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Eight-Vertex Model in Lattice Statistics and One-Dimensional Anisotropic Heisenberg Chain. II. Equivalence to a Generalized Ice-type Lattice Model*

RODNEY BAXTER†

Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790

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We establish an equivalence between the zero-field eight-vertex model and an Ising model (with four-spin interaction) in which each spin has L possible values, labeled $1,..., L$, and two adjacent spins must differ by one (to modulus L). Such an Ising model can also be thought of as a generalized ice-type model and we will later show that the eigenvectors of the transfer matrix can be obtained by a Bethe-type ansatz.

1. INTRODUCTION AND SUMMARY

This is the second paper of a series in which we intend to obtain the eigenvectors of the transfer matrix $\mathbf{T}$ of the zero-field eight-vertex model. Since $\mathbf{T}$ commutes with the Hamilton $\mathscr{H}$ of the one-dimensional anisotropic Heisenberg ring (for appropriate values of their parameters), these are also the eigenvectors of $\mathscr{H}$.

In the previous paper [1], Paper I, we found some special eigenvectors. Equations for all the eigenvalues have already been obtained from a functional matrix relation [2].

In this paper we consider the effect of slightly altering the special eigenvectors found in Paper I. We show that this leads us to construct a family of vectors such that if ψ is a vector of the family, then $\mathbf{T}\psi$ is a linear combination of vectors of the same family. We then observe that with respect to this basis $\mathbf{T}$ is the transfer matrix of an Ising model (with four-spin interactions) in which each spin can have L values, labeled $1,..., L$, where L is a positive integer. Two adjacent spins must differ by one, and we show from this property that we can regard the Ising model as a generalized ice-type model [3]. In the next paper of the series we shall obtain the general eigenvectors of $\mathbf{T}$ by a Bethe-type ansatz.

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† On leave of absence from the Research School of Physical Sciences, Australian National University, Canberra A.C.T. 2600, Australia.

2. The Transfer Matrix

In this section we recapitulate some of our notation and define the transfer matrix $\mathbf{T}$ of the eight vertex model.

Consider a vertex of the lattice and the surrounding four bonds. We place arrows on the bonds so that there are an even number of arrows pointing into (and out of) the vertex. There are then eight allowed configurations of arrows round the vertex, as shown in Fig. 1 of Paper I. We associate Boltzmann weights a, b, c, d with these configurations as in (I.4).

We associate parameters $\alpha, \beta, \lambda, \mu$ with the bonds as indicated in Fig. 1 of this paper, the parameters having value $+(-)$ if the arrow on the corresponding bond points up or to the right (down or to the left). We define a function R such that

$$R(\alpha, \beta \mid \lambda, \mu) = a, b, c, d \text{ or } 0, \tag{2.1}$$

according to whether the arrow configuration specified by $\alpha, \beta, \lambda, \mu$ is allowed and has Boltzmann weight a, b, c, d, or is not allowed (weight zero).

We can regard λ, μ as indices and define 2 by 2 matrices $\mathbf{R}(\alpha, \beta)$ such that

$$\mathbf{R}(\alpha, \beta) = \begin{pmatrix} R(\alpha, \beta \mid +, +) & R(\alpha, \beta \mid +, -) \\ R(\alpha, \beta \mid -, +) & R(\alpha, \beta \mid -, -) \end{pmatrix}. \tag{2.2}$$

Then

$$\begin{aligned} \mathbf{R}(+, +) &= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, & \mathbf{R}(+, -) &= \begin{pmatrix} 0 & d \\ c & 0 \end{pmatrix}, \\ \mathbf{R}(-, +) &= \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}, & \mathbf{R}(-, -) &= \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}. \end{aligned} \tag{2.3}$$

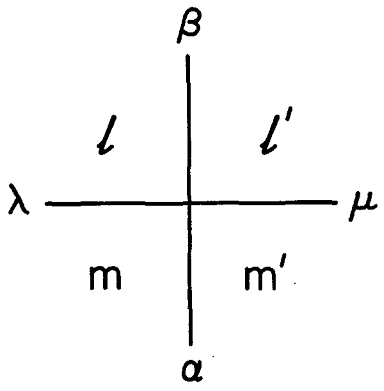

Fig. 1. Arrangement of bond parameters $\alpha, \beta, \lambda, \mu$ and spins l, l', m, m' round a vertex. The weight of this configuration in the Ising-like model is $W(m, m' \mid l, l')$.

We consider a rectangular lattice of N columns, would on a cylinder. The configuration of arrows on a row of vertical bonds is then specified by the set $\boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_N\}$, where α_J is a parameter associated with the bond in column J, having value $+$ for an up arrow, $-$ for a down arrow. The transfer matrix $\mathbf{T}$ is then defined as a 2^N by 2^N matrix with elements

$$T_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathrm{Tr}\{\mathbf{R}(\alpha_1, \beta_1)\,\mathbf{R}(\alpha_2, \beta_2) \cdots \mathbf{R}(\alpha_N, \beta_N)\}. \tag{2.4}$$

We use the notation ψ to denote a 2^N-dimensional vector in the space operated on by the matrix $\mathbf{T}$, and $[\psi]_\alpha$ to denote the element of ψ corresponding to the state $\boldsymbol{\alpha}$ ($\boldsymbol{\alpha}$ has 2^N values). We look in particular at vectors whose elements are products of functions of $\alpha_1, ..., \alpha_N$, respectively. For instance, if $N = 3$ we might consider a vector ψ whose elements are of the form

$$[\psi]_\alpha = f(\alpha_1)\, g(\alpha_2)\, h(\alpha_3) \tag{2.5}$$

for $\alpha_1 = \pm$, $\alpha_2 = \pm$, $\alpha_3 = \pm$.

We can regard $f(+)$ and $f(-)$ as the elements of a two-dimensional vector f, and similarly for g and h. We can then think of the 2^N-dimensional vector ψ defined by (2.5) as a direct product of N two-dimensional vectors and write (2.5) as

$$\psi = f \otimes g \otimes h. \tag{2.6}$$

We shall use this notation.

3. Reduction to an Ising-Like Model

In Appendix B we consider the effect of making small changes in the special eigenvectors of $\mathbf{T}$ that we found in Paper I. We show that this leads us to consider a family of 2^N-dimensional vectors, each a direct product like (2.6), a member being specified by a set of integers $l_1, ..., l_{N+1}$, where

$$l_{J+1} = l_J \pm 1, \qquad J = 1, ..., N. \tag{3.1}$$

We therefore write a typical vector of the family as $\psi(l_1, ..., l_{N+1})$. The form of this vector that we are led to consider is

$$\psi(l_1, ..., l_{N+1}) = \Phi_{l_1,l_2} \otimes \Phi_{l_2,l_3} \otimes \cdots \otimes \Phi_{l_N,l_{N+1}}, \tag{3.2}$$

where $\Phi_{l,l+1}$ and $\Phi_{l+1,l}$ (l an integer) are two sets of two-dimensional vectors, as yet arbitrary, with elements $\Phi_{l,l+1}(\alpha)$, $\Phi_{l+1,l}(\alpha)$ ($\alpha = +$ or $-$).

In Appendix B we also consider whether we can choose the $\Phi_{l,m}$ so that $\mathbf{T}\psi(l_1, ..., l_{N+1})$ is a linear combination of vectors of the family (3.2). We find that we can, provided there exist two other sets of two-dimensional vectors $z_{l,l+1}$, $z_{l+1,l}$, with elements $z_{l,l+1}(\lambda)$, $z_{l+1,l}(\lambda)$, and a set of coefficients $W(m, m' \mid l, l')$ such that

$$\sum_{\beta,\mu} R(\alpha, \beta \mid \lambda, \mu)\, \Phi_{l,l'}(\beta)\, z_{m',l'}(\mu) = \sum_{m} W(m, m' \mid l, l')\, \Phi_{m,m'}(\alpha)\, z_{m,l}(\lambda) \quad (3.3)$$

for $\alpha, \lambda = \pm$ and all integer values of l, l', m' such that $\mid l - l' \mid = \mid m' - l' \mid = 1$. The summation on the r.h.s. of (1.9) is over integer values of m such that $\mid m - m' \mid = 1$, $\mid m - l \mid = 1$. Since m' and l are either equal or differ by 2, there are only 2 or 1 allowed values of m, respectively, in this summation.

