Functions

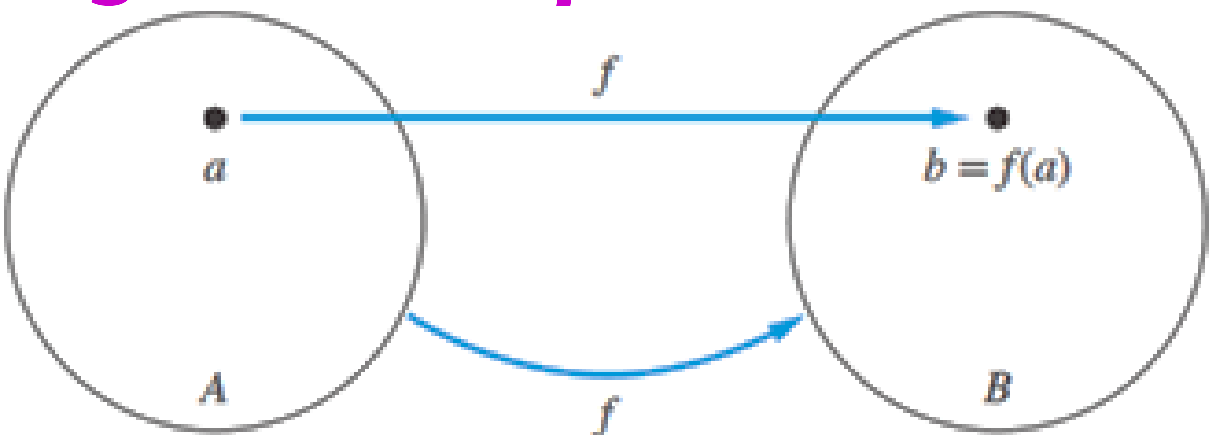
Let A and B be nonempty sets. A ***function*** f from A to B is an assignment of exactly one element of B to each element of A.

We write $\boldsymbol{f(a) = b}$ if **b is the unique element of B** assigned by the function f to the element a of A.

Functions

If f is a function from A to B, we write $f : A \to B$.

Functions are sometimes also called ***mappings*** or ***transformations***.



Functions

Domain & *Codomain*

Image & *Preimage*

If f is a function from A to B, we say that ***A* is the *domain*** of f and ***B* is the *codomain*** of f.

If ***f* (*a*) = *b***, we say that ***b* is the *image*** of a and ***a* is a *preimage*** of b.

Functions

Domain & Codomain

Image & Preimage

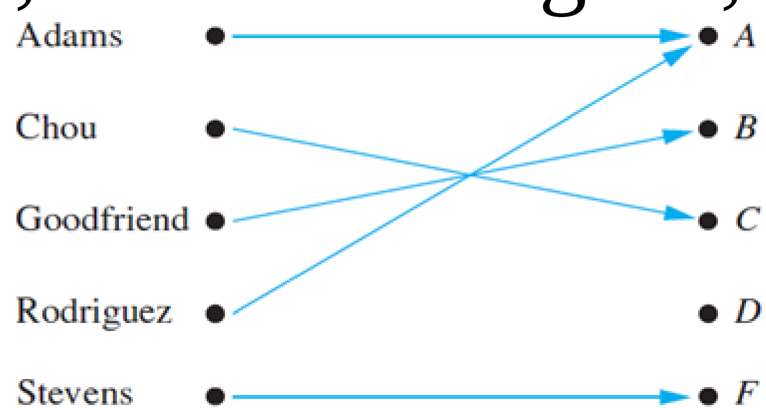
The ***range***, or ***image***, of f is the set of all images of elements of A.

Also, if f is a function from A to B, we say that ***f* *maps* *A* to *B***.

Functions

Examples of a Function

Suppose that each student in a discrete mathematics class is assigned a letter grade from the set $\{A, B, C, D, F\}$. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens.



Functions

Examples of a Function

Let G be the function that assigns a grade to a student in our discrete mathematics class.

Note that $G(\text{Adams}) = A$, for instance.

The domain of G is the set

{Adams, Chou, Goodfriend, Rodriguez, Stevens}, and

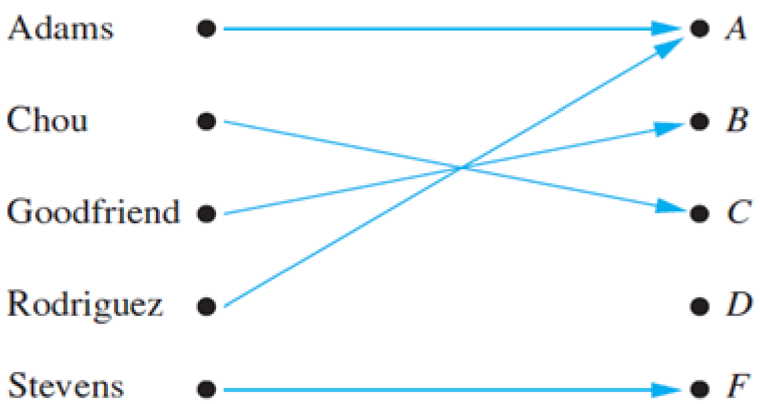
the codomain is the set $\{A, B, C, D, F\}$.

The range of G is the set

$\{A, B, C, F\}$, because each grade

except D is assigned to some

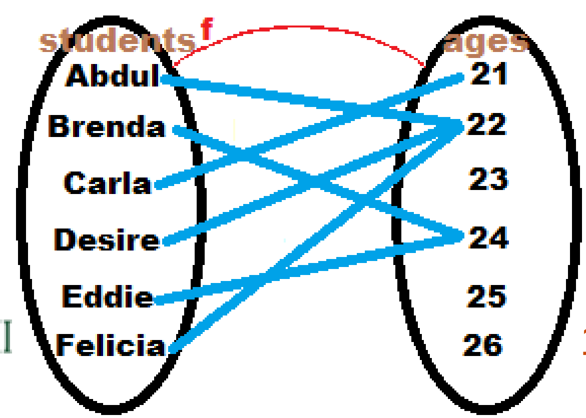
student.



Examples of a Function

Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

Solution:

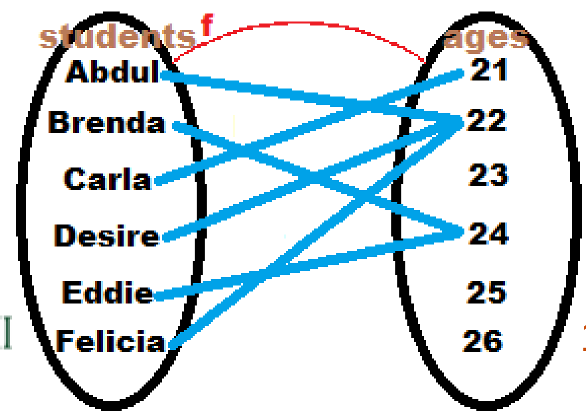


Functions

Examples of a Function

Solution: For the **domain**, we take the set **{Abdul, Brenda, Carla, Desire, Eddie, Felicia}**. We also need to specify a **codomain**, which needs to contain all possible ages of students. Because it is highly likely that all students are less than 26 years old, we can take **the set of positive integers less than 27** as the codomain.

The **range** of the function is the set **{21, 22, 24}.**



Examples of a Function

Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ assign the square of an integer to this integer.

Then, $f(x) = x^2$, where

the domain of f is the set of all integers,

the codomain of f is the set of all integers, and

the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$.

Functions

Real-Valued Function & Integer-Valued Function

A function is called ***real-valued*** if its codomain is the set of real numbers, and

it is called ***integer-valued*** if its codomain is the set of integers.

Two real-valued functions or two integer valued functions with the same domain can be added, as well as multiplied.

Functions

Properties of Real-Valued Function

Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 \,.\, f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$

$$(f_1 \,.\, f_2)(x) = f_1(x)\, f_2(x).$$

Examples of a Function

Let f_1 and f_1 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution: From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and $(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4$.

Examples of a Function

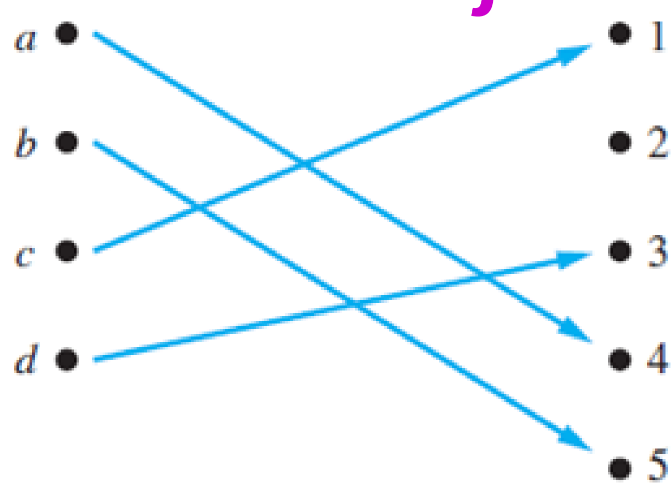
Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$.

The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

One-to-One Function or Injective Function

A function f is said to be ***one-to-one***, or an ***injection***, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

A function is said to be ***injective*** if it is one-to-one.



