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Eight-Vertex Model in Lattice Statistics and One-Dimensional Anisotropic Heisenberg Chain. III. Eigenvectors of the Transfer Matrix and Hamiltonian

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We obtain the eigenvectors of the transfer matrix of the zero-field eight vertex model. These are also the eigenvectors of the Hamiltonian of the corresponding one-dimensional anistropic Heisenberg chain.

1. INTRODUCTION AND SUMMARY

This is the third and final paper of a series in which we obtain the eigenvectors of the transfer matrix $\mathbf{T}$ of the zero-field eight-vertex model.

In Paper I (referred to as I) [1], we found some special eigenvectors. In Paper II (referred to as II) [2] we generalized these to form a basis set of vectors, with respect to which $\mathbf{T}$ becomes the transfer matrix of an Ising-like problem. In this problem each spin can have L values and the four spins round a square interact. Most importantly, two adjacent spins must differ by unity. From this last property it follows that the problem can be thought of as a generalized ice-type problem and we may hope to obtain the eigenvectors of $\mathbf{T}$ by an appropriate extension of the Bethe ansatz technique that works for the ice models [3]. We show here that this is so.

In Section 8 of I we showed that $\mathbf{T}$ commutes with the Hamiltonian $\mathscr{H}$ of an anisotropic one-dimensional Heisenberg chain. Thus the eigenvectors we construct here are also those of $\mathscr{H}$.

In this section we present our results for the eigenvectors and eigenvalues of $\mathbf{T}$, and show that the equations for the eigenvalues are the same as those we obtained in our original solution of the eight-vertex model [4]. This previous solution gave no information regarding the eigenvectors.

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The motivation of this series of papers, namely to obtain the eigenvectors, is something of a long-range one. In principle a knowledge of the eigenvectors should enable one to calculate correlations and spontaneous polarizations. However, our eigenvectors are more complicated than the eigenvectors found by Lieb [3] for the ice models (they contain them as special cases). As yet it has not been found possible to calculate the correlations and staggered polarizations of the ice models (e.g., the F model), due to the mathematical complexity of evaluating the required matrix elements. Thus our results must wait in hope for this to be done. The calculation of the correlations and spontaneous polarizations of the eight-vertex model should not then be far off.

Basic Vectors

We first quote the principal results of II. There we used the notation $|\, u\rangle$ to denote the two-dimensional vector

$$|\, u\rangle = \begin{pmatrix} H(u) \\ \Theta(u) \end{pmatrix}, \tag{1.1}$$

where $H(u)$, $\Theta(u)$ are the modified elliptic theta functions of argument u and modulus k, defined by (II.6.9). The modulus k, together with three other parameters ρ, η, v, are defined from the Boltzmann weights a, b, c, d of the 8 vertex model by the relations (II.6.1).

We then defined two sets of two-dimensional vectors $\Phi_{l,l+1}$ and $\Phi_{l+1,l}$ (l any integer) by

$$\Phi_{l,l+1} = |\, s + 2l\eta + \eta - v\rangle \tag{1.2a}$$

$$\Phi_{l+1,l} = |\, t + 2l\eta + \eta + v\rangle. \tag{1.2b}$$

The parameters s and t are arbitrary. (Except that the vectors $z_{l+1,l}$, and $z_{l-1,l}$ in (II.6.5) must be linearly independent for each value of l. In particular we exclude the case $s = t$.)

The next step is to define a family $\mathscr{F}$ of 2^N-dimensional vectors, a typical member being

$$\psi(l_1, \ldots, l_{N+1}) = \Phi_{l_1,l_2} \otimes \Phi_{l_2,l_3} \otimes \cdots \otimes \Phi_{l_N,l_{N+1}}, \tag{1.3}$$

where $l_1, \ldots, l_{N+1}$ are any integers such that

$$l_{J+1} = l_J \pm 1, \qquad J = 1, \ldots, N. \tag{1.4}$$

Pre-multiplying (1, 3) by the transfer matrix $\mathbf{T}$ of the eight-vertex model, as defined in (II.2.4), we found [Eq. (II.3.5)] that, providing a boundary condition was satisfied,

$$\mathbf{T}\psi(l_1, \ldots, l_{N+1}) = \sum \left\{ \prod_{J=1}^{N} W(m_J, m_{J+1} \mid l_J, l_{J+1}) \right\} \psi(m_1, \ldots, m_{N+1}), \tag{1.5}$$

where the coefficients W are defined by (II.7.1) and the summation is over all integers $m_1, \ldots, m_{N+1}$ such that

$$m_J = l_J \pm 1, \qquad J = 1, \ldots, N+1, \tag{1.6}$$

$$m_{J+1} = m_J \pm 1, \qquad J = 1, \ldots, N, \tag{1.7}$$

$$m_1 - l_1 = m_{N+1} - l_{N+1}. \tag{1.8}$$

The signs in (1.4), (1.6), (1.7) may be chosen arbitrarily for each J, except that for $J = 1$ and $N+1$ in (1.6) they must be the same. This ensures (1.8).

The boundary condition needed in the derivation of (1.5) is certainly satisfied if $l_{N+1} = l_1$. More generally, let K, K' be the complete elliptic integrals of moduli k, $k' = (1 - k^2)^{1/2}$, respectively, (Section 8.112 of Gradshteyn and Ryzhik [5], hereinafter referred to as GR). Then if there exist integers L, m_1, m_2 such that

$$L\eta = 2m_1 K + im_2 K', \tag{1.9}$$

then the modified theta functions $H(u)$, $\Theta(u)$ defined by (II.6.9) that we use in the working are periodic of period $2L\eta$. In this case it follows that the boundary condition is satisfied if

$$l_{N+1} - l_1 = L \times \text{integer}. \tag{1.10}$$

Note from (1.8) that if l_1 and l_{N+1} satisfy (1.10), then so do m_1 and m_{N+1}. Thus we see that we have defined a family $\mathscr{F}$ of 2^N-dimensional vectors such that pre-multiplying any member of the family by $\mathbf{T}$ gives a linear combination of vectors of the family.

We further point out in Section 4 of II that $\mathscr{F}$ breaks up into $N+1$ subfamilies $\mathscr{F}_0, \ldots, \mathscr{F}_N$, such that $\mathscr{F}_n$ consists of vectors of the form (1.3) with $N - n$ increasing steps in the sequence $l_1, \ldots, l_{N+1}$, and n decreasing steps [i.e. $N - n$ of the two-dimensional vectors on the r.h.s. of (1.3) are of the type (1.2a), the other n of the type (1.2b)]. Pre-multiplying a vector of $\mathscr{F}_n$ by $\mathbf{T}$ gives a linear combination of vectors of $\mathscr{F}_n$, provided the boundary condition (1.10) is satisfied. Since there are $N - n$ increasing steps in the sequence $l_1, \ldots, N+1$, and n decreasing steps, it follows that $l_{N+1} = l_1 + N - 2n$. Hence (1.10) is satisfied if

$$N - 2n = L \times \text{integer}. \tag{1.11}$$

We can therefore attempt to construct eigenvectors of $\mathbf{T}$ which are linear combinations of vectors of $\mathscr{F}_n$, for some fixed value of n. The case $n = 0$ is trivial and is discussed in I, and in Section 5 of II. In this paper we consider the general case when $0 < n < N$.

Bethe Ansatz

We can think of the decreasing steps (i.e. l_J followed by $l_J - 1$) in the sequence $l_1, \ldots, l_{N+1}$ as "down arrows" or "particles" (the arrow formulation is given in Section 4 of II). A basis vector of the family $\mathscr{F}_n$ can be specified by prescribing the value l of l_1 and the positions $x_1, \ldots, x_n$ of these decreasing steps. We then have

$$\begin{aligned} l_1 &= l, \\ l_{J+1} &= l_J + 1 \qquad \text{for} \quad J = 1, \ldots, N, \\ &\qquad\qquad \text{provided} \quad J \neq x_1, x_2, \ldots, \text{or } x_n, \\ l_{J+1} &= l_J - 1 \qquad \text{if} \quad J = x_1, x_2, \ldots, \text{or } x_n. \end{aligned} \tag{1.12}$$

Taking $x_1, \ldots, x_n$ to be arranged in increasing order and successively evaluating $l_1, l_2, \ldots, l_{N+1}$, it follows that

$$l_J = l + J - 1 - 2m \qquad \text{if} \quad x_m < J \leqslant x_{m+1} \tag{1.13}$$

(defining $x_0 = 0$, $x_{n+1} = N + 1$).

