Recitation 2 Guide: Graph Theory

Fall 2022

This week students will be learning about Breadth-First Search, Depth-First Search, and Dijkstra's algorithm. They have encountered these algorithms previously in CSCI 2270 Data Structures, although we would like to review them with more mathematical rigour. To that end, we should review the basic definitions and concepts of graph theory this week. There are many pages here but a lot of the beginning stuff is review and can be skimmed.

1 Sets

Definition 1.1. Let A be a set. We denote the set of unordered pairs of elements of A as

$$\binom{A}{2}$$

The reason for this notation is that

$$\left|\binom{A}{2}\right| = \binom{|A|}{2}$$

the number of unordered pairs of elements of A

i.e. if $|A| = n$ then $\left|\binom{A}{2}\right| = \binom{n}{2}$

Definition 1.2. Let A and B be sets. The cartesian product of A and B, denoted

$$A \times B$$

is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

The reason for this notation is that

$$|A \times B| = |A| \times |B|$$

the number of unordered pairs, which has one member from A, and the other from B.

i.e. if $|A| = k$ and $|B| = p$ then $|A \times B| = k \times p$

Noteworthy Distinction: $\binom{A}{2}$ is the set of unordered pairs of elements of A, whereas $(A \times A)$ is the set of ordered pairs of elements of A.

Please note that the answer could be different if using different definitions on graphs.

2 Graphs: Definitions

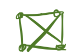
Definition 2.1. A (simple, undirected) graph is defined as a pair of sets $G = (V, E)$ where V is any set (commonly the set $[n] = \{1, 2, ..., n\}$) and $E \subseteq \binom{V}{2}$. V is called the set of vertices, and E is called the set of edges.

The edges of an undirected graph can be viewed as symmetric relationships between pairs of vertices (e.g. intersections being connected by a two-way road, accounts being friends on facebook, sports teams playing each other in a tournament, etc).

Question: what is the maximum number of edges that a simple undirected graph can have if it has n vertices?

$\binom{|V|}{2} = \binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$

$\frac{4!}{2!\,2!} = \frac{4\cdot 3}{2} = 6$



Definition 2.2. A directed graph (digraph) is defined as a pair of sets $G = (V, E)$ where V is any set (commonly the set $[n] = \{1, 2, ..., n\}$) and $E \subseteq (V \times V)$. Again, V is called the set of vertices, and E is called the set of edges.

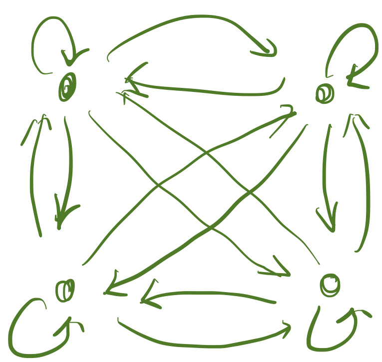
The edges of a directed graph can be viewed as asymmetric relationships between pairs of vertices (e.g. intersections being connected by a one-way road, one twitter account following another, one sports team beating another in a tournament, etc). Note that in this definition we allow edges from a vertex to itself, which is not necessarily always the case.

Question: what is the maximum number of edges that a digraph can have if it has n vertices?

$|V \times V| = |V| \times |V| = n^2$

Definition 2.3. (Adjacency) Let $G = (V, E)$ be a graph. For $v, w \in V$ we say that $v \sim w$ if v and w are connected by an edge, i.e. $(v, w) \in E$.

Definition 2.4. (Degree) Let $G = (V, E)$ be a graph. The degree $\deg(v)$ of a vertex $v \in V$ is the number of edges containing v as one of its endpoints



$|V \times V| = |V| \times |V|$

$= 4 \times 4$

$= 16$

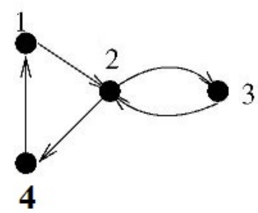
2.1 Example graph

A visual representation of the directed graph $G = (V, E)$ where

$$V = \{1, 2, 3, 4\}$$

and

$$E = \{(1 \to 2), (2 \to 3), (2 \to 4), (3 \to 2), (4 \to 1)\}$$



Vertex 1 is adjacent to vertex 2, but vertex 1 is not adjacent to vertex 3

$$1 \sim 2 \quad ; \quad 1 \not\sim 3$$

$$\deg(1) = 2 \quad ; \quad \deg(2) = 4 \quad ; \quad \deg(3) = 2 \quad ; \quad \deg(4) = 2$$

Question: Why is it that the sum of degrees is equal to twice the number of edges?

$$\sum_{v \in V} \deg(v) = 2|E|$$

Each edge has two endpoints.

