Recitation 1 Guide

Fall 2022

This week we are focusing on mathematical induction, so we should be going over the structure of inductive proofs and providing examples (and non-examples) of induction to the students.

1 Concrete Example

In class, students will have seen the inductive proof that, for all $n \in \mathbb{N}$

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$

We can call these propositions $\{P(n)\}_{n\in\mathbb{N}}$.

To solidify the understanding that students have of the mechanics of this proof, let's explicitly go through the proof that

$$P(5) \Rightarrow P(6)$$

which happens during the Inductive Step. In my experience, going through this has been very helpful for students who don't get the concept of induction at first.

When proving that $A \Rightarrow B$, we assume A is true, and then use that to prove B. Thus, in this instance, we will assume that $P(5)$ is true, and use that to prove $P(6)$.

ASK: Ask for a student to state what $P(5)$ is, and what $P(6)$ is.

Write out (5) and $P(6)$ on the board:

$$P(5): \quad 1 + 2 + 3 + 4 + 5 = \frac{5 \cdot 6}{2}$$

$$P(6): \quad 1 + 2 + 3 + 4 + 5 + 6 = \frac{6 \cdot 7}{2}$$

We can draw attention to how similar these propositions are, and then consider the expression

$$1 + 2 + 3 + 4 + 5 + 6$$

By assuming $P(5)$, we can rewrite this expression:

$$\underbrace{1 + 2 + 3 + 4 + 5}_{\frac{5 \cdot 6}{2}} + 6$$

$$= \frac{5 \cdot 6}{2} + 6$$

and then we can do some algebra:

$$= \frac{5 \cdot 6}{2} + \frac{2 \cdot 6}{2}$$

$$= \frac{5 \cdot 6 + 2 \cdot 6}{2}$$

$$= \frac{6 \cdot 7}{2}$$

so we have proven $P(6)$ by assuming that $P(5)$ is true.

2 Non-Examples of Induction

One important skill in being able to write correct inductive proofs is being able to spot flaws in inductive arguments and correct them if necessary. Below are some examples of flawed inductive arguments; present these to the students and ask them to discuss exactly where and what the errors are.

2.1 Every power of 5 is equal to 5

Proposition $P(n)$**:** $5^n = 5$

Proof. **Base case** $P(1)$**:**

$$5^1 = 5$$

We need to mention inductive hypothesis first.

Inductive Step $P(k) \Rightarrow P(k+1)$**:** Assume it is known to be true that

$$5^k = 5$$

Induction Hypothesis.

Let some $k \geq 1$, for each $1 \leq m \leq k$, assume $5^k = 5$

Then

$$5^{k+1} = 5^k \frac{5^k}{5^{k-1}}$$

$$= 5 \frac{5}{5}$$

(by strong inductive hypothesis)

$$= 5$$

$\square$

I.S.

$5^{k+1} = 5^k \cdot 5$

via IH $= 5 \cdot 5 = 25.$

This goes wrong for $k = 1$, since the expression

$$5^{k+1} = 5^k \frac{5^k}{5^{k-1}}$$

goes out of scope of quantification of the inductive proof: the base case is $P(1)$ but evaluating 5^{k-1} gives us 5^0 when $k = 1$.

The proposition is not correct.

2.2 All cats are the same colour

Proposition $P(n)$: given a set S of n cats, all cats in S are the same color.

Proof. **Base case $P(1)$:** For $n = 1$ cat, x_1, obviously all cats in the set $\{x_1\}$ have the same color

Inductive Step $P(k) \Rightarrow P(k+1)$: Suppose it is true that any set of k cats all have the same colour:

$$c(x_1) = c(x_2) = c(x_3) = ... = c(x_k)$$

Consider a set of cats

$$\{r_1, r_2, ..., r_k, r_{k+1}\}$$

Remove

$$r_{k+1}$$

and by inductive hypothesis, we have

$$c(r_1) = c(r_2) = c(r_3) = ... = c(r_k)$$

By strong inductive hypothesis, we also have

The equality holds only from the base case to case k. hence the following statement is invalid.

$$c(r_k) = c(r_{k+1})$$

and so by transitivity of equality, we have

$$c(r_1) = c(r_2) = c(r_3) = ... = c(r_k) = c(r_{k+1})$$

□

This goes wrong at $n = 2$ cats, since there is no overlapping subset of cats to which transitivity of equality can be applied.