Note that (3.3) is a local property of the lattice, involving only the allowed arrangements of arrows round a single vertex. We emphasize this because our working in terms of the transfer matrix tends to obscure this property.

The $\Phi_{l,l\pm1}(\alpha)$, $z_{l,l\pm1}(\lambda)$ and $W(m, m' \mid l, l')$ are now unknowns which must be chosen to satisfy (3.3).

Note that there are more equations than unknowns in (3.3). Taking $l' = 1, 2,..., p$ (p large) and all allowed values of α, λ, l and m', we see that we obtain $16p$ equations from (3.3). On the other hand, since any normalization factors of the 4 sets of vectors $\Phi_{l,m}$, $z_{l,m}$ can be incorporated into the 6 sets of coefficients W, there are only $10p$ independent unknowns. Thus we have no right to expect to be able to solve (3.3). Indeed for the eight-vertex model in the presence of ferroelectric fields [i.e., $\omega_1 \neq \omega_2$ or $\omega_3 \neq \omega_4$ in (I.4)], we cannot in general do so. However, for the zero-field eight-vertex model we can. We give the necessary formulas in Appendix C.

Equations (3.3) are the essential step in the working of this paper. Two side conditions that we also need are that the two vectors $z_{l-1,l}$ and $z_{l+1,l}$ be linearly independent of one another for each value of l, and that

$$z_{l_{N+1}\pm1,l_{N+1}} = z_{l_1\pm1,l_1} \quad (3.4)$$

for either choice of sign (the same on both sides) in (3.4). This latter condition comes from the fact that the lattice is wound cyclically onto a cylinder. Notice that it is certainly satisfied if $l_{N+1} = l_1$.

Using (3.3) and these conditions, we show in Appendix B that

$$\mathbf{T}\psi(l_1 ,..., l_{N+1}) = \sum \left\{ \prod_{J=1}^{N} W(m_J , m_{J+1} \mid l_J , l_{J+1}) \right\} \psi(m_1 ,..., m_{N+1}), \quad (3.5)$$

the summation being over all integers $m_1 ,..., m_{N+1}$ such that

$$m_{J+1} = m_J \pm 1, \qquad J = 1,..., N, \quad (3.6)$$

and

$$m_J = l_J \pm 1, \qquad J = 1,..., N + 1. \quad (3.7)$$

The choice of sign in (3.7) must be the same for $J = 1$ and $J = N + 1$. Thus if $l_{N+1} = l_1$, it follows that $m_{N+1} = m_1$.

It is therefore clear that we have constructed a family of vectors $\psi(l_1, ..., l_{N+1})$ such that $\mathbf{T}\psi(l_1, ..., l_{N+1})$ is a linear combination of vectors of the same family. Using (3.5), we can give a graphical interpretation of $\mathbf{T}$ with respect to this basis. Associate a "spin" $l_{I,J}$ with each face (I, J) of the lattice, where $l_{I,J}$ can take any integer value, and allow only configurations in which adjacent spins differ by unity. With each allowed configuration of the four spins l, l', m, m' round a vertex (as shown in Fig. 1) associate a "Boltzmann weight" $W(m, m' \mid l, l')$. Then with respect to the basis set of vectors $\psi(l_1, ..., l_{N+1})$, $\mathbf{T}$ is the row-to-row transfer matrix of this Ising-like spin problem. In this sense the zero-field eight-vertex problem and the spin problem are equivalent.

One rather unphysical feature of this equivalence is that the spins have an infinite number of possible values. However, we show in Section 6 that for certain values of a, b, c, d the vectors $\Phi_{l,l\pm1}$, $z_{l,l\pm1}$ are unchanged by incrementing l by some integer L. In fact we can find such special values of a, b, c, d arbitrarily close to any given values if we take L sufficiently large. In this case we can restrict the spins in the Ising-like problem to have values $1, ..., L$, and require that adjacent spins differ by one to modulus L.

The simplest such special case is when $ab = cd$, in which case $L = 4$. The eight-vertex model is then simply the superposition of two independent normal Ising models [4]. On the other hand, since L is even and adjacent spins must differ by one, we can restrict our Ising-like model to one in which only even spins (2 or 4) occur on one sublattice and odd spins (1 or 3) on the other. Thus each spin really has only two possible values. Thus the normal spin-$\frac{1}{2}$ (two values per spin) Ising model is equivalent to a spin-$\frac{1}{2}$ Ising model in which all four spins round a vertex interact, and not all configurations of these four spins are allowed. As we shall show in a later paper for the general case, this new Ising model can be solved by a Bethe-type ansatz.

4. Formulation as an Ice-Type Model

Returning to the general case, an important feature of the Ising-like problem we have obtained is that it can be thought of as a generalized "ice-type" problem. To see this, draw arrows on all vertical bonds, pointing up (down) if the spin to the right of the bond is one greater (less) than the spin to the left. Also draw arrows on all horizontal bonds, pointing to the right (left) if the spin below the bond is one greater (less) than the spin above.

For a given value of l (say), there are six possible sets of values of the spins l, l', m, m' round a vertex. We show these in Fig. 2, together with the corresponding arrow configurations on the four bonds. We see that in each case there are two arrows pointing into the vertex and two out, as in the ice model [3]. Thus the number n of down arrows in a row of vertical bonds is the same for each row of the lattice.

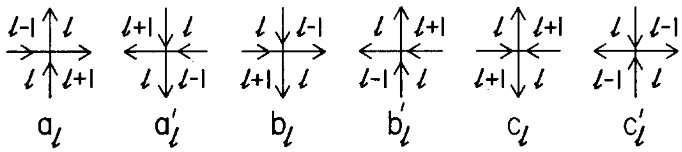

FIG. 2. The six types of vertex in the Ising-like model with their corresponding weights, e.g., $a_l = W(l, l+1 \mid l-1, l)$.

This means that we can split our family of vectors ψ of the form (3.2) into $N + 1$ subfamilies $\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_N$, such that if $\psi \in \mathscr{F}_n$, then in the definition (3.2) of ψ, n Φ's are of the type $\Phi_{l,l-1}$, while $N - n$ are of the type $\Phi_{l,l+1}$. Providing the boundary condition (3.4) is satisfied, it follows that $\mathbf{T}\psi$ is a linear combination of vectors belonging to $\mathscr{F}_n$.

We must study this boundary condition. Since there are n decreasing steps and $N - n$ increasing steps in the sequence $l_1, \ldots, l_{N+1}$, we see that if $\psi \in \mathscr{F}_n$, then

$$l_{N+1} = l_1 + N - 2n. \tag{4.1}$$

Thus in general (3.4) is satisfied if $N = 2n$. However, we can allow other values of n if we consider the special cases discussed above in which the vectors $z_{l\pm 1,l}$ are unchanged by incrementing l by L. The boundary condition (3.4) is then satisfied if

$$N - 2n = L \times \text{integer}. \tag{4.2}$$

5. The Case of $n = 0$

To fix our ideas, consider the case when $n = 0$, i.e., all arrows are up. The sequence $l_1, \ldots, l_{N+1}$ is then increasing and $l_J = l + J - 1$ for $J = 1, \ldots, N + 1$ (writing l for l_1). The boundary condition (3.4) is satisfied if

$$N = L \times \text{integer}. \tag{5.1}$$

(We are forced to restrict attention to the special values of a, b, c, d for which L exists.) From (3.2) we see that $\mathscr{F}_0$ is the family of vectors

$$\psi(l, l+1, \ldots, l+N) = \phi_l \otimes \phi_{l+1} \otimes \cdots \otimes \phi_{l+N-1}, \tag{5.2}$$

where

$$\phi_l = \Phi_{l,l+1}\,. \tag{5.3}$$

Incrementing l by L leaves the two-dimensional vector ϕ_l unchanged, so there are only L vectors in $\mathscr{F}_0$, corresponding to taking $l = 1,..., L$ in (5.2).

