CSCI3104
Spring 2023

University of Colorado Boulder

Recitation 8: Dynamic programming subproblems

Question 0. (Christos Papadimitriou, Sanjoy Dasgupta, and Umesh Vazirani) We are given a checkerboard which has 4 rows and n columns, and has an integer written in each square. We are also given a set of $2n$ pebbles, and we want to place some or all of these on the checkerboard (each pebble can be placed on exactly one square) so as to maximize the sum of the integers in the squares that are covered by pebbles. There is one constraint: for a placement of pebbles to be legal, no two of them can be on horizontally or vertically adjacent squares (diagonal adjacency is fine).

(a) Determine the number of legal patterns that can occur in any column (in isolation, ignoring the pebbles in adjacent columns) and describe these patterns.

Call two patterns compatible if they can be placed on adjacent columns to form a legal placement. Let us consider subproblems consisting of the first k columns $1 \leq k \leq n$. Each subproblem can be assigned a type, which is the pattern occurring in the last column.

(b) Using the notions of compatibility and type, give an $O(n)$-time dynamic programming algorithm for computing an optimal placement.

Answer: (a) There are 8 possible patterns: the empty pattern, the 4 patterns which each have exactly one pebble, and the 3 patterns that have exactly two pebbles (on the first and fourth squares, the first and third squares, and the second and fourth squares).

(b) For each pattern, there are a constant number of patterns that are compatible with it (for example, every pattern is compatible with the empty pattern). Define $c_j[i]$ to be the optimal value achievable by pebbling columns $1, \ldots, i$ such that the final column has pattern j. Then for any $j, c_j[i+1]$ is the value of the squares covered by j in column $i+1$, plus the maximum value of $c_{j'}[i]$, subject to j' being compatible with j. (This claim can be proven by a standard cut-and-paste argument: if not, then replace the first i columns with a higher-valued pebbling that still ends in pattern j', then pebble column $i+1$ with pattern j to get a higher-valued pebbling than the original.) The base cases are $c_j[0] = 0$ for all column patterns j. Furthermore, $c_j[i+1]$ can be computed in $O(1)$ time by examining the $O(1)$ values $c_{j'}[i]$ and adding the values of the $O(1)$ squares pebbled by j. From here, the dynamic programming algorithm is clear: keep 8 separate arrays (one for each column pattern) of n elements. For $i = 1, \ldots, n$, compute $c_j[i]$ for each of the 8 values of j as described above. To reconstruct the actual pebbling, find the maximum value c from $c_j[n]$, and pebble the nth column according to some j such that $c = c_j[n]$. Then subtract the value of the pebbled squares from c, and search for c among $c_j[n-1]$, etc. The running time of this algorithm is $O(n)$ because filling each of the (constant number of) arrays takes $O(n)$ time, and backtracking takes $O(1)$ time per column. Note that it is not sufficient to keep only the the largest value achievable by pebbling the first i columns (irrespective of the pattern in its final column). This is because it might be possible to get very high value from the $(i+1)$ st column, but only by using a pattern incompatible with the best pebbling of the first i columns.

Question 1. Suppose we have the standard 26-letter English alphabet, $\Sigma = \{a, b, \ldots, y, z\}$. Let W_n be the set of strings of length n which do not contain the word "yay":

$$W_n = \{\omega \in \Sigma^n : \omega_i\omega_{i+1}\omega_{i+2} \neq \text{ yay}, \forall i = 1, \ldots, n-2\}.$$

Write a recurrence for $f_n = |W_n|$, including base cases, to count the number of character strings of length n that do not contain the word yay. (The notation Σ^n means the set of any n characters from the alphabet Σ concatenated. So $\{x, y\}^3 = \{xxx, xxy, xyx, xyy, yxx, yxy, yyx, yyy\}$.)

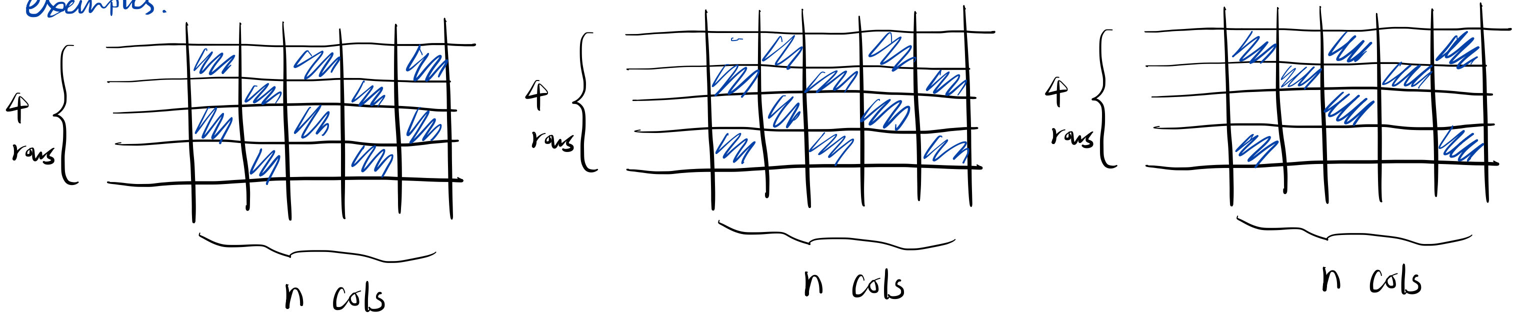
Question 0. (Christos Papadimitriou, Sanjoy Dasgupta, and Umesh Vazirani) We are given a checkerboard which has 4 rows and n columns, and has an integer written in each square. We are also given a set of $2n$ pebbles, and we want to place some or all of these on the checkerboard (each pebble can be placed on exactly one square) so as to maximize the sum of the integers in the squares that are covered by pebbles. There is one constraint: for a placement of pebbles to be legal, no two of them can be on horizontally or vertically adjacent squares (diagonal adjacency is fine).

