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**Homework-12**
**Dec 12**

# Problem 1

Consider the quadratic map:

$$x_{n+1} = x_n^2 + c$$

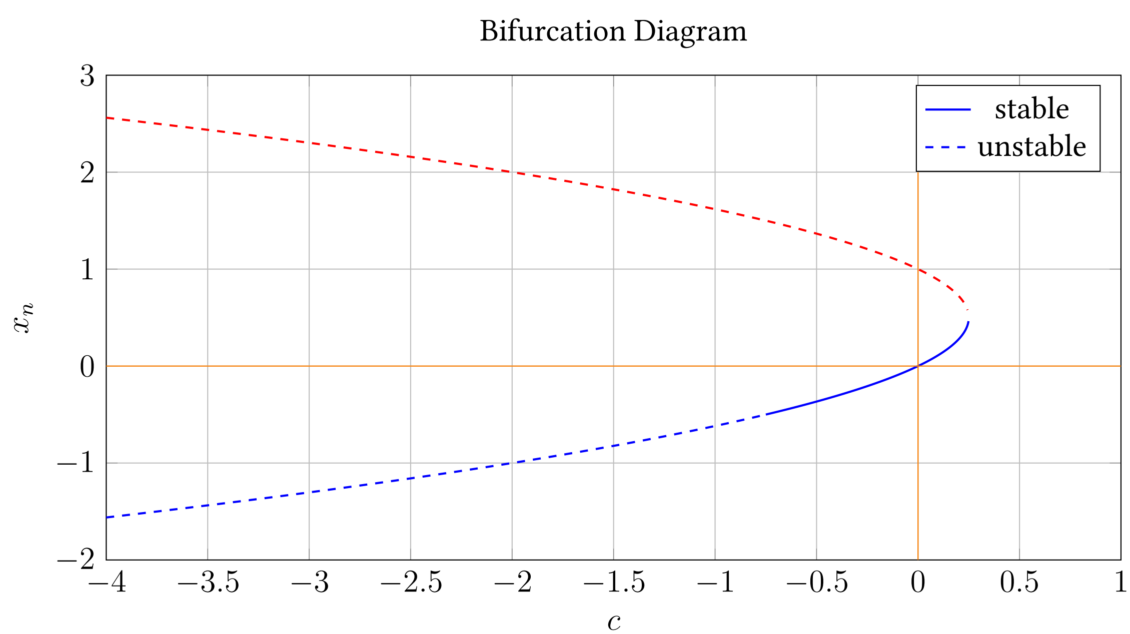
where $c$ is a constant.

1. Let $x_{n+1} = x_n$ we can find the fixed points are

$$x = \frac{1 \pm \sqrt{1-4c}}{2}$$

Then we find the $r$ with $|f'(x)| < 1$. Thus we have when $-3/4 < c < 1/4$, the $x = (1-\sqrt{1-4c})/2$ is stable, and $x = (1+\sqrt{1-4c})/2$ is unstable. When $c > \frac{1}{4}$, the two fixed points are not real.

2. Based on the result of (a), we can plot the bifurcation diagram as follows:




Thus we know that at $c = 1/4$ there is a saddle-node bifurcation.

3. For stable 2-cycle, we need to set $x^* = f^2(x^*)$, whichs is $x^* = x^4 + 2cx^2 + c^2 - x^* = 0$. We have two fixed points $x^* = \frac{1 \pm \sqrt{1-4c}}{2}$ which can be represented as $p, q$. Thus we have

$$x^4 + 2cx^2 + c^2 - x^* = (x-p)(x-q)(x^2 + bx + c) = 0.$$

Then we can use polynomial division to get $x^2 + bx + c$:

$$(x^2 + x + c + 1)(x - p)(x - q) = 0.$$

Thus we have last two fixed points $x^* = \frac{-1 \pm \sqrt{-3-4c}}{2}$ which can be represented as $r, s$. Thus we have

