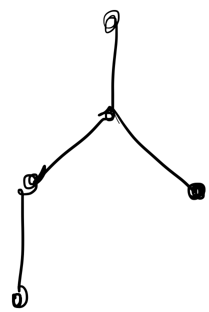
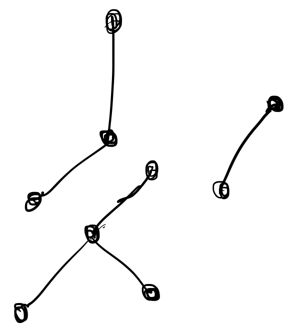
Tree and Forest.

In graph theory, a **tree** is an undirected graph in which any two vertices are connected by *exactly one* path, or equivalently a connected acyclic undirected graph.[1] A **forest** is an undirected graph in which any two vertices are connected by *at most one* path, or equivalently an acyclic undirected graph, or equivalently a disjoint union of trees.[2] [1]



Tree

Connected, Acyclic, Undireted Graph



Forest.

Acyclic, undirected Graph.

[1] Wiki: Tree

Minimum Spanning Tree.

Given a connected, directed graph $G = (V, E)$, the acyclic subset $T \subseteq E$ that connects all of the vertices and whose total weight

$$w(T) = \sum_{(u,v) \in T} w(u, v)$$

is minimized. [1]

```
GENERIC-MST(G, w)
A = Ø
while A does not form a spanning tree
    find an edge (u, v) that is safe for A
    A = A ∪ {(u, v)}
return A
```

This greedy strategy is captured by the following generic method, which grows the minimum spanning tree one edge at a time. The generic method manages a set of edges A, maintaining the following loop invariant:

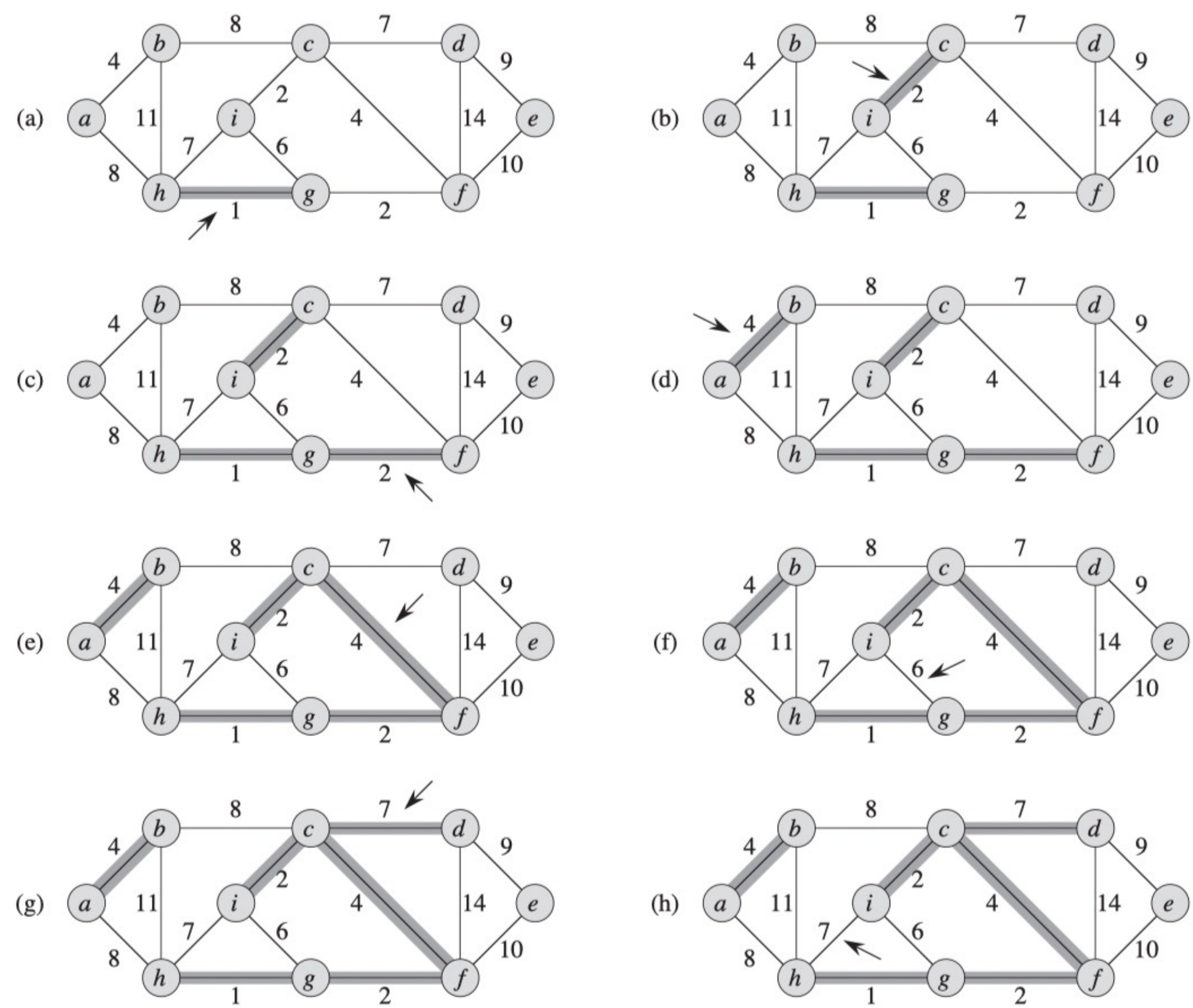
Prior to each iteration, A is a subset of some minimum spanning tree.

