Recitation 8: Recurrence Relations

This week we will be studying methods for finding the asymptotics of the runtime of an algorithm. The way we do this is by writing down a ***recurrence relation*** which relates a larger instance of a problem to a smaller instance of that problem. We can then unroll this recurrence relation down to its base case, and count the amount of unrolling that we need to do.

For example, the recurrence relation

$$T(n) = T(n-1) + 1$$

can be unrolled further:

$$T(n) = T(n-1) + 1 = T(n-2) + 2 = T(n-3) + 3 = ... = T(1) + (n-1) = n$$

0 Some basic pointers

a. We usually care about asymptotic upper bounds (i.e. big-O bounds) for recurrence relations, as opposed to exact solutions

b. When a recurrence describes the runtime of an algorithm, one can usually assume that the base case of the recurrence takes time $O(1)$, or even just 1

c. Repeated substitution should be the first tool that you try for solving a recurrence

1 Repeated Substitution Exercises

Unroll the following recurrences by using these steps:

a. Determine the number of times that we need to unroll the recurrence before reaching the base case

b. Write down several iterations of the recurrence

c. Identify the pattern

d. Simplify the result

1.1

$$T(n) = \begin{cases} 3T(n-4) + 2n & n > 4 \\ 4 & n \leq 4 \end{cases}$$

1. *Determine the number of times to unroll.* Each iteration reduces the input size by 4, so we solve $n - 4d \leq 4$ for d. We unroll $d = (n-4)/4$ times.

2. *Write out several iterations.*

$$\begin{aligned} T(n) &= 3T(n-4) + 2n \\ &= 9T(n-8) + 3 \cdot 2(n-4) + 2n \\ &= 27T(n-12) + 9 \cdot 2(n-8) + 3 \cdot 2(n-4) + 2n \end{aligned}$$

3. *Identify the pattern.* At the ith level of unrolling, we add $3^i 2(n-4i)$ to our runtme.

$$T(n) = \sum_{i=0}^{(n-4)/4} 3^i 2(n-4i)$$

4. *Simplify.* Use the identity $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$:

$$\begin{aligned} T(n) &= \sum_{i=0}^{(n-4)/4} 3^i 2(n-4i) \\ &= 2n \sum_{i=0}^{(n-4)/4} 3^i - 12 \sum_{i=0}^{(n-4)/4} i^2 \\ &= 2n \frac{(1 - 3^{(n-4)/4})}{1-3} - 2\left(\left(\frac{(n-4)}{4}\right)\left(\frac{(n-4)}{4} + 1\right)\left(\frac{(n-4)}{2} + 1\right)\right) \\ &= n\left(3^{(n-4)/4} - 1\right) - \left(\frac{(n-4)}{2}\right)\left(\frac{n}{4}\right)\left(\frac{(n-2)}{2}\right) \\ &= n\left(3^{(n-4)/4} - 1\right) - \frac{n(n-2)(n-4)}{16} \end{aligned}$$

1.2

$$T(n) = \begin{cases} 4T(n-5) + n/4 & n > 6 \\ 3 & n \le 6 \end{cases}$$

1. *Determine the number of times to unroll.* We unroll d times, until $n - 5d \le 6$. Solving for d, we get $d = (n-6)/5$.

2. *Write out several iterations.*

$$\begin{aligned} T(n) &= 4T(n-5) + n/4 \\ &= 16T(n-10) + 4(n-5)/4 + n/4 \\ &= 64T(n-15) + 16(n-10)/4 + 4(n-5)/4 + n/4 \end{aligned}$$

3. *Identify the pattern.* On the ith layer, we add $4^i(n-5i)/4$ to our runtime:

$$T(n) = \sum_{i=0}^{(n-6)/5} 4^i(n-5i)/4$$

4. *Simplify.*

$$\begin{aligned} T(n) &= \sum_{i=0}^{(n-6)/5} 4^i(n-5i)/4 \\ &= n \sum_{i=0}^{(n-6)/5} 4^i - \frac{5}{4} \sum_{i=0}^{(n-6)/5} i4^i \end{aligned}$$

Use the identity $\sum_{i=0}^{k} ix^i = \frac{x(kx^{k+1} - (k+1)x^k + 1)}{(x-1)^2}$:

$$\begin{aligned} T(n) &= n\frac{1 - 4^{(n-6)/5}}{1-4} - \frac{5}{(4-1)^2}\left(\frac{(n-6)}{5}4^{(n-6)/5+1} - \frac{(n-1)}{5}4^{(n-6)/5} + 1\right) \\ &= n\frac{4^{(n-6)/5} - 1}{3} - \left((n-6)4^{(n-6)/5+1} - (n-1)4^{(n-6)/5} + \frac{1}{9}\right) \\ &= n\frac{4^{(n-6)/5} - 1}{3} - \left((n-6)4^{(n-6)/5+1} - (n-1)4^{(n-6)/5} + \frac{1}{9}\right) \end{aligned}$$

1.3

$$T(n) = \begin{cases} 8T(n-2) + n^2 & n > 10 \\ 5 & n \leq 10 \end{cases}$$

1. *Determine the number of times to unroll.* We unroll d times, until $n - 2d \leq 10$. Solving for d, we get $d = (n-10)/2$.

