CSCI 3104 Fall 2022
Instructors: Prof. Grochow and Chandra Kanth Nagesh

Midterm S22

Due Date .... Saturday Nov 19, 2022 4pm MT
Name .... **Your Name**
Student ID .... **Your Student ID**
Quiz Code (enter in Canvas to get access to the LaTeX template) **bsuqP8j6Bi**

Contents

Instructions

- You may either type your work using this template, or you may handwrite your work and embed it as an image in this template. **If you choose to handwrite your work, the image must be legible, and oriented so that we do not have to rotate our screens to grade your work.** We have included some helpful LaTeX commands for including and rotating images commented out near the end of the LaTeX template.
- You should submit your work through the **class Gradescope page** only. Please submit one PDF file, compiled using this LaTeX template.
- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this document with no fewer pages than the blank template (or Gradescope has issues with it).
- You **may not collaborate with other students**. **Copying from any source is an Honor Code violation. Furthermore, all submissions must be in your own words and reflect your understanding of the material.** If there is any confusion about this policy, it is your responsibility to clarify before the due date.
- Posting to **any** service including, but not limited to Chegg, Discord, Reddit, StackExchange, etc., for help on an assignment is a violation of the Honor Code.
- You **must** virtually sign the Honor Code. Failure to do so will result in your assignment not being graded.

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Problem HC.

- My submission is in my own words and reflects my understanding of the material.
- Any collaborations and external sources have been clearly cited in this document.
- I have not posted to external services including, but not limited to Chegg, Reddit, StackExchange, etc.
- I have neither copied nor provided others solutions they can copy.

Agreed (signature here). □

22 Standard 22: Dynamic Programming: Writing Down Recurrence

Problem 22. The Placing Electrons on a Tree problem is defined as follows.

- Instance: A tree rooted tree T, with root $r \in V(T)$.
- Solution: A subset $S \subseteq V(T)$ such that there are no edges (s, s') for $s, s' \in S$, and such that $|S|$ is as large as possible.

The *idea* is that you are placing electrons on the vertices of a tree, but electrons repel each other, so you cannot place two electrons adjacent to one another, with the goal being to place as many electrons as possible.

For each vertex $v \in V(T)$, let $L[v]$ denote the maximum number of electrons that can be placed on the subtree rooted at v.

Write down a mathematical recurrence for $L[v]$. Clearly justify each case. *Hint:* your recurrence should involve the children and grandchildren of v. You may use notation such as "children(v)" to denote the set of children of v, and "grandchildren(v)" to denote the set of grandchildren of v. Remember to have base case(s).

Answer. For the problem of expressing $L[v]$, we claim to do so via

$$L[v] = \begin{cases} \max(\sum_{\hat{v} \in \text{child}(v)} L[\hat{v}], 1 + \sum_{\hat{v} \in \text{grand}(v)} L[\hat{v}]) & \text{child}(v) \neq \emptyset \\ 1 & \text{child}(v) = \emptyset \end{cases}. \tag{1}$$

Observe, given a root vertex v and assuming child$(v) \neq \emptyset$, we end up in two possible cases: (1) the value of $L[v]$ includes counting v or (2) the value of $L[v]$ does not include counting v.

Case 1, $L[v]$ includes counting v: If $L[v]$ includes counting v, then by definition of child(v) there exists edges $(v, \hat{v})$ for all $\hat{v} \in$ child(v). Hence, no vertices from child(v) can be included in the set of vertices which make up the count for $L[v]$. Thus, $L[v]$ must be the made up of the count of the optimal values of the trees made from the vertices of grand(v) and v itself in the count of $L[v]$. In all, this can be expressed by $1 + \sum_{\hat{v} \in \text{grand}(v)} L[\hat{v}])$.

Case 2, $L[v]$ does not include counting v: Leveraging the complimentary reasoning of Case 1, $L[v]$ must be the made up of the count of the optimal values of the trees made from the vertices of child(v) which can be expressed by $\sum_{\hat{v} \in \text{child}(v)} L[\hat{v}]$.

Given our objective of maximizing the cardinality of the chosen subset of vertices which satisfies our edge constraints, we choose $\max(\sum_{\hat{v} \in \text{child}(v)} L[\hat{v}], 1 + \sum_{\hat{v} \in \text{grand}(v)} L[\hat{v}])$ for a provided $L[v]$. Finally, in the case where child$(v) = \emptyset$, we are provided a single node so trivially, the max cardinality is one. Thus, proving our claim that equation 1 expresses $L[v]$. □