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Partition Function of the Eight-Vertex Lattice Model

RODNEY J. BAXTER

Research School of Physical Sciences, The Australian National University, Canberra, A.C.T. 2600, Australia

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The partition function of the zero-field "Eight-Vertex" model on a square M by N lattice is calculated exactly in the limit of M, N large. This model includes the dimer, ice and zero-field Ising, F and KDP models as special cases. In general the free energy has a branch point singularity at a phase transition, with an irrational exponent.

1. INTRODUCTION

There are very few models in statistical mechanics for which the partition function has been calculated exactly. The only models of multidimensional interacting systems that have been solved are certain two-dimensional lattice models. These can be classified into two types,

(a) those whose partition function can be expressed as a Pfaffian, notably the Ising, dimer and "free-fermion" models [1];

(b) the "ice-type" models which can be solved by a Bethe-type ansatz for the eigenvectors of the transfer matrix [2–4].

For the square lattice all of these (except for the ferroelectric models in the presence of electric fields) can be regarded as special cases of a more general zero-field "Eight-Vertex" model (c.f. [1], [5], and Appendix A of this paper). As previously reported [6], we have calculated the partition function of this model exactly in the limit of a large lattice. In this paper we present this calculation. The method is new, but is inspired by the results of the Bethe ansatz.

The model is defined in Section 2 and certain symmetry relations stated. The principal results are given in Sections 7 and 8. As far as possible, detailed working is left to the appendices.

2. DEFINITION OF THE MODEL

Consider a lattice of M rows (labelled $I = 1,..., M$) and N columns (labelled $J = 1,..., N$), with toroidal boundary conditions. Place arrows on the bonds of the

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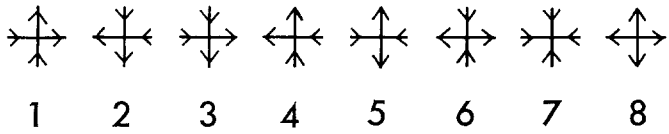

FIG. 1. The eight-arrow configurations allowed at a vertex.

lattice and allow only those configurations with an even number of arrows pointing into each vertex. Then there are eight possible configurations of arrows at each vertex (hence our name for the model), as shown in Fig. 1. Associating energies $\epsilon_1, \ldots, \epsilon_8$ with these vertex configurations, the problem is to evaluate the partition function

$$Z = \sum \exp\left(-\beta \sum_{j=1}^{8} N_j \epsilon_j\right), \tag{2.1}$$

where the summation is over all allowed configurations of arrows on the lattice, and N_j is the number of vertices of type j.

Clearly vertices of type 7 and 8 occur in pairs, being sources and sinks of arrows. Similarly, reversing all horizontal arrows we see that vertices of type 5 and 6 occur in pairs. Thus there is no loss of generality in setting

$$\epsilon_5 = \epsilon_6, \qquad \epsilon_7 = \epsilon_8. \tag{2.2a}$$

We further require that

$$\epsilon_1 = \epsilon_2, \qquad \epsilon_3 = \epsilon_4, \tag{2.2b}$$

so that the model is unchanged by reversing all arrows (in ferroelectric terminology this implies no electric fields). We can then write the vertex weights

$$\omega_j = \exp(-\beta\epsilon_j) \tag{2.3}$$

as

$$\begin{aligned} \omega_1 = \omega_2 = a, \qquad & \omega_3 = \omega_4 = b \\ \omega_5 = \omega_6 = c, \qquad & \omega_7 = \omega_8 = d. \end{aligned} \tag{2.4}$$

A related set of quantities that we shall use are

$$\begin{aligned} w_1 = \tfrac{1}{2}(c + d), \qquad & w_2 = \tfrac{1}{2}(c - d), \\ w_3 = \tfrac{1}{2}(a - b), \qquad & w_4 = \tfrac{1}{2}(a + b). \end{aligned} \tag{2.5}$$

It has been pointed out [1] that Z satisfies a number of symmetry relations. These are particularly simple if we think of Z as a function of w_1, w_2, w_3, w_4, for then they become (taking M, N to be even)

$$Z(w_1, w_2, w_3, w_4) = Z(\pm w_i, \pm w_j, \pm w_k, \pm w_l) \tag{2.6}$$

for any permutation (i, j, k, l) of $(1, 2, 3, 4)$. Thus Z is unaltered by negating or interchanging any of the w's.

In this paper we calculate Z for arbitrary values of a, b, c, d (or w_1, w_2, w_3, w_4) in the limit of M, N large. More precisely, we calculate the free energy per vertex f of an infinite lattice, given by

$$-\beta f = \lim_{M\to\infty} \lim_{N\to\infty} (MN)^{-1} \ln Z. \tag{2.7}$$

3. Transfer Matrix

We use the transfer matrix method (c.f. [2–4]). Look at some particular row of vertical bonds in the lattice and let $\alpha_J = +$ or $-$ according as whether there is an up or down arrow in column J. Let $\boldsymbol{\alpha}$ denote the set $\{\alpha_1, \ldots, \alpha_N\}$, so that $\boldsymbol{\alpha}$ defines the configuration of arrows on the whole row of vertical bonds and has 2^N possible values. Suppose $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$ correspond to the configurations of two successive rows and introduce the 2^N by 2^N transfer matrix $\mathbf{T}$, with elements

$$T_{\boldsymbol{\alpha}|\boldsymbol{\alpha}'} = \sum \exp\left(-\beta \sum_{j=1}^{8} n_j \epsilon_j\right), \tag{3.1}$$

where the sum is over allowed arrangements of arrows on the intervening row of horizontal bonds and n_j is the number of vertices of type j in this row. The partition function is then

$$\begin{aligned} Z &= \sum_{\alpha_1} \cdots \sum_{\alpha_M} T_{\alpha_1|\alpha_2} T_{\alpha_2|\alpha_3} \cdots T_{\alpha_M|\alpha_1}, \\ &= \mathrm{Tr}\{\mathbf{T}^M\}. \end{aligned} \tag{3.2}$$

Let $\lambda_J = +$ or $-$ according as whether there is a right- or left-pointing arrow on the horizontal bond between columns $J-1$ and J. Then (3.1) can be written more explicitly as

$$T_{\boldsymbol{\alpha}|\boldsymbol{\alpha}'} = \sum_{\lambda_1} \cdots \sum_{\lambda_N} \prod_{J=1}^{N} R(\alpha_J, \alpha_J' \mid \lambda_J, \lambda_{J+1}), \tag{3.3}$$

where if $\alpha, \alpha', \lambda, \lambda'$ specify the arrow configurations round a vertex as indicated in Fig. 2, then

$$R(\alpha, \alpha' \mid \lambda, \lambda') = 0 \quad \text{or} \quad \omega_j \tag{3.4}$$

according as whether this vertex configuration is not allowed, or is of type j.

Regarding the λ's as indices, (3.3) can in turn be written as

$$T_{\boldsymbol{\alpha}|\boldsymbol{\alpha}'} = \mathrm{Tr}\{\mathbf{R}(\alpha_1, \alpha_1')\, \mathbf{R}(\alpha_2, \alpha_2') \cdots \mathbf{R}(\alpha_N, \alpha_N')\}, \tag{3.5}$$

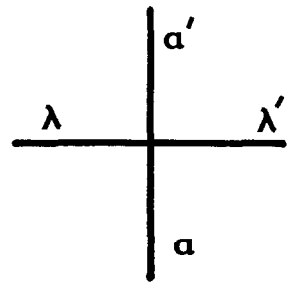

FIG. 2. The arrangement of the bond parameters α, α', λ, λ' round a vertex.

where the $\mathbf{R}(\alpha, \alpha')$ are 2 by 2 matrices

$$\begin{aligned} \mathbf{R}(+,+) &= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad \mathbf{R}(+,-) = \begin{pmatrix} 0 & d \\ c & 0 \end{pmatrix}, \\ \mathbf{R}(-,+) &= \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}, \qquad \mathbf{R}(-,-) = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}. \end{aligned} \tag{3.6}$$

An alternative formulation of R is to introduce the Pauli and unit matrices

$$\begin{aligned} \boldsymbol{\sigma}^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \boldsymbol{\sigma}^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\sigma}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.7}$$

From (2.4) and (3.6) we then find that

$$R(\alpha, \alpha' \mid \lambda, \lambda') = \sum_{j=1}^{4} w_j \sigma^j_{\alpha,\alpha'} \sigma^j_{\lambda,\lambda'} . \tag{3.8}$$

This formulation gives some insight into the symmetry properties (2.6).

4. Ice Model Results

We now seek to diagonalize the transfer matrix $\mathbf{T}$, thereby making it a simple matter to calculate the partition function from (3.2). To do this we are guided by some recent results [7] for an inhomogeneous system satisfying the "ice" condition $d = 0$. With fairly extensive changes in notation, these results can be summarized as follows:

(i) A generalized Bethe ansatz works provided on each row

$$\begin{aligned} a_J &= \rho_J \sin(v - v_J'' + \eta), \\ b_J &= \rho_J \sin(v - v_J'' - \eta), \\ c_J &= \rho_J \sin(2\eta), \\ d_J &= 0, \end{aligned} \tag{4.1}$$

where a_J, b_J, c_J, d_J are the vertex weights for the site on column J, the ρ_J are arbitrary normalization factors, η and $v_1'', \ldots, v_N''$ are the same for each row, but v can vary from row to row. Thus we can regard η and the ρ_J, v_J'' as constants, v as a variable, and write the transfer matrix associated with such a row as $\mathbf{T}(v)$.

(ii) When these conditions are satisfied the transfer matrices for any two rows commute, i.e., $\mathbf{T}(u)$, $\mathbf{T}(v)$ commute for all values of u and v.

(iii) The eigenvalues $T(v)$ of $\mathbf{T}(v)$ are entire functions of v and are given by the identity

$$T(v)\, Q(v) = \phi(v - \eta)\, Q(v + 2\eta) + \phi(v + \eta)\, Q(v - 2\eta), \tag{4.2}$$

where

$$\phi(v) = \prod_{J=1}^{N} [\rho_J \sin(v - v_J'')] \tag{4.3}$$

and $Q(v)$ can be written in the form

$$Q(v) = \prod_{j=1}^{n} \sin(v - v_j), \tag{4.4}$$

where n is the number of down arrows in each row of vertical bonds [this quantity is conserved in each row, so $\mathbf{T}(v)$ breaks up into diagonal blocks corresponding to $n = 0, 1, \ldots, N$].

The identity (4.2) is sufficient to determine $T(v)$, since on setting $v = v_1, \ldots, v_n$ the l.h.s. vanishes, giving n equations for $v_1, \ldots, v_n$. These can in principle be solved (there will be many solutions, corresponding to the different eigenvalues). $Q(v)$ can then be calculated from (4.4), and then $T(v)$ from (4.2).

Equation (4.2) was obtained using the Bethe ansatz for the eigenvectors of $\mathbf{T}(v)$. Note however that as $\mathbf{T}(u)$, $\mathbf{T}(v)$ commute, there exists a representation (independent of v) in which $\mathbf{T}(v)$ is diagonal for all v. Let $\mathbf{Q}(v)$ be a matrix which is also diagonal in this representation, with corresponding diagonal elements $Q(v)$. Then (4.2) can be thought of as a matrix equation in this diagonal representation. Returning to the

original representation, we see that there exists a matrix $\mathbf{Q}(v)$ which commutes with $\mathbf{T}(v)$ and whose elements are entire functions of v, such that

$$\mathbf{T}(v)\,\mathbf{Q}(v) = \phi(v-\eta)\,\mathbf{Q}(v+2\eta) + \phi(v+\eta)\,\mathbf{Q}(v-2\eta). \tag{4.5}$$

Thus instead of using the Bethe ansatz, we could attempt to construct $\mathbf{Q}(v)$ directly. This can be done: more generally, it can be done when $d \neq 0$ and the Bethe ansatz is not applicable. This is the method we use to solve the generalized problem.