One-to-One Function or Injective Function

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution: The function $f(x) = x^2$ is not one-to-one because, for instance,

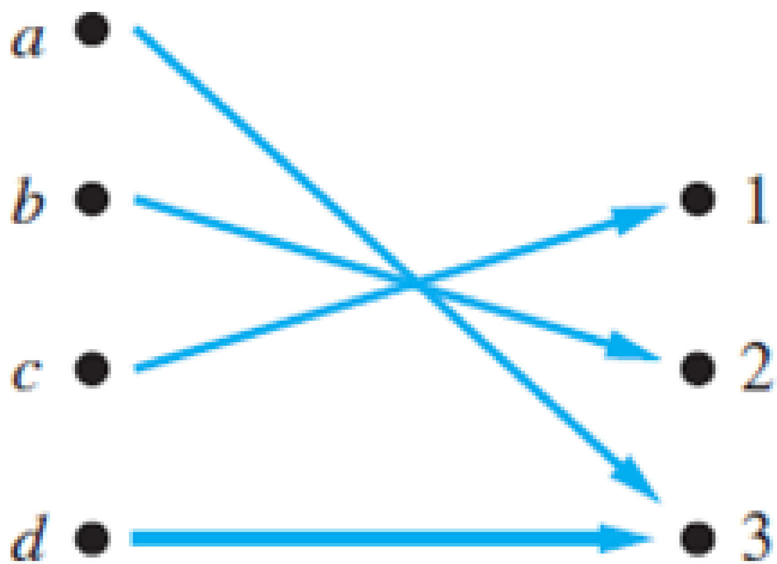
$f(1) = f(-1) = 1$, but $1 \neq -1$.

Increasing Function vs Decreasing Function

A function f whose domain and codomain are subsets of the set of real numbers is called ***increasing*** if $f(x) \leq f(y)$, and ***strictly increasing*** if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called ***decreasing*** if $f(x) \geq f(y)$, and ***strictly decreasing*** if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.

Onto Function or Surjective Function

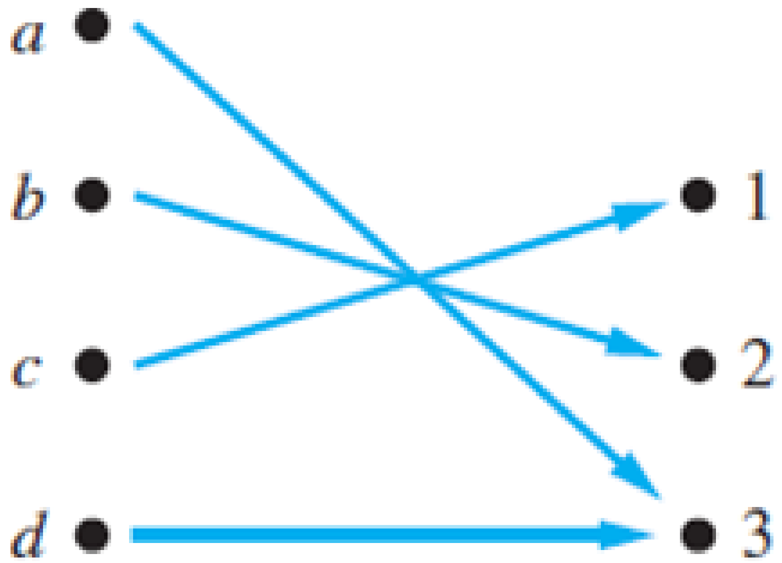
A function f from A to B is called ***onto***, or a ***surjection***, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called ***surjective*** if it is onto.



Functions

Onto Function or Surjective Function

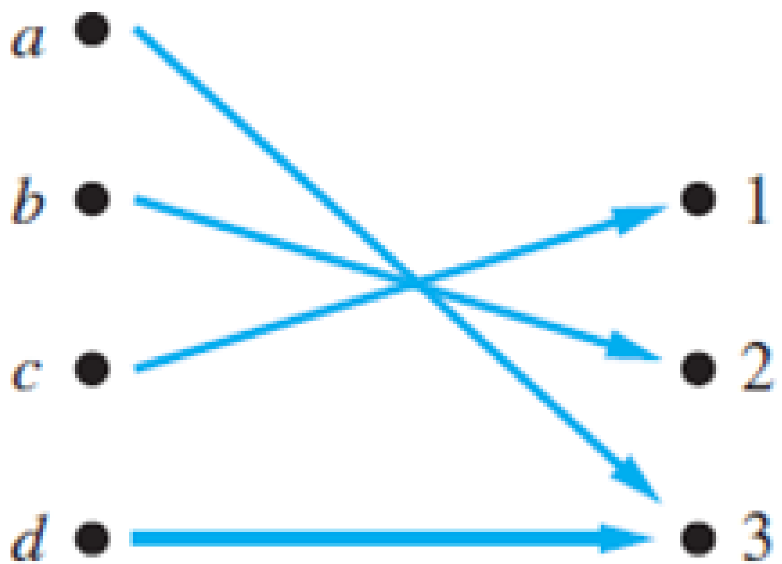
A function f is onto if $\forall y \exists x(f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.



Functions

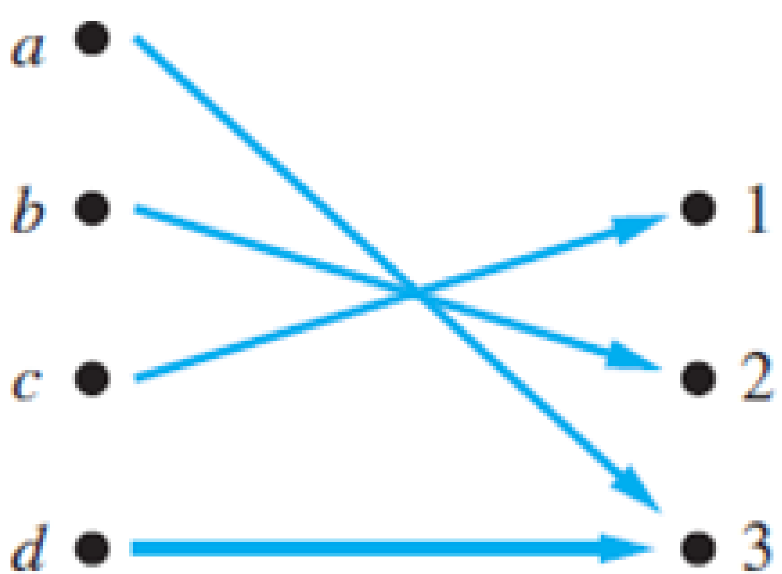
Onto Function or Surjective Function

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?



Onto Function or Surjective Function

Solution: Because all three elements of the codomain are images of elements in the domain, we see that f is onto.



Onto Function or Surjective Function

Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: The function f is not onto because there is no integer x with $x^2 = -1$, for instance.

Onto Function or Surjective Function

Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

Solution: This function is onto, because for every integer y there is an integer x such that $f(x) = y$.

To see this, note that $f(x) = y$ if and only if $x + 1 = y$, which holds if and only if $x = y - 1$. **(Note that $y - 1$ is also an integer, and so, is in the domain of f.)**

One-to-One Onto or Bijective Function

The function f is a ***one-to-one correspondence***, or a ***bijection***, if it is both one-to-one and onto. We also say that such a function is ***bijective***.

One-to-One Onto or Bijective Function

Suppose that each worker in a group of employees is assigned a job from a set of possible jobs, each to be done by a single worker.

In this situation, the function f that assigns a job to each worker is one-to-one. To see this, note that if x and y are two different workers, then $f(x) \neq f(y)$ because the two workers x and y must be assigned different jobs.

Consider the function f that assigns jobs to workers. The function f is onto if for every job there is a worker assigned this job. The function f is not onto when there is at least one job that has no worker assigned it.

One-to-One Onto or Bijective Function

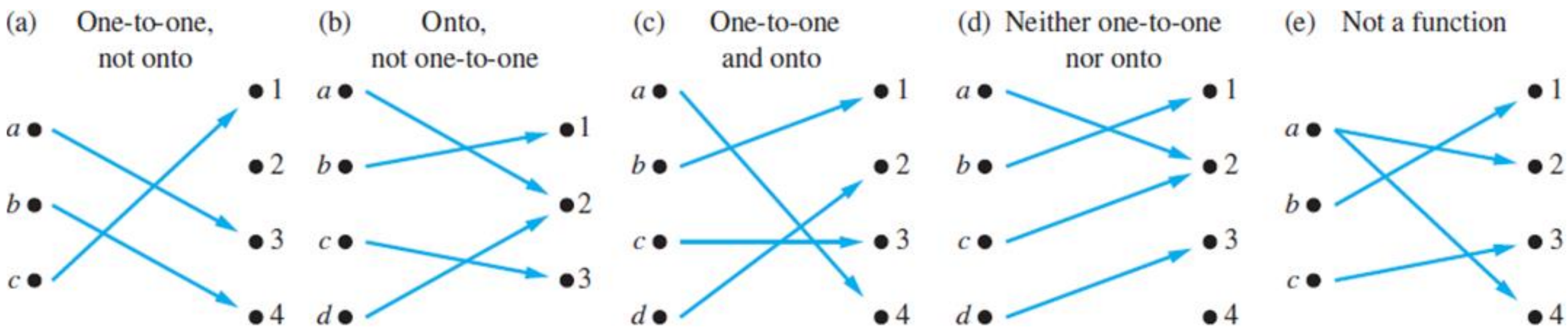
Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?

