

## Problem Set 5

---

Due Date ..... February 22  
Name ..... **Your Name**  
Student ID ..... **Your Student ID**  
Collaborators ..... **List Your Collaborators Here**

### Contents

<b>Instructions</b>	<b>1</b>
<b>1 Honor Code (Make Sure to Virtually Sign)</b>	<b>2</b>
<b>2 Standard 14 - Analyzing Code III: (Writing down recurrences)</b>	<b>3</b>
2.1 Problem 2(a) . . . . .	3
2.2 Problem 2(b) . . . . .	4
<b>3 Standard 15 - Unrolling</b>	<b>5</b>
<b>4 Standard 16 - Tree Method.</b>	<b>7</b>

### Instructions

- The solutions **must be typed**, using proper mathematical notation. We cannot accept hand-written solutions. Useful links and references on  $\text{\LaTeX}$  can be found here on Canvas.
- You should submit your work through the **class Canvas page** only. Please submit one PDF file, compiled using this  $\text{\LaTeX}$  template.
- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this document with no fewer pages than the blank template (or Gradescope has issues with it).
- You are welcome and encouraged to collaborate with your classmates, as well as consult outside resources. You must **cite your sources in this document**. **Copying from any source is an Honor Code violation. Furthermore, all submissions must be in your own words and reflect your understanding of the material.** If there is any confusion about this policy, it is your responsibility to clarify before the due date.
- Posting to **any** service including, but not limited to Chegg, Reddit, StackExchange, etc., for help on an assignment is a violation of the Honor Code.

- You **must** virtually sign the Honor Code (see Section Honor Code). Failure to do so will result in your assignment not being graded.

## 1 Honor Code (Make Sure to Virtually Sign)

**Problem 1.**     • My submission is in my own words and reflects my understanding of the material.

- Any collaborations and external sources have been clearly cited in this document.
- I have not posted to external services including, but not limited to Chegg, Reddit, StackExchange, etc.
- I have neither copied nor provided others solutions they can copy.

*Agreed (signature here).*



## 2 Standard 14 - Analyzing Code III: (Writing down recurrences)

**Problem 2.** Write down the recurrence relation for the runtime complexity of these algorithms. **Don't forget to include the base cases.**

### 2.1 Problem 2(a)

---

**Algorithm 1** Writing Recurrences 1

---

```
1: procedure Foo(Integer  $n$ )
2:   if  $n \leq 3$  then return
3:
4:   Foo( $n/3$ )
5:   Foo( $n/3$ )
6:   Foo( $n/3$ )
7:
8:   for  $i \leftarrow 1; i \leq 2 * n; i \leftarrow i + 1$  do
9:     print "Hi, Hi"
```

---

*Answer.*

$$T(n) = \begin{cases} 1 & : \text{if } n \leq 3, \\ 3T(n/3) + 2n & : \text{otherwise.} \end{cases}$$

□

## 2.2 Problem 2(b)

---

**Algorithm 2** Writing Recurrences 2

---

```
1: procedure Foo(Integer  $n$ )
2:   if  $n \leq 4$  then return
3:
4:   Foo( $n/4$ )
5:   Foo( $n/4$ )
6:   Foo( $n/3$ )
7:
8:   for  $i \leftarrow 1; i \leq n; i \leftarrow i * 2$  do
9:     print "Hi, Hi"
```

---

*Answer.*

$$T(n) = \begin{cases} 1 & : \text{if } n \leq 4. \\ 2T(n/4) + T(n/3) + \log_2 n & : \text{otherwise.} \end{cases}$$

□

### 3 Standard 15 - Unrolling

**Problem 3.** Consider the following recurrences and solve them using the unrolling method (i.e. find a suitable function  $f(n)$  such that  $T(n) \in \Theta(f(n))$ ).

(a)

$$T(n) = \begin{cases} 2 & : n < 3, \\ 2T(n-3) + 2 & : n \geq 3. \end{cases}$$

*Answer.*

$$\begin{aligned} T(n) &= 2T(n-3) + 2 \\ &= 2(2T(n-3-3) + 2) + 2 \\ &= 2(2(2T(n-3-3-3) + 2) + 2) + 2 \\ &\dots \\ &= 2^k T(n-3k) + 2k. \end{aligned}$$

The base case occurs when  $n - 3k < 3$  or

$$k = (n-3)/3$$

Then

$$\begin{aligned} T(n) &= 2^k \cdot T(n-3k) + 2k \\ &= 2^{(n-3)/3} \cdot T(n-3(n-3)/3) + 2(n-3)/3 \\ &= 2^{(n-3)/3} \cdot 2 + 2(n-3)/3. \\ &= 2^{(n-3)/3+1} + 2(n-3)/3. \end{aligned}$$

The exponential term dominates the asymptotic behavior and we have

$$T(n) \in \Theta(2^{(n-3)/3+1})$$

□

(b)

$$T(n) = \begin{cases} 3 & : n < 2, \\ T(n-2) + 4n & : n \geq 2. \end{cases}$$

*Answer.*

$$\begin{aligned} T(n) &= 2T(n-2) + 4n \\ &= 2(2T(n-2-2) + 4(n-2)) + 4n \\ &= 2(2(2T(n-2-2-2) + 4(n-2-2-2)) + 4(n-2-2) + 4(n-2)) + 4n \\ &\dots \\ &= 2^k T(n-2k) + \sum_{i=1}^k 4i. \end{aligned}$$

