莫比乌斯反演

1.数论函数

1)莫比乌斯函数

$$\mu(n)=\begin{cases}1,\ \ n=1\\ -1^k,\ \ n=\prod_{i=1}^{k}p_i\\ 0,\ \ n=else\end{cases}$$

性质

$$\mu(1)=1$$

$$\mu(p)=-1$$

$$i\%x=0\quad \mu(i\cdot x)=0$$

$$i\%x\neq 0\quad \mu(i\cdot x)=-\mu(i)$$

重要！用于条件转化：

$$\sum_{d|n}\mu(d)=[n=1]$$

2)欧拉函数

定义

$\varphi(n)$为1-n内与n互质的数的个数

$\varphi(n)=n\prod_{i=1}^{k}(1-\frac{1}{p_i})\quad n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$

性质

$\varphi(1)=1$

$\varphi(p)=p-1$

$i\%x=0\quad \varphi(i\cdot x)=\varphi(i)\cdot x$

$i\%x\neq 0\quad \varphi(i\cdot x)=\varphi(i)\cdot\varphi(x)$

重要！以下性质常用：

n的因子的欧拉函数值之和为n:

$\sum_{d|n}\varphi(d)=n\quad(\varphi * I=id)$

小于n且与n互质的数的和:

$$S(n)=\begin{cases}\frac{n\cdot\varphi(n)}{2}, & n\geqq 2\\ 0, & n=1\end{cases}$$

若a，m互质，则

$a^{\varphi(m)}\equiv 1\pmod m$

3)最小素因子出现的次数

性质

$F(1)=1$

$F(p)=1$

$i\%x=0\quad F(i\cdot x)=F(i)+1$

$i\%x\neq 0\quad F(i\cdot x)=1$

4)约数个数函数 lightoj - 1028

性质

$d(1)=1$

$$d(p)=2$$

$$i\%x=0 \quad d(i\cdot x)=d(i)\cdot\frac{F(i)+2}{F(i)+1}$$

$$i\%x\neq 0 \quad d(i\cdot x)=d(i)\cdot d(x)$$

5)最小素因子的0-k次幂之和

$$即(1+p_1+p_1^2+\cdots+p_1^{k_1}) \qquad x=p_1^{k1}p_2^{k_2}\cdots p_n^{k_n}$$

性质

$$pk(1)=0$$

$$pk(p)=p+1$$

$$i\%x=0 \quad pk(i\cdot x)=pk(i)\cdot x+1$$

$$i\%x\neq 0 \quad pk(i\cdot x)=x+1$$

6)约数和函数

性质

$$\sigma(1)=1$$

$$\sigma(p)=p+1$$

$$i\%x=0 \quad \sigma(i\cdot x)=\sigma(i)\cdot\frac{pk(i\cdot x)}{pk(i)}$$

$$i\%x\neq 0 \quad \sigma(i\cdot x)=\sigma(i)\cdot\sigma(x)$$

7)基础数论函数

$$I(n)=1$$
$$id(n)=n$$
$$e(n)=\begin{cases}1,n=1\\0,n\neq 0\end{cases}$$

8)数论函数线性筛模板(6个常用函数)

```
const int MAXN=1e6+5;
int prime[MAXN],pnum=0;
bool vis[MAXN];//是否是素数
int mu[MAXN];//莫比乌斯函数
int phi[MAXN];//欧拉函数
int F[MAXN];//最小素因子出现的次数
int d[MAXN];//约数个数
int pk[MAXN];//最小素因子的0-k次幂之和
int sigma[MAXN];//约数和
void sieve(){
    memset(vis,true,sizeof(vis));
```

```
    vis[0]=vis[1]=false;//x=1的特例
    mu[1]=1;
    phi[1]=1;
    F[1]=1;
    d[1]=1;
    pk[1]=0;
    sigma[1]=1;
    for(int i=2;i<MAXN;i++){
        if(vis[i]){
            prime[pnum++]=i;
            mu[i]=-1;
            phi[i]=i-1;
            F[i]=1;
            d[i]=2;
            pk[i]=i+1;
            sigma[i]=i+1;
        }
        for(int j=0;j<pnum&&1LL*i*prime[j]<MAXN;j++){
            int x=prime[j];
            vis[i*x]=false;
            if(i%x==0){//x是i的最小素因子
                mu[i*x]=0;
                phi[i*x]=phi[i]*x;
                F[i*x]=F[i]+1;
                d[i*x]=d[i]/(F[i]+1)*(F[i]+2);
                pk[i*x]=pk[i]*x+1;
                sigma[i*x]=sigma[i]/pk[i]*pk[i*x];
                break;
            }
            else{//i,x互质
                mu[i*x]=-mu[i];
                phi[i*x]=phi[i]*phi[x];
                F[i*x]=1;
                d[i*x]=d[i]*d[x];
                pk[i*x]=1+x;
                sigma[i*x]=sigma[i]*sigma[x];
            }
        }
    }
}
```

