Recitation 5: Flows & Cuts

This week we will be studying flow networks. A flow network is a specific type of weighted directed graph: it is a graph with a designated source vertex $s \in V$, and a designated target vertex $t \in V$. The weights of the edges are often called "capacities" instead of "weights", suggesting that we think of these edges as roads, pipelines, and other means of transportation. There are two relevant problems we study with flow networks: **Maximum Flow** and **Minimum Cut**.

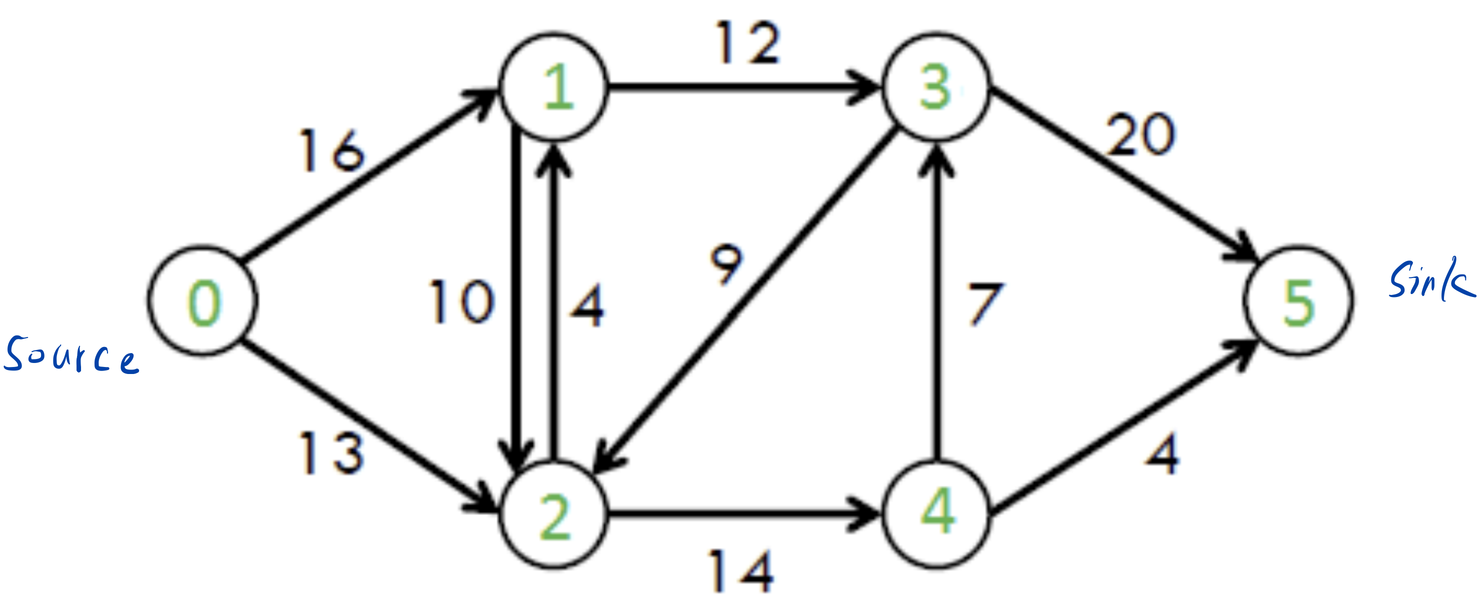
0 Definitions & Review

Definition 0.1. Formally, a flow network $G = (V, E, w, s, t)$ is a graph where

- V is a set of vertices
- $E \subseteq V \times V$ is a set of directed edges
- $c : E \to \mathbb{R}_{\geq 0}$ is a function assigning non-negative real-number weights (capacities) to the edges. Sometimes this function is called w instead of c.
- $s \in V$ is a designated source vertex, and may only have outgoing edges
- $t \in V$ is a designated target vertex, and may only have ingoing edges

Sink

Example 0.2. A flow network, with $c(e)$ written next to each edge e, e.g. $c(3 \to 5) = 20$. Here, the source s is vertex 0 and the target t is vertex 5.