$$p = \frac{1 + \sqrt{1 - 4c}}{2} \tag{1}$$

$$q = \frac{1 - \sqrt{1 - 4c}}{2} \tag{2}$$

$$r = \frac{-1 + \sqrt{-3 - 4c}}{2} \tag{3}$$

$$s = \frac{-1 - \sqrt{-3 - 4c}}{2} \tag{4}$$

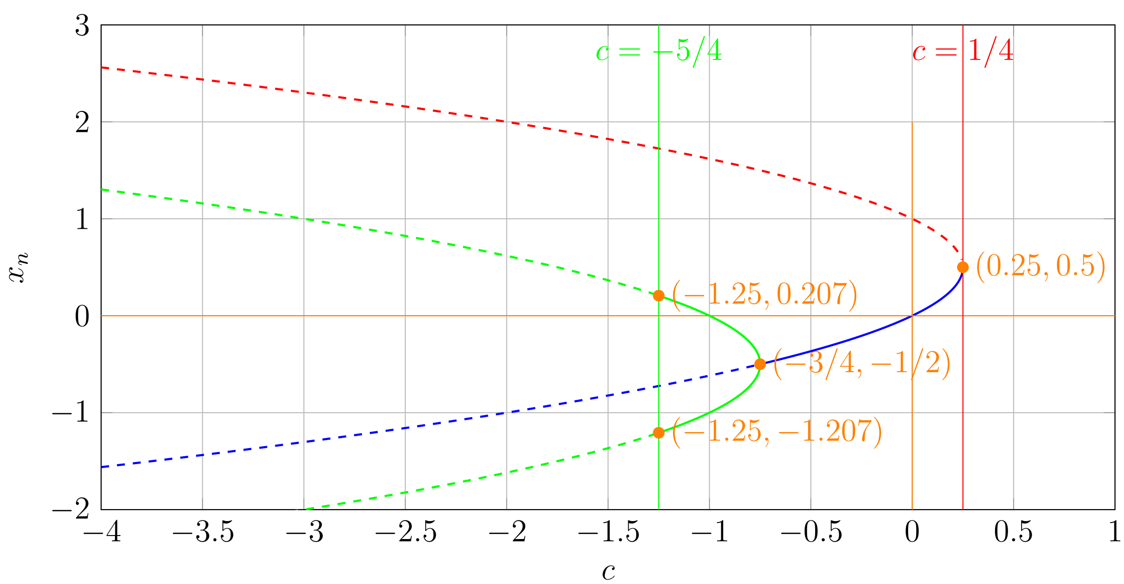
Then let's find the stability of 2-cycle. The derivative of $f(x)$ is $f'(x) = 2x$. Thus we have

$$f'(r)f'(s) = \frac{4}{4}(-1 + \sqrt{-3 - 4c})(-1 - \sqrt{-3 - 4c}) = 4c + 4.$$

So, linearly stable for $|4c + 4| < 1$ is $c \in (-5/4, -3/4)$.

4. For detials of the bifurcation diagram, we can see below:

Bifurcation Diagram




5. Given $y_n = ax_n + b$, we have $y_{n+1} = ax_{n+1} + b$ then substitute the logistic map $y_{n+1} = ry_n(1 - y_n)$. Then we have

$$ax_{n+1} + b = r(ax_n + b)(1 - (ax_n + b))$$
$$ax_{n+1} = r(ax_n + b)(1 - (ax_n + b)) - b$$
$$x_{n+1} = \frac{r(ax_n + b)(1 - (ax_n + b)) - b}{a}$$

Then we want to match the form of the quadratic map $x_{n+1} = x_n^2 + c$. Thus we have

$$x_{n+1} = -arx_n^2 + (r - 2br)x + \frac{rb - rb^2 - b}{a}.$$

Then we can match the coefficients to get

$$-ar = 1 \tag{5}$$
$$r - 2br = 0 \tag{6}$$
$$\frac{rb - rb^2 - b}{a} = c \tag{7}$$