Solution: The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.

Functions

Functions and their Types

Examples of different types of correspondence



Identity Function

Let A be a set. The ***identity function*** on A is the function $\iota_A\!:\!A\rightarrow A$, where $\iota_A(x)=x$ for all $x \in A$.

In other words, the identity function ιA is the function that assigns each element to itself.

The function ι_A is one-to-one and onto, so it is a bijection. (Note that ι is the Greek letter iota.)

Identification of a Function

Suppose that $f : A \longrightarrow B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

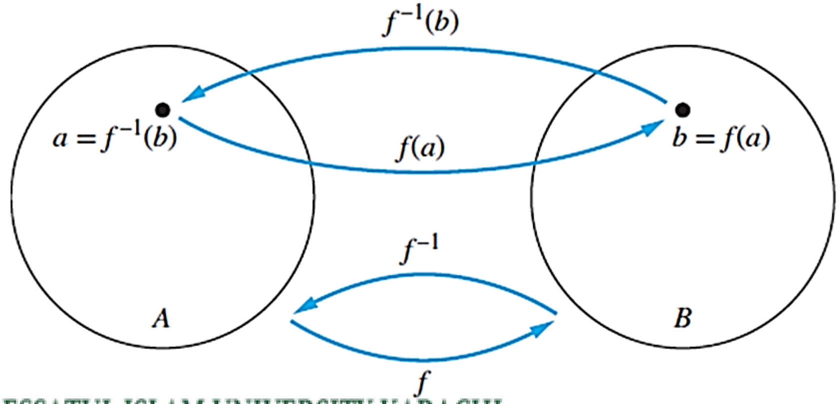
To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Functions

Inverse Function of a Function

Let f be a one-to-one correspondence from the set A to the set B. The ***inverse function*** of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.



Inverse Function of a Function

Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Inverse Function of a Function

Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

Inverse Function of a Function

Let f be the function from **R** to **R** with $f(x) = x^2$. Is f invertible?

Solution: Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible.

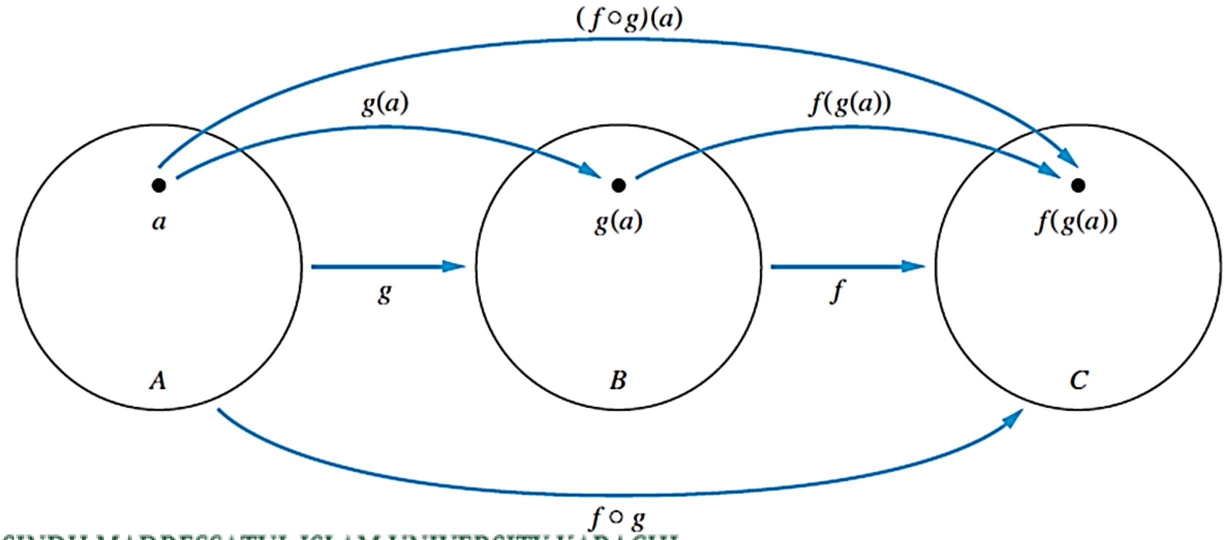
(Note we can also show that f is not invertible because it is not onto.)

Composition of the functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C.

The ***composition*** of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is the function from A to C defined by

$$(f \circ g)(a) = f(g(a)).$$



Composition of the functions

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

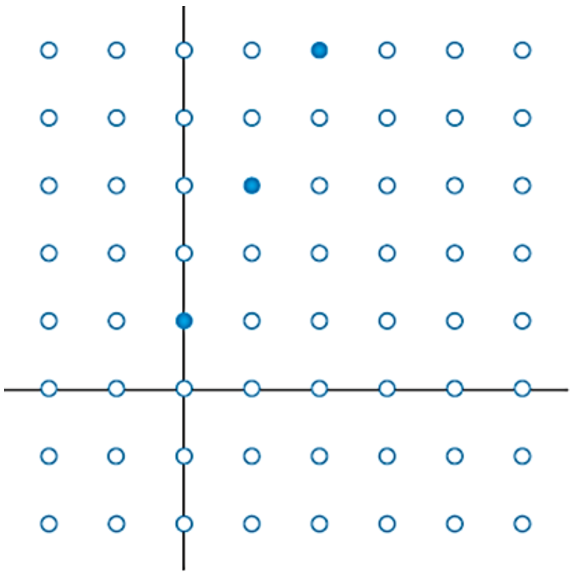
and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

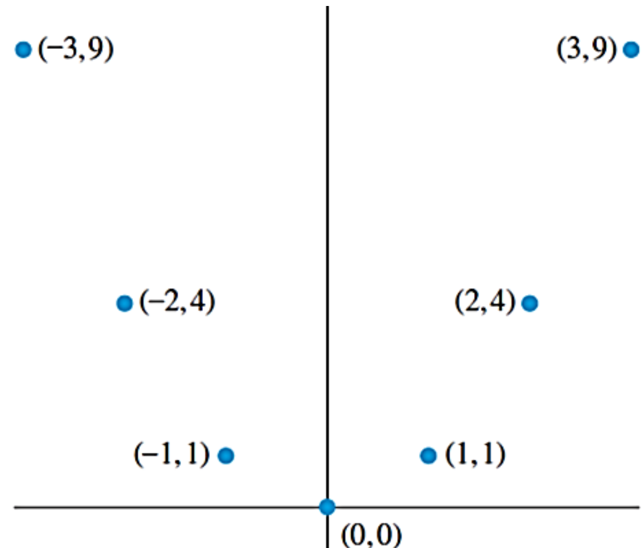
Functions

Graph of a Function

Let f be a function from the set A to the set B. The ***graph*** of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.



The graph of $f(n) = 2n + 1$ from Z to Z.

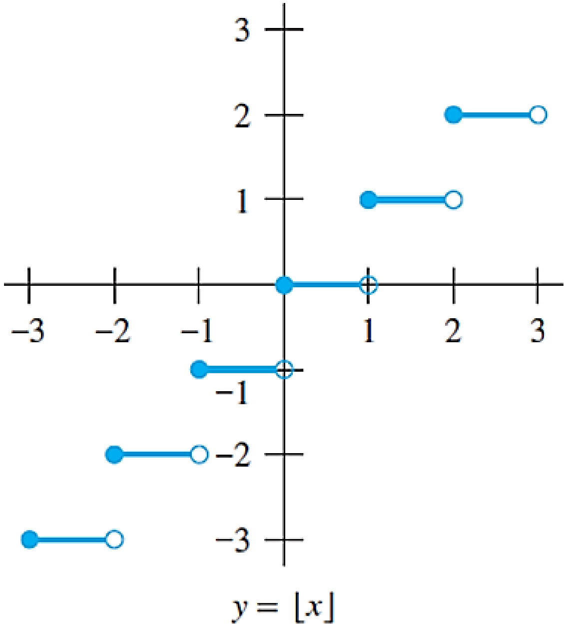


The graph of $f(x) = x^2$ from Z to Z.

Some Important Functions

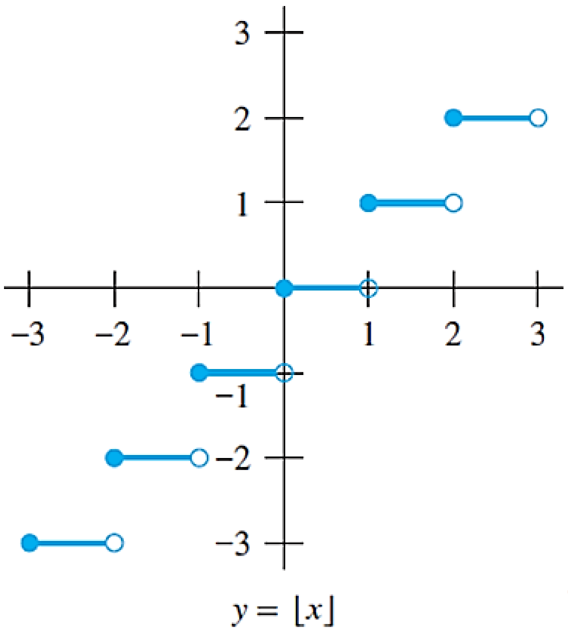
The ***floor function*** assigns to the real number x the largest integer that is less than or equal to x.