2.迪利克雷卷积

性质

1) 结合律 , $f*(g*h)=(f*g)*h$

2) 交换律 , $f*g=g*f$

3) 当 $\quad h=f*g \quad$ 时

可得 $\quad h(n)=\sum_{d|n} f(d)*g(\frac{n}{d}) \quad$ 或 $\quad h(n)=\sum_{n|d} f(d)*g(\frac{d}{n})$

4) 基本公式

$$\varphi * I = id$$
$$\mu * I = e$$
$$\mu * id = \varphi$$

3.杜教筛

用于求积性函数 f(i) 的前缀和

$$S(n) = \sum_{i=1}^{n} f(i)$$

构造h和g使得$h = f * g$

$\sum_{i=1}^{n} h(i)$

$= \sum_{i=1}^{n} \sum_{d|i} g(d) \cdot f(\frac{i}{d})$

$= \sum_{d=1}^{n} g(d) \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} f(i)$

$= \sum_{d=1}^{n} g(d) \cdot S(\lfloor \frac{n}{d} \rfloor)$

$= g(1) \cdot S(n) + \sum_{d=2}^{n} g(d) \cdot S(\lfloor \frac{n}{d} \rfloor)$

因此可得

$$g(1) \cdot S(n) = \sum_{i=1}^{n} h(i) - \sum_{d=2}^{n} g(d) \cdot S(\lfloor \frac{n}{d} \rfloor)$$

一般线性筛至$\sqrt{n}$左右，再用杜教筛。

1)模板题

①求$S(n) = \sum_{i=1}^{n} \mu(i)$　　luogu - 4213

取$g = I$，$h = \mu * I = e$

$\therefore I(1) \cdot S(n) = \sum_{i=1}^{n} e(i) - \sum_{d=2}^{n} I(d) \cdot S(\lfloor \frac{n}{d} \rfloor)$

$S(n) = 1 - \sum_{d=2}^{n} S(\lfloor \frac{n}{d} \rfloor)$

②求$S(n) = \sum_{i=1}^{n} \varphi(i)$　　luogu - 4213

取$g = I$，$h = \varphi * I = id$

$\therefore I(1) \cdot S(n) = \sum_{i=1}^{n} id(i) - \sum_{d=2}^{n} I(d) \cdot S(\lfloor \frac{n}{d} \rfloor)$

$S(n) = \frac{n(n+1)}{2} - \sum_{d=2}^{n} S(\lfloor \frac{n}{d} \rfloor)$

③求$S(n) = \sum_{i=1}^{n} i \cdot \varphi(i)$

取$g = id$，$h(n) = \sum_{d|n} d \cdot \varphi(d) \cdot \frac{n}{d} = n \sum_{d|n} \varphi(d) = n^2$

$$\therefore id(1)\cdot S(n)=\sum_{i=1}^{n} i^2-\sum_{d=2}^{n} id(d)\cdot S(\lfloor \frac{n}{d} \rfloor)$$

$$S(n)=\frac{n(n+1)(2n+1)}{6}-\sum_{d=2}^{n} d\cdot S(\lfloor \frac{n}{d} \rfloor)$$