We attempt to construct eigenvectors Ψ of $\mathbf{T}$ which are linear combinations of vectors of $\mathscr{F}_n$. Thus we define

$$\Psi = \sum_{l=1}^{L} \sum_{X} f(l \mid x_1, \ldots, x_n)\, \psi(l_1, \ldots, l_{N+1}), \tag{1.14}$$

where the X summation is over all positions $X = \{x_1, \ldots, x_n\}$ such that $1 \leqslant x_1 < x_2 < \cdots < x_n \leqslant N$, and $l_1, \ldots, l_{N+1}$ are given in terms of $l, x_1, \ldots, x_n$ by (1.12) and (1.13).

Using (1.5), we wish to choose the coefficients $f((l \mid X)$ so that

$$\mathbf{T}\Psi = \Lambda\Psi, \tag{1.15}$$

i.e., Ψ is an eigenvector of $\mathbf{T}$ with corresponding eigenvalue Λ.

The resulting equations for the $f(l \mid X)$ are given in Section 2. We see that they are very similar to the equations for the eigenvectors of the ice models [3]. By generalizing the technique that Lieb used for these models, we are led to try a generalized Bethe ansatz for $f(l \mid X)$, namely

$$f(l \mid X) = \sum_{P} A(P)\, g_{P1}(l, x_1)\, g_{P2}(l - 2, x_2) \cdots g_{Pn}(l - 2n + 2, x_n), \tag{1.16}$$

where the summation is over all $n!$ permutations $P = \{P1, \ldots, Pn\}$ of the integers $\{1, \ldots, n\}$, the $A(P)$ are $n!$ coefficients, and $g_1(l, x), \ldots, g_n(l, x)$ are n "single particle" functions of l and x.

These coefficients A and functions g are unknown. We attempt to choose them to satisfy the equations for $f(l \mid X)$ and Λ that follow from (1.14) and (1.15).

Results of the Bethe Ansatz

The working is given in Sections 2–6. After a series of mathematical flukes we find that the eigenvalue equations can indeed be satisfied. We are led to introduce n numbers (in general complex) $u_1, \ldots, u_n$ such that u_j is associated with the single particle function $g_j(l, x)$. We can also introduce n "wave numbers" $k_1, \ldots, k_n$ such that k_j is defined in terms of u_j by

$$\exp(ik_j) = h(u_j + \eta)/h(u_j - \eta) \tag{1.17}$$

for $j = 1, \ldots, n$. The function $h(u)$ is given by

$$h(u) = H(u)\,\Theta(-u). \tag{1.18}$$

It is an entire function, odd, periodic of period $2L\eta$.

The single-particle function $g_j(l, x)$ is then given by

$$g_j(l, x) = e^{ik_j x} h(w_{l+x-1} - \eta - u_j)/[h(w_{l+x-2})\, h(w_{l+x-1})] \tag{1.19a}$$

for $j = 1, \ldots, n$ and all integers l and x, where

$$w_l = \tfrac{1}{2}(s + t) - K + 2l\eta. \tag{1.19b}$$

The coefficients $A(P)$ are found to be given by

$$A(P) = \epsilon_P \prod_{1 \leqslant j < m \leqslant n} h(u_{Pj} - u_{Pm} + 2\eta), \tag{1.20}$$

where ϵ_P is the signature $(\pm)$ of the permutation $P = \{P1, \ldots, Pn\}$. The eigenvalue Λ is given by

$$\Lambda = \phi(v - \eta) \prod_{j=1}^{n} \frac{h(v - u_j + 2\eta)}{h(v - u_j)} + \phi(v + \eta) \prod_{j=1}^{n} \frac{h(v - u_j - 2\eta)}{h(v - u_j)}, \tag{1.21}$$

where the function $\phi(v)$ is defined to be

$$\phi(v) = [\rho\Theta(0)\, h(v)]^N. \tag{1.22}$$

We also get n equations which come from the cyclic boundary conditions we impose on the lattice model. They are

$$\exp(iNk_j) = -\prod_{m=1}^{n} [h(u_j - u_m + 2\eta)/h(u_j - u_m - 2\eta)] \tag{1.23}$$

for $j = 1,..., n$. These equations define $u_1 ,..., u_n$ (there will be many solutions, corresponding to different eigenvectors). Once these are known, the eigenvalue Λ of the transfer matrix can be evaluated from (1.21). The eigenvector Ψ can be obtained by evaluating the coefficients $A(P)$ and the single-particle functions $g_j(l, x)$ from (1.20) and (1.19), substituting these into (1.16) to give $f(l \mid X)$, and in turn substituting this result into (1.14) to give Ψ.

We remark that these results are completely analogous to those of the normal Bethe ansatz used by Lieb [3] and Yang and Yang [6]. The relation (1.17) between the wave numbers k_j and the parameters u_j is analogous to the difference-kernel transformation that takes k to α. The main difference is that in our working it is $u_1 ,..., u_n$, rather than $k_1 ,..., k_n$, that occur naturally.

Functional Equation for the Eigenvalues

We point out here that the equations (1.21), (1.23) are the same as those we originally obtained for the eigenvalues [4]. To see this, regard $u_1 ,..., u_n$ as known and define a function $Q(v)$ by

$$Q(v) = \prod_{j=1}^{n} h(v - u_j). \tag{1.24}$$

Using (1.17), the equations (1.21), (1.23) can be written as

$$\Lambda = [\phi(v - \eta)\, Q(v + 2\eta) + \phi(v + \eta)\, Q(v - 2\eta)]/Q(v), \tag{1.25}$$

and

$$\phi(u_j - \eta)\, Q(u_j + 2\eta) + \phi(u_j + \eta)\, Q(u_j - 2\eta) = 0 \tag{1.26}$$

for $j = 1,..., n$.

Note from (1.17) and (1.23) that $u_1 ,..., u_n$ depend on k and η, but not on v. From (1.19), (1.20) it follows that the coefficients $f(l \mid X)$ are independent of v. Also, from (1.2) and (1.3) we see that the basis vectors $\psi(l_1 ,..., l_{N+1})$ depend on s, t, v only through the linear combinations $s - v$, $t + v$. Since s and t are arbitrary, we can replace them by $s + v, t - v$. [This has no effect on (1.19).] The basis vectors are then independent of v.

It follows that the eigenvectors of $\mathbf{T}$ that we have constructed are independent of v. Assuming that these vectors form a complete set (i.e. span all 2^N-dimensional space), we can construct a 2^N by 2^N non-singular matrix $\mathbf{P}$ whose columns are the eigenvectors of $\mathbf{T}$. Exhibiting the dependence of $\mathbf{T}$ on v explicitly by writing it as $\mathbf{T}(v)$, it follows that

$$\mathbf{P}^{-1}\mathbf{T}(v)\mathbf{P} = \mathbf{T}_d(v), \tag{1.27}$$

where $\mathbf{T}_d(v)$ is the diagonal matrix whose diagonal elements are the 2^N eigenvalues $\Lambda(v)$ of $\mathbf{T}(v)$. The matrix $\mathbf{P}$ is independent of v.

Note that (1.27) implies that two transfer matrices $\mathbf{T}(u)$, $\mathbf{T}(v)$ commute (k and η are regarded here as constants, so must be the same for both). This can be proved directly, and was the first step in our original solution of the eight-vertex model [4].