For clarity we return to considering an homogeneous system (no variation of vertex weights from site to site) in the following working. The allowed extensions to inhomogeneous systems are straightforward and are outlined in Section 11.

5. Commuting Transfer Matrices

Guided by the results given in Section 4, we first consider under what circumstances two transfer matrices given by (3.3) and (3.8) (with different w's) commute. In Appendix B we show that they do so if

$$(w_j^2 - w_k^2)(w_l^2 - w_m^2) \tag{5.1}$$

is the same for both matrices, for all values 1, 2, 3, 4 of j, k, l and m. Thus a simple parametrization would be to set

$$w_j^2 = p(\xi - s_j), \qquad j = 1, 2, 3, 4. \tag{5.2}$$

All transfer matrices with the same values of s_1, s_2, s_3, s_4 would then commute, even if their values of p and ξ were different.

A more useful parametrization is derived in a natural way in Appendix B, namely,

$$w_1 : w_2 : w_3 : w_4 = \frac{\mathrm{cn}(V,l)}{\mathrm{cn}(\zeta,l)} : \frac{\mathrm{dn}(V,l)}{\mathrm{dn}(\zeta,l)} : 1 : \frac{\mathrm{sn}(V,l)}{\mathrm{sn}(\zeta,l)}, \tag{5.3}$$

where $\mathrm{sn}(u,l)$, $\mathrm{cn}(u,l)$, $\mathrm{dn}(u,l)$ are the Jacobian elliptic functions of modulus l defined in §8.14 of Gradshteyn and Ryzhik [8], hereafter referred to as GR. (At this stage we need only consider ratios of the w's, and of a, b, c, d, since multiplying them by a normalization factor p simply multiplies the transfer matrix by p^N.)

Using the formulae

$$\begin{aligned} \mathrm{sn}^2(u,l) + \mathrm{cn}^2(u,l) &= 1, \\ l^2\,\mathrm{sn}^2(u,l) + \mathrm{dn}^2(u,l) &= 1, \end{aligned} \tag{5.4}$$

it is easy to verify that all the expressions (5.1) depend only on ζ and l. Thus if we regard ζ, l as constants and V as a variable, all transfer matrices with w's satisfying (5.3) commute for all values of V.

For the purposes of the next two sections it is convenient to work with elliptic functions of modulus

$$k = (1 - l)/(1 + l). \tag{5.5}$$

Defining v, η by

$$v = iV/(1 + k), \qquad \eta = i\zeta/(1 + k), \tag{5.6}$$

and using §§8.152, 8.153 of GR [8], we can verify from (2.4) and (5.3) that

$$\begin{aligned} a : b : c : d = {} & \operatorname{sn}(v + \eta, k) : \operatorname{sn}(v - \eta, k) : \operatorname{sn}(2\eta, k) : \\ & k \operatorname{sn}(2\eta, k) \operatorname{sn}(v - \eta, k) \operatorname{sn}(v + \eta, k). \end{aligned} \tag{5.7}$$

Note that when $k = 0$, (5.7) becomes the ice model parametrization (4.1) (with $v_J'' = 0$).

6. Matrix Equation for $\mathbf{T}(v)$

Throughout this section and in Appendices C and D, we work with elliptic functions of modulus k, given by (5.5). We therefore adopt the convention that any elliptic function or integral is to be interpreted as of this modulus, unless explicitly shown or stated to be otherwise.

From §8.191 of GR [8] we have the formula

$$\operatorname{sn} u = k^{-1/2} H(u)/\Theta(u), \tag{6.1}$$

where $H(u)$, $\Theta(u)$ are the elliptic theta functions. Thus from (5.7) we can write

$$\begin{aligned} a &= \rho\Theta(2\eta)\, \Theta(v - \eta)\, H(v + \eta), \\ b &= \rho\Theta(2\eta)\, H(v - \eta)\, \Theta(v + \eta), \\ c &= \rho H(2\eta)\, \Theta(v - \eta)\, \Theta(v + \eta), \\ d &= \rho H(2\eta)\, H(v - \eta)\, H(v + \eta), \end{aligned} \tag{6.2}$$

where ρ is some normalization constant.

Regarding ρ, k, η as constants and v as a variable, we can write the transfer matrix associated with the vertex weights (6.2) as $\mathbf{T}(v)$. Then from Section 5 we see that two matrices $\mathbf{T}(u)$, $\mathbf{T}(v)$ commute for any values of u and v. Further, since the theta functions are entire, all the elements of $\mathbf{T}(v)$ must be entire functions of v.

Thus far we have generalized part (ii) of Section 4. We now look for a matrix equation of the type (4.5).

In Appendix C we construct a 2^N by 2^N matrix $\mathbf{Q}(v)$ such that (4.5) is satisfied, only now we have

$$\phi(v) = [\rho\Theta(0)\, H(v)\, \Theta(v)]^N. \tag{6.3}$$

We also find that any two matrices $\mathbf{Q}(u)$, $\mathbf{Q}(v)$ commute, and that $\mathbf{Q}(v)$ commutes with $\mathbf{T}(v)$. Thus there exists a representation in which all these matrices are diagonal (for all v). Writing the diagonal elements of $\mathbf{T}(v)$, $\mathbf{Q}(v)$ in this representation (i.e., their eigenvalues) as $T(v)$, $Q(v)$, we regain the scalar equation (4.2).

We still need to generalize (4.4). To do this, first note that there are two elementary ways in which the matrix $\mathbf{T}(v)$, and hence $\mathbf{Q}(v)$, can be broken up into diagonal blocks or subspaces. We associate "quantum numbers" with these subspaces as follows:

$$\nu' = 0 \text{ or } 1 \text{ if the number of down arrows in each row of vertical bonds is even or odd, respectively;} \tag{6.4}$$

$$\nu'' = 0 \text{ or } 1 \text{ in the subspace symmetric or antisymmetric, respectively, with respect to reversing all arrows.} \tag{6.5}$$

Let K be the complete elliptic integral of the first kind of modulus k, and K' the same integral of the complementary modulus $k' = (1 - k^2)^{1/2}$ (§§8.110–8.112 of GR [8]). Let

$$q = \exp(-\pi K'/K). \tag{6.6}$$

Then the elliptic theta functions satisfy the quasiperiodic conditions

$$H(u + 2K) = -H(u), \qquad \Theta(u + 2K) = \Theta(u), \tag{6.7}$$

$$\begin{aligned} H(u + iK') &= iq^{-1/4} \exp(-\tfrac{1}{2}i\pi u/K)\, \Theta(u), \\ \Theta(u + iK') &= iq^{-1/4} \exp(-\tfrac{1}{2}i\pi u/K)\, H(u). \end{aligned} \tag{6.8}$$

Using these, it follows from Eqs. (C2), (C17), (C19) of Appendix C that (taking N to be even)

$$\begin{aligned} \mathbf{Q}(v + 2K) &= (-1)^{\nu'}\, \mathbf{Q}(v), \\ \mathbf{Q}(v + iK') &= (-1)^{\nu''}\, q^{-N/4} \exp(-\tfrac{1}{2}iN\pi v/K)\, \mathbf{Q}(v). \end{aligned} \tag{6.9}$$

It is also apparent from Appendix C that the elements of $\mathbf{Q}(v)$ are entire functions of v. Hence its eigenvalues are also entire functions and satisfy the conditions (6.9).

It follows that it must be possible to factorize them in the form

$$Q(v) = \exp(-\tfrac{1}{2}iv\pi v/K) \prod_{j=1}^{\frac{1}{2}N} \{H(v - v_j)\, \Theta(v - v_j)\}, \tag{6.10}$$

where the integer ν and the parameters $v_1, \ldots, v_{N/2}$ satisfy

$$\nu + \nu' + \tfrac{1}{2}N = \text{(even integer)}, \tag{6.11}$$

$$K^{-1}\left\{\tfrac{1}{2}ivK' - \sum_{j=1}^{\frac{1}{2}N} v_j\right\} = \nu'' + \tfrac{1}{2}N + \text{(even integer)}. \tag{6.12}$$

We can now in principle use (4.2), (6.3) and (6.10) to calculate the eigenvalues of the transfer matrix in the same way as for the ice-type models. Namely, we set $v = v_1, \ldots, v_{N/2}$ in (4.2): the l.h.s. vanishes from (6.10), giving $N/2$ equations for $v_1, \ldots, v_{N/2}$. If these can be solved then $Q(v)$ is given by (6.10) and $T(v)$ by (4.2).

It is interesting to note that if $4\eta = 2m_1K + im_2K'$, where m_1 and m_2 are integers, then the equations for $v_1, \ldots, v_{N/2}$ split up into $N/2$ independent equations and are easy to solve. This is the situation for the Ising, dimer and free-fermion models, all of which can alternatively be solved by the Pfaffian method.

7. Free Energy

From (2.7) and (3.2) we see that the free energy f of the infinite lattice is given by

$$-\beta f = \lim_{N\to\infty} N^{-1} \ln[T(v)]_{\max}, \tag{7.1}$$

where $[T(v)]_{\max}$ is the maximum eigenvalue of the transfer matrix $\mathbf{T}(v)$.

We consider the regime

$$w_1 > w_2 > w_3 > |\, w_4 \,|. \tag{7.2}$$

[Strictly speaking this regime is unphysical, since from (2.5) it implies that b is negative. However, negating b does not alter the partition function or the following discussion.] From (5.3) or Appendix B the parameters l, ζ, V are given by

$$l = [(w_1^2 - w_4^2)(w_2^2 - w_3^2)/(w_1^2 - w_3^2)(w_2^2 - w_4^2)]^{1/2}, \tag{7.3}$$

$$\mathrm{sn}(\zeta, l) = [(w_1^2 - w_3^2)/(w_1^2 - w_4^2)]^{1/2}, \tag{7.4}$$

$$\mathrm{sn}(V, l) = w_3^{-1} w_4\, \mathrm{sn}(\zeta, l). \tag{7.5}$$

Thus when (7.2) is satisfied we can choose l, ζ, V to be real and such that

$$0 < l < 1, \qquad |V| < \zeta < K_l, \tag{7.6}$$

where K_l is the complete elliptic integral of the first kind of modulus l.

When this is so we show in Appendix D that we can calculate the maximum eigenvalue of $\mathbf{T}(v)$ from the equation (4.2), using a perturbation-type technique that neglects certain terms that become relatively exponentially small as N becomes large. We find that there are actually two numerically largest eigenvalues, equal in magnitude but positive or negative according to whether the associated eigenvector is symmetric or antisymmetric with respect to reversing all arrows on the lattice (i.e., whether $\nu'' = 0$ or 1). This indicates that we are considering an ordered state of the antiferroelectric type, which agrees with the observation that in the regime (7.2) c is the numerically largest vertex weight. (Vertices of type 5 and 6 are favoured, as in the F-model of Lieb [3].)

