Exercise 1

1.

From the definition of the probability density function for gaussian distribution, we have:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)$$
$$f_Y(z-x) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(z-x-\mu_2)^2}{2\sigma_2^2}\right)$$

$$f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(z-x-\mu_2)^2}{2\sigma_2^2}\right) dx$$
$$= \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(z-x-\mu_2)^2}{2\sigma_2^2}\right) dx$$
$$= \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} \int_{-\infty}^{\infty} \exp\left(-\left[\frac{(x-\mu_1)^2}{2\sigma_1^2} + \frac{(z-x-\mu_2)^2}{2\sigma_2^2}\right]\right) dx$$

$$= \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} \int_{-\infty}^{\infty} \exp\left(-\left[x^2\left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2}\right) - x\left(\frac{\mu_1}{\sigma_1^2} + \frac{z-\mu_2}{\sigma_2^2}\right) + \left(\frac{\mu_1^2}{2\sigma_1^2} + \frac{(z-\mu_2)^2}{2\sigma_2^2}\right)\right]\right) dx$$

let:

$$A = \left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2}\right), \quad B = \left(\frac{\mu_1}{\sigma_1^2} + \frac{z-\mu_2}{\sigma_2^2}\right), \quad C = \left(\frac{\mu_1^2}{2\sigma_1^2} + \frac{(z-\mu_2)^2}{2\sigma_2^2}\right)$$

we get:

$$= \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} \int_{-\infty}^{\infty} \exp\left(-Ax^2 + Bx - C\right) dx$$

Now complete the square in the exponent:

$$-Ax^2 + Bx - C = -A\left(x - \frac{B}{2A}\right)^2 + \frac{B^2}{4A} - C$$

Thus the integral becomes:

$$f_Z(z) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} \exp\left(\frac{B^2}{4A} - C\right) \int_{-\infty}^{\infty} \exp\left(-A\left(x - \frac{B}{2A}\right)^2\right) dx$$

We now rewrite the exponent to match the standard Gaussian integral form:

$$-A\left(x - \frac{B}{2A}\right)^2 = -\frac{(x - \frac{B}{2A})^2}{2b^2} \quad \text{where} \quad b^2 = \frac{1}{2A}$$

Then:

$$\int_{-\infty}^{\infty} \exp\left(-A\left(x - \frac{B}{2A}\right)^2\right) dx = \int_{-\infty}^{\infty} \exp\left(-\frac{(x-a)^2}{2b^2}\right) dx = \sqrt{2\pi b^2}$$

Substitute $b^2 = \frac{1}{2A} \Rightarrow \sqrt{2\pi b^2} = \sqrt{\frac{\pi}{A}}$, so:

$$f_Z(z) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} \cdot \sqrt{2\pi b^2} \cdot \exp\left(\frac{B^2}{4A} - C\right) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} \cdot \sqrt{2\pi \cdot \frac{1}{2A}} \cdot \exp\left(\frac{B^2}{4A} - C\right)$$

Here:

$$\frac{1}{2A} = \frac{1}{2\left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2}\right)} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$$\sqrt{2\pi b^2} = \sqrt{2\pi \cdot \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}} = \sigma_1\sigma_2 \cdot \sqrt{\frac{2\pi}{\sigma_1^2 + \sigma_2^2}}$$

$$\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} \cdot \sqrt{2\pi b^2} = \frac{1}{2\pi\sigma_1\sigma_2} \cdot \left(\sigma_1\sigma_2 \cdot \sqrt{\frac{2\pi}{\sigma_1^2 + \sigma_2^2}}\right) = \frac{1}{2\pi} \cdot \sqrt{\frac{2\pi}{\sigma_1^2 + \sigma_2^2}}$$

$$= \frac{\sqrt{2\pi}}{2\pi} \cdot \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}}$$

Now we get:

$$f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \cdot \exp\left(\frac{B^2}{4A} - C\right)$$

$$\frac{B^2}{4A} - C = -\frac{(z - (\mu_1 + \mu_2))^2}{2(\sigma_1^2 + \sigma_2^2)}$$

Substitute, we get:

$$f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \cdot \exp\left(-\frac{(z - (\mu_1 + \mu_2))^2}{2(\sigma_1^2 + \sigma_2^2)}\right)$$

This is equivalent to:

$$Z = X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

‘

2.

Because of padding happens on all four sides, the new dimensions become:

$$H_{\text{pad}} = H + 2p, \quad W_{\text{pad}} = W + 2p$$

The filter of size $K_H \times K_W$ moves over the padded image with a stride of s. This means the filter slides over the image in steps of s pixels. Then, this becomes

$$H_{\text{out}} = \left\lfloor \frac{H_{\text{pad}} - K_H}{s} \right\rfloor + 1 = \left\lfloor \frac{H + 2p - K_H}{s} \right\rfloor + 1, \quad W_{\text{out}} = \left\lfloor \frac{W_{\text{pad}} - K_W}{s} \right\rfloor + 1 = \left\lfloor \frac{W + 2p - K_W}{s} \right\rfloor + 1$$