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Homework-7
Nov 7

Problem 1

For glider motion, we have non-linear system

$$\begin{cases} m\dot{v} = -mg\sin\theta - \frac{1}{2}\rho C_D S v^2 \\ mv\dot{\theta} = -mg\cos\theta + \frac{1}{2}\rho C_L S v^2 \end{cases}$$

where v is the velocity, θ is the angle of attack, m is the mass, g is the gravity, ρ is the air density, C_D is the drag coefficient, C_L is the lift coefficient, and S is the wing area.

(a) Let $v = x/v_0, T = v_0/g, t = \tau T$, then we can find the dimensionless form of the system

$$\begin{aligned} \frac{dv}{dt} &= \frac{dv}{d\tau}\cdot\frac{d\tau}{dt} \\ \frac{dv}{dt} &= \frac{dv}{d\tau}\cdot\frac{g}{v_0} \\ \frac{mg}{v_0}\frac{dv}{d\tau} &= -mg\sin\theta - \frac{1}{2}\rho C_D S v^2 \\ \frac{mg}{v_0^2}\frac{dx}{d\tau} &= -mg\sin\theta - \frac{\rho C_D S}{2v_0^2}x^2 \\ \frac{dx}{d\tau} &= -v_0^2\sin\theta - \frac{\rho C_D S}{2mg}x^2 \end{aligned}$$

here we know $v_0^2 = 1$, $2mg/\rho S = C_L$, and

$$\begin{aligned} mv\frac{d\theta}{dt} &= -mg\cos\theta + \frac{1}{2}\rho C_L S v^2 \\ \frac{mvg}{v_0}\frac{d\theta}{d\tau} &= -mg\cos\theta + \frac{1}{2}\rho C_L S v^2 \\ \frac{mgx}{v_0^2}\frac{d\theta}{d\tau} &= -mg\cos\theta + \frac{1}{2v_0^2}\rho C_L S x^2 \\ mgx\frac{d\theta}{d\tau} &= -mgv_0^2\cos\theta + \frac{1}{2}\rho C_L S x^2 \\ x\frac{d\theta}{d\tau} &= -\cos\theta + x^2 \end{aligned}$$

Thus, we have the dimensionless form of the system

$$\begin{cases} \frac{dx}{d\tau} &= -\sin\theta - Dx^2 \\ x\frac{d\theta}{d\tau} &= -\cos\theta + x^2 \end{cases}$$

where $D = C_D/C_L$.

(b) When $D = 0$, we have

(i) For $E(x, \theta) = x^3 - 3x\cos\theta$, we can take derivative with respect to τ and get

$$\begin{aligned}\frac{d}{d\tau}E(x,\theta) &= 3x^2\frac{dx}{d\tau} - 3\cos\theta\frac{dx}{d\tau} + 3x\sin\theta\frac{d\theta}{d\tau}\\ &= 3x^2(-\sin\theta) - 3\cos\theta(-\sin x) + 3x\sin\theta(\frac{-\cos\theta}{x} + x)\\ &= 0\end{aligned}$$

Thus, $E(x, \theta)$ is a constant.

(c) Find and classify all the fixed points. When $\dot{x} = 0$, $\sin\theta = 0$, thus $\theta = n\pi$, $n \in \mathbb{Z}$. When $\dot{\theta} = 0$, $\cos\theta = x^2$, thus $\theta = \pm\arccos x^2 + 2n\pi$, $n \in \mathbb{Z}$. Thus, we have fixed points $(x, \theta) = (\pm 1, 2n\pi)$, $n \in \mathbb{Z}$. Since this system is conservative, the fixed points are centers.