Provided M is even this choice of sign does not affect (7.1), so using the result of Appendix D we obtain

$$-\beta f = \ln(w_1 + w_2) + 2\sum_{n=1}^{\infty} \frac{\sinh^2[(\tau - \lambda)\, n]\{\cosh(n\lambda) - \cosh(n\alpha)\}}{n \sinh(2n\tau) \cosh(n\lambda)}, \tag{7.7}$$

where

$$\tau = \pi K_l/K_l', \qquad \lambda = \pi\zeta/K_l', \qquad \alpha = \pi V/K_l', \tag{7.8}$$

and K_l' is the complete elliptic integral of the first kind of modulus $l' = (1 - l^2)^{1/2}$.

For tunately there is no need to repeat the working for any other cases, since from the symmetry relations (2.6) we can always arrange w_1, w_2, w_3, w_4 so as to lie in the regime (7.2) (or its boundaries: as f is continuous these can be handled by taking an appropriate limit). Thus we have solved the general problem.

8. Phase Transitions

Suppose now that the vertex configuration energies $\epsilon_1, \ldots, \epsilon_8$ are fixed and satisfy (2.2), and we vary the temperature T from 0 to ∞. Then w_1, w_2, w_3, w_4, given by (2.5) and negated or arranged where necessary to satisfy (7.2), are entire functions of T. Further, so long as (7.2) is satisfied, the free energy f given by (7.7) is an entire function of the w's, and hence of T. A phase transition (i.e., a singularity in the function $f(T)$) can therefore occur only when the w's cross a boundary of the regime (7.2).

In general this will correspond to just two of the w's becoming numerically equal (if three are equal, or two pairs are equal, we have a more complicated situation which we do not discuss here: the F and KDP models [3, 4] are of this type).

Across the $w_1 = w_2$ or $w_3 = |\, w_4 \,|$ boundaries of the regime (7.2) we find that f is the same analytic function of T on either side, so there is no phase transition.

The behaviour when $w_2 - w_3$ becomes zero is discussed in Appendix E [and alternative expressions for f given which are more suitable near this boundary than (7.7)]. In general this will correspond to $w_2 - w_3$ having a simple zero at some value T_c of the temperature T. We find that f can be written as the sum of an analytic and a singular function of T. The analytic part is the same on both sides of T_c, while near T_c the singular part is proportional to

$$\cot(\tfrac{1}{2}\pi^2/\mu)\, |\, T - T_c\, |^{\pi/\mu}, \tag{8.1a}$$

or if $\pi/(2\mu) = m =$ integer, to

$$\pi^{-1}2(T - T_c)^{2m} \ln |\, T - T_c\, |, \tag{8.1b}$$

where

$$\mu = \pi\zeta/K_l\,. \tag{8.2}$$

The constant of proportionality which multiplies (8.1) is positive and is the same on both sides of T_c. From (7.6) we see that

$$0 < \mu < \pi. \tag{8.3}$$

Also, from (7.3) we see that $l = 0$ when $w_2 = w_3$, so $K_l = \pi/2$, $\mathrm{sn}(u, l) = \sin u$, and from (7.4) we find that at the transition temperature

$$\cos\mu = (2w_2^2 - w_1^2 - w_4^2)/(w_1^2 - w_4^2). \tag{8.4}$$

Thus the 8-vertex model undergoes a phase transition if the middle two w's, arranged in numerically decreasing order, cross one another as the temperature T is varied. (Unless $\pi/\mu =$ odd integer greater than one, in which case all singularities in f disappear.) This verifies the conjecture of Sutherland [5]. At the critical temperature the free energy has a branch point singularity, and it is interesting to note that the exponent π/μ of this singularity can be varied continuously (by appropriate choices of the interaction energies) from one to infinity. Thus it is not confined to any simple set of rational values. Rather it is the special cases previously solved that correspond to such special values of the exponent.

To see this, define three intermediate parameters A, B, C such that

$$\begin{aligned} A : B : C &= (-w_1^2 + w_2^2 + w_3^2 - w_4^2) : \\ &\quad (w_1^2 - w_2^2 + w_3^2 - w_4^2) : (w_1^2 + w_2^2 - w_3^2 - w_4^2) \\ &= -ab - cd : cd - ab : \tfrac{1}{2}(c^2 + d^2 - a^2 - b^2) \end{aligned} \tag{8.5}$$

[using (2.5)]. From (7.3) and (7.4) we see that

$$A : B : C = \mathrm{cn}(2\zeta, l) : \mathrm{dn}(2\zeta, l) : 1, \tag{8.6}$$

and when the restrictions (7.2) are satisfied

$$C/B > 1 > |A/B|. \tag{8.7}$$

Interchanging the w's simply rearranges A, B, C, and possibly negates two of them. Their ratios occur fairly naturally in the above working [e.g., in Eq. (C11)], and can be thought of as generalizations of the parameter Δ used by Lieb [3, 4] for the F and KDP models.

When one of the terms on the r.h.s. of (8.5) vanishes the 8-vertex model can be solved by Pfaffians. This is the case for the dimer, Ising and free-fermion models. Thus for such models one of the parameters A, B, C must be zero, and after rearranging the w's to satisfy (7.2) we see from (8.7) that we must have $A = 0$. From (8.6) it follows that $\mathrm{cn}(2\zeta, l) = 0$ and, using the restriction (7.6), $\zeta = \frac{1}{2}K_l$. From (8.2) we therefore see that $\mu = \pi/2$, so the singularity in the free energy is therefore of the type (8.1b) with $m = 1$. Clearly this is a very special case.

From (7.8) we also see that for the Pfaffian models $\lambda = \frac{1}{2}\tau$, so the denominator in the series in (7.7) simplifies. One can verify (after a rather lengthy calculation) that the expression (7.7) for the free energy is then the same as those previously obtained (c.f. [1, 9]).

For the ice-type models $d = 0$, so (8.5) can be written

$$A : B : C = 1 : 1 : \Delta, \tag{8.8}$$

where

$$\Delta = (a^2 + b^2 - c^2)/(2ab) \tag{8.9}$$

(this is the Δ used by Lieb). In addition we define

$$\Delta_1 = [(a^2 - b^2)^2 - 4c^2(a^2 + b^2)]/(8abc^2) \tag{8.10}$$

[this is obtained by inverting the w's in the definition (8.5), (8.8) of Δ, which from (5.3) is equivalent to interchanging V and ζ].

When $d = 0$ we see from (2.5) that $w_1 = w_2$, so it is impossible to arrange the w's to satisfy (7.2). Nevertheless we can arrange them to lie on a boundary of the region (7.2), and then take an appropriate limit in the formula (7.7) for the free energy. We find there are only three cases to consider, so for completeness we list the results,

(i) $$\begin{aligned} &\Delta_1 > \Delta > 1 : l = 1, \qquad \tau = \lambda = \alpha = \infty, \\ &\mu = U = \pi, \qquad \Delta = \cosh(\tau - \lambda), \\ &\Delta_1 = \cosh(\tau - \alpha), \qquad -\beta f = \ln a. \end{aligned} \tag{8.11}$$

(ii) $$-1 \leqslant \Delta_1 < \Delta < 1 : l = 0, \qquad \tau = 0,$$

$$\lambda = \alpha = 0, \qquad \Delta = -\cos \mu, \qquad \Delta_1 = -\cos U,$$

$$-\beta f = \ln \left\{ \frac{a + b + c}{2} \right\} + \int_{-\infty}^{\infty} dx \, \frac{\sinh^2[(\pi - \mu)\, x][\cosh(\mu x) - \cosh(Ux)]}{x \sinh(2\pi x) \cosh(\mu x)}. \tag{8.12}$$

(iii) $$-1 \geqslant \Delta_1 > \Delta : l = 1, \qquad \tau = \infty,$$

$$\mu = U = 0, \qquad \Delta = -\cosh \lambda, \qquad \Delta_1 = -\cosh \alpha,$$

$$-\beta f = \ln c + \sum_{n=1}^{\infty} \frac{e^{-2n\lambda}[\cosh(n\lambda) - \cosh(n\alpha)]}{n \cosh(n\lambda)}. \tag{8.13}$$

(The parameter U is the one used in Appendix E, and λ, α, μ, U must satisfy the inequalities $|\alpha| \leqslant \lambda \leqslant \tau$, $|U| \leqslant \mu \leqslant \pi$.) These results agree with those of Lieb [3, 4] for the F and KDP models.

Phase transitions occur when $\Delta = \pm 1$. Since more than two w's are then numerically equal, the formulae (8.1) no longer apply (see Lieb [3, 4] for discussions of the transition behaviour). Nevertheless it is interesting to note that $\mu = 0$ or π at such a transition, which again correspond to very special values of π/μ. Indeed for the F-model transition $\Delta = -1$, $\mu = 0$, and we can think of this as a limiting case in which the singularity (8.1) becomes of infinitely high order.

9. Generalized F-Model

As an example we consider a generalized F-model in which the interaction energies have the values

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon > 0,$$
$$\epsilon_5 = \epsilon_6 = 0, \qquad \epsilon_7 = \epsilon_8 = \epsilon' > 0. \tag{9.1}$$

Let

$$p = p(T) = \tfrac{1}{2}(1 + e^{-\epsilon'/kT}), \tag{9.2}$$

$$q = q(T) = \tfrac{1}{2}(1 - e^{-\epsilon'/kT}), \tag{9.3}$$

$$r = r(T) = e^{-\epsilon/kT}. \tag{9.4}$$

Then from (2.5) we see that

$$w_1, w_2, w_3, w_4 = p, q, 0, r. \tag{9.5}$$

Now arrange w_1, w_2, w_3, w_4 in numerically decreasing order. Since $p > q > 0$, there are at most three cases to consider, according as whether $r > p$, $p > r > q$, or $q > r$.

When $\epsilon' > 2\epsilon$ all three cases occur as T varies from 0 to ∞. Define two temperatures T_1, T_2 by

$$r(T_1) = q(T_1), \tag{9.6}$$

$$r(T_2) = p(T_2). \tag{9.7}$$

Then $0 < T_1 < T_2$ and the three cases are

(i) $0 < T < T_1 : q > r$,

$$w_1, w_2, w_3, w_4 = p, q, r, 0. \tag{9.8}$$

(ii) $T_1 < T < T_2 : p > r > q$,

$$w_1, w_2, w_3, w_4 = p, r, q, 0. \tag{9.9}$$

(iii) $T > T_2 : r > p$,

$$w_1, w_2, w_3, w_4 = r, p, q, 0. \tag{9.10}$$

In each case the restrictions (7.2) are satisfied and the free energy $f(T)$ can be calculated from Eqs. (7.3)–(7.7). The function $f(T)$ is analytic inside each region and across the $T = T_2$ boundary (largest two w's crossing). At $T = T_1$ the middle two w's cross, so there is a phase transition and $f(T)$ has a branch point singularity of the form (8.1), with exponent π/μ. At this temperature T_1 we see from (8.4) that μ is given by

$$\cos \mu = (2q^2 - p^2)/p^2. \tag{9.11}$$

When $\epsilon' \leqslant 2\epsilon$, $p > r$ at all temperatures, so the region (iii) no longer occurs. Nevertheless there is still a transition at the temperature T_1 defined by (9.6), with μ given by (9.11).

