\rightsquigarrow} \quad \begin{array}{c}\mathfrak{D}\\ +\,A\end{array}$$

$$\cfrac{\cfrac{\begin{array}{c}x\\ +\,A\end{array} \qquad \begin{array}{c}\mathfrak{D}\\ -\,A\end{array}}{\bot}\,(\text{Nc.})}{-\,A}\,(\mathrm{R}_-)^{x} \quad \overset{(\eta_-)}{\rightsquigarrow} \quad \begin{array}{c}\mathfrak{D}\\ -\,A\end{array}$$

The ξ-reductions take a special role on the question of reduction preservation. Let us consider the ξ-reductions in BI-ND$_{\text{FULL}}$: the contractum requires introduction rules with the inverse sign compared to that of the redex.

In Bi-ND, we omit either the positive or the negative operational rules of a connective. Thus, while half of the ξ-redexes are preserved in Bi-ND, their contractums are not. For reduction preservation, we need to embed the contractums and we need to apply the definitions of the introduction rules. From here on the reduction relation of Bi-ND is extended by $(\xi_\circ^{AK})$ rules, so that we can reduce these kinds of redexes and prove the preservation of reducability.

Using these embeddings, the complex structure of ξ-reductions in λterms (see Fig. 2.3) become evident. To give one example, we need to apply the embedding of $(\to I_-)$ in $(\xi_\to)$ for the extension of Bi-ND with $(\xi_\circ^{AK})$:

$$\dfrac{\dfrac{\begin{matrix}\dot{x}\\ -\ A_0\to A_1\\ \mathfrak{D}_0\\ \bot\end{matrix}}{+\ A_0\to A_1} \quad \begin{matrix}\mathfrak{D}_1\\ +\ A_0\end{matrix}}{+\ A_1} \quad \overset{(\xi^{AK}_{\to})}{\rightsquigarrow}$$

$$\left(\dfrac{\begin{matrix}\dfrac{\begin{matrix}\mathfrak{D}_1\\ +\ A_0\end{matrix}\quad \begin{matrix}\dot{x}_n\\ -\ A_1\end{matrix}}{-\ A_0\to A_1}\,(\to I_-)\\ \mathfrak{D}_0\\ \bot\end{matrix}}{+\ A_1}\,\dot{x}_n\right)^{AK}$$

$$\overset{\text{def}}{=} \quad \dfrac{\begin{matrix}\dfrac{\dfrac{\dfrac{\begin{matrix}x\\ +\ A_0\to A_1\end{matrix}\quad\begin{matrix}\mathfrak{D}_1^{AK}\\ +\ A_0\end{matrix}}{+\ A_1}\quad\begin{matrix}\dot{x}_n\\ -\ A_1\end{matrix}}{\bot}}{-\ A_0\to A_1}\,x\\ \mathfrak{D}_0^{AK}\\ \bot\end{matrix}}{+\ A_1}\,\dot{x}_n$$

The derivation of $-\ A_0 \to A_1$ in the embedded contractum corresponds through the Curry-Howard isomorphism to the term, which we substitute into the configuration of $(\xi_\to)$, $\mu x. \langle xE \mid \dot{x}_n\rangle$ (see Fig. 2.3).

With these new reduction steps the following relations hold:

Proposition 1. *For all derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ in* Bi-ND$_{\text{FULL}}$*, it holds:*

1. *if $\mathfrak{D}_1 \overset{\beta}{\rightsquigarrow} \mathfrak{D}_2$, then $\mathfrak{D}_1^{AK} \overset{\beta\mu\eta}{\rightsquigarrow} \mathfrak{D}_2^{AK}$,*
2. *if $\mathfrak{D}_1 \overset{\iota}{\rightsquigarrow} \mathfrak{D}_2$, then $\mathfrak{D}_1^{AK} \overset{\beta\mu}{\rightsquigarrow} \mathfrak{D}_2^{AK}$,*

3. *if* $\mathfrak{D}_1 \overset{\xi}{\rightsquigarrow} \mathfrak{D}_2$, *then* $\mathfrak{D}_1^{AK} \overset{\xi^{AK}\mu}{\rightsquigarrow} \mathfrak{D}_2^{AK}$,

4. *if* $\mathfrak{D}_1 \overset{\gamma}{\rightsquigarrow} \mathfrak{D}_2$, *then* $\mathfrak{D}_1^{AK} \overset{\xi^{AK}\mu}{\rightsquigarrow} \mathfrak{D}_2^{AK}$,

5. *if* $\mathfrak{D}_1 \overset{\beta}{\rightsquigarrow} \mathfrak{D}_2$, *then* $\mathfrak{D}_1^{LC} \overset{\iota\eta}{\rightsquigarrow} \mathfrak{D}_2^{LC}$,

6. *if* $\mathfrak{D}_1 \overset{\xi}{\rightsquigarrow} \mathfrak{D}_2$, *then* $\mathfrak{D}_1^{LC} \overset{\mu}{\rightsquigarrow} \mathfrak{D}_2^{LC}$, *and*

7. *if* $\mathfrak{D}_1 \overset{\gamma}{\rightsquigarrow} \mathfrak{D}_2$, *then* $\mathfrak{D}_1^{LC} \overset{\mu}{\rightsquigarrow} \mathfrak{D}_2^{LC}$.

Proof. We only look at the preservation of $(\iota_\to)$, $(\beta_{\vee+})$ and one instance of the γ-reductions in BI-ND. The other cases are similar.

- First, we look at $(\iota_\to)$:

$$\left(\dfrac{\dfrac{\begin{matrix}x\\ +A\\ \mathfrak{D}_0\\ +B\end{matrix}}{+A\to B}(\to I_+)^x \qquad \dfrac{\begin{matrix}\mathfrak{D}_1\\ +A\end{matrix}\quad \begin{matrix}\mathfrak{D}_2\\ -B\end{matrix}}{-A\to B}(\to I_-)}{\bot}(\mathrm{Nc.})\right)^{AK}$$

$$= \dfrac{\dfrac{\begin{matrix}x\\ +A\\ \mathfrak{D}_0^{AK}\\ +B\end{matrix}}{+A\to B}(\to I_+)^x \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}y\\ +A\to B\end{matrix}\quad \begin{matrix}\mathfrak{D}_1^{AK}\\ +A\end{matrix}}{+B}(\to E_+)\quad \begin{matrix}\mathfrak{D}_2^{AK}\\ -B\end{matrix}}{\bot}(\mathrm{Nc.})}{-A\to B}(\mathrm{R}_-)^y}{\bot}(\mathrm{Nc.})$$

$$\overset{(\mu_R)}{\rightsquigarrow} \dfrac{\dfrac{\dfrac{\begin{matrix}x\\ +A\\ \mathfrak{D}_0^{AK}\\ +B\end{matrix}}{+A\to B}(\to I_+)^x \quad \begin{matrix}\mathfrak{D}_1^{AK}\\ +A\end{matrix}}{+B}(\to E_+) \quad \begin{matrix}\mathfrak{D}_2^{AK}\\ -B\end{matrix}}{\bot}(\mathrm{Nc.})$$

$$\overset{(\beta_\to)}{\rightsquigarrow} \dfrac{\begin{matrix}\mathfrak{D}_1^{AK}\\ +A\\ \mathfrak{D}_0^{AK}\\ +B\end{matrix}\quad \begin{matrix}\mathfrak{D}_2^{AK}\\ -B\end{matrix}}{\bot}(\mathrm{Nc.})$$

$$= \left(\dfrac{\begin{matrix}\mathfrak{D}_1\\ +A\\ \mathfrak{D}_0\\ +B\end{matrix}\quad \begin{matrix}\mathfrak{D}_2\\ -B\end{matrix}}{\bot}(\mathrm{Nc.})\right)^{AK}$$

- The second case is $(\beta_{\vee +})$, where we restrict the conclusion to signed formulas. The case for $\bot$ is analogous.[18]

$$\left(\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_i\\ +\,A_i\end{matrix}}{+\,A_0 \vee A_1}(\vee I_+) \qquad \begin{matrix}x\\ +\,A_0\\ \mathfrak{D}_2\\ s\;C\end{matrix} \qquad \begin{matrix}y\\ +\,A_1\\ \mathfrak{D}_3\\ s\;C\end{matrix}}{s\;C}(\vee E_+)^{x,y} \right)^{AK}$$

$$= \dfrac{\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_i^{AK}\\ +\,A_i\end{matrix}}{\color{blue}{+\,A_0 \vee A_1}}(\vee I_+) \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}x\\ +\,A_0\\ \mathfrak{D}_2^{AK}\\ s\;C\end{matrix} \quad \begin{matrix}x_n\\ \bar{s}\;C\end{matrix}}{\bot}(\mathrm{Nc.})}{-\,A_0}(\mathrm{R}_-)^x \qquad \dfrac{\dfrac{\begin{matrix}y\\ +\,A_1\\ \mathfrak{D}_3^{AK}\\ s\;C\end{matrix} \quad \begin{matrix}x_n\\ \bar{s}\;C\end{matrix}}{\bot}(\mathrm{Nc.})}{-\,A_1}(\mathrm{R}_-)^y}{\color{blue}{-\,A_0 \vee A_1}}(\vee I_-)}{\bot}(\mathrm{Nc.})}{s\;C}(\mathrm{R}_s)$$

$$\overset{(\beta_{\vee I})}{\rightsquigarrow} \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_i^{AK}\\ +\,A_i\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}x/y\\ +\,A_i\\ \mathfrak{D}_{2/3}^{AK}\\ s\;C\end{matrix} \quad \begin{matrix}x_n\\ \bar{s}\;C\end{matrix}}{\bot}(\mathrm{Nc.})}{\color{blue}{-\,A_i}}(\mathrm{R}_-)}{\bot}(\mathrm{Nc.})}{s\;C}(\mathrm{R}_s) \quad \overset{(\mu_R)}{\rightsquigarrow} \quad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_i^{AK}\\ +\,A_i\\ \mathfrak{D}_{2/3}^{AK}\\ s\;C\end{matrix} \quad \begin{matrix}x_n\\ \bar{s}\;C\end{matrix}}{\bot}(\mathrm{Nc.})}{s\;C}(\mathrm{R}_s)$$

$$\overset{\color{red}{(\eta_s)}}{\rightsquigarrow} \begin{matrix}\mathfrak{D}_i^{AK}\\ +\,A_i\\ \mathfrak{D}_{2/3}^{AK}\\ s\;C\end{matrix}$$

$$= \left(\begin{matrix}\mathfrak{D}_i\\ +\,A_i\\ \mathfrak{D}_{2/3}\\ s\;C\end{matrix} \right)^{AK}$$

- The last case is the following instance of γ-reductions:

[18] The first reduction step should be understood as an abbreviation for the preserved reduction of $(\beta_{\vee I})$.

$$\left(\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_i\\ +\,A_0\vee A_1\end{array}\quad \begin{array}{c}x\\ +\,A_0\\ \mathfrak{D}_2\\ +\,C_1\wedge C_2\end{array}\quad \begin{array}{c}y\\ +\,A_1\\ \mathfrak{D}_3\\ +\,C_1\wedge C_2\end{array}}{+\,C_1\wedge C_2}\,(\vee E_+)^{x,y}}{+\,C_i}\,(\wedge E_+)\right)^{AK}$$

$$=\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_i^{AK}\\ +\,A_0\vee A_1\end{array}\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}x\\ +\,A_0\\ \mathfrak{D}_2^{AK}\\ +\,C_1\wedge C_2\end{array}\quad \begin{array}{c}\dot{x}_n\\ -\,C_1\wedge C_2\end{array}}{\bot}\,(\mathrm{Nc.})}{-\,A_0}\,(\mathrm{R}_-)^x\quad \dfrac{\dfrac{\begin{array}{c}y\\ +\,A_1\\ \mathfrak{D}_3^{AK}\\ +\,C_1\wedge C_2\end{array}\quad \begin{array}{c}\dot{x}_n\\ -\,C_1\wedge C_2\end{array}}{\bot}\,(\mathrm{Nc.})}{-\,A_1}\,(\mathrm{R}_-)^y}{-\,A_0\vee A_1}\,(\vee I_-)}{\bot}\,(\mathrm{Nc.})}{+\,C_1\wedge C_2}\,(\mathrm{R}_+)^{\dot{x}_n}}{+\,C_i}\,(\wedge E_+)$$

