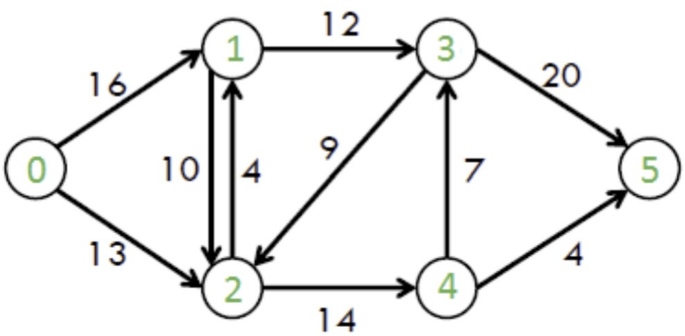
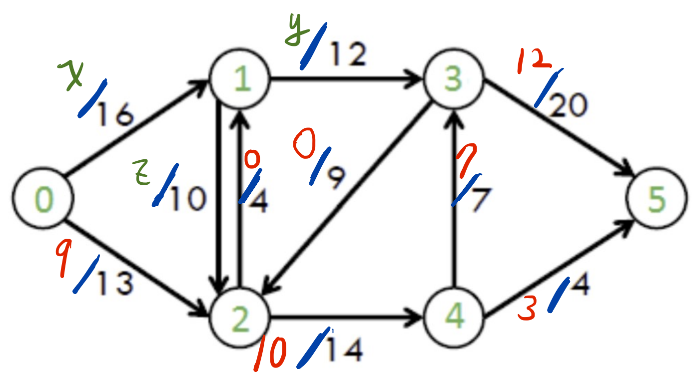
0 Definitions & Review

Definition 0.1. Formally, a flow network $G = (V, E, w, s, t)$ is a graph where

- V is a set of vertices
- $E \subseteq V \times V$ is a set of directed edges
- $c : E \to \mathbb{R}_{\geq 0}$ is a function assigning non-negative real-number weights (capacities) to the edges. Sometimes this function is called w instead of c.
- $s \in V$ is a designated source vertex, and may only have outgoing edges
- $t \in V$ is a designated target vertex, and may only have ingoing edges

1. What are the flows of x, y and z?





2. What is the value of the flow?

Definition 0.3. A **flow** on a flow network is a function

$$f : E \to \mathbb{R}$$

satisfying certain constraints:

1. For every edge e, it holds that $f(e) \leq c(e)$.

   This says that the flow on an edge cannot exceed its capacity (e.g. a road cannot carry more traffic than its capacity. a wire cannot carry more current than its capacity. a pipeline cannot carry more oil than its capacity)

2. For every vertex $v \in (V \setminus \{s, t\})$, Kirchhoff's Law holds:

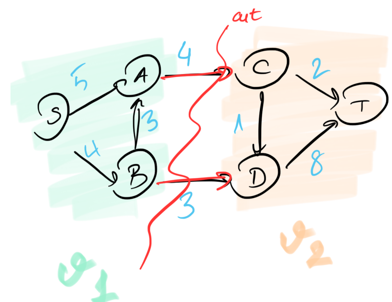
$$\sum_{e \in in(v)} f(e) = \sum_{e \in out(v)} f(e)$$

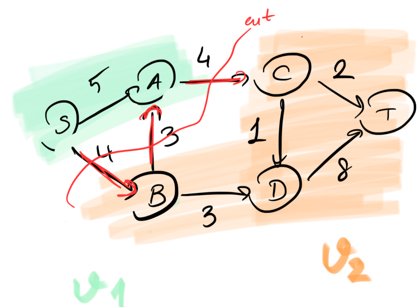
   This says that for every vertex besides s and t, the amount of flow going into that vertex must be equal to the amount of flow coming out of it. This is a conservation law.

   The **value** of a flow is defined as either the amount of flow coming out of s, or the amount of flow going into t. These quantities are always equal to each other.

- A **cut** in a flow network G is a partition of V into two disjoint sets $V = V_1 \oplus V_2$ (that is, $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \varnothing$) where $s \in V_1$ and $t \in V_2$.
- Or equivalently, a **cut** in G is a subset of edges $E' \subseteq E$ such that removing the edges in E' disconnects s from t.