Definition 0.3. A **flow** on a flow network is a function

$$f : E \to \mathbb{R}$$

satisfying certain constraints:

1. For every edge e, it holds that $f(e) \leq c(e)$.

   This says that the flow on an edge cannot exceed its capacity (e.g. a road cannot carry more traffic than its capacity. a wire cannot carry more current than its capacity. a pipeline cannot carry more oil than its capacity)

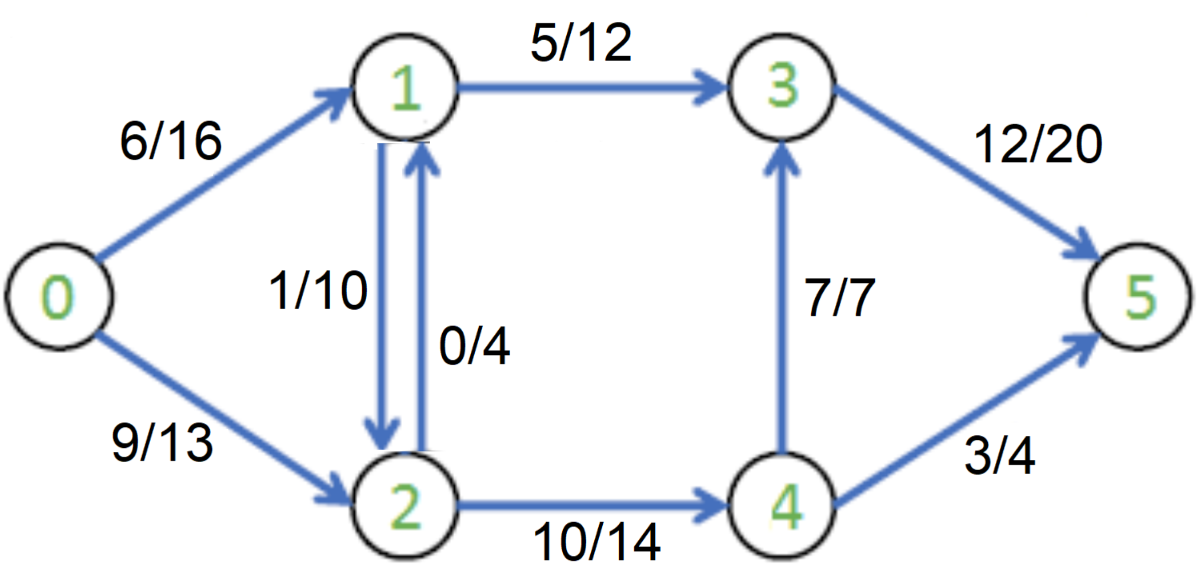
2. For every vertex $v \in (V \setminus \{s, t\})$, Kirchhoff's Law holds:

$$\sum_{e \in in(v)} f(e) = \sum_{e \in out(v)} f(e)$$

   This says that for every vertex besides s and t, the amount of flow going into that vertex must be equal to the amount of flow coming out of it. This is a conservation law.

   The **value** of a flow is defined as either the amount of flow coming out of s, or the amount of flow going into t. These quantities are always equal to each other.

Example 0.4. An example of a flow on the above flow network would be:



The numerators denote the flow on each edge, and the denominators denote the capacity of each edge. Notice how the flow on each edge does not exceed that edge's capacity, and that every non-source non-target vertex has an equal amount of flow coming in and going out. The value of this flow is 15.

1 Maximum Flow

The flow in the network above does not have the greatest possible value: we are only pushing 15 units of flow from s to t, whereas we could be pushing more. The general problem is, given a flow network, find the maximum value flow on that network.

