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Eight-Vertex Model in Lattice Statistics and One-Dimensional Anisotropic Heisenberg Chain. I. Some Fundamental Eigenvectors*

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We obtain some simple eigenvectors of the transfer matrix of the zero-field eight-vertex model. These are also eigenvectors of the Hamiltonian of the one-dimensional anisotropic Heisenberg chain. We also obtain new equations for the matrix $\mathbf{Q}(v)$ introduced in earlier papers.

1. INTRODUCTION AND SUMMARY

In two previous papers [1, 2] (the results of which were anounced earlier [3, 4]), we obtained equations for the eigenvalues of the transfer matrix $\mathbf{T}$ of the two-dimensional zero-field eight-vertex lattice model, and of the Hamiltonian $\mathscr{H}$ of the one-dimensional anisotropic Heisenberg ring (the "XYZ model"). We were thus able to calculate the partition function of the eight-vertex model, and the ground-state energy of $\mathscr{H}$, for infinitely large systems.

In this and a subsequent paper we shall further obtain expressions for the eigenvectors of $\mathbf{T}$. Since $\mathscr{H}$ and $\mathbf{T}$ commute (for appropriate values of their parameters), these are also the eigenvectors of $\mathscr{H}$.

In this present paper, we obtain some eigenvectors which have a particularly simple form. In a later paper we shall show that we can generalize these vectors to give a basis in which $\mathbf{T}$ breaks up into diagonal blocks. The general eigenvectors can then be obtained by a Bethe ansatz similar to that used for the "ice-type" models [5]. The eigenvectors we shall now obtain can be regarded as akin to the $n = 0$ case of these ice models (but see the end of this section for a fuller discussion of this point).

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As a by-product of the working, in Section 6 we obtain new equations for the matrix $\mathbf{Q}(v)$ which was introduced in [1]. In particular, for the ice-type models these enable us to write down explicit expressions for the elements of $\mathbf{Q}(v)$ in the $n = \frac{1}{2}N$ subspace.

We define the model and state our main result straight away. Consider a square N by N lattice, wound on a torus, and place an arrow on each bond of the lattice so that at each vertex there are an even number of arrows pointing in (or out). Then there are eight possible configurations of arrows at each vertex, as shown in Fig. 1.

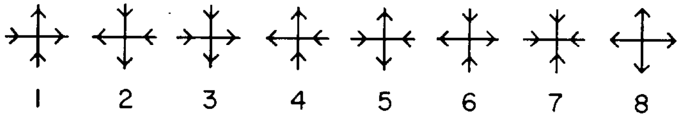

FIG. 1. The eight arrow configurations allowed at a vertex.

Associating energies $\epsilon_1, \ldots, \epsilon_8$ with these vertex configurations, the partition function is

$$Z = \sum \exp\left(-\beta \sum_{j=1}^{8} N_j \epsilon_j\right), \tag{1}$$

where the summation is over all allowed configurations of arrows on the lattice and N_j is the number of vertices of type j.

We impose the "zero field" condition

$$\begin{aligned} \epsilon_1 &= \epsilon_2, \qquad & \epsilon_3 &= \epsilon_4, \\ \epsilon_5 &= \epsilon_6, \qquad & \epsilon_7 &= \epsilon_8. \end{aligned} \tag{2}$$

We can then write the vertex weights

$$\omega_j = \exp(-\beta\epsilon_j) \tag{3}$$

as

$$\begin{aligned} \omega_1 &= \omega_2 = a, \qquad & \omega_3 &= \omega_4 = b, \\ \omega_5 &= \omega_6 = c, \qquad & \omega_7 &= \omega_8 = d. \end{aligned} \tag{4}$$

Look at some particular row of vertical bonds in the lattice and let $\alpha_J = +$ or $-$ according to whether there is an up or down arrow in column J ($J = 1, \ldots, N$). Let α denote the set $\{\alpha_1, \ldots, \alpha_N\}$, so that α defines the configuration of arrows on the whole row of bonds and has 2^N possible values. Then in Section 3 of [1] we show that

$$Z = \operatorname{Tr} \mathbf{T}^N \tag{5}$$

where $\mathbf{T}$ is the 2^N by 2^N transfer matrix. Its elements are

$$T_{\alpha,\alpha'} = \mathrm{Tr}\{\mathbf{R}(\alpha_1, \alpha_1')\, \mathbf{R}(\alpha_2, \alpha_2') \cdots \mathbf{R}(\alpha_N, \alpha_N')\}, \tag{6}$$

where the $\mathbf{R}(\alpha_j, \alpha_j')$ are 2 by 2 matrices:

$$\begin{aligned} \mathbf{R}(+,+) &= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad & \mathbf{R}(+,-) &= \begin{pmatrix} 0 & d \\ c & 0 \end{pmatrix}, \\ \mathbf{R}(-,+) &= \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}, \quad & \mathbf{R}(-,-) &= \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}, \end{aligned} \tag{7}$$

From (5) we see that if we can calculate the eigenvalues of $\mathbf{T}$, then we can calculate the partition function of the model. This has been done (for an infinite lattice) in [1]. A related problem (not solved in [1]) is to calculate the eigenvectors, and this is the problem to which we address ourselves. A knowledge of the eigenvectors is a first step towards calculating correlations and spontaneous polarizations of the model, but we hasten to add that these further calculations are very complicated and have not yet yielded tractable results.

We show that it is convenient to introduce parameters ρ, k, η, v (in general complex numbers), related to a, b, c, d by the equations

$$\begin{aligned} a &= \rho\Theta(-2\eta)\, \Theta(\eta - v)\, H(\eta + v), \\ b &= -\rho\Theta(-2\eta)\, H(\eta - v)\, \Theta(\eta + v), \\ c &= -\rho H(-2\eta)\, \Theta(\eta - v)\, \Theta(\eta + v), \\ d &= \rho H(-2\eta)\, H(\eta - v)\, H(\eta + v). \end{aligned} \tag{8}$$

For the purpose of discussing the matrix $\mathbf{Q}(v)$ (Section 6), it is sufficient to take $H(u)$ and $\Theta(u)$ to be the elliptic theta functions of argument u and modulus k (Section 8.192 of [6], hereinafter referred to as GR). The parametrization (8) is then the same as that of Ref. [1].

However, for our main purpose of discussing eigenvectors, it is necessary to suppose that there exist integers L, m_1, m_2 such that

$$L\eta = 2m_1K + im_2K', \tag{9}$$

where K, K' are the complete elliptic integrals of moduli k, $k' = (1 - k^2)^{1/2}$, respectively. It is then convenient to redefine $H(u)$, $\Theta(u)$ as

$$\begin{aligned} H(u) &= H_{Jb}(u) \exp[i\pi m_2(u - K)^2/(4KL\eta)], \\ \Theta(u) &= \Theta_{Jb}(u) \exp[i\pi m_2(u - K)^2/(4KL\eta) , \end{aligned} \tag{10}$$

where $H_{Jb}(u)$, $\Theta_{Jb}(u)$ are the normal Jacobian elliptic theta functions given in GR. The effect of this is to ensure that

$$H(u + 2L\eta) = H(u), \qquad \Theta(u + 2L\eta) = \Theta(u), \tag{11}$$

and simply to renormalize a, b, c, d.

Note that (9) is quite a mild restriction, since we can approach arbitrarily close to any desired value of η by taking sufficient large integer values of L, m_1 and m_2.