2. *Write out several iterations.*

$$...$$

3. *Identify the pattern.* On the ith layer, we add $8^i(n-2i)^2$ to our runtime:

$$T(n) = \sum_{i=0}^{(n-10)/2} 8^i(n-2i)^2$$

4. *Simplify.*

$$\begin{aligned} T(n) &= \sum_{i=0}^{(n-10)/2} 8^i(n-2i)^2 \\ &= \sum_{i=0}^{(n-10)/2} 8^i(n^2 - 4ni + 4i^2) \\ &= n^2 \sum_{i=0}^{(n-10)/2} 8^i - 4n \sum_{i=0}^{(n-10)/2} i8^i + 4 \sum_{i=0}^{(n-10)/2} i^2 8^i \end{aligned}$$

Use the identites from the previous part, and also

$$\sum_{i=0}^{k} i^2 x^i = \frac{(x(1 + x - x^k(1 - 2k(-1 + x) + k^2(-1 + x)^2 + x)))}{(1-x)^3}$$

2 Tree Method

You may have noticed that the recurrences we solved by unrolling were of the form

$$T(n) = aT(n - b) + cn^d$$

and each step can be unrolled from the previous one directly. Sometimes this is not the case, especially in recurrences of the form

$$T(n) = aT\left(\frac{n}{b}\right) + cn^d$$

These recurrences can be more easily solved by drawing a tree. The root of this tree corresponds to $T(n)$, and it branches into a children, each corresponding to an instance $T(n/b)$. A cost of cn^d is added to each node.

2.1

Try to solve the following recurrence by drawing a tree:

$$T(n) = \begin{cases} 3T(n/6) + n^2 & n > 3 \\ 5 & n \leq 3 \end{cases}$$

a. Draw the first few layers of the tree

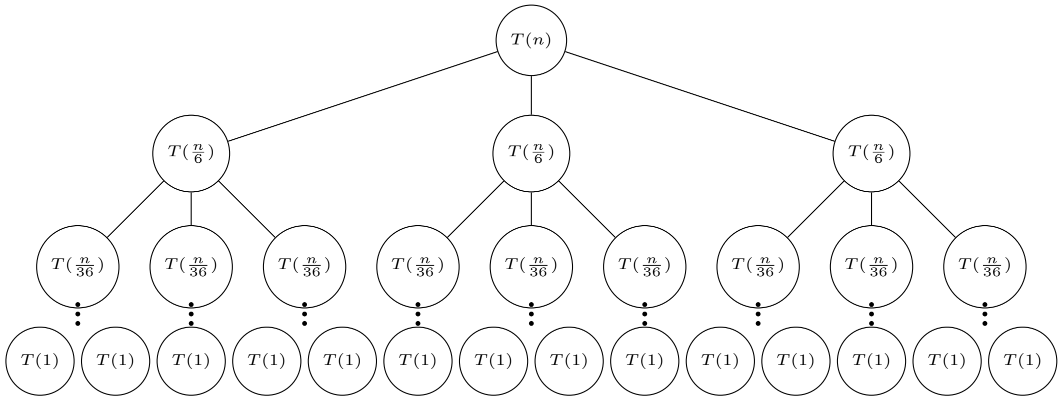
b. How many iterations does it take to reach the base case?

c. Write down a few iterations by hand

d. Identify the pattern

e. Simplify the result

1. *Draw the tree.* We draw the tree of *input sizes* for each vertex, rather than the amount of work done at each vertex (which is also acceptable).



2. *Determine the depth.* We reach our base case at the dth layer, when $n/6^d \leq 3$. Thus, we have depth $d \geq \log_6 n - \log_6 3$.

3. *Sum the work.* At layer i, there are 3^i vertices, each of which takes time $(n/6^i)^2 = n^2/6^{2i}$. For each base case, we do 5 work. Thus,

$$T(n) = \sum_{i=1}^{\log_6 n - \log_6 3} 3^i n^2/6^{2i} + 3^i \cdot 5$$

4. *Simplify.* We simplify this expression:

$$\begin{aligned}
T(n) &= n^2 \sum_{i=0}^{\log_6 n - \log_6 3} \left(\frac{1}{12}\right)^i \\
&= n^2 \left(\frac{1 - 12^{-\log_6 n + \log_6 3 - 1}}{1 - 12^{-1}}\right) \\
&= \frac{12}{11} n^2 \left(1 - 12^{-\log_6 n} 12^{\log_6 3 - 1}\right) \\
&= \frac{12}{11} n^2 \left(1 - \frac{12^{\log_6 3 - 1}}{12^{\frac{\log_{12} n}{\log_{12} 6}}}\right) = \frac{12}{11} n^2 \left(1 - \frac{12^{\log_6 3 - 1}}{n^{\frac{1}{\log_{12} 6}}}\right)
\end{aligned}$$