Let $u_1, \ldots, u_n$ be a solution of (1.23), or equivalently (1.26). Hopefully there will be 2^N independent solutions (allowing all possible values of n), corresponding to the eigenvectors of $\mathbf{T}(v)$. For each solution (1.24) and (1.25) define an eigenvalue $\Lambda(v)$ of $\mathbf{T}(v)$ and a function $Q(v)$. Let $\mathbf{Q}_d(v)$ be a diagonal matrix with these functions $Q(v)$ as diagonal elements, arranged in the same order as the $\Lambda(v)$ in $\mathbf{T}_d(v)$. From (1.25) it is then apparent that the 2^N by 2^N diagonal matrices $\mathbf{T}_d(v)$, $\mathbf{Q}_d(v)$ satisfy the relation

$$\mathbf{T}_d(v)\,\mathbf{Q}_d(v) = \phi(v-\eta)\,\mathbf{Q}_d(v+2\eta) + \phi(v+\eta)\,\mathbf{Q}_d(v-2\eta). \tag{1.28}$$

Now define a matrix $\mathbf{Q}(v)$ by

$$\mathbf{Q}(v) = \mathbf{P}\mathbf{Q}_d(v)\,\mathbf{P}^{-1}. \tag{1.29}$$

From (1.27)–(1.29) we see that the matrices $\mathbf{T}(v)$, $\mathbf{Q}(u)$, $\mathbf{Q}(v)$ all commute for any values of u and v, and that

$$\mathbf{T}(v)\,\mathbf{Q}(v) = \phi(v-\eta)\,\mathbf{Q}(v+2\eta) + \phi(v+\eta)\,\mathbf{Q}(v-2\eta). \tag{1.30}$$

This is the functional matrix relation we derived directly in [4]. Given this relation and the commutation properties one can reason backwards to (1.25). The equations (1.26) for $u_1, \ldots, u_n$ then follow simply as a consequence of the fact that the elements, and from (1.27) the eigenvalues $\Lambda(v)$, of $\mathbf{T}(v)$ are entire functions of v. Thus when $v = u_1, \ldots, u_n$ the numerator of the r.h.s. of (1.25) must vanish, giving (1.26).

Our present results agree precisely with those of [4] provided we take $n = \frac{1}{2}N$ in (1.24). Remember that this is the case when the boundary condition is automatically satisfied. In this case the re-normalization (II.6.9) of the theta functions does not affect our final equations.

For the cases when η satisfies (1.9) we see from (1.11) that we also allow n in (1.24) to have the values (taking N to be even):

$$n = \tfrac{1}{2}N + L \times \text{integer}. \tag{1.31}$$

It seems likely that these solutions can be regarded as aberrations of the $n = \frac{1}{2}N$ case, in which some of the u_j form complete "strings," each string being made up of L u_j's of the form $u_j = \text{constant} + 2j\eta$ for $j = 1, \ldots, L$. Such strings will cancel out of the equations (1.25), (1.26).

Dependence of Eigenvectors on s and t

We are still free to choose the parameters s and t arbitrarily. From (1.17) and (1.21)–(1.23) we see that these do not enter into the equations for $u_1, \ldots, u_n$ and the eigenvalue Λ. They do however occur in the expressions (1.2), (1.19), (1.20) that are used in constructing the eigenvectors.

Let choose some solution $u_1, \ldots, u_n$ of (1.23) and then vary s and t. The eigenvectors that we form will all have the same eigenvalue and must lie in some sub-space of 2^N-dimensional space. In particular, from the Frobenius theorem the eigenvector corresponding to the maximum eigenvalue should be unique. Thus in this case varying s and t can only alter the eigenvector by a multiplicative normalization factor.

That this should be so is by no means obvious from the above equations. It is possible that we can obtain further insight into the structure of the eigenvector by using this requirement.

Summary of Remaining Sections

We have now stated our results and discussed them. In the following Sections 2 to 6 we go through the mathematical working required to obtain the coefficients $f(l \mid X)$ in the expression (1.14) for the eigenvector Ψ. In Section 7 we summarize this working and in Section 8 we indicate an allowed extension to a certain class of inhomogeneous lattices.

2. Transfer Matrix Equations

From (1.5), (1.14), (1.15) the transfer matrix equations for Λ and $f(l \mid X)$ are:

$$\Lambda f(l \mid X) = \sum \left\{ \prod_{J=1}^{N} W(l_J, l_{J+1} \mid m_J, m_{J+1}) \right\} f(m \mid Y), \tag{2.1}$$

where the $l_1, \ldots, l_{N+1}$ are given in terms of l and $x_1, \ldots, x_n$ by (1.13); $m_1, \ldots, m_{N+1}$ are similarly expressed in terms of m and $y_1, \ldots, y_n$. The summation is over all $m_1, \ldots, m_{N+1}$ such that $m_J = l_J \pm 1$, $m_{J+1} = m_J \pm 1$.

We can think of this summation graphically. Remember that $W(l, l' \mid m, m')$ is the weight of the configuration of spins round a vertex shown in Fig. 1. Since adjacent spins must differ by unity, there are six sets of possible spin configurations at a vertex. We show these in Fig. 2. In this figure we have drawn broken lines on vertical bonds (horizontal bonds) across which the spins increase from left to right (down to up), and we have drawn solid lines on bonds for the other cases. In the arrow notation of Paper II, broken lines are arrows pointing up or to the right, solid lines are arrows pointing down or to the left.

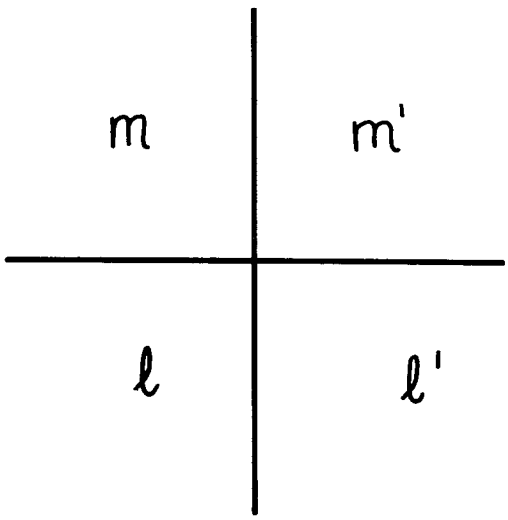

FIG. 1. Arrangements of spins round a vertex.

The weights a_l ,..., c_l' shown in Fig. 2 are the appropriate weights $W(l, l' \mid m, m')$. They are given explicitly in (II.7.2). Note in particular that a_l , b_l' are independent of l.

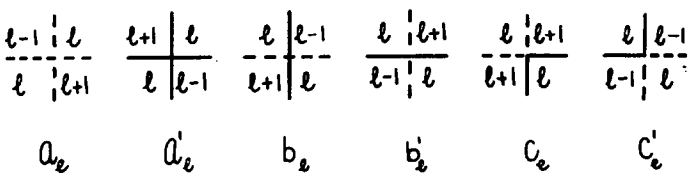

FIG. 2. The six types of allowed configurations at a vertex.

Now look at (2.1). We see that this is a transfer matrix equation relating two rows of the lattice. Note that l and $X = \{x_1, ..., x_n\}$ specify the state of the lower row, while m and Y specify the state of the upper. Since x_1 ,..., x_n are the positions of the decreasing steps in the sequence l_1 ,..., l_{N+1} , they are the positions of the solid lines in the lower row, $l = l_1$ is the value of the first spin (the one to the left of bond 1) in the lower row.

Similarly, y_1 ,..., y_n are the positions of the solid lines in the upper row, m is the value of the left-hand upper spin.

Let us draw the second vertex in Fig. 2 as in Fig. 3.

We see then that a sequence of solid lines follows a continuous and unique path

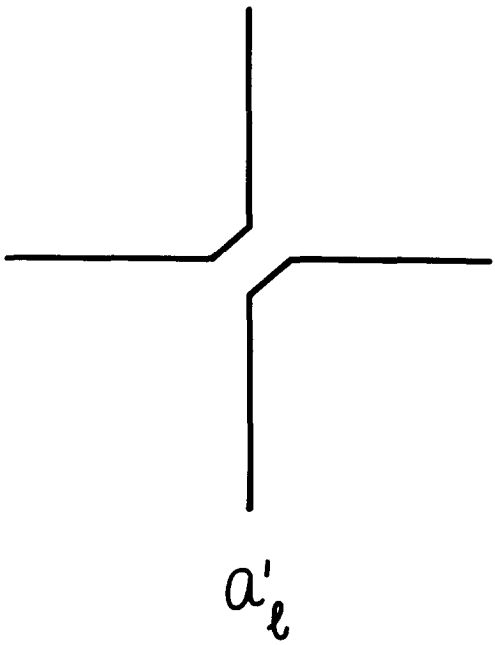

FIG. 3. The second configuration of Fig. 2.

through the lattice starting at the top right, at any stage such a path is moving down or to the left. Paths never cross.