In Kruskal's algorithm, the set A is a forest whose vertices are all those of the given graph. The safe edge added to A is always a least-weight edge in the graph that connects two distinct components.

In Prim's algorithm, the set A forms a single tree. The safe edge added to A is always a least-weight edge connecting the tree to a vertex not in the tree.

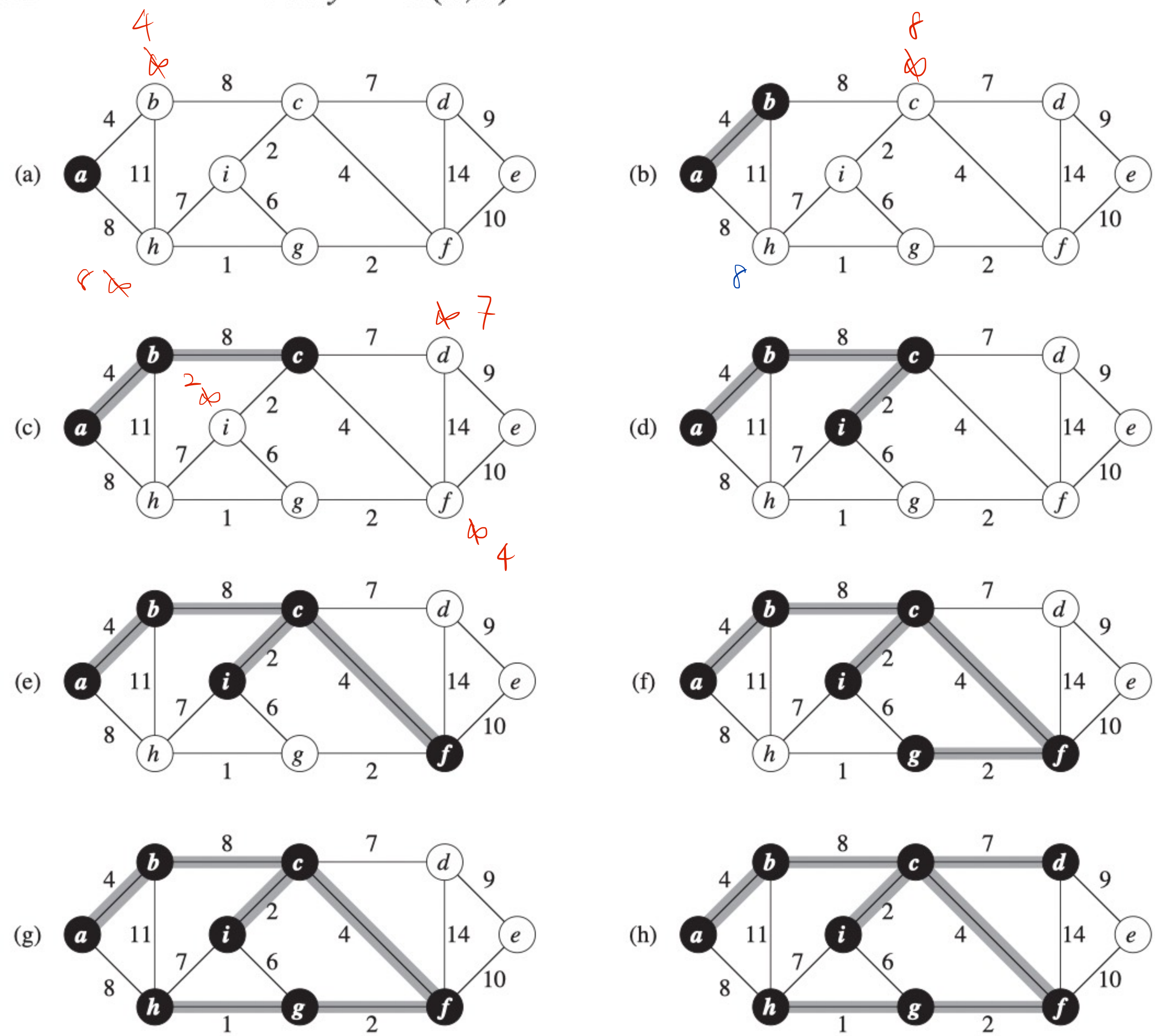
[1]. Intro to algorithm., chapter 23

```
MST-KRUSKAL(G, w)
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
sort the edges of G.E into nondecreasing order by weight w
for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
    if FIND-SET(u) ≠ FIND-SET(v)
        A = A ∪ {(u, v)}
        UNION(u, v)
return A
```



