CSCI 3104: Algorithms
Spring 2022
Recitation #3 - Graph Notation

A *Graph* is a set of *vertices* V connected to one another by *edges* E. We usually write this as the graph $G = (V, E)$, and edges as $e = (v_i, v_j)$, for vertices v_i and v_j in V. We generally refer to the number of vertices as $n = |V|$ and the number of edges as $m = |E|$.

If a graph is *directed*, then we distinguish between the edges (v_i, v_j) and (v_j, v_i), and there can be an edge to v_j from v_i but not an edge from v_i to v_j.

Problem 1

Draw the undirected, unweighted graph with vertices $V = \{1, 2, 3, 4\}$ and edges $E = \{(1, 2), (2, 3), (3, 4), (4, 1), (2, 4)\}$.

Problem 2

Determine whether each of the following graphs are bipartite. Justify your answer. (You may use, without proof, the fact that a graph G is bipartite if and only if G does not have any cycles of odd length.)

(a) A *tree* is a graph $T(V, E)$ that is connected and acyclic (that is, T does not contain any cycles).

(b) For which values of $n \geq 3$ is the cycle graph C_n bipartite?

(c) For which values of $n \geq 4$ is the wheel graph W_n bipartite?

(d) For which values of $n \geq 1$ is the complete graph K_n bipartite?

Problem 3

Consider the *hypercube* graph of dimension d, denoted Q_d. The vertex set of Q_d the set of strings $\{0, 1\}^d$ (*i.e.*, the set of binary strings of length d). Two vertices v_1, v_2 are adjacent if and only if they differ in precisely one position (*e.g.* the vertices $\{0, 0\}$ and $\{0, 1\}$ are adjacent in Q_2, but $\{0, 0\}$ and $\{1, 1\}$ are not). Answer the following:

(a) How many vertices belong to Q_d?

(b) How many edges belong to Q_d?

(c) Suppose that a string $\omega \in \{0,1\}^d$ has exactly i digits that are 1's. Explain why ω's neighbors in Q_d only have either exactly $(i+1)$ or exactly $(i-1)$ digits that are 1's.

(d) Prove that the graph Q_d is bipartite by giving an explicit bipartition of $\{0,1\}^d$. [**Hint:** Use part (c) to construct your bipartition.]

Problem 4

Do the following.

(a) Let $d \geq 1$. Show how to construct Q_d using two copies of Q_{d-1}.

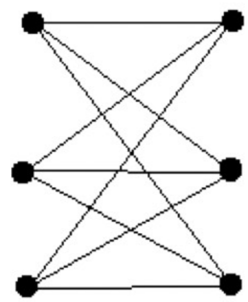
(b) A *Hamiltonian cycle* in a graph G is a cycle graph that includes every vertex of G. Prove by induction that Q_d has a Hamiltonian cycle for all $d \geq 2$. [**Hint:** Use part (a) in the inductive step.]

Problem 5

Suppose we have n couples at a party. Each person shakes hands with everyone else, with the exception of their partner. Note that a person does not shake hands with themselves. How many handshakes occurred at this party?

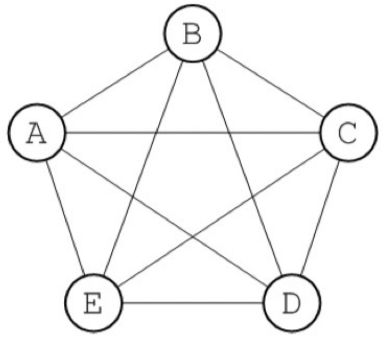
Definition 227 (Bipartite Graph)**.** A bipartite graph $G(V, E)$ has a vertex set $V = X \dot{\cup} Y$, with edge set $E \subseteq \{xy : x \in X, y \in Y\}$. That is, no two vertices in the same part of V are adjacent. So no two vertices in X are adjacent, and no two vertices in Y are adjacent.

Example 228. A common class of bipartite graphs include even-cycles C_{2n}. The complete bipartite graph is another common example. We denote the complete bipartite graph as $K_{m,n}$ which has vertex partitions $X \dot{\cup} Y$ where $|X| = m$ and $|Y| = n$. The edge set $E(K_{m,n}) = \{xy : x \in X, y \in Y\}$. The graph $K_{3,3}$ is pictured below.



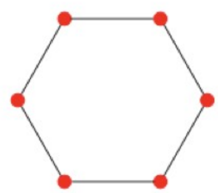
Definition 219 (Complete Graph)**.** The complete graph, denoted K_n, has the vertex set $V = \{1, 2, \ldots, n\}$ and edge set E which consists of **all** two-element subsets of V. That is, K_n has all possible edges between vertices.

Example 220. The complete graph on five vertices K_5 is pictured below.



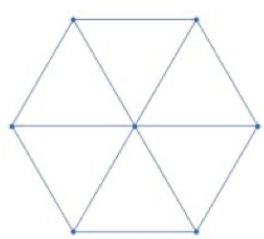
Definition 223 (Cycle Graph)**.** Let $n \geq 3$. The cycle graph, denoted C_n, has the vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E = \{\{i, i+1\} : 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$.

Example 224. Intuitively, C_n can be thought of as the regular n-gon. So C_3 is a triangle, C_4 is a quadrilateral, and C_5 is a pentagon. The graph C_6 is pictured below.



Definition 225 (Wheel Graph)**.** Let $n \geq 4$. The wheel graph, denoted W_n, is constructed by joining a vertex n to each vertex of C_{n-1}. So we take $C_{n-1} \dot{\cup} n$ and add the edges vn for each $v \in [n-1]$.

Example 226. The wheel graph on seven vertices W_7 is pictured below.



Quoted from. Michaeles note.

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curly brackets

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round brackets

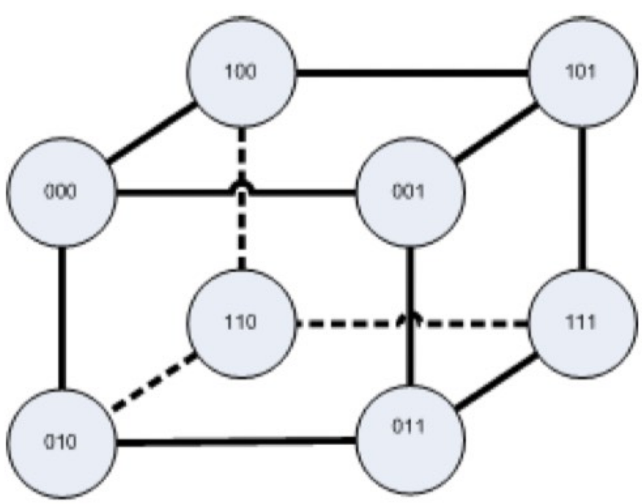
(a) How many vertices belong to Q_d?

(b) How many edges belong to Q_d?

(c) Suppose that a string $\omega \in \{0,1\}^d$ has exactly i digits that are 1's. Explain why ω's neighbors in Q_d only have either exactly $(i+1)$ or exactly $(i-1)$ digits that are 1's.

Suppose $i > 1$ here.

(d) Prove that the graph Q_d is bipartite by giving an explicit bipartition of $\{0,1\}^d$. **[Hint:** Use part (c) to construct your bipartition.]



Q_3

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