$\overset{(\xi_{i\wedge}^{AK})}{\rightsquigarrow}$

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_i^{AK}\\ +\,A_0\vee A_1\end{array}\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}x\\ +\,A_0\\ \mathfrak{D}_2^{AK}\\ +\,C_1\wedge C_2\end{array}\quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\dot{y}_n\\ +\,C_1\wedge C_2\end{array}}{+\,C_i}\,(\wedge E_+)\quad \begin{array}{c}\dot{z}_n\\ -\,C_i\end{array}}{\bot}\,(\mathrm{Nc.})}{-\,C_1\wedge C_2}\,(\mathrm{R}_-)^{\dot{y}_n}}{}}{\bot}\,(\mathrm{Nc.})}{-\,A_0}\,(\mathrm{R}_-)^x\quad \dfrac{\dfrac{\begin{array}{c}y\\ +\,A_1\\ \mathfrak{D}_3^{AK}\\ +\,C_1\wedge C_2\end{array}\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\dot{y}_n\\ +\,C_1\wedge C_2\end{array}}{+\,C_i}\,(\wedge E_+)\quad \begin{array}{c}\dot{z}_n\\ -\,C_i\end{array}}{\bot}\,(\mathrm{Nc.})}{-\,C_1\wedge C_2}\,(\mathrm{R}_-)^{\dot{y}_n}}{\bot}\,(\mathrm{Nc.})}{-\,A_1}\,(\mathrm{R}_-)^y}{-\,A_0\vee A_1}\,(\vee I_-)}{\bot}\,(\mathrm{Nc.})}{+\,C_i}\,(\mathrm{R}_+)^{\dot{z}_n}$$

$\overset{(\mu_R)}{\rightsquigarrow}$

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_i^{AK}\\ +\,A_0\vee A_1\end{array}\quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}x\\ +\,A_0\\ \mathfrak{D}_2^{AK}\\ +\,C_1\wedge C_2\end{array}}{+\,C_i}\,(\wedge E_+)\quad \begin{array}{c}\dot{z}_n\\ -\,C_i\end{array}}{\bot}\,(\mathrm{Nc.})}{-\,A_0}\,(\mathrm{R}_-)^x\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}y\\ +\,A_1\\ \mathfrak{D}_3^{AK}\\ +\,C_1\wedge C_2\end{array}}{+\,C_i}\,(\wedge E_+)\quad \begin{array}{c}\dot{z}_n\\ -\,C_i\end{array}}{\bot}\,(\mathrm{Nc.})}{-\,A_1}\,(\mathrm{R}_-)^y}{-\,A_0\vee A_1}\,(\vee I_-)}{\bot}\,(\mathrm{Nc.})}{+\,C_i}\,(\mathrm{R}_+)^{\dot{z}_n}$$

$$=\left(\dfrac{\begin{array}{c}\mathfrak{D}_i\\ +\,A_0\vee A_1\end{array}\quad \dfrac{\begin{array}{c}x\\ +\,A_0\\ \mathfrak{D}_2\\ +\,C_1\wedge C_2\end{array}}{+\,C_i}\,(\wedge E_+)\quad \dfrac{\begin{array}{c}y\\ +\,A_1\\ \mathfrak{D}_3\\ +\,C_1\wedge C_2\end{array}}{+\,C_i}\,(\wedge E_+)}{+\,C_i}\,(\vee E_+)^{x,y}\right)^{AK}$$

□

Remark. This proof shows how the pay-off for fewer inference rules leads to more complex reduction rules and reduction sequences in the fragments. For readability and brevity, one might wish to use the additional inference rules and reductions, however we aim to prove the normalization theorem, where fewer rules lead to shorter proofs with fewer cases to distinguish.

Chapter 4

Normalization of the fragments

In this chapter we characterize the normal forms of the introduced fragments Bi-ND and Bi-ND$_{\text{INT}}$ and prove the normalization theorems for these fragments.

4.1 Normal forms of Bi-ND

We begin by characterizing the normal forms of derivations in Bi-ND. Derivations were inductively defined, consequently we can give an inductive definition of their normal forms. Before doing so, we can ask, when derivations $\mathfrak{D}$ in Bi-ND are reducible. This is the case if at least one of the three redex kinds occurs in $\mathfrak{D}$. Each kind of redex comes with pairs of inference rules. These rule pairs need to be observed. Both β- and ξ-redexes end with an elimination rule. μ-redexes end with (Nc.). By contraposition, we can infer the definition of normal forms inductively, if we avoid these rule pairs.

Definition 7 (Normal forms of Bi-ND)**.** A derivation in *normal form* is inductively defined as follows:

1. Every assumption of $s\ A$ is in normal form.

2. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are in normal form, then the following are derivations in normal form as well:

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \bot\end{matrix}}{\text{s A}}\,(\mathrm{R}_s) \quad \text{and} \quad \dfrac{\begin{matrix}\mathfrak{D}_1\\ s\ A\end{matrix}}{s\ B}\,(\circ I_s) \quad \text{and} \quad \dfrac{\begin{matrix}\mathfrak{D}_1\\ s\ A\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ s\ B\end{matrix}}{s\ C}\,(\circ I_s)$$

3. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are in normal form, and both do not end with (R_s), then the following is in normal form as well:

$$\dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2 \\ +\ A & -\ A\end{array}}{\bot}\,(\text{Nc.})$$

4. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are in normal form, and $\mathfrak{D}_1$ does not end with (R_s) or an introduction rule, then the following are in normal form as well:

$$\dfrac{\begin{array}{c}\mathfrak{D}_1 \\ s\ A\end{array}}{s\ B}\,(\circ E_s) \quad \text{and} \quad \dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2 \\ s\ A & s\ B\end{array}}{s\ C}\,(\circ E_s)$$

Remark. As previously stated, Bi-ND is isomorphic to λterms. Hence, the complete grammar describing normal terms is given in Fig. 4.1.

$\tilde{E}_{in}$	$::=$	$x \mid \lambda x.\tilde{E} \mid \left\langle \tilde{E}, \tilde{E} \right\rangle \mid \text{not}\left(\tilde{C}\right)$	normal intro expr.
$\tilde{E}_{el}$	$::=$	$\tilde{E}_{el}\tilde{E} \mid \tilde{E}_{el}.1^{\wedge} \mid \tilde{E}_{el}.2^{\wedge}$	normal elim expr.
$\tilde{E}_{R}$	$::=$	$\mu\dot{x}.\tilde{D}$	normal reductio expr.
$\tilde{E}$	$::=$	$\tilde{E}_{in} \mid \tilde{E}_{el} \mid \tilde{E}_{R}$	normal expr.
$\tilde{C}_{in}$	$::=$	$\dot{x} \mid \lambda\dot{x}.\tilde{C} \mid \left[\tilde{C}, \tilde{C}\right]$	normal intro cont.
$\tilde{C}_{el}$	$::=$	$\tilde{C}_{el}\tilde{C} \mid \tilde{C}_{el}.1^{\vee} \mid \tilde{C}_{el}.2^{\vee} \mid \text{unpack}\left(\tilde{E}_{el}\right)$	normal elim cont.
$\tilde{C}_{R}$	$::=$	$\mu x.\tilde{D}$	normal reductio cont.
$\tilde{C}$	$::=$	$\tilde{C}_{in} \mid \tilde{C}_{el} \mid \tilde{C}_{R}$	normal cont.
$\tilde{E}_{nR}$	$::=$	$\tilde{E}_{in} \mid \tilde{E}_{el}$	normal non-reductio expr.
$\tilde{C}_{nR}$	$::=$	$\tilde{C}_{in} \mid \tilde{C}_{el}$	normal non-reductio cont.
$\tilde{D}$	$::=$	$\left\langle \tilde{E}_{nR} \mid \tilde{C}_{nR} \right\rangle$	normal configuration
N	$::=$	$\tilde{E} \mid \tilde{C} \mid \tilde{D}$	normal terms

Figure 4.1: Normal forms for λterms.

4.2 Normalization for Bi-ND

We begin by defining metrics, which then combine to a *rank*. The rank is the final metric to prove that all derivations in Bi-ND can be reduced into normal forms.

Definition 8.

- For any derivation $\mathfrak{D}$ its *degree* $\deg\mathfrak{D}$ is the maximum degree of all its redexes or zero, if $\mathfrak{D}$ is in normal form.
- $\#_\alpha\mathfrak{D}$ counts the number of α-redexes of maximum degree in $\mathfrak{D}$ with $\alpha \in \{\beta, \xi, \mu, \iota\}$.

Definition 9. The *rank* of a derivation $\mathfrak{D}$ in BI-ND is the tuple $\langle \deg \mathfrak{D}, \#_\mu \mathfrak{D}, \#_\xi \mathfrak{D}, \#_\beta \mathfrak{D} \rangle$. We order ranks lexicographically:
$\langle d, m, x, b \rangle < \langle d', m', x', b' \rangle \Leftrightarrow d < d'$ or ($d = d'$ and $m < m'$) or ($d = d', m = m'$ and $x < x'$) or ($d = d', m = m', x = x'$ and $b < b'$).

Remark. The rank counts the numbers of all possible redexes of maximum degree in BI-ND. Consequently, $\#_\iota \mathfrak{D}$ is excluded from consideration, as it cannot be generated within the BI-ND fragment. A derivation is in normal form, if and only if its rank is $\langle 0, 0, 0, 0 \rangle$. Hence, we prove the normalization by induction over the rank.

Theorem 2 (Normalization)**.** *Every derivation $\mathfrak{D}$ in*BI-ND *is normalizable.*

Proof. Let $\mathfrak{D}$ be a derivation in BI-ND with rank $r = \langle d, m, x, b \rangle$, for arbitrary $d, m, x, b \in \mathbb{N}_0$. We prove the normalization of $\mathfrak{D}$ by strong induction over the rank r.
Induction base:

- Let $r = \langle 0, 0, 0, 0 \rangle$. Then $\mathfrak{D}$ is already in normal form. Therefore, the assertion is vacuously true.

Induction hypothesis: The assertion holds for all derivations $\mathfrak{D}'$, if the rank k of $\mathfrak{D}'$ is smaller than $\langle d + 1, m, x, b \rangle$.
Induction step:
Let $\mathfrak{D}$ be a derivation in BI-ND, with rank $r = \langle d + 1, m, x, b \rangle$ for arbitrary $d, m, x, b \in \mathbb{N}_0$. We distinguish three cases.

1. Let $m = x = 0$ and $b > 0$. Among these β-redexes of degree $d + 1$, we chose non-deterministically a redex, where the redex is conclusion of $(\circ I_s)$ and the main premise of $(\circ E_s)$, and the subderivation above all *minor* premises of $(\circ E_s)$ have no redex of degree $d + 1$.[1] We can always find such a redex. Assume two redexes on the same path from the conclusion to the leaves. Then, the redex further away from the conclusion is the one we choose, as this one will fulfil the conditions from above.

   There are five kinds of β-redexes that might occur:

   1.1. Let $\circ = \to$. For the relevant subderivation of $\mathfrak{D}$ it holds:

$$\dfrac{\dfrac{\begin{matrix} x \\ +\,A_0 \\ \mathfrak{D}_0 \\ +\,A_1 \end{matrix}}{+\,A_0 \to A_1}\,(\to I_+)^x \qquad \begin{matrix} \mathfrak{D}_1 \\ +\,A_0 \end{matrix}}{+\,A_1}\,(\to E_+) \qquad \overset{(\beta_\to)}{\rightsquigarrow} \qquad \begin{matrix} \mathfrak{D}_1 \\ +\,A_0 \\ \mathfrak{D}_0 \\ +\,A_1 \\ \mathfrak{D}_2 \end{matrix}$$
$$\mathfrak{D}_2$$

[1]The latter is only relevant, if there is a *minor* premise of $(\circ E_s)$.