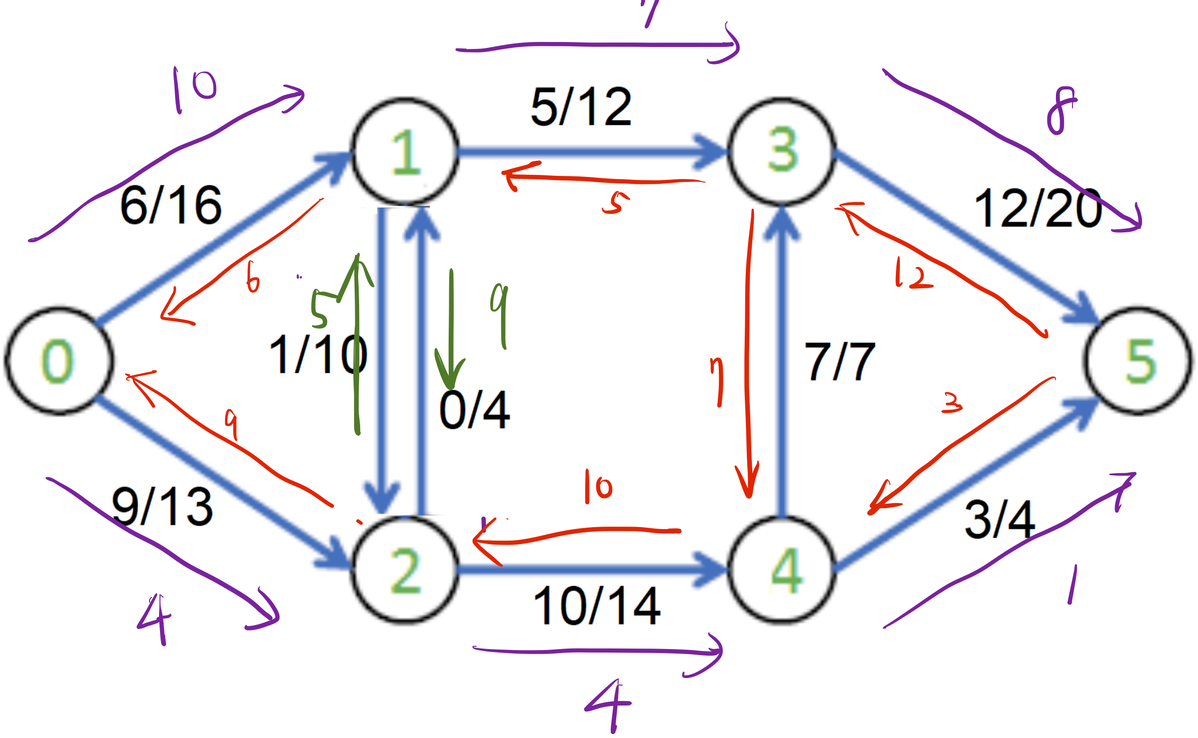
1.1 Augmenting Paths

The strategy for finding a maximum network flow is a greedy one: we will simply start looking for paths, along whose edges we can increase the flow by some constant amount.

Definition 1.1. An augmenting path is a path $s \to v_1 \to v_2 \to \cdots \to v_k \to t$ where each edge $(v_i, v_i + 1)$ has positive difference between its capacity and its flow.

The algorithmic strategy for finding a max flow is to start with a non-maximum flow, find an augmenting path p, update the flow by increasing the flow along p, and then repeating this process.

Exercise 1.2. Find an augmenting path in the following flow network:



Exercise 1.3. Update the flow network to reflect the augmenting path you found

Exercise 1.4. Find the maximum flow on this network by repeatedly finding augmenting paths

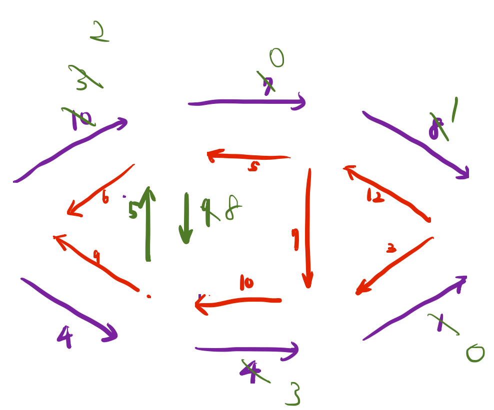
Augmenting paths

$0 \to 1 \to 3 \to 5$ 7 unit

$0 \to 1 \to 2 \to 4 \to 5$ 1 unit

$0 \to 2 \to 1 \to 3 \to 5$ 4 unit

$0 \to 2 \to 4 \to 5$ 1 unit



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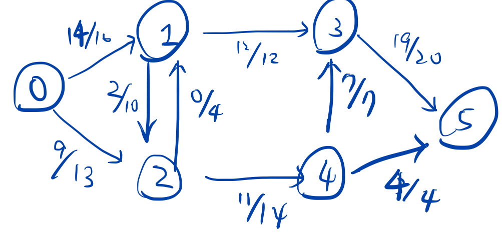