Using the definitions (10), we further define two functions:

$$\xi(+ \mid u) = H(u), \qquad \xi(- \mid u) = \Theta(u). \tag{12}$$

The set of numbers $\alpha = \{\alpha_1, ..., \alpha_N\}$ labels the 2^N possible states of the arrows on the N vertical bonds in a row of the lattice (we can think of these as N spins, each of which can be either up or down). We use the notation Ψ to denote a vector in this 2^N-dimensional space, and $[\Psi]_\alpha$ to denote the element of Ψ corresponding to the state α. Define a vector Ψ and a function $g(v)$ by

$$[\Psi]_\alpha = \sum_{j=1}^{L} \omega^j \prod_{J=1}^{N} \xi[\alpha_J \mid s + 2(J-j)\eta], \tag{13}$$

$$g(v) = \rho\Theta(0)\, H(v)\, \Theta(-v), \tag{14}$$

where ω is an L-th root of unity. Provided

$$N = L \times \text{integer} \tag{15}$$

we find that

$$\mathbf{T}\Psi = [\omega^{-1} g^N(v - \eta) + \omega g^N(v + \eta)]\Psi. \tag{16}$$

Thus Ψ is an eigenvector of the transfer matrix $\mathbf{T}$. There are more than one such eigenvectors, since s is arbitrary and there are L choices of ω. However, we show in Section 7 that there are at most $2N$ linearly independent eigenvectors that can be formed by taking linear combinations of Ψ's.

It is instructive to look at the case when $k \to 0$, η and v being held finite. In this case $d \to 0$ and we regain the ice-type models. The transfer matrix then breaks up into diagonal blocks connecting states with the same number of n of down arrows (spins). Since $K \to \frac{1}{2}\pi$ and $K' \to \infty$, to satisfy the restriction (9) we must set $m_2 = 0$, when it becomes

$$L\eta = m_1\pi. \tag{17}$$

The most general limiting form of (13) is obtained by choosing $s \pm \frac{1}{2}iK'$ to remain finite as $k \to 0$. From Section 8.192 of GR we then have

$$\xi(\alpha_J \mid s) \sim \exp\{\tfrac{1}{2}i(1 + \alpha_J)[\tfrac{1}{2}iK' \pm (s - K)]\} \tag{18}$$

and by substituting this expression into (13) we find that any Ψ is a sum of $2N$ linearly independent eigenvectors $\Psi_0, \Psi_{1,\pm}, \ldots, \Psi_{N-1,\pm}, \Psi_N$, given by

$$[\Psi_{n,\pm}]_\alpha = 0, \text{ if } \alpha_1 + \cdots + \alpha_N \neq N - 2n; = \exp\left\{\mp i\eta \sum_{J=1}^{N} J(1 - \alpha_J)\right\} \tag{19}$$

if $\alpha_1 + \cdots + \alpha_N = N - 2n$. In the latter case there must be n negative α_J's (down arrows). Writing $x_1, \ldots, x_n$ for the positions (J-values) of these negative α_J's, we see from (19) that

$$[\Psi_{n,\pm}]_\alpha = \exp[\mp 2i\eta(x_1 + \cdots + x_n)]. \tag{20}$$

Each such $\Psi_{n,\pm}$ satisfies the eigenvalue Eq. (16), the corresponding value of ω being

$$\omega = \exp[\mp 2inm_1\pi/L] \tag{21}$$

[using the restrictions (15) and (17)]. Thus in the ice-type limit there are precisely $2N$ linearly independent eigenvectors of the form (13). We surmise that the same is true for general values of k.

At first sight an eigenvector of the form (20) appears to contradict the results of Lieb [5] for the ice-type models, and Yang [7] for the related Heisenberg chain. However, this is not of course so. Both authors introduce a parameter Δ, which in our notation is given by

$$\begin{aligned} \Delta &= (a^2 + b^2 - c^2)/(2ab) \\ &= \cos(2\eta). \end{aligned} \tag{21}$$

Our form (20) of Ψ is equivalent to choosing $k_1 = \cdots = k_n = \pm 2\eta$ in [5] and [7]. In this case we see that

$$2\Delta e^{ik_j} - 1 - e^{i[k_j + k_l]} = 0 \tag{22a}$$

and hence the equations relating the coefficients A_P are automatically satisfied. Also, from (15) and (17) we see that

$$\exp(iNk_j) = 1, \tag{22b}$$

so the cyclic boundary conditions are satisfied.

We point out that values of η satisfying (17) for quite small values of L do explicitly occur in certain models. For instance, the XY model [8] is the Heisenberg chain with $\Delta = 0$, so from (21) we see that $\eta = \pi/4$. Also, the ice model [5] has vertex weights $a = b = c = 1$, hence $\Delta = \frac{1}{2}$ and $\eta = \pi/6$.

2. Simplification of $\mathbf{T}\psi$

We wish to find the eigenvectors of the matrix $\mathbf{T}$ defined by (6). This expression suggests first trying vectors whose elements are products of single-spin functions, i.e., a vector ψ with elements

$$[\psi]_\alpha = \phi_1(\alpha_1)\,\phi_2(\alpha_2)\cdots\phi_N(\alpha_N) \tag{23}$$

for all 2^N choices of $\alpha = \{\alpha_1,\ldots,\alpha_N\}$.

Such a vector is not translation invariant, i.e., replacing each α_J by α_{J+1} does not leave it unchanged. However, we can if we wish form a translation-invariant vector by performing all N cyclic permutations of the functions $\phi_1,\ldots,\phi_N$ in (23) and summing the N vectors thereby formed.

The product $\mathbf{T}\psi$ is of course a vector with elements

$$[\mathbf{T}\psi]_\alpha = \sum_{\alpha'} T_{\alpha,\alpha'}[\psi]_{\alpha'}, \tag{24}$$

where the summation is over all 2^N values of $\alpha' = \{\alpha_1',\ldots,\alpha_N'\}$, i.e., over $\alpha_1' = \pm,\ldots,\alpha_N' = \pm$. The point in looking at vectors of the type (23) is that (24) then has a fairly simple form. From (6) we find that

$$[\mathbf{T}\psi]_\alpha = \mathrm{Tr}\{\mathbf{U}_1(\alpha_1)\,\mathbf{U}_2(\alpha_2)\cdots\mathbf{U}_N(\alpha_N)\}, \tag{25}$$

where

$$\mathbf{U}_J(\alpha_J) = \sum_{\alpha_J'} \mathbf{R}(\alpha_J, \alpha_J')\,\phi_J(\alpha_J'). \tag{26}$$

Thus $\mathbf{U}_J(+)$ and $\mathbf{U}_J(-)$ are 2 by 2 matrices. From (7) we see that

$$\begin{aligned}\mathbf{U}_J(+) &= \begin{pmatrix} a\phi_J(+) & d\phi_J(-) \\ c\phi_J(-) & b\phi_J(+)\end{pmatrix},\\ \mathbf{U}_J(-) &= \begin{pmatrix} b\phi_J(-) & c\phi_J(+) \\ d\phi_J(+) & a\phi_J(-)\end{pmatrix}.\end{aligned} \tag{27}$$