MST-PRIM(G, w, r)

```
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```

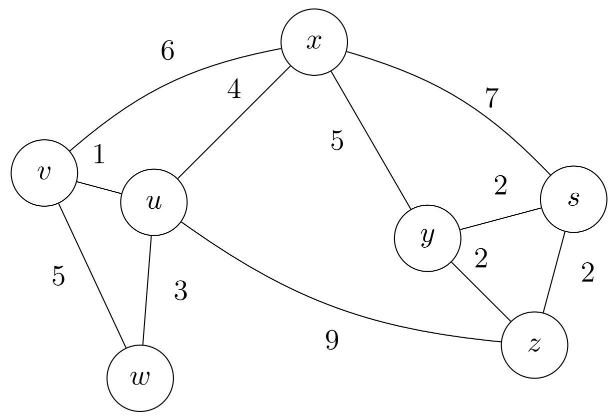


CSCI 3104: Algorithms
Spring 2022
Recitation #4 - MSTs

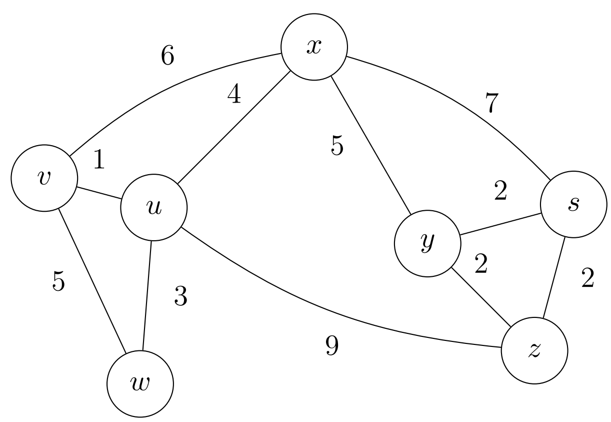
Problem 1

Prim's and Kruskal's are two algorithms for finding a Minimum Spanning Tree on a graph. In this problem, you will practice

a. Use Prim's Algorithm to construct a MST on the following graph, and state the total weight.



b. Use Kruskal's Algorithm to find a MST on the same graph:



c. Are the MSTs produced by the two algorithms the same? Why or why not? Are there any other MSTs?

Problem 2

For each of the following statements about MSTs on an undirected graph $G = (V, E)$ with (not necessarily distinct) edge weights $w_e > 0$ for each $e \in E$, either prove the statement or disprove it with a counterexample.

a. Assume there is an edge e^* which has a strictly smaller weight than all other edges in E. Then e^* is in every minimum spanning tree of G.

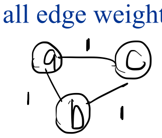
This is true. We argue by contradiction. Assume there is a minimum spanning tree T that does not contain $e*$.
Then T $\cup$ { $e*$ } contains a cycle C, and $e*$ is not the most expensive edge in C.
Remove the most expensive edge e' in C. This new graph T $\cup$ { $e*$ } \ { e' } costs less than T, and is also a spanning tree.
This contradicts the assumption that T was an MST.

b. Assume there is an edge e^* which has a strictly larger weight than all other edges in E. Then e^* is not in any minimum spanning tree of G.

c. If an edge e is in a minimum spanning tree T of G, then there exists some cut $(S, V \setminus S)$ such that e is the (not necessarily unique) cheapest edge crossing $(S, V \setminus S)$.

d. For any minimum spanning tree T and pair of vertices u and v, T contains a (weighted) shortest path from u to v in G.

This is false. Consider C_3, the cycle on three vertices a, b, and c, with all edge weights 1. The MST { (a, b), (b, c) } does nor contain the shortest path from a to c.



e. Note that in a graph with nondistinct edge weights, there may be multiple minimum-cost spanning trees. Let $T \subseteq E$ be a spanning tree. If each edge $e \in T$ belongs to *some* MST in G, then T is also a MST in G.

This is false. Consider the cycle graph C_3, the cycle on three vertices a, b, and c, with edge weights w (a,b) = 2, w (a,c) = 2, and w (b,c) = 1. { (a, b), (b, c) } and { (a, c), (b, c) } are both MSTs with weight 3, but the spanning tree { (a, b), (a, c) } has weight 4.