When $\epsilon' = \infty$ we regain the original F-model [3]. If we vary ϵ' from 0 to ∞, then T_1 decreases from ∞ to the F-model transition temperature, while μ decreases from π to 0. Thus again we see that we can think of the F-model transition as a limiting case in which $\mu \to 0$, giving an infinitely high-order singularity. When $\epsilon' = 2\epsilon$ we find that μ has the Pfaffian value $\pi/2$, and indeed in this case the model becomes the modified F-model of Wu [9], which is soluble by the Pfaffian method.

It is intriguing to note that for certain special values of ϵ', namely, those corresponding to π/μ being an odd integer greater than one, the phase transition disappears, $f(T)$ being analytic at T_1.

10. Inhomogeneous Systems

As a final point we mention that we can handle a restricted class of inhomogeneous 8-vertex models by the above methods (in the same way as the Bethe ansatz can be used for a restricted class of ice-type models).

Suppose a, b, c, d, and hence w_1, w_2, w_3, w_4, can vary from site to site of the lattice. At each site define l, ζ, V, and hence k, η, v, by (5.3), (5.7) or (6.2). Note that the interaction between columns of the lattice enters the above working only via the matrices **R**, **P**, **X**, **Y** of Appendices B and C, and that these depend only on l, ζ (or k, η) and the difference between two V's or v's. Using this, we find that we can generalize the results of these Appendices provided

(i) The parameters l, ζ, and hence k, η, are the same for each site of the lattice.

(ii) There exist parameters $V_1', \ldots, V_M'$, $V_1'', \ldots, V_N''$ such that at the site (I, J) V has the value

$$V = V_I' - V_J'', \tag{10.1}$$

and similarly, using (5.6),

$$v = v_I' - v_J''. \tag{10.2}$$

The normalization factors ρ in (6.2) can be varied arbitrarily from site to site. However, the effect of this is a trivial variation in the normalization of the partition function, so without loss of generality we can regard ρ as a constant. Then V_I', or v_I', is the only parameter that can vary from row to row, so we can write the transfer matrix of row I as $\mathbf{T}(v_I')$.

All such transfer matrices commute, and again we can construct **Q** so that (4.5) is satisfied, only now

$$\phi(v) = \prod_{J=1}^{N} \{\rho\Theta(0)\, H(v - v_J'')\, \Theta(v - v_J'')\}. \tag{10.3}$$

We can still use the technique of Appendix D to sum a perturbation expansion about the purely ordered F-model state in the limit of M and N large. We find that the total free energy F of the lattice is given by

$$\begin{aligned} -\beta F &= \ln Z, \\ &= -\beta \sum_{I=1}^{M} \sum_{J=1}^{N} f_{I,J}, \end{aligned} \tag{10.4}$$

where $f_{I,J}$ is given by (7.7), using the parameters w_1, w_2, α appropriate to the site (I, J). This is a remarkably simple result, and it is certainly true when each α is not

large compared with unity and λ, $\tau - \lambda$ are sufficiently large (i.e., the system is sufficiently close to the purely ordered state). However, since $z_1, \ldots, z_r$ of Appendix D no longer lie on the unit circle it is not clear under what precise conditions the perturbation expansion is convergent and the result (10.4) is valid.

11. Summary

In the absence of fields the interaction energies of the 8-vertex model can be chosen to satisfy (2.2). The free energy f in the limit of a large lattice is then given by the following procedure:

(i) Calculate w_1, w_2, w_3, w_4 from (2.3)–(2.5).

(ii) Replace each w_j by its absolute value and arrange them in nonincreasing order, so that

$$w_1 \geqslant w_2 \geqslant w_3 \geqslant w_4 \geqslant 0.$$

(iii) Calculate l, ζ, V from (7.3)–(7.6).

(iv) Calculate τ, λ, α from (7.8).

(v) Calculate f from (7.7). Alternatively, if the series in (7.7) converges only slowly it may be more convenient to calculate f from equation (E6) of Appendix E.

In general there is a phase transition when, and only when, the middle two w's cross. At this transition the free energy has the branch point singularity (8.1), with exponent π/μ defined by (8.4). This exponent can be varied continuously (by varying the interaction energies) from one to infinity. It is not confined to any simple set of rational values.

It would be very interesting to solve the 8-vertex model in the presence of fields, i.e., to remove the restriction (2.2b), since then one could solve the staggered F-model [10] and the Ising model with first- and second-neighbour interactions. Unfortunately we have not been able to do this. In this respect our method appears at a disadvantage compared with the Pfaffian and Bethe ansatz techniques, which when applicable can be used in the presence of fields.

Appendix A

Clearly the zero-field F, KDP and free-fermion models are special cases of the zero-field eight-vertex model. The same is true of the nearest-neighbour Ising model and the dimer model, as we now show.

The required transformation of the Ising model is discussed by Wu [9]. For

completeness we also give the argument here. Consider a square lattice with spins $\sigma_{i,j} = \pm 1$ on sites (i, j), with total energy

$$E = -\sum_i \sum_j \{J\sigma_{i,j}\sigma_{i+1,j+1} + J'\sigma_{i,j}\sigma_{i+1,j-1}\}. \tag{A1}$$

Thus we allow only second-neighbour interactions (crossed bonds). However, as there is no interaction between sites on the A sublattice ($i + j$ even) and sites on the B sublattice ($i + j$ odd), the partition function factorizes into the product of the partition functions of the A and B sublattices. On each of these sublattices we have an Ising system with *first*-neighbour interactions J, J', so that in the limit of a large lattice

$$f_{\text{crossed bonds}} = f_{\text{nearest neighbour}}, \tag{A2}$$

where f is the free energy per spin.

Now transform to the dual of the complete lattice, so the spins are associated with faces, rather than sites. Draw right- or up-pointing arrows on bonds between adjacent faces with the same spin, and left- or down-pointing arrows between adjacent faces with opposite spins. Then only the arrow configurations shown in Fig. 1 are allowed at a vertex. With each such vertex configuration we associate the interaction energy of the spins on the faces surrounding the vertex. The ϵ_j of Section 2 are then given by

$$\begin{aligned} \epsilon_1 &= \epsilon_2 = -J - J', & \epsilon_3 &= \epsilon_4 = J + J', \\ \epsilon_5 &= \epsilon_6 = J' - J, & \epsilon_7 &= \epsilon_8 = J - J'. \end{aligned} \tag{A3}$$

Noting that there is a simple 2 to 1 correspondence between spin orientations on the faces of the lattice and allowed configurations of arrows on the bonds, for a large lattice it follows that

$$f = f_{\text{Ising}}, \tag{A4}$$

where f is the free energy per vertex of the zero-field 8-vertex model specified by (A3), and f_{Ising} is the free energy per spin of an Ising model with first-neighbour interactions J, J'.

To express the close-packed dimer problem on the square lattice as an 8-vertex model, draw the lattice diagonally and add to it "superbonds" as shown in Fig. 3. At each vertex of the original lattice draw an arrow on the superbond passing through it, pointing right or left (up or down) according to whether the dimer at that vertex lies right or left of (above or below) that vertex.

Each vertex of the superbond lattice is surrounded by four bonds of the original lattice. Considering possible dimer arrangements on these four bonds, we see that only the arrow configurations 1, 2, 3, 4, 7, 8 of Fig. 1 can occur at a superbond

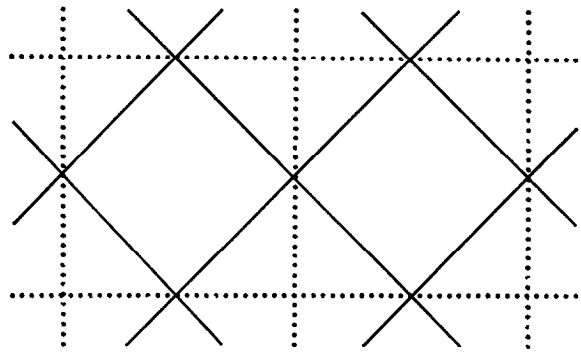

FIG. 3. The lattice transformation of Appendix A. A dimer covering of the original lattice (solid lines) is equivalent to an eight-vertex model on the superbond lattice (dotted lines).

vertex. With each such configuration we associate as a weight the product of the activities of the dimers on the surrounding four bonds. Noting that there are two dimer arrangements corresponding to configuration 7, this gives

$$\omega_1 = \omega_2 = z, \qquad \omega_3 = \omega_4 = z', \tag{A5}$$

$$\omega_5 = \omega_6 = 0, \tag{A6}$$

$$\omega_7 = z^2 + z'^2, \qquad \omega_8 = 1, \tag{A7}$$

where z, z' are the activities of dimers on the horizontal and vertical bonds, respectively, of the original lattice.

Clearly vertices of type 7 and 8 occur in pairs (being sinks and sources of arrows). Thus the partition function is unaffected if we replace (A7) by

$$\omega_7 = \omega_8 = (z^2 + z'^2)^{1/2}. \tag{A8}$$

Summing over allowed arrow configurations on the superbond lattice with these weights is equivalent to summing over dimer coverings of the original lattice, so we have reduced the dimer problem to a zero-field 8-vertex model.

APPENDIX B

Here we obtain the conditions under which two transfer matrices commute. Let $\mathbf{T}$ be the transfer matrix defined by (3.3) and (3.8), and let $\mathbf{T}'$ be similarly defined, but with the w_j replaced by w_j'. Then from (3.3)

$$[\mathbf{T}\,\mathbf{T}']_{\alpha|\beta} = \mathrm{Tr}\left\{\prod_{J=1}^{N} \mathbf{S}(\alpha_J, \beta_J)\right\}, \tag{B1}$$

where the matrix product over $J = 1,..., N$ is to be ordered in the same way as in (3.5), and the $\mathbf{S}(\alpha, \beta)$ are 4 by 4 matrices with elements

$$S^{\alpha,\beta}_{\lambda,\mu|\lambda',\mu'} = \sum_{\gamma} R(\alpha, \gamma \mid \lambda, \lambda')\, R'(\gamma, \beta \mid \mu, \mu') = \sum_{j=1}^{4}\sum_{k=1}^{4} w_j w_k' (\boldsymbol{\sigma}^j \boldsymbol{\sigma}^k)_{\alpha,\beta}\, \sigma^j_{\lambda,\lambda'} \sigma^k_{\mu,\mu'} \tag{B2}$$

[using (3.8)].

Similarly,

$$[\mathbf{T}'\,\mathbf{T}]_{\alpha|\beta} = \mathrm{Tr}\left\{\prod_{J=1}^{N} \mathbf{S}'(\alpha_J, \beta_J)\right\}, \tag{B3}$$

where the $\mathbf{S}'(\alpha, \beta)$ are given by interchanging the primed and unprimed w's in (B2). If we also interchange λ with μ, and λ' with μ' [this has no effect on (B3)], and j with k, the elements of $\mathbf{S}'(\alpha, \beta)$ are

$$\sum_{j=1}^{4}\sum_{k=1}^{4} w_j w_k' (\boldsymbol{\sigma}^k \boldsymbol{\sigma}^j)_{\alpha,\beta}\, \sigma^j_{\lambda,\lambda'} \sigma^k_{\mu,\mu'}\,. \tag{B4}$$

For $\mathbf{T}$ and $\mathbf{T}'$ to commute, the right-hand sides of (B1), (B3) must be the same. This will be so if there exists a 4 by 4 nonsingular matrix $\mathbf{R}$ such that

$$\mathbf{S}'(\alpha, \beta) = \mathbf{R}\mathbf{S}(\alpha, \beta)\, \mathbf{R}^{-1}, \tag{B5}$$

or, more conveniently,

$$\mathbf{S}'(\alpha, \beta)\mathbf{R} = \mathbf{R}\mathbf{S}(\alpha, \beta), \tag{B6}$$

for $\alpha = \pm$ and $\beta = \pm$.