The next few steps of our argument closely parallel those of Appendix C of [1]. Note that (25) is unaffected if we replace each $\mathbf{U}_J(\pm)$ by

$$\mathbf{U}_J^*(\pm) = \mathbf{M}_J^{-1}\mathbf{U}_J(\pm)\,\mathbf{M}_{J+1} \tag{28}$$

for $J = 1,\ldots, N$, provided

$$\mathbf{M}_{N+1} = \mathbf{M}_1 . \tag{29}$$

The $\mathbf{M}_J$ are of course 2 by 2 matrices. The product (28) is most conveniently handled if we introduce p_J, p_J', r_J, r_J', related to the elements of $\mathbf{M}_J$ in such a way that

$$\mathbf{M}_J = \begin{pmatrix} r_J' & r_J p_J \\ r_J' p_J' & r_J \end{pmatrix}. \tag{30}$$

The trace in (25) will simplify to the sum of two products if we can choose the $\phi_J(\pm)$ and $\mathbf{M}_J$ so that all $\mathbf{U}_J{}^*(+)$ and $\mathbf{U}_J{}^*(-)$ are lower left-triangular matrices, i.e., their top right elements vanish. Evaluating the matrix product in (28), we find that this is so if

$$\begin{aligned} (ap_{J+1} - bp_J)\, \phi_J(+) + (d - cp_J p_{J+1})\, \phi_J(-) &= 0, \\ (c - dp_J p_{J+1})\, \phi_J(+) + (bp_{J+1} - ap_J)\, \phi_J(-) &= 0, \end{aligned} \tag{31}$$

for $J = 1,\ldots, N$.

Clearly (31) is a pair of homogeneous linear equations for $\phi_J(+)$ and $\phi_J(-)$, so the determinant of the coefficients must vanish, i.e.,

$$(a^2 + b^2 - c^2 - d^2)\, p_J p_{J+1} = ab(p_J{}^2 + p_{J+1}^2) - cd[1 + p_J{}^2 p_{J+1}^2] \tag{32}$$

for $J = 1,\ldots, N$.

Given p_J, (32) is a quadratic equation for p_{J+1}. Thus if we take p_1 to be given, we can construct the entire sequence $p_1 ,\ldots, p_{N+1}$, at each stage having a choice of two alternatives. To satisfy (29) we require that $p_{N+1} = p_1$, i.e., the sequence closes cyclically. It is obviously important to find the simplest possible parametrization of this sequence. In the next section we find that this leads us to introduce elliptic functions.

3. Parametrization in Terms of sn Functions

We seek to parametrize a, b, c, d, $p_1 ,\ldots, p_{N+1}$ so as to be readily able to handle the Eqs. (32).

Let

$$S = (a^2 + b^2 - c^2 - d^2)/(2ab), \tag{33}$$

$$X = cd/(ab). \tag{34}$$

Solving (32) for p_{J+1}, we find

$$p_{J+1} = (Sp_J \pm X^{1/2} D_J)/(1 - Xp_J{}^2), \tag{35}$$

where

$$D_J = \{1 - Yp_J^2 + p_J^4\}^{1/2} \tag{36}$$

and

$$Y = X + X^{-1} - X^{-1}S^2. \tag{37}$$

Note that the discrimant D_J is the square root of a quartic polynomial in p_J. Such expressions can be parametrized in terms of the Jacobian elliptic sn(u), cn(u), dn(u) functions, of argument u and modulus k, by using the properties

$$\begin{aligned} \mathrm{cn}^2(u) &= 1 - \mathrm{sn}^2(u), \\ \mathrm{dn}^2(u) &= 1 - k^2\,\mathrm{sn}^2(u) \end{aligned} \tag{38}$$

(Sections 8.141–8.144 of GR). In our case it is sufficient to define the modulus k so that

$$k + k^{-1} = Y \tag{39}$$

and a parameter u_J such that

$$p_J = k^{1/2}\,\mathrm{sn}(u_J). \tag{40}$$

From (36) and (38) it is then apparent that

$$D_J = \mathrm{cn}(u_J)\,\mathrm{dn}(u_J). \tag{41}$$

The elliptic functions satisfy a number of addition theorems. In particular, from Sections 8.156.1 of GR we have

$$\mathrm{sn}(u \pm v) = \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v \pm \mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1 - k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v}. \tag{42}$$

Using (40) and (41), we see that the r.h.s. of (35) is of the same form as (42). This suggests defining a further parameter η such that

$$X = k\,\mathrm{sn}^2(2\eta). \tag{43}$$

From (37) and (39) we can then verify that

$$S = \{1 - YX + X^2\}^{1/2} = \mathrm{cn}(2\eta)\,\mathrm{dn}(2\eta). \tag{44}$$

The complete elliptic integrals K, K' of moduli k, $k' = (1 - k^2)^{1/2}$ are defined in Sections 8.112 of GR. Incrementing η by K leaves (43) unchanged but negates the r.h.s. of (44) (Sections 8.151.2 of GR). Thus whatever the sign of S, there exists an η such that (43) and (44) are simultaneously satisfied.

Using (40), (41), (43), (44) in (35), we see from (42) that

$$p_{J+1} = k^{1/2}\,\mathrm{sn}(u_J \pm 2\eta). \tag{45}$$

Comparing (40) and (45), we see that the parametrization of the sequence $p_1, \ldots, p_{N+1}$ is now simple. We define $\sigma_1, \ldots, \sigma_N$ such that

$$\sigma_J = \pm 1 \tag{46}$$

for $J = 1, \ldots, N$ (some may be positive, some negative), and set

$$u_J = t + 2(\sigma_1 + \cdots + \sigma_{J-1})\eta \tag{47}$$

for $J = 1, \ldots, N+1$ (t is arbitrary). Then p_J is given by (40) for $J = 1, \ldots, N+1$ and the Eqs. (32) are satisfied identically.

The requirement that $p_{N+1} = p_1$ leads to some complications that we wish to postpone. For the moment note that it is certainly satisfied if

$$\sigma_1 + \cdots + \sigma_N = 0. \tag{48}$$

The Eqs. (32) followed from (31). In order to handle these we further need to parametrize a, b, c, d so as to satisfy (43) and (44) [and hence (39)]. To do this, note from (34) and (43) that we can introduce a parameter z such that

$$\begin{aligned} c &= az^{-1}k^{1/2}\,\mathrm{sn}(2\eta) \\ d &= bzk^{1/2}\,\mathrm{sn}(2\eta). \end{aligned} \tag{49}$$

Equation (43) is then identically satisfied. Substituting these forms for c and d into (33) and (44), we get

$$[1 - z^2k\,\mathrm{sn}^2(2\eta)]b^2 - 2\,\mathrm{cn}(2\eta)\,\mathrm{dn}(2\eta)ab + [1 - z^{-2}k\,\mathrm{sn}^2(2\eta)]a^2 = 0. \tag{50}$$

This is a quadratic equation for b/a, with a discriminant that is a quartic in z. As with the equation relating p_{J+1} and p_J, we find that it is natural to introduce a further parameter y such that

$$z = k^{1/2}\,\mathrm{sn}(y). \tag{51}$$

Using the formulas (38) and (42), it then follows that we can choose

$$b/a = \mathrm{sn}(y - 2\eta)/\mathrm{sn}(y). \tag{52}$$

We can now obtain all the ratios $a : b : c : d$ from (49), (51) and (52). We find it convenient to set

$$y = v + \eta \tag{53}$$

and we then find that

$$a : b : c : d = \mathrm{sn}(v + \eta) : \mathrm{sn}(v - \eta) : \mathrm{sn}(2\eta) : k\,\mathrm{sn}(2\eta)\,\mathrm{sn}(v - \eta)\,\mathrm{sn}(v + \eta). \tag{54}$$

With this parametrization, Eqs. (39), (43) and (44) are satisfied identically.