The base case occurs when  $n - 2k < 2$  or

$$k = (n-2)/2$$

Then

$$\begin{aligned} T(n) &= 2^k \cdot T(n-2k) + \sum_{i=1}^k 4i. \\ &= 2^{(n-2)/2} \cdot T(n-2(n-2)/2) + \sum_{i=1}^{(n-2)/2} 4i. \\ &= 2^{(n-2)/2} \cdot 3 + \sum_{i=1}^{(n-2)/2} 4i. \\ &= 2^{(n-2)/2} \cdot 3 + 4 \cdot \frac{(n-2)/2((n-2)/2-1)}{2} \end{aligned}$$

The exponential term dominates the asymptotic behavior and we have

$$T(n) \in \Theta(2^{(n-2)/2})$$

□

## 4 Standard 16 - Tree Method.

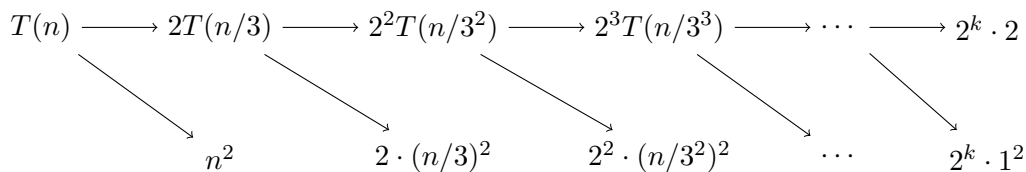
**Problem 4.** Consider the recurrence  $T(n)$  below. Using the **tree method**, determine a suitable function  $f(n)$  such that  $T(n) \in \Theta(f(n))$ . Clearly show all steps. Note the following:

- You may assume, without loss of generality, that  $n$  is a power of 3 (i.e.,  $n = 3^k$  for some integer  $k \geq 0$ ).
- You may hand-draw your tree and embed it, provided it is legible and we do not have to rotate our screens to read it. However, **all your calculations must be typed**.

$$T(n) = \begin{cases} 2, & n < 3 \\ 2T(n/3) + n^2, & \text{otherwise.} \end{cases}$$

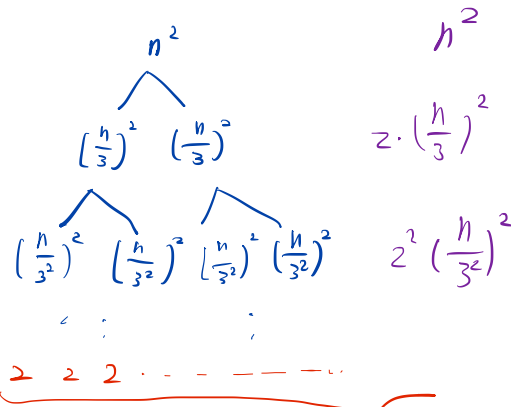
$$\frac{n}{3^k} < 3 \Rightarrow k = \log_3 n - 1$$

Answer. Graph:



The general formula can be expressed as

$$\sum_{i=1}^k 2^{i-1} \cdot \left(\frac{n}{3^i}\right)^2 + 2^{k+1} = \sum_{i=1}^k \frac{2^{i-1}}{3^{2i}} \cdot n^2 + 2^{k+1} = C \cdot n^2 + 2^{k+1}$$



We reach a base case when  $n/3^k < 3$  or

$$k = \log_3 n - 1$$

$$\begin{aligned} T(n) &= C \cdot n^2 + 2^{k+1} \\ &= C \cdot n^2 + 2^{\log_3 n - 1 + 1} \\ &= C \cdot n^2 + 2^{\log_3 n} \\ &= C \cdot n^2 + 2^{\log_2 n / \log_2 3} \\ &= C \cdot n^2 + n^{1/\log_2 3} \end{aligned}$$

$$\begin{aligned} &= n^2 + 2 \left(\frac{n}{3}\right)^2 + 2^2 \left(\frac{n}{3^2}\right)^2 + \dots \\ &= \sum_{i=0}^k 2^i \left(\frac{n}{3^i}\right)^2 \\ &= n^2 \sum_{i=0}^k \left(\frac{2}{9}\right)^i \\ &= n^2 \left( \frac{1 - \left(\frac{2}{9}\right)^{k+1}}{1 - \frac{2}{9}} \right) \\ &= n^2 \left( \frac{1 - \left(\frac{2}{9}\right)^{\log_3 n}}{\frac{7}{9}} \right) \\ &= \frac{9}{7} n^2 \left( 1 - \frac{n^{\log_3 2}}{n^1} \right) \\ &= \Theta(n^2) \end{aligned}$$

The term with the largest exponent dominates and  $1/\log_2 3 < 1$ , so

$$T(n) \in \Theta(n^2)$$

$$\left(\frac{2}{9}\right)^{\log_3 n} = n^{\log_3 \frac{2}{9}}$$

$$= n^{\log_3 2 - \log_3 9} = \frac{n^{\log_3 2}}{n^2}$$

$$\Rightarrow n^2 + n^{\log_3 2} = \Theta(n^2)$$