The two general cases of what can happen across a horizontal row are shown for $n = 2$ in Fig. 4. Special cases can arise when lines go straight down (e.g., $y_2 = x_1$ in the first figure), or touch (e.g., $y_1 = x_1$ in the first figure).

In any event we see that in the first case we must have $1 \leqslant y_1 \leqslant x_1 \leqslant y_2 \leqslant x_2$, in the second case $x_1 \leqslant y_1 \leqslant x_2 \leqslant y_2 \leqslant N$. ($N$ is the number of columns of the lattice.)

It follows that for $n = 2$ we can write (2.1) more explicitly as

$$\begin{aligned} \Lambda f(l \mid x_1, x_2) = &\sum_{y_1=1}^{x_1} \sum_{y_2=x_1}^{x_2 *} D_L(l, X, Y) f(l+1 \mid y_1, y_2) \\ &+ \sum_{y_1=x_1}^{x_2} \sum_{y_2=x_2}^{N *} D_R(l, X, Y) f(l-1 \mid y_1, y_2), \end{aligned} \tag{2.2}$$

where the * on the summation indicates that the cases $y_1 = x_1 = y_2$, $y_1 = x_2 = y_2$ are excluded. The factors D_L, D_R are the products of the weights of the vertices

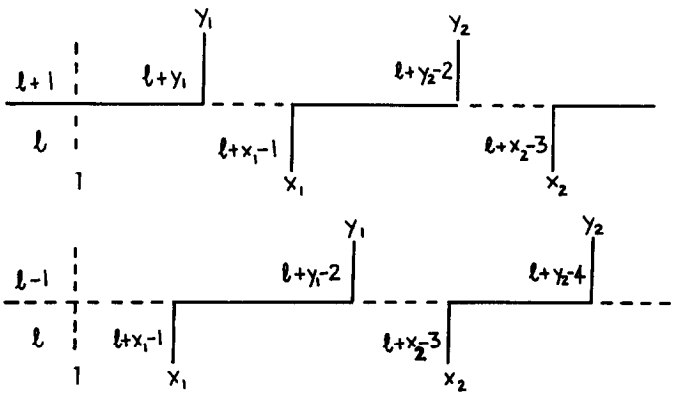

FIG. 4. General arrangements of solid lines between two rows for $n = 2$. Most of the broken lines on vertical bonds are omitted. The spins shown are those of the face immediately to the left of the vertical bond drawn.

in the row. For instance, for the configurations shown in Fig. 4 we see by using Fig. 2 that

$$\begin{aligned} D_L(l, X, Y) &= b'^{y_1-1}c'_{l+y_1}a^{x_1-y_1-1}c_{l+x_1-2}b'^{y_2-x_1-1}c'_{l+y_2-2}a^{x_2-y_2-1}c_{l+x_2-4}b'^{N-x_2}, \\ D_R(l, X, Y) &= a^{x_1-1}c_{l+x_1-2}b'^{y_1-x_1-1}c'_{l+y_1-2}a^{x_2-y_1-1}c_{l+x_2-4}b'^{y_2-x_2-1}c'_{l+y_2-4}a^{N-y_2}. \end{aligned} \tag{2.3}$$

(Since a_l, b_l' are independent of l we have written them simply as a, b': they are not to be confused with the vertex weights of the original eight-vertex model.)

Substituting these expressions into (2.2), we see that this equation simplifies if we define

$$\tilde{f}(l \mid x_1, x_2) = \left(\frac{b'}{a}\right)^{x_1+x_2} \frac{c'_{l+x_1-1}c'_{l+x_2-3}}{a^2} f(l \mid x_1, x_2). \tag{2.4}$$

The equation (2.2) then becomes

$$\begin{aligned} \Lambda\tilde{f}(l \mid x_1, x_2) = r_{l+x_1}r_{l+x_2-2} \Big\{ & b'^N \sum_{y_1=1}^{x_1} \sum_{y_2=x_1}^{x_2 *} \tilde{D}_L(l, X, Y)\tilde{f}(l+1 \mid y_1, y_2) \\ & + a^N \sum_{y_1=x_1}^{x_2} \sum_{y_2=x_2}^{N *} \tilde{D}_R(l, X, Y)\tilde{f}(l-1 \mid y_1, y_2) \Big\}. \end{aligned} \tag{2.5}$$

where

$$r_l = (c_{l-2}c'_{l-1})/(ab') \tag{2.6}$$

The factors $\tilde{D}_L(l, X, Y)$, $\tilde{D}_R(l, X, Y)$ are now unity for the configuratons shown in Fig. 4. Considering the cases when a solid line goes straight down (third diagram in Fig. 2), or two lines touch (Fig. 3), we see that the $\tilde{D}$ are the products of modified vertex weights on the row, the modified vertex weights corresponding to the configurations shown in Fig. 2 being

$$\begin{aligned} \tilde{a}_l &= \tilde{b}_l' = \tilde{c}_l = \tilde{c}_l' = 1, \\ \tilde{a}_l' &= (aa_l')/(c_{l-1}c_{l+1}') = p_l, \\ \tilde{b}_l &= (b'b_l)/(c_lc_l') = q_l. \end{aligned} \tag{2.7}$$

Considering the cases when a y equals an x in (2.5), it is now fairly easy to see that

$$\begin{aligned} \tilde{D}_L(l, X, Y) &= U(l+1 \mid 0, y_1, x_1)\, U(l-1 \mid x_1, y_2, x_2), \\ \tilde{D}_R(l, X, Y) &= U(l-1 \mid x_1, y_1, x_2)\, U(l-3 \mid x_2, y_2, N+1), \end{aligned} \tag{2.8}$$

where the function $U(l \mid x, y, x')$ is defined for $x < x'$, $x \leqslant y \leqslant x'$ by:

$$\begin{aligned} U(l \mid x, y, x') &= q_{l+x-1} \quad \text{if} \quad y = x \\ &= 1 \quad \text{if} \quad x < y < x' \\ &= p_{l+x'-2} \quad \text{if} \quad y = x'. \end{aligned} \tag{2.9}$$

For clarity we have considered the case $n = 2$. The extension to arbitrary n is straightforward: $\tilde{f}$ becomes $\tilde{f}(l \mid x_1, \ldots, x_n)$, the r-factor in (2.5) becomes $r_{l+x_1}r_{l+x_2-2}r_{l+x_3-4} \cdots r_{l+x_n-2n+2}$, the two summations are over $y_1, \ldots, y_n$ such that $1 \leqslant y_1 \leqslant x_1$, $x_1 \leqslant y_2 \leqslant x_2, \ldots, x_{n-1} \leqslant y_n \leqslant x_n$ and $x_1 \leqslant y_1 \leqslant x_2, \ldots, x_{n-1} \leqslant y_{n-1} \leqslant x_n$, $x_n \leqslant y_n \leqslant N$ (but no two y's can be equal), and the D are given by

$$\begin{aligned} \tilde{D}_L(l, X, Y) &= \prod_{j=1}^{n} U(l+1-2j \mid x_{j-1}, y_j, x_j), \\ \tilde{D}_R(l, X, Y) &= \prod_{j=1}^{n} U(l-1-2j \mid x_j, y_j, x_{j+1}), \end{aligned} \tag{2.10}$$

where we adopt the conventions $x_0 = 0$, $x_{n+1} = N + 1$.

