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CALCULUS

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In this lecture we will discuss about

- Increasing and decreasing function.
- Relative maxima and minima
- Absolute Maxima and Minima

Increasing and Decreasing Functions

- Increasing function on an interval means that as we move from left to right in the x-direction, the y-values increase in magnitude.
- Decreasing function on an interval means that as we move from left to right in the x-direction, the y-values decrease in magnitude.

Definition :

Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$

(b) f is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$

(c) f is constant on the interval if $f(x_1) = f(x_2)$ for all x_1 and x_2

Theorem

Let f be a function that is continuous on a closed interval [a,b] and differentiable on the open interval (a,b)

(a) If $f'(x) > 0$ for every value of x in (a,b), then f is increasing on [a,b].

(b) If $f'(x) < 0$ for every value of x in (a,b), then f is decreasing on [a,b].

(c) If $f'(x) = 0$ for every value of x in (a,b), then f is constant on [a,b].

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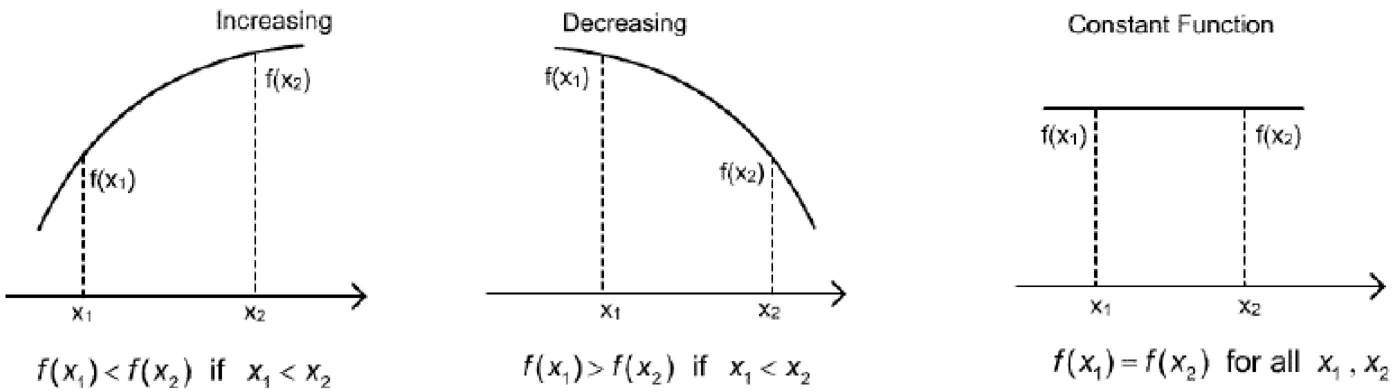
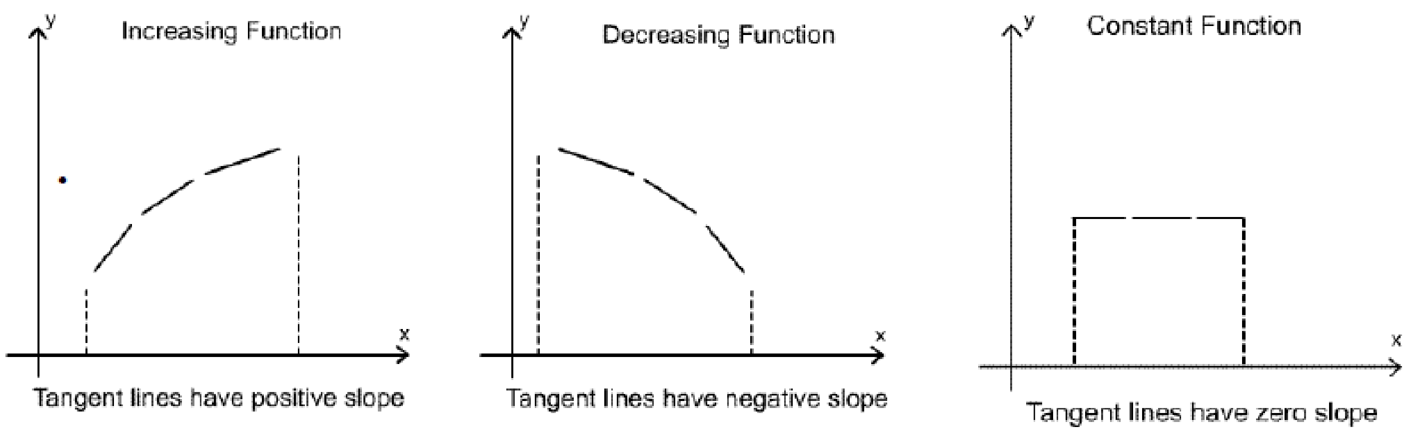
As shown in the figures below