The value of the floor function at x is denoted by $\lfloor x \rfloor$.



Some Important Functions

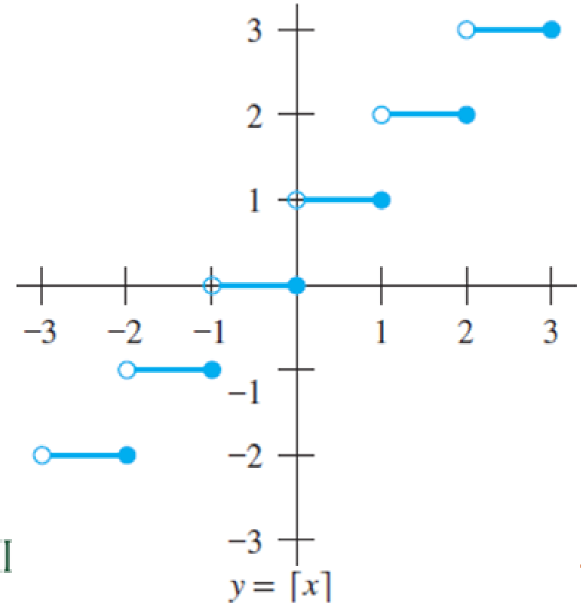
Remark: The ***floor function*** is often also called the greatest integer function. It is often denoted by [x].



Functions

Some Important Functions

The ***ceiling function*** assigns to the real number *x* the smallest integer that is greater than or equal to *x*. The value of the ceiling function at *x* is denoted by $\lceil x \rceil$.



Functions

Properties of Floor and Ceiling Functions

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

n is an integer and ***x*** is a real number

Examples of a Floor and Ceiling Functions

These are some values of the floor and ceiling functions:

$$\left\lfloor \frac{1}{2} \right\rfloor = 0, \qquad \left\lceil \frac{1}{2} \right\rceil = 1,$$

$$\left\lfloor -\frac{1}{2} \right\rfloor = -1, \qquad \left\lceil -\frac{1}{2} \right\rceil = 0,$$

$$\lfloor 3.1 \rfloor = 3, \qquad \lceil 3.1 \rceil = 4,$$

$$\lfloor 7 \rfloor = 7, \qquad \lceil 7 \rceil = 7.$$

Examples of a Floor and Ceiling Functions

Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

Solution:

Number of bits to encode = 100

Number of bits in one byte = 8

Examples of a Floor and Ceiling Functions

Solution:

Number of bits to encode = 100

Number of bits in one byte = 8

To determine the number of bytes needed, we determine the smallest integer that is at least as large as the quotient when 100 is divided by 8, the number of bits in a byte.

Consequently, $\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$ bytes are required.

Examples of a Floor and Ceiling Functions

In Asynchronous Transfer Mode (ATM) ***(a communications protocol used on backbone networks)***, data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 1 minute over a connection that transmits data at the rate of 500 kilobits per second?

Solution: Data transmission,

in 1 second = 500 kilobits = 500,000 bites

in 1 minute = $500{,}000 \times 60$ = 30,000,000 bits.

Each ATM cell = 53 bytes = 53×8 = 424 bits long.

Examples of a Floor and Ceiling Functions

Solution: Data transmission,

in 1 second = 500 kilobits = 500,000 bites

in 1 minute = $500{,}000 \times 60 = 30{,}000{,}000$ bits.

Each ATM cell = 53 bytes = $53 \times 8 = 424$ bits long.

To determine the number of cells that can be transmitted in 1 minute, we determine the largest integer not exceeding the quotient when 30,000,000 is divided by 424.

Consequently, $\lfloor 30{,}000{,}000/424 \rfloor = 70{,}754$ ATM cells can be transmitted in 1 minute over a 500 kilobit per second connection

Factorial Function

Factorial Function $f : N \to Z^+$, denoted by $f(n) = n!$.

The value of $f(n) = n!$ is the product of the first n positive integers, so

$$\boldsymbol{f(n) = 1 \cdot 2 \cdots (n-1) \cdot n}$$

$$[\boldsymbol{and\ f(0) = 0! = 1}]$$

Factorial Function

We have

$f(1) = 1! = 1,$

$f(2) = 2! = 1 \cdot 2 = 2,$

$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720,$ and

$$f(20) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20 =$$

$2{,}432{,}902{,}008{,}176{,}640{,}000.$

Partial and Total Functions

A ***partial function*** *f* from a set *A* to a set *B* is an assignment to each element *a* in a subset of *A*, called the ***domain of definition of f***, of a unique element *b* in *B*.

The sets *A* and *B* are called the *domain* and *codomain* of *f*, respectively.

We say that *f* is *undefined* for elements in *A* that are not in the domain of definition of *f* .

When the domain of definition of *f* equals *A*, we say that *f* is a ***total function***.

Functions

Partial and Total Functions

The function $f : \mathbf{Z} \rightarrow \mathbf{R}$,

where $f(n) = \sqrt{n}$ is a ***partial function*** from $\mathbf{Z}$ to $\mathbf{R}$,

where the ***domain of definition*** is the set of nonnegative integers.

Note that f is undefined for negative integers.

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 2

Functions - Exercise

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Functions - Exercise

2.3 Exercise – Page #: 161 to 164

1. Why is *f* not a function from $\boldsymbol{R}$ to $\boldsymbol{R}$ if

a) $f(x) = 1/x$?

f **(0) is not defined.**

b) $f(x) = \sqrt{x}$?

f **(*x*) is not defined for *x* < 0.**

c) $f(x) = \pm\sqrt{x^2 + 1}$?

f **(*x*) is not well defined because there are two distinct values assigned to each *x*.**

2.3 Exercise – Page #: 161 to 164

2. Determine whether f is a function from $\boldsymbol{Z}$ to $\boldsymbol{R}$ if

a) $f(n) = \pm n.$

***f* (*x*) is not well defined because there are two distinct values assigned to each *x*.**

b) $f(n) = \sqrt{n^2 + 1}.$

It is a function.

c) $f(n) = 1/(n^2 - 4).$

***f* (2) is not defined.**

2.3 Exercise – Page #: 161 to 164

4. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) the function that assigns to each nonnegative integer its last digit.

Domain = Set of nonnegative integers.

Range = {0, 1, 2, 3, …, 9}

2.3 Exercise – Page #: 161 to 164

4. b) the function that assigns the next largest integer to a positive integer

Domain = {1, 2, 3, …}

Range = {2, 3, 4, …}

c) the function that assigns to a bit string the number of one bits in the string

Domain = Set of bitstrings = {0, 1}

Range = Set the natural numbers = {1, 2, 3, …}.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

4.

d) the function that assigns to a bit string the number of bits in the string

Domain = Set of bitstrings = {0, 1}

Range = Set the natural numbers = {1, 2, 3, …}.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

5. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) the function that assigns to each bit string the number of ones in the string minus the number of zeros in the string

Domain = The set of bit strings;

Range = The set of integers

2.3 Exercise – Page #: 161 to 164

b) the function that assigns to each bit string twice the number of zeros in that string

Domain = The set of bit strings;

Range = The set of even nonnegative integers

c) the function that assigns the number of bits left over when a bit string is split into bytes (which are blocks of 8 bits)

Domain = The set of bit strings;

Range = The set of nonnegative integers not exceeding

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

d) the function that assigns to each positive integer the largest perfect square not exceeding this integer

Domain = The set of integers;

Range = The set of set of perfect square

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

6. Find the domain and range of these functions.

a) the function that assigns to each pair of positive integers the first integer of the pair

Domain = The set of ordered pair of integers;

Range = The set of integers

2.3 Exercise – Page #: 161 to 164

b) the function that assigns to each positive integer its largest decimal digit

Domain = The set of all positive integers;

Range = {1, 2, 3, …, 9}

c) the function that assigns to a bit string the number of ones minus the number of zeros in the string

Domain = The set of bitstrings;

Range = The set of natural numbers

2.3 Exercise – Page #: 161 to 164

d) the function that assigns to each positive integer the largest integer not exceeding the square root of the integer

Domain = The set of all positive integers;

Range = The set of square roots of all positive integers minus one.

e) the function that assigns to a bit string the longest string of ones in the string

Domain = The set of bitstrings;

Range = The set of natural numbers

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

7. Find the domain and range of these functions.

a) the function that assigns to each pair of positive integers the maximum of these two integers

Domain = The set of ordered pair of integers;

Range = The set of integers

b) the function that assigns to each positive integer the number of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 that do not appear as decimal digits of the integer

Domain = The set of all positive integers;

Range = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

c) the function that assigns to a bit string the number of times the block 11 appears

Domain = The set of bit strings;

Range = The set of natural numbers

d) the function that assigns to a bit string the numerical position of the first 1 in the string and that assigns the value 0 to a bit string consisting of all 0s

Domain = The set of bit strings;