2)杜教筛模板(①+②)

```
typedef long long ll;
const int MAXN=5000010;//线性筛到最大的2/3次方
int prime[MAXN],pnum=0;
bool vis[MAXN];
int phi[MAXN],mu[MAXN];//欧拉函数和莫比乌斯函数
ll sum1[MAXN],sum2[MAXN];//存MAXN以下的前缀和
void sieve(){//线性筛
    memset(vis,true,sizeof(vis));
    vis[0]=vis[1]=false;
    mu[1]=1;
    phi[1]=1;
    for(int i=2;i<MAXN;i++){
        if(vis[i]){
            prime[pnum++]=i;
            mu[i]=-1;
            phi[i]=i-1;
        }
        for(int j=0;j<pnum&&1ll*i*prime[j]<MAXN;j++){
            int x=prime[j];
            vis[i*x]=false;
            if(i%x==0){
                mu[i*x]=0;
                phi[i*x]=phi[i]*x;
                break;
            }
            else{
                mu[i*x]=-mu[i];
                phi[i*x]=(x-1)*phi[i];
            }
        }
    }
    for(int i=1;i<=MAXN;i++){//求一部分前缀和
        sum1[i]=sum1[i-1]+phi[i];
        sum2[i]=sum2[i-1]+mu[i];
    }
}
unordered_map<int,ll>sumphi;//数组下标放不下
ll djsphi(int x){
    if(x<=MAXN)
        return sum1[x];
    if(sumphi[x])
        return sumphi[x];
    ll ans=1ll*(x+1)*x/2;
    for(int l=2,r;l<=x;l=r+1){
        r=x/(x/l);
        ans-=1ll*(r-l+1)*djsphi(x/l);
    }
    return sumphi[x]=ans;//记忆化保存前缀和
}
```

```
unordered_map<int,int>summu;
ll djsmu(int x){
    if(x<=MAXN)
        return sum2[x];
    if(summu[x])
        return summu[x];
    ll ans=1;
    for(int l=2,r;l<=x;l=r+1){
        r=x/(x/l);
        ans-=1ll*(r-l+1)*djsmu(x/l);
    }
    return summu[x]=ans;
}
```

4.例题

1.$\sum_{i=1}^{n}\sum_{j=1}^{m}[gcd(i,j)=1]$

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m}[gcd(i,j)=1]\\=&\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{d|gcd(i,j)}\mu(d)\\=&\sum_{d=1}^{min(n,m)}\mu(d)\sum_{i=1}^{\lfloor\frac{n}{d}\rfloor}\sum_{j=1}^{\lfloor\frac{m}{d}\rfloor}\\=&\sum_{d=1}^{min(n,m)}\mu(d)\cdot\lfloor\frac{n}{d}\rfloor\cdot\lfloor\frac{m}{d}\rfloor\end{aligned}$$

2.$\sum_{i=1}^{n}\sum_{j=1}^{m}[gcd(i,j)=k]$ hdu - 1695

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m}[gcd(i,j)==k]\\=&\sum_{i=1}^{n}\sum_{j=1}^{m}[gcd(\frac{i}{k},\frac{j}{k})=1]\\=&\sum_{i=1}^{\lfloor\frac{n}{k}\rfloor}\sum_{j=1}^{\lfloor\frac{m}{k}\rfloor}[gcd(i,j)=1]\\=&\sum_{i=1}^{\lfloor\frac{n}{k}\rfloor}\sum_{j=1}^{\lfloor\frac{m}{k}\rfloor}\sum_{d|gcd(i,j)}\mu(d)\\=&\sum_{d=1}^{min(\lfloor\frac{n}{k}\rfloor,\lfloor\frac{m}{k}\rfloor)}\mu(d)\sum_{i=1}^{\lfloor\frac{n}{kd}\rfloor}\sum_{j=1}^{\lfloor\frac{m}{kd}\rfloor}\\=&\sum_{d=1}^{min(\lfloor\frac{n}{k}\rfloor,\lfloor\frac{m}{k}\rfloor)}\mu(d)\cdot\lfloor\frac{n}{kd}\rfloor\cdot\lfloor\frac{m}{kd}\rfloor\end{aligned}$$