3. Bethe Ansatz and Resulting Equations

It is apparent that the transfer matrix equation (2.5) is of the same type (though considerably more complicated) as the equations obtained by Lieb for the ice-models (cf. [3], in particular Eq. (2.9) of Lieb's first paper). We might hope to solve it by the same general techniques, and indeed it turns out that we can.

We therefore need to find an appropriately generalized Bethe ansatz for f. Some inspection suggests that we try

$$\tilde{f}(l \mid X) = \sum_P A(P)\, \tilde{g}_{P1}(l, x_1)\, \tilde{g}_{P2}(l-2, x_2) \cdots \tilde{g}_{Pn}(l-2n+2, x_n), \qquad (3.1)$$

where the summation is over all permutations $P = \{P1,..., Pn\}$ of the integers $1,..., n$. The $A(P)$ are $n!$ coefficients, to be chosen appropriately, and the $g_j(l, x)$ $(j = 1,..., n)$ can be thought of as n "single-particle" functions associated with a solid line in our graph. They are also at our disposal.

Again we focus attention on the case $n = 2$. As a first step we set

$$\tilde{f}(l \mid x_1, x_2) = \tilde{g}_1(l, x_1)\, \tilde{g}_2(l-2, x_2) \qquad (3.2)$$

on the r.h.s. of (2.5). We can perform the summations explicitly by first allowing $y_1 = y_2$ and then subtracting off the spurious terms thereby introduced (Lieb calls these "diagonal" terms). Leaving out the r, b'^N and a^N factors, the first summation in (2.5) gives

$$\begin{aligned}&[F_1(l+1, x_1) - C_1(l+1)][F_2(l-1, x_2) - G_2(l-1, x_1)] \\ &\quad - p_{l+x_1-1} q_{l+x_1-2}\, \tilde{g}_1(l+1, x_1)\, \tilde{g}_2(l-1, x_1), \end{aligned} \qquad (3.3)$$

while the second gives

$$\begin{aligned}&[F_1(l-1, x_2) - G_1(l-1, x_1)][C_2'(l-3) - G_2(l-3, x_2)] \\ &\quad - p_{l+x_2-3} q_{l+x_2-4}\, \tilde{g}_1(l-1, x_2)\, \tilde{g}_2(l-3, x_2). \end{aligned} \qquad (3.4)$$

The functions $F_j(l, x)$, $G_j(l, x)$, $C_j'(l)$ are defined by

$$F_j(l, x) = C_j(l) + \sum_{y=1}^{x-1} \tilde{g}_j(l, y) + p_{l+x-2}\, \tilde{g}_j(l, x), \qquad (3.5)$$

$$G_j(l, x) = C_j(l) + \sum_{y=1}^{x} \tilde{g}_j(l, y) - q_{l+x-1}\, \tilde{g}_j(l, x),$$

$$C_j'(l) = C_j(l) + \sum_{y=1}^{N} \tilde{g}_j(l, y). \qquad (3.6)$$

The parameters $C_j(l)$ are at our disposal.

Expanding the products in (3.3) and (3.4), we can classify the resulting terms into:

(i) *Wanted terms*: These are products of n F's, or n G's. Each x_j occurs once and only once in the product.

(ii) *Unwanted internal terms*: These are products in which there are n F, G or $\tilde{g}$ functions, but one of the x_j occurs as an argument in two of the functions simultaneously. Note that the "diagonal" terms are included in this category.

(iii) *Unwanted boundary terms*: These are terms which contain a C or a C'.

We endeavour to choose the *wanted terms* so as to cancel with the l.h.s. of the transfer matrix equation (2.5). For instance, the wanted term in (3.3) is $F_1(l+1, x_1) F_2(l-1, x_2)$. Multiplying this by the factor $r_{l+x_1} r_{l+x_2-2}$ in (2.5), we see that it is of the same form as the r.h.s. if there exist $\lambda_1, \ldots, \lambda_n$ such that

$$\lambda_j \tilde{g}_j(l, x) = r_{l+x} F_j(l+1, x) \tag{3.7}$$

for $j = 1, \ldots, n$ and all integers l, x. Similarly, the wanted term $G_1(l-1, x_1)$ $G_2(l-3, x_2)$ in (3.4) will be of the same form as the l.h.s. if there exist $\mu_1, \ldots, \mu_n$ such that

$$-\mu_j \tilde{g}_j(l, x) = r_{l+x} G_j(l-1, x). \tag{3.8}$$

Taken together, the two wanted terms on the r.h.s. of (2.5) will then cancel with the l.h.s. if

$$\Lambda = b'^N \lambda_1, \ldots, \lambda_n + a^N \mu_1, \ldots, \mu_n . \tag{3.9}$$

Note that these equations (3.7)–(3.9), together with the definitions (3.5), are unaffected if we permute the n single-particle functions $\tilde{g}_1(l, x), \ldots, \tilde{g}_n(l, x)$ in (3.2). Thus if we can find solutions of (3.7) and (3.8) we are then free to use the general ansatz (3.1) and to attempt to use the coefficients $A(P)$, together with any remaining freedom in the single-particle functions, to cancel the unwanted terms on the r.h.s. of (2.5).

The *unwanted internal terms* in (3.3) can be written as $-B_{1,2}(l, x_1)$, where

$$B_{i,j}(l, x) = F_i(l+1, x)\, G_j(l-1, x) + p_{l+x-1} q_{l+x-2} \tilde{g}_i(l+1, x)\, \tilde{g}_j(l-1, x) \tag{3.10}$$

for $i, j = 1, \ldots, n$ and any integers l, x. Interchanging the suffixes 1 and 2 on the single-particle functions in (3.3) and using the general ansatz (3.1), we see that these will cancel if

$$B_{1,2}(l, x)\, A(1, 2) + B_{2,1}(l, x)\, A(2, 1) = 0 \tag{3.11}$$

for all integers l, x.

The unwanted internal terms in (3.4) will also cancel if (3.11) is satisfied. More generally, for arbitrary n we find the unwanted internal terms cancel if

$$B_{i,j}(l, x)\, A(P) + B_{j,i}(l, x)\, A(Q) = 0 \tag{3.12}$$

for all integers l, x and all permutations $P = \{\ldots, i, j, \ldots\}$, $Q = \{\ldots, j, i, \ldots\}$. ($P$ and Q differ only in interchanging adjacent elements i and j.)

Note that these requirements imply that the ratio $B_{i,j}(l, x)/B_{j,i}(l, x)$ must be independent of l and x for $i = 1,..., n$ and $j = 1,..., n$. This is a very strong condition.

Interchanging the function suffixes 1 and 2 in (3.4), we see that the *unwanted boundary terms* in (3.3) and (3.4) are of similar type. They can be made to cancel by requiring that

$$\left(\frac{b'}{a}\right)^N C_1(l + 1)\, A(1, 2) = C_1'(l - 3)\, A(2, 1). \tag{3.13}$$

For arbitrary n we find that they cancel if

$$\left(\frac{b'}{a}\right)^N C_{P1}(l + 1)\, A(P1,..., Pn) = C'_{P1}(l + 1 - 2n)\, A(P2, P3,..., Pn, P1) \tag{3.14}$$

for all integers l and all permutations $P = \{P1,..., Pn\}$.

To summarize the working so far: The Bethe ansatz (3.1) will work if we can satisfy (3.7), (3.8), (3.12) and (3.14). The functions $\tilde{g}_j(l, x)$, the coefficients $A(P)$, and the parameters $C_j(l)$, λ_j, μ_j are at our disposal. Nevertheless we have many more equations than unknowns, and these equations cannot be satisfied for arbitrary weights a_l, a_l', b_l, b_l', c_l, c_l'. However, for the weights we are interested in, namely those given by (II.7.1), it turns out that they are soluble.

4. Single Particle Equations

We look first at the equations (3.7) and (3.8), together with the subsidiary definitions (3.5). To handle these we define one more function:

$$\Gamma_j(l, x) = C_j(l) + \sum_{y=1}^{x} \tilde{g}_j(l, y), \tag{4.1}$$

$j = 1,..., n$, all integers l, x.