4. Parametrization in Terms of Theta Functions

Up to now we have considered only the requirements (31) that the topright elements of $\mathbf{U}_J{}^*(\pm)$ vanish. We shall of course need to calculate the diagonal elements of these matrices. To do this it turns out to be convenient to normalize a, b, c, d and $\phi_J(\pm)$ in such a way as to introduce the Jacobi elliptic functions $H(u)$, $\Theta(u)$ defined in Section 8.192 of GR. These are related to $\mathrm{sn}(u)$ by

$$k^{1/2}\,\mathrm{sn}(u) = H(u)/\Theta(u) \tag{55}$$

(Sections 8.191.1 of GR).

Noting that $\mathrm{sn}(u)$ is an odd function, it follows from (54) and (55) that we can choose the normalization of a, b, c, d so that they are given by (8), ρ being regarded as a normalization constant. The theta functions are entire functions in the complex plane and satisfy the quasi-periodic conditions

$$\begin{aligned} H(u + 2K) &= -H(u), \qquad \Theta(u + 2K) = \Theta(u), \\ H(u + 2iK') &= -H(u)\exp[\pi(K' - iu)/K], \\ \Theta(u + 2iK') &= -\Theta(u)\exp[\pi(K' - iu)/K] \end{aligned} \tag{56}$$

(Sections 8.182.1–4 of GR).

Also, $H(u)$ has only simple zeros, occurring at

$$u = 2mK + 2inK', \quad m, n \text{ integers} \tag{57}$$

[Section 8.151.1 of GR, using (55)].

Since $\mathrm{sn}(u)$ is an odd function, we can easily deduce from (55) the identity

$$H(u)\,\Theta(-u) + H(-u)\,\Theta(u) = 0. \tag{58}$$

These properties (56)–(58) are sufficient for us to be able to prove two addition theorems which are all we need to determine $\phi_J(\pm)$ and the diagonal elements of $U_J{}^*(\pm)$.

They are

$$\Theta(u)\,\Theta(v)\,H(w)\,H(u+v+w)+H(u)\,H(v)\,\Theta(w)\,\Theta(u+v+w)$$
$$=\Theta(0)\,\Theta(u+v)\,H(v+w)\,H(w+u), \tag{59}$$

$$\Theta(u)\,\Theta(v)\,\Theta(w)\,\Theta(u+v+w)+H(u)\,H(v)\,H(w)\,H(u+v+w)$$
$$=\Theta(0)\,\Theta(u+v)\,\Theta(v+w)\,\Theta(w+u) \tag{60}$$

for all values of u, v, w.

To prove (59), regard u and v as constants. We can then think of the ratio of the l.h.s. to the r.h.s. as a function $F(w)$. From (56) we can verify that this function is doubly periodic, with periods $2K$ and $2iK'$. Within a given period rectangle the r.h.s. has zeros (simple) only at $w = -u$ and $w = -v$, but by using (58) we see that these are also zeros of the l.h.s. Thus $F(w)$ is entire and doubly periodic. It is therefore bounded, and by the Cauchy–Liouville theorem it must be a constant. Setting $w = 0$, we find that this constant is unity. Hence $F(w) \equiv 1$, which proves (59).

The proof of (60) is similar, except that is convenient to first rearrange the equation so that all Θ functions occur on the l.h.s., all H functions on the r.h.s.

5. Evaluation of $\mathbf{T}\psi$

We are now in a position to calculate $\phi_J(\pm)$, $U_J{}^*(\pm)$, ψ and $\mathbf{T}\psi$. From (40) and (55) we see that

$$p_J = H(u_J)/\Theta(u_J) \tag{61}$$

where $u_1, \ldots, u_{N+1}$ are given by (47), in particular,

$$u_{J+1} = u_J + 2\sigma_J\eta, \tag{62}$$

where $\sigma_J = \pm 1$. The vertex weights a, b, c, d we take to be given by (8).

Using these formulations together with the formulas (58)–(60), we can evaluate all the expressions occurring in (31). For instance,

$$ap_{J+1} - bp_J = \rho\Theta(-2\eta)\{\Theta(\eta - v)\,\Theta(u_J)\,H(\eta + v)\,H(u_{J+1})$$
$$+\,H(\eta - v)\,H(u_J)\,\Theta(\eta + v)\,\Theta(u_{J+1})\}/[\Theta(u_J)\,\Theta(u_{J+1})]. \tag{63}$$

We can simplify this expression by using the identity (59). If $\sigma_J = +1$ we replace u, v, w in (59) by $\eta - v$, u_J, $\eta + v$; if $\sigma_J = -1$ we replace them by $\eta + v$, u_{J+1}, $\eta - v$. In either case we obtain

$$ap_{J+1} - bp_J = \rho\Theta(0)\,H(2\eta)\,\Theta(-2\eta)\,H[u_J + \sigma_J(\eta + v)]$$
$$\times\,\Theta[u_J + \sigma_J(\eta - v)]/[\Theta(u_J)\,\Theta(u_{J+1})]. \tag{64}$$

Similarly,

$$d - cp_J p_{J+1} = \rho\Theta(0)\, H(-2\eta)\, H[u_J + \sigma_J(\eta + v)]\, \Theta(2\eta) \\ \times H[u_J + \sigma_J(\eta - v)]/[\Theta(u_J)\, \Theta(u_{J+1})], \tag{65}$$

so from the first of the Eqs. (31), using (58), we see that we can choose

$$\begin{aligned} \phi_J(+) &= H[u_J + \sigma_J(\eta - v)], \\ \phi_J(-) &= \Theta[u_J + \sigma_J(\eta - v)]. \end{aligned} \tag{66}$$

It is convenient to introduce the two functions $\xi(\pm \mid u)$ defined by (12). Equation (66) can then be written as

$$\phi_J(\pm) = \xi[\pm \mid u_J + \sigma_J(\eta - v)]. \tag{67}$$

Equations (31) ensure that $\mathbf{U}_J{}^*(\pm)$ is a 2 by 2 matrix of the form

$$\mathbf{U}_J{}^*(\pm) = \begin{pmatrix} A_J(\pm) & 0 \\ C_J(\pm) & B_J(\pm) \end{pmatrix}. \tag{68}$$

The diagonal elements can be simplified by using (31). For instance, from Eqs. (27)–(30) we find that

$$A_J(+) = r'_{J+1}(F_J + p'_{J+1}G_J)/[r_J'(1 - p_J p_J')], \tag{69}$$

where

$$\begin{aligned} F_J &= a\phi_J(+) - cp_J\phi_J(-), \\ G_J &= d\phi_J(-) - bp_J\phi_J(+). \end{aligned} \tag{70}$$

The first of equations (31) can be written as

$$p_{J+1}F_J + G_J = 0; \tag{71}$$

so (69) is equivalent to

$$A_J(+) = r'_{J+1}(1 - p_{J+1}p'_{J+1})F_J/[r_J'(1 - p_J p_J')]. \tag{72}$$

Using (8), (61) and (66), F_J can also be evaluated by (59), giving

$$F_J = \rho\Theta(0)\, H(v - \eta)\, \Theta(\eta - v)\, \Theta(u_{J+1})\, H[u_J - \sigma_J(\eta + v)]/\Theta(u_J). \tag{73}$$