With the definition of this case, setting $\mathfrak{D}_1$ on top of the discharged assumptions $+\ A_0$ does not create new redexes of degree $d+1$. First, because in the subderivation of the minor premise, there is no redex of degree $d+1$. Hence, there cannot be multiplications of such redexes through multiplications of $\mathfrak{D}_1$ by setting $\mathfrak{D}_1$ over multiple occurrences of assumptions $+\ A_0$. Second, any new redex would be $+\ A_0$ or $+\ A_1$, which have strictly lower degrees than $+\ A_0 \to A_1$.

Let $\tilde{\mathfrak{D}}$ be the contractum above. By cutting out a β-redex of degree $d+1$, the rank of $\tilde{\mathfrak{D}}$ is $\langle d+1, m, x, b-1\rangle$ or $\langle d, m', x', b'\rangle$, with arbitrary $m', x', b' \in \mathbb{N}_0$. The latter is the case if $b = 1$. Nevertheless, the rank of $\tilde{\mathfrak{D}}$ is lower than $\langle d+1, m, j\rangle$, and we can apply the induction hypothesis such that the assertion holds.

1.2. Let $\circ = \leftarrow$. This case is dual to the case for implication.

1.3. Let $\circ = \land$. For the relevant subderivation of $\mathfrak{D}$ it holds:

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 \\ +\ A_1\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2 \\ +\ A_2\end{matrix}}{+\ A_1 \land A_2}\,(\land I_+)}{+\ A_i}\,(\land E_+) \\ \mathfrak{D}_3\end{matrix} \quad \overset{(\beta_{i\land})}{\rightsquigarrow} \quad \begin{matrix}\mathfrak{D}_i \\ +\ A_i \\ \mathfrak{D}_3\end{matrix}$$

By cutting out one β-redex of degree $d+1$ we get at most one new redex $+\ A_i$ of degree d. Thus, the rank reduced and the assertion holds by the induction hypothesis.

1.4. Let $\circ = \lor$. This case is dual to the case for conjunction.

1.5. Let $\circ = \neg$. For the relevant subderivation of $\mathfrak{D}$ it holds:

$$\begin{matrix}\mathfrak{D}_1 \\ \dfrac{\dfrac{-\ A}{+\ \neg A}\,(\neg I_+)}{-\ A}\,(\neg E_+) \\ \mathfrak{D}_2\end{matrix} \quad \overset{(\beta_{\neg})}{\rightsquigarrow} \quad \begin{matrix}\mathfrak{D}_1 \\ +\ A \\ \mathfrak{D}_2\end{matrix}$$

By cutting out one β-redex of degree $d+1$ we get at most one new redex $+\ A$ of degree d. Thus, the rank reduced and the assertion holds by the induction hypothesis.

2. Let $m = 0$ and $x > 0$. Among those ξ-redexes with degree $d+1$, we can choose one non-deterministically such that the redex is conclusion of (R_s) and main premise of $(\circ E_s)$, where the subderivations above all premises of $(\circ E_s)$ have no redexes of degree $d+1$. Again, we can always find such a redex with the same strategy of the first case above.

There are five kinds of ξ-redexes that might occur:

2.1. Let $\circ = \to$. In a $(\xi^{AK}_{\to})$ step, $\mathfrak{D}$ reduces to $\mathfrak{D}'$ as follows:

$$\dfrac{\dfrac{\begin{array}{c}\dot{x}\\ -\ A_0\to A_1\\ \mathfrak{D}_0\\ \bot\end{array}}{+\ A_0\to A_1}\,\dot{x} \qquad \begin{array}{c}\mathfrak{D}_1\\ +\ A_0\end{array}}{\begin{array}{c}+\ A_1\\ \mathfrak{D}_2\end{array}} \quad \overset{(\xi^{AK}_{\to})}{\rightsquigarrow} \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}x_n\\ +\ A_0\to A_1\end{array} \qquad \begin{array}{c}\mathfrak{D}_1\\ +\ A_0\end{array}}{+\ A_1} \qquad \begin{array}{c}\dot{x}_n\\ -\ A_1\end{array}}{\bot}}{-\ A_0\to A_1}\,x_n\\ \mathfrak{D}_0\\ \dfrac{\bot}{+\ A_1}\,\dot{x}_n\\ \mathfrak{D}_2\end{array}$$

The rank of $\mathfrak{D}'$ is then $\langle d+1, m', x-1, b\rangle$, where $m' \geq 0$.[2] We will now prove with a subinduction on m' that $\mathfrak{D}'$ reduces to $\mathfrak{D}''$, which has a lower rank than $\mathfrak{D}$. Then, we can finally apply the initial *induction hypothesis* on $\mathfrak{D}''$ and infer the normalization of $\mathfrak{D}''$ (and with that of $\mathfrak{D}$).

Induction base:

For $m' = 0$: The rank of $\mathfrak{D}'$: $\langle d+1, 0, x-1, b\rangle$ is already lower than the rank of $\mathfrak{D}$.

Induction hypothesis on m':

The assertion holds for all k, if $k < m'+1$.

Induction step:

For $m'+1$: The form of $\mathfrak{D}_0$ can be WLOG depicted in the following way:[3]

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathfrak{D}_{0.1}\\ +\ A_0\to A_1\end{array} \qquad \begin{array}{c}\dot{x}\\ -\ A_0\to A_1\end{array}}{\bot}\,(\text{Nc.})\\ \mathfrak{D}_{0.2}\\ \bot\end{array}$$

In $\mathfrak{D}_0$ further assumptions of the form $-\ A_0 \to A_1$ may occur in $\mathfrak{D}_{0.1}$ and $\mathfrak{D}_{0.2}$. However, it is important to note that in $\mathfrak{D}_{0.1}$ these assumptions are not premises of (Nc.), else we would have chosen that assumption for our deliberations.

Let $\mathfrak{D}^\star$ be:[4]

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}x\\ +\ A_0\to A_1\end{array} \qquad \begin{array}{c}\mathfrak{D}_1\\ +\ A_0\end{array}}{+\ A_1} \qquad \begin{array}{c}\dot{x}_n\\ -\ A_1\end{array}}{\bot}}{-\ A_0\to A_1}\,(\text{R}_-)^x$$

[2]In Bi-ND every connective has either positive or negative inference rules. Thus, the assumption $-\ A_0 \to A_1$ of $\mathfrak{D}_0$ cannot become a ξ-redex in $\mathfrak{D}'$. This analogously holds for the other ξ-reductions in Bi-ND.

[3]From here on, i replace $\mathfrak{D}_n$ with $\mathfrak{D}_{n.m}$ to indicate that the latter are subderivations of the former.

[4]The derivation of $\mathfrak{D}^\star$ is the embedding of $(\to I_-)$ in Bi-ND (see Definition 5).

Furthermore, we want to define $\mathfrak{D}^\star_n$ as a modification of $\mathfrak{D}_n$, such that $\mathfrak{D}^\star$ is set above any open assumption $-\ A_0 \to A_1$ with variable $\dot{x}$ in $\mathfrak{D}_n$.

Finally, the form of $\mathfrak{D}'$ can be depicted with:

$$\dfrac{\begin{array}{c}\mathfrak{D}^\star_{0.1}\\ +\ A_0 \to A_1\end{array} \qquad \begin{array}{c}\mathfrak{D}^\star\\ \color{blue}{-\ A_0 \to A_1}\end{array}}{\begin{array}{c}\perp\\ \mathfrak{D}^\star_{0.2}\\ \dfrac{\perp}{+\ A_1}\,\dot{x}_n\\ \mathfrak{D}_2\end{array}}\,(\mathrm{Nc.})$$

Here the blue marked $-\ A_0 \to A_1$ became a μ-redex, since $\mathfrak{D}^\star$ ends with (R_-). Thus, we can apply (μ_R) to reduce $\mathfrak{D}'$ to $\mathfrak{D}''$:

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}^\star_{0.1}\\ +\ A_0 \to A_1\end{array} \qquad \begin{array}{c}\mathfrak{D}^\square_1\\ +\ A_0\end{array}}{+\ A_1} \qquad \begin{array}{c}\dot{x}_n\\ -\ A_1\end{array}}{\begin{array}{c}\perp\\ \mathfrak{D}^\star_{0.2}\\ \dfrac{\perp}{+\ A_1}\,\dot{x}_n\\ \mathfrak{D}_2\end{array}}$$

$\mathfrak{D}^\square_1$ is $\mathfrak{D}_1$ modified, such that $\mathfrak{D}^\star_{0.1}$ is set above any open assumption $+\ A_0 \to A_1$ with variable x in $\mathfrak{D}_1$.

Comparing the ranks of $\mathfrak{D}'$ and $\mathfrak{D}''$, the rank reduced from $\langle d+1, m'+1, x-1, b\rangle$ to $\langle d+1, m', x-1, b'\rangle$, where $b' = b$ or $b' > b$. The latter is the case, if the conclusion $+\ A_0 \to A_1$ of $\mathfrak{D}^\star_{0.1}$ is the conclusion of an introduction rule and therefore, some of its *copies* might became β-redexes in $\mathfrak{D}''$. Nevertheless, we can apply the induction hypothesis of m' on $\mathfrak{D}''$. Hence, $\mathfrak{D}''$ reduces to a derivation with lower rank than $\mathfrak{D}$. Thereafter, we can apply the initial induction hypothesis and infer that $\mathfrak{D}$ is, by the transitivity of the reduction relation, normalizable.

2.2. Let $\circ = \leftarrow$. This behaves dual to the previous case, since the rules are dual.

2.3. Let $\circ = \wedge$. In a $(\xi^{AK}_{(j+1)\wedge})$ step, $\mathfrak{D}$ reduces to $\mathfrak{D}'$ as follows:

$$
\begin{array}{c} \dot{x} \\ - A_0 \wedge A_1 \\ \mathfrak{D}_0 \\ \dfrac{\dfrac{\perp}{+ A_0 \wedge A_1}\,\dot{x}}{+ A_j} \\ \mathfrak{D}_1 \end{array}
\quad \overset{(\xi_{(j+1)\wedge})}{\rightsquigarrow} \quad
\begin{array}{c} \dfrac{\dfrac{\dfrac{\overset{x_n}{+ A_0 \wedge A_1}}{+ A_j} \qquad \overset{\dot{x}_n}{- A_j}}{\perp}}{- A_0 \wedge A_1}\,x_n \\ \mathfrak{D}_0 \\ \dfrac{\perp}{+ A_j}\,\dot{x}_n \\ \mathfrak{D}_1 \end{array}
$$

The rank of $\mathfrak{D}'$ is then $\langle d+1, m', x-1, b\rangle$, where $m' \geq 0$. We will now prove with a subinduction on m' that $\mathfrak{D}'$ reduces to $\mathfrak{D}''$, which has a lower rank than $\mathfrak{D}$. Then, we can finally apply the initial *induction hypothesis* on $\mathfrak{D}''$ and infer the normalization $\mathfrak{D}''$ (and with that of $\mathfrak{D}$).

Induction base:

For $m' = 0$: The rank of $\mathfrak{D}'$ with $\langle d+1, 0, x-1, b\rangle$ is already lower than the rank of $\mathfrak{D}$.

Induction hypothesis on m':

The assertion holds for all k, if $k < m' + 1$.