Let us write $\psi(l, l+1,..., l+N)$ simply as ψ_l. Then from (3.5) we can verify that

$$\mathbf{T}\psi_l = g_1\psi_{l+1} + g_2\psi_{l-1}\,, \tag{5.4}$$

where

$$\begin{aligned} g_1 &= \prod_{J=1}^{N} W(l+J, l+J+1 \mid l+J-1, l+J), \\ g_2 &= \prod_{J=1}^{N} W(l+J-2, l+J-1 \mid l+J-1, \quad +J). \end{aligned} \tag{5.5}$$

When we calculate the weights W that occur in (5.5) we find that g_1 and g_2 are constants, independent of l. Also, $\psi_{l+L} = \psi_l$ for any integer l. Thus, if ω is the Lth root of unity and

$$\Psi = \sum_{l=1}^{L} \omega^l \psi_l\,, \tag{5.6}$$

we see from (5.4) that

$$\mathbf{T}\Psi = [\omega^{-1}g_1 + \omega g_2]\Psi, \tag{5.7}$$

i.e., Ψ is an eigenvector of the transfer matrix. These are the special eigenvectors that we found in Paper I. In our next paper we shall consider the cases when $n > 0$ and show that we can then obtain the eigenvectors of $\mathbf{T}$ by a generalized Bethe ansatz.

6. The Vectors $\Phi_{l,m}$, $z_{l,m}$

So far we have discussed the general nature of our results, deliberately avoiding the mathematical details of the solution of (3.3). We now state what we find the two-dimensional vectors $\Phi_{l,l\pm1}$, $z_{l,l\pm1}$ and the coefficients $W(m, m' \mid l, l')$ to be. (The derivation is explained in Appendix C.)

We find that to solve (3.3) it is first convenient to introduce parameters ρ, k, η, v which are related to the Boltzmann weights a, b, c, d of the eight-vertex model by

$$\begin{aligned} a &= \rho\Theta(-2\eta)\,\Theta(\eta - v)\,H(\eta + v), \\ b &= -\rho\Theta(-2\eta)\,H(\eta - v)\,\Theta(\eta + v), \\ c &= -\rho H(-2\eta)\,\Theta(\eta - v)\,\Theta(\eta + v), \\ d &= \rho H(-2\eta)\,H(\eta - v)\,H(\eta + v), \end{aligned} \tag{6.1}$$

where $H(u)$, $\Theta(u)$ are related to the elliptic theta functions of argument u and modulus k (Section 8.192 of [5], hereinafter referred to as GR). We leave their precise definition for the moment and note only that we require that their ratio be the same as that of the theta functions, i.e.,

$$H(u)/\Theta(u) = k^{1/2}\,\mathrm{sn}(u), \tag{6.2}$$

where $\mathrm{sn}(u)$ is the elliptic sine-amplitude function of argument u and modulus k (Sections 8.14, 8.191.1 of GR).

Eliminating ρ between any two of Eqs. (6.1), we see that only such ratios of $H(u)$ and $\Theta(u)$ occur, so (6.1) and (6.2) are sufficient to define k, η, v.

Remember that the $\Phi_{l,m}$ ($m = l \pm 1$) are two-dimensional vectors with elements $\Phi_{l,m}(\alpha)$, i.e.,

$$\Phi_{l,m} = \begin{pmatrix} \Phi_{l,m}(+) \\ \Phi_{l,m}(-) \end{pmatrix} \tag{6.3}$$

(and similarly for the $z_{l,m}$). To define these vectors it is convenient to introduce a notation borrowed from quantum mechanics and write $|\, u\rangle$ for the two-dimensional vector

$$|\, u\rangle = \begin{pmatrix} H(u) \\ \Theta(u) \end{pmatrix} \tag{6.4}$$

(u any complex number). Then we find that

$$\begin{aligned} \Phi_{l,l+1} &= |\, s + 2l\eta + \eta - v\rangle, \\ \Phi_{l+1,l} &= |\, t + 2l\eta + \eta + v\rangle, \\ z_{l+1,l} &= |\, s + 2l\eta\rangle, \\ z_{l-1,l} &= h(w + 2l\eta)\,|\, t + 2l\eta\rangle, \end{aligned} \tag{6.5}$$

for all integers l, where s and t are some constants which can be chosen arbitrarily, the function $h(u)$ is defined by

$$h(u) = H(u)\,\Theta(-u), \tag{6.6}$$

and

$$w = \tfrac{1}{2}(s + t - 2K), \tag{6.7}$$

K and K' being the complete elliptic integrals of moduli k and $k' = (1 - k^2)^{1/2}$ (Sections 8.110–112 of GR).

Note that (6.2) and (6.4) define $|\, u\rangle$ to within a normalization factor. The function $\mathrm{sn}(u)$ is doubly periodic, with periodic, with periods $4K$, $2iK'$.

Thus if there exist integers L, m_1, m_2 such that

$$L\eta = 2m_1K + im_2K', \tag{6.8}$$

then $\mathrm{sn}(u)$ is periodic of period $2L\eta$. It follows that to within a normalization factor the vectors $z_{l,l\pm1}$ defined in (6.5) are periodic functions of l, with period L. As we remarked above, this periodicity is convenient to our working (though not strictly essential) since it enables us to consider values of n other than $\frac{1}{2}N$.

We write the usual Jacobian elliptic theta functions defined in GR as $H_{Jb}(u)$ and $\Theta_{Jb}(u)$. Like $\mathrm{sn}(u)$, they are periodic of period $4K$, but they are only quasi-periodic of period $2iK'$. If we wish to discuss the special values of η given by (6.8) (with $m_2 \neq 0$), it is therefore convenient to define $H(u)$, $\Theta(u)$ in the above equations to be given by

$$\begin{aligned} H(u) &= H_{Jb}(u) \exp[i\pi m_2(u - K)^2/(4KL\eta)], \\ \Theta(u) &= \Theta_{Jb}(u) \exp[i\pi m_2(u - K)^2/(4KL\eta)]. \end{aligned} \tag{6.9}$$

This ensures that $H(u)$, $\Theta(u)$ are completely periodic of period $2L\eta$. Hence the vectors $z_{l,l\pm1}$ are periodic of period l.

We emphasize that this renormalization of the theta functions is purely a matter of mathematical convenience. The reader who finds it confusing can focus attention on the case $m_2 = 0$, when the theta functions used here are the same as those of Jacobi. Alternatively, he can consider the case $n = \frac{1}{2}N$, when the restriction (6.8) on η is unnecessary and the re-normalization (6.9) is irrelevant, affecting none of our equations.

7. The Coefficients W

It remains only to quote the results for the coefficients W in (3.3). The working is outlined in Appendix C. We find that for all integers l:

$$\begin{aligned} W(l, l+1 \mid l-1, l) &= a_l = \rho' h(v+\eta), \\ W(l, l-1 \mid l+1, l) &= a_l' = \rho' h(v+\eta)\, h(w_{l+1})/h(w_l), \\ W(l+1, l \mid l, l-1) &= b_l = \rho' h(v-\eta)\, h(w_{l-1})/h(w_l), \\ W(l-1, l \mid l, l+1) &= b_l' = \rho' h(v-\eta), \\ W(l+1, l \mid l, l+1) &= c_l = \rho' h(2\eta)\, h(w_l+\eta-v)/[h(w_l)\, h(w_{l+1})], \\ W(l-1, l \mid l, l-1) &= c_l' = \rho' h(2\eta)\, h(w_l-\eta+v), \end{aligned} \tag{7.1}$$

where

$$\rho' = \rho\Theta(0), \tag{7.2}$$

$$w_l = \tfrac{1}{2}(s + t) - K + 2l\eta, \tag{7.3}$$

and the function $h(u)$ is defined by (6.6).