Similarly, we can calculate the other diagonal elements. We find that if we choose r_J, r_J' so that

$$\begin{aligned} r_J &= \Theta(u_J), \\ \det \mathbf{M}_J &= r_J r_J'(1 - p_J p_J') = 1, \end{aligned} \tag{74}$$

then

$$\begin{aligned} A_J(\pm) &= g(v-\eta)\, \xi[\pm \mid u_J - \sigma_J(\eta + v)], \\ B_J(\pm) &= g(v+\eta)\, \xi[\pm \mid u_J + \sigma_J(3\eta - v)], \end{aligned} \tag{75}$$

where the function $g(v)$ is defined by

$$g(v) = \rho\Theta(0)\, H(v)\, \Theta(-v). \tag{76}$$

We shall not need $C_J(\pm)$ in this paper, but for future reference record that, using (27)–(30) and (74),

$$\begin{aligned} C_J(+) &= r_J' r_{J+1}'\{(bp_{J+1}' - ap_J')\, \phi_J(+) + (c - dp_J' p_{J+1}')\, \phi_J(-)\}, \\ C_J(-) &= r_J' r_{J+1}'\{(d - cp_J' p_{J+1}')\, \phi_J(+) + (ap_{J+1}' - bp_J')\, \phi_J(-)\}. \end{aligned} \tag{77}$$

From (23), (47) and (67) we see that the vector ψ we have constructed depends on v, t and $\sigma_1, \ldots, \sigma_N$. Exhibiting this dependence explicitly, we can write ψ as a vector function $\psi(v \mid t, \sigma)$, where σ denotes the set $\{\sigma_1, \ldots, \sigma_N\}$. The element of this vector corresponding to the state $\alpha = \{\alpha_1, \ldots, \alpha_N\}$ is

$$[\psi(v \mid t, \sigma)]_\alpha = \prod_{J=1}^{N} \xi[\alpha_J \mid t + 2(\sigma_1 + \cdots + \sigma_{J-1})\eta + \sigma_J(\eta - v)]. \tag{78}$$

Note that the arguments of the functions ξ in (75) differ from those in (67) only by replacing v by $v \pm 2\eta$. Replacing each $\mathbf{U}_J(\pm)$ in (25) by $\mathbf{U}_J^*(\pm)$, and expanding the trace by using (68), it follows that

$$\mathbf{T}\psi(v \mid t, \sigma) = g^N(v-\eta)\, \psi(v + 2\eta \mid t, \sigma) + g^N(v+\eta)\, \psi(v - 2\eta \mid t, \sigma). \tag{79}$$

These Eqs. (78) and (79) are true for all complex numbers v, t, and all allowed values ± 1 of $\sigma_1, \ldots, \sigma_N$, provided the boundary condition (29) is satisfied.

6. The Matrix $\mathbf{Q}(v)$

This section is not strictly relevant to our main aim of discussing the eigenvectors of $\mathbf{T}$, but sheds some new light on the method previously used for obtaining the eigenvalues.

We suppose that N is even and

$$\sigma_1 + \cdots + \sigma_N = 0. \tag{80}$$

Thus $\frac{1}{2}N$ of the σ_J are $+1$, the other $\frac{1}{2}N$ are -1. From (47) we have $u_{N+1} = u_1$, and the boundary condition (29) is automatically satisfied (for *all* values of η).

Let $\mathbf{Q}_R(v)$ be a 2^N by 2^N matrix whose columns are all vectors $\psi(v \mid t, \sigma)$, using different values of t or σ for different columns. From (79) we then have

$$\mathbf{T}\mathbf{Q}_R(v) = g^N(v-\eta)\,\mathbf{Q}_R(v+2\eta) + g^N(v+\eta)\,\mathbf{Q}_R(v-2\eta). \tag{81}$$

From (7) and (8) we can verify that replacing v by $2K - v$ is equivalent to transposing $\mathbf{T}$. Replacing v by $2K - v$ in (81) and transposing, we obtain

$$\mathbf{Q}_L(v)\mathbf{T} = g^N(v-\eta)\,\mathbf{Q}_L(v+2\eta) + g^N(v+\eta)\,\mathbf{Q}_L(v-2\eta), \tag{82}$$

where

$$\mathbf{Q}_L(v) = \mathbf{Q}'(2K - v), \tag{83}$$

the prime denoting transposition. We have used (76) and the relations

$$H(2K - v) = H(v), \qquad \Theta(2K - v) = \Theta(v).$$

By using methods similar to those used to derive (59) and (60), we can show that there exist single-valued functions $f_1(u), f_2(u)$, periodic of period $4K$, such that

$$H(u)\,H(v) + \Theta(u)\,\Theta(v) = f_1(u+v)\,f_2(u-v). \tag{84}$$

From (78) we can then deduce that the scalar product

$$\psi'(2K - u \mid t', \sigma')\,\psi(v \mid t, \sigma)$$

is a symmetric function of u and v, for any values of t', σ', t, σ. (The proof is not completely trivial; we note that all allowed values of $\{\sigma_1',..., \sigma_N'\}$ are permutations of one another, consider the effect of interchanging σ_J' and σ_{J+1}', and obtain a proof by recursion).

From this symmetry and the definitions of $\mathbf{Q}_R(v)$, $\mathbf{Q}_L(v)$, it follows that

$$\mathbf{Q}_L(u)\,\mathbf{Q}_R(v) = \mathbf{Q}_L(v)\,\mathbf{Q}_R(u). \tag{85}$$

Suppose we can construct $\mathbf{Q}_R(v)$, $\mathbf{Q}_L(v)$ so that they are nonsingular at some value v_0 of v. Then from (85) we can further define a matrix $\mathbf{Q}(v)$ such that

$$\mathbf{Q}(v) = \mathbf{Q}_R(v)\,\mathbf{Q}_R^{-1}(v_0) = \mathbf{Q}_L^{-1}(v_0)\,\mathbf{Q}_L(v). \tag{86}$$

Post-multiplying (81) by $\mathbf{Q}_R^{-1}(v_0)$, and pre-multiplying (82) by $\mathbf{Q}_L^{-1}(v_0)$, we therefore find

$$\mathbf{TQ}(v) = g^N(v - \eta)\,\mathbf{Q}(v + 2\eta) + g^N(v + \eta)\,\mathbf{Q}(v - 2\eta) \tag{87}$$

$$= \mathbf{Q}(v)\mathbf{T}. \tag{88}$$

Similarly, from (85) we obtain

$$\mathbf{Q}(u)\,\mathbf{Q}(v) = \mathbf{Q}(v)\,\mathbf{Q}(u). \tag{89}$$

These functional matrix relations (87)–(89) were derived in [1]. Together with the analyticity and quasi-periodic properties of $\mathbf{Q}(v)$ (which can be obtained from our construction), they enable one to obtain equations for the eigenvalues of the transfer matrix $\mathbf{T}$.