3 Example: All trees on n vertices have $n-1$ edges

Definition 3.1. Recall that a tree is a connected graph with no cycles.

Definition 3.2. A **flag** in a graph is a pair (v, e) where v is a vertex and e is an edge incident on v.

The purpose of this example is to illustrate some specifics about how the inductive hypothesis is applied. Namely, that we must first fix a graph on $k+1$ vertices, remove a leaf flag to get a tree on k vertices, and THEN apply the inductive hypothesis. If we do the proof by adding a flag to a tree on k vertices, we have not proven a universal statement (*for all* trees on $k+1$ vertices). We have simply exhibited one tree on $k+1$ vertices for which the theorem holds, and this is not sufficient. When there are multiple objects of each size k, we need to take an approach like this.

Base case: consider the single-node graph, which has 1 node and 0 edges. $P(1)$ is true.

Inductive Step $P(k) \Rightarrow P(k+1)$: Let $T = (V, E)$ be any tree on $k+1$ vertices (*any* tree meaning we are proving a statement FOR ALL trees on $k+1$ vertices). We remove a leaf flag from T to create a graph T'. (Why can we always find a leaf?)

T' has k vertices, and so by the inductive hypothesis, T' has $k-1$ edges. Now, we may add the flag back to T', which gives us T.

By construction, T has one more vertex and one more edge than T'. Thus T satisfies the theorem as well, since it has $k+1$ vertices and $(k-1)+1 = k$ edges.

3.2 Problem 3

Problem 3. Consider the sequence T_n, $n \geq 1$ defined by the following recurrence: $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$.

Prove by induction that $T_n < 2^n$ for all $n \geq 1$.

Proof. The proof is by induction on $n \geq 1$.

- **Base Case:** This holds for when n is 1, 2, 3 or 4. For $n = 1$, $T_1 = 1 < 2^1$. For $n = 2$, $T_2 = 1 < 2^2$. For $n = 1$, $T_3 = 1 < 2^3$. Furthermore, for $n = 4$,

$$\begin{aligned} T_4 &= T_1 + T_2 + T_3 \\ &= 1 + 1 + 1 \\ &= 3 \\ &< 16 \\ &= 2^4 \end{aligned}$$

- **Inductive Hypothesis:** Let some $k \geq 1$, for each $1 \leq m \leq k$, assume $T_m \leq 2^m$.

- **Inductive Step:** We show it also holds for $k + 1$. We write the LHS for the k+1 case:

$$\begin{aligned} T_{k+1} &= T_{k-2} + T_{k-1} + T_k && (5) \\ &< 2^{k-2} + 2^{k-1} + 2^k && (6) \\ &= 7 * 2^{k-2} && (7) \\ &< 8 * 2^{k-2} && (8) \\ &= 2^{k+1} && (9) \end{aligned}$$

where we apply IH to obtain equation (5) from (6).

□

3.3 Problem 4

Problem 4. The complete, balanced ternary tree of depth d, denoted $\mathcal{T}(d)$, is defined as follows.

- $\mathcal{T}(0)$ consists of a single vertex.
- For $d > 0$, $\mathcal{T}(d)$ is obtained by starting with a single vertex and setting each of its three children to be copies of $\mathcal{T}(d-1)$.

Prove by induction that $\mathcal{T}(d)$ has 3^d leaf nodes. To help clarify the definition of $\mathcal{T}(d)$, illustrations of $\mathcal{T}(0), \mathcal{T}(1)$, and $\mathcal{T}(2)$ are on the next page. [**Note:** $\mathcal{T}(d)$ is a tree and **not** the number of leaves on the tree. Avoid writing $\mathcal{T}(d) = 3^d$, as these data types are incomparable: a tree is not a number.]

Proof. The proof is by induction on $n \geq 1$.

- **Base Case:** This holds for when $d = 0$. $\mathcal{T}(0)$ has 1 leaf node and $3^0 = 1$.
- **Inductive Hypothesis:** For some k, assume $\mathcal{T}(k)$ has 3^k leaf nodes.
- **Inductive Step:** We show it also holds for $k+1$. According to the definition, $\mathcal{T}(k+1)$ has a single vertex as the new root and three children, which are copies of $\mathcal{T}(k)$. The single vertex is not a leaf node. From the IH, we know every children $\mathcal{T}(k)$ has 3^k leaf nodes. Thus $\mathcal{T}$ has $3(3^k) = 3^{k+1}$ leaf nodes. The properties still hold.

$\square$