Math 415
Homework-10
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Problem 1

Consider the van der Pol oscillator $\ddot{x} + \mu(x^2-1)\dot{x} + x = a$ for real parameters μ and a. Find the curves in the (μ, a)-space at which a Hopf bifurcation occurs, for what $(x, \dot{x})$ does it occur at?

Solution:

Let $y = \dot{x}$, then we have the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = a - x - \mu(x^2 - 1)y \end{cases}$$

When $\dot{x} = \dot{y} = 0$, we can find nullclines $y = a - x/\mu(x^2 - 1)$ and $y = 0$. The intersection of these two nullclines is $(x, y) = (a, 0)$ which is the only fixed point of the system. The Jacobian matrix of the system is

$$J(x, y) = \begin{bmatrix} 0 & 1 \\ -1 - 2\mu xy & -\mu(x^2 - 1) \end{bmatrix}_{(a,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu(a^2 - 1) \end{bmatrix}$$

The $\tau = \text{tr}(J) = -\mu(a^2 - 1)$ and $\Delta = \det(J) = 2a\mu y + 1$. The Hopf bifurcation occurs when $\tau = 0$ and $\Delta > 0$. Thus, the Hopf bifurcation occurs at $a = \pm 1$. Thus the Hopf bifurcation occurs at $(x, y) = (\pm 1, 0)$.

Problem 2

Consider the system

$$\begin{cases} \dot{x} = x[x(1 - x) - y] \\ \dot{y} = y(x - a) \end{cases}$$

where $x \geq 0$ is the dimensionless population of the prey, $y \geq 0$ is the dimensionless population of the predator, and $a \geq 0$ is a control parameter.

(a) Sketch the nullclines in the first quadrant $(x, y \geq 0)$.
Here is the nullclines in the first quadrant.

When $\dot{x} = \dot{y} = 0$, we have nullclines $x = 0$, $y = 0$, $x = 1$, and $x = a$. Thus we have three fixed points $(x, y) = (0, 0)$, $(1, 0)$, and $(a, a - a^2)$. The Jacobian matrix of the system is

$$J(x, y) = \begin{bmatrix} 2x(1 - x) - y & -x \\ y & x - a \end{bmatrix}$$

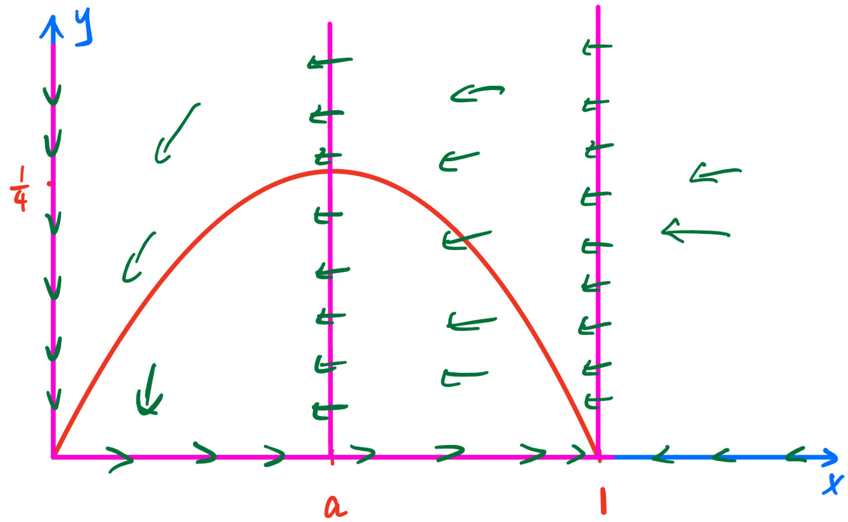

Figure 1: Nullclines in the first quadrant

(b) Show that the fixed points and classify them. From the Jacobian matrix, we have

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -a \end{bmatrix}, \quad J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & 1-a \end{bmatrix}, \quad J(a, a-a^2) = \begin{bmatrix} a-2a^2 & -a \\ a-a^2 & 0 \end{bmatrix}$$

(i) $(0,0)$: $\tau = -a$, $\Delta = 0$, thus the fixed point is line of stable nodes.

(ii) $(1,0)$: $\tau = -a$, $\Delta = a-1$. When $a < 1$, the fixed point is a saddle point. When $a = 1$, the fixed point is a line of stable nodes. When $a > 1$, the fixed point is a stable node.

(iii) $(a, a-a^2)$: $\tau = a - 2a^2$, $\Delta = a^2 - 2a^3$. When $a = 0$, the fixed point is uniform motion. When $0 < a < 1/2$, the fixed point is an unstable spiral. When $a = 1/2$, the fixed point is also uniform motion. When $a > 1/2$, the fixed point is a saddle point.

(c) Sketch the phase portrait for $a > 1$. What happens to the predators as $t \to \infty$?
Here is the phase portrait for $a > 1$ when $a = 2$.