We can express $\tilde{g}_j(l, x)$, $F_j(l, x)$, $G_j(l, x)$ as linear combinations of $\Gamma_j(l, x)$ and $\Gamma_j(l, x - 1)$. Doing this, the equations (3.7), (3.8) become:

$$\begin{aligned}
&\lambda_j[\Gamma_j(l, x) - \Gamma_j(l, x - 1)] \\
&\quad = r_{l+x}[p_{l+x-1}\Gamma_j(l + 1, x) + p'_{l+x-1}\Gamma_j(l + 1, x - 1)], \\
&-\mu_j[\Gamma_j(l, x) - \Gamma_j(l, x - 1)] \\
&\quad = r_{l+x}[q'_{l+x-2}\Gamma_j(l - 1, x) + q_{l+x-2}\Gamma_j(l - 1, x - 1)],
\end{aligned} \tag{4.2}$$

where

$$p_x' = 1 - p_x, \qquad q_x' = 1 - q_x. \tag{4.3}$$

Note that we can consider some particular value of j. Also, the coefficients r, p, p', q, q' depend on l and x only via $l + x$. Under quite general conditions it follows that the solution of (4.2) must be of the form

$$\Gamma_j(l, x) = \exp(ik_j'x)\,\gamma_j(l + x), \tag{4.4}$$

where i is now the square root of -1, k_j' is an unknown wave-number, and $\gamma_j(x)$ is some function of the integer variable x.

Substituting (4.4) into (4.2), replacing $l + x$ in the first equation by y, in the second by $y + 1$, we get two equations, homogeneous and linear in the three unknowns $\gamma_j(y - 1), \gamma_j(y), \gamma_j(y + 1)$. Solving them we get, dropping the suffixes j,

$$\begin{aligned} &e^{-ik'}\gamma(y - 1) : \gamma(y) : e^{ik'}\gamma(y + 1) \\ &\quad = \lambda'\mu' - \mu' r_y + R_y : \lambda'\mu' - S_y : \lambda'\mu' - \lambda' r_{y+1} + R_y', \end{aligned} \tag{4.5}$$

where

$$\lambda' = e^{ik'}\lambda, \qquad \mu' = e^{-ik'}\mu, \tag{4.6}$$

$$\begin{aligned} R_y &= r_y r_{y+1} p_{y-1} q_{y-1}', \\ S_y &= r_y r_{y+1} p_{y-1} q_{y-1}, \\ R_y' &= r_y r_{y+1} p_{y-1}' q_{y-1}. \end{aligned} \tag{4.7}$$

The ratio of the first two terms in (4.5) must be equal to the ratio of the second two with y replaced by $y - 1$. This gives the equation

$$\begin{aligned} &\lambda'\mu'(\lambda' + \mu') - \lambda'\mu'(R_{y-1}' + R_y + S_{y-1} + S_y + r_y^2)/r_y \\ &\quad + \lambda' R_y + \mu' R_{y-1}' + (S_{y-1}S_y - R_{y-1}' R_y)/r_y = 0. \end{aligned} \tag{4.8}$$

This equation must be satisfied for all integers y and for n values of λ', μ', corresponding to $j = 1, \ldots, n$. We should therefore like the coefficients in this equation (regarded as a biquadratic equation relating λ' and μ') to be independent of y. Using (II.7.1) we find that this is indeed the case. The main tool that we need in the working is an addition formula that is satisfied by our elliptic functions $h(u)$, defined by (1.18), namely:

Formula

$$\begin{aligned} &h(x + y)\,h(x - y)\,h(z + t)\,h(z - t) - h(x + t)\,h(x - t)\,h(z + y)\,h(z - y) \\ &\quad = h(y - t)\,h(y + t)\,h(x + z)\,h(x - z) \end{aligned} \tag{4.9}$$

for all complex numbers x, y, z, t.

The proof is simple: $h(u)$ is an entire function, odd, and for any integers m, n there exist coefficients $d_{m,n}$, $e_{m,n}$ such that

$$h(u + 2mK + inK') = h(u) \exp[d_{m,n}u + e_{m,n}]. \tag{4.10}$$

for all complex numbers u. [This is a very weak way of writing the quasi-periodic properties of $h(u)$: (4.10) is satisfied whether we use the Jacobi theta functions in (1.18), or our modified theta functions defined by (II.6.9).]

We regard x as a complex variable and y, z, t as constants. Define a function $f(x)$ as the ratio of the l.h.s. of (4.9) to the r.h.s. Then $f(x)$ is the ratio of two entire functions. From (4.10) it is doubly periodic, with periods $2K$, iK'. The r.h.s. has zeros, which are simple, only at $x = \pm z + 2mK + inK'$. When $x = \pm z$ it is clear that the l.h.s. also vanishes, so it must do so at all zeros of the r.h.s. Thus $f(x)$ is entire and doubly periodic. It is therefore bounded, and by the Cauchy–Liouville theorem must be a constant. Setting $x = t$ we find that this constant is unity, so $f(x) \equiv 1$. This proves the formula.

We return to looking at (4.8). From (II.7.1), (2.6) and (2.7) we have

$$r_l = \frac{h^2(2\eta)\, h(w_l - 3\eta + v)\, h(w_l - 3\eta - v)}{h(v - \eta)\, h(v + \eta)\, h(w_l - 4\eta)\, h(w_l - 2\eta)}, \tag{4.11}$$

$$p_l = \frac{h^2(v + \eta)\, h(w_l - 2\eta)\, h(w_l + 2\eta)}{h^2(2\eta)\, h(w_l - \eta - v)\, h(w_l + \eta + v)}, \tag{4.12}$$

$$q_l = \frac{h^2(v - \eta)\, h(w_l - 2\eta)\, h(w_l + 2\eta)}{h^2(2\eta)\, h(w_l + \eta - v)\, h(w_l - \eta + v)}, \tag{4.13}$$

where

$$w_l = \tfrac{1}{2}(s + t) - K + 2l\eta. \tag{4.14}$$

(Remember that s, t, K, η are constants, so $w_{l+1} = w_l + 2\eta$.)

From (4.3), (4.12), (4.13) we find, using our formula (4.9):

$$p_l' = -\frac{h(v - \eta)\, h(v + 3\eta)\, h^2(w_l)}{h^2(2\eta)\, h(w_l - \eta - v)\, h(w_l + \eta + v)} \tag{4.15}$$

$$q_l' = -\frac{h(v + \eta)\, h(v - 3\eta)\, h^2(w_l)}{h^2(2\eta)\, h(w_l - \eta + v)\, h(w_l + \eta - v)}. \tag{4.16}$$

Substituting these expressions into (4.7) gives

$$\begin{aligned} R_y &= -h(v - 3\eta)\, h(v + \eta)/h^2(v - \eta), \\ S_y &= h(w_y - 4\eta)\, h(w_y)/h^2(w_y - 2\eta), \\ R_y' &= -h(v + 3\eta)\, h(v - \eta)/h^2(v + \eta). \end{aligned} \tag{4.17}$$

We see immediately that the coefficients R_y , R'_{y-1} in (4.8) are independent of y. By using (4.9) once more we can also show that the last coefficient is independent of y. The coefficient of $\lambda'\mu'$ is more difficult, but we can show that it is an entire, doubly-periodic function of w_y , and hence a constant.