Let's take a few points on the 3 graphs in above figures and make tangent lines on these points. This gives



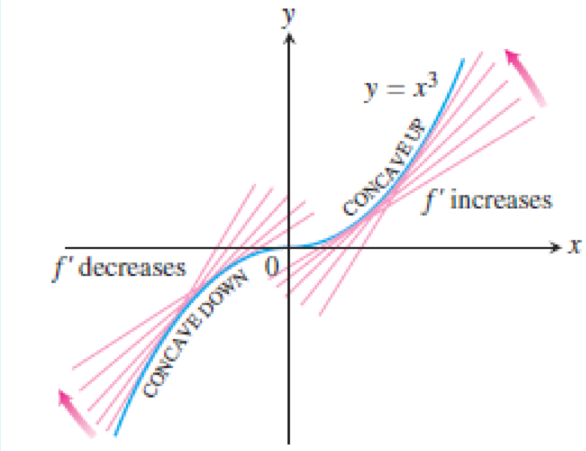
Note that incase where the graph was increasing, we get tangent line with positive slopes, decreasing we get negative slope, and constant gives 0 slope.

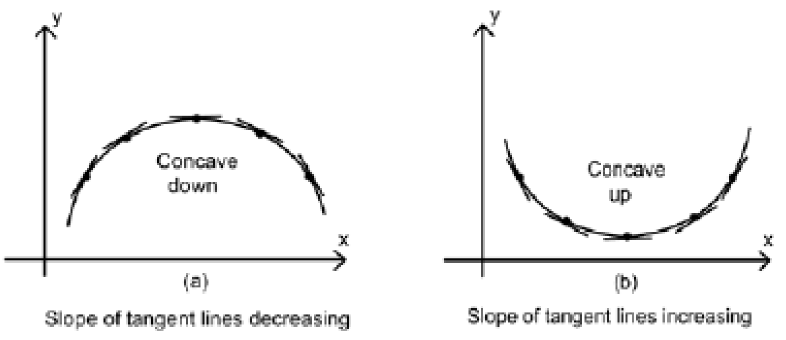
Definition

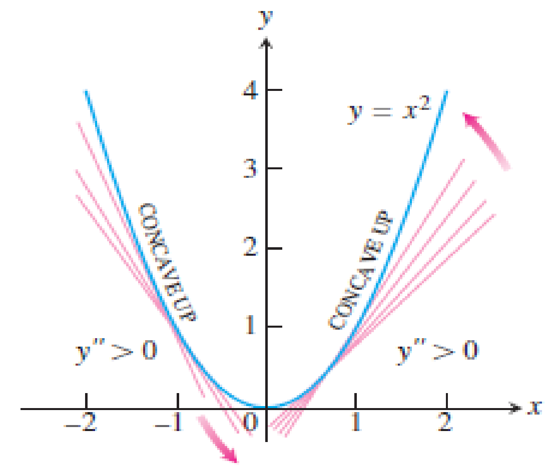
Let f be differentiable on an interval

(a) f is called concave up on the interval if f is increasing on the interval.

(b) f is called concave down on the interval if f is decreasing on the interval.