We have introduced the coefficients a_l, a_l', b_l, b_l', c_l, c_l' into (7.1) simply as a shorthand way of writing the coefficients W. They can be thought of as the Boltzmann weights: of the ice-type vertices 1,..., 6, respectively, shown in Fig. 2. Note that they are not to be confused with the Boltzmann weights a, b, c, d of the original eight-vertex model.

We have remarked above that changing the normalizations of the vectors $\Phi_{l,l\pm1}$, $z_{l,l\pm1}$ simply changes the coefficients W. The normalizations we use in (6.5) are chosen so as to simplify the coefficients W as far as possible. In particular, the extra normalization factor $h(w + 2l\eta)$ in $z_{l-1,l}$ is introduced so as to ensure that both a_l and b_l' are independent of l. This turns out to be a help in handling the Bethe ansatz equations for the eigenvectors of $\mathbf{T}$.

8. An Inhomogeneous Lattice

Equation (3.5) can be regarded as a similarity transformation of $\mathbf{T}$ which converts the eight-vertex model transfer matrix to that of the Ising-like model. This equation is derived in Appendix B from the local property (3.3) and the boundary conditions (3.4). In this derivation the only interaction between columns of the lattice occurs through the two-dimensional vectors $z_{l\pm1,l}$. From (6.5) we note that these depend on k, η and the arbitrary parameters s, t, but they do not depend on v. Thus the working still goes through if we suppose the Boltzmann weights a, b, c, d to be given by (6.1), where k and η are the same for all sites of the lattice but v can vary from column to column. The only change is that in evaluating the coefficients W and the vectors $\Phi_{l,l\pm1}$ from (7.1) and (6.5), one must first consider to which column of the lattice they belong and insert the appropriate value of v.

This generalization can be helpful in studying the structure of the eigenvectors of the transfer matrix that we shall obtain.

9. Summary

We have shown that the zero-field eight-vertex model can be translated into an Ising-like model with four spin interactions. This model can in turn be thought of as a generalized ice-type problem. In a subsequent paper we shall show that we can obtain the eigenvectors of the transfer matrix by a method similar to that used for the ice models [3], namely, by a generalized Bethe ansatz.

The essential step in our working is the fact that Eqs. (3.3) can be solved for the coefficients $W(m, m' \mid l, l')$ and the two-dimensional vectors $\Phi_{l,l\pm1}$, $z_{l,l\pm1}$. This is a purely local property of the lattice. We can in fact establish the equivalence of the eight-vertex model and the Ising-like model directly from (3.3) by imagining the lattice to be built up successively from top to bottom and right to left, associating as we go appropriate combinations of vectors $\Phi_{l,l\pm1}$, $z_{l,l\pm1}$ with the bonds of the lattice.

We have left the detailed working of this paper to the Appendices. In Appendix A we summarize the results of Paper I, in which we obtained some special eigenvectors of $\mathbf{T}$. In Appendix B we show how perturbing these leads us to consider vectors of the form (3.2) and to look for the relations (3.3). We go on to show that if these relations are satisfied, together with the boundary conditions (3.4), then Eq. (3.5) (which is the main result of this paper) follows. In Appendix C we show that (3.3) can indeed be satisfied, and end by showing that it is a corollary of two mathematical identities that involve elliptic theta functions.

Appendix A

We summarize here some results of Paper I concerning certain special eigenvectors of the transfer matrix $\mathbf{T}$. We make a few notational changes to suit our present purpose.

Replace j and s in (I.111) by $-l$ and $s - \eta - v$. Using the notation outlined above in Eqs. (2.5) and (2.6), the 2^N-dimensional vector ψ_j defined by (I.111) can then be written as (dropping the suffix j):

$$\psi = \phi_l \otimes \phi_{l+1} \otimes \cdots \otimes \phi_{l+N-1}, \tag{A.1}$$

where for any integer l, ϕ_l is a two-dimensional vector with elements

$$\begin{aligned} \phi_l(+) &= H(s + 2l\eta + \eta - v), \\ \phi_l(-) &= \Theta(s + 2l\eta + \eta - v). \end{aligned} \tag{A.2}$$

From the form (2.4) of the elements of the transfer matrix $\mathbf{T}$, it then follows that

$$[\mathbf{T}\psi]_\alpha = \mathrm{Tr}\{\mathbf{U}_l(\alpha_1)\, \mathbf{U}_{l+1}(\alpha_2) \cdots \mathbf{U}_{l+N-1}(\alpha_N)\}, \tag{A.3}$$

where

$$\mathbf{U}_l(\alpha) = \sum_\beta \mathbf{R}(\alpha, \beta)\, \phi_l(\beta) \tag{A.4}$$

for $\alpha = +$ and $-$, and for all integers l. The summation in (A.4) is over the two values $+$ and $-$ of β.

Note that our suffix l plays a different role to the suffix J used in (I.26)–(I.32), denoting the vector ϕ_l used in (A.4) rather than the position of a particular $\mathbf{U}$ in the product on the r.h.s. of (A.3).

The essence of the working of Paper I is that the r.h.s. of (A.3) is unchanged if we replace each $\mathbf{U}_l(\alpha)$ by

$$\mathbf{U}_l^*(\alpha) = \mathbf{M}_l^{-1}\mathbf{U}(\alpha)\,\mathbf{M}_{l+1} \tag{A.5}$$

(all integers l and $\alpha = +$ and $-$), provided that the boundary condition

$$\mathbf{M}_{l+N} = \mathbf{M}_l \tag{A.6}$$

is satisfied. Further, we can choose the 2 by 2 matrices $\mathbf{M}_l$ so as to make each matrix $\mathbf{U}_l^*(\alpha)$ lower-left triangular.

The working is formally equivalent to setting $\sigma_J = 1$ in (I.67)–(I.75), replacing u_J by $s + 2l\eta$, and suffixes J by l. The only result that we need here is that

$$\mathbf{U}_l^*(\alpha) = \begin{pmatrix} b_l'\phi_{l-1}(\alpha) & 0 \\ c_l\tau_l(\alpha) & a_{l+1}\phi_{l+1}(\alpha) \end{pmatrix} \tag{A.7}$$

for $\alpha = \pm$ and all integers l. We leave the coefficients a_l, b_l', c_l and the two-dimensional vectors τ_l undefined. We shall derive them in Appendix C, using a more direct method than that of Paper I.

Note that the coefficients a_l, b_l', c_l are not to be confused with the Boltzmann weights a, b, c, d of the eight-vertex model.

Replacing each $\mathbf{U}_l(\alpha)$ in (A.3) by $\mathbf{U}_l^*(\alpha)$, given by (A.7), we see that the r.h.s. of (A.3) becomes the trace of a product of lower-left triangular matrices. It is therefore easy to evaluate and by doing this we can construct the special eigenvectors discussed in Paper I, and in Section 5 of this paper.

Appendix B

Here we consider the effect of slightly altering the special eigenvectors found in Paper I. We show that this leads us to consider a family of vectors of the type (3.2), and to see if we can choose the two-dimensional vectors $\Phi_{l,l\pm1}$, $z_{l,l\pm1}$ so that (3.3) is satisfied. We show that if this can be done and the boundary condition (3.4) satisfied, then Eq. (3.5) is valid.