3.$\sum_{i=1}^{n}\sum_{j=1}^{m}[gcd(i,j)=prime]$ luogu - 2257

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m}[gcd(i,j)=prime]\\=&\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{p=2}^{min(n,m)}[gcd(i,j)=p]\\=&\sum_{p=2}^{min(n,m)}\sum_{i=1}^{\lfloor\frac{n}{p}\rfloor}\sum_{j=1}^{\lfloor\frac{m}{p}\rfloor}[gcd(i,j)=1]\\=&\sum_{p=2}^{min(n,m)}\sum_{i=1}^{\lfloor\frac{n}{p}\rfloor}\sum_{j=1}^{\lfloor\frac{m}{p}\rfloor}\sum_{d|gcd(i,j)}\mu(d)\end{aligned}$$

$= \sum_{p=2}^{min(n,m)} \sum_{d=1}^{min(\lfloor \frac{n}{p} \rfloor, \lfloor \frac{m}{p} \rfloor)} \mu(d) \cdot \lfloor \frac{n}{dp} \rfloor \cdot \lfloor \frac{n}{dp} \rfloor$

令 x = dp

$= \sum_{x=2}^{min(n,m)} \lfloor \frac{n}{x} \rfloor \cdot \lfloor \frac{m}{x} \rfloor \sum_{p|x} \mu(\lfloor \frac{x}{p} \rfloor)$

令 $f(x) = \sum_{p|x} \mu(\lfloor \frac{x}{p} \rfloor)$，线性筛$f(x)$

$f(1) = 0$

$f(p) = 1$

$i\%x = 0 \quad f(i \cdot x) = \mu(x)$

$i\%x \neq 0 \quad f(i \cdot x) = \mu(x) - f(x)$

整除分块$(\sqrt{n})$ 解决

4.$\sum_{i=1}^{n} \sum_{j=1}^{m} lcm(i,j)$ bzoj - 2154

$\sum_{i=1}^{n} \sum_{j=1}^{m} lcm(i,j)$

$= \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{i \cdot j}{gcd(i,j)}$

$= \sum_{d=1}^{min(n,m)} \sum_{i=1}^{n} \sum_{j=1}^{m} [gcd(i,j) = d] \frac{i \cdot j}{d}$

$= \sum_{d=1}^{min(n,m)} \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} \sum_{j=1}^{\lfloor \frac{m}{d} \rfloor} [gcd(i,j) = 1] i \cdot j$

$= \sum_{d=1}^{min(n,m)} d \cdot \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} i \cdot \sum_{j=1}^{\lfloor \frac{m}{d} \rfloor} j \cdot \sum_{x|gcd(i,j)} \mu(x)$

$= \sum_{d=1}^{min(n,m)} d \cdot \sum_{x=1}^{min(\lfloor \frac{n}{d} \rfloor, \lfloor \frac{m}{d} \rfloor)} \mu(x) \cdot \sum_{i=1}^{\lfloor \frac{n}{dx} \rfloor} i \cdot x \sum_{j=1}^{\lfloor \frac{m}{dx} \rfloor} j \cdot x$

$= \sum_{d=1}^{min(n,m)} d \cdot \sum_{x=1}^{min(\lfloor \frac{n}{d} \rfloor, \lfloor \frac{m}{d} \rfloor)} \mu(x) \cdot x^2 \sum_{i=1}^{\lfloor \frac{n}{dx} \rfloor} i \cdot \sum_{j=1}^{\lfloor \frac{m}{dx} \rfloor} j$

$= \sum_{d=1}^{min(n,m)} d \cdot \sum_{x=1}^{min(\lfloor \frac{n}{d} \rfloor, \lfloor \frac{m}{d} \rfloor)} \mu(x) \cdot x^2 \cdot \frac{\lfloor \frac{n}{dx} \rfloor \cdot (\lfloor \frac{n}{dx} \rfloor + 1)}{2} \cdot \frac{\lfloor \frac{m}{dx} \rfloor \cdot (\lfloor \frac{m}{dx} \rfloor + 1)}{2}$