Induction step:

For $m'+1$: The form of $\mathfrak{D}_0$ can be WLOG depicted in the following way:

$$
\begin{array}{c} \dfrac{\overset{\mathfrak{D}_{0.1}}{+ A_0 \wedge A_1} \qquad \overset{\dot{x}}{- A_0 \wedge A_1}}{\perp}\,(\text{Nc.}) \\ \mathfrak{D}_{0.2} \\ \perp \end{array}
$$

In $\mathfrak{D}_0$ further assumptions of the form $- A_0 \to A_1$ may occur in $\mathfrak{D}_{0.1}$ and $\mathfrak{D}_{0.2}$. However, it is important to note that in $\mathfrak{D}_{0.1}$ these assumptions are not premises of (Nc.), else we would have chosen that assumption for our deliberations.

For an easier comparison, we define $\mathfrak{D}^\star$ and $\mathfrak{D}^\star_n$ analogous to case (2.1). Let $\mathfrak{D}^\star$ be this time:[5]

$$
\dfrac{\dfrac{\dfrac{\overset{x_n}{+ A_0 \wedge A_1}}{+ A_j} \qquad \overset{\dot{x}_n}{- A_j}}{\perp}}{- A_0 \wedge A_1}\,x_n
$$

$\mathfrak{D}^\star_n$ is the modification of $\mathfrak{D}_n$, such that $\mathfrak{D}^\star$ is set above any open assumption $- A_0 \wedge A_1$ with variable $\dot{x}$ in $\mathfrak{D}_n$.

[5]The derivation of $\mathfrak{D}^\star$ is now the embedding of $(\wedge I_-)$ in BI-ND(see Definition 5).

Finally, the form of $\mathfrak{D}'$ can be depicted with:

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathfrak{D}^\star_{0.1}\\ +\,A_0\wedge A_1\end{array} \qquad \begin{array}{c}\mathfrak{D}^\star\\ \color{blue}{-\,A_0\wedge A_1}\end{array}}{\bot}\,(\text{Nc.})\\ \mathfrak{D}^\star_{0.2}\\ \dfrac{\bot}{+\,A_1}\,\dot{x}_n\\ \mathfrak{D}_1\end{array}$$

Here the blue marked $-\ A_0 \to A_1$ became a μ-redex, since $\mathfrak{D}^\star$ ends with (R_-). Thus, we can apply (μ_R) to reduce $\mathfrak{D}'$ to $\mathfrak{D}''$:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}^\star_{0.1}\\ +\,A_0\wedge A_1\end{array}}{+\,A_j} \qquad \begin{array}{c}\dot{x}_n\\ -\,A_j\end{array}}{\bot}\\ \mathfrak{D}^\star_{0.2}\\ \dfrac{\bot}{+\,A_j}\,\dot{x}_n\\ \mathfrak{D}_1\end{array}$$

Comparing the ranks of $\mathfrak{D}'$ and $\mathfrak{D}''$, the rank reduced from $\langle d+1, m'+1, x-1, b\rangle$ to $\langle d+1, m', x-1, b'\rangle$, where $b' = b$ or $b' = b+1$. The latter is the case if the conclusion $+\ A_0 \wedge A_1$ of $\mathfrak{D}^\star_{0.1}$ is the conclusion of an introduction rule and therefore, became a β-redex. Nevertheless, we can apply the induction hypothesis of m' on $\mathfrak{D}''$. Hence, $\mathfrak{D}''$ reduces to a derivation with lower rank than $\mathfrak{D}$. Thereafter, we can apply the initial induction hypothesis and infer that $\mathfrak{D}$ is, by the transitivity of the reduction relation, normalizable.

2.4. Let $\circ = \vee$. This behaves dual to the previous case.

2.5. Let $\circ = \neg$. This is analogous to (2.1.) and (2.3.).

3. Let $m > 0$. Among those μ-redexes with degree $d+1$, we can choose one non-deterministically, such that the μ-redex is conclusion of (R_s) and premise of (Nc.), where the subderivations above both premises of (Nc.) have no redexes of degree $d+1$.[6] Analogous to the other two cases, we can find such a μ-redex.

   We have two kinds of μ-redexes that might occur, depending on whether the redex is conclusion of (R_s). The cases are analogous, we will therefore consider just one of them. The reduction of a single μ-redex might

[6] While the non-deterministic redex choice of β- and ξ-redexes can be summarized as *top-most and right-most*, the choice of μ-redexes is only *top-most*, because both critical μ-redexes can be considered as *equally important* premises of (Nc.).

generate new μ-redexes. Therefore, we prove this case by subinduction on the number of newly generated μ-redexes. In order for a new μ-redex to be generated, the considered redex must be a critical μ-redex. Observe however, a critical μ-redex is a necessary, but not sufficient condition for new μ-redexes to arise.

3.1. Let the redex $-\ A$ be the conclusion and a premise of (R_-) and (Nc.), respectively. In a (μ_R) step, $\mathfrak{D}$ reduces to $\mathfrak{D}'$ as follows:

$$\dfrac{\begin{array}{c}\mathfrak{D}_0\\ +\,A\end{array} \qquad \dfrac{\begin{array}{c}x\\ +\,A\\ \mathfrak{D}_1\\ \bot\end{array}}{-\,A}\,(\mathrm{R}_-)^x}{\begin{array}{c}\bot\\ \mathfrak{D}_2\end{array}}\,(\mathrm{Nc.}) \qquad \overset{(\mu_R)}{\rightsquigarrow} \qquad \begin{array}{c}\mathfrak{D}_0\\ +\,A\\ \mathfrak{D}_1\\ \bot\\ \mathfrak{D}_2\end{array}$$

The rank of $\mathfrak{D}'$ is then at most $\langle d+1, m-1+m', x+x', b+b'\rangle$, where $m', x', b' \geq 0$.[7] We will now prove with a subinduction on m' that $\mathfrak{D}'$ reduces to $\mathfrak{D}''$, which has a lower rank than $\mathfrak{D}$. Then, we can finally apply the initial induction hypothesis on $\mathfrak{D}''$ and infer that $\mathfrak{D}''$ and by the transitivity of the reduction relation $\mathfrak{D}$ are normalizable.
Induction base:
For $m' = 0$: The rank of $\mathfrak{D}'$ with $\langle d+1, m-1, x+x', b+b'\rangle$ is already lower than the rank of $\mathfrak{D}$.
Induction hypothesis: The assertion holds for all k, if $k < m'+1$.
Induction step:
For $m'+1$: The form of $\mathfrak{D}_0$ and $\mathfrak{D}_1$ can be WLOG depicted in the following way:

$$\dfrac{\begin{array}{c}\dot{x}\\ -\,A\\ \mathfrak{D}_{0.1}\\ \bot\end{array}}{+\,A}\,(\mathrm{R}_+)^{\dot{x}} \qquad \text{and} \qquad \dfrac{\begin{array}{c}x\\ +\,A\end{array} \qquad \begin{array}{c}\mathfrak{D}_{1.1}\\ -\,A\end{array}}{\begin{array}{c}\bot\\ \mathfrak{D}_{1.2}\\ \bot\end{array}}\,(\mathrm{Nc.})$$

In $\mathfrak{D}_1$ further assumptions of the form $+\ A$ may occur in $\mathfrak{D}_{1.1}$ and $\mathfrak{D}_{1.2}$. However, it is important to note that in $\mathfrak{D}_{1.1}$ these assumptions are not premises of (Nc.), else we would have chosen that assumption for our deliberations. The conclusion of $\mathfrak{D}_{1.1}$ cannot be a μ-redex, else we would have chosen that subderivation of $\mathfrak{D}$ to begin with.

[7] In this case the assertion $+\ A$ can be any redex in the contractum, but $b' > 0$ implies that $m' = x' = 0$: for $b' > 0$ we need $\mathfrak{D}_0$ to end with an introduction rule. Analog, if $m' > 0$ or $x' > 0$, then $b' = 0$ holds.

Furthermore, we want to define $\mathfrak{D}_n^\star$ as a modification of $\mathfrak{D}_n$, such that $\mathfrak{D}_0$ is set above any open assumption $+ A$ with variable x in $\mathfrak{D}_n$. The form of $\mathfrak{D}'$ can be depicted with:

$$
\cfrac{\cfrac{\begin{matrix}\dot{x} \\ -\ A \\ \mathfrak{D}_{0.1} \\ \bot\end{matrix}}{\color{magenta}{+\ A}}\ (\mathrm{R}_+)^{\dot{x}} \qquad \begin{matrix}\mathfrak{D}^\star_{1.1} \\ -\ A\end{matrix}}{\begin{matrix}\bot \\ \mathfrak{D}^\star_{1.2} \\ \bot \\ \mathfrak{D}_2\end{matrix}}\ (\mathrm{Nc.})
$$

Here the magenta marked $+ A$ became a μ-redex in this subderivation, among other possible new redexes in $\mathfrak{D}^\star_{1.1}$ and $\mathfrak{D}^\star_{1.2}$. We can apply (μ_L) on the magenta marked redex, to reduce $\mathfrak{D}'$ to $\mathfrak{D}''$:

$$
\begin{matrix}\mathfrak{D}^\star_{1.1} \\ -\ A \\ \mathfrak{D}_{0.1} \\ \bot \\ \mathfrak{D}^\star_{1.2} \\ \bot \\ \mathfrak{D}_2\end{matrix}
$$

Comparing the ranks of $\mathfrak{D}'$ and $\mathfrak{D}''$, the rank reduced from

$$
\begin{gathered}
\langle d+1, m-1+(m'+1), x+x', b+b'\rangle \\
\text{to } \langle d+1, m-1+m', x+x'+x'', b+b'+b''\rangle,
\end{gathered}
$$

where $x'', b'' > 0$, if the copies of the conclusion of $\mathfrak{D}^\star_{1.1}$ became ξ- or β-redexes. Nevertheless, we can apply the induction hypothesis of m' on $\mathfrak{D}''$. Hence, $\mathfrak{D}''$ reduces to a derivation with lower rank than $\mathfrak{D}$. Thereafter, we can apply the initial induction hypothesis over the rank on $\mathfrak{D}''$ and infer that $\mathfrak{D}$ is, by the transitivity of the reduction relation, normalizable.

□

Remark. The reduction strategy that arises from this proof can be summarized to *μ-redex over ξ-redex over β-redex, and highest degree first, top-most, right-most*. If more than one redex fulfils this, we choose one among those redexes non-deterministically.

Furthermore, *right-most* is omitted by the choice of μ-redexes. The reduction strategy that follows from this proof is not confluent (see Example 7).

With this reduction strategy, we reduce non-deterministically one of the two critical μ-redexes. As we have seen in Example 7 this might lead to different normal forms.

In Proposition 1, we considered additional reduction steps (η_+) and (η_-). Extending Bi-ND by them preserves the normalization of the reduction relation. Then, we would prove by induction over the extended rank $r = \langle \deg \mathfrak{D}, \#_\mu \mathfrak{D}, \#_\xi \mathfrak{D}, \#_\beta \mathfrak{D}, \#_\eta \mathfrak{D} \rangle$. Furthermore, we would apply those η-reductions after every other kind of redex of highest degree was reduced before and show that η-reductions do not increase the number of any of the redexes of degree $\deg \mathfrak{D}$. The proof is left to the reader.

Examples 5 to 7 are reduced into normal form with this reduction strategy. A comparison to Abe and Kimura's proof is given in Chapter 6.

4.3 Normal forms of Bi-ND$_{\text{INT}}$

We will follow Lovas and Crary's description of normal forms [LC06, p.8f]. Redexes in Bi-ND$_{\text{INT}}$ are either ι- or μ-redexes. Thus, a redex is always the premise of (Nc.), and analyzing derivations that end on (Nc.) is sufficient to describe normal forms.

If at least one of the premises of (Nc.) is the conclusion of (R$_s$), we have a μ-redex. If both premises of (Nc.) are derived by introduction rules, we have ι-redexes. From these two kinds of redexes, we can infer, if none of the premises are conclusions of (R$_s$) and at least one of them is not the conclusion of an introduction rule, then both premises are not redexes.