(d) By using nullcline, we briefly plot the phase portrait as follows

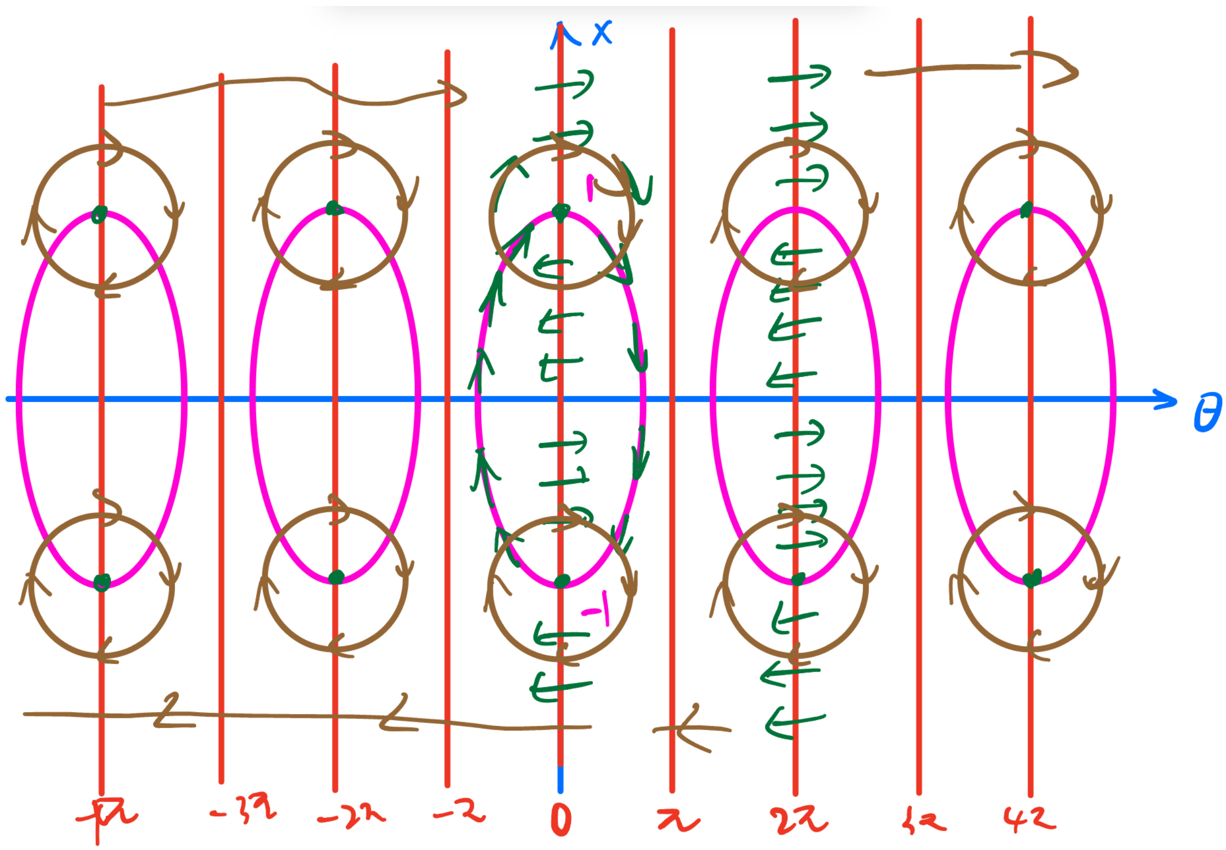

Figure 1: Phase portrait of the system

(e) When $\theta(0) = 0, x(0) = 1/2$ the glider will fly in a circle with radius $1/2$.
When $\theta(0) = 0, x(0) = 2$ the glider will fly along around the trajectory $x = 2$ and keep flying far away.

Problem 2

For the system

$$\begin{cases} \dot{x} = \sin y \\ \dot{y} = \sin x \end{cases}$$

(a) Show that the system is invariant under the mappings

(i) $(t, x, y) \mapsto (-t, x, -y)$. Let $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$, then we have

$$\begin{cases} f(x, -y) = \sin -y = -\sin y = -f(x, y) \\ g(x, -y) = \sin x = g(x, y) \end{cases}$$

Thus, the system is invariant under the mapping $(t, x, y) \mapsto (-t, x, -y)$.

(ii) $(t, x, y) \mapsto (t, -x, y)$. Let $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$, then we have

$$\begin{cases} f(-x, y) = \sin y = f(x, y) \\ g(-x, y) = \sin -x = -\sin x = -g(x, y) \end{cases}$$

Thus, the system is invariant under the mapping $(t, x, y) \mapsto (t, -x, y)$.

(b) Find all the fixed points of the system.
When $\dot{x} = 0$, $\sin y = 0$, thus $y = n\pi$, $n \in \mathbb{Z}$. When $\dot{y} = 0$, $\sin x = 0$, thus $x = n\pi$, $n \in \mathbb{Z}$. Thus, we have fixed points $(x, y) = (n\pi, m\pi)$, $n, m \in \mathbb{Z}$. The Jacobian matrix is

$$J = \begin{bmatrix} 0 & \cos y \\ \cos x & 0 \end{bmatrix}$$

Since this system is time-reversible, we only consider the first quadrant where all the $x, y \geq 0$. When $x = y$, we can get $\tau = 0$, $\Delta = -\cos x^2 < 0$ and the fixed point is a saddle. When $(x, y) = (0, n\pi)$, we can get $\tau = 0$, $\Delta = (-1)^n < 0$ when n is odd and $\Delta = (-1)^n > 0$ when n is even. Thus, the fixed point is a saddle when n is odd and a center when n is even. Thus, we have the following phase portrait