Thus we have

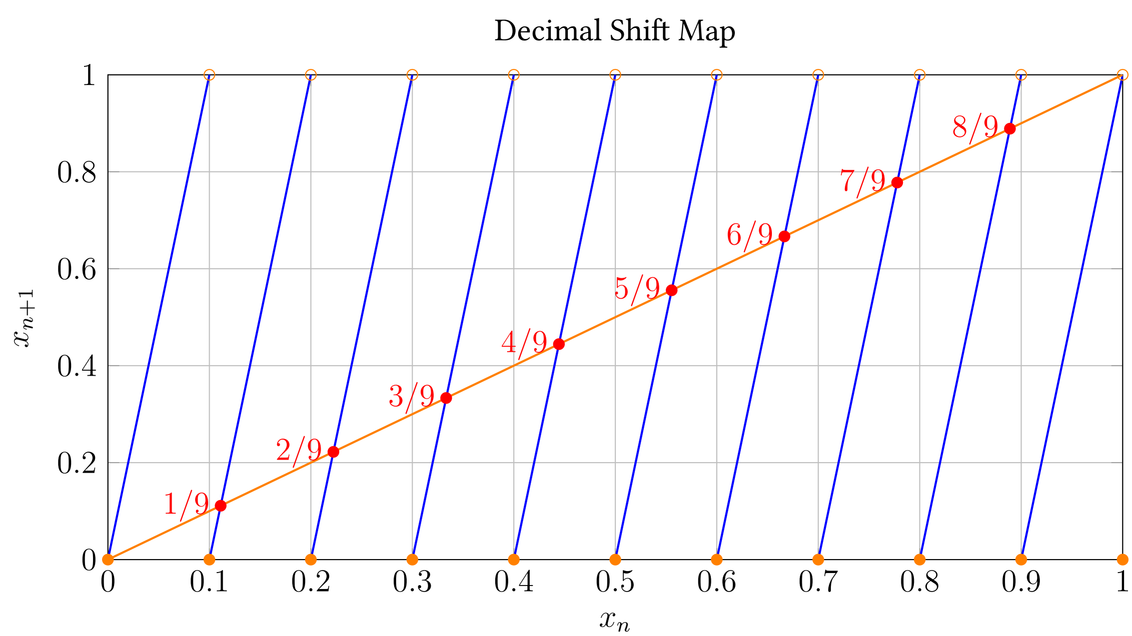
$$b = \frac{1}{2}, \quad a = -\frac{1}{r}, \quad c = \frac{r}{2} - \frac{r^2}{4}.$$

# Problem 2

Consider the decimal shift map which maps $[0, 1]$ onto $[0, 1]$:

$$x_{n+1} = 10x_n \pmod 1.$$

1. By listing out every $x_n$ we can find that there is a period relationship between every $0.1$ distance. They such rate is $10$ and the period is $0.1$. Thus we can use this infomation to draw the graph




2. The fixed points has been pointed out in (a) grpah. They are $0.aaaaaaa...$ where $a \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Thus we have $9$ fixed points. Since $f'(x) = 10 > 1$, all the fixed points are unstable.

3. For a period-$p$ cycle where $p > 1$, we can find a $x_n = x_{n+p}$ such this condition. Thus we have
$$0.(a_0 a_1 \dots a_p) ... (a_0 a_1 \dots a_p) ... (a_0 a_1 \dots a_p)$$
Then we can find the period-$p$ cycle is $0.a_0 a_1 ... a_p$. Thus we have
$$x_{n+p} = 10^p x_n \pmod 1.$$
Since $f'(x) = 10^p > 1$, all the period-$p$ cycles are unstable.

4. When $x_n$ is an irrational number, then we know that $x_n$ is aperiodic. For example $\pi$.

5. The Liapunov exponent is defined as
$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|.$$
Then we can calculate the Liapunov exponent for the decimal shift map. Since $f'(x) = 10$, we have
$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |10| = \ln 10.$$
Thus we have $\lambda = \ln 10 \approx 2.30$. Thus we know that the decimal shift map is chaotic.