(a) Determine the number of legal patterns that can occur in any column (in isolation, ignoring the pebbles in adjacent columns) and describe these patterns.

Call two patterns compatible if they can be placed on adjacent columns to form a legal placement. Let us consider subproblems consisting of the first k columns $1 \leq k \leq n$. Each subproblem can be assigned a type, which is the pattern occurring in the last column.

(b) Using the notions of compatibility and type, give an $O(n)$-time dynamic programming algorithm for computing an optimal placement.

examples.



(a)

Place 2 stones :

2A : 1st, 3rd

2B : 2nd, 4th

2C : 1st, 4th

1 stone :

1A : 1st

1B : 2nd

1C : 3rd

1D : 4th

In addition, we can also place 0 stones. 0A.

There are 8 patters.

2A is compatible with 2B, 1B, 1D, 0A

2B is compatible with 2A, 1A, 1C, 0A

2C is compatible with 1B, 1C, 0A.

1A is compatible with 2B, 1B, 1C, 1D, 0A

1B is = 2A, 2C, 1A, 1C, 1D, 0A

1C is = 2B, 2C,

	2A	2B	2C	1A	1B	1C	1D	0A
2A		▨			▨		▨	▨
2B	▨			▨		▨		▨
2C					▨	▨		▨
1A		▨			▨	▨	▨	▨
1B	▨		▨	▨		▨	▨	▨
1C		▨	▨	▨	▨		▨	▨
1D	▨			▨	▨	▨		▨
0A	▨	▨	▨	▨	▨	▨	▨	▨

$$
\begin{array}{ll}
OPT(n) = \max \{ & OPT(n-1, 2A) + val(Neighbor(2A)), \\
& OPT(n-1, 2B + val(\ " \ (2B)), \\
& \ " \ , 2C + val(\ " \ (1A)), \\
& \ " \ , 1A + val(\ " \ (1B)), \\
& , 1B + val \\
& \ " \ , 1D + val(\ " \ (1D)), \\
& \ " \ , 0A + val(\ " \ (0A)), \}
\end{array}
$$

$$OPT(n-1, x) = \left(\max_{y \in N(x)} OPT(n-2, y) \right) + val(x)$$

	1	2	3	....	n
2A	val(2A)				
2B	val(2B)				
1A	⋮				
1B	⋮				
1C	⋮				
1D	⋮				
0A					

Answer: We begin by determining some base cases: If $n = 0$, then $W_n = \{\varepsilon\}$, where ε denotes the empty string, and we say that $f_0 = 1$. (Note that sometimes it can be notationally helpful to define base cases such as $n = -1$, which doesn't make sense interpreted as a string length but may be meaningful in a recursion.)

If $n = 1$, then we have 26 possible values for the first character and $W_1 = \Sigma, f_1 = 26$.

If $n = 2$, then we still have no restrictions on the characters and $W_2 = \Sigma^2$ and $f_2 = 26^2 = 676$.

We now consider $n \geq 3$. We denote a string in Σ^n by $\omega_1\omega_2 \ldots \omega_{n-1}\omega_n$, and break the problem into the first characters plus the rest of the string.

Here we consider several cases:

$\omega_1 \neq$ y. There are 25 cases in which this happens, and in each one we have no restrictions on the rest of the string, so there are $25\,|W_{n-1}|$ strings in which this happens.

$\omega_1 =$ y. In this case, we cannot count all possible strings of length $n - 1$ that do not contain yay as possible postfixes to ω_1 : if we contatenate y with aye we get yaye which is not a valid string in W_n, despite aye being a valid word in W_{n-1}. Instead, we break this into several additional cases:

$\omega_2 \neq$ a, y. There are 24 cases in which the second letter is not an a or y, and in each case we have no restrictions on the remaining $n - 2$ letters. Thus, there are $24\,|W_{n-2}|$ strings in which this case happens.

