ENGN4627: Aligned RMSE

This document gives instructions on how to compute the aligned RMSE between two tuples of points, $(\hat{p}_i)_{i=1}^n$ and $(p_i)_{i=1}^n$, defined as

$$\text{aRMSE} = \min_{S \in \mathbf{SE}(2)} \sqrt{\frac{1}{n}\sum_{i=1}^{n} \|R_S^\top(\hat{p}_i - x_S) - p_i\|^2}. \tag{1}$$

Note that the indices must correspond: p_i is matched to $\hat{p}_i$ for each $i = 1, ..., n$. In order to compute the aRMSE, it is necessary to compute the $S \in \mathbf{SE}(2)$ that minimises the above expression. This can be done following the Umeyama's method [1]. First, define the following terms:

$$\begin{aligned}
\hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} \hat{p}_i, \\
\mu &= \frac{1}{n}\sum_{i=1}^{n} p_i, \\
\Sigma &= \frac{1}{n}\sum_{i=1}^{n} (p_i - \mu)(\hat{p}_i - \hat{\mu})^\top.
\end{aligned}$$

Let $UDV^\top = \Sigma$ be the singular value decompositions (SVD) of Σ. In MATLAB, this may be computed using `[U,D,V] = svd(Sigma)`. Define the matrix A by

$$A = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \det(\Sigma) \geq 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \det(\Sigma) < 0 \end{cases}$$

Then the rotation matrix R_S and the translation vector x_S are given by

$$\begin{aligned}
R_S &= VAU^\top, \\
x_S &= \hat{\mu} - R_S \mu.
\end{aligned}$$

The aligned RMSE is now calculated by substituting these values into (1).

References

[1] Umeyama, Shinji. "Least-squares estimation of transformation parameters between two point patterns." *IEEE Transactions on Pattern Analysis & Machine Intelligence* 4 (1991): 376-380.