Maxima & Minima

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

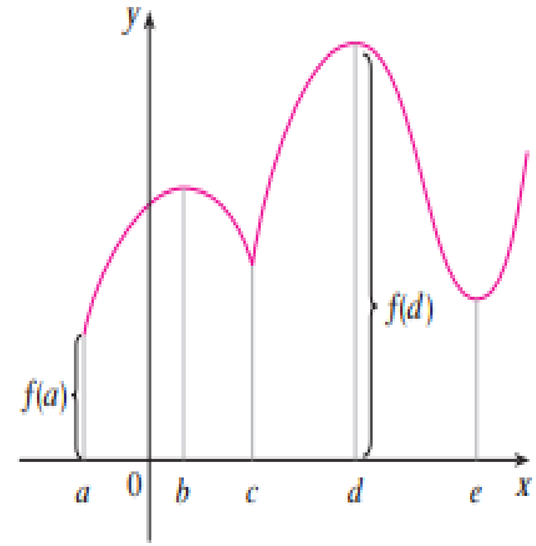


FIGURE 2

Abs min $f(a)$, abs max $f(d)$,
loc min $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$

Maxima & Minima Examples

In every day life, one faces the problem of finding the best way to do something. For example;

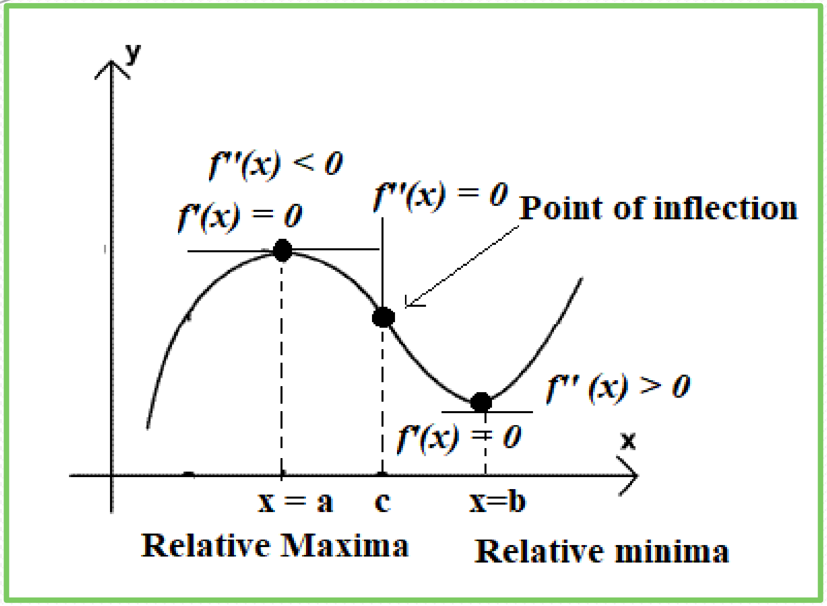
- A doctor wishes to select the minimum dosage of drugs to cure his patient.
- A manufacturer is always to get maximum profit.

All such problems relate to minimize or maximizing. In calculus, maxima and minima is a powerful tool to solve such problems.

Test for Maxima and Minima:

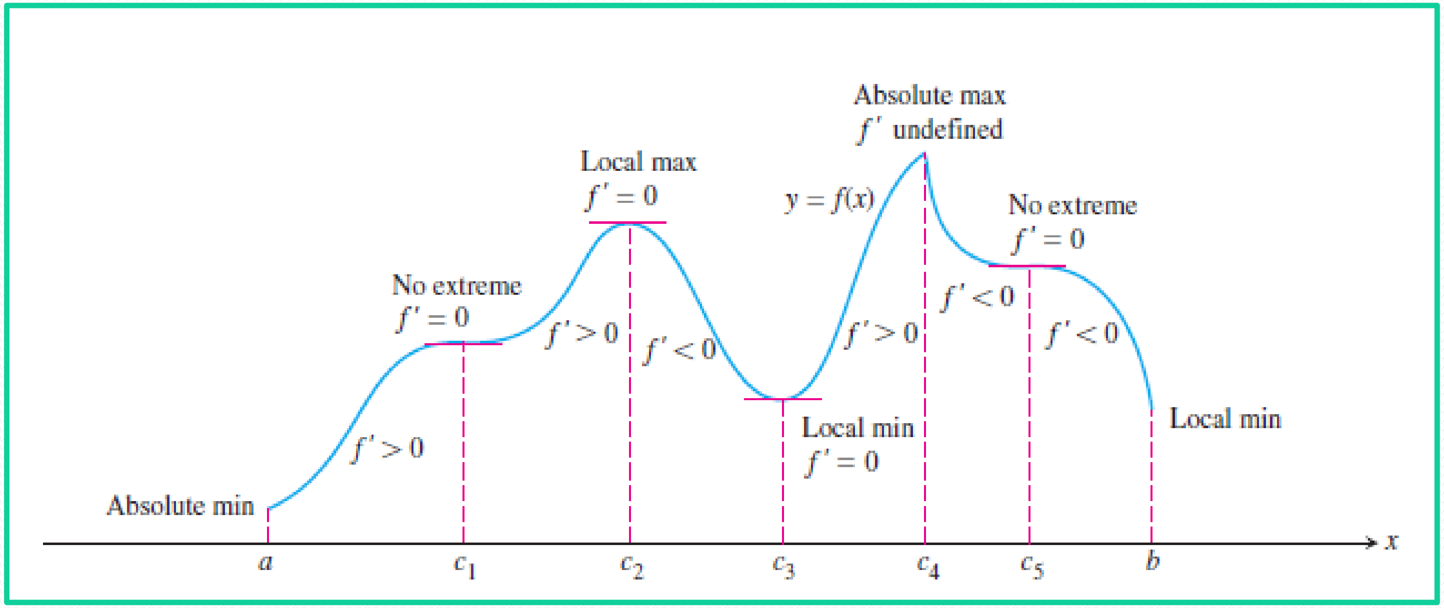
Test for Maxima and Minima:

- $Suppose\ f\ is\ a\ function\ whose\ f'and\ f''exist\ at\ each\ x \in$ $(a,b).\ If\ for\ any$
- $c \in (a,b),$
- $(i)\ f'(c) = 0, and\ f''(c) < 0 \Rightarrow f\ have\ maximum\ value\ at\ x = c.$
- $(ii)\ f'(c) = 0, and\ f''(c) > 0 \Rightarrow f\ have\ minimum\ value\ at\ x = c.$
- $(iii)\ f'(c) = 0, and\ f''(c) = 0 \Rightarrow$ $f\ has\ no\ maximum\ or\ minimum\ value\ at\ x = c.$
- $This\ point\ is\ called\ inflection\ point.$



DEFINITION: Stationary / Critical Point:
A point where gradient of a curve is zero is called **critical** or **stationary point**.

DEFINITION: Point of Inflection:
A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.



- Most of the graphs we have seen have ups and downs, much like Hills and valleys on earth.
- The Ups or the Hills are called relative Maxima.
- The downs or the Valleys are called relative Minima
- The reason we use the word relative is that just like a given Hill in a mountain range need not necessarily be the Highest point in the range. Similarly a given maxima in a graph need not be the maximum possible value in the graph.

Same goes for the relative minima.

In general, we may say that a given Hill is the highest one in some area. Look at relative maxima in a given interval. Again, same is true for valleys and relative minima.

So when we talk about relative maxima and relative minima, we talk about them in the context of some interval.

Example:1 Locate the relative extrema and point of inflection of the curve:

$$f(x) = 2x^3 - 15x^2 + 36x + 10$$

Solution:

$f(x) = 2x^3 - 15x^2 + 36x + 10 \qquad (1)$

$f'(x) = 6x^2 - 30x + 36 \qquad (2)$

$f''(x) = 12x - 30 \qquad (3)$

<u>At critical Point:</u> $\boldsymbol{f'(x) = 0}$

$\boldsymbol{Eq.(2)} \Rightarrow 6x^2 - 30x + 36 = 0$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x^2 - 2x - 3x + 6 = 0$