From this, it follows that two kinds of normal derivations can be premises of (Nc.), if one premise of (Nc.) is a premise. The first kind is a normal derivation ending on an application of an introduction rule. The second kind are assumptions. Every other combination of normal derivations of the premises leads to redexes. Therefore, derivations ending on introductions and assumptions, if already normal, are called *neutral* or *in neutral form* [LC06, p.8].

For all other derivations ending on a rule other than (Nc.) it holds that, they are normal, if their respective subderivations are normal. These considerations lead to the following mutually inductive definition of neutral and normal forms.

Definition 10 (Normal forms of Bi-ND$_{\text{INT}}$)**.** A derivation in *normal form* is mutually inductively defined with *neutral forms* as follows:

1. Every assumption of $s\ A$ is in neutral form.

2. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are in normal form, then the following are derivations in neutral form:

$$\begin{array}{c}\mathfrak{D}_1\\ \dfrac{s\ A}{s\ B}\,(\circ I_s)\end{array} \quad \text{and} \quad \begin{array}{c}\mathfrak{D}_1 \qquad \mathfrak{D}_2\\ \dfrac{s\ A \qquad s\ B}{s\ C}\,(\circ I_s)\end{array}$$

3. If $\mathfrak{D}$ is in normal form, then the following is a derivation in normal form:

$$\begin{array}{c}\mathfrak{D}\\ \dfrac{\perp}{\text{s A}}\,(\mathrm{R}_s)\end{array}$$

4. If $\mathfrak{D}$ is in neutral form, the following are in normal form:

$$\begin{array}{c}\mathfrak{D} \qquad\quad \dot{x}\\ \dfrac{+\ A \qquad -\ A}{\perp}\,(\text{Nc.})\end{array} \quad \text{and} \quad \begin{array}{c}x \qquad\quad \mathfrak{D}\\ \dfrac{+\ A \qquad -\ A}{\perp}\,(\text{Nc.})\end{array}$$

5. Every neutral derivation is normal.

4.4 Normalization for Bi-ND$_{\text{INT}}$

We prove the normalization of Bi-ND$_{\text{INT}}$ following an approach uniform to Bi-ND. First, we introduce a *rank* that counts the μ- and ι-redexes, which are the sole redex types in Bi-ND$_{\text{INT}}$, as per the limitation to introduction rules. Then, a proof strategy that is uniform to Theorem 2 is applied to Bi-ND$_{\text{INT}}$.

Definition 11. The *rank* of a derivation $\mathfrak{D}$ in Bi-ND$_{\text{INT}}$ is the tuple $\langle \deg \mathfrak{D}, \#_\mu \mathfrak{D}, \#_\iota \mathfrak{D} \rangle$. We order ranks lexicographically, analogous to Definition 9.

Theorem 3. *Every derivation $\mathfrak{D}$ in* Bi-ND$_{\text{INT}}$ *is normalizable.*

Proof. Let $\mathfrak{D}$ be a derivation in Bi-ND$_{\text{INT}}$ with rank $r = \langle d, m, j \rangle$, for arbitrary $d, m, j \in \mathbb{N}_0$. We prove the normalization of $\mathfrak{D}$ by strong induction over the rank r.
Induction base:

- Let $r = \langle 0, 0, 0 \rangle$. Then $\mathfrak{D}$ is already in normal form. Therefore, the assertion is vacuously true.

Induction hypothesis: The assertion holds for all derivations $\mathfrak{D}'$, if the rank k of $\mathfrak{D}'$ is smaller than $\langle d+1, m, j \rangle$.
Induction step:
Let $\mathfrak{D}$ be a derivation in Bi-ND$_{\text{INT}}$, with rank $r = \langle d+1, m, j \rangle$ for arbitrary $d, m, j \in \mathbb{N}_0$. We distinguish two cases.

1. Let $m = 0$ and $j > 0$. Among the ι-redexes of degree $d+1$, we chose non-deterministically a ι-redex, where the redex consists of $(\circ I_s)$ and (Nc.), where the subderivations above both premises of (Nc.) have no redexes of degree $d + 1$.

   We can always find such a redex. Assume two redexes on the same path from the conclusion to the leaves. Then, the redex further away from the conclusion is the one we choose, as this one will fulfill the conditions from above.

   We have five kinds of ι-redexes that might occur:

   1.1. Let $\circ = \to$. For the relevant subderivation of $\mathfrak{D}$ it holds:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}x\\ +\ A_0\\ \mathfrak{D}_0\\ +\ A_1\end{array}}{+\ A_0 \to A_1}(\to I_+)^x \qquad \dfrac{\begin{array}{c}\mathfrak{D}_1\\ +\ A_0\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ -\ A_1\end{array}}{-\ A_0 \to A_1}(\to I_-)^x}{\perp}(\text{Nc.})\\ \mathfrak{D}_3\end{array}$$

$$\overset{(\iota_\to)}{\rightsquigarrow} \quad \begin{array}{c}\dfrac{\begin{array}{c}\mathfrak{D}_1\\ +\ A_0\\ \mathfrak{D}_0\\ +\ A_1\end{array} \qquad \begin{array}{c}\mathfrak{D}_2\\ -\ A_1\end{array}}{\perp}(\text{Nc.})\\ \mathfrak{D}_3\end{array}$$

   With the definition of this case, we know that setting $\mathfrak{D}_1$ on top of the discharged assumptions $+\ A_0$ does not create new redexes of degree $d + 1$. First, because in the subderivation of the premise of (Nc.), there is no redex of degree $d + 1$. Hence, there cannot be multiplications of such redexes through multiplications of $\mathfrak{D}_1$ by setting $\mathfrak{D}_1$ over multiple occurrences of assumptions $+\ A_0$. Second, any new redex would be $+\ A_0$ or $+\ A_1$, which have strictly lower degrees than $s\ A_0 \to A_1$.

   Let $\tilde{\mathfrak{D}}$ be the contractum above. From these two arguments, and the fact that two ι-redexes are cut out, it follows that $\tilde{\mathfrak{D}}$ has lower rank than $\mathfrak{D}$. Thus, we can apply the induction hypothesis, such that the assertion holds.

   1.2. Let $\circ = \leftarrow$. This case is analogous to $\to$.

   1.3. Let $\circ = \wedge$. For the relevant subderivation of $\mathfrak{D}$ it holds:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_3\\ +\ A_1\end{matrix}\quad\begin{matrix}\mathfrak{D}_4\\ +\ A_2\end{matrix}}{+\ A_1 \wedge A_2}\,(\wedge I_+)\qquad \dfrac{\begin{matrix}\mathfrak{D}_i\\ -\ A_i\end{matrix}}{-\ A_1 \wedge A_2}\,(\wedge I_-)}{\begin{matrix}\bot\\ \mathfrak{D}_5\end{matrix}}\,(\text{Nc.})\quad \overset{(\beta_{i\wedge})}{\rightsquigarrow}$$

$$\dfrac{\begin{matrix}\mathfrak{D}_{i+2}\\ +\ A_i\end{matrix}\qquad\begin{matrix}\mathfrak{D}_i\\ -\ A_i\end{matrix}}{\begin{matrix}\bot\\ \mathfrak{D}_5\end{matrix}}\,(\text{Nc.})$$

We cut out two ι-redexes of degree $d+1$ and get at most two new redex $s\ A_i$ of degree d. Thus, the rank reduced and the assertion holds by the induction hypothesis.

1.4. Let $\circ = \vee$. This case is analogous to $\wedge$.

1.5. Let $\circ = \neg$. For the relevant subderivation of $\mathfrak{D}$ it holds:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ -\ A\end{matrix}}{+\ \neg A}\,(\neg I_+)\qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ +\ A\end{matrix}}{-\ \neg A}\,(\neg I_-)}{\bot}\,(\text{Nc.})\quad \overset{(\iota_\neg)}{\rightsquigarrow}$$

$$\dfrac{\begin{matrix}\mathfrak{D}_2\\ +\ A\end{matrix}\qquad\begin{matrix}\mathfrak{D}_1\\ -\ A\end{matrix}}{\bot}\,(\text{Nc.})$$

We cut out two ι-redexes of degree $d+1$ and get at most two new redexes $s\ A$ of degree d. Thus, the rank reduced and the assertion holds by the induction hypothesis.

2. Let $m > 0$. This case is almost the same as case (3) in Theorem 2. There is only one difference. We cannot get additional ξ-redexes through (μ_L) and (μ_R) steps, which makes the proof slightly easier in Bi-ND$_{\text{INT}}$.

□

Remark. Our uniform proof strategy and similar rank to Bi-ND and Theorem 2 leads to the following reduction strategy: *μ-redex over ι-redex, and highest degree first, top-most.* If more than one redex fulfils this, we choose one non-deterministically. In contrast to Bi-ND we omit the *right-most* aspect of the reductions strategy, as both ι- and μ-redex end with (Nc.), where both premises are in a sense *main premises.*

A comparison to Lovas and Crary's proof is given in Chapter 6.

Chapter 5

Normalization of Bi-ND$_{\mathbf{FULL}}$

In the last chapter, we proved the normalization of the fragments in a uniform manner. This uniform proof strategy will also be used to prove the normalization of Bi-ND$_{\text{FULL}}$, in this chapter. However, proving the normalization of Bi-ND$_{\text{FULL}}$ is a more intricate task due to γ-reductions and segments, which were not present in the fragments. Thus, we need a few more definitions and a more complex rank before proving the normalization theorem. After proving the normalization theorem, we end this chapter with a proof of the subformula property of Bi-ND$_{\text{FULL}}$.

5.1 Preliminaries

We begin by defining a metric that counts the length of a redex segment and an adjusted counter of redexes.

Definition 12. Let $\alpha \in \{\beta, \xi, \mu, \iota\}$:

- The *length* of an α-redex is the number of formula occurrences in the segment of the redex or one, if the redex consists of a single formula occurrence.
- $\#_\alpha\mathfrak{D}$ is the sum of all lengths of α-redexes, which are of maximum degree, in $\mathfrak{D}$. $\#_\alpha\mathfrak{D}$ is zero, if there is no α-redex of maximum degree in $\mathfrak{D}$.

Remark. β- and ι-redexes cannot be segments. Therefore, $\#_\beta\mathfrak{D}$ and $\#_\iota\mathfrak{D}$ count the number of these redexes of maximal degree, analogous to the definitions in the fragments. Thus, a comparison of the proofs for β- and ι-redexes will be straightforward.

Definition 13. The *rank* of a derivation $\mathfrak{D}$ in Bi-ND$_{\text{FULL}}$ is the tuple $\langle \deg \mathfrak{D}, \#_\mu\mathfrak{D}, \#_\xi\mathfrak{D}, \#_\iota\mathfrak{D}, \#_\beta\mathfrak{D} \rangle$. We order the rank analogues to Definition 9, lexicographically.

5.2 Normalization theorem

Theorem 4. *Every derivation $\mathfrak{D}$ in* BI-ND$_{\text{FULL}}$ *is normalizable.*

Proof. Let $\mathfrak{D}$ be a derivation in BI-ND$_{\text{FULL}}$ with rank $r = \langle d, m, x, j, b\rangle$, for arbitrary $d, m, x, j, b \in \mathbb{N}_0$. We prove the normalization of $\mathfrak{D}$ by strong induction over the rank r.
Induction base:

- Let $r = \langle 0, 0, 0, 0, 0\rangle$. Then $\mathfrak{D}$ is already in normal form. Therefore, the assertion is vacuously true.

Induction hypothesis: The assertion holds for all derivations $\mathfrak{D}'$, if the rank k of $\mathfrak{D}'$ is smaller than $\langle d+1, m, x, j, b\rangle$.
Induction step:
Let $\mathfrak{D}$ be a derivation in BI-ND$_{\text{FULL}}$, with the rank $r = \langle d+1, m, x, j, b\rangle$ for arbitrary $d, m, x, j, b \in \mathbb{N}_0$. We distinguish four cases.