2 Cuts

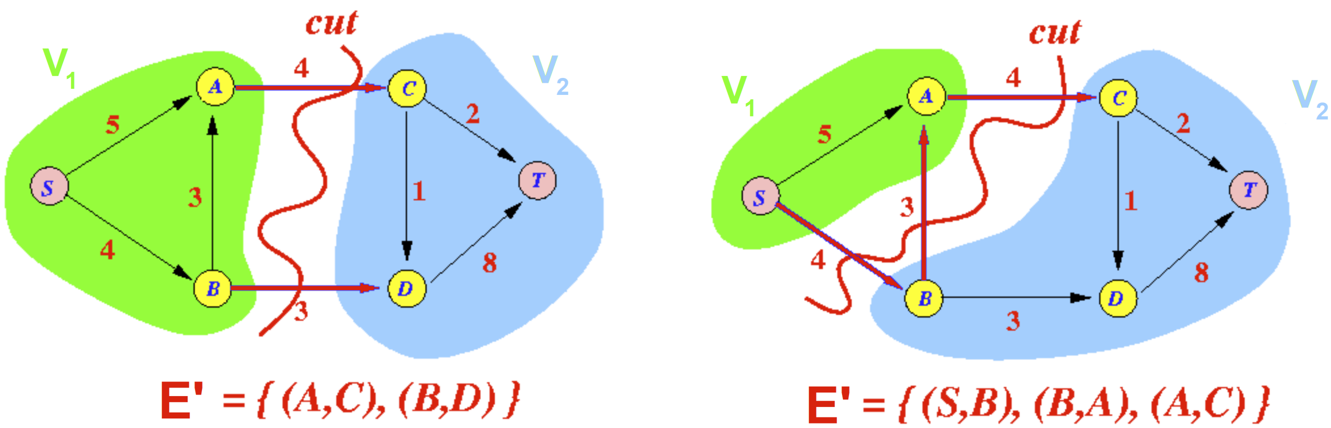
There are two ways to define a cut of a flow network:

Definition 2.1. Let $G = (V, E, c, s, t)$ be a flow network. A **cut** in G is a partition of V into two disjoint sets $V = V_1 \oplus V_2$ (that is, $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \varnothing$) where $s \in V_1$ and $t \in V_2$.

Definition 2.2. Let $G = (V, E, c, s, t)$ be a flow network. A **cut** in G is a subset of edges $E' \subseteq E$ such that removing the edges in E' disconnects s from t.

These definitions are equivalent: if you can find a partition of V into $V_1 \oplus V_2$, then E' is going to be the set of edges starting in V_1 and ending in V_2. Conversely, from E' one can recover a unique partition of V. You can think of cuts in either way you prefer.

Example 2.3. Two different cuts on the same flow network:



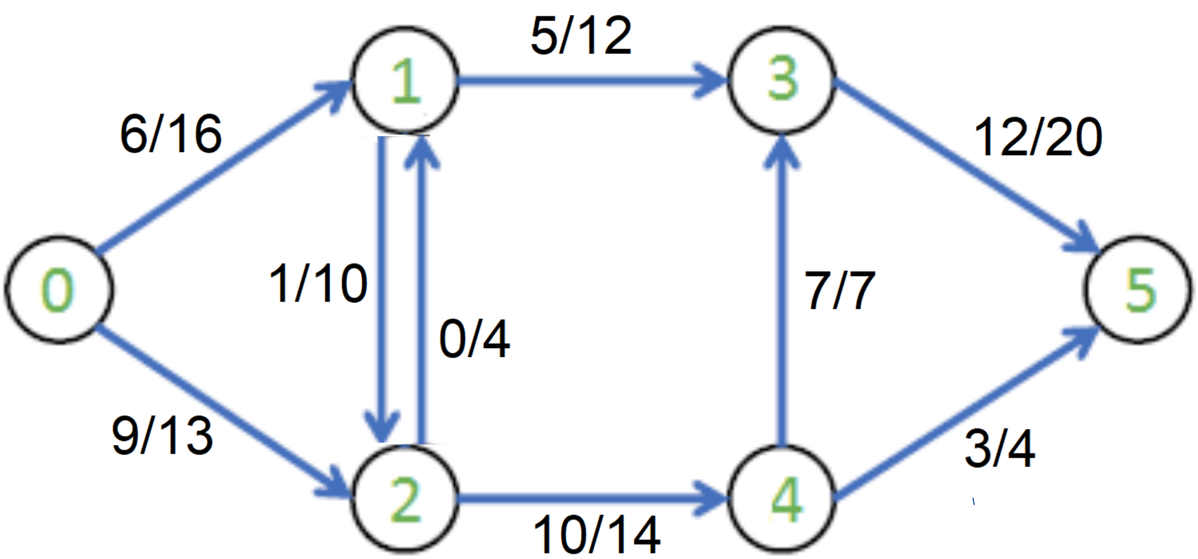
Definition 2.4. The **value** of a cut is the sum of the capacities of edges in E'. In the image above, the left cut has value 7 and the right cut has value 11.

