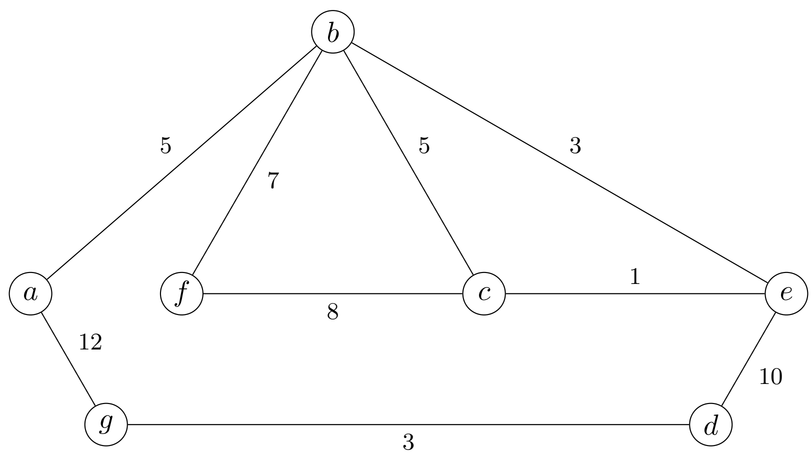
3 Standard 3 – Dijkstra's Algorithm

3.1 Problem2

Problem 2. Consider the undirected weighted graph $G(V, E, w)$ pictured below. Work through Dijkstra's algorithm on the following graph, using the source vertex a. **Note:** In order to get full credits consider the following:

- Clearly include the contents of the priority queue, the distance from a and the parent of each vertex at each iteration.
- If you use a table to store the distances, clearly label the keys according to the vertex names rather than numeric indices (i.e., `dist['B']` is more descriptive than `dist['1']`).
- You do **not** need to draw the graph at each iteration, though you are welcome to do so. [This may be helpful scratch work, which you do not need to include.]
- Finally represent the shortest path graph.



Answer. We proceed as follows.

(a) Our source node is a. We begin by setting the found distances for each node other than a to ∞. We also set a as processed.

(b) We begin by examining a's neighbors, updating their found distances and placing into the priority queue, ordering by found distance. So:

$$Q = [(b, 5), (g, 12)].$$

$nodes$	a	b	c	d	e	f	g
$distances$	∞	5	∞	∞	∞	∞	12
$parent$	NA	a	NA	NA	NA	NA	a

(c) We begin by examining b's neighbors, updating their found distances and placing into the priority queue, ordering by found distance. So:

$$Q = [(e, 8), (c, 10), (g, 12), (f, 12)].$$

$nodes$	a	b	c	d	e	f	g
$distances$	∞	5	10	∞	8	12	12
$parent$	NA	a	b	NA	b	b	a

(d) We begin by examining e's neighbors, updating their found distances and placing into the priority queue, ordering by found distance. So:

$$Q = [(c, 9), (g, 12), (f, 12), (d, 18)].$$

$nodes$	a	b	c	d	e	f	g
$distances$	∞	5	9	18	8	12	12
$parent$	NA	a	e	e	b	b	a

(e) We begin by examining c's neighbors, updating their found distances and placing into the priority queue, ordering by found distance. So:

$$Q = [(g, 12), (f, 12), (d, 18)].$$

$nodes$	a	b	c	d	e	f	g
$distances$	∞	5	9	18	8	12	12
$parent$	NA	a	e	e	b	b	a

(f) We begin by examining g's neighbors, updating their found distances and placing into the priority queue, ordering by found distance. So:

$$Q = [(f, 12), (d, 15)].$$

$nodes$	a	b	c	d	e	f	g
$distances$	∞	5	9	15	8	12	12
$parent$	NA	a	e	g	b	b	a

(g) We begin by examining f's neighbors, updating their found distances and placing into the priority queue, ordering by found distance. So:

$$Q = [(d, 15)].$$

$nodes$	a	b	c	d	e	f	g
$distances$	∞	5	9	15	8	12	12
$parent$	NA	a	e	g	b	b	a

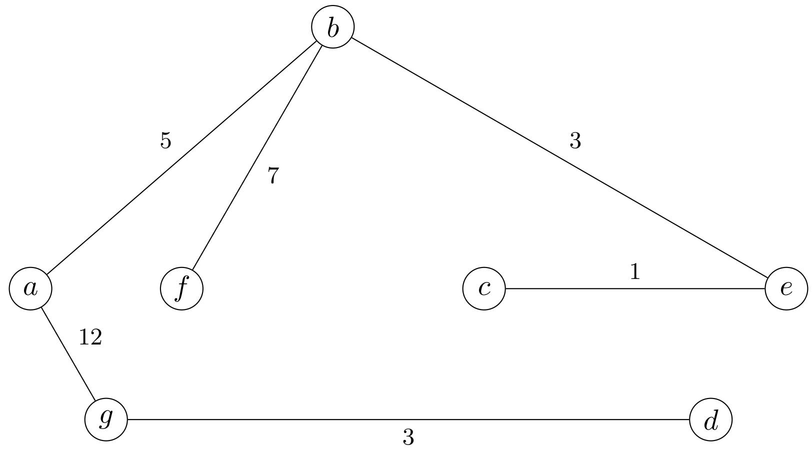
(h) We begin by examining d's neighbors, updating their found distances and placing into the priority queue, ordering by found distance. So:

$$Q = [].$$

$nodes$	a	b	c	d	e	f	g
$distances$	∞	5	9	15	8	12	12
$parent$	NA	a	e	g	b	b	a

(i) Finally, we see that the Priority Queue is empty, which is when we exit the algorithm.

The final Dijkstra's shorted path graph:



$\square$

Procedure B001 (Integer n).

for $i \leftarrow 1$; $i \leq n$; $i \leftarrow i * 3$ do

for $j \leftarrow i$; $j \leq n$; $j \leftarrow j+2$ do

print "Hi"

Inner loop has J iterations:

$$i + 2J > n \Rightarrow J \approx \frac{n-i}{2}$$

Each iteration of inner loop has 4 computations, including condition checking $(j \leq n)$, increment step $(j+2)$, updation step $(j \leftarrow j+2)$, printing step (print "Hi").

Hence, the inner loop has $\left(\frac{n-i}{2}\right) \cdot 4 = 2(n-i)$ executions.

Outer loop has K iterations.

$$1 \cdot 3^K > n \Rightarrow K \approx \lg_3 n$$

(At the iteration k of outer loop, $i = 3^{k-1}$)

Each iteration of outer loop requries executing the inner loop and also 3 computions, including condition checking, increment, and updation.

$$\sum_{k=1}^{\lg_3 n} 3 + 2(n-i) \overset{i=3^{k-1}}{=} \sum_{k=1}^{\lg_3 n} 3 + 2(n-3^{k-1})$$

$$= 3\lg_3 n + 2n \cdot \lg_3 n - \sum_{k=1}^{\lg_3 n} 3^{k-1}$$

$$a + ar + ar^2 + \cdots ar^{m-1} = \frac{a(1-r^m)}{1-r}$$

$$3^0 + 3^1 + \cdots + 3^{\lg_3 n - 1} = \frac{3^0(1-3^{\lg_3 n})}{1-3}$$

$$= 3\lg_3 n + 2n\lg_3 n - \frac{1(1-3^{\lg_3 n})}{1-3}$$

$$= 3\lg_3 n + 2n\lg_3 n - \frac{1-n^{\lg_3 3}}{-2}$$

$$= 3\lg_3 n + 2n\lg_3 n + \frac{1}{2} - \frac{n}{2}$$

$$= \Theta(n\lg n).$$

Hence the nested loop has $\Theta(n\lg n)$ executions.