1. Let $m = x = j = 0$ and $b > 0$. Among these β-redexes of degree $d+1$, we chose non-deterministically a redex, which is the conclusion of $(\circ I_s)$ and the main premise of $(\circ E_s)$, and the subderivation above the premises of $(\circ E_s)$ have no redex of degree $d+1$ and all minor premises are not part of a redex of degree $d+1$.[1] We can always find such a redex. Assume two redexes on the same path from the conclusion to the leaves, then the redex further away from the conclusion is the one we choose, as this one will satisfy the conditions from above. If the main and minor premises have all degree $d+1$, we choose the right-most.

   The proof for the reduction of β-redexes is now analogous to the proof of those in case (1.) of Theorem 2. The only significant differences are: in BI-ND$_{\text{FULL}}$ we have more cases and the new redexes of degree d constructed in the contractum have arbitrary segment length.

   Here we only consider the case where the β-redex is conclusion of $(\vee I_+)$ and main premise of $(\vee E_+)$. For the relevant subderivation of $\mathfrak{D}$ it holds:

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_0\\ +\ A_i\end{array}}{+\ A_1 \vee A_2}\,(\vee I_+) \qquad \begin{array}{c}x\\ +\ A_1\\ \mathfrak{D}_1\\ F^\perp\end{array} \qquad \begin{array}{c}y\\ +\ A_2\\ \mathfrak{D}_2\\ F^\perp\end{array}}{\begin{array}{c}F^\perp\\ \mathfrak{D}_3\end{array}}\,(\vee E_+)$$

[1]The latter is only relevant, if there is a *minor* premise of $(\circ E_s)$.

$$\overset{\beta}{\rightsquigarrow} \quad \begin{array}{c} \mathfrak{D}_0 \\ + A_i \\ \mathfrak{D}_2 \\ F^{\perp} \\ \mathfrak{D}_3 \end{array}$$

By the choice of the redex, we know that setting $\mathfrak{D}_0$ on top of the discharged assumptions $+ A_i$ does not create new redexes of degree $d+1$. First, because in $\mathfrak{D}_0$, there is no redex of degree $d+1$. Hence, there cannot be multiplications of such redexes through multiplications $\mathfrak{D}_0$ by setting this over multiple occurrences of assumptions $+ A_i$. Second, any new redex would be $+ A_i$ or $F^{\perp}$, but $+ A_i$ has strictly lower degree than $+ A_1 \vee A_2$.

For $F^{\perp}$ it holds: it cannot be a part of a redex of degree $d+1$ in $\mathfrak{D}$,[2] due to our *right-most* redex choice. Furthermore, for $F^{\perp}$ to become a part of a new redex in the contractum, it is a necessary precondition to be already a part of a redex in $\mathfrak{D}$. Thus, $F^{\perp}$ cannot be a part of a new redex of degree $d+1$ in the contractum. Finally, the rank is reduced and we can apply the induction hypothesis.

2. Let $m = x = 0$ and $j > 0$. The proof for the reduction of ι-redexes is analogue to the proof of those in the first case (1.) of Theorem 3. The only significant difference here is: in Bi-ND$_{\text{FULL}}$ the new redexes of degree d constructed in the contractum might have arbitrary segment length.

3. Let $m = 0$ and $x > 0$. We cannot refer to Theorem 2 to reduce ξ-redexes. First, because the contractums of ξ-reductions in Bi-ND use embeddings of the introduction rules and second, Bi-ND has no segments and therefore, no γ-reductions can be applied in the fragment.

   Thus, we need to distinguish between γ-reductions of ξ-redexes and (actual) ξ-reductions. We choose the ξ-redex analogous to the non-deterministic choice of β-redexes from above.

   3.1. Let the ξ-redex in $\mathfrak{D}$ be a segment $A_1, \ldots, A_n$. Furthermore, $\{s\ B\}$ indicates possible minor premises. In a γ-reduction step, $\mathfrak{D}$ reduces to $\mathfrak{D}'$ as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_0\\ + D \vee E\end{array} \quad \begin{array}{c}\mathfrak{D}_1\\ s\ A_{n-1}\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ s\ A_{n-1}\end{array}}{s\ A_n}(\vee E_+) \quad \begin{array}{c}\mathfrak{D}_k\\ \{s\ B\}\end{array}}{\begin{array}{c} s\ C \\ \mathfrak{D}_4\end{array}}(\circ E_s) \quad \overset{\gamma}{\rightsquigarrow}$$

[2]Here "be a part of a redex" is an abbreviation for "part of a redex segment or to be a redex that is a formula occurrence".

$$\begin{matrix}\dfrac{\begin{matrix}\mathfrak{D}_0\\ +\ D \vee E\end{matrix} \qquad \dfrac{\begin{matrix}\mathfrak{D}_1\\ s\ A_{n-1}\end{matrix} \qquad \begin{matrix}\mathfrak{D}_k\\ \{s\ B\}\end{matrix}}{s\ C}(\circ E_s) \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ s\ A_{n-1}\end{matrix} \qquad \begin{matrix}\mathfrak{D}_k\\ \{s\ B\}\end{matrix}}{s\ C}(\circ E_s)}{s\ C}\\ \mathfrak{D}_4\end{matrix}$$

By the choice of the redex, we know that the (possible) minor premises $\{s\ B\}$ are not part of redexes of maximal degree and in the subderivations of the minor premises are no redexes of maximal degree. Hence, duplicating these in $\mathfrak{D}'$ does not increase the rank. Furthermore, the assumption $s\ C$ of $\mathfrak{D}_4$ is not part of a new redex of degree $d+1$ in $\mathfrak{D}'$: In the case where $\{s\ B\}$ is empty or singular, we know by the structure of the rules that $s\ C$, as well as $s\ B$ are both subformulas of $s\ A_n$ and therefore, of lower degree; if $\{s\ B\}$ consists of two formulas,[3] $s\ C$ could only become a new redex of degree $d+1$, if it was already such a redex in $\mathfrak{D}$, which contradicts our choice of the redex. Therefore, we can observe that no new redexes arise by this reduction step and the rank decreases.

3.2. Let the ξ-redex in $\mathfrak{D}$ be a formula occurrence $s\ A$. Here eight cases arise, i.e. for each of the four connectives a case for each sign. We consider two of them. All other are analogous to these.

i. Let $s\ A = +\ A_1 \to A_2$. In a $(\xi_\to)$ step, $\mathfrak{D}$ reduces to $\mathfrak{D}'$ as follows:

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\dot{x}\\ -\ A_1 \to A_2\\ \mathfrak{D}_0\\ \bot\end{matrix}}{+\ A_1 \to A_2}\dot{x} \qquad \begin{matrix}\mathfrak{D}_1\\ +\ A_1\end{matrix}}{+\ A_2}\\ \mathfrak{D}_2\end{matrix} \quad \overset{(\xi_\to)}{\rightsquigarrow} \quad \begin{matrix}\dfrac{\begin{matrix}\mathfrak{D}_1\\ +\ A_1\end{matrix} \qquad \begin{matrix}\dot{x}_n\\ -\ A_2\end{matrix}}{-\ A_1 \to A_2}\\ \mathfrak{D}_0\\ \dfrac{\bot}{+\ A_2}\dot{x}_n\\ \mathfrak{D}_2\end{matrix}$$

The rank of $\mathfrak{D}'$ is either $\langle d+1, 0, x-1, j', b'\rangle$ or $\langle d', m', x', j', b'\rangle$. $\mathfrak{D}'$ has rank $\langle d+1, 0, x-1, j', b'\rangle$, with $j' \geq j$ and $b' \geq b$, if new redexes $-\ A_1 \to A_2$ that are the conclusions of $(\to I_-)$ arise. $\mathfrak{D}'$ has rank $\langle d', m', x', j', b'\rangle$, with $d' \leq d$ and $m', x', j', b' \in \mathbb{N}_0$, if the considered redex is the only redex of degree $d+1$ and no new redex was created by this reduction step in $\mathfrak{D}'$. By induction hypothesis $\mathfrak{D}'$ is normalizable.

ii. Let $s\ A = +\ A_1 \vee A_2$ and let WLOG the conclusion of $(\vee E_+)$ be $+\ C$. In a $(\xi_{\vee+})$ step, $\mathfrak{D}$ reduces to $\mathfrak{D}'$ as follows:

[3] This can only be the case, if $(\circ E_s) = (\vee E_+)$ or $(\circ E_s) = (\wedge E_-)$.

$$
\dfrac{\dfrac{\begin{array}{c}\dot{x}\\ -\ A_1 \vee A_2\\ \mathfrak{D}_0\\ \bot\end{array}}{+\ A_1 \vee A_2}\,(\mathrm{R}_+)^{\dot{x}} \qquad \begin{array}{c}x\\ +\ A_1\\ \mathfrak{D}_1\\ +\ C\end{array} \qquad \begin{array}{c}y\\ +\ A_2\\ \mathfrak{D}_2\\ +\ C\end{array}}{\begin{array}{c}+\ C\\ \mathfrak{D}_3\end{array}}\,(\vee E_+)^{x,y} \quad \overset{(\xi_{\vee +})}{\rightsquigarrow}
$$

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\begin{array}{c}x\\ +\ A_1\\ \mathfrak{D}_1\\ +\ C\end{array} \qquad \begin{array}{c}\dot{x}_n\\ -\ C\end{array}}{\bot}}{-\ A_1}\,x \qquad \dfrac{\dfrac{\begin{array}{c}y\\ +\ A_2\\ \mathfrak{D}_2\\ +\ C\end{array} \qquad \begin{array}{c}\dot{x}_n\\ -\ C\end{array}}{\bot}}{-\ A_2}\,y}{-\ A_1 \vee A_2}\\
\mathfrak{D}_0\\
\dfrac{\bot}{+\ C}\,(\mathrm{R}_+)^{\dot{x}_n}\\
\mathfrak{D}_3
\end{array}
$$

As usual, we need to consider every possible new redex in $\mathfrak{D}'$. By the choice of our redex, the assumption $+\ C$ of $\mathfrak{D}_3$ cannot be a redex of degree $d+1$ in $\mathfrak{D}$. Thus, it cannot be a redex in $\mathfrak{D}'$ either.[4] For the conclusions of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ the situation is more complicated. We might create new μ-redexes, depending on the structure of $\mathfrak{D}_1$ and $\mathfrak{D}_2$. Thus, we prove that $\mathfrak{D}'$ with rank $\langle d+1, m', x-1, j', b'\rangle$ is normalizable by induction on m'.[5]

Induction base:

For $m' = 0$: The rank of $\mathfrak{D}'$ with $\langle d+1, 0, x-1, j', b'\rangle$ is already lower than the rank of $\mathfrak{D}$.

Induction hypothesis on m':

The assertion holds for all k, if $k < m' + 1$.

Induction step:

For $m' + 1$: The form of $\mathfrak{D}_1$ can be WLOG depicted in one of two kinds. Either it ends with an application of (R_+) or with a segment. If it ends with (R_+), $\mathfrak{D}_1$ can be depicted as:

[4] The assumption $+\ C$ of $\mathfrak{D}_3$ can only become a redex in the contractum, if the assumption is part of a redex segment in $\mathfrak{D}$. We choose our considered redex as the right-most, wherefore this cannot be the case.