We have summarized the results of Paper I that we need in Appendix A. Suppose we alter the last vector, ϕ_{l+N-1}, in (A.1). Most of the reasoning still goes through, but now when we replace each $\mathbf{U}_l(\alpha)$ in (A.3) by $\mathbf{U}_l^*(\alpha)$, we find that the first $N-1$ matrices on the r.h.s. are lower-left triangular, while the last is not. We can still expand the trace explicitly, only now we get $N+1$ terms, a typical one being proportional to

$$\phi_{l+1} \otimes \phi_{l+2} \otimes \phi_{l+3} \otimes \tau_{l+3} \otimes \phi_{l+3} \otimes \phi_{l+4} \otimes \phi_{l+5} \otimes \cdots. \tag{B.1}$$

Typically we get a break in the sequence, characterized by a ϕ_{l+J} followed by a τ_{l+J} followed by a ϕ_{l+J}. For the moment we ignore what happens at the end of the sequence.

Now consider the effect of premultiplying the vector (B.1) by the transfer matrix $\mathbf{T}$. From (2.4) we see that the product is a 2^N-dimensional vector with elements

$$\operatorname{Tr}\{\mathbf{U}_{l+1}(\alpha_1)\,\mathbf{U}_{l+2}(\alpha_2)\,\mathbf{U}_{l+3}(\alpha_3)\,\mathbf{V}_{l+3}(\alpha_4)\,\mathbf{U}_{l+3}(\alpha_5)\,\mathbf{U}_{l+4}(\alpha_6)\cdots\}, \tag{B.2}$$

where $\mathbf{U}_l(\alpha)$ is given by (A.4) and

$$\mathbf{V}_l(\alpha) = \sum_\beta \mathbf{R}(\alpha, \beta)\,\tau_l(\beta) \tag{B.3}$$

for any integer l and $\alpha = +$ or $-$.

Again we can apply the transformation (A.5) so as to replace each $\mathbf{U}_l(\alpha)$ by $\mathbf{U}_l^*(\alpha)$ in (B.2). We see that we must then replace $\mathbf{V}_l(\alpha)$ (for any integer l) by

$$\mathbf{V}_l^*(\alpha) = \mathbf{M}_{l+1}^{-1}\mathbf{V}_l(\alpha)\,\mathbf{M}_l\,. \tag{B.4}$$

Let us write the 2 by 2 matrices $\mathbf{V}_l^*(\alpha)$ explicitly as

$$\mathbf{V}_l^*(\alpha) = \begin{pmatrix} f_l(\alpha) & g_l(\alpha) \\ h_l(\alpha) & r_l(\alpha) \end{pmatrix} \tag{B.5}$$

for all integers l and $\alpha = \pm$. Replacing each $\mathbf{U}_l(\alpha)$, $\mathbf{V}_l(\alpha)$ in (B.2) by $\mathbf{U}_l^*(\alpha)$, $\mathbf{V}_l^*(\alpha)$ and evaluating the trace, using (A.7) and (B.5), we get a number of terms, some typical ones being proportional to

$$\begin{aligned}
&\phi_l \otimes \phi_{l+1} \otimes \phi_{l+2} \otimes f_{l+3} \otimes \phi_{l+2} \otimes \phi_{l+3} \otimes \cdots, \\
&\phi_{l+2} \otimes \tau_{l+2} \otimes \phi_{l+2} \otimes g_{l+3} \otimes \phi_{l+4} \otimes \phi_{l+5} \otimes \cdots, \\
&\phi_{l+2} \otimes \phi_{l+3} \otimes \phi_{l+4} \otimes h_{l+3} \otimes \phi_{l+2} \otimes \phi_{l+3} \otimes \cdots, \\
&\phi_{l+2} \otimes \phi_{l+3} \otimes \phi_{l+4} \otimes r_{l+3} \otimes \phi_{l+4} \otimes \phi_{l+5} \otimes \cdots.
\end{aligned} \tag{B.6}$$

[We are using the notation outlined in (2.5) and (2.6).]

We should like the vectors (B.6) to be of the same general type as (B.1), since then **T** would map a vector of this family onto the space spanned by the family and we might hope to use this closure to calculate the eigenvectors of **T**. Comparing (B.1) and (B.6), we see that necessary conditions for this to be so are that

$$\begin{aligned} f_l &\propto \tau_{l-1}, & g_l &\propto \phi_l, \\ h_l &= 0, & r_l &\propto \tau_{l+1}. \end{aligned} \tag{B.7}$$

Thus we should like to be able to choose the two-dimensional vectors ϕ_l, τ_l and the 2 by 2 matrices **M** so that (A.7) is satisfied and

$$\mathbf{V}_l^*(\alpha) = \begin{pmatrix} a_l'\tau_{l-1}(\alpha) & c_{l+1}'\phi_l(\alpha) \\ 0 & b_{l+1}\tau_{l+1}(\alpha) \end{pmatrix} \tag{B.8}$$

for all integers l and $\alpha = \pm$, where a_l', b_l, c_l' are some coefficients.

These requirements can be written more explicitly if we define two 2-dimensional vectors x_l, y_l, with elements $x_l(\pm)$, $y_l(\pm)$, such that

$$\mathbf{M}_l = \begin{pmatrix} y_l(+) & x_l(+) \\ y_l(-) & x_l(-) \end{pmatrix}. \tag{B.9}$$

Premultiplying Eqs. (A.5), (B.4) by $\mathbf{M}_l$, $\mathbf{M}_{l+1}$, respectively, and performing the matrix multiplications explicitly, using (2.2), (A.4), (A.7), (B.3), (B.8) and (B.9), we find that these requirements are equivalent to four sets of equations, one of them being

$$\sum_{\beta,\mu} R(\alpha, \beta \mid \lambda, \mu)\, \phi_l(\beta)\, x_{l+1}(\mu) = a_{l+1}\phi_{l+1}(\alpha)\, x_l(\lambda) \tag{B.10}$$

for all integers l and $\alpha = \pm$, $\lambda = \pm$. The summation on the r.h.s. is over $\beta = \pm$ and $\mu = \pm$.

The structure of these equations becomes clearer if we use an abbreviated notation which utilizes the fact that we already think of $\phi_l(+)$, $\phi_l(-)$ as the elements of a two-dimensional vector ϕ_l (similarly for x_l). We write (B.10) as

$$\mathbf{R}\{\phi_l \otimes x_{l+1}\} = a_{l+1}\phi_{l+1} \otimes x_l. \tag{B.11a}$$

Using this notation, the other three sets of equations that we obtain can be written as

$$\begin{aligned} \mathbf{R}\{\phi_l \otimes y_{l+1}\} &= b_l'\phi_{l-1} \otimes y_l + c_l\tau_l \otimes x_l, \\ \mathbf{R}\{\tau_l \otimes x_l\} &= c_{l+1}'\phi \otimes y_{l+1} + b_{l+1}\tau_{l+1} \otimes x_{l+1}, \\ \mathbf{R}\{\tau_l \otimes y_l\} &= a_l'\tau_{l-1} \otimes y_{l+1} \end{aligned} \tag{B.11b}$$

for all integers l.

The first two of these sets of equations (B.11) are equivalent to requiring that $\mathbf{U}_l^*(\alpha)$ be given by (A.7), the second two to requiring that $\mathbf{V}_l^*(\alpha)$ be given by (B.8). We know from Paper I that we can satisfy (A.7), and hence the first two equations (B.11). It is far from obvious that we can also satisfy the second two. Nevertheless it turns out that we can. The working is given in Appendix C.

It remains to see whether Eqs. (B.11) are sufficient to obtain the required closure, namely, that premultiplying a 2^N-dimensional vector of the form (B.1) by $\mathbf{T}$ gives a linear combination of vectors of the same general form.