Range = The set of natural numbers

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

8. Find these values.

a) $\lfloor 1.1 \rfloor = \mathbf{1}$

b) $\lceil 1.1 \rceil = \mathbf{2}$

c) $\lfloor -0.1 \rfloor = \mathbf{-1}$

d) $\lceil -0.1 \rceil = \mathbf{0}$

e) $\lceil 2.99 \rceil = \mathbf{3}$

f) $\lceil -2.99 \rceil = \mathbf{-2}$

g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor = \lfloor \frac{1}{2} + \mathbf{1} \rfloor = \lfloor \mathbf{\frac{3}{2}} \rfloor = \mathbf{1}$

h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil = \lceil \mathbf{0} + \mathbf{1} + \frac{1}{2} \rceil = \lceil \mathbf{\frac{3}{2}} \rceil = \mathbf{2}$

2.3 Exercise – Page #: 161 to 164

9. Find these values.

a) $\lceil \frac{3}{4} \rceil = \mathbf{1}$

b) $\lfloor \frac{7}{8} \rfloor = \mathbf{0}$

c) $\lceil -\frac{3}{4} \rceil = \mathbf{0}$

d) $\lfloor -\frac{7}{8} \rfloor = \mathbf{-1}$

e) $\lceil 3 \rceil = \mathbf{3}$

f) $\lfloor -1 \rfloor = \mathbf{-1}$

g) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor = \lfloor \frac{1}{2} + \mathbf{2} \rfloor = \lfloor \mathbf{\frac{5}{2}} \rfloor = \mathbf{2}$

h) $\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rfloor = \lfloor \frac{1}{2} \cdot \mathbf{2} \rfloor = \lfloor \mathbf{1} \rfloor = \mathbf{1}$

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

10. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

11. Which functions in Exercise 10 are onto?

a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$

One-to-one onto

b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$

Neither one to one nor onto

c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

Neither one to one nor onto

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

12. Determine whether each of these functions from **Z** to **Z** is one-to-one.

13. Which functions in Exercise 12 are onto?

a) $f(n) = n - 1$ **One-to-one onto**

b) $f(n) = n^2 + 1$ **Neither one-to-one nor onto**

c) $f(n) = n^3$ **One-to-one**

d) $f(n) = \lceil n/2 \rceil$ **Onto**

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

14. Determine whether $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = 2m - n$. **Onto**

b) $f(m, n) = m^2 - n^2$. **Not onto**

c) $f(m, n) = m + n + 1$. **Onto**

d) $f(m, n) = |m| - |n|$. **Not onto**

e) $f(m, n) = m^2 - 4$. **Not onto**

2.3 Exercise – Page #: 161 to 164

15. Determine whether the function $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = m + n$. **Onto**

b) $f(m, n) = m^2 + n^2$. **Not onto**

c) $f(m, n) = m$. **Onto**

d) $f(m, n) = |n|$. **Not onto**

e) $f(m, n) = m - n$. **Onto**

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

16. Consider these functions from the set of students in a discrete mathematics class. Under what conditions is the function one-to-one if it assigns to a student his or her

18. Specify a codomain for each of the functions in Exercise 16. Under what conditions is each of these functions with the codomain you specified onto?

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

a) mobile phone number.

If there are no two students with a common phone, the function that assigns to a student his or her mobile phone number is one-to-one.

The set of all mobile phone numbers, there are some which didn't assign to any students, so it is not onto.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

b) student identification number.

Since the student identification number is unique, and hence there are no two students with the same identification number, the function that assigns to a student his or her identification number is one-to-one.

The set of all student identification numbers, there are some which didn't assign to any students, so it is not onto.

2.3 Exercise – Page #: 161 to 164

c) final grade in the class.

Two student can have the same final grade in the class. Therefore, in general case this function is not one-to-one. It can be one-to-one if the number of students is less or equal to the number of final grades and different students have different final grades.

It may one-to-one onto but depend upon the number of students.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

d) home town.

This function in general case is not one-to-one because of two students can live in the same town. It can be injective if different students have different home town.

It is onto, if students belongs to all town of the city.

2.3 Exercise – Page #: 161 to 164

17. Consider these functions from the set of teachers in a school. Under what conditions is the function one-to-one if it assigns to a teacher his or her

19. Specify a codomain for each of the functions in Exercise 17. Under what conditions is each of the functions with the codomain you specified onto?

a) office.

Depends on whether teachers share offices, it may not one-to-one. There is many offices at the school other than teachers' offices; probably not onto.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

b) assigned bus to chaperone in a group of buses taking students on a field trip.

One-to-one assuming only one teacher per bus

Set of buses going on the trip; onto, assuming every bus gets a teacher chaperone

c) salary.

Most likely not one-to-one, especially if salary is set by a collective bargaining agreement. Set of real numbers; not onto

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

d) social security number.

One-to-one because every teacher has unique social security number. Set of strings of nine digits with hyphens after third and fifth digits; not onto.

2.3 Exercise – Page #: 161 to 164

20. Give an example of a function from **N** to **N** that is

a) one-to-one but not onto.

A function from N to N that is $f(n) = 2n$ is one-to-one because of $f(a) = f(b)$ implies $2a = 2b$, and hence $a = b$,

but it is not onto because of the preimage of the odd number 11 is empty set.

2.3 Exercise – Page #: 161 to 164

b) onto but not one-to-one.

A function from N to N that is

$$f(n) = \begin{cases} 1, & if\ n = 1 \\ n - 1, & if\ n > 1 \end{cases}$$

is not one-to-one onto because of the preimage of each natural number nn contains $n + 1$, but it is not one-to-one as $f(1) = 1 = f(2)$.

2.3 Exercise – Page #: 161 to 164

c) both onto and one-to-one (but different from the identity function).

A function from N to N

$$f(n) = \begin{cases} 1, & if\ n = 2 \\ 2, & if\ n = 1 \\ n, & if\ n \geq 1 \end{cases}$$

obviously is onto and one-to-one, and it is different from the identity function.

2.3 Exercise – Page #: 161 to 164

d) neither one-to-one nor onto.

A function from N to N

$$f(n) = 2022$$

is neither one-to-one nor onto.

Indeed, since $f(1) = f(2) = f(3) = 2021$,

we conclude that it is not one-to-one.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

21. Give an explicit formula for a function from the set of integers to the set of positive integers that is

a) one-to-one, but not onto.

The function $f(x)$ with

$$\boldsymbol{f(x) = \begin{cases} 3x+1, & if\ x \geq 0 \\ -3x+1, & if\ n < 0 \end{cases}}$$

b) onto, but not one-to-one.

The function $f(x)$ with

$$\boldsymbol{f(x) = |x| + 1}$$

2.3 Exercise – Page #: 161 to 164

c) one-to-one and onto.

The function $f\,(x)$ with

$$\boldsymbol{f(x) = \begin{cases} 2x + 1, & if\ x \geq 0 \\ -2x, & if\ n < 0 \end{cases}}$$

d) neither one-to-one nor onto.

The function $f\,(x)$ with

$$\boldsymbol{f\,(x) = x^2 + 1}$$

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

22. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = -3x + 4$ **Yes**

b) $f(x) = -3x^2 + 7$ **No**

c) $f(x) = \frac{(x+1)}{(x+2)}$ **No**

d) $f(x) = x^5 + 1$ **Yes**

2.3 Exercise – Page #: 161 to 164

23. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = 2x + 1$ **Yes**

b) $f(x) = x^2 + 1$ **No**

c) $f(x) = x^3$ **Yes**

d) $f(x) = \frac{(x^2 + 1)}{(x^2 + 2)}$ **No**

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

30. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

a) $f(x) = 1$. $\quad f(S) = \{1\}$

b) $f(x) = 2x + 1$. $\quad f(S) = \{0, 1, 5, 9, 15\}$

c) $f(x) = \lceil \frac{x}{5} \rceil$. $\quad f(S) = \{0, 1, 2\}$

d) $f(x) = \lfloor \frac{x^2+1}{3} \rfloor$. $\quad f(S) = \{0, 1, 5, 16\}$

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

31. Let $f(x) = \lfloor \frac{x^2}{3} \rfloor$. Find $f(S)$ if

a) $S = \{-2, -1, 0, 1, 2, 3\}$. $f(S) = \{0, 1, 3\}$

b) $S = \{0, 1, 2, 3, 4, 5\}$. $f(S) = \{0, 1, 3, 5, 8\}$

c) $S = \{1, 5, 7, 11\}$. $f(S) = \{0, 8, 16, 40\}$

d) $S = \{2, 6, 10, 14\}$. $f(S) = \{1, 12, 33, 65\}$

2.3 Exercise – Page #: 161 to 164

32. Let $f(x) = 2x$ where the domain is the set of real numbers. What is

a) $f(\mathbf{Z})$?

$$\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$$

b) $f(\mathbf{N})$?

$$\{2, 4, 6, \ldots\}$$

c) $f(\mathbf{R})$?

$$\mathbb{R}$$

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

38. Find $\boldsymbol{f} \circ \boldsymbol{g}$ and $\boldsymbol{g} \circ \boldsymbol{f}$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from **R** to **R**.