The value of a cut is the sum of the capacities of edges in E'. Now consider the following two cuts in the same flow network:





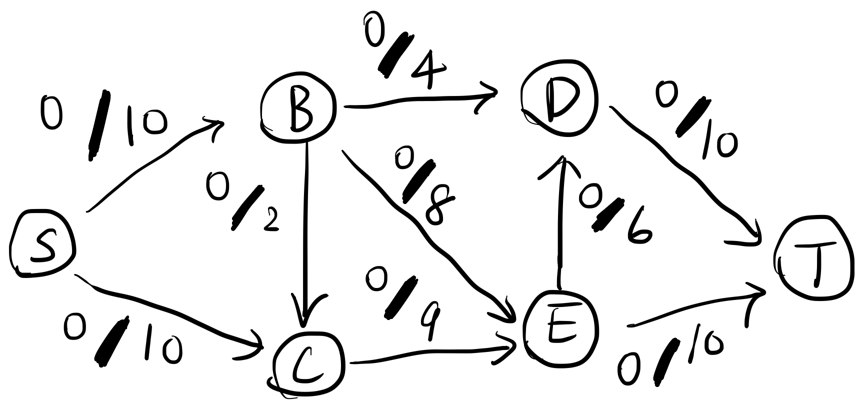
In the first cut above, $E' = ((A \to C), (B \to D))$ with value 7 and in the second one $E' = ((A \to C), (B \to A), (S \to B))$ with value 11. We are especially interested in finding the cuts with a minimum value since (i) this problem is connected to many optimization problems we would like to solve, (ii) it is mathematically dual (equivalent) to another problem known as "maximum flow" problem.

Min-cut problem is the problem of finding the cut with minimum value. You can intuitively think of this problem as trying to disconnect a given network with the smallest amount of obstruction. **Max-flow problem** is the problem of finding the the maximum amount of flow that we can send from s to t, constrained at each edge by its maximum capacity. Similarly, you can intuitively think of this problem as trying send as much as water / current / bits of information from a source to target.

To solve the max-flow problem by itself, we use the following algorithmic approach: Find paths along which we can move positive flow from a source to a sink. We call such paths flow-**augmenting**. While we can always push flow in the forward direction along an edge, we can also re-route existing flow back along an edge. Precisely, let (u, v) be an edge, and let $f(u, v)$ be the current flow that is being pushed from $u \to v$. We may push at most $c(u, v) - f(u, v)$ additional units of flow from $u \to v$. Alternatively, we may push $f(u, v)$ units of flow back from $v \to u$. We then re-route this flow at v. Iteratively,

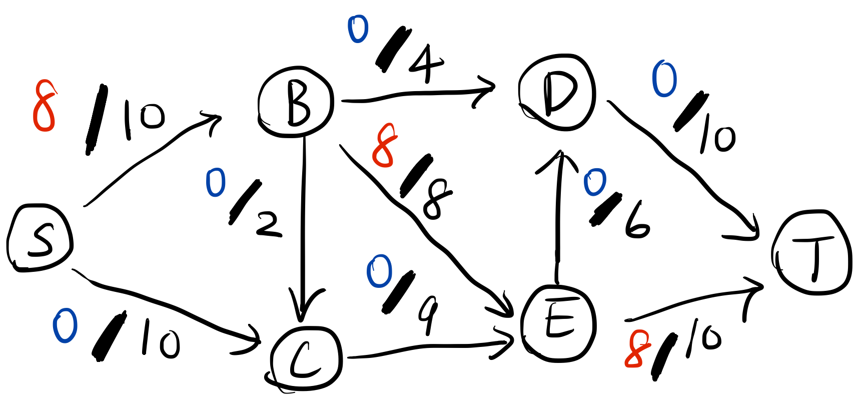
```
procedure FordFulkerson(FlowNetwork N)
     while N has a flow-augmenting path do
     Let P be a flow-augmenting path of N
     Push as much flow as possible along P
```

Question 1. Find the maximum flow on the following network:

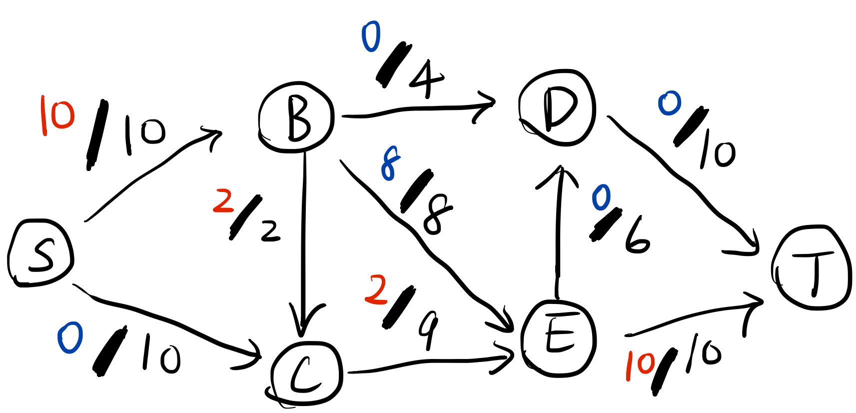


Answer.