Some inspection shows that if such a matrix $\mathbf{R}$ exists it must have elements of the form

$$R_{\lambda,\mu|\lambda',\mu'} = \sum_{j=1}^{4} x_j \sigma^j_{\lambda,\lambda'} \sigma^j_{\mu,\mu'}\,. \tag{B7}$$

Substituting the forms (B2), (B4), (B7) into (B6), performing the matrix multiplications, we find that (B6) is satisfied provided

$$w_m w_l' x_j - w_l w_m' x_k + w_k w_j' x_l - w_j w_k' x_m = 0 \tag{B8}$$

for all permutations (j, k, l, m) of $(1, 2, 3, 4)$.

There are six such equations (B8). Regarding them as linear homogeneous equations for x_1, x_2, x_3, x_4, we find that they have a nontrivial solution provided

$$(w_j^2 - w_k^2)/(w_l^2 - w_m^2) = (w_j'^2 - w_k'^2)/(w_l'^2 - w_m'^2) \tag{B9}$$

for all values 1, 2, 3, 4 of j, k, l and m. Thus these are the conditions for $\mathbf{T}$ and $\mathbf{T}'$ to commute. There are only two such independent conditions, and an obvious parametrization that satisfies (B9) is to set

$$\begin{aligned} w_j^2 &= p(\xi - s_j), \\ w_j'^2 &= p'(\xi' - s_j) \end{aligned} \tag{B10}$$

for $j = 1,\ldots, 4$.

When (B9) is satisfied, we can verify from (B8) that

$$(x_j^2 - x_k^2)/(x_l^2 - x_m^2) = (w_j^2 - w_k^2)/(w_l^2 - w_m^2) \tag{B11}$$

for all values of j, k, l, m. Thus we can also introduce p'', ξ'' such that

$$x_j^2 = p''(\xi'' - s_j). \tag{B12}$$

Clearly p, p', p'' are arbitrary normalization factors that cancel out of (B8). Regarding s_1, s_2, s_3, s_4 as constants, any of the equations (B8) can be regarded as defining ξ'' as a function of ξ and ξ'. Differentiating, we find that

$$\frac{\partial \xi''}{\partial \xi} \Big/ \frac{\partial \xi''}{\partial \xi'} = \frac{-g(\xi)}{g(\xi')}, \tag{B13}$$

where

$$g(\xi) = \text{constant} \times \prod_{j=1}^{4} (\xi - s_j)^{-1/2}. \tag{B14}$$

The factorization of the r.h.s. of (B13) is interesting, since it implies that if we transform from ξ, ξ' to new variables V, V' such that

$$dV/d\xi = g(\xi), \qquad dV'/d\xi' = g(\xi'), \tag{B15}$$

then ξ'' will be a function only of $V - V'$. (This transformation is related to the transformation to a difference kernel that is used in the Bethe ansatz for the ice models.) The integrations in (B15) can be performed using elliptic functions (§3.147 of GR [8]), and with an appropriate choice of the constant in (B14) we find that

$$\xi = [(s_3 - s_1)\, s_4 - s_3(s_4 - s_1)\, \mathrm{sn}^2(V, l)]/[s_3 - s_1 - (s_4 - s_1)\, \mathrm{sn}^2(V, l)], \tag{B16}$$

where $\mathrm{sn}(u, l)$ is the elliptic sine-amplitude function (§8.143 of GR [8]) of modulus

$$l = [(s_3 - s_2)(s_4 - s_1)/(s_4 - s_2)(s_3 - s_1)]^{1/2}. \tag{B17}$$

(We implicitly consider the case $\xi > s_4 > s_3 > s_2 > s_1$, when V is real and $0 < l < 1$. However, the parametrization is not limited to this region.) Adding primes to ξ and V in (B16), we also obtain the relation between ξ' and V'.

Substituting the expression (B16) into (B10) and defining a parameter ζ such that

$$\mathrm{sn}(\zeta, l) = [(s_3 - s_1)/(s_4 - s_1)]^{1/2}, \tag{B18}$$

it follows that we can choose the w's so that

$$w_1 : w_2 : w_3 : w_4 = \frac{\mathrm{cn}(V, l)}{\mathrm{cn}(\zeta, l)} : \frac{\mathrm{dn}(V, l)}{\mathrm{dn}(\zeta, l)} : 1 : \frac{\mathrm{sn}(V, l)}{\mathrm{sn}(\zeta, l)}, \tag{B19}$$

where $\mathrm{cn}(u, l)$, $\mathrm{dn}(u, l)$ are the other Jacobian elliptic functions (§8.14 of GR [8]). This is the parametrization (5.3) quoted in the text.

Note that l, ζ are defined by s_1, s_2, s_3, s_4, but V depends also on ξ. Thus any two transfer matrices with the same values of ζ and l, but different values of V, must commute.

Appendix C

Using the parametrization (5.7), or more explicitly (6.2), of the vertex weights, we look for a matrix $\mathbf{Q}(v)$ which satisfies an equation of the form (4.5).

As a first step, we attempt to construct a 2^N by 2^N matrix $\mathbf{Q}_R$ such that

$$\mathbf{T}\mathbf{Q}_R = \text{sum of two } Q\text{-type matrices.} \tag{C1}$$

From the form (3.5) of $\mathbf{T}$, we see that the most general matrix $\mathbf{Q}_R$ that we can conveniently handle is one with elements

$$[\mathbf{Q}_R]_{\alpha|\beta} = \mathrm{Tr}\left\{\prod_{J=1}^{N} \mathbf{S}(\alpha_J, \beta_J)\right\}, \tag{C2}$$

where the $\mathbf{S}(\alpha, \beta)$ are matrices of some dimension L, and the matrix product over $J = 1,..., N$ is to be ordered in the same way as in (3.5).

From (3.5), (3.6) and (C2), we see that

$$[\mathbf{T}\,\mathbf{Q}_R]_{\alpha|\beta} = \mathrm{Tr}\left\{\prod_{J=1}^{N} \mathbf{U}(\alpha_J, \beta_J)\right\}, \tag{C3}$$

where the $\mathbf{U}(\alpha, \beta)$ are $2L$ by $2L$ matrices and

$$\begin{aligned} \mathbf{U}(+, \beta) &= \begin{pmatrix} a\mathbf{S}(+, \beta) & d\mathbf{S}(-, \beta) \\ c\mathbf{S}(-, \beta) & b\mathbf{S}(+, \beta) \end{pmatrix}, \\ \mathbf{U}(-, \beta) &= \begin{pmatrix} b\mathbf{S}(-, \beta) & c\mathbf{S}(+, \beta) \\ d\mathbf{S}(+, \beta) & a\mathbf{S}(-, \beta) \end{pmatrix}. \end{aligned} \tag{C4}$$

The r.h.s. of (C3) will decompose into the sum of two matrices if we can find a $2L$ by $2L$ matrix $\mathbf{M}$ (independent of α and β) such that

$$\mathbf{M}^{-1}\mathbf{U}(\alpha, \beta)\,\mathbf{M} = \begin{pmatrix} \mathbf{A}(\alpha, \beta) & \mathbf{0} \\ \mathbf{C}(\alpha, \beta) & \mathbf{B}(\alpha, \beta) \end{pmatrix}, \tag{C5}$$

since substituting this form for $\mathbf{U}(\alpha, \beta)$ into (C3) gives

$$\mathbf{T}\mathbf{Q}_R = \mathbf{H}_1 + \mathbf{H}_2\,, \tag{C6}$$

where

$$\begin{aligned} [\mathbf{H}_1]_{\alpha|\beta} &= \mathrm{Tr}\left\{\prod_{J=1}^{N} \mathbf{A}(\alpha_J\,, \beta_J)\right\}, \\ [\mathbf{H}_2]_{\alpha|\beta} &= \mathrm{Tr}\left\{\prod_{J=1}^{N} \mathbf{B}(\alpha_J\,, \beta_J)\right\}. \end{aligned} \tag{C7}$$

The form of the requirement (C5) is unaffected by postmultiplying $\mathbf{M}$ by a lower blocktriangular matrix, so we can in general choose

$$\mathbf{M} = \begin{pmatrix} \mathbf{E} & \mathbf{P} \\ \mathbf{0} & \mathbf{E} \end{pmatrix}, \tag{C8}$$

where $\mathbf{E}$ is the identity matrix and $\mathbf{P}$ is some L by L matrix. Further, (C2) and the form of (C5) are unaffected by applying a similarity transformation to the $\mathbf{S}(\alpha, \beta)$ and $\mathbf{P}$, so we can in general choose $\mathbf{P}$ to be diagonal, with elements

$$P_{m,n} = p_m \delta_{m,n} \qquad (m, n = 1,..., L). \tag{C9}$$

Writing the elements of $\mathbf{S}(\alpha, \beta)$ as $s_{m,n}^{\alpha,\beta}$, using (C4), (C8), (C9), we find that the upper right block of the matrix on the l.h.s. of (C5) vanishes provided

$$\begin{aligned} (ap_n - bp_m)\, s_{m,n}^{+,\beta} + (d - cp_m p_n)\, s_{m,n}^{-,\beta} &= 0, \\ (c - dp_m p_n)\, s_{m,n}^{+,\beta} + (bp_n - ap_m)\, s_{m,n}^{-,\beta} &= 0 \end{aligned} \tag{C10}$$

for $m = 1,..., L$ and $n = 1,..., L$.

For given values of m, n, β, (C10) is a pair of homogeneous linear equations. They have a nontrivial solution provided the determinant of the coefficients vanishes, i.e.,

$$(a^2 + b^2 - c^2 - d^2)\, p_m p_n = ab(p_m{}^2 + p_n{}^2) - cd(1 + p_m{}^2 p_n{}^2). \tag{C11}$$

This can only happen for certain values of m and n. For all other values we must have

$$s_{m,n}^{\alpha,\beta} = 0. \tag{C12}$$

Using the parametrization (5.7) of a, b, c, d we find that

$$\frac{a^2 + b^2 - c^2 - d^2}{ab} = 2\,\mathrm{cn}(2\eta)\,\mathrm{dn}(2\eta),$$
$$cd/ab = k\,\mathrm{sn}^2(2\eta) \tag{C13}$$

(note that these quantities are independent of v). It follows that if $p_m = k^{1/2}\,\mathrm{sn}\,u$, then (C11) can be solved to give $p_n = k^{1/2}\,\mathrm{sn}(u \pm 2\eta)$. This suggests a very natural ordering of $p_1, \ldots, p_L$, namely,

$$p_m = k^{1/2}\,\mathrm{sn}[K + (2m - 1)\eta], \qquad m = 1, \ldots, L, \tag{C14}$$

where K is the complete elliptic integral of the first kind of modulus k.

We have chosen (C14) so that (formally) $p_0 = p_1$. Thus (C11) is satisfied for $m = n = 1$ as well as $n = m \pm 1$. Hence the matrices $\mathbf{S}(\alpha, \beta)$ can have nonzero elements in the upper left diagonal and one-off diagonal positions. We can also require that the bottom right diagonal element be nonzero, which implies that p_{L+1} given by (C14) be such that $p_{L+1} = p_L$. Using the periodicity and oddness properties of sn u (§8. 151of GR [8]) this will be so if there exist integers m_1, m_2 such that

$$2L\eta = 2m_1K + im_2K', \tag{C15}$$

where K' is the complete elliptic integral of the first kind of modulus $k' = (1 - k^2)^{1/2}$.