$$\boldsymbol{f \circ g = f\big(g(x)\big) = f(x+2)}$$

$$\boldsymbol{f(x) = x^2 + 1 \text{ then}}$$

$$\boldsymbol{f(x+2) = (x+2)^2 + 1}$$

$$\boldsymbol{f(x+2) = x^2 + 4x + 4 + 1}$$

$$\boldsymbol{f(x+2) = x^2 + 4x + 5}$$

2.3 Exercise – Page #: 161 to 164

38. Find $\boldsymbol{f} \circ \boldsymbol{g}$ and $\boldsymbol{g} \circ \boldsymbol{f}$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from **R** to **R**.

$$\boldsymbol{g \circ f = g(f(x)) = g(x^2 + 1)}$$

$\boldsymbol{g(x) = x + 2}$ **then**

$$\boldsymbol{g(x^2 + 1) = (x^2 + 1) + 2}$$

$$\boldsymbol{g(x^2 + 1) = x^2 + 3}$$

2.3 Exercise – Page #: 161 to 164

40. Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $\boldsymbol{f \circ g = g \circ f}$.

$$\boldsymbol{f \circ g = f\big(g(x)\big) = f(cx + d)}$$

$$f(x) = ax + b$$

$$f(cx + d) = a(cx + d) + b = acx + ad + b$$

$$\boldsymbol{g \circ f = g\big(f(x)\big) = g(ax + b)}$$

$$g(x) = cx + d$$

$$g(ax + b) = c(ax + b) + d = acx + bc + d$$

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

$$\boldsymbol{f \circ g = f\big(g(x)\big) = f(cx+d)}$$

$$f(x) = ax + b$$

$$f(cx+d) = a(cx+d) + b = acx + ad + b$$

$$\boldsymbol{g \circ f = g\big(f(x)\big) = g(ax+b)}$$

$$g(x) = cx + d$$

$$g(ax+b) = c(ax+b) + d = acx + bc + d$$

We have $\boldsymbol{f \circ g = g \circ f}$

$$acx + ad + b = acx + bc + d$$

Therefore $\boldsymbol{b = d}$ and $\boldsymbol{a = c}$

2.3 Exercise – Page #: 161 to 164

44. Let f be the function from **R** to **R** defined by $f(x) = x^2$. Find

Therefore $\boldsymbol{f^{-1}(x) = \pm\sqrt{x}}$

a) $f^{-1}(\{1\}) = \boldsymbol{\{-1, 1\}}$

b) $f^{-1}(\{x|0<x<1\}) = \boldsymbol{\{x| -1 < x < 0 \vee 0 < x < 1\}}$.

c) $f^{-1}(\{x|x>4\}) = \boldsymbol{\{x|x > 2 \vee x < -2\}}$.

2.3 Exercise – Page #: 161 to 164

45. Let $g(x) = \lfloor x \rfloor$. Find

a) $g^{-1}(\{0\}) = \{x \mid 0 \leq x < 1\}$.

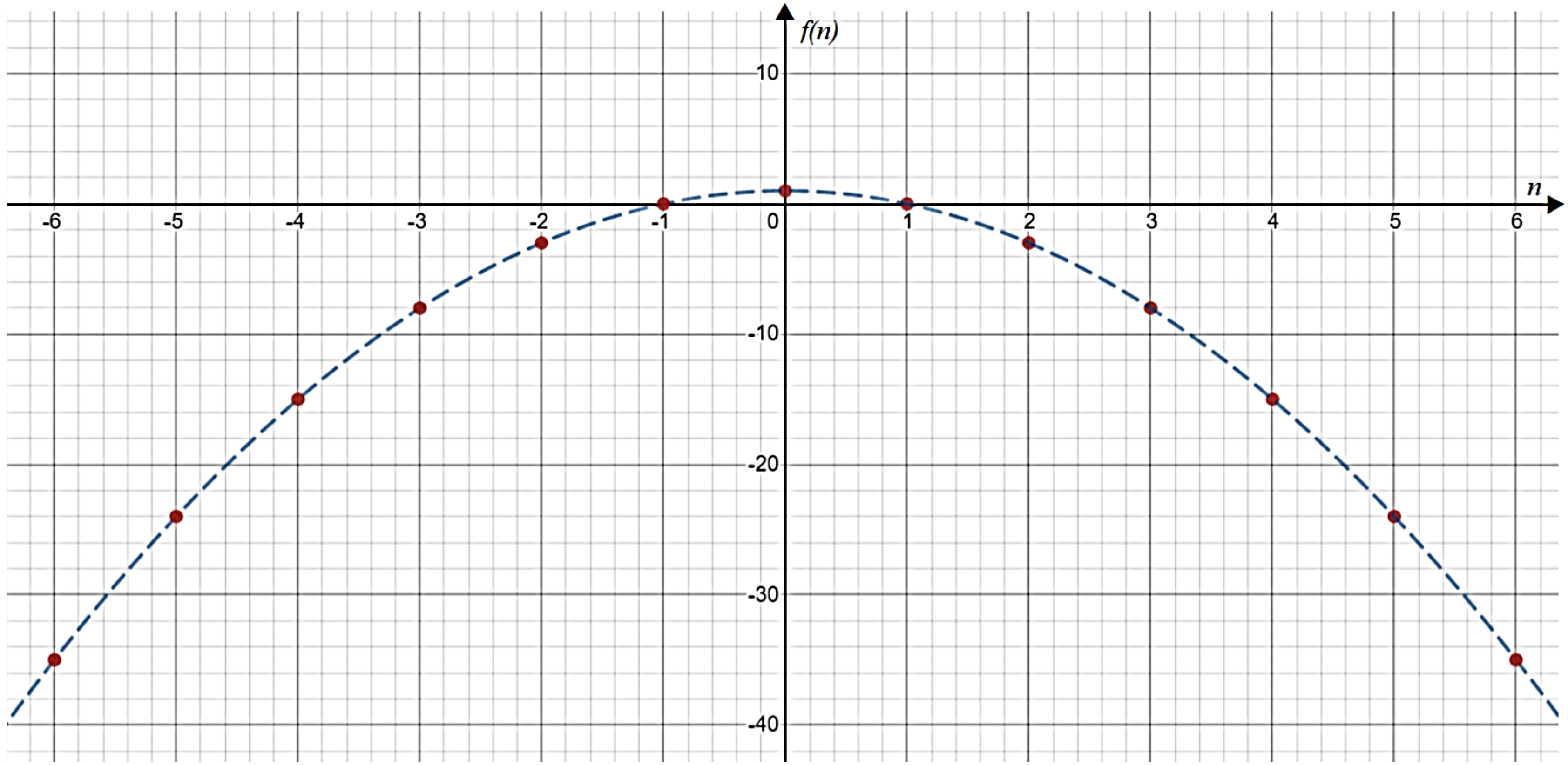
b) $g^{-1}(\{-1, 0, 1\}) = \{x \mid -1 \leq x < 2\}$.

c) $g^{-1}(\{x \mid 0 < x < 1\}) = \emptyset$.