To do this we must characterize such vectors more precisely. Allowing the possibility of more than one break in the sequence in (B.1), we see that the rules for constructing such vectors are, for $J = 1,..., N-1$:

(i) a two-dimensional vector ϕ_l in position J must be followed by either ϕ_{l+1} or τ_l in position $J+1$,

(B.12)

(ii) a vector τ_l in position J must be followed by either ϕ_l or τ_{l-1} in position $J+1$.

(The last possibility follows from considering the effect of having two τ-breaks in the increasing ϕ-sequence, and progressively moving them to adjacent positions.)

These rules can be stated more simply if we adopt a notation that is already suggested by (A.5) and (B.4), and define two sets of two-dimensional vectors $\Phi_{l,l+1}$, $\Phi_{l+1,l}$ such that

$$\Phi_{l,l+1} = \phi_l\,, \qquad \Phi_{l+1,l} = \tau_l \tag{B.13}$$

for all integers l. The rules (B.12) then become

a vector $\Phi_{l\pm 1,l}$ in position J must be followed by either $\Phi_{l,l+1}$ or $\Phi_{l,l-1}$ in position $J+1$ ($J = 1,..., N-1$). (B.14)

It follows that any vector of the form (B.1) can be specified by a set of integers $l_1,..., l_{N+1}$ such that $l_{J+1} = l_J \pm 1$ for $J = 1,..., N$, and is given explicitly by (3.2).

Now premultiply (3.2) by the transfer matrix $\mathbf{T}$. Using (2.4), (A.4), (B.3) and (B.13), we see that

$$[\mathbf{T}\psi(l_1,..., l_{N+1})]_\alpha = \operatorname{Tr}\{\mathbf{S}_{l_1,l_2}(\alpha_1)\,\mathbf{S}_{l_2,l_3}(\alpha_2)\cdots \mathbf{S}_{l_N,l_{N+1}}(\alpha_N)\}, \tag{B.15}$$

where

$$\mathbf{S}_{l,l+1}(\alpha) = \mathbf{U}_l(\alpha), \qquad \mathbf{S}_{l+1,l}(\alpha) = \mathbf{V}_l(\alpha) \tag{B.16}$$

for all integers l and $\alpha = \pm$.

We now make the tsansformations (A.5), (B.4), i.e., we replace each $\mathbf{S}_{l,m}(\alpha)$ in (B.15) by

$$\mathbf{S}^*_{l,m}(\alpha) = \mathbf{M}_l^{-1}\mathbf{S}_{l,m}(\alpha)\,\mathbf{M}_m\,, \qquad m = l \pm 1. \tag{B.17}$$

From (A.5) and (B.4) we see that

$$\mathbf{S}^*_{l,l+1}(\alpha) = \mathbf{U}_l{}^*(\alpha), \qquad \mathbf{S}^*_{l+1,l} = \mathbf{V}_l{}^*(\alpha). \tag{B.18}$$

We now substitute the explicit forms (A.7), (B.8) of $\mathbf{U}_l{}^*(\alpha)$, $\mathbf{V}_l{}^*(\alpha)$ into (B.18) and use (B.13) to replace ϕ_l, τ_l by $\Phi_{l,l+1}$, $\Phi_{l+1,l}$ (for any integer l). A little inspection then shows that the element (λ, μ) of the matrix $\mathbf{S}^*_{l,l'}(\alpha)$ can be written as

$$[\mathbf{S}^*_{l,l'}(\alpha)]_{\lambda,\mu} = W(l-\lambda, l'-\mu \mid l, l')\, \Phi_{l-\lambda,l'-\mu}(\alpha) \tag{B.19}$$

for $\lambda, \mu, \alpha = \pm 1$, and $l' = l-1$ or $l+1$. The coefficients $W(m, m' \mid l, l')$ are related to the a_l, a_l', b_l, b_l', c_l, c_l' as in (7.1), and

$$W(l-1, l+2 \mid l, l+1) = W(l+2, l-1 \mid l+1, l) = 0. \tag{B.20}$$

(This last equation follows from the vanishing elements of $\mathbf{U}_l{}^*(\alpha)$, $\mathbf{V}_l{}^*(\alpha)$.)

Using (B.17), we see that we can replace each $\mathbf{S}_{l,m}(\alpha)$ in (B.15) by $\mathbf{S}^*_{l,m}(\alpha)$ provided that

$$\mathbf{M}_{l_{N+1}} = \mathbf{M}_{l_1}. \tag{B.21}$$

Doing this, writing the trace explicitly and using (B.19), we find that

$$\begin{aligned}[\mathbf{T}\psi(l_1, ..., l_{N+1})]_\alpha = \sum_{\lambda_1,\ldots,\lambda_N} \left\{ \prod_{J=1}^{N} W(l_J - \lambda_J, l_{J+1} - \lambda_{J+1} \mid l_J, l_{J+1}) \right\} \\ \times\, \psi(l_1 - \lambda_1, ..., l_{N+1} - \lambda_{N+1}),\end{aligned} \tag{B.22}$$

where $\lambda_{N+1} \equiv \lambda_1$ and the summation is over $\lambda_1 = \pm 1, ..., \lambda_N = \pm 1$.

Setting

$$m_J = l_J - \lambda_J, \qquad J = 1, ..., N+1, \tag{B.23}$$

we see that we can replace the summation in (B.22) by a summation over the integers $m_1, ..., m_{N+1}$, subject to the restrictions

$$\begin{aligned} m_J &= l_J \pm 1, \qquad J = 1, ..., N+1, \\ m_1 - l_1 &= m_{N+1} - l_{N+1}. \end{aligned} \tag{B.24}$$

Since l_J and l_{J+1} differ by one, it follows that m_J and m_{J+1} differ by one or three. However, the latter case gives a zero contribution since from (B.20) the summand then vanishes. Thus we can impose the further restriction

$$m_{J+1} = m_J \pm 1, \qquad J = 1, ..., N. \tag{B.25}$$

Equations (B.22)–(B.25) then become the result (3.5)–(3.7) quoted in the text. Thus Eqs. (B.11), together with the boundary condition (B.21) and the requirement that each matrix $\mathbf{M}_l$ be non-singular, are sufficient to establish the closure property (3.5).

We can use (B.13) to combine the four equations (B.11) into one. Define two sets of two-dimensional vectors $z_{l+1,l}$ and $z_{l-1,l}$ by

$$z_{l+1,l} = x_l\,, \qquad z_{l-1,l} = y_l \tag{B.26}$$

for all integers l. Using (B.13) and (B.26), we then find that all four equations (B.11) can be written as

$$\mathbf{R}\{\Phi_{l,l'} \otimes z_{m',l'}\} = \sum_m W(m, m' \mid l, l')\, \Phi_{m,m'} \otimes z_{m,l} \tag{B.27}$$

for all integer values of l, l', m' such that $| l - l' | = | m' - l' | = 1$. The summation on the r.h.s. is over integer values of m such that $| m - m' | = 1$, $| m - l | = 1$. The coefficients $W(m, m' \mid l, l')$ are again related to $a_l ,..., c_l'$ by (7.1).

We note that (B.27) is simply Eq. (3.3) written in abbreviated notation. Thus our conditions (B.11) are the conditions (3.3) quoted in the text.

From (B.9) and (B.26) we see that the two column vectors of the matrix $\mathbf{M}_l$ are $z_{l-1,l}$ and $z_{l+1,l}$. Thus $\mathbf{M}_l$ is nonsingular if these are linearly independent. Also, the boundary conditions (B.21) and (3.4) are equivalent. This complete the proof that the conditions (3.3)–(3.4) are sufficient to ensure the closure property (3.5)–(3.7).

Appendix C

Here we show that we can solve (3.3), or equivalently (B.11). The solution leads us to introduce elliptic functions.

As we remarked in Appendix B, the first two sets of Eqs. (B.11) have already been solved in Paper I. There we attempted to show in some detail how the parametrization in terms of elliptic functions arises. We shall therefore not stress this aspect, referring the reader to Paper I for complete details.

