CSCI 3104: Algorithms
Spring 2022
Recitation #5 - Asymptotics

It may be helpful to recall the following:

(Limit Comparison Test) If the limit $L := \lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists, then:

- if $0 < L < \infty$, then $f(x) \in \Theta(g(x))$
- if $L = 0$, then $f(x) \in O(g(x))$, but $f(x) \notin \Theta(g(x))$
- if $L = \infty$, then $g(x) \in O(f(x))$, but $g(x) \notin \Theta(f(x))$

(L'Hôpital's rule) Suppose g and f are both differentiable functions, with either

a. $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$; or

b. $\lim_{x\to\infty} f(x) = \pm\infty$ and $\lim_{x\to\infty} g(x) = \pm\infty$

If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$.

Problem 1

Order the following functions in increasing growth rate, using Big-O and Big-Theta notation. (e.g., $n \in O(n^2)$, $n^2 \in \Theta(n^2+1)$, $n^2+1 \in O(n^3)$)

$$n^3 + 8n^2, \quad 2n^2\sqrt{n}, \quad 6n^3 - n^2, \quad n^2 + 300n$$

Problem 2

Simplify the following expressions such that a) all logarithms are base 2 and b) terms are expressed as sums and multiplication where possible (remove exponents). It may be helpful to recall the following:

$$\log_b(mn) = \log_b(m) + \log_b(n), \qquad \log_b\left(\frac{m}{n}\right) = \log_b(m) - \log_b(n)$$
$$\log_b(n^p) = p\log_b(n), \qquad \log_b(n) = \frac{log_p(n)}{log_p(b)}$$

a. $\log_3(x^2y^3) - \log_2(\sqrt{z})$

b. $\log_3\left(\frac{(t+5)}{(t^4)}\right)$

c. $\log_2(2^x) + \log_4(\frac{x}{5y})$

Problem 3

Order the following problems in increasing growth rate, using Big-O and Big-Theta notation.

$$n \ln(2n), \quad 2^{2n}, \quad n^3 + 7, \quad n^3 + 2^{5n}$$

It may be helpful to remember the product rule: $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$.

Problem 4

Analyze the worst-case runtime of the following algorithms. Derive the runtime complexity $T(n)$ and give an asymptotic upper bound $f(n)$ such that $T(n) = O(f(n))$.

a.

```
procedure algs_are_fun_1(integer n):
  for i = n, i >= 1
      i = i - 1
      print ‘outer’
      for j = 1, j <= 500
          j = j + 2
          print ‘inner 1’
      for k = 1, k <= 2i
          k = k + 1
           print ‘inner 2’
```

b.

```
procedure algs_are_fun_2(integer n):
  for i = 1, i <= n
      i = i + 2
      print ‘outer’
      for j = 1, j <= i
          j = 2j
          print ‘inner’
```

Problem 1

Order the following functions in increasing growth rate, using Big-O and Big-Theta notation. (e.g., $n \in O(n^2)$, $n^2 \in \Theta(n^2+1)$, $n^2+1 \in O(n^3)$)

$$n^3 + 8n^2, \quad 2n^2\sqrt{n}, \quad 6n^3 - n^2, \quad n^2 + 300n$$

$$n^3 + 8n^2 \in \Theta(6n^3 - n^2)$$

$$2n^2\sqrt{n} \in O(6n^3 - n^2)$$

$$n^2 + 300n \in O(2n^2\sqrt{n})$$

$$\lim_{n\to\infty} \frac{n^2+300n}{2n^2\sqrt{n}} = \lim_{n\to\infty} \frac{2n+300}{2(2.5)n^{1.5}} = \lim_{n\to\infty} \frac{2}{5(1.5)n^{0.5}} = 0$$

$$\lim_{n\to\infty} \frac{2n^2\sqrt{n}}{6n^3-n^2} = \lim_{n\to\infty} \frac{2 \cdot n^{0.5}}{6n-1} = \lim_{n\to\infty} \frac{1}{n^{0.5}(6)} = 0$$

$$\lim_{n\to\infty} \frac{n^3+8n^2}{6n^3-n^2} = \lim_{n\to\infty} \frac{n+8}{6n-1} = \lim_{n\to\infty} \frac{1}{6} = \frac{1}{6}$$

