CSCI 3104: Algorithms
Spring 2022
Recitation #2 - Greedy

Problem 1

Consider the problem of "making change." We have an infinite supply of pennies (worth 1 cent), nickels (worth 5 cents), dimes (worth 10 cents), and quarters (worth 25 cents). We take as input an integer $n \geq 0$. Our goal is to make n cents in coins, using the fewest number of coins possible.

The greedy algorithm chooses as many quarters as possible, followed by as many dimes as possible, then as many nickels as possible. Finally, the greedy algorithm uses pennies to finish making change.

We will use a greedy exchange argument to prove that this greedy solution is optimal.

a. For any n, how many coins of each type can appear in an optimal solution?

- What is the maximum number of pennies (1 cent)?

  4, otherwise 5 pennies can be replaced by 1 nickels.

- What is the maximum number of nickels (5 cents)?

  1, otherwise 2 nickels " by 1 dimes.

- What is the maximum number of dimes (10 cents)?

  2, otherwise 3 dimes " by 1 quarters and 1 nickels.

b. Given some collection of coins C worth n cents, devise some rules by which coins in C can be *exchanged* to obtain a smaller collection C' also worth n cents (e.g. "if there are 5 pennies in C, replace them with 1 nickel"). Will you always be able to apply these rules?

c. Argue that the collection C' obtained by applying your rules until none of them can be applied further (e.g. there are no more pennies) is identical to the solution S obtained by the greedy algorithm.

d. Why does c. imply that the greedy algorithm produces an optimal solution?

b. Given some collection of coins C worth n cents, devise some rules by which coins in C can be *exchanged* to obtain a smaller collection C' also worth n cents (e.g. "if there are 5 pennies in C, replace them with 1 nickel"). Will you always be able to apply these rules?

(a) 5 pennies → 1 nickel

(b) 2 nickels → 1 dime

(c) 2 dimes and 1 nickels → 1 quarters

(d) 3 dimes → 1 quarters and 1 nickels

We can apply rules to reduce the number of coins until there is no more than 5 pennies / 2 nickels / 2 dimes and 1 nickel / 3 dimes.

c. Argue that the collection C' obtained by applying your rules until none of them can be applied further (e.g. there are no more pennies) is identical to the solution S obtained by the greedy algorithm.

The greedy algorithm chooses as many quarters as possible, followed by as many dimes as possible, then as many nickels as possible. Finally, the greedy algorithm uses pennies to finish making change.

Let the optimal number of pennies p^*
nickels m^*
dimes d^*
quarters q^*

$\Rightarrow$

$25 > 10d^* + 5m^* + p^*$ —— ①

$10 > 5m^* + p^*$ —— ②

$5 > p^*$ —— ③

(a) $5p \Rightarrow 1m$
(b) $2m \Rightarrow 1d$
(c) $2d + 1m \Rightarrow 1q$
(d) $3d \Rightarrow 1q + 1m$

Rules.

Assume $C' \neq S$, that means at least one inequalities ①, ②, ③ would be violated when only applying rules (a), (b), (c), (d).

Apply rule (a), the inequality ③ can be satisfied.

In addition, we can assume that $4 \geq p^*$

Thus, $10 > 5m^* + 4 \geq 5m^* + p^*$

$\Rightarrow \quad 6 > 5m^* \Rightarrow 2 > m^*$ —— ④

Apply rule (b), the inequality ④ can be satisfied.
(With rule (a), (b), the inequality ② can be satisfied).

$25 > 10d^* + 5m^* + 4 \geq 10d^* + 5m^* + p^*$

$\Rightarrow 21 > 10d^* + 5m^*$ —— ⑤

Apply rule (c) and (d), the inequality ⑤ can be satisfied.
(With rule (a), (b), (c), (d), the inequality ① can be satisfied.

Therefore, applying rules can let the inequalities be satisfied.
That's contradiction.

d. Why does c. imply that the greedy algorithm produces an optimal solution?

Consider starting from an optimal solution C. Since C is optimal, it has the minimum size and so we can't apply any rules to shrink it anymore. Since none of rules can be applied, C must be identical to S, which is the solution applying the greedy algorithm. In other words, the greedy algorithm produce an optimal solution!

2c statement.