Functions - Exercise

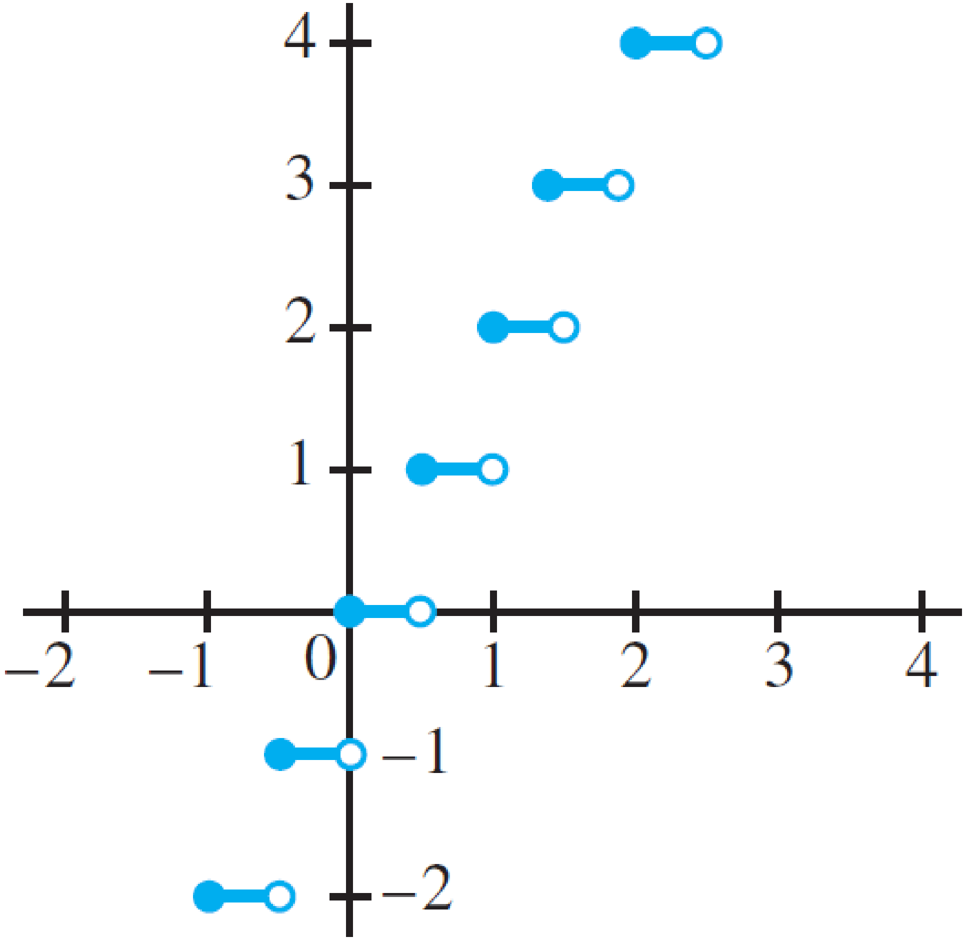
2.3 Exercise – Page #: 161 to 164

64. Draw the graph of the function $f(n) = 1 - n^2$ from **Z** to **Z**.



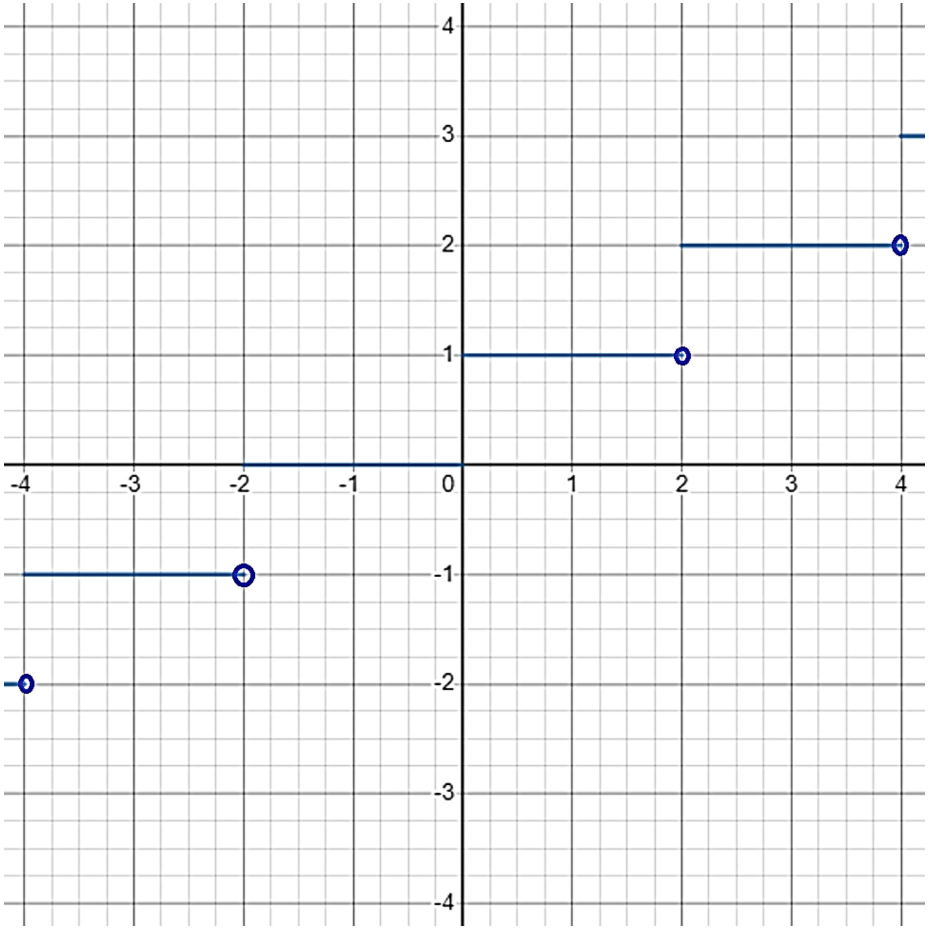
2.3 Exercise – Page #: 161 to 164

65. Draw the graph of the function $f(x) = \lfloor 2x \rfloor$ from **R** to **R**.



2.3 Exercise – Page #: 161 to 164

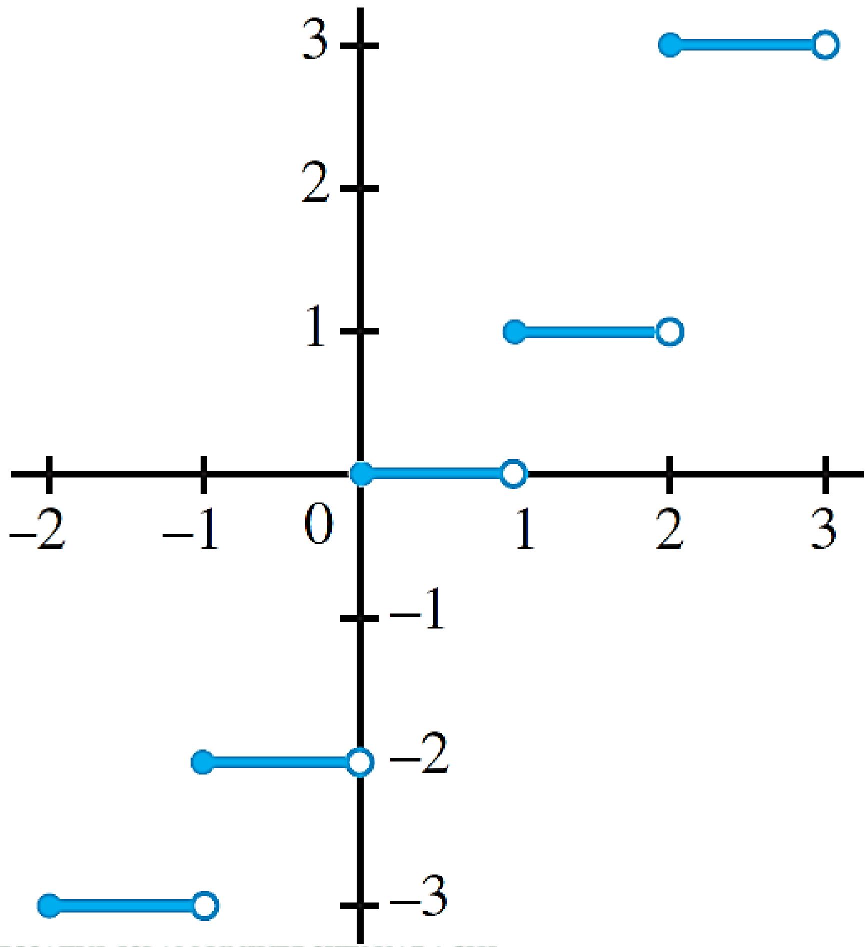
66. Draw the graph of the function $f(x) = \lfloor x/2 \rfloor$ from $\mathbf{R}$ to $\mathbf{R}$.