Problem 2.5. The **Minimum Cut Problem** is the problem of finding the cut with minimum value. This is a saboteur's problem: disconnect the network with the smallest amount of obstruction.

2.1 Exercises

{(3→5), (4,3), {2,4} }

1. In the following network, find a cut with value 41
2. Find a cut with value 29 {(0,1), (0,2)}
3. Find a cut with value 23 {(1,3), (4,3), (4,5)}
4. Can you find a cut with value less than 23?



We might be interested in proving that there is no cut with value less than 23. This brings us to the Max-Flow-Min-Cut theorem.

Theorem 2.6. *Let $G = (V, E, c, s, t)$ be a flow network. Let f be a flow on G, and let E' be a cut on G. Then*

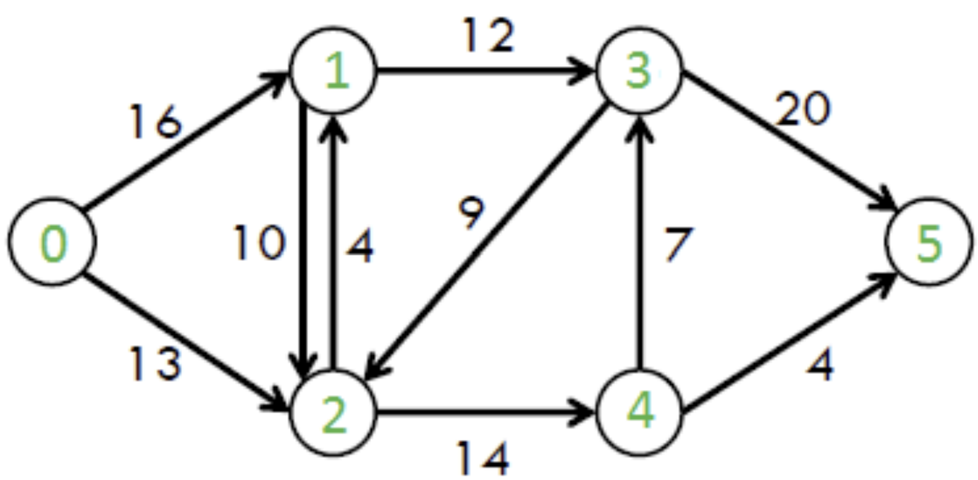
$$value(f) \leq value(E')$$

This theorem tells us that we can prove that there is no cut with value less than 23 by exhibiting a flow with value 23.

3 Correspondence & Duality

Maximum Flow and **Minimum Cut** are problems which are **dual** to each other. When we find a maximum flow, we implicitly find a minimum cut, and vice versa. We will practice finding a minimum cut, given a maximum flow.

Consider the flow network below, in which we found a maximum flow.



Exercise 3.1. How might we find a cut, given the maximum flow we found? Consider that, if the flow is maximum, then there is no augmenting path from s to t. How could we make use of this fact to define a partition of V?

Exercise 3.2. Find a maximum flow, and the corresponding minimum cut, in the following graph: