Question 1. (i) $-u''+u=f$ $v(-u''+u)=f\cdot v$ then integrate from 0 to 1:

$$\int_0^1 -u''\cdot v\,dx + \int_0^1 uv\,dx = \int_0^1 f\cdot v\,dx \quad ①$$

$$\int_0^1 -u''v\,dx = -u'v\Big|_0^1 + \int_0^1 u'v'\,dx$$

$$= \int_0^1 u'v'\,dx \qquad (v(0)=0)$$

$$① \rightarrow \int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 f\cdot v\,dx$$

$$\int_0^1 (u'v'+uv)\,dx = \int_0^1 f(x)v(x)\,dx$$

(ii) $-u''=f$, $-u''v=fv$ then integrate from 0 to 1:

$$\int_0^1 -u''v\,dx = \int_0^1 fv\,dx$$

$$\int_0^1 u'v'\,dx = \int_0^1 fv\,dx \qquad (v(0)=v(1)=0)$$

(iii) $-u''+u=f$ $v(-u''+u)=v\cdot f$

$$\int_0^1 (-u''v+uv)\,dx = \int_0^1 f\cdot v\,dx$$

$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx$$

$$\int_0^1 (u'v'+uv)\,dx = \int_0^1 fv\,dx.$$

Q2: let $e(x) = u(x) - u_1(x)$. Since it's linear, $e''(x) = u''(x)$

$$\int_{x_{i-1}}^{x_i} (u-u_1)^2 dx \le c(x_i - x_{i-1})^4 \int_{x_{i-1}}^{x_i} (u'')^2 dx \quad ①$$

$$\Rightarrow \int_{x_{i-1}}^{x_i} e(x)^2 dx \le c(x_i - x_{i-1})^4 \int_{x_{i-1}}^{x_i} e''(x)^2 dx \quad ②$$

let $\tilde{e}(\tilde{x}) = e\left(x_{i-1} + \tilde{x}(x_i - x_{i-1})\right)$

② becomes $\int_0^1 \tilde{e}(\tilde{x})^2 d\tilde{x} \le c \int_0^1 \tilde{e}''(\tilde{x})^2 dx$

Since $u(x_i) = u_1(x_i)$, $u(x_{i-1}) = u_1(x_{i-1})$,

$\tilde{e}(0) = 0, \tilde{e}(1) = 0$

$\therefore \left|\tilde{e}(x')\right| = \left|\int_0^{x'} \tilde{e}'(t)\, dt\right| \le \left|\int_0^{x'} 1\, dt\right|^{\frac{1}{2}} \left|\int_0^{x'} \tilde{e}'(t)^2 dt\right|^{\frac{1}{2}}$
$(x' \in [0,1])$

$= \left|\int_0^{x'} \tilde{e}'(t)^2 dt\right|^{\frac{1}{2}}$

$\therefore \tilde{e}(x')^2 \le \left|\int_0^{x'} \tilde{e}'(t)^2 dt\right|$

$\tilde{e}'(t)^2 \ge 0,\ x' \in [0,1]$

$\Rightarrow \tilde{e}(x')^2 \le \left|\int_0^1 \tilde{e}'(t)^2 dt\right|$

$\le c\left|\int_0^1 \tilde{e}''(t)^2 dt\right|$

$$\Rightarrow \int_0^1 \tilde{e}(x')^2 dx' \le c \int_0^1 \left|\int_0^1 \tilde{e}''(t)^2 dt\right| = c\left|\int_0^1 \tilde{e}''(t)^2 dt\right|$$

$$\therefore \int_{x_{i-1}}^{x_i} (u(x) - u_1(x))^2 dx \le c(x_i - x_{i-1})^4 \int_{x_{i-1}}^{x_i} (u''(x))^2 dx$$

$$\therefore \int_0^1 (u-u_1)^2 dx \le ch^4 \int_0^1 u''^2 dx \quad \therefore \|u - u_1\| \le c' h^2 \|u''\|$$

Q.3

$$V(x) = V(0) + \int_0^x v'(t)\,dt$$

$$= \int_0^x v'(t)\,dt \overset{\text{C-S inequality}}{\leq} \left(\int_0^x dt\right)^{\frac{1}{2}} \left(\int_0^x v'(t)^2 dt\right)^{\frac{1}{2}}$$

$$\leq \left(\int_0^x v'(t)^2 dt\right)^{\frac{1}{2}} \leq \left(\int_0^1 v'(t)^2 dt\right)^{\frac{1}{2}}$$

$$= a(v,v)^{\frac{1}{2}}$$

$$\therefore \quad \|v\|^2 + \|v'\|^2 \leq \|v\|_\infty^2 + \|v'\|^2$$

$$\leq a(v,v) + a(v,v) = 2a(v,v)$$

Q. 4

$$a(u-u_s, u-u_s) = a(u, u-u_s) - a(u_s, u-u_s)$$

$$= f(u-u_s) - a(u_s, u-u_s)$$

$$= f(u-u_s) - a(u, u_s) + a(u_s, u_s)$$

$$= f(u-u_s) - f(u_s) + f(u_s)$$

$$= f(u-u_s)$$

Q.5 $\phi_i(x) = \begin{cases} \frac{x-x_i}{h}, & x_{i-1} \le x \le x_i \\ \frac{x_{i+1}-x}{h}, & x_i \le x \le x_{i+1} \\ 0, & \text{otherwise} \end{cases}$ $\quad \phi_i'(x) = \begin{cases} \frac{1}{h} & x_{i-1} \le x \le x_i \\ -\frac{1}{h} & x_i \le x \le x_{i+1} \\ 0 & \text{otherwise} \end{cases}$

1. $A_{ji} = \int_0^1 \phi_i' \phi_j' dx, \quad F_j = \int_0^1 f(x)\phi_j(x)\,dx$

1) $A_{ii} = \int_0^1 \phi_i' \phi_i' \, dx = \int_{x_{i-1}}^{x_{i+1}} \phi_i'(x)^2 dx = \int_{x_{i-1}}^{x_{i+1}} \frac{1}{h^2} dx$

$= \frac{2}{h}$

2) $A_{i+1,i} = \int_{x_i}^{x_{i+1}} \phi_i'(x)\, \phi_{i+1}'(x)\, dx = \int_{x_i}^{x_{i+1}} -\frac{1}{h^2} dx = -\frac{1}{h}$

3) $A_{i-1,i} = -\frac{1}{h}$

4) otherwise, $A_{ji} = 0$

$F_j = \int_0^1 f(x)\phi_j(x)\,dx = \int_0^1 \phi_j(x)\,dx = \int_{x_{j-1}}^{x_{j+1}} \phi_j(x)\,dx = h$

$\therefore F = \begin{bmatrix} h \\ h \\ \vdots \\ h \end{bmatrix}$

2. $\int_0^1 (x-x^2)/2\,dx = \frac{1}{2}\int_0^1 (x-x^2)\,dx = \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{12}$

See attached files. We draw the log error - log h.

We can see that gradient is 2.

$\therefore \frac{\text{error}}{h^2} = $ costant.