[5] The degree of $+\ C$ might be higher than $d+1$, then the degree of $\mathfrak{D}'$ increases as well, but $+\ C$ would be the only redex of this degree and no new redexes of that higher degree would arise in the subinduction. Furthermore, all redexes of this higher degree are removable, such that the degree of the derivation will directly drop back to $d+1$. Thus, WLOG we assume $\deg s\ C = d+1$.

$$\dfrac{\begin{array}{c}\dot{y}_n\\ -\,C\\ \mathfrak{D}_{1.1}\\ \bot\end{array}}{+\,C}\,(\mathrm{R}_+)$$

Then, the form of $\mathfrak{D}'$ can be depicted with:

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\dot{y}_n\\ -\,C\\ \mathfrak{D}_{1.1}\\ \bot\end{array}}{\color{magenta}{+\,C}}\,(\mathrm{R}_+) \qquad \begin{array}{c}\dot{x}_n\\ -\,C\end{array}}{\bot}}{-\,A_1}\,x \qquad \dfrac{\dfrac{\begin{array}{c}y\\ +\,A_2\\ \mathfrak{D}_2\\ +\,C\end{array} \qquad \begin{array}{c}\dot{x}_n\\ -\,C\end{array}}{\bot}}{-\,A_2}\,y}{-\,A_1 \vee A_2}\\ \mathfrak{D}_0\\ \dfrac{\bot}{+\,C}\,(\mathrm{R}_+)^{\dot{x}_n}\\ \mathfrak{D}_3\end{array}$$

Here the magenta marked $+\ C$ becomes a μ-redex. We can apply (μ_L) on the magenta marked redex, to reduce $\mathfrak{D}'$ to $\mathfrak{D}''$:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\dot{x}_n\\ -\,C\\ \mathfrak{D}_{1.1}\\ \bot\end{array}}{-\,A_1}\,x \qquad \dfrac{\dfrac{\begin{array}{c}y\\ +\,A_2\\ \mathfrak{D}_2\\ +\,C\end{array} \qquad \begin{array}{c}\dot{x}_n\\ -\,C\end{array}}{\bot}}{-\,A_2}\,y}{-\,A_1 \vee A_2}\\ \mathfrak{D}_0\\ \dfrac{\bot}{+\,C}\,(\mathrm{R}_+)^{\dot{x}_n}\\ \mathfrak{D}_3\end{array}$$

Comparing the ranks of $\mathfrak{D}'$ and $\mathfrak{D}''$, the rank reduced from $\langle d+1, m'+1, x-1, j', b'\rangle$ to $\langle d+1, m', x', j', b'\rangle$. Then, we can apply the induction hypothesis of m' on $\mathfrak{D}''$. Hence, $\mathfrak{D}''$ reduces to a derivation with lower rank than $\mathfrak{D}$. Thereafter, we can apply the (initial) induction hypothesis and infer that $\mathfrak{D}$ is normalizable.

Now assuming $\mathfrak{D}_1$ ends with a segment $C_1, \ldots, C_k$, it can be depicted as:

$$\dfrac{\begin{array}{c}\mathfrak{D}_{1.1}\\ s\ D \circ E\end{array} \qquad \begin{array}{c}\mathfrak{D}_{1.2}\\ +\,C_{k-1}\end{array} \qquad \begin{array}{c}\mathfrak{D}_{1.3}\\ +\,C_{k-1}\end{array}}{+\,C_k}$$

$C_j = C$ and it might be the case that neither of the minor premises are part of a segment of length k. However, this is negligible, as the following argumentation is going to show. The form of $\mathfrak{D}'$ can be depicted with:

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_{1.1}\\ s\ D\circ E\end{array} \quad \begin{array}{c}\mathfrak{D}_{1.2}\\ +\,C_{k-1}\end{array} \quad \begin{array}{c}\mathfrak{D}_{1.3}\\ +\,C_{k-1}\end{array}}{+\,C_k} \qquad \begin{array}{c}\dot{x}_n\\ -\,C\end{array}}{\bot}}{-\,A_1}\,x \qquad \dfrac{\dfrac{\begin{array}{c}y\\ +\,A_2\\ \mathfrak{D}_2\\ +\,C\end{array} \qquad \begin{array}{c}\dot{x}_n\\ -\,C\end{array}}{\bot}}{-\,A_2}\,y}{-\,A_1 \vee A_2}\\
\mathfrak{D}_0\\
\dfrac{\bot}{+\,C}\ (\mathrm{R}_+)^{\dot{x}_n}\\
\mathfrak{D}_3
\end{array}
$$

The magenta marked segments become μ-redexes. We can apply a γ-reduction step on these segments, to reduce $\mathfrak{D}'$ to $\mathfrak{D}''$:

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_{1.1}\\ s\ D\circ E\end{array} \quad \dfrac{\begin{array}{c}\mathfrak{D}_{1.2}\\ +\,C_{k-1}\end{array} \quad \begin{array}{c}\dot{x}_n\\ -\,C\end{array}}{\bot} \quad \dfrac{\begin{array}{c}\mathfrak{D}_{1.3}\\ +\,C_{k-1}\end{array} \quad \begin{array}{c}\dot{x}_n\\ -\,C\end{array}}{\bot}}{\bot}}{-\,A_1}\,x \qquad \dfrac{\dfrac{\begin{array}{c}y\\ +\,A_2\\ \mathfrak{D}_2\\ +\,C\end{array} \qquad \begin{array}{c}\dot{x}_n\\ -\,C\end{array}}{\bot}}{-\,A_2}\,y}{-\,A_1 \vee A_2}\\
\mathfrak{D}_0\\
\dfrac{\bot}{+\,C}\ (\mathrm{R}_+)^{\dot{x}_n}\\
\mathfrak{D}_3
\end{array}
$$

Comparing the ranks of $\mathfrak{D}'$ and $\mathfrak{D}''$, we observe that the rank reduced from $\langle d+1, m'+1, x-1, j', b'\rangle$ to at least $\langle d+1, m'-l, x', j', b'\rangle$, where $l > 1$ is the number of siblings, that end in $+\,C_k$. Then, we can apply the induction hypothesis of m' on $\mathfrak{D}''$. Hence, $\mathfrak{D}''$ reduces to a derivation with a lower rank than $\mathfrak{D}$. Thereafter, we can apply the (initial) induction hypothesis and infer that $\mathfrak{D}$ is normalizable.

4. Let $m > 0$. Among those μ-redexes with degree $d+1$, we can choose one non-deterministically such that the μ-redex is conclusion of (R_s) and premise of (Nc.) or a segment ending with an application of (Nc.). In both cases, we choose such that the immediate subderivations of (Nc.) have no redexes of degree $d+1$. Analogous to the other cases, we can find such a μ-redex.

Proving this last case of the normalization is complex. Therefore, we want to summarize the idea behind the following subcases and compare this proof with the normalization proof of Bi-ND (see Theorem 2). For Bi-ND, we had to perform a subinduction. The inductive case arises when the reduction of a critical μ-redex generates new μ-redexes. In Bi-ND$_{\text{FULL}}$, the definition of μ-redexes is extended by a case for segments, thus we have three combinations for critical μ-redexes:

- two critical μ-redexes, where each of them is a formula occurrence,
- two critical μ-redexes, where each of them is a segment ending on the same (Nc.) application, or
- two critical μ-redexes, where one of them is a formula occurrence and the other is a segment.[6]

We begin our deliberations with μ-redexes, where the other premise is not a μ-segment. This was already considered in the proof of case (3.1) in Theorem 2 and is analogue here. Thus, we omit it here.

What is left are the other two combinations of critical μ-redexes and the case where a μ-segment is not part of a pair of critical μ-redexes. For all three, a single γ-reduction step in combination with the induction hypothesis over the rank is enough.

4.1. Let the μ-redex be a segment $A_1, \ldots, A_n$, with the condition that the other premise of the relevant (Nc.) is not a μ-redex. Assume further WLOG that this redex is the right premise of the relevant (Nc.) application and the segment ends with $(\vee E_+)$.[7] In a γ-reduction step, $\mathfrak{D}$ reduces to $\mathfrak{D}'$ as follows:

$$\dfrac{\begin{matrix}\mathfrak{D}_3\\ +\,B\end{matrix} \qquad \dfrac{\begin{matrix}\mathfrak{D}_0\\ +\,D \vee E\end{matrix} \qquad \begin{matrix}\mathfrak{D}_1\\ -\,A_{n-1}\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ -\,A_{n-1}\end{matrix}}{-\,A_n}\,(\vee E_+)}{\begin{matrix}\bot\\ \mathfrak{D}_4\end{matrix}}\,(\text{Nc.}) \quad \overset{(\mu_R)}{\rightsquigarrow}$$

$$\dfrac{\begin{matrix}\mathfrak{D}_0\\ +\,D \vee E\end{matrix} \qquad \dfrac{\begin{matrix}\mathfrak{D}_3\\ +\,B\end{matrix} \qquad \begin{matrix}\mathfrak{D}_1\\ -\,A_{n-1}\end{matrix}}{\bot}\,(\text{Nc.}) \qquad \dfrac{\begin{matrix}\mathfrak{D}_3\\ +\,B\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ -\,A_{n-1}\end{matrix}}{\bot}\,(\text{Nc.})}{\begin{matrix}\bot\\ \mathfrak{D}_4\end{matrix}}\,(\vee E_+)$$

The rank of $\mathfrak{D}'$ is then at most $\langle d+1, m-2, x+x', j+j', b+b' \rangle$, with $x', j', b' \geq 0$. By our redex choice, $\mathfrak{D}_3$ avoids μ-redex duplication within $\mathfrak{D}'$.

[6] The latter pair of critical μ-redexes is also called *mixed critical μ-pair*.

[7] WLOG, new parts of segments will also be created with $(\vee E_+)$.

4.2. Now consider the case where the analyzed segment $A_1, \ldots, A_n$ is part of a *mixed* critical μ-pair. Furthermore, assume that the segment is like case (4.1.). In a γ-reduction step, $\mathfrak{D}$ reduces to $\mathfrak{D}'$ as follows:

$$\dfrac{\dfrac{\begin{matrix}\dot{x}\\ -B\\ \mathfrak{D}_{3.1}\\ \perp\end{matrix}}{+B}\,(\mathrm{R}_+)^{\dot{x}} \qquad \dfrac{\begin{matrix}\mathfrak{D}_0\\ +D\vee E\end{matrix} \quad \begin{matrix}\mathfrak{D}_1\\ -A_{n-1}\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ -A_{n-1}\end{matrix}}{-A_n}\,(\vee E_+)}{\begin{matrix}\perp\\ \mathfrak{D}_4\end{matrix}}\,(\mathrm{Nc.}) \quad \overset{(\mu_R)}{\rightsquigarrow}$$

$$\dfrac{\begin{matrix}\mathfrak{D}_0\\ +D\vee E\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\dot{x}\\ -B\\ \mathfrak{D}_{3.1}\\ \perp\end{matrix}}{+B}\,(\mathrm{R}_+)^{\dot{x}} \quad \begin{matrix}\mathfrak{D}_1\\ -A_{n-1}\end{matrix}}{\perp}\,(\mathrm{Nc.}) \qquad \dfrac{\dfrac{\begin{matrix}\dot{x}\\ -B\\ \mathfrak{D}_{3.1}\\ \perp\end{matrix}}{+B}\,(\mathrm{R}_+)^{\dot{x}} \quad \begin{matrix}\mathfrak{D}_2\\ -A_{n-1}\end{matrix}}{\perp}\,(\mathrm{Nc.})}{\begin{matrix}\perp\\ \mathfrak{D}_4\end{matrix}}\,(\vee E_+)$$

The rank of $\mathfrak{D}'$ is at most $\langle d+1, m-2+1, x+x', j+j', b+b'\rangle$, with $x', j', b' \geq 0$. Although the μ-redex $+\,B$ is duplicated here, the rank of the derivation decreases by at least two. This is due to our definition of μ-redexes in the case of segments. A μ-redex segment comes always with at least one sibling.