Functions - Exercise

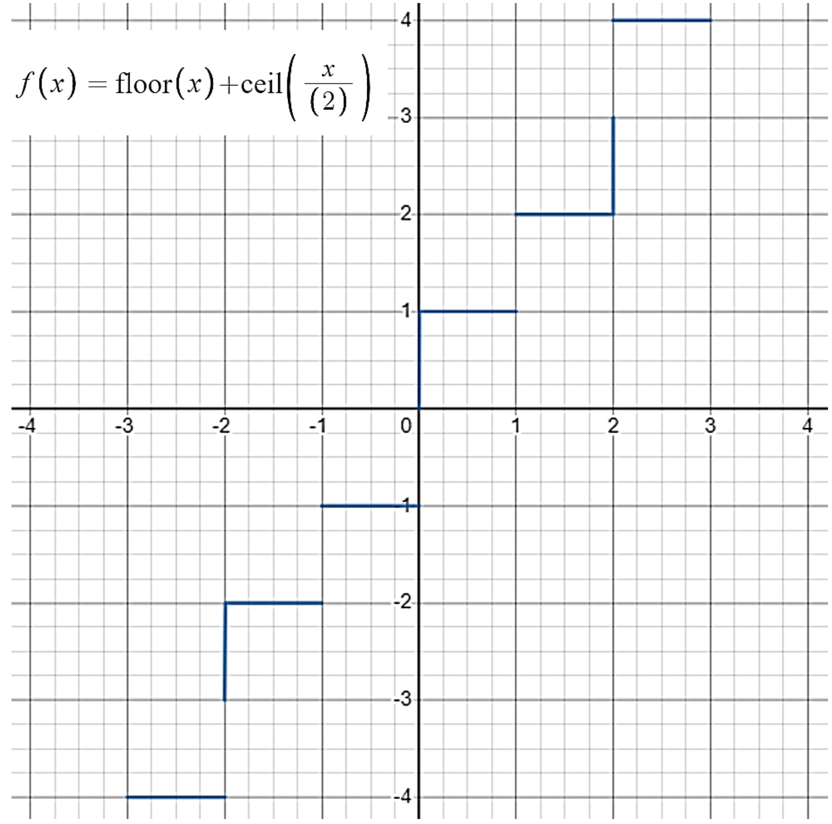
2.3 Exercise – Page #: 161 to 164

67. Draw the graph of the function $f(x) = \lfloor x \rfloor + \lfloor x/2 \rfloor$ from **R** to **R**.



2.3 Exercise – Page #: 161 to 164

68. Draw the graph of the function $f(x) = [x] + \lfloor x/2 \rfloor$ from **R** to **R**.



https://www.desmos.com/

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

69. Draw graphs of each of these functions.

a) $f(x) = \lfloor x + 12 \rfloor$

b) $f(x) = \lfloor 2x + 1 \rfloor$

c) $f(x) = \lceil x/3 \rceil$

d) $f(x) = \lceil 1/x \rceil$

e) $f(x) = \lceil x - 2 \rceil + \lfloor x + 2 \rfloor$

f) $f(x) = \lfloor 2x \rfloor \lceil x/2 \rceil$

g) $f(x) = \lceil \lfloor x - 12 \rfloor + 12 \rceil$

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

70. Draw graphs of each of these functions.

a) $f(x) = \lceil 3x - 2 \rceil$

b) $f(x) = \lceil 0.2x \rceil$

c) $f(x) = \lfloor -1/x \rfloor$

d) $f(x) = \lfloor x2 \rfloor$

e) $f(x) = \lceil x/2 \rceil \lfloor x/2 \rfloor$

f) $f(x) = \lfloor x/2 \rfloor + \lceil x/2 \rceil$

g) $f(x) = \lfloor 2 \lceil x/2 \rceil + 12 \rfloor$

2.3 Exercise – Page #: 161 to 164

71. Find the inverse function of $f(x) = x^3 + 1$.

$$f(x) = x^3 + 1$$

$$y = x^3 + 1$$

$$y - 1 = x^3$$

$$x^3 = y - 1$$

$$x = (y - 1)^{\frac{1}{3}}$$

$$x = \sqrt[3]{(y - 1)}$$

$$f^{-1}(x) = \sqrt[3]{(x - 1)}$$

Introduction to Functions

Chapter Reading

Book: Kenneth H. Rosen,

Discrete Mathematics and Its Applications (Eight Edition)

Exercise for Practice

Section 2.3: Functions

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Week - 7 and Lecture – 1 & 2

Inclusion–Exclusion

SINDH MADRESSATUL ISLAM UNIVERSITY KARACHI
Chartered by Government of Sindh, Recognized by HEC

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 1

Principle of Inclusion–Exclusion

SINDH MADRESSATUL ISLAM UNIVERSITY KARACHI

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Principle of Inclusion–Exclusion

APPLICATIONS OF INCLUSION - EXCLUSION

- To find the number of primes less than a positive integer.
- Counting the number of onto functions from one finite set to another.
- The well-known hatcheck problem can be solved using the principle of inclusion–exclusion.

Principle of Inclusion–Exclusion

INTRODUCTION

A discrete mathematics class contains 30 women and 50 sophomores.

How many students in the class are either women or sophomores?

This question cannot be answered unless more information is provided.

Principle of Inclusion–Exclusion

INTRODUCTION

Adding the number of women in the class and the number of sophomores probably does not give the correct answer, because women sophomores are counted twice.

This observation shows that the number of students in the class that are either sophomores or women is the sum of the number of women and the number of sophomores in the class minus the number of women sophomores.

Principle of Inclusion–Exclusion

DEFINITION

If a task can be done in either n_1 ways or n_2 ways,

then the number of ways to do the task is

$n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The is known as the ***subtraction rule*** or ***principle of inclusion–exclusion***, especially when it is used to count the number of elements in the union of two sets.

Principle of Inclusion–Exclusion

DEFINITION

Suppose that A_1 and A_2 are sets. Then,

$|A_1|$ ways to select an element from A_1 and

$|A_2|$ ways to select an element from A_2.

The number of ways to select an element *from A_1 or from A_2*, that is, the number of ways to select an element from their union $|A_1 \cup A_2|$,
is $|A_1| + |A_2|$, the sum of the number of ways to select an element from A_1 and the number of ways to select an element from A_2,
minus $|A_1 \cap A_2|$, the number of ways to select an element that is *in both A_1 and A_2*.

Because there are $|A_1 \cup A_2|$ ways to select an element in either A_1 or in A_2, and $|A_1 \cap A_2|$ ways to select an element common to both sets,
we have $$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Principle of Inclusion–Exclusion

PROBLEM

Question: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: We have a bit string of length eight then,

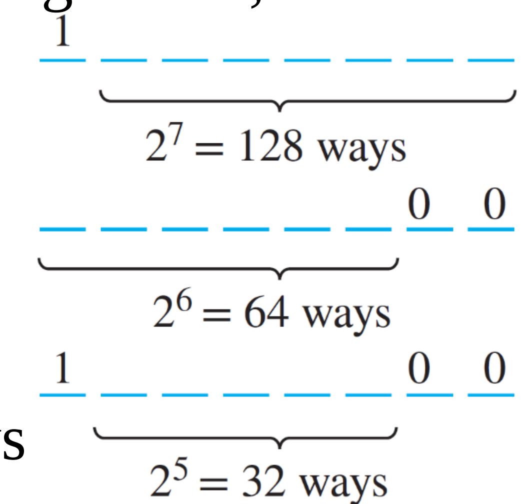
There are two choices for each place 0 or 1.

Number of ways,

begin with a 1 = 2^7 = **128** ways

ending with the two bits 00 = 2^6 = **64** ways

begin with 1 and end with a 00 = 2^5 = **32** ways



If either start with a 1 bit or end with the two bits 00

= **128** + **64** – **32** = **160** ways

Principle of Inclusion–Exclusion

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Q. 50. How many bit strings of length seven either begin with two 0s or end with three 1s?

$$\underline{0}\ \underline{0}\ \underbrace{\_\ \_\ \_\ \_\ \_}_{2^5 = 32\ Ways}$$

$$\underbrace{\_\ \_\ \_\ \_}_{2^4 = 16\ Ways}\ \underline{1}\ \underline{1}\ \underline{1}$$

$$\underline{0}\ \underline{0}\ \underbrace{\_\ \_}_{2^2 = 4\ Ways}\ \underline{1}\ \underline{1}\ \underline{1}$$

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Q. 51. How many bit strings of length 10 either begin with three 0s or end with two 0s?

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Q. 52. How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Solution: Let's break down the problem:

The answer for this is simply 2^{10} because you have 10 slots of bits xxxxxxxxxx.

In each slot u can either put 0 or 1 so the total possible strings

$= 2\times2\times2\times2\times2\times2\times2\times2\times2\times2 = 2^{10}$

Principle of Inclusion–Exclusion

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Bit strings of length 10 with 5 consecutive 1's.

Total 6 possible positions exists. Note that extra 0 is added from the 2nd position to avoid counting duplicate bit strings twice.

11111_ _ _ _ _ (5 slots ⇨ 2^5) 011111_ _ _ _ (4 slots ⇨ 2^4)

_ 011111_ _ _ (4 slots ⇨ 2^4) _ _ 011111 _ _ (4 slots ⇨ 2^4)

_ _ _ 011111_ (4 slots ⇨ 2^4) _ _ _ _ 011111 (4 slots ⇨ 2^4)

Total bit strings of length 10 contain 5 consecutive ones

$= 2^5 + 5 \times 2^4 = 112$

Principle of Inclusion–Exclusion

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Total bit strings of length 10 contain 5 consecutive ones

$= 2^5 + 5 \times 2^4 = 112$

Similarly,

Total bit strings of length 10 contain 5 consecutive zeros

$= 2^5 + 5 \times 2^4 = 112$

A = bit strings of length 10 contain either five consecutive 1s

B = bit strings of length 10 contain either five consecutive 0s

“Or” means union

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

By set theory we know that A ∪ B = A + B - A ∩ B

Here A ∩ B means that the bit string contains 5 consecutive ones and 5 consecutive zeroes.

1111100000 and ***0000011111***

There are only two such possibilities = 112 + 112 - 2 = 222

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Q. 53. How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?

Solution: Let's break down the problem:

Note that extra 1 is added from the 2nd position to avoid counting duplicate bit strings twice.

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Bit strings of length 8 with three consecutive 0's. The possible positions are:

000_ _ _ _ _ (5 slots ⇨ 2^5)

1000_ _ _ _ (4 slots ⇨ 2^4)

_ 1000_ _ _ (4 slots ⇨ 2^4)

_ _ 1000 _ _ (4 slots ⇨ 2^4)

_ _ _ 1000_ (4 slots ⇨ 2^4)

_ _ _ _ 1000 (4 slots ⇨ 2^4)

Total bit strings of length 8 contain three consecutive zeroes

$= 2^5 + 5 \times 2^4 = 112$

Principle of Inclusion–Exclusion

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Similarly, bit strings of length 8 with four consecutive 1's. The possible positions are:

1111 _ _ _ _ (5 slots ⇨ 2^4)

01111 _ _ _ (4 slots ⇨ 2^3)

_ 01111 _ _ (4 slots ⇨ 2^3)

_ _ 01111 _ (4 slots ⇨ 2^3)

_ _ _ 01111 (4 slots ⇨ 2^3)

Total bit strings of length 8 contain four consecutive ones

$= 2^4 + 4 \times 2^3 = 48$

Principle of Inclusion–Exclusion

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

A = bit strings of length 8 contain 000, B = bit strings of length 8 contain 1111

"Or" means union, we know that **A ∪ B = A + B - A ∩ B**

Here A ∩ B means that the common bit strings, those are:

00001111,	11110000,	11111000,	00011111,
00011110,	10001111,	01111000,	11110001,
00011000,	00010001,	00010000,	00001000,
10001000.	**A ∪ B = A + B - A ∩ B**		

There are only two such possibilities = 112 + 48 - 13 = ***147***