COMP4670 Assignment 1 Theory Answers

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Exercise 1 Properties of Independent Variables

1

By definition of expectation of a continuous random variable:

$$\mathbf{E}_X = \int_\Omega X(\omega)dP(\omega), where\, x, y, x+y \in \Omega$$

Therefore,

$$\begin{aligned}\mathbf{E}_{X,Y}[x+y] &= \iint (x+y)f(x,y)dxdy, \text{ f is the probability density function of x,y} \\ &= \iint xf(x,y)dxdy + \iint yf(x,y)dxdy \\ &= \mathbf{E}_X[x] + \mathbf{E}_Y[y] \quad \text{Q.E.D.}\end{aligned}$$

2

By definition of expectation of a continuous random variable:

$$\mathbf{E}_X = \int_\Omega X(\omega)dP(\omega), where\, x, y, xy \in \Omega$$

Therefore,

$$\mathbf{E}_{X,Y}[xy] = \iint xyf(x,y)dxdy, \text{ f is the probability density function of x,y}$$
$$\text{Because X and Y are independent from each other, we have} \quad f(x,y) = f(x)f(y)$$

Then we have

$$\begin{aligned}\mathbf{E}_{X,Y}[xy] &= \iint xyf(x)f(y)dxdy \\ &= \int xf(x)dx \int yf(y)dy \\ &= \mathbf{E}_X\mathbf{E}_Y \quad Q.E.D.\end{aligned}$$

3

By definition of convariance:

$$cov[x,y] = \mathbf{E}_{X,Y}[x - \mathbf{E}_X[x])(y - \mathbf{E}_Y[y])]$$

Therefore,

$$cov[x,y] = \mathbf{E}[xy - y\mathbf{E}_X[x] - x\mathbf{E}_Y[y] + \mathbf{E}_X\mathbf{E}_Y]$$

From the proof mentioned in 1.1,1.2:

$$\begin{gathered}\mathbf{E}_{X,Y}[xy] = \mathbf{E}_X\mathbf{E}_Y \quad (1) \\ \mathbf{E}_{X,Y}[x+y] = \mathbf{E}_X[x] + \mathbf{E}_Y[y] \quad (2)\end{gathered}$$

we have

$$\begin{aligned}cov[x,y] &= \mathbf{E}[xy] - \mathbf{E}[\mathbf{E}_X[x]y] - \mathbf{E}[\mathbf{E}_Y[y]x] + \mathbf{E}[\mathbf{E}_X\mathbf{E}_Y] \\ &= \mathbf{E}_{X,Y} - \mathbf{E}_X\mathbf{E}_Y - \mathbf{E}_Y\mathbf{E}_X + \mathbf{E}_{X,Y} \\ &= \mathbf{E}_{X,Y} - \mathbf{E}_X\mathbf{E}_Y \\ &= 0 \quad \text{Q}.E.D.\end{aligned}$$

Exercise 2 Beta Priors

1

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx, \text{ where } a,b \geq 1$$

$$\begin{aligned}
&\text{Use Integration by substitution,} \\
&= \frac{1}{a}\int_0^1 (1-x)^{b-1}d(x^a) \\
&\text{Use Integration by parts,} \\
&= (x^{a-1}(1-x)^{b-1})\Big|_0^1 - \frac{1}{b}\int_0^1 x^{a-1}d(1-x)^b \\
&= 0 + \frac{1}{b}\int_0^1 x^{a-1}dx^b \\
&= \int_0^1 x^{a-1}x^{b-1}dx \\
&= \int_0^1 x^{a+b-2}dx \\
&= \frac{1}{a+b-1}x^{a+b-1}\Big|_0^1 \\
&= \frac{1}{a+b-1}(1-0) \\
&a+b-1 \geq 1, \text{ given } a,b \geq 1 \\
&\text{Therefore, } \frac{1}{a+b-1} \in (0,1] \quad \text{Q.E.D.}
\end{aligned}$$

2

First of all ,to prove $Beta(x|a,b)$ is a valid probability distribution, we need to prove:

$$\begin{aligned}
&(1) \quad \forall x \in [0,1], Beta(x|a,b) > 0 \\
&(2) \quad \int_0^1 Beta(x|a,b) = 1
\end{aligned}$$

Proof of (1):

$Beta(x|a,b) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$

Also from 2.1, $B(a,b) = \frac{1}{a+b-1}$

Therefore, $Beta(x|a,b) = (a+b-1)x^{a-1}(1-x)^{b-1}$

Since $a, b \geq 1$

Then $a+b-1 \geq 1 \geq 0$

Because $x \in [0,1]$

$\forall m \in \mathcal{R} \quad x^m > 0$

(except for 0, where 0^0 can be defined as 1 or remain undefined)

Hence $x^{a-1} > 0$, $(1-x)^{b-1} > 0$

Therefore, $Beta(x|a,b) = (a+b-1)x^{a-1}(1-x)^{b-1} \geq 0$ because 3 multipliers are all non-negative.

Proof of (2):
According to Exercise 2.1,

$$\begin{aligned}
\int_0^1 Beta(x|a,b) &= \frac{1}{B(a,b)} \int_0^1 x^{a-1}(1-x)^{b-1} dx \\
&\text{because } \frac{1}{B(a,b)} \text{ is a constant according to 2.1} \\
&= \frac{1}{a+b-1} * (a+b-1) \\
&= 1
\end{aligned}$$

Therefore, the integral of PDF $Beta(x|a,b)$ is equal to 1.
Hence this PDF is a valid probability distribution.

Secondly, we need to prove $Beta(x|a,b)$ is well defined.
Because $B(a,b)$ (which is the denominator) is larger than 1, the only potential singular point for $Beta(x|a,b)$ is when a = 1, x = 0 or b = 1, x = 1 which makes a 0^0.
However, usually 0^0 can be defined as 1.
Hence this PDF is a well defined probability distribution.

From the proofs above, we can say that $Beta(x|a,b)$ is a well defined and valid probability distribution.

3

By definition of Gamma function and Beta function by Gamma function,

$$\begin{aligned}
\text{LEFT} &= \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} \\
&= \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)} \\
&= \frac{a}{a+b}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\
&= \text{RIGHT} \quad \text{Q.E.D.}
\end{aligned}$$

Exercise 3 Coin Flips in Generality

1

Exercise 4 Noisy Coin Flips