令 $D = dx$

$= \sum_{D=1}^{min(n,m)} \cdot \frac{\lfloor \frac{n}{D} \rfloor \cdot (\lfloor \frac{n}{D} \rfloor + 1)}{2} \cdot \frac{\lfloor \frac{m}{D} \rfloor \cdot (\lfloor \frac{m}{D} \rfloor + 1)}{2} \cdot \sum_{x|D} \frac{D}{x} \cdot x^2 \cdot \mu(x)$

$= \sum_{D=1}^{min(n,m)} \cdot \frac{\lfloor \frac{n}{D} \rfloor \cdot (\lfloor \frac{n}{D} \rfloor + 1)}{2} \cdot \frac{\lfloor \frac{m}{D} \rfloor \cdot (\lfloor \frac{m}{D} \rfloor + 1)}{2} \cdot \sum_{x|D} D \cdot x \cdot \mu(x)$

令 $f(D) = \sum_{x|D} D \cdot x \cdot \mu(x)$ 积性函数，线性筛$f(D)$

$f(1) = 1$

$f(p) = p - p^2$

$i\%x = 0 \quad f(i \cdot x) = f(i) \cdot x$

$i\%x \neq 0 \quad f(i \cdot x) = f(i) \cdot f(x) = f(i) \cdot (x - x^2)$

整除分块$(\sqrt{n})$ 解决

5.$\sum_{i=1}^{A} \sum_{j=1}^{B} \sum_{k=1}^{C} \varphi(gcd(i, j^2, k^3))$ hdu - 6428

$\sum_{i=1}^{A}\sum_{j=1}^{B}\sum_{k=1}^{C}\varphi(gcd(i,j^2,k^3))$

$=\sum_{x=1}^{A}\varphi(k)\sum_{i=1}^{A}\sum_{j=1}^{B}\sum_{k=1}^{C}[gcd(i,j^2,k^3)=x]$

令 $f(x)=\sum_{i=1}^{A}\sum_{j=1}^{B}\sum_{k=1}^{C}[gcd(i,j^2,k^3)=x]$ ， $g(x)=\sum_{i=1}^{A}\sum_{j=1}^{B}\sum_{k=1}^{C}[x|gcd(i,j^2,k^3)]$

$\therefore g(x)=\sum_{x|d}f(d)$, 即 $g=f*I$

$\because \mu*I=e$

$\therefore f=g*\mu$

$f(x)=\sum_{x|d}\mu(\frac{d}{x}))\cdot g(d)$

$=\sum_{x|d}\mu(\frac{d}{x})\sum_{i=1}^{A}\sum_{j=1}^{B}\sum_{k=1}^{C}[d|gcd(i,j^2,k^3)]$

$=\sum_{x|d}\mu(\frac{d}{x})\sum_{i=1}^{A}[d|i]\sum_{j=1}^{B}[d|j^2]\sum_{k=1}^{C}[d|k^3]$

$=\sum_{x=1}^{A}\varphi(x)\cdot f(x)$

$=\sum_{x=1}^{A}\varphi(x)\cdot\sum_{x|d}\mu(\frac{d}{x})\sum_{i=1}^{A}[d|i]\sum_{j=1}^{B}[d|j^2]\sum_{k=1}^{C}[d|k^3]$

$=\sum_{d=1}^{A}\quad\sum_{x|d}\mu(\frac{d}{x})\cdot\varphi(x)\quad\sum_{i=1}^{A}[d|i]\quad\sum_{j=1}^{B}[d|j^2]\quad\sum_{k=1}^{C}[d|k^3]$

预处理 $\sum_{x|d}\mu(\frac{d}{x})\cdot\varphi(x)$ ， $\sum_{i=1}^{A}[d|i]$ ， $\sum_{j=1}^{B}[d|j^2]$ 和 $\sum_{k=1}^{C}[d|k^3]$

令 $h(x)=\sum_{x|d}\mu(\frac{d}{x})\cdot\varphi(x)$ 积性函数，线性筛$h(x)$

$h(1)=1$

$h(p)=p-2$

$h(p^k)=(p-1)^2\cdot p^{k-2}$

$i\%x\neq 0\quad h(i\cdot x)=h(i)\cdot h(x)$

$i\%x=0\quad h(i\cdot x)=h(\frac{i}{x^{a_1}})\cdot h(x^{a_1+1})=h(\frac{i}{x^{a_1}})\cdot(x-1)^2\cdot x^{a_1-1}\quad(i=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k})$