4.3. Let the μ-redex be a segment $A_1, \ldots, A_n$, with the condition that the other premise of the relevant (Nc.) is a μ-redex segment $B_1, \ldots, B_k$. Furthermore, assume that both segments are like case (4.1.). In a γ-reduction step, $\mathfrak{D}$ reduces to $\mathfrak{D}'$ as follows:

$$\dfrac{\begin{matrix}\mathfrak{D}_3\\ +B_k\end{matrix} \qquad \dfrac{\begin{matrix}\mathfrak{D}_0\\ +D\vee E\end{matrix} \quad \begin{matrix}\mathfrak{D}_1\\ -A_{n-1}\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ -A_{n-1}\end{matrix}}{-A_n}\,(\vee E_+)}{\begin{matrix}\perp\\ \mathfrak{D}_4\end{matrix}}\,(\mathrm{Nc.}) \quad \overset{(\mu_R)}{\rightsquigarrow}$$

$$\dfrac{\begin{matrix}\mathfrak{D}_0\\ +D\vee E\end{matrix} \qquad \dfrac{\begin{matrix}\mathfrak{D}_3\\ +B_k\end{matrix} \quad \begin{matrix}\mathfrak{D}_1\\ -A_{n-1}\end{matrix}}{\perp}\,(\mathrm{Nc.}) \qquad \dfrac{\begin{matrix}\mathfrak{D}_3\\ +B_k\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ -A_{n-1}\end{matrix}}{\perp}\,(\mathrm{Nc.})}{\begin{matrix}\perp\\ \mathfrak{D}_4\end{matrix}}\,(\vee E_+)$$

In $\mathfrak{D}'$, we know that both subderivations of the minor premises of $(\vee E_+)$ have a lower rank than $\mathfrak{D}$. This is because both consist of single occurrences of $\mathfrak{D}_3$ and $\mathfrak{D}_2$ or $\mathfrak{D}_3$ and $\mathfrak{D}_1$, respectively. Let the minor premises in $\mathfrak{D}'$ be called $\mathfrak{D}'_{MP-L}$ and $\mathfrak{D}'_{MP-R}$. By induction hypothesis, we can reduce $\mathfrak{D}'_{MP-L}$ and $\mathfrak{D}'_{MP-R}$ into normal forms $\mathfrak{D}''_{MP-L}$ and $\mathfrak{D}''_{MP-R}$. Let $\mathfrak{D}''$ be the result of the normalizations of these minor premises. In $\mathfrak{D}''$, there are no μ-redexes of degree $d+1$ left: in $\mathfrak{D}_0$, there are no μ-redexes by choice of $\mathfrak{D}$; $\mathfrak{D}''_{MP-L}$ and $\mathfrak{D}''_{MP-R}$ are in normal form, thus without μ-redexes. As a final thought, the conclusions $\perp$ of $\mathfrak{D}''_{MP-L}$ and $\mathfrak{D}''_{MP-R}$ cannot become new μ-redexes, since $\perp$ cannot be applied on (Nc.). Conclusively, by the induction hypothesis $\mathfrak{D}''$, and by transitivity of the reduction relation, $\mathfrak{D}$ can be normalized.

□

Remark. As previously stated, the proof strategy here is uniform to those of Theorem 2 and 3. The only significant difference is the possible occurrence of segments in BI-ND$_{\text{FULL}}$. This is especially of interest in the case of μ-redexes, where the addition of segments leads to a refined case distinction over critical μ-redexes. Nevertheless, we were able to outsource the technical details of β- and ι-redexes, as well as parts of μ-redexes, into the previous theorems. Outsourcing was possible, since all three introduced ranks count the same, if we exclude segments.

To give a brief summary of the reduction strategy that arises from this proof: We choose μ-redexes over ξ-redexes over ι-redexes over β-redexes non-deterministically.[8] Among those redexes, if we choose a μ-redex, it is always *top-most*, for all other redex kinds we choose *top-most* and *right-most*. For *mixed* critical μ-pairs: we choose μ-segments over μ-redexes that are formula occurrences. In all other critical μ-pairs, we choose one of them non-deterministically.

5.3 Subformula property

This section will follow Troelstra and Schwichtenberg's structure from [TS00, Sec. 6.2]. We begin by defining *main* and *minor premises* for (Nc.) in normal derivations. Thereafter, we define *tracks* and *ranks of tracks*. We end this section by proving two characteristics of tracks in normal derivations and by proving the subformula property for BI-ND$_{\text{FULL}}$.

Consider derivations in normal form ending with (Nc.). None of the immediate subderivations of (Nc.) end with (R$_s$) or (Nc.). Hence, only introduction and elimination rules are left as possible endings of the subderivations of (Nc.). One of the immediate subderivations might end with an introduction rule, but not both, else we would have ι-redexes. Thus, at

[8] We could have also chosen β-redexes over ι-redexes.

least one of the subderivations needs to end with an elimination rule. We call the premise $s\ A$ of (Nc.) the *main premise*, if $s\ A$ is the conclusion of an elimination rule. We call the premise $s\ A$ of (Nc.) *minor premise*, if $s\ A$ is the conclusion of an introduction rule. It is to be observed that both premises can be main premises.

Definition 14. A *track* of a derivation $\mathfrak{D}$ is a sequence of formula occurrences $F_0^\perp, \ldots, F_n^\perp$ such that

1. $F_0^\perp = s \overset{x}{A}_0 \in FV(\mathfrak{D})$ and $F_0^\perp$ is not discharged by $(\vee E_+)$ or $(\wedge E_-)$.
2. if $j < n$, then $F_j^\perp$ is not a minor premise of $(\to E_+), (\leftarrow E_-)$ or (Nc.), and either
   (a) $F_j^\perp$ is not the main premise of $(\vee E_+)$ or $(\wedge E_-)$ and $F_{j+1}^\perp$ is directly below $F_j^\perp$, or
   (b) $F_j^\perp$ is the main premise of $(\vee E_+)$ or $(\wedge E_-)$ and $F_{j+1}^\perp$ is an assumption discharged by that $(\vee E_+)$ or $(\wedge E_-)$.
3. $F_n^\perp$ is either
   (a) a minor premise of an elimination rule, or
   (b) the minor premise of (Nc.), or
   (c) the conclusion of $\mathfrak{D}$, or
   (d) the main premise of an instance of $(\vee E_+)$ or $(\wedge E_-)$, which has no assumptions discharged.

Remark. Let $F_0^\perp, \ldots, F_n^\perp$ be a track π in $\mathfrak{D}$. We can regroup the track to $\sigma_0, \ldots, \sigma_k$, where σ_j is a formula occurrence or a segment in the track π and the order of the track is unchanged.

If σ_j is a segment, then we call the rule of which the last formula of the segment is a premise the *premise of* σ_j. Analogous for the rule of which the first formula of the segment is a premise.

Proposition 2. *Let $\mathfrak{D}$ be a normal derivation in* Bi-ND$_{\text{FULL}}$*, and let $\sigma_1, \ldots, \sigma_n$ be a track π in $\mathfrak{D}$. Then, there is a segment σ_k (or formula occurrence) in the track, the* minimum segment *or* minimum part *of π, which separates two (possibly empty) parts of π, called* E-part *(elimination part) and the* I-part *(introduction part) of π such that:*

1. *for each σ_j in the E-part one has $j < k$. σ_j is a main premise of an elimination rule and σ_{j+1} is a subformula of σ_j or σ_j is a main premise of (Nc.), but then $\sigma_{j+1} = \perp$ is the minimum part σ_k;*

2. *for each σ_j in the I-part one has $k < j$. If $j \neq n$, then σ_j is premise of an introduction rule and σ_j is a subformula of σ_{j+1};*

3. *if $k \neq n$, then σ_k is a premise of an introduction rule or a premise of (R_s) and σ_k is a subformula of σ_1.*

Proof. By induction on the structure of normal derivations. □

Definition 15. A *track of order* 0, or *main track*, in a normal derivation $\mathfrak{D}$ is a track ending in the conclusion of $\mathfrak{D}$. A *track of order* $n+1$ is a track ending in a minor premise, with main premise belonging to a track of order n.

Proposition 3. *In a normal derivation each formula occurrence belongs to some track.*

Proof. By induction on the structure of normal derivations. □

Theorem 5 (Subformula property)**.** *Let $\mathfrak{D}$ be a normal derivation in* BI-ND$_{\text{FULL}}$ *with conclusion $s\ A$. Then, each formula in $\mathfrak{D}$ is a subformula of a formula in $FV(\mathfrak{D}) \cup \{A\}$ or it is $\bot$.*

Proof. By induction on n for tracks of order n. □

Chapter 6

Conclusion

We began by presenting an extension of a symmetric λ-calculus formulated by Abe and Kimura. This should particularly demonstrate how proofs of rejections can be interpreted as programs and what their respective interplay with expressions looks like.

Thereafter, we presented Rumfitt's bilateral natural deduction system Bi-ND$_{\text{FULL}}$ extended by the primitive but-not connective. We embedded this full system into two of its fragments. The first fragment Bi-ND corresponds to the symmetric λ-calculus from Chapter 2. The second fragment Bi-ND$_{\text{INT}}$ corresponds to Lovas and Crary's symmetric λ-calculus.

Then we proved the normalization theorems of all three systems uniform by induction over proper ranks. The difficult cases were almost always solved by subinductions that counted possible new redexes, which might arise from a reduction step. For each of the three systems Bi-ND$_{\text{FULL}}$, Bi-ND and Bi-ND$_{\text{INT}}$, there are three similar systems in the literature for which also normalization theorems were proven. We briefly compare our proofs to those proofs given in the literature.

For Bi-ND$_{\text{INT}}$ without implication and but-not, Lovas and Crary proved the normalization in a substantially different manner. They analyzed the normal forms and used the relation between neutral and normal derivations in Bi-ND$_{\text{INT}}$ to prove the normalization by induction on the structure of derivations. This path needs more distinctions than our proof strategy needed. Thus, our proof is significantly shorter.

Kürbis published a normalization proof for Bi-ND$_{\text{FULL}}$ that had a more relaxed rank than ours, where the rank counted the number of all redexes, without distinguishing them as we did [Kü21a, p.542ff]. For critical μ-redexes, he chose to always reduce the right one, so that his reduction strategy implements some key elements of the call-by-name reduction strategy mentioned in Example 7. It was later pointed out by Pedro del Valle-Inclan that Kürbis normalization strategy does not always decrease his rank [Kü21b, p.2257f]. In his *Addenda*, Kürbis addresses possible suggestions to solve this problem [Kü21b, p.2257f], but these suggestions require further investigation.

For the fragment of Bi-ND without conjunction, disjunction and negation, Abe and Kimura gave a proof that uses—implicitly—the same rank as Kürbis. Their presentation keeps the technical details of the proof implicit. However, the case of critical μ-redexes is handled essentially in the same way as we handled it. Although their proof might scale to the full system Bi-ND$_{\text{FULL}}$, the rank that we choose to assign to derivations (according to which the different kinds of redexes are ordered, beginning with μ-redexes and then continuing with ξ- and β-redexes) yields a much clearer overall structure in the proof.

In addition, we were able to connect the uniform proofs of normalizations for Bi-ND, Bi-ND$_{\text{INT}}$ and Bi-ND$_{\text{FULL}}$. The proof of the full system uses the same argumentation as the fragments in almost all cases that are already in the fragments. This relation might be useful for future investigations on the strong normalization of bilateral natural deduction systems. For now, strong normalization for all three systems presented here is an open question.

At last, we proved the subformula property. Rumfitt gives a proof for the *separability* in [Rum00, p.809f]. Separability follows from the subformula property, but the subformula property does not follow from separability. Thus, our result is stronger. The subformula property is a significant difference to standard natural deduction systems for classical logic. To give a prominent example: consider Pierce's Law. In standard natural deduction systems, we need negations of subformulas to prove it. In bilateral systems, negations can be avoided by negative signs, if a negation is the main connective of the formula. This can also be observed in our proof of Pierce's Law (see Example 4). The subformula property might also be a chance for natural deduction to become a viable option in automated proof search for algorithmic treatment, possibly rivalling tableaux, sequent calculus and resolution [DGM20, p.342]. D'Agostino et. al. present a different bilateral natural deduction system to achieve this [DGM20]. They also prove the normalization and the subformula property of their system [DGM20].

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