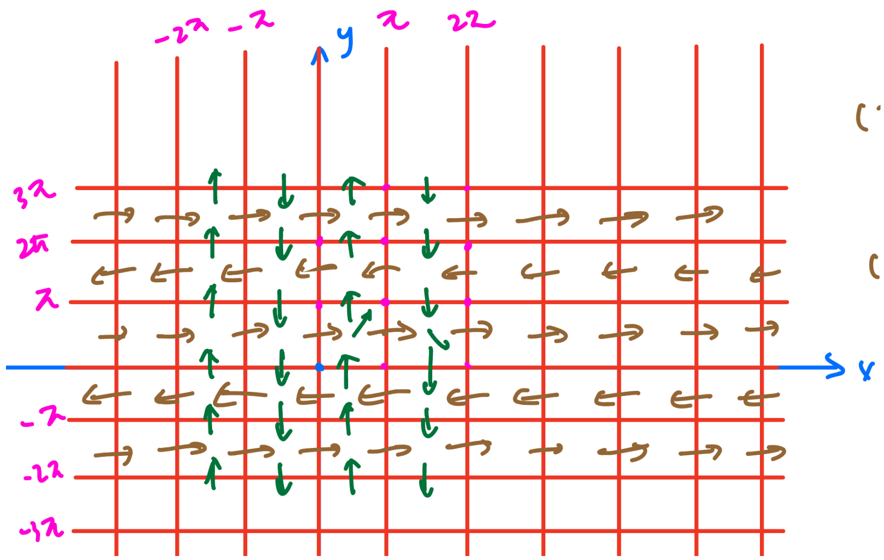

Figure 2: Phase portrait of the system

Problem 3

Determine the index of the following fixed points at $(0,0)$:

(a) $\dot{x} = y - x, \quad \dot{y} = x^2$
We have

$$J = \begin{bmatrix} -1 & 1 \\ 2x & 0 \end{bmatrix}$$

Thus, we have fixed point $(0,0)$

$$J(0,0) = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, $\tau = -1$, $\Delta = 0$. The trajectory around $(0,0)$ cannot be determined by the linearized system. Let's try to use nullcline to plot the phase portrait then choose the unit circle around $(0,0)$ as closed curve C.

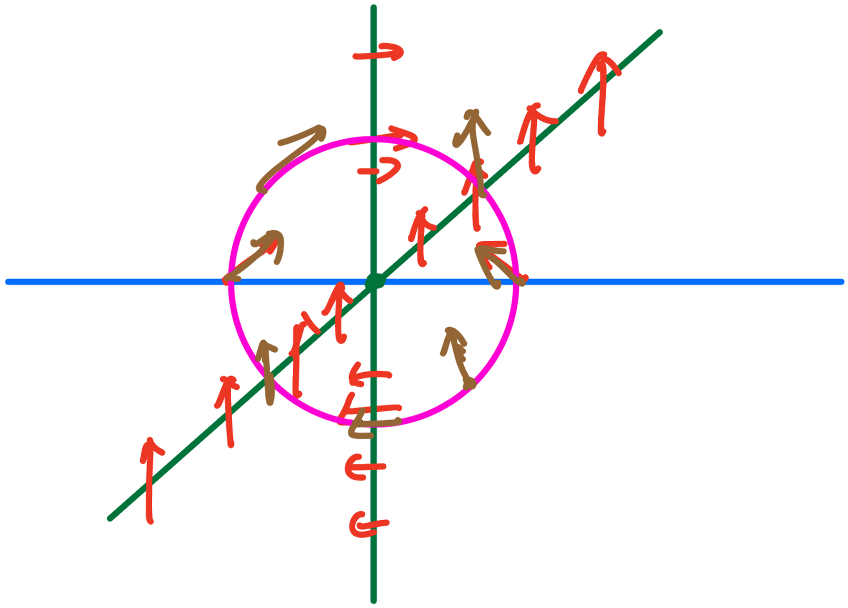

Figure 3: Phase portrait of the system

By plot the variation of the vector field of that unit circle, we can see that the index of $(0, 0)$ is 0.

(b) $\dot{x} = y^3, \quad \dot{y} = x$
We have

$$J = \begin{bmatrix} 0 & 3y^2 \\ 1 & 0 \end{bmatrix}$$

Thus, we have fixed point $(0, 0)$

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Thus, $\tau = 0$, $\Delta = 0$. The trajectory around $(0, 0)$ cannot be determined by the linearized system. Let's try to use nullcline to plot the phase portrait then choose the unit circle around $(0, 0)$ as closed curve C.