First look at the first of the Eqs. (B.11), which is written explicitly in (B.10). Taking $\alpha = \pm$, $\lambda = \pm$, we obtain four equations (for a given value of l) which are homogeneous and linear in the unknowns $\phi_l(+)$, $\phi_l(-)$, $\phi_{l+1}(+)$, $\phi_{l+1}(-)$. The determinant of the coefficients must therefore vanish, giving

$$(a^2 + b^2 - c^2 - d^2)\, p_l p_{l+1} = ab(p_l{}^2 + p_{l+1}^2) - cd[1 + p_l{}^2 p_{l+1}^2], \tag{C.1}$$

where

$$p_l = x_l(+)/x_l(-) \tag{C.2}$$

for all integers l.

This symmetric quadratic recursion relation between p_l and p_{l+1} was discussed in Paper I [Eqs. (I.32)–(I.54)]. We showed that it led us to introduce parameters k, η, v, defined in terms of the Boltzmann weights a, b, c, d by

$$a : b : c : d = \mathrm{sn}(v + \eta) : \mathrm{sn}(v - \eta) : \mathrm{sn}(2\eta) : k\,\mathrm{sn}(2\eta)\,\mathrm{sn}(v - \eta)\,\mathrm{sn}(v + \eta), \tag{C.3}$$

where $\mathrm{sn}(u)$ is the elliptic sine-amplitude function of argument u and modulus k (8.14 of GR). It followed that if

$$p_l = k^{1/2}\,\mathrm{sn}(u_l) \tag{C.4}$$

for some value of l, then

$$p_{l+1} = k^{1/2}\,\mathrm{sn}(u_l \pm 2\eta). \tag{C.5}$$

Equations (B.10) can now be solved for $\phi_l(\pm)$, $\phi_{l+1}(\pm)$. Using the relation (6.2) and the addition formulae (I.59), (I.60), we find that

$$\begin{aligned} \phi_l(+)/\phi_l(-) &= k^{1/2}\,\mathrm{sn}[u_l \pm (\eta - v)], \\ \phi_{l+1}(+)/\phi_{l+1}(-) &= k^{1/2}\,\mathrm{sn}[u_l \pm (3\eta - v)]. \end{aligned} \tag{C.6}$$

Equations (C.4)–(C.6) apply to some particular value of l and the same choice of sign must be made in each equation. To satisfy them for each value of l it appears that we must make the same choice of sign for all values of l, say positive. We then see that (C.4)–(C.6) are satisfied by setting

$$p_l = k^{1/2}\,\mathrm{sn}(s + 2l\eta), \tag{C.7}$$

$$\phi_l(+)/\phi_l(-) = k^{1/2}\,\mathrm{sn}(s + 2l\eta + \eta - v) \tag{C.8}$$

for all integers l. The parameter s is arbitrary, but the same for all l.

In an exactly similar way we can solve the last of Eqs. (B.11) and obtain

$$y_l(+)/y_l(-) = k^{1/2}\,\mathrm{sn}(t + 2l\eta), \tag{C.9}$$

$$\tau_l(+)/\tau_l(-) = k^{1/2}\,\mathrm{sn}(t + 2l\eta + \eta + v), \tag{C.10}$$

for all integers l; t is arbitrary.

We are free to choose any convenient normalization of the vectors x_l, ϕ_l, y_l, τ_l and to write a, b, c, d in any way that satisfies (C.3). We find that the coefficients a_l,

a_l' in (B.11) are greatly simplified if we define a, b, c, d by (6.1) and for the moment choose

$$\begin{aligned} x_l(-) &= \Theta(s + 2l\eta), \\ \phi_l(-) &= \Theta(s + 2l\eta + \eta - v), \\ y_l(-) &= \Theta(t + 2l\eta), \\ \tau_l(-) &= \Theta(t + 2l\eta + \eta + v). \end{aligned} \tag{C.11}$$

Using (6.2) and the fact that $\mathrm{sn}(u)$ is an odd function, we see that

$$H(u)\,\Theta(-u) + H(-u)\,\Theta(u) = 0 \tag{C.12}$$

for all complex numbers u. It follows that (6.1) implies (C.3). Also, from (6.2), (C.2), (C.4), (C.5), (C.9)–(C.11), we see that the elements $x_l(+)$, $\phi_l(+)$, $y_l(+)$, $\tau_l(+)$ are given by replacing the minus signs on the left of (C.11) by plus signs, and the functions $\Theta(u)$ on the right by $H(u)$.

Using the notation (6.4), it follows that for the moment we are choosing the vectors x_l, ϕ_l, y_l, τ_l to be given by

$$\begin{aligned} x_l &= |\, s + 2l\eta\rangle, & \phi_l &= |\, s + 2l\eta + \eta - v\rangle, \\ y_l &= |\, t + 2l\eta\rangle, & \tau_l &= |\, t + 2l\eta + \eta + v\rangle. \end{aligned} \tag{C.13}$$

With these normalizations, the coefficients a_l, a_l' in the first and last of Eqs. (B.11) are found to be

$$a_l = a_l' = \rho' h(v + \eta) \tag{C.14}$$

for all integers l, where the function $h(u)$ is defined by (6.6), i.e.,

$$h(u) = H(u)\,\Theta(-u) \tag{C.15}$$

and, as in (7.2),

$$\rho' = \rho\Theta(0). \tag{C.16}$$

To summarize our results so far: Given the parametrization (6.1) of a, b, c, d in terms of ρ, k, η, v, the first and last of Eqs. (B.11) are satisfied by (C.13) and (C.14). The parameters s and t are arbitrary.

It is quite simple to verify this directly, using only the properties (I.59), (I.60), (C.12) of the functions $H(u)$, $\Theta(u)$. These properties are certainly satisfied by elliptic theta functions of Jacobi (Section 8.192 of GR), which we write as $H_{Jb}(u)$, $\Theta_{Jb}(u)$. More generally, they are also satisfied by the modified theta functions:

$$\begin{aligned} H(u) &= H_{Jb}(u) \exp\{A(u - K)^2\}, \\ \Theta(u) &= \Theta_{Jb}(u) \exp\{A(u - K)^2\}, \end{aligned} \tag{C.17}$$

for any values of the constant A. Throughout this appendix we take $H(u)$, $\Theta(u)$ to be defined by (C.17) for some value of A, which is not yet defined. We use this extra degree of freedom in Section 6 to ensure that under certain circumstances [i.e., when (6.8) is satisfied] the functions $H(u)$, $\Theta(u)$ can be chosen to have a convenient periodicity property.

All the mathematical formulas written in this appendix, as well as (I.59) and (I.60), apply to these generalized theta functions, for any value of A.

It remains to check if we can satisfy the middle two Eqs. (B.11), using the forms (C.13) of the vectors. Remember that these equations are written in the same short-hand notation that abbreviates (B.10) to (B.11a). Thus for a given value of l each is actually four equations, corresponding to taking $\alpha = \pm$, $\lambda = \pm$.

One way to check the second equation is to hold l fixed, regard the vectors ϕ_l, y_{l+1}, y_l, x_l as known, given by (C.13), and to solve for the four unknowns $b_l'\phi_{l-1}(\pm)$, $c_l\tau_l(\pm)$. This involves only solving two independent pairs of linear homogeneous equations in two unknowns. For instance, using (2.2) and (2.3) to write the l.h.s. explicitly and solving for $c_l\tau_l(+)$, we find that

$$[x_l(+)\, y_l(-) - x_l(-)\, y_l(+)]\, c_l\tau_l(+) = F_l\phi_l(+) + G_l\phi_l(-), \tag{C.18}$$

where

$$\begin{aligned} F_l &= ay_{l+1}(+)\, y_l(-) - by_{l+1}(-)\, y_l(+), \\ G_l &= dy_{l+1}(-)\, y_l(-) - cy_{l+1}(+)\, y_l(+). \end{aligned} \tag{C.19}$$

All the terms in this equation are regarded as known, given by (6.1) and (C.13), except for $c_l\tau_l(+)$, which we are to evaluate.