The above methods provide a different (though obviously related) definition of $\mathbf{Q}(v)$ to that of [1], which may help us to understand $\mathbf{Q}(v)$ a little better. From (86) we have

$$\mathbf{Q}(v)\,\mathbf{Q}_R(v_0) = \mathbf{Q}_R(v). \tag{90}$$

Each column of $\mathbf{Q}_R(v)$ is a ψ-vector. Thus a typical column of the matrix Eq. (90) is

$$\mathbf{Q}(v)\,\psi(v_0 \mid t, \sigma) = \psi(v \mid t, \sigma). \tag{91}$$

This result must be true for all values of t and σ used in the column vectors of $\mathbf{Q}_R(v)$. However, from the reasoning of Section 6 of [1], we see that (87)–(89) define the eigenvectors of $\mathbf{Q}(v)$ (the same as of $\mathbf{T}$), and each eigenvalue to within a multiplicative constant. The definition (86) fixes these constants so as to ensure that $\mathbf{Q}(v)$ is the identity matrix when $v = v_0$. Thus $\mathbf{Q}(v)$ is then uniquely defined and (91) must be true whatever values of t and σ we select.

This is quite a startling result, since (91) gives an infinite number of equations as t, σ are varied, and $\mathbf{Q}(v)$ has to be chosen so as to satisfy all of them. Nevertheless we conjecture that this can be done.

It is instructive to consider the case when $k \to 0$, η and v being held finite during this limiting procedure. In this case $d \to 0$ and we regain the ice-type "six-vertex" models [5].

The most general form for ψ is obtained if we hold not t, but $t - \frac{1}{2}iK'$ finite. In this limit we can deduce from Sections 8.112, 8.192 of GR that

$$K \to \tfrac{1}{2}\pi, \qquad K' \to \infty, \tag{92}$$

$$\begin{aligned} H(t) &\sim \exp[i(K + \tfrac{1}{2}iK' - t)], \\ \Theta(t) &\sim 1. \end{aligned} \tag{93}$$

Using (12), we can combine these last two formulas as

$$\xi(\alpha_J \mid t) \sim \exp\{\tfrac{1}{2}i(1+\alpha_J)(K+\tfrac{1}{2}iK'-t)\}. \tag{94}$$

Substituting this into (78), we then obtain

$$\psi(v \mid t, \sigma) = \sum_{n=0}^{N} \Phi_n(v \mid \sigma) \exp\{i(N-n)(K+\tfrac{1}{2}iK'-t)\}, \tag{95}$$

where $\Phi_n(v \mid \sigma)$ is a vector whose elements are zero except for states with n negative α_J (down arrows) and $N-n$ positive α_J (up arrows). For such states its elements are

$$[\Phi_n(v \mid \sigma)]_\alpha = \exp\left\{-\tfrac{1}{2}i \sum_{J=1}^{N} (1+\alpha_J)[2(\sigma_1 + \cdots + \sigma_{J-1})\eta + \sigma_J(\eta - v)]\right\}. \tag{96}$$

Thus for any value of t, $\psi(v \mid t, \sigma)$ lies in a subspace spanned by $\Phi_0(v \mid \sigma),..., \Phi_N(v, \sigma)$. Further, any $\Phi_n(v \mid \sigma)$ can be formed by Fourier analyzing $\psi(v \mid t, \sigma)$. Thus (91) is equivalent to

$$\mathbf{Q}(v)\, \Phi_n(v_0 \mid \sigma) = \Phi_n(v \mid \sigma) \tag{97}$$

for $n = 0,..., N$ and all σ satisfying (80). Hence $\mathbf{Q}(v)$ breaks up into $N+1$ diagonal blocks connecting states with the same number n of down arrows. This is to be expected, since we know the transfer matrix $\mathbf{T}$ has this property for the ice-type models [5].

Look at some particular value of n in (97). The corresponding diagonal block of $\mathbf{Q}(v)$ is of dimension $\binom{N}{n}$ by $\binom{N}{n}$, while there are $\binom{N}{\frac{1}{2}N}$ values of $\sigma_1 ,..., \sigma_N$ that satisfy (80). Thus in general there are more equations obtainable from (97) than there are elements of $\mathbf{Q}(v)$. However, we expect that for general values of v_0 they can be solved (they can for $n = 1$).

When $n = \frac{1}{2}N$ we have just the right number of equations for the number of unknowns. In this case (which is interesting since this subspace contains the maximum eigenvalue of $\mathbf{T}$) we can obtain an explicit expression for $\mathbf{Q}(v)$ by letting $v_0 \to -i\infty$. From (96) we then find after dividing by a constant factor $\exp(\frac{1}{2}iNv_0)$ that

$$[\Phi_n(v_0 \mid \sigma)]_\alpha = 0 \tag{98}$$

unless $\sigma_1 = \alpha_1$, $\sigma_2 = \alpha_2 ,..., \sigma_N = \alpha_N$. From (97) it follows that the element of $\mathbf{Q}(v)$ between states α and β is given by

$$[\mathbf{Q}(v)]_{\alpha,\beta} = [\Phi_n(v) \mid \beta)]_\alpha / [\Phi_n(v_0 \mid \beta)]_\beta\,. \tag{99}$$

In the $n = \frac{1}{2}N$ subspace α, β satisfy

$$\alpha_1 + \cdots + \alpha_N = \beta_1 + \cdots + \beta_N = 0. \tag{100}$$

Using this restriction on β, it follows from (96) and (99) that

$$[\mathbf{Q}(v)]_{\alpha,\beta} = \exp\left\{\tfrac{1}{2}i\eta \sum_{1\leqslant J<K\leqslant N}\sum (\alpha_J\beta_K - \alpha_K\beta_J) + \tfrac{1}{2}iv\sum_{J=1}^{N}\alpha_J\beta_J\right\} \tag{101}$$

[removing the constant factor $\exp(-\frac{1}{2}iNv_0)$].

This is a comparitively simple expression for the elements of $\mathbf{Q}(v)$. Since $\mathbf{T}$ commutes with $\mathbf{Q}(v)$ and hence has the same eigenvectors, it is possible that some new insights into the structure of the transfer matrix could be obtained from (101).

It seems appropriate to give a word or warning regarding the choice of v_0. For symmetry reasons it is tempting to try $v_0 = \pm\eta$, 0 or K. However, for $n = 2$ and $N = 4$ we can verify from (101) that at these values $\mathbf{Q}(v_0)$ is singular and (86) cannot be satisfied. This seems likely to be true for arbitrary n, N. Thus either such values of v_0 should be avoided or the above definitions should be modified.

7. Eigenvectors of T

We return to our prime aim, which is to find vectors of the type $\psi(v \mid t, \sigma)$, or simple linear combinations of these, which are eigenvectors if $\mathbf{T}$. To do this we see from (79) that we should like $\mathbf{T}\psi(v \pm 2\eta \mid t, \sigma)$ to simplify into the sum of two terms in the same way as $\mathbf{T}\psi(v \mid t, \sigma)$ does, i.e. given t, we should like there to exist t', σ' such that

$$\psi(v + 2\eta \mid t, \sigma) = \psi(v \mid t', \sigma') \tag{102}$$

[Similarly for $\psi(v - 2\eta \mid t, \sigma)$]. From (78) we can see that this will in general only be so if the σ_J are all equal.