Having established that (4.5) is internally consistent, we can go on to solve it. The equation that led to (4.8) is

$$(\lambda'\mu' - S_y)(\lambda'\mu' - S_{y-1}) = (\lambda'\mu' - \lambda' r_y + R')(\lambda'\mu' - \mu' r_y + R) \quad (4.18)$$

(dropping the redundant suffixes on R, R'). Guided by the form (4.17) of S_y , we introduce a parameter u such that

$$\lambda'\mu' = h(u - 2\eta)\, h(u + 2\eta)/h^2(u). \quad (4.19)$$

The integer y enters (4.18) only through the complex number w_y that occurs in (4.17). Thus we expect (4.18) to be identically satisfied by any complex number w_y . Comparing (4.17) and (4.19) we see that $\lambda'\mu'$ S_y vanishes if $w_y = 2\eta + u$. Thus one of the factors on the r.h.s., say the first, must then vanish. (Choosing the second is simply equivalent to replacing u by $-u$.) This gives us an equation for λ', namely

$$\lambda' = [(\lambda'\mu' + R')/r_y]_{w_y = 2\eta + u} . \quad (4.20)$$

We can evaluate this by using (4.19), (4.17), (4.11) and the formula (4.9). We find that

$$\lambda' = zh(u - 2\eta)/h(u), \quad (4.21)$$

where

$$z = [h(v - \eta)\, h(v + \eta + u)]/[h(v + \eta)\, h(v - \eta + u)]. \quad (4.22)$$

From (4.19) we therefore have

$$\mu' = z^{-1}h(u + 2\eta)/h(u). \quad (4.23)$$

Substituting these results into (4.5) and making repeated use of the formula (4.9), we find that (4.5) becomes

$$e^{-ik'}\gamma(y - 1) : \gamma(y) : e^{ik'}\gamma(y + 1)$$
$$= z^{-1}[h(w_{y-2} - u)/h(w_{y-2})] : h(w_{y-1} - u)/h(w_{y-1}) : z[h(w_y - u)/h(w_y)] \quad (4.24)$$

It is apparent that these equations are internally consistent.

We still have some freedom in our choice of the single-particle functions (in addition to the choice of the parameter u), since we can multiply each function $\Gamma_j(l, x)$ by a factor $\exp(\chi_j l)$, where χ_1 ,..., χ_n are some constants. This re-defines λ_j ,

μ_j in (4.2), and k_j', $\gamma_j(y)$ in (4.4). Put another way, the choice of k_j' is at our disposal.

To remove this arbitrariness, we note that for those values of η such that the (modified) elliptic functions are periodic of period $2L\eta$, we expect $f(l \mid X)$ to be a periodic function of l, of period L. From (2.4), (3.1), (4.1) and (4.4) we therefore expect $\tilde{g}_j(l, x)$, $\Gamma_j(l, x)$ and $\gamma_j(l)$ to be periodic functions of l, with period L. From (4.24) and (4.14) it follows that

$$e^{iLk'} = z^L. \tag{4.25}$$

Incrementing u by $2K$ or iK' leaves the definition (4.19) unchanged, but multiplies the definition (4.22) of z by an Lth root of unity. It follows that we can always choose

$$e^{ik_j'} = z_j\,. \tag{4.26}$$

(We now restore the suffixes j to λ, μ, $\gamma(y)$, and to the related parameters u, z.)

From (4.4) and (4.24) we can now see that

$$\Gamma_j(l, x) = \tau_j e^{ik_j' x} h(w_{l+x-1} - u_j)/h(w_{l+x-1}), \tag{4.27}$$

where $\tau_1, \ldots, \tau_n$ are arbitrary constants, independent of l and x. From (4.1)

$$\tilde{g}_j(l, x) = \Gamma_j(l, x) - \Gamma_j(l, x - 1). \tag{4.28}$$

Using (4.27), (4.26), (4.22) and the formula (4.9), this gives

$$\tilde{g}_j(l, x) = -\tau_j e^{ik_j' x}\, \frac{h(2\eta)\, h(u_j)\, h(w_{l+x-1} + v - \eta)\, h(w_{l+x-1} - \eta - u_j - v)}{h(v - \eta)\, h(v + u_j + \eta)\, h(w_{l+x-2})\, h(w_{l+x-1})}. \tag{4.29}$$

From the transformation (2.4) we see that the single-particle functions $g_j(l, x)$ in (1.16) are related to the functions $\tilde{g}_j(l, x)$ in (3.1) by

$$g_j(l, x) = (a/b')^x\, [a/c'_{l+x-1}]\, \tilde{g}_j(l, x). \tag{4.30}$$

Using (4.29), the definitions (II.7.1) of a, b', c_x', and the definitions (4.26), (4.22) of $\exp(ik_j')$, it follows that

$$g_j(l, x) = \left\{\frac{h(u_j + v + \eta)}{h(u_j + v - \eta)}\right\}^x \frac{h(w_{l+x-1} - u_j - v - \eta)}{h(w_{l+x-2})\, h(w_{l+x-1})}, \tag{4.31}$$

where we have chosen

$$\tau_j = -[h(v - \eta)\, h(v + u_j + \eta)]/[h(v + \eta)\, h(u_j)]. \tag{4.32}$$

For each $j = 1,..., n$, we have now obtained the single-particle functions in terms of one unknown parameter u_j. The corresponding single-particle eigenvalues λ_j, μ_j are given by (4.6), (4.21), (4.23), (4.22) and (4.26) to be

$$\lambda_j = h(u_j - 2\eta)/h(u_j), \qquad \mu_j = h(u_j + 2\eta)/h(u_j). \tag{4.33}$$

Hence from (3.9) the corresponding eigenvalue Λ of the transfer matrix I is

$$\Lambda = [\rho' h(v - \eta)]^N \prod_{j=1}^{n} \frac{h(u_j - 2\eta)}{h(u_j)} + [\rho' h(v + \eta)]^N \prod_{j=1}^{n} \frac{h(u_j + 2\eta)}{h(u_j)}. \tag{4.34}$$

5. Equations Arising from Unwanted Internal Terms

We now look at the equations (3.12), together with the definition (3.10). These determine the coefficients $A(P)$.

From (3.7), (3.8), (4.11), (4.29) and (4.33) we find that

$$F_j(l + 1, x) = -\tau_j e^{ik_j' x} \frac{h(v + \eta)\, h(u_j - 2\eta)\, h(w_{l+x-1} - \eta - u_j - v)}{h(2\eta)\, h(u_j + v + \eta)\, h(w_{l+x-1} - \eta - v)}, \tag{5.1}$$

$$G_j(l - 1, x) = \tau_j e^{ik_j' x} \frac{h(v + \eta)\, h(u_j + 2\eta)\, h(w_{l+x-1} - \eta - u_j - v)}{h(2\eta)\, h(u_j + v + \eta)\, h(w_{l+x-1} - \eta - v)}. \tag{5.2}$$

Substituting these expressions into (3.10), using also (4.12), (4.13) and (4.29), we find that we can again use the formula (4.9) and obtain

$$B_{j,m}(l, x) = \tau_j \tau_m e^{i(k_j' + k_m')x} \times \frac{h^2(v + \eta)\, h(w_{l+x-1} - \eta - u_j - u_m - v)\, h(u_m - u_j + 2\eta)}{h(2\eta)\, h(w_{l+x-1} - \eta - v)\, h(u_j + v + \eta)\, h(u_m + v + \eta)}. \tag{5.3}$$

We note that interchanging u_j and u_m affects only the last term in the numerator of (5.3). Thus

$$B_{j,m}(l, x)/B_{m,j}(l, x) = h(u_m - u_j + 2\eta)/h(u_j - u_m + 2\eta). \tag{5.4}$$

This ratio is therefore independent of l and x and the equations (3.12) are consistent. They determine $A(P)$ to within a normalization constant and are satisfied by

$$A(P) = \epsilon_P \prod_{1 \leqslant j < m \leqslant n} h(u_{P_j} - u_{P_m} + 2\eta), \tag{5.5}$$

where ϵ_P is the signature of the permutation P (+ for an even number of interchanges—for an odd number).

6. Equations Arising from Unwanted Boundary Terms

The final step in the solution of the Bethe ansatz is to look at the equations (3.14). From (4.1) and (3.6) we see that

$$C_j(l) = \Gamma_j(l, 0), \tag{6.1}$$

$$C_j'(l) = \Gamma_j(l, N). \tag{6.2}$$

From (4.27) it follows that

$$\frac{C_j'(l + 1 - 2n)}{C_j(l + 1)} = e^{iNk_j'} \frac{h(w_l)\, h[w_l - u_j + (N - 2n)\,\eta]}{h[w_l + (N - 2n)\,\eta]\, h[w_l - u_j]}. \tag{6.3}$$

However, we have imposed the boundary condition that either $n = \frac{1}{2}N$, or if there exists an integer L such that $2L\eta$ is a period of the elliptic functions, then $N - 2n = L \times$ integer. In either case we see that the functions h on the r.h.s. of (6.3) cancel, so the result is independent of l. This is necessary for the equations (3.14) to be consistent.