Clearly (C15) is a restriction on the possible values of η. However, as the positive integer L is arbitrary, we can approach arbitrarily close to any desired value.

We can now solve (C10) and (C5) for $\mathbf{S}(\alpha, \beta)$, $\mathbf{A}(\alpha, \beta)$, $\mathbf{B}(\alpha, \beta)$. All these matrices are of the form

$$\begin{pmatrix} z_0 & z_{-1} & 0 & 0 & \cdot & 0 \\ z_1 & 0 & z_{-2} & 0 & \cdot & \cdot \\ 0 & z_2 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & z_{1-L} \\ 0 & \cdot & \cdot & 0 & z_{L-1} & z_L \end{pmatrix}. \tag{C16}$$

Using (6.2), we find the following values of z_m ($m = 1 - L, \ldots, L$):

$$\mathbf{S}(\alpha, \beta) : z_m = q(\alpha, \beta, m \mid v),$$
$$\mathbf{A}(\alpha, \beta) : z_m = \rho\Theta(0)\,H(v - \eta)\,\Theta(v - \eta)(x_m/x_{m+1})\,q(\alpha, \beta, m \mid v + 2\eta), \tag{C17}$$
$$\mathbf{B}(\alpha, \beta) : z_m = \rho\Theta(0)\,H(v + \eta)\,\Theta(v + \eta)(x_{m+1}/x_m)\,q(\alpha, \beta, m \mid v - 2\eta),$$

where

$$x_m = \Theta[K + (2m - 1)\eta] \tag{C18}$$

and the functions $q(\alpha, \beta, m \mid v)$ of v are defined by

$$\begin{aligned} q(+, \beta, m \mid v) &= H(v + K + 2m\eta)\, \tau_{\beta,m}\,, \\ q(-, \beta, m \mid v) &= \Theta(v + K + 2m\eta)\, \tau_{\beta,m}\,, \end{aligned} \tag{C19}$$

the parameters $\tau_{\beta,m}$ being arbitrary. To derive (C17) we have used (6.1), the fact that sn u and $H(u)$ are odd functions while $\Theta(u)$ is an even function, and the general formulae

$$\begin{aligned} \text{sn } A \text{ sn } B - \text{sn } C \text{ sn } D &= \frac{\Theta(0)\, \Theta(A + B)\, H(A - D)\, H(B - D)}{k\Theta(A)\, \Theta(B)\, \Theta(C)\, \Theta(D)}\,, \\ 1 - k^2 \text{ sn } A \text{ sn } B \text{ sn } C \text{ sn } D &= \frac{\Theta(0)\, \Theta(A + B)\, \Theta(A - D)\, \Theta(B - D)}{\Theta(A)\, \Theta(B)\, \Theta(C)\, \Theta(D)}\,, \end{aligned} \tag{C20}$$

when $A + B = C + D$.

From (C17) it is apparent that we can regard $\mathbf{S}(\alpha, \beta)$ as a function of v. Exhibiting this dependence explicitly by writing the matrix as $\mathbf{S}(\alpha, \beta \mid v)$, we see that

$$\begin{aligned} \mathbf{A}(\alpha, \beta) &= \rho\Theta(0)\, H(v - \eta)\, \Theta(v - \eta)\, \mathbf{X}^{-1}\mathbf{S}(\alpha, \beta \mid v + 2\eta)\, \mathbf{X}, \\ \mathbf{B}(\alpha, \beta) &= \rho\Theta(0)\, H(v + \eta)\, \Theta(v + \eta)\, \mathbf{X}\mathbf{S}(\alpha, \beta \mid v + 2\eta)\, \mathbf{X}^{-1}, \end{aligned} \tag{C21}$$

where $\mathbf{X}$ is an L by L diagonal matrix with elements $x_m\delta_{m,n}$.

Clearly $\mathbf{T}$ and $\mathbf{Q}_R$ are also functions of v. Exhibiting this dependence explicitly, substituting the expressions (C21) into (C7) and using (C2), the equation (C6) becomes

$$\mathbf{T}(v)\, \mathbf{Q}_R(v) = \phi(v - \eta)\, \mathbf{Q}_R(v + 2\eta) + \phi(v + \eta)\, \mathbf{Q}_R(v - 2\eta), \tag{C22}$$

where

$$\phi(v) = \{\rho\Theta(0)\, H(v)\, \Theta(v)\}^N. \tag{C23}$$

[The matrices $\mathbf{X}$ cancel out of (C7).]

Commutation Relations

We have obviously come a long way towards finding the generalization of (4.5). It remains to construct a matrix $\mathbf{Q}(v)$ which satisfies (C22) and also commutes with $\mathbf{T}(v)$.

To do this we first repeat the above working, replacing $\mathbf{Q}_R$ by $\mathbf{Q}_L$ and forming

the product $\mathbf{Q}_L\mathbf{T}$ in (C3). The effect of this is to interchange c and d in (C4) and (C10), but the matrix $\mathbf{P}$ remains the same. We find that

$$\mathbf{Q}_L(v)\,\mathbf{T}(v) = \phi(v-\eta)\,\mathbf{Q}_L(v+2\eta) + \phi(v+\eta)\,\mathbf{Q}_L(v-2\eta), \tag{C24}$$

where

$$[Q_L(v)]_{\alpha|\beta} = \mathrm{Tr}\left\{\prod_{J=1}^{N} \mathbf{S}'(\alpha_J, \beta_J \mid v)\right\} \tag{C25}$$

and the $\mathbf{S}'(\alpha, \beta \mid v)$ are L by L matrices of the form (C16) with

$$\begin{aligned} &\mathbf{S}'(\alpha, + \mid v) : z_m = \Theta(v - K - 2m\eta)\,\tau'_{\alpha,m}\,, \\ &\mathbf{S}'(\alpha, - \mid v) : z_m = H(v - K - 2\eta)\,\tau'_{\alpha,m}\,, \end{aligned} \tag{C26}$$

the parameters $\tau'_{\alpha,m}$ being arbitrary.

Now note that (C22), (C24) are unaffected by post- and pre-multiplying $\mathbf{Q}_R(v)$, $\mathbf{Q}_L(v)$, respectively, by constant matrices. Thus to construct a matrix $\mathbf{Q}(v)$ which commutes with $\mathbf{T}(v)$, it is sufficient to show that there exist nonsingular constant matrices $\mathbf{F}$, $\mathbf{G}$ such that

$$\mathbf{Q}(v) = \mathbf{Q}_R(v)\,\mathbf{F} = \mathbf{G}\mathbf{Q}_L(v). \tag{C27}$$

We also expect any two matrices $\mathbf{Q}(u)$, $\mathbf{Q}(v)$ to commute. From (C27) this will be so if

$$\mathbf{Q}_L(u)\,\mathbf{Q}_R(v) = \mathbf{Q}_L(v)\,\mathbf{Q}_R(u). \tag{C28}$$

We can prove this relation. From (C2) and (C25) we see that

$$[\mathbf{Q}_L(u)\,\mathbf{Q}_R(v)]_{\alpha|\beta} = \mathrm{Tr}\left\{\prod_{J=1}^{N} \mathbf{W}(\alpha_J, \beta_J \mid u, v)\right\}, \tag{C29}$$

where the $\mathbf{W}(\alpha, \beta \mid u, v)$ are L^2 by L^2 matrices, with elements

$$[\mathbf{W}(\alpha, \beta \mid u, v)]_{m,m'|n,n'} = \sum_{\gamma=\pm} [\mathbf{S}'(\alpha, \gamma \mid u)]_{m,n}\,[\mathbf{S}(\gamma, \beta \mid v)]_{m',n'}\,. \tag{C30}$$

Using (C17), (C19), (C26) and the identity

$$H(A)\,\Theta(B) + \Theta(A)\,H(B) = f(A+B)\,g(A-B), \tag{C31}$$

where $f(u)$, $g(u)$ are functions such that

$$g(u) = g(-u) = -g(u+4K), \tag{C32}$$

we find that

$$[\mathbf{W}(\alpha, \beta \mid u, v)]_{m,m'|n,n'} = y_{m,m'}[\mathbf{W}(\alpha, \beta \mid v, u)]_{m,m'|n,n'}/y_{n,n'}, \tag{C33}$$

where

$$y_{m,m'} = t_{m+m'}t_{m-m'+1} \tag{C34}$$

and the t_m are defined by the recurrence relation

$$t_m/t_{m+2} = g(u - v + 2K + 2m\eta)/g(v - u + 2K + 2m\eta). \tag{C35}$$

Equation (C33) can be written in matrix form as

$$\mathbf{W}(\alpha, \beta \mid u, v) = \mathbf{Y}\mathbf{W}(\alpha, \beta \mid v, u)\,\mathbf{Y}^{-1}, \tag{C36}$$

where $\mathbf{Y}$ is a diagonal L^2 by L^2 matrix with elements $y_{m,m'}\delta_{m,n}\delta_{m',n'}$. Substituting the expression (C36) into (C29), the matrices $\mathbf{Y}$ cancel and we obtain the required identity (C28).

Although we have not been able to construct a proof, it appears that the matrices $\mathbf{Q}_R(v)$, $\mathbf{Q}_L(v)$ are in general nonsingular (certainly this is so for $N = 1$ and 2). Assuming this, it follows from (C28) that we can define

$$\mathbf{Q}(v) = \mathbf{Q}_R(v)\,\mathbf{Q}_R^{-1}(v_0) = \mathbf{Q}_L^{-1}(v_0)\,\mathbf{Q}_L(v), \tag{C37}$$

where v_0 is some fixed value of v. From (C22), (C24) and (C28) it then follows that

$$\begin{aligned}
&\mathbf{T}(v)\,\mathbf{Q}(v) = \phi(v - \eta)\,\mathbf{Q}(v + 2\eta) + \phi(v + \eta)\,\mathbf{Q}(v - 2\eta),\\
&\mathbf{T}(v)\,\mathbf{Q}(v) = \mathbf{Q}(v)\,\mathbf{T}(v),\\
&\mathbf{Q}(u)\,\mathbf{Q}(v) = \mathbf{Q}(v)\,\mathbf{Q}(u),
\end{aligned} \tag{C38}$$

for all values of v and u. These are the required generalizations of the equations for the ice-type models.

Note that if the eigenvalues of $\mathbf{Q}(v)$ are nondegenerate (as appears to be the case), then (C38) is sufficient to establish that any two transfer matrices $\mathbf{T}(u)$, $\mathbf{T}(v)$ commute. Thus in this sense Appendix B is redundant. Nevertheless we have included it, partly for completeness but also because it provides a natural way of starting the calculation.

Appendix D

We here use (4.2), (6.3), (6.10) to obtain the maximum eigenvalue $T(v)$ of $\mathbf{T}(v)$ in the limit of N large. We could employ the method used by Yang [11] and Lieb [2–4] for the Bethe ansatz, namely to look at the equations for $v_1, \ldots, v_{N/2}$, assume

that they tend to a continuous distribution along some line segment in the complex plane, and thus obtain an integral equation for the distribution function. Instead we adopt a perturbation expansion approach similar to that used by Baxter [7], which we believe makes some of the properties of the equation (4.2) more transparent. In addition, it can be applied to the inhomogeneous system discussed at the end of this paper.

