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CALCULUS

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In this lecture we will discuss about

- Integration

Integration

Integration and differentiation are intimately connected. The nature of the connection is one of the most important ideas in all mathematics, and its independent discovery by Leibniz and Newton still constitutes one of the greatest technical advances of modern times.

Definitions

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f. The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign.** The function f is the **integrand** of the integral and x is the **variable of integration.**

$$\int f(x)\,dx = F(x) + C. \qquad (1)$$

Why we include C? The derivative of a constant is 0. However, when you integrate, you should consider that there is a possible constant involved, but we don't know what it is for a particular problem. Therefore, you can just use C to represent the value.

Definite Integral

Terminology

There is a fair amount of terminology associated with the symbol $\int_a^b f(x)\,dx$.

The function is the **integrand.**

Upper limit of integration

x is the **variable of integration.**

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\textbf{Integral}\text{ of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have **evaluated** the integral.

The value of the definite integral of a function over any particular interval depends on the function and not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\,dt \qquad \text{or} \qquad \int_a^b f(u)\,du \qquad \text{instead of} \qquad \int_a^b f(x)\,dx.$$

Power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

C = Constant of integration

u = Function

n = Power

du = Derivative

Table Integral formulas

Indefinite integral	Reversed derivative formula
1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1,\ n$ rational	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
$\int dx = \int 1\,dx = x + C$ (special case)	$\frac{d}{dx}(x) = 1$
2. $\int \sin kx\,dx = -\frac{\cos kx}{k} + C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$
3. $\int \cos kx\,dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$
4. $\int \sec^2 x\,dx = \tan x + C$	$\frac{d}{dx} \tan x = \sec^2 x$
5. $\int \csc^2 x\,dx = -\cot x + C$	$\frac{d}{dx}(-\cot x) = \csc^2 x$
6. $\int \sec x \tan x\,dx = \sec x + C$	$\frac{d}{dx} \sec x = \sec x \tan x$
7. $\int \csc x \cot x\,dx = -\csc x + C$	$\frac{d}{dx}(-\csc x) = \csc x \cot x$

Evaluate the following indefinite integrals.

$$(1)\int (x^2+4x+13)dx$$

$$I=\int x^2 dx+\int 4x\,dx+\int 13\,dx$$

$$I=\frac{x^3}{3}+\frac{4x^2}{2}+13x+c$$

$$I=\frac{x^3}{3}+2x^2+13x+c$$

$$2.\ \int (3x^4 - 5x^3 - 4x^2 - 2)dx$$

$$I = \int 3x^4 dx - \int 5x^3\ dx - \int 4x^2\, dx - \int 2\, dx$$

$$I = \frac{3x^5}{5} - \frac{5x^4}{4} - \frac{4x^3}{3} - 2x + c$$

Evaluate the following integral.

$$(1)\int (4x^3 - 12x^2 - 4x + 12)dx$$

$$(2)\int x^2(x^2 - 4)dx$$

$$(3)\int (x^3 - 6x^2 + 11x - 6)dx$$

$$(4)\int (12x^2 - 4x + 12)dx$$

Substitution Method

The substitution method turns an unfamiliar integral into one that can be evaluated. In other words, substitution gives a simpler integral involving the variable u.

Let's review the five steps for integration by substitution.

Step 1: Choose a new variable *u*
Step 2: Determine the value *dx*
Step 3: Make the substitution
Step 4: Integrate resulting integral
Step 5: Return to the initial variable x

$$(1)\int (2x+3)^4\,dx$$

Step 1: Choose the substitution function u

The substitution function is $u = 2x+3$

Step 2: Determine the value of dx

$$du = (2x+3)'\,dx$$

$$du = 2dx$$

$$dx = \frac{1}{2}du$$

Step 3: Do the substitution

$$\int u^4 . \frac{1}{2} du$$

Step 4: Integrate the resulting integral

$$\int (2x+3)^4 dx = \int u^4 . \frac{1}{2} du = \frac{u^5}{10} + c$$

Step 5: Return to the initial variable: x

$$\frac{u^5}{10} + c = \frac{(2x+3)^5}{10} + c$$

2. $\int (1-x)^8\, dx$ Let $u = 1 - x$

$$\frac{du}{dx} = -1 \quad \Rightarrow \quad -du = dx$$

So, $$\int (1-x)^8\, dx = \int -u^8\, du$$

$$= -\frac{u^9}{9} + C$$

$$= -\frac{(1-x)^9}{9} + C$$

3. $\int x(1+x^2)^4\,dx$ Let $u = 1 + x^2$

$$\frac{du}{dx} = 2x$$

So, $\int x(1+x^2)^4\,dx$ $= \int xu^4 \frac{du}{2x}$

$$= \int \frac{u^4}{2} du$$

$$= \frac{u^5}{10} + C = \frac{(1+x^2)^5}{10} + C$$

TABLE OF TRIGONOMETRIC SUBSTITUTIONS

- Here, list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

TABLE OF TRIGONOMETRIC SUBSTITUTIONS

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Integration by Parts

Integration by Parts is a special method of integration that is often useful when two functions are multiplied together, but is also helpful in other ways.

$$\int u.v\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)dx$$

u is the function u(x)

v is the function v(x)

$$\int x \sin x \, dx$$

$$\int u.v dx = u \int v dx - \left\{ \int u^{'} \int v dx \right\} dx$$

$$u = x, \;\; v = \sin x$$

$$u^{'} = 1$$

$$x \int \sin x \, dx - \left(\int -\cos x \, dx \right).(1) dx$$

$$I = -x \cos x - (-\sin x) + c$$

$$I = -x \cos x + \sin x + c$$

Integration by Partial Fractions

If the integrand (the expression after the integral sign) is in the form of an algebraic fraction and the integral cannot be evaluated by simple methods, the fraction needs to be expressed in partial fractions before integration takes place.

The steps needed to decompose an algebraic fraction into its partial fractions results from a consideration of the reverse process – addition (or subtraction).

$$\frac{f(x)}{(x+a)(x+b)} \equiv \frac{A}{x+a} + \frac{B}{x+b}$$

Example. Evaluate the following integral.

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

$$\int \frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$\int \frac{3x+11}{(x-3)(x+2)} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$$x=-2,\ x=3$$

$$3x+11=A(x+2)+B(x-3)$$

$$3(-2)+11=A(-2+2)+B(-2-3)$$

$$5=-5B, \quad B=-1$$

$$3(3)+11=A(3+2)+B(3-3)$$

$$5A=20, \quad A=4$$

$$\int \frac{3x+11}{x^2-x-6}\, dx=\int \frac{4}{x-3}-\frac{1}{x+2}\, dx$$

$$\int \frac{4}{x-3}-\int \frac{1}{x+2}\, dx$$

$$4\ln|x-3|-\ln|x+2|+c$$

Expanding the quotients in partial fraction.

(1) $$\frac{5x-13}{(x-3)(x-2)}$$

(2) $$\frac{5x-7}{x^2-3x+2}$$