3 Examples of Graphs

3.1 Paths

Definition 3.1. The undirected path of length n, P_n, is a graph whose vertex set is

$$V(P_n) = [n+1] = \{1, 2, ..., n, (n+1)\}$$

and whose edges are

$$E(P_n) = \{(i, i+1) \mid 1 \leq i \leq n\}$$

Example 3.2. P_3, the path of length 3

with edges $(1, 2), (2, 3), (3, 4)$

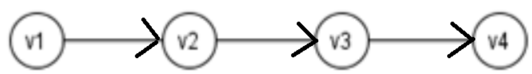
Definition 3.3. The directed path of length n, $\vec{P}_n$, is a graph whose vertex set is

$$V(\vec{P}_n) = [n+1] = \{1, 2, ..., n, (n+1)\}$$

and whose edges are

$$E(\vec{P}_n) = \{i \to i+1 \mid 1 \leq i \leq n\}$$

Example 3.4. $\vec{P}_3$, the directed path of length 3



with edges $(1 \to 2), (2 \to 3), (3 \to 4)$

3.2 Cycles

The undirected and directed cycle graphs are defined analogously:

Definition 3.5. The directed cycle of length n, C_n, is a graph whose vertex set is

$$V(C_n) = [n] = \{1, 2, ..., n\}$$

and whose edges are

$$E(C_n) = \{(i, i+1 \mod n) \mid 1 \leq i \leq n\}$$

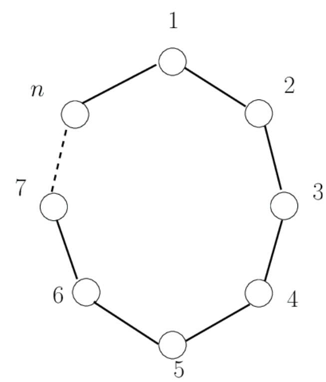
Definition 3.6. The undirected cycle of length n, $\vec{C_n}$, is a graph whose vertex set is

$$V(\vec{C_n}) = [n] = \{1, 2, ..., n\}$$

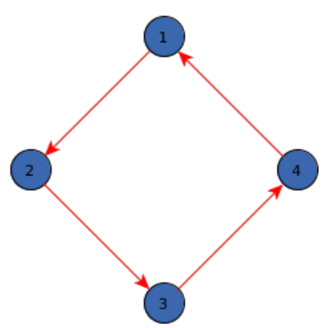
and whose edges are

$$E(\vec{C_n}) = \{i \to i + 1 \mod n \mid 1 \leq i \leq n\}$$

Example 3.7. C_n

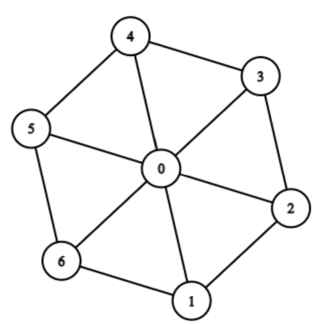


Example 3.8. $\vec{C_4}$



3.3 Wheels

The wheel graph W_n consists of a cycle graph C_n, with an additional vertex which is connected to every other vertex. Below is a drawing of W_5 :

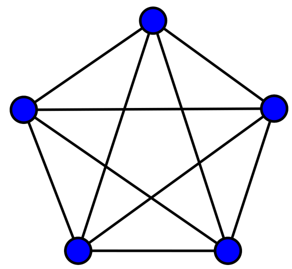


3.4 Complete Graphs

A complete graph is a graph which has an edge between every possible pair of vertices. The complete directed and undirected graphs on n vertices are denoted K_n and $\vec{K}_n$, respectively.

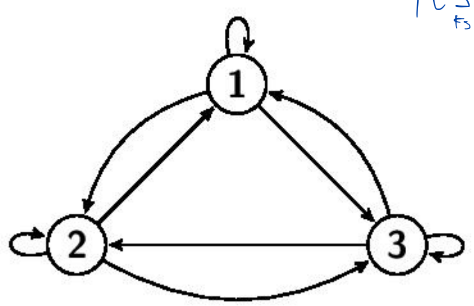
Example 3.9. K_5

$|E_{K5}| = \binom{5}{2} = \frac{5 \cdot 4}{2} = 10$



Example 3.10. $\vec{K}_3$

$|E_{\vec{K}_3}| = 3 \cdot 3 = 9$



4 Subgraphs

We are often interested in finding paths, cycles, wheels, and complete graphs INSIDE of other graphs – e.g. we might want to know if there is a path between two vertices, and if so, what the shortest path might be. Finding paths and cycles in graphs is generally a pretty easy problem, but finding complete subgraphs (a.k.a. cliques) of a given graph is computationally hard, as we will see later in the course.

