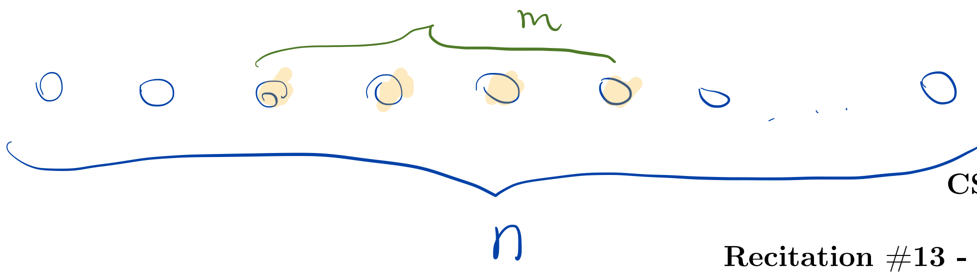


**CSCI 3104: Algorithms**
**Spring 2022**
**Recitation #13 - Midterm 2 Review II**

**SOLUTIONS**

## Problem 1

A mean wizard has turned you into a goose. Your new flock is preparing to migrate south for the winter, and you've been tasked with using your algorithm design skills to plan the best route.

Your migration path can be described as $n$ ponds, with pond $i$ having some amount $a_i > 0$ of yummy plants to eat. Each day, you can fly south from your current pond to any of the next $m$ ponds, where you will stop for the night and eat all $a_i$ plants. Each day of flying consumes some amount of energy $C$. Assume we start at pond 0, with $a_0 = 0$.

Your goal is to design a route that maximizes plants consumed less energy used flying. That is, you want to choose a set $P$ of ponds to rest at such that each resting place is at most $m$ ponds apart, the last resting place is pond $n$, and the energy surplus $\sum_{i \in P} a_i - C|P|$ is maximized.

a. Identify a subproblem for your flock.

> At pond $i$, you choose which of the next $m$ ponds to fly to the next day.

b. Define a recurrence for the maximum energy surplus you can achieve if you land at pond $i$.

> We work backward from pond $n$, considering the maximum surplus you can achieve at pond $i$ to be the maximum amount between ponds $i, \dots, n$. You can also work forward from pond 0.
>
> Define the optimal recurrence $\textsc{Opt}(i)$ = maximum energy surplus we can get starting at pond $i$ and going to pond $n$. We write $\textsc{Opt}(i)$ as
>
> $$\textsc{Opt}(i) = \begin{cases} a_n & \text{if } i = n \\ a_i - C + \max_{i < j \le \min(i+m, n)} \textsc{Opt}(j) & \text{if } i < n; \end{cases}$$
>
> That is, if we're already at pond $n$ we get $a_n$ plants and don't move. Otherwise, $\textsc{Opt}(i)$ is the energy we get from stopping at pond $i$ less the energy we expend flying to the next pond $j$, plus the maximum amount of energy we can get starting from pond $j$.

c. Use your recurrence to find the optimal energy surplus for your migration route for the following ponds, when $m = 3$ and $C = 5$. What route would you take?

| $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 14 | 10 | 4 | 11 | 5 | 3 | 6 | 14 | 7 |

We construct $\textsc{Opt}$ as follows, adding our base case when $i = n = 9$:

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\textsc{Opt}(i) =$ | | | | | | | | | | 7 |

We use our recurrence to fill out the table, moving from right to left:

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\textsc{Opt}(i) =$ | 32 | 37 | 28 | 22 | 23 | 17 | 15 | 17 | 16 | 7 |
| next stop | 1 | 2 | 4 | 4 | 5 or 7 | 7 | 7 | 8 | 9 | – |

By stopping at ponds 0, 1, 2, 4, 7, 8, and 9, we can achieve an energy surplus for our journey of 32. Note that while we have an entry with higher energy surplus, we cannot achieve that surplus while starting from pond 0.

d. **(Bonus.)** How might you modify your recurrence if your flock also insisted the migration take at most $d$ days?

We can add a dimension to our recurrence to represent how many days we've used so far. We define $\textsc{Opt}(i, j)$ = the maximum energy surplus achievable flying from pond $i$ to $n$ in at most $j$ days, and write the recurrence

$$\textsc{Opt}(i,j) = \begin{cases} a_n & \text{if } j = 0, i = n \\ -\infty & \text{if } j = 0, i < n \\ a_i - C + \max_{i < k \leq \min(i+m,n)} \textsc{Opt}(k, j-1) & \text{otherwise} \end{cases}$$

We use $-\infty$ to indicate an infeasible migration route: we cannot make it from any pond other than pond $n$ to pond $n$ in 0 days. Our base cases are $-\infty$ for all cases of 0 days, except when we start at pond $n$ in which case we have exactly $a_n$ plants to eat. Otherwise, we find the optimal surplus of routes which take $j - 1$ days we can reach from our starting pond.

Our maximal energy surplus will be $\textsc{Opt}(0, d)$, the maximum energy surplus achievable in at most $d$ days, starting from pond 0.