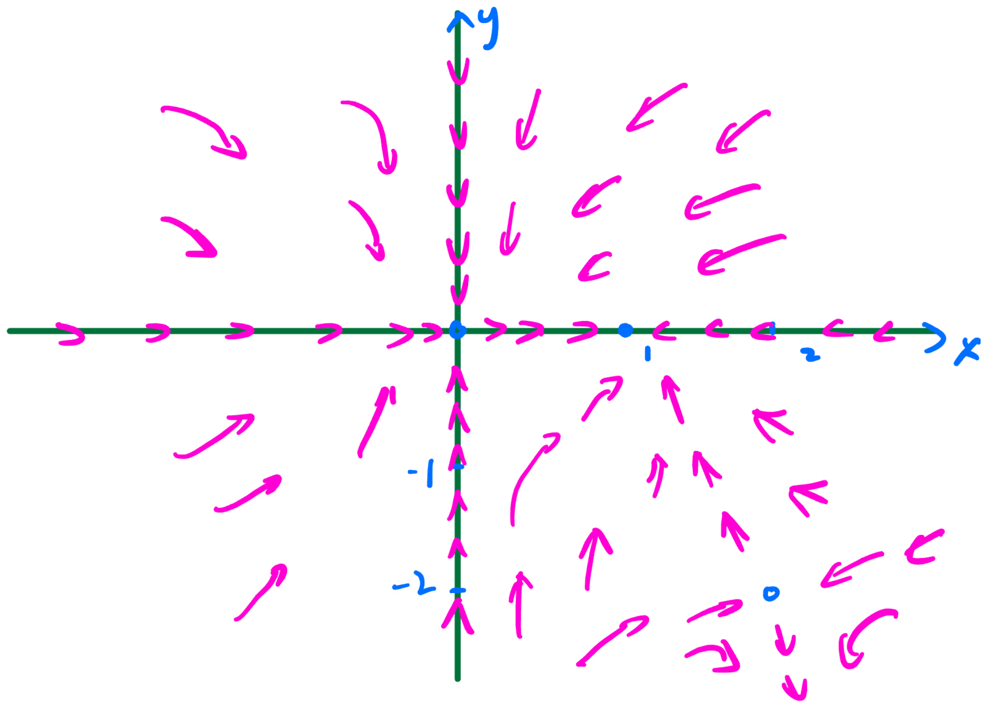

Figure 2: Phase portrait for $a = 2$

As $t \to \infty$, the predators will go to extinction and the prey will go to 1.

(d) What type of bifurcation occurs at $a = 1$?
At $a = 1$, the fixed point $(1, 0)$ changes from a saddle point to a stable node. Thus, the bifurcation is a saddle-node bifurcation.

(e) At what value of a does the Hopf bifurcation occur?

$$\tau = a - 2a^2 = 0 \implies a = 0, \frac{1}{2}$$

Problem 3

Consider the Lorenz system

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(r - z) - y \\ \dot{z} = xy - bz \end{cases}$$

1. First find the Jacobian matrix $J(x, y, z)$ of the system at fixed point.

$$J(x, y, z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}_{(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)}$$

Then

$$\begin{bmatrix} -\sigma & \sigma & 0 \\ 0 & -1 & \pm\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{bmatrix}$$

Thus, the $\det(J - \lambda I) = 0$ is

$$\begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ 0 & -1-\lambda & \pm\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b-\lambda \end{vmatrix} = 0$$

By the general formula of characteristic polynomial

$$\lambda^3 - Tr(J)\lambda^2 + (J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{12}J_{21} - J_{13}J_{31} - J_{23}J_{32})\lambda - \det(J) = 0$$

we have

$$\lambda^3 + (\sigma + 1 + b)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r-1) = 0$$

2. For eigenvalue $\lambda = i\omega$, we can plug into the characteristic polynomial and get

$$(i\omega)^3 + (\sigma + 1 + b)(i\omega)^2 + (r + \sigma)b(i\omega) + 2b\sigma(r-1) = 0$$

That gives us in imaginary part and real part equal to zero

$$[2b\sigma(r-1) - (\sigma + 1 + b)\omega^2] + [b(r+\sigma)\omega - i\omega^3] = 0$$

Since imaginary part and real part are equal to zero, we have

$$2b\sigma(r-1) = (\sigma + 1 + b)\omega^2, \quad b(r+\sigma)\omega = \omega^3$$

Thus, we have

$$\frac{2b\sigma(r-1)}{1+\sigma+b} = \omega^2, \quad b(r+\sigma) = \omega^2$$

Then we can solve for r,

$$r = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

when $\sigma > b + 1$. Since this pair of eigenvalues are zero, Hopf bifurcation occurs at $r_H = \sigma(\sigma + b + 3)/(\sigma - b - 1)$.

3. Since the τ of the system is $-\sigma - 1 - b$ and the sum of two pairs of eigenvalues is zero, thus the third eigenvalue is $\lambda_3 = -(\sigma + 1 + b)$.