Using the definitions (4.26), (4.22) of $\exp(ik_j)$, together with the definitions (II.7.1) of a, b', we see that

$$\left(\frac{a}{b'}\right)^N \frac{C_j'(l + 1 - 2n)}{C_j(l + 1)} = \left\{\frac{h(u_j + v + \eta)}{h(u_j + v - \eta)}\right\}^N. \tag{6.4}$$

Using this result, together with the result (5.5) for $A(P)$, in (3.14), we get the n equations

$$\left\{\frac{h(u_j + v + \eta)}{h(u_j + v - \eta)}\right\}^N = -\prod_{m=1}^{n} \frac{h(u_j - u_m + 2\eta)}{h(u_j - u_m - 2\eta)} \tag{6.5}$$

for $j = 1,\ldots, n$. (We have used the fact that $h(u)$ is an odd function.)

These equations determine $u_1, \ldots, u_n$, which up to now have been at our disposal.

7. Summary of Sections 2 to 6

In Section 2 we set up the eigenvalue equations for Λ and $f(l \mid X)$, finding it convenient to transform from $f(l \mid X)$ to $\tilde{f}(l \mid X)$ as in Eq. (2.4). In Section 3 we made the Bethe ansatz (3.1) for $\tilde{f}(l \mid X)$. This is equivalent to the Bethe ansatz (1.16) for $f(l \mid X)$ if we relate the single-particle functions g and $\tilde{g}$ by (4.30). We found that this worked provided we could satisfy (3.7)–(3.9), (3.12) and (3.14). In Sections 4–6 we have shown that we can indeed do so, despite the fact that at each stage we have more equations than unknowns.

The results are contained in equations (6.5), (5.5), (4.34) and (4.31). The equations (6.5) determine $u_1, \ldots, u_n$. The coefficients $A(P)$, the eigenvalue Λ and the single-particle functions $g_j(l, x)$ are then given by (5.5), (4.34) and (4.31) respectively.

Using the definition (4.14) and $\rho' = \rho\Theta(0)$ [Eq. (II.7.2)], we see that these results are the same as those quoted in Eqs. (1.17)–(1.23) of Section 1, provided we replace each u_j by $u_j - v$.

8. Inhomogeneous System

We end by remarking that all the above working still goes through if we consider an eight-vertex model with weights a, b, c, d given by (II.6.1), in which k and η (and trivially ρ) are the same for each site of the lattice, but v can vary from column to column.

Let v_x be the value of v on column x, where $x = 1, 2, \ldots, N$. The equations (1.2), (1.3) for our basis vectors $(l_1, \ldots, l_{N+1})$ become

$$\psi(l_1, \ldots, l_{N+1}) = \Phi^{(1)}_{l_1, l_2} \otimes \Phi^{(2)}_{l_2, l_3} \otimes \cdots \otimes \Phi^{(N)}_{l_N, l_{N+1}}, \tag{8.1}$$

where

$$\begin{aligned} \Phi^{(x)}_{l, l+1} &= |\, s + 2l\eta + \eta - v_x \rangle, \\ \Phi^{(x)}_{l+1, l} &= |\, t + 2l\eta + \eta + v_x \rangle, \end{aligned} \tag{8.2}$$

for $x = 1, \ldots, N$ and all integers l.

The transfer matrix equation (1.5) in this basis becomes

$$\mathbf{T}\psi(l_1, \ldots, l_{N+1}) = \sum \left\{ \prod_{J=1}^{N} W^{(J)}(m_J, m_{J+1} \mid l_J, l_{J+1}) \right\} \psi(m_1, \ldots, m_{N+1}), \tag{8.3}$$

where $W^{(J)}(m, m' \mid l, l')$ is given by (II.7.1) with v replaced by v_J $(J = 1, \ldots, N)$.

The eight-vertex model therefore transforms to an Ising-like model whose weights a_l, a_l', b_l, b_l', c_l, c_l' (shown in Fig. 2) depend on which column (J or x) is being considered. We therefore write them as $a_l(J), \ldots, c_l'(J)$, where $J = 1, \ldots, N$. They are given by (II.7.1) with v replaced by v_J.

The transformation (2.4) becomes

$$\tilde{f}(l \mid x_1, x_2) = s_l(x_1)\, s_{l-2}(x_2)\, f(l \mid x_1, x_2), \tag{8.4}$$

where

$$s_l(x) = \left\{ \prod_{y=1}^{x-1} b'(y) \right\} c'_{l+x-1}(x) \left\{ \prod_{y=x+1}^{N} a(y) \right\} \tag{8.5}$$

(b' and a are still independent of l). We make the same ansatz (1.16), which is equivalent to (3.1) if

$$\tilde{g}_j(l, x) = s_l(x)\, g_j(l, x). \tag{8.6}$$

The definitions (2.6), (2.7) apply for each column separately of the lattice, so $r_l(x)$, $p_l(x)$, $q_l(x)$ are given by (4.11) and (4.12) with v replaced by v_x. In the Bethe ansatz equations (3.5)–(3.14) each p_l, q_l, r_l is replaced by $p_l(x)$, $q_l(x)$, $r_l(x)$, and a^N, b'^N by

$$\prod_{y=1}^{N} a(y), \qquad \prod_{y=1}^{N} b'(y). \tag{8.7}$$

Similarly, the coefficients r_{l+x}, p_{l+x-1}, p'_{l+x-1}, q_{l+x-2}, q'_{l+x-2} in (4.2) all have argument x (i.e., they are evaluated from (4.11) and (4.12) with v replaced by v_x). We can no longer use (4.4) to simplify (4.2), but by considering a particular value of x and using the previous results of Section 4 it is not difficult to see that (4.2) is satisfied by (4.27), (4.29) and (4.33), with $\exp(ik_j'x)$ replaced by

$$\prod_{y=1}^{x} \frac{h(v_y - \eta)\, h(v_y + \eta + u_j)}{h(v_y + \eta)\, h(v_y - \eta + u_j)}, \tag{8.8}$$

and v in (4.29) by v_x.

From (8.5), (8.6) and (4.29) it follows that (with an appropriate choice of τ_j):

$$g_j(l, x) = \left\{\prod_{y=1}^{x-1} h(u_j + v_y + \eta)\right\} \frac{h(w_{l+x-1} - \eta - u_j - v_x)}{h(w_{l+x-2})\, h(w_{l+x-1})} \prod_{y=x+1}^{N} h(u_j + v_y - \eta). \tag{8.9}$$

This defines the single-particle functions. Using the rule (8.7), the equation (4.34) for the eigenvalue Λ becomes:

$$\Lambda = \left\{\prod_{J=1}^{N} \rho' h(v_J - \eta)\right\}\left\{\prod_{j=1}^{n} \frac{h(u_j - 2\eta)}{h(u_j)}\right\} + \left\{\prod_{J=1}^{N} \rho' h(v_J + \eta)\right\}\left\{\prod_{j=1}^{n} \frac{h(u_j - 2\eta)}{h(u_j)}\right\}. \tag{8.10}$$

The equation (5.5) for the coefficients $A(P)$ is the same for the inhomogeneous system as for the homogeneous, i.e.,

$$A(P) = \epsilon_P \prod_{1 \leqslant j < m \leqslant n} h(u_{P_j} - u_{P_m} + 2\eta). \tag{8.11}$$

The equations (6.5) for $u_1, \ldots, u_n$ become

$$\prod_{y=1}^{N} \frac{h(u_j + v_y + \eta)}{h(u_j + v_y - \eta)} = -\prod_{m=1}^{n} \frac{h(u_j - u_m + 2\eta)}{h(u_j - u_m - 2\eta)}, \qquad j = 1, \ldots, n. \tag{8.12}$$

These results (8.9)–(8.12) are formally not much more complicated than those for the homogeneous lattice. In some ways they make the properties of the solution clearer: for instance it is apparent from (8.9) that each single-particle function is a product of N factors, one from each column of the lattice.

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