Definition 4.1. (Embedding) Given graphs $H = (V_H, E_H)$ and $G = (V_G, E_G)$, we say that H embeds in G if there is a function

$$f : V_H \to V_G$$

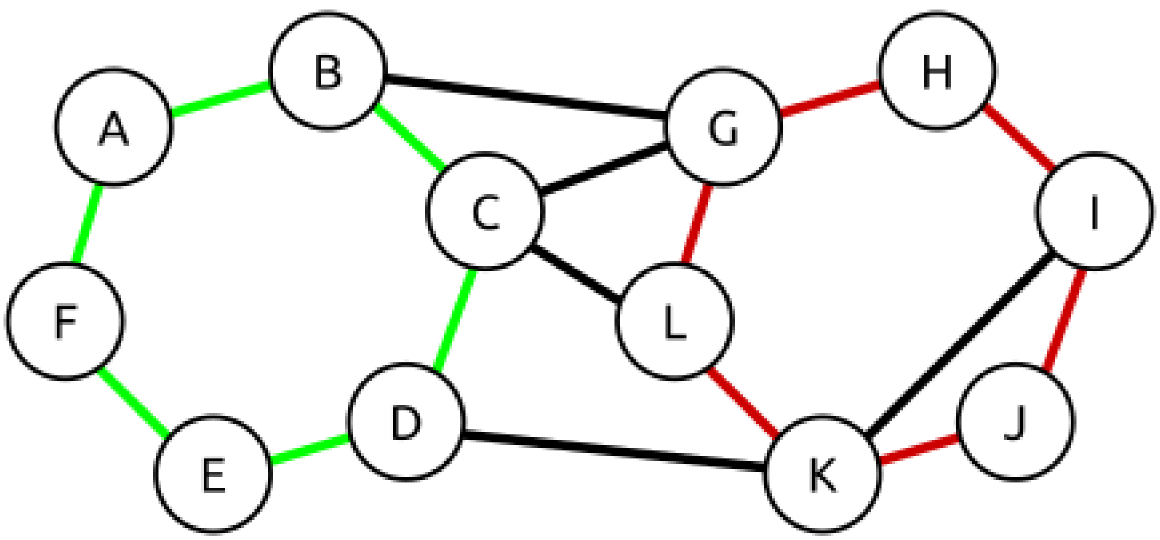
such that

$$(v, w) \in E_H \Longrightarrow \big(f(v), f(w)\big) \in E_G$$

In other words, if v is adjacent to w in H, then $f(v)$ should be adjacent to $f(w)$ in G.

Example 4.2. C_6 embeds in the following graph, with

$$f(1) = A, f(2) = B, f(3) = C, f(4) = D, f(5) = E, f(6) = F$$



This function is an embedding because $1 \sim 2$ so $f(1) \sim f(2)$, and $2 \sim 3$ so $f(2) \sim f(3)$, and so on...

Question: Can you find an embedding of C_4 into the above graph?

$f(1) = C$, $f(2) = L$, $f(3) = K$, $f(4) = D$

How about C_3's ? embeddings?

Definition 4.3. (Path in G) We say that a graph G contains a path of length n if P_n embeds in G. We say that there is a path from v to w if P_n embeds in G with $f(1) = v$ and $f(n+1) = w$.

Because P_n has edges $(i, i+1)$, this allows us to think of a path P_n in G as a sequence of vertices

$$f(1), f(2), ...f(n+1)$$

such that $f(i)$ is adjacent to $f(i+1)$ in G for all i.

Example 4.4. There is a path of length 4 from B to J in the above graph because P_4 embeds in G with $f(1) = B$ and $f(5) = J$. In fact, there are two paths of length 4 from B to J. What are they? (B, C), (C, L), (L, K), (K, J) and ?

Definition 4.5. We say that a graph G contains a cycle of length n if C_n embeds in G.

Definition 4.6. We say that a graph G contains a complete graph on n vertices if K_n embeds in G.