$a_1=F(i)$，即最小质因子的次数，之前筛过

令$f_k(x)=\prod_{i=1}^{n}p_i^{\lceil\frac{a_i}{k}\rceil}\quad(x=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k})$ 积性函数

$\therefore\sum_{i=1}^{A}[d|i]=\lfloor\frac{A}{f_1(d)}\rfloor=\lfloor\frac{A}{d}\rfloor$

$\sum_{j=1}^{B}[d|j^2]=\lfloor\frac{B}{f_2(d)}\rfloor$

$\sum_{k=1}^{C}[d|k^3]=\lfloor\frac{C}{f_3(d)}\rfloor$

利用$F(i)$，线性筛$f_2(x)$和$f_3(x)$

$f_k(1)=1$

$f_k(p)=p$

$i\%x=0\quad f_k(i\cdot x)=f_k(i)\cdot x^{\lceil\frac{a_1+1}{k}\rceil-\lceil\frac{a_1}{k}\rceil}$

$i\%x\neq 0\quad f_k(i\cdot x)=f_k(i)\cdot f_k(x)$

6.$\sum_{i=1}^{n}\sum_{j=1}^{i}gcd(i^a-j^a,i^b-j^b)[gcd(i,j)=1]$ hdu - 6706

$\sum_{i=1}^{n}\sum_{j=1}^{i} gcd(i^a-j^a,i^b-j^b)[gcd(i,j)=1]$

$=\sum_{i=1}^{n}\sum_{j=1}^{i}(i-j)[gcd(i,j)=1]$

$=\sum_{i=1}^{n}\sum_{j=1}^{i} j\cdot[gcd(i,j)=1]$

利用欧拉函数性质

小于n且与n互质的数的和：$S(n)=\begin{cases}\frac{n\cdot\varphi(n)}{2}，& n\geqq 2\\ 0，& n=1\end{cases}$

$=\sum_{i=2}^{n} S(i)+S(1)$

$=\sum_{i=1}^{n}\frac{i\cdot\varphi(i)}{2}-\frac{1}{2}$

$=\frac{1}{2}\sum_{i=1}^{n}(i\cdot\varphi(i))-\frac{1}{2}$

令$f(n)=\sum_{i=1}^{n} i\cdot\varphi(i)$

杜教筛模板，求得$f(n)=\frac{n(n+1)(2n+1)}{6}-\sum_{d=2}^{n} d\cdot f(\lfloor\frac{n}{d}\rfloor)$

7.$\sum_{i=1}^{N}\sum_{j=1}^{N} lcm(a_i,a_j)$ AtCoder - 5200

$\sum_{i=1}^{N}\sum_{j=1}^{N} lcm(a_i,a_j)$

$=\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{a_ia_j}{gcd(a_i,a_j)}$

$=\sum_{g=1}^{1e6}\sum_{i=1}^{N}\sum_{j=1}^{N}[gcd(a_i,a_j)=g]\cdot\frac{a_ia_j}{g}$

$=\sum_{g=1}^{1e6}\frac{1}{g}\cdot\sum_{i=1}^{N}\sum_{j=1}^{N} a_ia_j\cdot\sum_{d|gcd(\frac{a_i}{g},\frac{a_j}{g})}\mu(d)$

$=\sum_{g=1}^{1e6}\frac{1}{g}\cdot\sum_{i=1}^{N}\sum_{j=1}^{N} a_ia_j\cdot\sum_{d|\frac{a_i}{g},d|\frac{a_j}{g}}\mu(d)$

$=\sum_{g=1}^{1e6}\frac{1}{g}\cdot\sum_{d=1}^{\frac{1e6}{g}}\mu(d)\cdot\sum_{gd|a_i}\sum_{gd|a_j} a_ia_j$

$=\sum_{gd=1}^{1e6}\sum_{d|gd}\mu(d)\cdot\frac{1}{id}(\frac{gd}{d})\cdot(\sum_{gd|a_i} a_i)^2$