We consider the regime

$$w_1 > w_2 > w_3 > |w_4|. \tag{D1}$$

In this case it follows from (5.3) (or Appendix B) that we can choose l, V, ζ so that

$$0 < l < 1, \qquad |V| < \zeta < K_l, \tag{D2}$$

K_l being the complete elliptic integral of the first kind of modulus l. Using (5.5), (5.6) and the formulae (§8.126 of GR [8])

$$\begin{aligned} K_l &= \tfrac{1}{2}(1 + k)\, K_{k'}, \\ K_l' &= (1 + k)\, K_k, \end{aligned} \tag{D3}$$

we can therefore define real parameters τ, λ, α such that

$$\tau = \pi K_{k'}/(2K_k) = \pi K_l/K_l', \tag{D4}$$

$$\lambda = -i\pi\eta/K_k = \pi\zeta/K_l', \tag{D5}$$

$$\alpha = -i\pi v/K_k = \pi V/K_l', \tag{D6}$$

and

$$0 < k < 1, \qquad |\alpha| < \lambda < \tau. \tag{D7}$$

We further define

$$q = e^{-2\tau}, \qquad x = e^{-\lambda}, \qquad z = e^{-\alpha}, \tag{D8}$$

so from (D7) we see that

$$q < x^2 < 1, \tag{D9}$$

$$x < z < x^{-1}, \tag{D10}$$

These parameters occur quite naturally when we expand the elliptic theta functions of modulus k in (6.3) and (6.10) as infinite products, using the general formulae (§8.181 of GR [8])

$$\begin{aligned} H(v) &= 2i\gamma q^{1/4} \sinh(\tfrac{1}{2}\alpha) \prod_{n=1}^{\infty} (1 - q^{2n}z)(1 - q^{2n}z^{-1}), \\ \Theta(v) &= \gamma \prod_{n=1}^{\infty} (1 - q^{2n-1}z)(1 - q^{2n-1}z^{-1}), \end{aligned} \tag{D11}$$

where α, q, z are defined in terms of k and v by (D6) and (D8), and

$$\gamma = \prod_{n=1}^{\infty} (1 - q^{2n}). \tag{D12}$$

We find that it is convenient also to define the quantities

$$\xi = i\rho\Theta(0)\,\gamma^2 q^{1/4} x^{-1}, \tag{D13}$$

$$r = \tfrac{1}{2}N, \tag{D14}$$

$$z_j = \exp(i\pi v_j/K_k), \qquad j = 1,\ldots, r. \tag{D15}$$

(We take N to be even, so r is an integer.)

First Order Approximation

As a first step consider the extreme case $q \ll x^2 \ll 1$ and suppose that $|\, z_j \,| \sim 1$ for $j = 1,\ldots, r$. Then the condition (6.12) implies that the integer ν is zero and gives the *exact* equation

$$z_1 \cdots z_r = (-1)^{r+\nu''}. \tag{D16}$$

Although v must be pure imaginary for the vertex weights a, b, c, d to be real, the equation (4.2) is an algebraic identity, valid for all complex values of v, and hence z. Using (6.3), (6.10), (D11) to expand the terms in (4.2) as infinite products, we find that when $|\, z \,| \sim 1$ each of the products on the r.h.s. is dominated by a single term. Retaining only such dominant terms, the equation (4.2) becomes

$$T(v) \prod_{j=1}^{r} (z - z_j) \simeq \xi^N\{(-1)^r\, z_1 \cdots z_r z^r + 1\}. \tag{D17}$$

This equation is an identity, true for all values of z close to the unit circle. As $T(v)$ is an entire function, it follows that the r.h.s. is a polynomial of degree r, with zeros $z = z_1, \ldots, z_r$. Thus

$$\prod_{j=1}^{r} (z - z_j) \simeq z^r + (-1)^r/(z_1 \cdots z_r). \tag{D18}$$

Equating final coefficients on both sides of (D18), we see that

$$(z_1 \cdots z_r)^2 = 1. \tag{D19}$$

This agrees with the exact equation (D16). There are two solutions, corresponding to which square root of (D19) we take, i.e., to whether $\nu'' = 0$ or 1 in (D16). We also note that (D18) implies that $z_1, \ldots, z_r$ lie on the unit circle, as was assumed.

The polynomials on both sides of (D17) therefore cancel. Using (D16) we obtain

$$T(v) \simeq (-1)^{\nu''}\, \xi^N. \tag{D20}$$

Having located $z_1, \ldots, z_r$, we can now let v, and hence z, assume any value in (4.2). There are many cases to be considered, corresponding to which terms dominate in the various product expansions, but in particular we find that $T(v)$ is given by (D20) when

$$\max[x^2, q^{1/2}] < |z| < \min[x^{-2}, q^{-1/2}]. \tag{D21}$$

Restricting attention to real values of a, b, c, d, and therefore z, we also find that when (D10) is satisfied, $c \simeq \xi$ and c is the dominant vertex weight. The energetically favoured configurations of the lattice are therefore the two completely ordered F-model states, in which vertices on one sublattice are of type 5, those on the other of type 6. Thus in the region (D10), which is contained in (D21), the above working does indeed give the two largest eigenvalues (D20) of $\mathbf{T}(v)$, equal in magnitude and opposite in sign.

Outside (D10) the maximum eigenvalue is not given by (D20). Since all matrices $\mathbf{T}(v)$ commute, and hence have the same eigenvectors, it follows that the eigenvectors which correspond to the largest eigenvalues of $\mathbf{T}(v)$ inside (D10) do not do so outside, and vice versa. This does not violate any theorems of the Frobenius type, since at the boundaries of (D10) $v = \pm\eta$. Thus from (5.7) we see that two of the weights a, b, c, d become zero and $\mathbf{T}(v)$ becomes a matrix with mostly zero elements, breaking up into many diagonal blocks. The restriction (D10) on the validity of the formula for the maximum eigenvalue is therefore a very natural one.

Perturbation Expansion

We now implicitly consider a perturbation expansion about the solution (D20) and show that if we neglect certain terms that tend to zero as N becomes large, then we can solve (4.2) for $T(v)$.

To do this we define the functions

$$A(z) = \prod_{n=0}^{\infty} (1 - q^n z)^N, \tag{D22}$$

$$F(z) = \prod_{j=1}^{r} \prod_{n=0}^{\infty} (1 - q^n z/z_j), \tag{D23}$$

$$G(z) = \prod_{j=1}^{r} \prod_{n=0}^{\infty} (1 - q^n z_j/z). \tag{D24}$$

[Thus $A(z)$ is a known function. The functions $F(z)$, $G(z)$ are to be determined.] Using the expansions (D11), we can then write (4.2) exactly as

$$(-1)^r (z_1 \cdots z_r)^{-1} T(v) F(qz) G(q^{-1}z) \prod_{j=1}^{r} (z - z_j)$$
$$= \xi^N A(x^{-1}qz) A(x^{-1}q/z) F(x^2 z) G(x^{-2}z) \Gamma(z), \tag{D25}$$

where

$$\Gamma(z) = z^r A(x/z)\, G(q^{-1}x^2z)/[A(x^{-1}q/z)\, G(x^{-2}z)] + (-1)^r(z_1 \cdots z_r)^{-1}\, A(xz)\, F(x^{-2}qz)/[A(x^{-1}qz)\, F(x^2z)]. \tag{D26}$$

We assume again that $|z_j| = 1$, $j = 1,\ldots, r$. Then when $|z| = 1$ and the restrictions (D9) are satisfied, we can develop convergent expansions of the functions on the r.h.s. of (D26) in increasing powers of x and $x^{-2}q$. Neglecting terms of order x^r, $x^{-2r}q^r$, or smaller, we find that $\Gamma(z)$ is a polynomial of degree r, with leading coefficient unity. Further, from (D25) we see that $\Gamma(z)$ has zeros $z = z_1,\ldots, z_r$, so to this order of approximation we have the identity

$$\prod_{j=1}^{r} (z - z_j) \equiv \Gamma(z). \tag{D27}$$

Thus the parameters $z_1,\ldots, z_r$ can in principle be calculated by equating coefficients on both sides of (D27). In particular, the last such equation gives (D19), in agreement with the exact result (D16). We therefore again have two solutions, corresponding to whether we take $\nu'' = 0$ or 1 in (D16).

From (D27) we see that the last terms on both sides of (D25) cancel. Using (D16), we obtain

$$T(v) = (-1)^{\nu''}\, \xi^N[A(x^{-1}qz)\, A(x^{-1}q/z)\, F(x^2z)\, G(x^{-2}z)/F(qz)\, G(q^{-1}z)]. \tag{D28}$$

Rather than evaluate $z_1,\ldots, z_r$ directly, we see from (D28) that it is sufficient to determine $F(z)$ for $|z| \leqslant x^2$ and $G(z)$ for $|z| \geqslant x^{-2}$. We can do this from (D26) and (D27).

Since (D27) is an identity, it must be true for all values of z, so long as we replace $\Gamma(z)$ by its truncated expansion, or ensure that any terms neglected, or extra terms introduced, are negligible. When $|z| \geqslant x^{-2}$, to order x^N only the first term on the r.h.s. of (D26) contributes to this expansion. Using (D24), we can then write (D27) as

$$G(z)/G(q^{-1}z) = A(x/z)\, G(q^{-1}x^2z)/[A(x^{-1}q/z)\, G(x^{-2}z)]. \tag{D29}$$

This can be thought of as a recurrence relation for $G(z)$. It is greatly simplified if we note that it is equivalent to

$$H(z) = H(q^{-1}x^2z), \tag{D30}$$

where

$$H(z) = G(z)\, G(x^{-2}z)/A(x/z). \tag{D31}$$

The function $H(z)$ is Laurent expandable in inverse powers of z with leading term unity, and $q^{-1}x^2 > 1$. Expanding both sides of (D30) and equating coefficients, it follows that

$$H(z) = 1. \tag{D32}$$

Thus from (D31) the recurrence relation for $G(z)$ simplifies to

$$G(z) = A(x/z)/G(x^{-2}z). \tag{D33}$$

Solving this by iteration, using the fact that $G(z) \to 1$ as $z \to \infty$, we obtain

$$G(z) = \prod_{m=0}^{\infty} \frac{A(x^{4m+1}/z)}{A(x^{4m+3}/z)} \tag{D34}$$

provided $|z| \geqslant x^{-2}$.

Similarly, when $|z| \leqslant x^2$ only the second term on the r.h.s. of (D26) is significant and from (D23) and (D27) we can deduce that

$$F(z) = \prod_{m=0}^{\infty} \frac{A(x^{4m+1}z)}{A(x^{4m+3}z)}. \tag{D35}$$

Substituting these forms for the functions $G(z)$ and $F(z)$ into (D28), using (D22) and noting from (6.1), (6.2), (D11) and (D13) that

$$c/\xi = \prod_{n=1}^{\infty} \{(1 - q^{2n-1})^2 (1 - q^{2n-2}x^2)(1 - q^{2n}x^{-2}) \times (1 - q^{2n-1}xz)(1 - q^{2n-1}x^{-1}z)(1 - q^{2n-1}xz^{-1})(1 - q^{2n-1}x^{-1}z^{-1})\}, \tag{D36}$$

we find that $(-1)^{\nu''} c^{-N}T(v)$ is a product of terms of the form $1 - \alpha$, where $|\alpha| < 1$. Taking logarithms and Taylor expanding each $\ln(1 - \alpha)$ term, the various m and n summations can be performed to give

$$N^{-1} \ln[(-1)^{\nu''} T(v)] = \ln c + \sum_{j=1}^{\infty} \frac{x^{-j}(x^{2j} - q^j)^2 (x^j + x^{-j} - z^j - z^{-j})}{j(1 - q^{2j})(1 + x^{2j})}. \tag{D37}$$

Using the definitions (D8), this is the result quoted in the text.