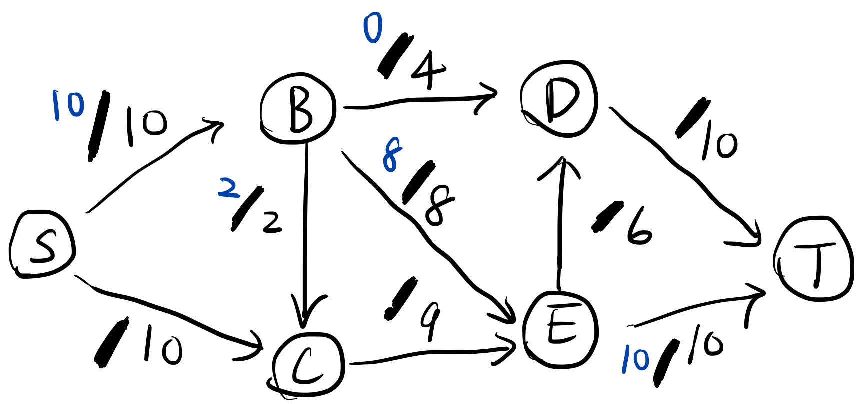
Iteration 1: $S \to B \to E \to t$



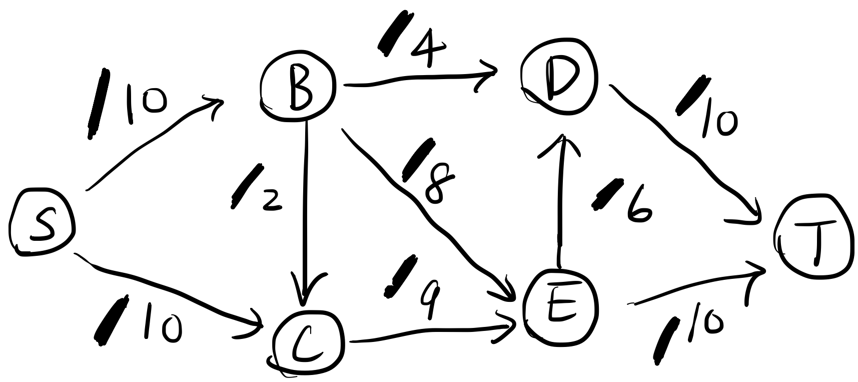
Iteration 2: $S \to B \to C \to E \to t$



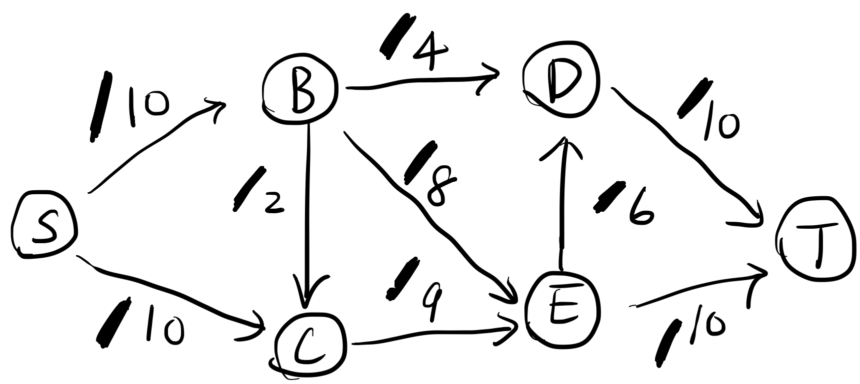
Iteration 3: $S \to C \to E \to D \to t$



Iteration 4: $S \to C \to B \to D \to t$



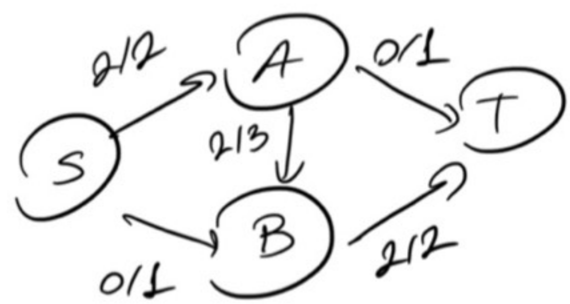
Iteration 5: ?