Problem 2

Simplify the following expressions such that a) all logarithms are base 2 and b) terms are expressed as sums and multiplication where possible (remove exponents). It may be helpful to recall the following:

$$\log_b(mn) = \log_b(m) + \log_b(n), \quad \log_b\left(\frac{m}{n}\right) = \log_b(m) - \log_b(n)$$

$$\log_b(n^p) = p\log_b(n), \quad \log_b(n) = \frac{log_p(n)}{log_p(b)}$$

a. $\log_3(x^2y^3) - \log_2(\sqrt{z})$

b. $\log_3\left(\frac{(t+5)}{(t^4)}\right)$

c. $\log_2(2^x) + \log_4(\frac{x}{5y})$

Let replest lg_2 as lg in the following.

ⓐ $lg_3(x^2y^3) - lg_2(\sqrt{z})$

$= \dfrac{lg\, x^2y^3}{lg\, 3} - \frac{1}{2} lg\, z = \dfrac{1}{lg\, 3}(2 lg\, x + 3 lg\, y) - \frac{1}{2} lg\, z$

ⓑ. $lg_3\left(\dfrac{t+5}{t^4}\right)$

$= lg_3(t+5) - lg_3 t^4 = lg_3(t+5) - 4\, lg_3 t$

$= \dfrac{1}{lg\, 3}(lg(t+5) - 4\, lg\, t)$

ⓒ. $lg\, 2^x + lg_4\left(\dfrac{x}{5y}\right)$

$= x + \dfrac{lg_2\left(\frac{x}{5y}\right)}{lg_2 4} = x + \frac{1}{2}(lg\, x - lg\, 5y)$

Problem 3

Order the following problems in increasing growth rate, using Big-O and Big-Theta notation.

$$n\ln(3n),\quad 2^{2n},\quad n^3+7,\quad n^3+2^{5n}$$

It may be helpful to remember the product rule: $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$.

$$2^{2n} \in O(n^3+2^{5n}), \qquad n^3+7 \in O(2^{2n}),$$

$$n\ln(2n) \in O(n^3+7)$$

$$\lim_{n\to\infty} \frac{n\ln(3n)}{n^3+7} = \lim_{n\to\infty} \frac{\ln(3n) + \cancel{n}\cdot\frac{1}{\cancel{3n}}\cdot\cancel{3}}{2n+7} = \lim_{n\to\infty} \frac{1}{2n} = 0$$

$$\lim_{n\to\infty} \frac{n^3+7}{2^{2n}} = \lim_{n\to\infty} \frac{n^3+7}{e^{2n\ln 2}} = \lim_{n\to\infty} \frac{3n^2}{e^{2n\ln 2}\cdot(2\ln 2)}$$

$$= \lim_{n\to\infty} \frac{6n}{e^{2n\ln 2}(2\ln 2)^2}$$

$$= \lim_{n\to\infty} \frac{6}{e^{2n\ln 2}(2\ln 2)^3} = 0$$

$$\lim_{n\to\infty} \frac{n^3+2^{5n}}{2^{2n}} = \lim_{n\to\infty} \frac{n^3}{2^{2n}} + \frac{2^{5n}}{2^{2n}}$$

$$= 0 + \lim_{n\to\infty} 2^{3n} = \infty$$

Problem 4

Analyze the worst-case runtime of the following algorithms. Derive the runtime complexity $T(n)$ and give an asymptotic upper bound $f(n)$ such that $T(n) = O(f(n))$.

a.

```
procedure algs_are_fun_1(integer n):
    for i = n, i >= 1
        i = i - 1
        print 'outer'
        for j = 1, j <= 500
            j = j + 2
            print 'inner 1'
        for k = 1, k <= 2i
            k = k + 1
            print 'inner 2'
```

Answer the following Questions for each loop.