$\omega_2 =$ a. In this case, we have $\omega_1\omega_2 =$ ya, so we know that $\omega_3 \neq$ y. There are 25 remaining possibilities for ω_3, none of which run the risk of spelling out yay in $\omega_3\omega_4\omega_5$ for any choice of ω_4 and ω_5, so we have $25\,|W_{n-3}|$ possible strings starting from ω_3.

$\omega_2 =$ y. In this case, we must be careful not to pick a string beginning ay.

We recurse again, paying attention again to cases where $\omega_3 =$ a, y:

$\omega_3 \neq$ a,y. As above, there are 24 characters that ω_3 can be where we don't have to care about what comes next. This case can happen in $24\,|W_{n-3}|$ ways.

$\omega_3 =$ a. Again, we must ensure that $\omega_4 \neq$ y, and have no further restrictions. This contributes $25\,|W_{n-4}|$ strings.

$\omega_3 =$ y. We have to recurse again in this case ensuring the next character is not an a, and if it is a y selecting it carefully. We continue this recurrence for each time we get another y.

We repeat this pattern until we end with y and are safe to pick anything else: when $\omega_{n-2} =$ y, we have $25f_1$ choices when $\omega_{n-1} \neq a$ and 25 choices when $\omega_{n-1} = a$. We can now write a recurrence for $n \geq 3$:

$$\begin{aligned} f_n &= 25f_{n-1} + 25f_{n-2} + 25f_{n-3} + 24f_{n-3} + \cdots + 25f_2 + 24f_2 + 25f_1 + 25 \\ &= 25f_{n-1} + 49\sum_{i=2}^{n-2} f_i + 25f_1 + 25 \end{aligned}$$

we leave this as-is, since computing this recurrence is beyond the scope of this class. Thus, our full recurrence is

$$f_n = \begin{cases} 1 & n = 0 \\ 26 & n = 1 \\ 26^2 & n = 2 \\ 25f_{n-1} + 24f_{n-2} + 49\sum_{i=2}^{n-3} f_i + 25f_1 + 25 & n \geq 3 \end{cases}$$

Question 2. You've decided to leave CS to pursue a career in train robbery as an adventurer. You've been observing the train schedules in the Boulder area, and have a pretty good idea of what trains will be running in the next month, and the approximate value of each train's cargo.

Over the next month, you know there will be n trains running in your target area, with train i carrying cargo worth some value v_i. Unfortunately, you expect the law to be close on your heels; you've decided after each heist it's best to lay low and leave the next 2 trains alone to avoid getting caught.

Give a dynamic programming algorithm to determine the maximum amount of loot you'll be able to make off with in the next month.

i) Identify the subproblem to solve.

ii) Define a recurrence for V_i, the total value of loot you can boost over trains $i, i+1, \ldots, n$. Include your base cases.

iii) Say there are 12 trains running this month, with values

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
20	18	6	8	15	8	4	23	7	9	13	16

Use your recurrence to compute the maximum loot value you can get this month. What is the maximum value? How could you modify this to give a schedule for your train robbery, as well as your optimal value?

Answer:

i) On the i th train, we choose whether to 1) rob the train or 2) let it go by.

ii) First, we define the value we get from each of the two choices we can make in our subproblem:

$$\begin{array}{ll} V_{i+1} & \text{if we don't hold up train } i \\ v_i + V_{i+3} & \text{if we do hold up train } i \end{array}$$

Thus, the best we can do on the last i trains is

$$V_i = \begin{cases} \max\left(V_{i+1}, v_i + V_{i+3}\right) & 1 \leq i \leq n \\ 0 & i > n, \end{cases}$$

since we get 0 value from robbing trains that don't run (trains after the nth train).

iii) We create an array with an entry for each train we might rob, with the i th entry storing V_i. We also add entries for V_{13}, V_{14}, and V_{15} for our base cases, which we initialize to 0 according to our recurrence:

$i =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$V_i =$													0	0	0

(1)

We fill out our lookup table working backward from $i = n$ according to our recurrence: $V_{12} = \max(0, 16 + 0) = 16, V_{11} = \max(16, 13+0) = 16, V_{10} = \max(16, 9+0) = 16$, $V_9 = \max(16, 7+16) = 23$, etc. This gives us the table

$i =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$V_i =$	74	72	54	54	54	39	39	39	23	16	16	16	0	0	0

(2)

and we read off the maximum value we can rob from $V_1 = 74$. We can modify the table to add an indicator rob(i) showing whether or not we chose to rob train i:

$i =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$V_i =$	74	72	54	54	54	39	39	39	23	16	16	16	0	0	0
rob(i) =	Y	Y	N	N	Y	N	N	Y	Y	N	N	Y	N	N	N

(3)

Using these indicators, we can find an ordering of heists to commit by starting at $i = 1$: if rob(i) = Y, we add train i to our to-rob list and skip to $i + 3$. If rob(i) = N, we move to $i + 1$. This gives us the robbery schedule of trains $1, 5, 8$, and 12.

Question 4. If time permits, then see Levet lecture notes pp. 107-110.