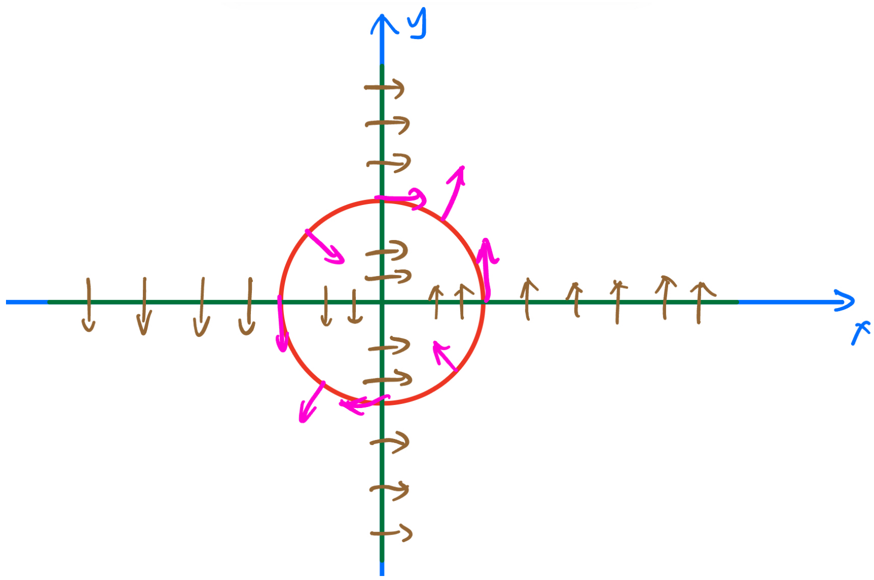

Figure 4: Phase portrait of the system

By testing the variation of the vector field of that unit circle, we can see that the index of $(0, 0)$ is -1.

(c) $\dot{x} = xy$, $\dot{y} = x + y$.
We have

$$J = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$$

Thus, we have fixed point $(0, 0)$

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

The two vectors are not linearly independent, thus we cannot determine the index of $(0, 0)$ by using the linearized system. Let's try to use nullcline to plot the phase portrait then choose the unit circle around $(0, 0)$ as closed curve C.

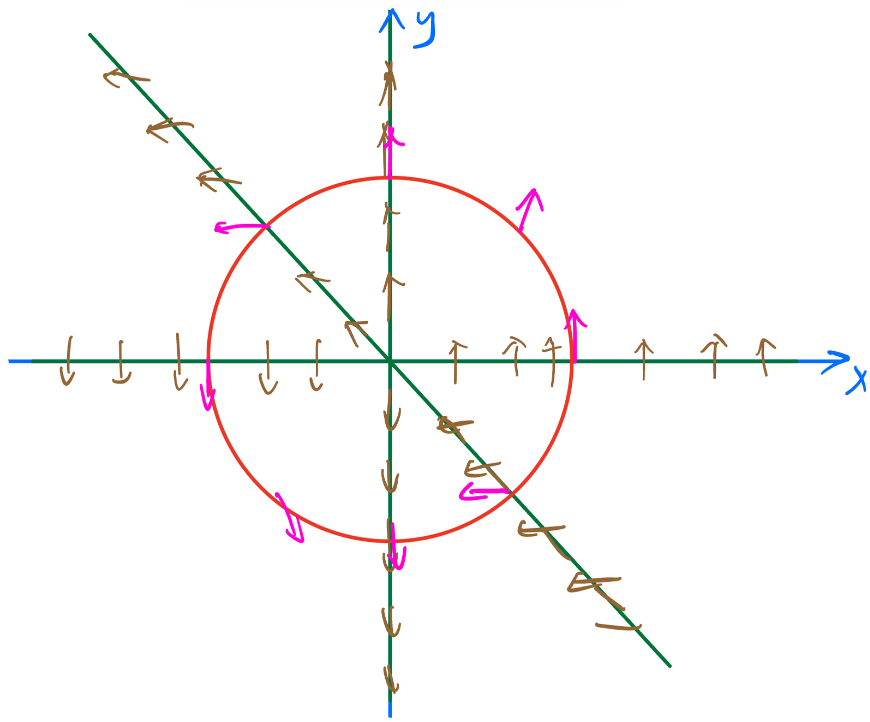

Figure 5: Phase portrait of the system

By testing the variation of the vector field of that unit circle, we can see that the index of $(0, 0)$ is 0.

Problem 4

Proof. By theorem 6.8.2 in Strogatz, any closed orbit in the phase plane must enclose fixed points whose indices sum to $+1$. Thus, we have

$$NI_N + FI_F + CI_C + SI_S = 1$$

Since the The S is saddles the index is -1, we have

$$NI_N + FI_F + CI_C - S = 1$$

Thus, we have

$$N + F + C = 1 + S.$$

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