1. How many iterations before finishing the loop?
2. How many steps to do in each iteration?

loop J:

$1 + S_j * 2 > 500 \Rightarrow S_j = 250$

5 steps for each iteration in loop J.

loop k:

$1 + S_k > 2i \Rightarrow S_k = 2i$

5 steps for each iteration in loop K.

loop i:

$n - S_i < 1 \Rightarrow n - 1 < S_i \Rightarrow S_i = n$

4 steps + 250 loop J + $2i$ loop k for each iteration in loop i.

$$\sum_{i=1}^{n} 4 + 250(4) + 2i(4)$$

$$= 4(251)n + 4\sum_{i=1}^{n} 2i$$

$$= 4(251)n + 4\cdot\cancel{2}\frac{(1+n)(n)}{\cancel{2}} = 4n^2 + 1256n$$

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$= \Theta(n^2)$, which implies $O(n^2)$.

$$\lim_{n\to\infty} \frac{4n^2 + 1256n}{n^2} = \lim_{n\to\infty} \frac{8n + 1256}{2n} = \lim_{n\to\infty} \frac{8}{2} = 4$$

b.

```
procedure algs_are_fun_2(integer n):
  for i = 1, i <= n
      i = i + 2
      print 'outer'
      for j = 1, j <= i
          j = 2j
          print 'inner'
```

loop j:

$1 * 2^{s_j} > i \Rightarrow s_j > \lg_2 i \Rightarrow s_j = \lfloor \lg_2 i \rfloor + 1$

4 steps for each iteration in loop j

loop i:

$1 + 2 s_i > n \Rightarrow s_i > \frac{n-1}{2} \Rightarrow s_i = \lfloor \frac{n-1}{2} \rfloor + 1$

4 steps $+ (\lfloor \lg_2 i \rfloor + 1)$ loop j

$$\sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor + 1} 4 + (\lfloor \lg_2 i \rfloor + 1) 4$$

$$= 10 \left(\lfloor \frac{n-1}{2} \rfloor + 1 \right) + 4 \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor + 1} \lfloor \lg_2 i \rfloor$$

$$= 10 + 10 \lfloor \frac{n-1}{2} \rfloor + 4 \lg_2 \left(\prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor + 1} i \right)$$

$$= 10 + 10 \lfloor \frac{n-1}{2} \rfloor + 4 \lg_2 \left[\left(\lfloor \frac{n-1}{2} \rfloor + 1 \right)! \right]$$

Let assume $\lfloor \frac{n-1}{2} \rfloor + 1 = m$

$$= 10m + 4\lg_2 (m)!$$

$$\approx 10m + 4\lg_2\left[\left(\frac{m}{e}\right)^m \sqrt{2\pi m}\right]$$ using stirling's approximation

$$= 10m + \underline{4m\lg_2\left(\frac{m}{e}\right)} + \frac{1}{2}\lg_2(m) + \frac{1}{2}\lg_2(2\pi)$$

$$\lim_{m\to\infty} \frac{4m\lg_2\left(\frac{m}{e}\right) + 10m + \frac{1}{2}\lg_2(m) + \frac{1}{2}\lg_2(2\pi)}{m\lg m}$$

$$= \lim_{m\to\infty} \frac{4\lg_2\left(\frac{m}{e}\right) + 4\cancel{m}\cdot\frac{e}{\cancel{m}} + 10 + \frac{1}{2}\cdot\frac{1}{m}}{\lg m + 1}$$

$$= \lim_{m\to\infty} \frac{\frac{4e}{m} - \frac{1}{2}m^{-2}}{\frac{1}{m}}$$

$$= \lim_{m\to\infty} \frac{8em - 1}{2m}$$

$$= \lim_{m\to\infty} \frac{8e - 1}{2} = \frac{8e-1}{2} \Rightarrow$$ It's a constant.

$$\Theta(m\lg m) = \Theta\left(\left(\left\lfloor\frac{n-1}{2}\right\rfloor + 1\right)\lg\left(\left\lfloor\frac{n-1}{2}\right\rfloor + 1\right)\right)$$

$$= \Theta(n\lg n)$$