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Homework-8
Nov 16

Problem 1

Consider the nonlinear oscillator

$$\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$$

where a is in $(0, 2)$.

(a) Let $y = \dot{x}$. Show that the system can be written as a first order system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -ay(x^2 + y^2 - 1) - x \end{cases}.$$

To find the fixed points, we need to solve the system when $\dot{x} = \dot{y} = 0$. So we have

$$\begin{cases} y = 0 \\ -ay(x^2 + y^2 - 1) - x = 0 \end{cases}$$

which implies $x = 0$ and $y = 0$ thus the fixed point is $(0, 0)$.

(b) Let $(x, y) = (r\cos\theta, r\sin\theta)$. Show that the system can be written as

$$\begin{cases} \dot{r} = -ar(\sin\theta)^2(r^2 - 1) \\ \dot{\theta} = -1 - a\sin\theta\cos\theta(r^2 - 1) \end{cases}$$

thus this is a polar system.
When $\dot{r} = 0$, we have $r = 0$ or $r = \pm 1$. Since the radius cannot be negative and zero, so when $r = 1$ we have a circular limit cycle. The amplitude of the limit cycle is 1 and period is -2π.

(c) By linearizing the radius about 1, we have $r(t) = A + \delta(t) = 1 + \delta(t)$. Then we have

$$\dot{\delta} = -a(1 + \delta)(\sin\theta)^2((\delta + 1)^2 - 1)$$

Since $\delta \ll 1$, we can ignore the higher order terms in Taylor expansion. Thus we can approximate the $(\delta + 1)^2 \approx 1 + 2\delta$. Then we have

$$\dot{\delta} \approx -a(1 + \delta)(\sin\theta)^2(2\delta) = -2ar\sin^2\theta\delta$$

Now we can see that $\dot{\delta}$ is linear and less than zero, so the limit cycle is stable.

Problem 2

Show the following system are gradient systems

(a) For the first system

$$\dot{x} = y^2 + y\cos x, \quad \dot{y} = 2xy + \sin y.$$

Now we test $\partial V/\partial x = -f(x,y)$ we have

$$\frac{\partial V}{\partial x} = -y^2 - y\cos x$$
$$\int \frac{\partial V}{\partial x} = -y^2 x - y\sin x + C$$

Then we test $\partial V/\partial y = -g(x,y)$ we have

$$\frac{\partial V}{\partial y} = -2xy - \sin y$$
$$\int \frac{\partial V}{\partial y} = -y^2 x - y\sin x + C$$

Thus we can find a potential function $V(x,y) = -y^2 x - y\sin x$ such the conditions. So the system is a gradient system.

(b) For the second system

$$\dot{x} = 3x^2 - 1 - e^{2y}, \quad \dot{y} = -2xe^{2y}$$

Now we test $\partial V/\partial x = -f(x,y)$ we have

$$\frac{\partial V}{\partial x} = -3x^2 + 1 + e^{2y}$$
$$\int \frac{\partial V}{\partial x} = -x^3 + x + e^{2y}x + C$$

Then we test $\partial V/\partial y = -g(x,y)$ we have

$$\frac{\partial V}{\partial y} = 2xe^{2y}$$
$$\int \frac{\partial V}{\partial y} = xe^{2y} + C_y$$

Where the $C_y = -x^3 + x$. Thus we can find a potential function $V(x,y) = -x^3 + x + e^{2y}x$ such the conditions. So the system is a gradient system.

Problem 3

Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits.

Proof. By constructing a Lyapunov function $V(x,y) = ax^2 + by^2$, we can form a suitable a, b for the system. Then we have $\dot{V}(x,y) = 2ax\dot{x} + 2by\dot{y}$, by substituting the system we have

$$\begin{aligned}\dot{V}(x,y) &= 2ax(y - x^3) + 2by(-x - y^3)\\ &= 2axy - 2ax^4 - 2bxy - 2by^4\\ &= 2xy(a-b) - 2x^4a - 2y^4b\end{aligned}$$

When $a = b > 0$, we have $\dot{V}(x,y) = -2x^4a - 2y^4b < 0$. Thus the system is stable. Since the system is stable, so there is no closed orbits. □

Problem 4

Consider the system $\dot{x} = x^2 - y - 1$, $\dot{y} = y(x-2)$.

(a) Let $\dot{x} = 0$, $\dot{y} = 0$, we have $x^2 - y - 1 = 0$ and $y(x-2) = 0$. Thus we have $y = 0$ or $x = 2$. When $y = 0$, we have $x = \pm 1$. When $x = 2$, we have $y = 3$. Thus the fixed points are $(-1,0)$, $(1,0)$ and $(2,3)$. Then let's find the jacobian matrix of the system

$$J = \begin{bmatrix} 2x & -1 \\ y & x-2 \end{bmatrix}$$

Then we can evaluate the jacobian matrix at the fixed points. For three fixed points $(-1,0)$, $(1,0)$ and $(2,3)$, we have the jacobian matrix respectively

$$J_{(-1,0)} = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix}, \quad J_{(1,0)} = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}, \quad J_{(2,3)} = \begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix}$$

Based on the jacobian matrix, we can find the eigenvalues of the fixed points. For $(-1,0)$, we have $\lambda_1 = -2$ and $\lambda_2 = -3$. For $(1,0)$, we have $\lambda_1 = 2$ and $\lambda_2 = -1$. For $(2,3)$, we have $\lambda_1 = 3$ and $\lambda_2 = 1$. Thus we can see that $(1,0)$ is a saddle point, $(-1,0)$ is a stable node and $(2,3)$ is a unstable node.