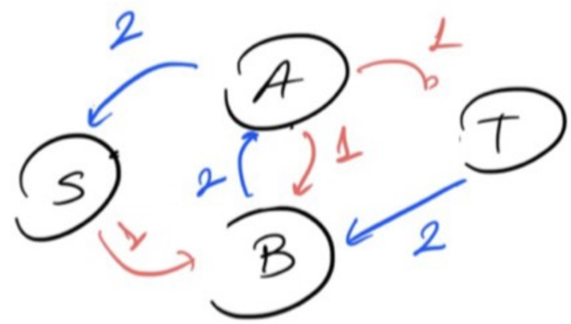
What is the total flow ?

Question 2. Find the minimum cut corresponding to maximum flow from Question 1.

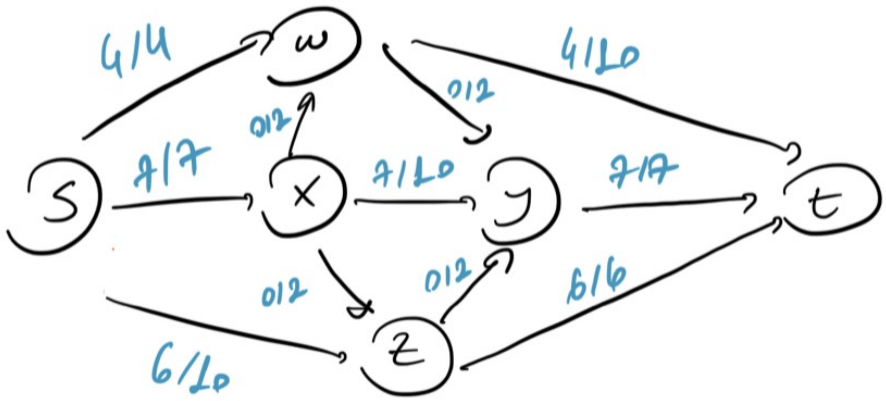
Question 3. Construct the residual network for the following network:



Answer:



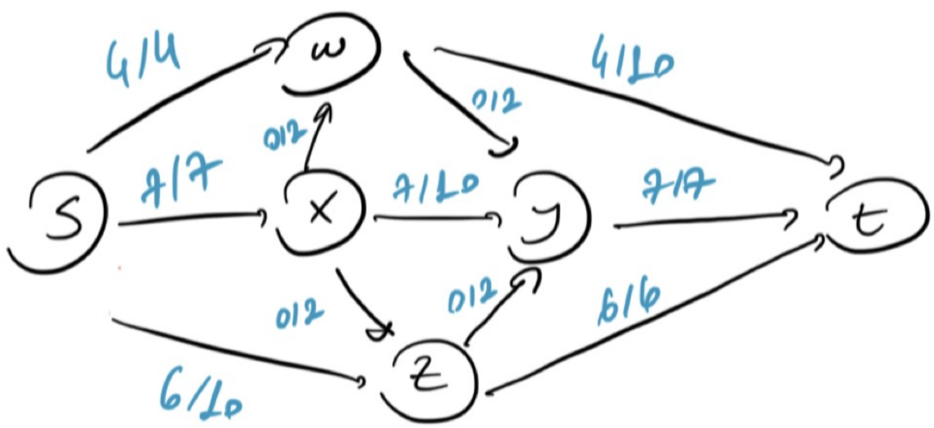
Question 4. Consider the following flow network with an $s-t$ flow:

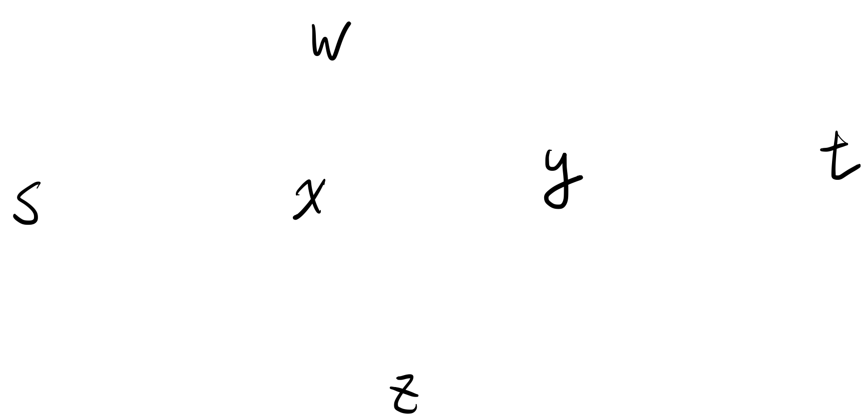


(i) What is the value of this flow?

(ii) Is this a maximum $s-t$ flow in this graph? If not, find a maximum $s-t$ flow using the residual network.

(iii) Find a minimum s - t cut using the residual network. (Specify which vertices belong to the sets of the cut.)





Question 5. How many possible cuts are there in the graph below? Can you find the minimum cuts?