Example 4.7. Breadth-First Search (BFS) and Depth-First Search (DFS) both answer the question "Is there a path between v and w?". Equivalently, "is there some n such that P_n embeds in G, with $f(1) = v$ and $f(n) = w$?"

Example 4.8. Dijkstra's algorithm solves the problem of finding the smallest-length path between a given pair of vertices v and w.

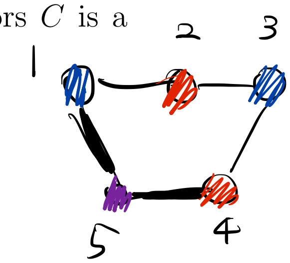
5 Graph Coloring

A coloring of a graph is an assignment of colours to its vertices such that no two adjacent vertices receive the same color. A problem of algorithmic interest is coloring a graph with the minimum possible number of colors.

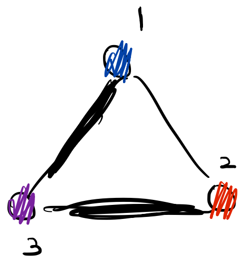
Definition 5.1. Formally, a colouring of a graph $G = (V, E)$ with a set of colors C is a function

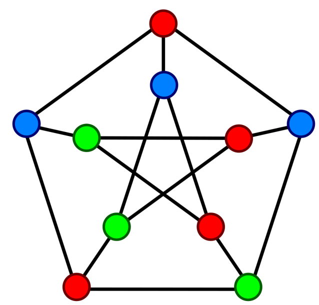
$$f : V \to C$$

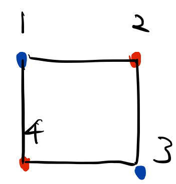
such that for all $(u, v) \in E$, we have $f(u) \neq f(v)$.

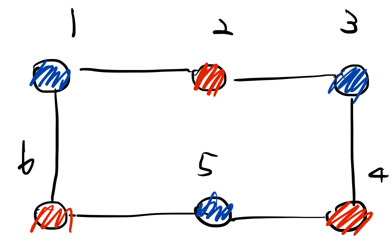


Example 5.2. The following graph is correctly coloured with three colors.





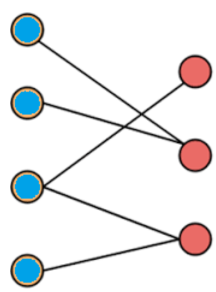


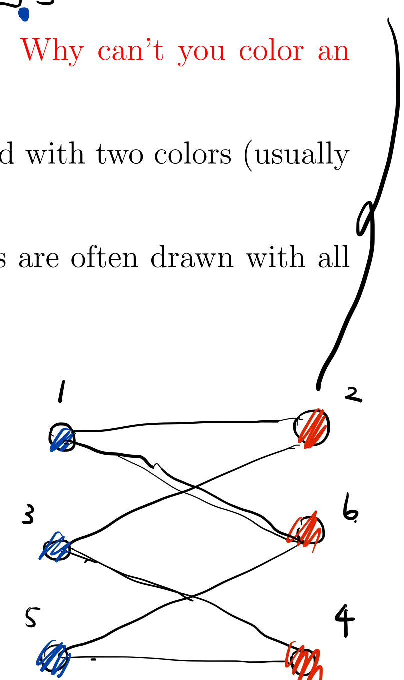


Question: Can you color an even-length cycle with two colors? Why can't you color an odd-length cycle with two colors?

Definition 5.3. A bipartite graph is a graph which can be colored with two colors (usually red and blue).

Example 5.4. The following graph is bipartite. Bipartite graphs are often drawn with all the blue vertices on one side and the red vertices on the other.





Note that there are no edges between the blue vertices and each other, and there are no edges between the red vertices and each other.

① Let's label nodes with increasing indexes starting from 1. and label them following clockwise order. ② Then we color blue on the nodes with odd indexes and red on the nodes with even indexes

② If the number of nodes is even, then all nodes' neighbors will be colored in different color. However, if the number of nodes is odd, Then the node with the starting index and the node with the last index will be colored the same and they are neighbors of each other.

Theorem 5.5. *A graph G is bipartite (2-colorable) if and only if it contains no odd-length cycle*

This theorem is an example of something called a **good characterization**, which links the existence of one thing (a 2-coloring) to the nonexistence of something else (the embedding of an odd cycle). That is, a graph can be 2-coloured if and only if it contains no odd cycles. This is a useful concept, since proving the nonexistence of one thing amounts to exhibiting the existence of the other thing.