Problem 3.

Proof: Let $p = \langle v_1 \dots v_k \rangle$ be a k-length path in the DAG, $G(l)$ be the sub-DAG induced by $v \in V$, where $\rho(v) = l$.

For each $d \in \{0, \dots, \log\Delta\}$, subsets $Q = \{u \in U \mid |N(u) \cap U| \le 2^d\}$ are removed exactly r times. Note that l is assigned $\underset{= \rho(v)}{i}$ in the SLL-Priority-Assignment, whilst i is ~~increm~~ incremented by 1 r times in the ~~inner~~ second-layer for-loop, We have $\lfloor l/r \rfloor = d$.

For $v_i \in P \cap G(l)$, it always holds that

$$\rho(v_{i-1}) \le \rho(v_i) = l.$$

Thus, $v_{i-1} \in G(l')$, where $l' \le l$.

Decompose p into a set of paths:

$$p = \langle p_{\lceil \log\Delta \cdot r \rceil} \cdots p_1 \rangle.$$

(Note that l can never be 0 as i starts from 1)

In each path p_l, the degree Δ_l can be no bigger than $2^d = 2^{\lfloor l/r \rfloor}$.

By the ~~IP-R~~ Some Theorem:

$$E[|p_l|] = \cancel{O(\log V + \lfloor l/r \rfloor \cdot \min\{\sqrt{E}, 2^{\lfloor l/r \rfloor} +} \; O(2^{\lfloor l/r \rfloor} + \lfloor l/r \rfloor \lg V / \lg\lg V) \; \cancel{\lfloor l/r \rfloor \log V / \log(\log V)\})}$$

$$\text{Thus } E[|p|] = \sum_{l=1}^{\lceil \log\Delta \cdot r \rceil} E[|p_l|] = \sum_{l=1}^{\lceil \lg\Delta \rceil} O(2^{\lfloor l/r \rfloor} + \lfloor l/r \rfloor \lg V / \lg\lg V)$$

$$= O(\Delta + \lg^2\Delta \lg V / \lg\lg V).$$

Observe that at most $E/2^{\lfloor l/r \rfloor}$ vertices have degree at least $2^{\lfloor l/r \rfloor}$, we have that the longest path in $G(l)$ is shorter than $E/2^{\lfloor l/r \rfloor}$, for $l > \lceil r \lg \sqrt{E} \rceil$.

Thus: $$E[|p|] \le \sum_{l=0}^{\lceil r \lg \sqrt{E} \rceil} O(2^{\lfloor l/r \rfloor}) + \sum_{l > \lceil r \lg \sqrt{E} \rceil}^{\lceil r \lg \Delta \rceil} E/2^{\lfloor l/r \rfloor}$$
$$+ \sum_{l=0}^{\lceil r \lg \Delta \rceil} O(\lfloor l/r \rfloor \lg V / \lg \lg V)$$
$$= O(\sqrt{E} + \lg^2 \Delta \lg V / \lg \lg V).$$

To summarize: $E[|p|] = O(\min\{\sqrt{E}, \Delta\} + \lg^2 \Delta \lg V / \lg \lg V)$.