$=\sum_{n=1}^{1e6}\sum_{d|n}\mu(d)\cdot\frac{1}{id}(\frac{n}{d})\cdot(\sum_{n|a_i} a_i)^2$

令$f=\mu*\frac{1}{id}$，$f(n)=\sum_{d|n}\mu(d)\cdot\frac{1}{id}(\frac{n}{d})=\sum_{d|n}\mu(d)\cdot\frac{d}{n}$

$f(1)=1$

$f(p)=\frac{1}{p}-1$

$i\%x=0\quad f(i\cdot x)=\frac{f(i)}{x}$

$i\%x\neq 0\quad f(i\cdot x)=\frac{f(i)}{x}-f(i)$

令$g(n)=(\sum_{n|a_i} a_i)^2$　预处理$g(n)$，$O(nlog^n)$

8.$\sum_{i=1}^{n}\sum_{j=1}^{m}\mu(lcm(i,j))$ hdu - 6715

$\sum_{i=1}^{n}\sum_{j=1}^{m}\mu(lcm(i,j))$

其中$\mu(lcm(i,j))=\mu(i)\cdot\mu(j)\cdot\mu(gcd(i,j))$

证明:

①当 $\mu(i)$ 中 $\mu(j)$ 有一个等于0时，$\mu(lcm(i,j))$ 必然等于0，等式成立

②当 $\mu(i)\cdot\mu(j)\neq 0$ 时，说明 $\frac{i}{gcd(i,j)}$，$\frac{j}{gcd(i,j)}$，$gcd(i,j)$ 三者必然互相互质

$\therefore \mu(lcm(i,j))=\mu(\frac{ij}{gcd(i,j)})=\mu(\frac{i}{gcd(i,j)})\cdot\mu(\frac{j}{gcd(i,j)})\cdot\mu(gcd(i,j))$

当 $\mu(gcd(i,j))=0$ 时，左式和右式都等于0，等式成立

当 $\mu(gcd(i,j))\neq 0$ 时，$\mu^2(gcd(i,j))=1$

$\therefore \mu(lcm(i,j))=\mu(\frac{i}{gcd(i,j)})\cdot\mu(\frac{j}{gcd(i,j)})\cdot\mu^3(gcd(i,j))=\mu(i)\cdot\mu(j)\cdot\mu(gcd(i,j))$，等式成立

$=\sum_{i=1}^{n}\sum_{j=1}^{m}=\mu(i)\cdot\mu(j)\cdot\mu(gcd(i,j))$

$=\sum_{d=1}^{min(n,m)}\sum_{i=1}^{n}\sum_{j=1}^{m}[gcd(i,j)=d]\mu(i)\cdot\mu(j)\cdot\mu(d)$

$=\sum_{d=1}^{min(n,m)}\mu(d)\cdot\sum_{i=1}^{\lfloor\frac{n}{d}\rfloor}\sum_{j=1}^{\lfloor\frac{m}{d}\rfloor}[gcd(i,j)=1]\mu(id)\cdot\mu(jd)$

$=\sum_{d=1}^{min(n,m)}\mu(d)\cdot\sum_{k=1}^{min(\lfloor\frac{n}{d}\rfloor,\lfloor\frac{m}{d}\rfloor)}\mu(k)\cdot\sum_{i=1}^{\lfloor\frac{n}{kd}\rfloor}\mu(ikd)\cdot\sum_{j=1}^{\lfloor\frac{m}{kd}\rfloor}\mu(jkd)$

令$x=kd$

$=\sum_{x=1}^{min(n,m)}\quad\sum_{d|x}\mu(d)\cdot\mu(\frac{x}{d})\quad\sum_{i=1}^{\lfloor\frac{n}{x}\rfloor}\mu(ix)\quad\sum_{j=1}^{\lfloor\frac{m}{x}\rfloor}\mu(jx)$

预处理 $\sum_{d|x}\mu(d)\cdot\mu(\frac{x}{d})$，$\sum_{i=1}^{\lfloor\frac{n}{x}\rfloor}\mu(ix)$，$\sum_{j=1}^{\lfloor\frac{m}{x}\rfloor}\mu(jx)$ $\qquad 3\cdot O(nlog^n)$