CSCI 3104 Fall 2021
Instructors: Profs. Grochow and Waggoner

Problem Set 12

Due Date .... TODO
Name .... **Your Name**
Student ID .... **Your Student ID**
Collaborators .... **List Your Collaborators Here**

Contents

1 Instructions

- The solutions **must be typed**, using proper mathematical notation. We cannot accept hand-written solutions. Here's a short intro to LaTeX.
- You should submit your work through the **class Canvas page** only. Please submit one PDF file, compiled using this LaTeX template.
- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this document with no fewer pages than the blank template (or Gradescope has issues with it).
- You are welcome and encouraged to collaborate with your classmates, as well as consult outside resources. You must **cite your sources in this document. Copying from any source is an Honor Code violation. Furthermore, all submissions must be in your own words and reflect your understanding of the material.** If there is any confusion about this policy, it is your responsibility to clarify before the due date.

- Posting to **any** service including, but not limited to Chegg, Reddit, StackExchange, etc., for help on an assignment is a violation of the Honor Code.
- You **must** virtually sign the Honor Code (see Section 2). Failure to do so will result in your assignment not being graded.

2 Honor Code (Make Sure to Virtually Sign)

Problem 1.

- My submission is in my own words and reflects my understanding of the material.
- I have not collaborated with any other person.
- I have not posted to external services including, but not limited to Chegg, Discord, Reddit, StackExchange, etc.
- I have neither copied nor provided others solutions they can copy.

Agreed (signature here). □

3 Standard 26- Computational Complexity: Formulating Decision Problems

Problem 2. Recall the Maximum Bipartite Matching problem from class, where we took as input a bipartite graph $G(L\dot{\cup}R, E)$ and asked for a maximum cardinality matching. Formulate the decision variant of this problem using the Instance/Decision format from class. [**Note:** See Example 172 of M. Levet's Lecture Notes.]

Answer. $\square$

4 Standard 27- Computational Complexity: Problems in P

Problem 3. Consider the decision variant of the Interval Scheduling problem.

- Instance: Let $\mathcal{I} = \{[s_1, f_1], \ldots, [s_n, f_n]\}$ be our set of intervals, and let $k \in \mathbb{N}$.
- Decision: Does there exist a set $S \subseteq \mathcal{I}$ of at least k pairwise-disjoint intervals?

Show that the decision variant of the Interval Scheduling problem belongs to P. You are welcome and encouraged to cite algorithms we have previously covered in class, including known facts about their runtime. [**Note:** To gauge the level of detail, we expect your solutions to this problem will be 2-4 sentences. We are not asking you to analyze an algorithm in great detail.]

Answer. □

5 Standard 28- Computational Complexity: Problems in NP

Problem 4. We consider an algebraic structure $Q = (S, \star)$, where S is our set of elements and $\star : S \times S \to S$ is our "multiplication" operation. That is, for any $i, j \in S$, the product $i \star j \in S$. When S is finite, we can write out its "multiplication table": the rows are indexed by the elements of S, the columns are indexed by the elements of S, and the entry in the (i, j) position in the table is $i \star j$. We say that $(S, \star)$ is a *quasigroup* if the multiplication table is a Latin square; that is, if each element of S appears exactly once in each row and exactly once in each column.

The Latin Square Isotopy problem is defined as follows.

- Decision: Let $Q_1 = (S_1, \star)$ and $Q_2 = (S_2, \diamond)$ be quasigroups, where S_1, S_2 are n-element sets. Suppose that $(S_1, \star)$ and $(S_2, \diamond)$ are given by their multiplication tables.
- Decision: Do there exist one-to-one functions $\alpha, \beta, \gamma : S_1 \to S_2$ such that for all $x, y \in S$:

$$\alpha(x) \diamond \beta(y) = \gamma(x \star y).$$

  Here, $\alpha(x) \diamond \beta(y)$ is a product being considered in Q_2, while $x \star y$ is a product being considered in Q_1.

Show that the Latin Square Isotopy problem belongs to NP. *

Answer. □

*It remains open whether the Latin Square Isotopy problem is in P. The Latin Square Isotopy problem, as well as closely related algebraic problems such as Group Isomorphism in the multiplication table model, serve as key barriers for placing Graph Isomorphism into P. The best known algorithm for Graph Isomorphism, due to Babai in 2016, runs in time $n^{\Theta(\log^2(n))}$. Babai's algorithm combines combinatorial techniques such as color-refinement (Weisfeiler–Leman) and algebraic techniques (e.g., permutation group algorithms, the Classification of Finite Simple Groups). This is a very significant result. There has been considerable work on Group Isomorphism in the last 10 years. Isomorphism testing is an active area of research in our CS Theory group!

2 Standard 26 - Showing problems belong to P

Problem 1. Consider the Shortest Path problem that takes as input a graph $G = (V, E)$ and two vertices $v, t \in V$ and returns the shortest path from v to t. The shortest path decision problem takes as input a graph $G = (V, E)$, two a vertices $v, t \in V$, and a value k, and returns True if there is a path from v to t that is at most k edges and False otherwise. Show that the shortest path decision problem is in P. You are welcome and encouraged to cite algorithms we have previously covered in class, including known facts about their runtime. [**Note:** To gauge the level of detail, we expect your solutions to this problem will be 2-4 sentences. We are not asking you to analyze an algorithm in great detail.]

Answer. □

A path is simple if all vertices in the path are distinct.

3 Standard 27 - Showing problems belong to NP

Problem 2. Consider the Simple Shortest Path decision problem that takes as input a directed graph $G = (V, E)$, a cost function $c(e) \in \mathbb{Z}$ for $e \in E$, and two vertices $v, t \in V$. The problem returns True if there is a simple path from v to t with edge weights that sum to at most k, and False otherwise. Show this problem is in NP.

page 1170 of intro to alg.

Answer.

Problem 3. Indiana Jones is gathering n artifacts from a tomb, which is about to crumble and needs to fit them into 5 cases. Each case can carry up to W kilograms, where W is fixed. Suppose the weight of artifact i is the positive integer w_i. Indiana Jones needs to decide if he is able to pack all the artifacts. We formalize the Indiana Jones decision problem as follows.

- Instance: The weights of our n items, $w_1, \ldots, w_n > 0$.
- Decision: Is there a way to place the n items into different cases, such that each case is carrying weight at most W?

Show that Indiana Jones $\in$ NP.

Answer. $\square$