2.5 Problem 5

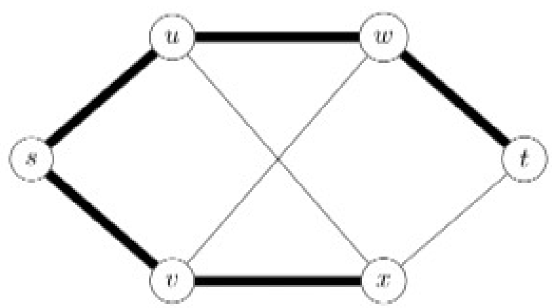
Problem 5. Give an example of a simple, undirected, and unweighted graph $G(V, E)$ that has a single source shortest path tree which a **breadth-first traversal** will not return for any ordering of its vertices. Your answer must

(a) Provide a drawing of the graph G. [**Note:** We have provided TikZ code below if you wish to use LaTeX to draw the graph. Alternatively, you may hand-draw G and embed it as an image below, provided that (i) your drawing is legible and (ii) we do not have to rotate our screens to grade your work.]

(b) Specify the single source shortest path tree $T = (V, E_T)$ by specifying E_T and also specifying the root $s \in V$. [**Note:** You may again hand-draw this tree. If you wish, you may clearly mark the edges of T on your drawing of G. Please make it easy on the graders to identify the edges of T.]

(c) Include a clear explanation of why the breadth-first search algorithm we discussed in class will never produce T for any orderings of the vertices.

Answer.



We go with the example graph with some modifications to make wide edges. So $V = \{s, t, u, v, w, x\}$ and:

$$E = \{\{s, u\}, \{s, v\}, \{u, w\}, \{u, x\}, \{v, w\}, \{v, x\}, \{w, t\}, \{x, t\}\}.$$

We have chosen $E_T = \{\{s, u\}, \{s, v\}, \{u, w\}, \{v, x\}, \{w, t\}\}$ indicated with bold edges. We assert that the given T is a spanning tree from s as all the vertices can be reached by a path originating from s. This is because T is connected and includes all of the vertices of G. Moreover, T is a single-source shortest-path spanning tree with source s: it is clear that there are no shorter paths to u and v since they are already at distance 1 from s in T. Likewise for w and x, they are not adjacent to s and so the shortest paths must have length at least 2. Finally, we see that there is no path of length 2 from s to t.

E_T can never be given by BFS starting from s. There are two cases.

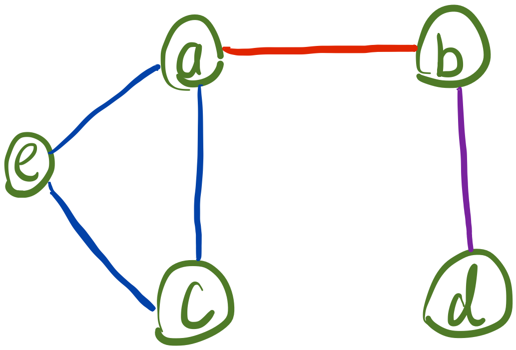
- **Case 1:** If $u < v$ in the ordering we are given, then we will first explore the vertices adjacent to u, which means that we would reach x from u, and T would need to include the edge $\{u, x\}$.
- **Case 2:** If $v < u$ in the ordering we are given, then we will first explore the vertices adjacent to v, which means that we would reach w from v, and T would need to include the edge $\{v, w\}$.

□

Let’s think about a class schedule problem. Assume there are 5 classes, which are a, b, c, d, e and 3 students, X, Y and Z. The course registry information is shown below.

Student X takes class a, b
Student Y takes class a, c, e
Student Z takes class b, d

Assume all the classes take 1 hour, then how many separate periods do we need to schedule courses?



Assume that we don’t color the same for vertices, who are neighbors to each other. Then, . . .

Could we color them using 5 colors?
Could we color them using 4 colors?
Could we color them using 3 colors?
Could we color them using 2 colors?
Could we color them using 1 color?

```
HUFFMAN(C)
n = |C|
Q = C
for i = 1 to n - 1
    allocate a new node z
    z.left = x = EXTRACT-MIN(Q)
    z.right = y = EXTRACT-MIN(Q)
    z.freq = x.freq + y.freq
    INSERT(Q, z)
return EXTRACT-MIN(Q)      // return the root of the tree
```