We must verify the assumption that $z_1, \ldots, z_r$ lie in the unit circle. To do this, note that each z_j is a zero of $\Gamma(z)$. Using the forms (D34) and (D35) of $G(z)$ and $F(z)$ in (D26), it follows that $z_1, \ldots, z_r$ are roots of the equation

$$Z^r = -(-1)^{\nu''} \prod_{m=0}^{\infty} \left(\frac{(1 - x^{4m+1}z)(1 - x^{4m+3}z^{-1})}{(1 - x^{4m+3}z)(1 - x^{4m+1}z^{-1})} \right)^N. \tag{D38}$$

(All terms explicitly involving q cancel out.) This equation can be written quite nearly in terms of elliptic functions of modulus k_1, with associated elliptic integrals K_1, K_1' such that

$$x = \exp(-\pi K_1'/K_1). \tag{D39}$$

Define u so that

$$z = \exp(i\pi u/K_1), \tag{D40}$$

then (D38) becomes

$$\exp\{iN \operatorname{am}(u, k_1)\} = -(-1)^{\nu''}, \tag{D41}$$

where $\operatorname{am}(u, k_1)$ is the elliptic amplitude function of argument u and modulus k_1 (§8.141 of GR [8]). Thus when $z = z_j$, $\operatorname{am}(u, k_1)$ is real. Hence u is real and z_j lies on the unit circle, as assumed.

In deriving (D37) we have neglected only terms of order x^r, $x^{-2r}q^r$, or smaller terms. Thus in the limit of N and r large we expect it to be exact, provided (D9) is satisfied. Also, although we took $|z| = 1$ in obtaining (D37), having located $z_1, \ldots, z_r$ this condition can be relaxed and we find that $T(v)$ is given by (D37) throughout the region (D21). However, from the above discussion we expect it to give the largest eigenvalues of $\mathbf{T}(v)$ only in the restricted region (D10).

To summarize: the numerically largest eigenvalues $T(v)$ of the transfer matrix $\mathbf{T}(v)$ are given by (D37), provided the restrictions (D9) and (D10) are satisfied. These restrictions are equivalent to (D7), to (D2), and to (D1).

Appendix E

We here discuss the behaviour of the free energy f, given by (7.7), across the boundary $w_2 = w_3$ of the regime (7.2).

When $w_2 \to w_3$ we see from (7.3)–(7.5) that $l \to 0$, so that $K_l \to \pi/2$, $K_l' \to +\infty$, and ζ and V tend in general to finite nonzero values. From (7.8) it follows that $\tau, \lambda, \alpha \to 0$, and hence in this limit the summation in (7.7) becomes an integral.

This suggests that to consider the behaviour about this limit we should make a Poisson transformation of the series in (7.7). Define

$$\mu = \pi\zeta/K_l\,, \qquad U = \pi V/K_l\,, \tag{E1}$$

$$g(x) = \sinh^2[(\pi - \mu)\,x]\{\cosh(\mu x) - \cosh(Ux)\}/x \sinh(2\pi x) \cosh(\mu x), \tag{E2}$$

$$h(s) = \int_{-\infty}^{\infty} e^{isx} g(x)\, dx. \tag{E3}$$

Noting that $g(x)$ is an even function and $g(0) = 0$, the equation (7.7) can be written as

$$-\beta f = \ln(w_1 + w_2) + \pi^{-1}\tau \sum_{n=-\infty}^{\infty} g(\pi^{-1}\tau n). \tag{E4}$$

Expressing $g(x)$ in terms of its Fourier transform $h(s)$ by inverting the relation (E3) and substituting the result in (E4), the summation and integration can be interchanged. The summation can then be performed to give a periodic delta function. Performing the integration then gives

$$\begin{aligned} -\beta f &= \ln(w_1 + w_2) + \sum_{m=-\infty}^{\infty} h(2\pi^2\tau^{-1}m) \\ &= \ln(w_1 + w_2) + h(0) + 2\sum_{m=1}^{\infty} h(2\pi K_l' m/K_l) \end{aligned} \tag{E5}$$

[using (7.8) and the evenness of $h(s)$].

When s is positive the integration in (E3) can be closed round the upper halfplane, giving $h(s)$ as a sum of residues. Substituting the result into (E5), the summation over m can be performed for each residue, giving

$$\begin{aligned} -\beta f = \psi + h(0) &- 2\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \frac{\sin^2(n\mu)[\cos(n\mu) - \cos(nU)]}{n\cos(n\mu)} \\ &+ 4\sum_{n=1}^{\infty} \frac{(-)^n q^{(2n-1)\pi/\mu}}{1-q^{(2n-1)\pi/\mu}} \frac{1}{2n-1} \\ &\times \cot\left(\frac{(2n-1)\pi^2}{2\mu}\right) \cos\left\{\frac{(2n-1)\pi U}{2\mu}\right\}, \end{aligned} \tag{E6}$$

where

$$q = \exp(-\pi K_l'/K_l) \tag{E7}$$

and

$$\begin{aligned} \psi = \ln(w_1 + w_2) &+ 4\sum_{n=1}^{\infty} \frac{q^{2n-1}}{(2n-1)[1-q^{2n-1}]} \cos\left(\frac{2n-1}{2}\mu\right) \\ &\times \left\{\cos\left(\frac{2n-1}{2}\mu\right) - \cos\left(\frac{2n-1}{2}U\right)\right\}. \end{aligned} \tag{E8}$$

(Note that this q is *not* the same as the q of Appendix D, being defined in terms of the modulus l, rather than k.)

The series in (E8) can be summed, using the Fourier series for $\ln[\mathrm{sn}\, u/\mathrm{cn}\, u\, \mathrm{dn}\, u]$ (§8.146 of GR [8]). After some manipulation, using (5.3), we find that

$$\psi = \frac{1}{2}\ln\left\{\frac{\pi}{2l'^{1/2}K_l}\tan\left(\frac{1}{2}\mu\right)\frac{\cos(\frac{1}{2}U)+\cos(\frac{1}{2}\mu)}{\cos(\frac{1}{2}U)-\cos(\frac{1}{2}\mu)}\right\}$$
$$+\frac{1}{8}\ln\{(w_1^2-w_2^2)(w_1^2-w_3^2)(w_2^2-w_4^2)(w_3^2-w_4^2)\}. \tag{E9}$$

As $q \to 0$ when $w_2 \to w_3$, the remaining series in (E6) are uniformly convergent in some neighbourhood of $w_2 = w_3$. Also, from the definitions (7.3)–(7.5) we can verify that l', K_l, q, μ, U are analytic functions of the w's in some such neighbourhood, that q has a simple zero at $w_2 = w_3$, but the other parameters are nonzero. Using (E3) and (E9), it follows that the first three terms on the r.h.s. of (E6) are analytic in this neighbourhood, but the last is a sum of functions with branch-point singularities, the dominant singular contribution to βf near $w_2 = w_3$ being

$$(\beta f)_{\mathrm{sing}} = 4\cos(\pi U/2\mu)\cot(\pi^2/2\mu)\, q^{\pi/\mu}. \tag{E10}$$

As q has a simple zero at $w_2 = w_3$, this singular term is proportional to

$$(w_2 - w_3)^{\pi/\mu}. \tag{E11}$$

This result breaks down if $\pi/\mu = 2m$, where m is an integer. In this case some of the poles of the integrand of (E3) coincide and we find that

$$(\beta f)_{\mathrm{sing}} = \pi^{-1}\, 8\cos(mU)\, q^{2m}\ln q, \tag{E12}$$

which is proportional to

$$(w_2 - w_3)^{2m}\ln(w_2 - w_3). \tag{E13}$$

To summarize the working so far: the free energy f is given by the above equations when the restrictions (7.2) are satisfied, in particular when $w_2 > w_3$. The above equations also define an analytic continuation of f across the $w_2 = w_3$ boundary, and we see that this function has a branch-point singularity.

However, when $w_2 < w_3$ the true free energy is not given by the above equations. Rather we see from the symmetry relations (2.6) that it is given by the above equations when w_2 and w_3 are interchanged in the definitions (7.3)–(7.5). Let us denote any variable obtained by interchanging w_2 and w_3 in these equations by a suffix 1. Then we must still compare the analytic continuation of f into the region $w_2 < w_3$ with the true free energy f_1 in this region. For instance the analytic parts of these functions might be different, leading to a discontinuity in the free energy or its derivatives at $w_2 = w_3$. Such a discontinuity might be more important in the critical region than the branch point singularity already found.

From (7.3) we see that $l' = (1 - l^2)^{1/2}$ is given by

$$l' = [(w_1^2 - w_2^2)(w_3^2 - w_4^2)/(w_1^2 - w_3^2)(w_2^2 - w_4^2)]^{1/2}. \tag{E14}$$

Interchanging w_2 and w_3 simply inverts the r.h.s. of this equation. Thus

$$l_1' = 1/l'. \tag{E15}$$

From §§8.126, 8.128 of GR [8], it follows that

$$K_{l_1} = l' K_l , \tag{E16}$$

$$K_{l_1}' = l'(K_l' + iK_l), \tag{E17}$$

and hence from (E7) that

$$q_1 = -q. \tag{E18}$$

From §§8.152.1, 8.153 of GR [8], we can also deduce the transformation formula

$$\mathrm{sn}(l'u, l_1) = l' \,\mathrm{sn}(u, l)/\mathrm{dn}(u, l). \tag{E19}$$

From (7.3) and (7.4),

$$\begin{aligned} l' \,\mathrm{sn}(\zeta, l)/\mathrm{dn}(\zeta, l) &= [(w_1^2 - w_2^2)/(w_1^2 - w_4^2)]^{1/2}, \\ &= \mathrm{sn}(\zeta_1 , l_1). \end{aligned} \tag{E20}$$

Hence

$$\zeta_1 = l'\zeta, \tag{E21}$$

and similarly we can show that

$$V_1 = l'V. \tag{E22}$$

From these results and (E1) it follows that

$$l_1^{1/2} K_{l_1} = l'^{1/2} K_l , \qquad \mu_1 = \mu, \qquad U_1 = U. \tag{E23}$$

Using (E18) and (E23), together with (E2), (E3), (E9), we see that the first three terms on the r.h.s. of (E6) are unaltered by interchanging w_2 and w_3. Thus the analytic part of the true free energy is in fact the same function of w_1, w_2, w_3, w_4 on both sides of the $w_2 = w_3$ boundary and there are no discontinuities of the type mentioned above.

We also see from (E18) and (E23) that the dominant singular contribution to the free energy on *either* side of the $w_2 = w_3$ boundary is given by (E10) or (E12), with q replaced by $|\, q \,|$. Thus if $w_2 - w_3$, and hence q, has a simple zero at some value T_c of the temperature T, the singularity is of the form (8.1) of the text.

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