Consider the case when

$$\sigma_1 = \sigma_2 = \cdots = \sigma_N = 1. \tag{103}$$

From (47) we see that

$$u_{N+1} = u_1 + 2N\eta. \tag{104}$$

We can satisfy the boundary condition (29) by using the periodic or quasi-periodic properties of the elliptic functions. Suppose there exist intergers L, m_1, m_2 such that

$$L\eta = 2m_1K + im_2K' \tag{105}$$

(as we pointed out in the Introduction, this is quite a weak condition); then since $\mathrm{sn}(u)$ is periodic of periods $4K$, $2iK'$, we have

$$\mathrm{sn}(u + 2L\eta) = \mathrm{sn}(u). \tag{106}$$

Thus if

$$N = L \times \text{integer}, \tag{107}$$

we see from (104) and (40) that

$$p_{N+1} = p_1 \,. \tag{108}$$

This goes part of the way toward satisfying the boundary conditions. In particular it ensures that $\mathbf{M}_{N+1}^{-1}\mathbf{M}_1$ is diagonal. In fact this is sufficient to ensure that the trace in (25) decomposes into the sum of two products, but we get some irritating extra factors due to our normalization in terms of theta functions, which in general are only quasi-periodic over a period $2L$.

The easiest way out of these problems seems to be to change our definitions of the theta functions, renormalizing them to ensure the required periodicity. From now on we denote the normal Jacobi theta functions defined in GR as $H_{Jb}(u)$, $\Theta_{Jb}(u)$. We use $H(u)$, $\Theta(u)$ to denote the renormalized theta functions defined by (10). Since we can establish from (56) that

$$\begin{aligned} H_{Jb}(u + 4mK) &= H_{Jb}(u), \\ H_{Jb}(u + 2inK') &= (-1)^n\, H_{Jb}(u) \exp[\pi(n^2K' - inu)/K] \end{aligned} \tag{109}$$

for all u and all integers m, n, it follows that the function $H(u)$ defined by (10) is periodic of period $2L\eta$. Similarly, so is $\Theta(u)$.

We can verify that this renormalization of the theta functions leaves the identities (58)–(60) unchanged. Thus the working of Section 4, notably Eqs. (78) and (79), remains true. The condition (107) is now sufficient to ensure that the boundary condition (29) is completely satisfied.

From (78) and (103) we see that $\psi(v \mid t, \sigma)$ involves t and v only via their difference $t - v$. It follows that if we define a vector

$$\psi_j = \psi(v \mid s + v + (1 - 2j)\eta, \sigma), \tag{110}$$

with elements

$$[\psi_j]_\alpha = \prod_{J=1}^{N} \xi[\alpha_J \mid s + 2(J - j)\eta], \tag{111}$$

then (79) implies that

$$\mathbf{T}\psi_j = g^N(v - \eta)\, \psi_{j+1} + g^N(v + \eta)\, \psi_{j-1} \,. \tag{112}$$

From the periodicity of our modified theta functions we see that

$$\psi_{j+L} = \psi_j \,; \tag{113}$$

so if ω is an L-th root of unity and

$$\Psi = \sum_{j=1}^{L} \omega^j \psi_j \,, \tag{114}$$

then

$$\mathbf{T}\Psi = [\omega^{-1} g^N(v - \eta) + \omega g^N(v + \eta)]\Psi. \tag{115}$$

Thus Ψ is an eigenvector of the transfer matrix $\mathbf{T}$, as stated in the Introduction.

By varying s and making different choices of ω, we can construct an infinite number of eigenvectors. Clearly only a finite number of these can be linearly independent; we now show that there at most $2N$ such linearly independent eigenvectors.

To do this we need the quasiperiodic properties of our renormalized theta functions. From the definitions (9), (10) and the properties (56) of the Jacobi functions, we find that

$$\begin{aligned} &H(u + 4pK + 2iqK') \\ &\quad = H(u) \exp\{2\pi i (pm_2 - qm_1)[u + (2p-1)K + iqK']/(L\eta)\} \end{aligned} \tag{116}$$

for all integers p, q. The same relation holds if we replace the function H by Θ.

From (111) it is apparent that there exists a vector function $\Phi(s)$ such that

$$\psi_j = \Phi(s - 2j\eta) \tag{117}$$

for all values of s and j. The elements of $\Phi(s)$ are products of our renormalized theta functions and differ only in whether we use an H or Θ function in a given position. Since (116) applies to both these functions, we can verify, using (107), that the vector function $\Phi(s)$ satisfies the quasi-periodic relation

$$\begin{aligned} &\Phi(s + 4pK + 2iqK') \\ &\quad = \Phi(s) \exp\{2\pi i N(pm_2 - qm_1)[s + (2p-1)K + iqK']/(L\eta)\} \end{aligned} \tag{118}$$

for all integers p and q.

Setting $p = m_1$, $q = m_2$ and using (9), it is apparent that $\Phi(s)$ is periodic of period $2L\eta$. More generally, if l is the highest common factor of m_1 and m_2, then $\Phi(s)$ is periodic of period $2L\eta/l$. Hence there exists a Fourier expansion

$$\Phi(s) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n \exp[i\pi nls/(L\eta)], \tag{119}$$

where the $\mathbf{x}_n$ are fixed vectors, independent of s.

Without loss of generality we can suppose that the integers L, m_1 and m_2 in (9) have no common factors, and hence L and l have no common factor. Substituting the form (119) of $\Phi(s)$ into (117) and (114), it follows that Ψ is an eigenvector of $\mathbf{T}$ for all values of s if and only if each $\mathbf{x}_n$ is an eigenvector. The eigenvalue corresponding to $\mathbf{x}_n$ is, using (115),

$$\omega_n^{-1} g^N(v - \eta) + \omega_n g^N(v + \eta), \tag{120}$$

where ω_n is the L-th root of unity given by

$$\omega_n = \exp(2\pi i n l/L). \tag{121}$$

We appear to have constructed an infinite set of eigenvectors $\mathbf{x}_n$. However, substituting the form (119) of $\Phi(s)$ into (118) and equating Fourier coefficients, we find that

$$\mathbf{x}_{n'} = \mathbf{x}_n \exp\{\pi i l[(n + n')(2pK + iqK') + (n - n')K]/(L\eta)\} \tag{122}$$

for

$$n' = n + 2N(pm_2 - qm_1)/l \tag{123}$$

and all integer values of n, p and q.

Since l is a factor of m_1 and m_2, we see from (123) that n' is an integer, as it should be. Also, as it is the highest such factor, for any integer r we can choose p and q so that

$$(pm_2 - qm_1)/l = r. \tag{124}$$

It therefore follows from (122) that the vectors $\mathbf{x}_n$, $\mathbf{x}_{n\pm 2n}$, $\mathbf{x}_{n\pm 4N}$, etc. are simply scalar multiples of one another. With appropriate normalization they therefore give the same eigenvector.

Thus there are at most $2N$ distinct eigenvectors $\mathbf{x}_n$, namely, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{2N}$. From (120) and (121) we see that they form L sets of $2N/L$ eigenvectors, all members of a set having the same eigenvalue.

We have discussed these vectors in the "ice" limit $k \to 0$ in the Introduction and shown that they are then linearly independent. It seems reasonable to suppose that this will also be so for nonzero k.

8. Heisenberg Chain

We can regard the transfer matrix $\mathbf{T}$ defined by Eqs. (6)–(8) as a function $\mathbf{T}(v)$ of v (regarding ρ, k, η as constants). Then in [1] we showed that two matrices $\mathbf{T}(u)$, $\mathbf{T}(v)$ commute. (Note that the eigenvectors we have constructed are independent of v, in agreement with this property.)