(b) Let's call $A = (-1,0)$, $B = (1,0)$ and $C = (2,3)$. Then let's find lines AB, BC and AC. By simple calculation, we have $l_{AB} : y = 0$, $l_{BC} : y = 3x - 3$ and $l_{AC} : y = x + 1$. Then we can examine these lines see if they are invariant.

(i) For l_{AB}, we can substitute $y = 0$ into the system and we have $\dot{x} = x^2 - 1$ and $\dot{y} = 0$. Then we can see that $\dot{x} < 0$ when $x \in [-1,1]$. Thus the line l_{AB} is invariant.

(ii) For l_{BC}, we can substitute $y = 3x - 3$ into the system and we have $\dot{x} = x^2 - 3x + 2 = (x-2)(x-1)$ and $\dot{y} = 3(x-1)(x-2)$. Thus $\dot{y} = 3\dot{x}$. Since the slope of $\dot{y}$ is same as y, we know that every point on the line l_{BC} will move along the line l_{BC}. Thus the line l_{BC} is invariant.

(iii) For l_{AC}, we can substitute $y = x + 1$ into the system and we have $\dot{x} = x^2 - x - 2 = (x-2)(x+1)$ and $\dot{y} = (x-2)(x+1)$. Thus $\dot{y} = \dot{x}$. Since the slope of $\dot{y}$ is same as y, we know that every point on the line l_{AC} will move along the line l_{AC}. Thus the line l_{AC} is invariant.

(c) Here is the phase portrait of the system. The red circle is the closed curve at $(-1,0)$, the blue circle is the closed curve at $(1,0)$ and the green circle is the closed curve at $(2,3)$.

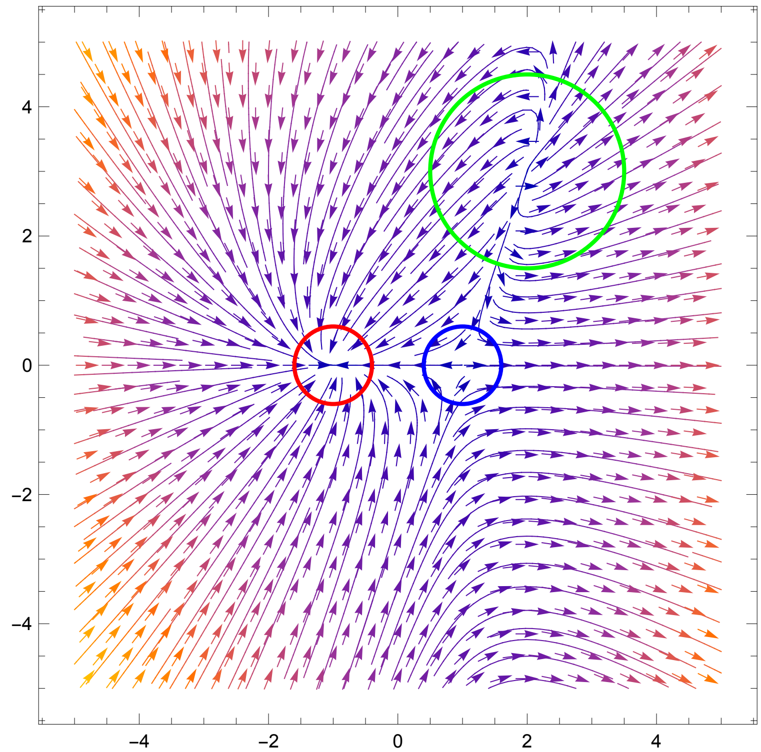

Figure 1: Phase portrait of the system

For those three closed curves are all violates the rule that trajectories cannot cross becasue there is always straight line between two fixed points. Thus the system cannot had a closed orbit.

Problem 5

Consider the system

$$\begin{cases} \dot{x} = x - y - x(x^2 + 5y^2) \\ \dot{y} = x + y - y(x^2 + y^2) \end{cases}$$

(a) Let $\dot{x} = 0$, $\dot{y} = 0$, we have $x - y - x(x^2 + 5y^2) = 0$ and $x + y - y(x^2 + y^2) = 0$. Thus we have $(x^*, y^*) = (0,0)$. Then let's find the jacobian matrix of the system at $(0,0)$

$$J = \begin{bmatrix} 1 - 3x^2 - 5y^2 & -1 - 10xy \\ 1 - 2xy & 1 - x^2 - 3y^2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Then we can find the eigenvalues of the fixed point $(0,0)$, we have $\lambda_1 = 1+i$ and $\lambda_2 = 1-i$. Thus we can see that $(0,0)$ is a unstable spiral node.

(b) Let $(x, y) = (r\cos\theta, r\sin\theta)$. Show that the system can be written as

$$\begin{cases} \dot{r} = r - r^3 - 2r^3 \sin^2\theta \cos^2\theta \\ \dot{\theta} = 1 + 4r^2 \sin^3\theta \cos\theta \end{cases}$$

thus this is a polar system.

(c) Consider r_1 as the radius for inner circle, r_2 as the radius for outer circle. Then for r_1 must such that $\dot{r} > 0$ and for r_2 must such that $\dot{r} < 0$. Thus we have

$$\begin{cases} r_1 - r_1^3 - 2r_1^4 \sin^2\theta \cos^2\theta > 0 \\ r_2 - r_2^3 - 2r_2^4 \sin^2\theta \cos^2\theta < 0 \end{cases}$$

Then we can solve the inequality and we have $r_1 \in (1/\sqrt{3}, 1)$ and $r_2 \in (1/\sqrt{3}, \infty)$. Thus the largest r_1 is 1 and the smallest r_2 is $1/\sqrt{3}$.