Our choice of grouping the terms on the r.h.s. of (C.18) is dictated by the fact that F and G can be simplified by using the addition theorem (I.59). Substituting the forms for a, b, c, d, $y_l(\pm)$, $y_{l+1}(\pm)$ given by (6.1) and (C.13) [or (C.11) and its analogue], we find from (I.59) that

$$\begin{aligned} F_l &= \rho\Theta(0)\, H(2\eta)\, \Theta(-2\eta)\, H(t + 2l\eta + \eta + v)\, \Theta(t + 2l\eta + \eta - v), \\ G_l &= \rho\Theta(0)\, H(-2\eta)\, \Theta(2\eta)\, H(t + 2l\eta + \eta + v)\, H(t + 2l\eta + \eta - v). \end{aligned} \tag{C.20}$$

Using (C.12)–(C.16), it follows that the r.h.s. of (C.18) is

$$\begin{aligned} &\rho' h(2\eta)\, H(t + 2l\eta + \eta + v)[\Theta(t + 2l\eta + \eta - v)\, H(s + 2l\eta + \eta - v) \\ &\quad - H(t + 2l\eta + \eta - v)\, \Theta(s + 2l\eta + \eta - v)]. \end{aligned} \tag{C.21}$$

Also, the bracketted factor on the l.h.s. of (C.18) is [using (C.13)]

$$H(s + 2l\eta)\, \Theta(t + 2l\eta) - \Theta(s + 2l\eta)\, H(t + 2l\eta). \tag{C.22}$$

To proceed further it is clearly necessary to obtain some mathematical identity which simplifies the expression $H(u)\,\Theta(v) - \Theta(u)\,H(v)$. Such an identity can be obtained by noting that the zeros of this expression occur at $u - v = 4mK + 2inK'$, $u + v = (4m + 2)K + 2inK'$, for any integers m, n. From the analyticity and quasi-periodicity of the theta functions, it follows that there must exist two entire functions $f(u)$, $g(u)$ such that

$$H(u - v)\,\Theta(u + v) - \Theta(u - v)\,H(u + v) = f(u)\,g(v) \tag{C.23}$$

for all complex numbers u, v.

Setting $u = 0$ and using (C.5), it follows that we can choose

$$g(v) = H(v)\,\Theta(-v) = h(v), \tag{C.24}$$

where $h(u)$ is the function defined by (6.6) and (C.15). To obtain $f(u)$ we set $v = K$ in (C.23) and use the symmetry relations

$$H(2K - u) = H(u), \qquad \Theta(2K - u) = \Theta(u) \tag{C.25}$$

together with (C.5) and (C.24). This gives

$$f(u) = 2h(u - K)/h(K). \tag{C.26}$$

Thus we have established the identity

$$H(u - v)\,\Theta(u + v) - \Theta(u - v)\,H(u + v) = 2h(u - K)\,h(v)/h(K), \tag{C.27}$$

where $h(u)$ is the function defined by (6.6) and (C.15). This identity applies for any value of the parameter A in the definition (C.17) of our generalized theta functions.

We now use this identity to simplify the expressions (C.21) and (C.22). We find that they both have a factor $2h[(s - t)/2]/h(K)$. Substituting the resulting expressions into (C.18), these common factors cancel, leaving

$$c_l\tau_l(+) = \rho' h(2\eta)\,h(w_l + \eta - v)\,H(t + 2l\eta + \eta + v)/h(w_l), \tag{C.28}$$

where

$$w_l = \tfrac{1}{2}(s + t) - K + 2l\eta. \tag{C.29}$$

We now go through similar working to solve the second of the equations (B.11) for $c_l\tau_l(-)$. We find that the result is the same as the r.h.s. of (C.28), except that $H(t + 2l\eta + \eta + v)$ is replaced by $\Theta(t + 2l\eta + \eta + v)$. Taking the ratios of these results and using (6.2), we see that the vector τ_l we have obtained satisfied (C.10), and hence this first test of the consistency of the equations is satisfied. (Note that s

and t can still be chosen arbitrarily.) Using the given normalization (C.13) of the vector τ_l, we see from (C.28) that

$$c_l = \rho' h(2\eta)\, h(w_l + \eta - v)/h(w_l). \tag{C.30}$$

The next step is to solve for $b_l'\phi_{l-1}(\pm)$. The same techniques work, we find that the result is consistent with (C.13) and that

$$b_l' = \rho' h(v - \eta)\, h(w_{l+1})/h(w_l). \tag{C.31}$$

This verifies the second of the Eqs. (B.11).

The verification of the third can be done in exactly the same way, solving now for $c_{l+1}'\phi_l$ and $b_{l+1}\tau_{l+1}$. We find that it is satisfied provided that

$$c_{l+1}' = \rho' h(2\eta)\, h(w_l + \eta + v)/h(w_{l+1}), \tag{C.32}$$

$$b_{l+1} = \rho' h(v - \eta)\, h(w_l)/h(w_{l+1}). \tag{C.33}$$

These equations are valid for all integers l.

Having gone through all this working, we find that we have actually proved two mathematical identities. Combining the notations used in (B.11) and (6.4), these can be written as

$$\mathbf{R}\{| s \pm (\eta - v)\rangle \otimes | s \pm 2\eta\rangle\} = r_1 \,| s \pm (3\eta - v)\rangle \otimes | s\rangle, \tag{C.34a}$$

$$\begin{aligned}\mathbf{R}\{| s \pm (\eta - v)\rangle \otimes | t \pm 2\eta\rangle\} = {} & r_2 \,| s \mp (\eta + v)\rangle \otimes | t\rangle \\ & + r_3 \,| t \pm (\eta + v)\rangle \otimes | s\rangle,\end{aligned} \tag{C.34b}$$

where a, b, c, d are given by (6.1) and the coefficients r_1, r_2, r_3 by

$$\begin{aligned} r_1 &= \rho' h(v + \eta), \\ r_2 &= \rho' h(v - \eta)\, h(w \pm 2\eta)/h(w), \\ r_3 &= \rho' h(2\eta)\, h[w \pm (\eta - v)]/h(w), \end{aligned} \tag{C.35}$$

$$w = \tfrac{1}{2}(s + t - 2K). \tag{C.36}$$

These equations are mathematical identities, satisfied for all values, real or complex, of s and t. The choice of sign (upper or lower) must be made consistently throughout either of Eqs. (C.34) and the subsidiary definitions (C.35).

These identities are sufficient to establish Eqs. (B.11), and hence (3.3). The first and last of the Eqs. (B.11) are obtained by choosing the upper and lower signs in (C.34a) and replacing s by $s + 2l\eta$, $t + 2(l + 1)$, respectively. The second (third) of Eqs. (B.11) is obtained by choosing the upper (lower) signs in (C.34b) and replacing s, t by $s + 2l\eta$, $t + 2l\eta$ ($t + 2l\eta + 2\eta$, $s + 2l\eta + 2\eta$).

We end this Appendix by noting from (B.11) and (C.31) that we can arrange that b_l' becomes independent of l by renormalizing the vector y_l to be given by

$$y_l = h(w_l) \mid t + 2l\eta\rangle \tag{C.37}$$

for all integers l. The vectors x_l, ϕ_l, τ_l remain given by (C.13). Using (B.11) to make the appropriate adjustments in a_l, a_l', b_l, b_l', c_l, c_l', we find from (C.14) and (C.30)–(C.33) that these coefficients are given by (7.1). This renormalization turns out to be a help when we come to handling the Bethe ansatz equations for the eigenvectors of the transfer matrix.

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