In [2] we further showed that the logarithmic derivative of $\mathbf{T}(v)$ at $v = \eta$ was related to the Hamiltonian of the fully anisotropic one-dimensional Ising-Heisenberg ring (the "XYZ model"). Using our present notation and renormalized theta functions, we can verify from (6)–(8) that

$$[\mathbf{T}(\eta)]^{-1}\,\mathbf{T}'(\eta) = \tfrac{1}{2}\mathscr{H}/\mathrm{sn}(2\eta) + Np\mathbf{E}, \tag{125}$$

where

$$p = \frac{1}{2}\left\{\frac{H'(2\eta)}{H(2\eta)} + \frac{\Theta'(2\eta)}{\Theta(2\eta)}\right\} - \frac{\Theta'(0)}{\Theta(0)}, \tag{126}$$

$$\begin{aligned}\mathscr{H} = \sum_{J=1}^{N} \{&[1 + k\,\mathrm{sn}^2(2\eta)]\,\sigma_J{}^x\sigma_{J+1}^x + [1 - k\,\mathrm{sn}^2(2\eta)]\,\sigma_J{}^y\sigma_{J+1}^y \\ &+ \mathrm{cn}(2\eta)\,\mathrm{dn}(2\eta)\,\sigma_J{}^z\sigma_{J+1}^z\},\end{aligned} \tag{127}$$

$\sigma_J{}^x$, $\sigma_J{}^y$, $\sigma_J{}^z$ being the 2 by 2 Pauli matrices acting on the spin (arrow) at site J. [We have used (55), (58) and Sections 8.158.1 of GR.]

From (127) it is apparent that $\mathscr{H}$ is the Hamiltonian of the anisotropic Heisenberg ring, while from (125) we see that it must commute with $\mathbf{T}(v)$ and hence have the same eigenvectors. Thus the vectors $\boldsymbol{\Psi}$ defined by (13) are also eigenvectors of $\mathscr{H}$. Substituting the form given by (16) for the eigenvalue of $\mathbf{T}(v)$ into (125), we find that the corresponding eigenvalue of $\mathscr{H}$ is

$$\Lambda = N\,\mathrm{sn}(2\eta)\left\{\frac{2\Theta'(0)}{\Theta(0)} - \frac{H'(-2\eta)}{H(-2\eta)} - \frac{\Theta'(-2\eta)}{\Theta(-2\eta)}\right\}. \tag{128}$$

Note that this is independent of the choice of ω in (16). Thus all the $2N$ linearly independent eigenvectors that can be formed by varying Ψ are degenerate for the Heisenberg chain, all having the same eigenvalue.

We remark that we can deduce directly that each ψ_j given by (111) (and hence each Ψ) is an eigenvector of $\mathscr{H}$. To do this we write, in obvious notation,

$$\mathscr{H} = \sum_{J=1}^{N} e_1 \otimes \cdots \otimes e_{J-1} \otimes H_{J,J+1} \otimes e_{J+2} \otimes \cdots \otimes e_N, \tag{129}$$

$$\psi_j = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_N, \tag{130}$$

(e_J being the identity 2 by 2 matrix operating on the spin J), and show that there exists a scalar λ_J and a 2-dimensional vector s_J such that

$$(H_{J,J+1} - \lambda_J)\,\phi_J \otimes \phi_{J+1} = s_J \otimes \phi_{J+1} - \phi_J \otimes s_{J+1}. \tag{131}$$

It then follows that, provided $s_{N+1} = s_1$,

$$\mathscr{H}\psi_j = (\lambda_1 + \cdots + \lambda_N)\,\psi_j\,. \tag{132}$$

Using this method, we find an alternative expression for the eigenvalue Λ, namely,

$$\Lambda = N\{\mathrm{cn}(2\eta)\,\mathrm{dn}(2\eta) + 2k^2\,\mathrm{sn}^2(2\eta)F\}, \tag{133}$$

where

$$F = N^{-1} \sum_{J=1}^{N} \mathrm{sn}[t + 2J\eta]\,\mathrm{sn}[t + 2(J+1)\eta]. \tag{134}$$

Remembering that we require N and η to be chosen so that $\mathrm{sn}(u)$ is periodic of period $2N\eta$, we can verify that F is an entire function of the parameter t that occurs in its definition. Since it is doubly periodic it is therefore bounded, and hence a constant, by the Cauchy–Liouville theorem. Thus in fact F does not depend on t, despite appearances.

This makes the mathematical identity implied by the equivalence of the two formulas (128) and (133) more reasonable, though still not obvious. Equating the two formulas, the identity can be written as

$$[\Theta'(0)/\Theta(0)] - [\Theta'(-2\eta)/\Theta(-2\eta)] - k^2\,\mathrm{sn}(2\eta)F = 0. \tag{135}$$

We have managed to prove this identity when $m_2 = 0$. In this case the fact that F is independent of t means that we can replace (134) by its average over all real t, namely,

$$F = (2K)^{-1} \int_0^{2K} dt\,\mathrm{sn}(t)\,\mathrm{sn}(t + 2\eta). \tag{136}$$

The identity (135) must then be true for all real η. We prove it by analytically continuing the l.h.s. of (135) to the whole complex η-plane and showing that the resulting function is doubly periodic and analytic in a domain containing the period parallelogram. By the Cauchy–Liouville theorem it is therefore a constant, and since it vanishes when $\eta = 0$, it is zero. This proves the identity (135).

When $m_2 \neq 0$ the situation is more complicated, but presumably a proof of (135) can be constructed in a similar manner.

9. Inhomogeneous Lattice Model

Note that the recurrence relation (32) involves the vertex weights a, b, c, d only via the expressions S and X defined in (33) and (34). From (43) and (44) we see that these are functions of k and η, but not of v.

It follows that all the above working (except for the Heisenberg chain) can be performed if v has a different value v_J on each column J of the lattice (but k and η are the same for all columns). In particular, we can still construct the eigenvectors Ψ satisfying an equation of the form (16), only now

$$[\Psi]_\alpha = \sum_{j=1}^{L} \omega^j \prod_{J=1}^{N} \xi[\alpha_J \mid s + 2(J - j)\eta - v_J], \tag{137}$$

and

$$\mathbf{T}\Psi = [\omega^{-1}\tau_1 + \omega\tau_2]\Psi, \tag{138}$$

where

$$\begin{aligned} \tau_1 &= \sum_{J=1}^{N} g(v_J - \eta), \\ \tau_2 &= \sum_{J=1}^{N} g(v_J + \eta). \end{aligned} \tag{139}$$

From (137) we see that adding the same constant to each $v_1, \ldots, v_N$ is equivalent to subtracting that constant from the arbitrary parameter s, and therefore does not affect the set of eigenvectors that can be obtained by varying s. This agrees with the observation in Section 10 of [1] that two transfer matrices commute if their v_J's for a given column differ by a constant which is the same for all columns.

We do not claim that varying the v_J in this way from column to column is of any great physical significance, but we do feel it aids in understanding the mathematics of the solution. In a subsequent paper we intend to show how by starting with Ψ as a "zero-particle" eigenvector we can introduce dislocations or "particles" into its definition and obtain further eigenvectors by a Bethe-type ansatz. Any mathematical insight we can gain into such eigenvectors is welcome.

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