$\Rightarrow x(x-2) - 3(x-2) = 0$

$\Rightarrow (x-2)(x-3) = 0$

$\Rightarrow x = 2, \quad 3$

<u>For Point of Inflection</u>: $f''(x) = 0$

$Eq.(3) \Rightarrow 12x - 30 = 0$

$\Rightarrow 12x = 30$

$\Rightarrow x = \frac{30}{12} = \frac{5}{2}$

At last put x =2, 3 in equation (1) to find maximum and minimum point

<u>For local maxima and minima:</u>

$At\ x = 2 \qquad Eq.(3) \Rightarrow f''(2) = 12(2) - 30 = 24 - 30 = -6 < 0$

$Therefore, f(x) has\ maximum\ value\ at\ x = 2$

$At\ x = 3 \qquad Eq.(3) \Rightarrow f''(3) = 12(3) - 30 = 36 - 30 = 6 > 0$

$Therefore, f(x) has\ minimum\ value\ at\ x = 3$

Example:2 Locate the relative extrema and point of inflection of the curve:

$$f(x) = 3x^4 - 4x^3 + 5$$

Solution:

$$f(x) = 3x^4 - 4x^3 + 5 \qquad (1)$$

$$f'(x) = 12x^3 - 12x^2 \qquad (2)$$

$$f''(x) = 36x^2 - 24x \qquad (3)$$

At critical Point: $\boldsymbol{f'(x) = 0}$

$$\boldsymbol{Eq.(2)} \Rightarrow 12x^3 - 12x^2 = 0$$

$$\Rightarrow 12x^2(x-1) = 0$$

$$\Rightarrow x^2 = 0 \text{ or } x - 1 = 0$$

$$\Rightarrow x = 0, \qquad 1$$

For Point of Inflection: $f''(x) = 0$

$$Eq.(3) \Rightarrow 36x^2 - 24x = 0$$

$$\Rightarrow 12x(3x-2) = 0$$

$$\Rightarrow x = 0, \qquad x = \frac{3}{2}$$

At last point x =0, 1, 3/2 in equation (1) to find minimum point and point of inflection.

For local maxima and minima:

$At\ x = 0 \qquad Eq.(3) \Rightarrow f''(0) = 36(0) - 24(0) = 0 \Rightarrow inflection\ point$

$Therefore, f(x) has\ no\ maximum\ or\ minimum\ value\ at\ x = 0.$

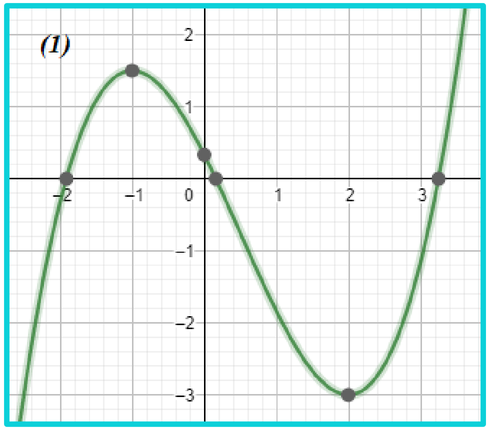
$At\ x = 1 \qquad Eq.(3) \Rightarrow f''(1) = 36(1) - 24(1) = 36 - 24 = 12 > 0$

$Therefore, f(x) has\ minimum\ value\ at\ x = 1$

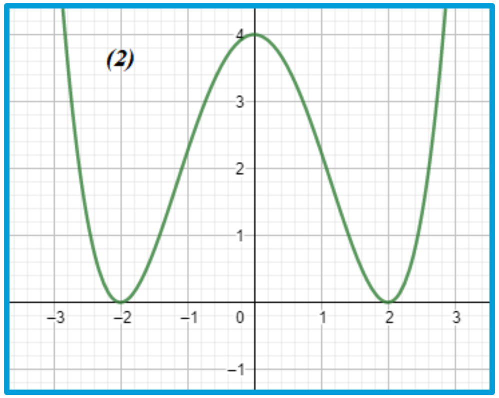
Exercise

Identify the inflection point and local maxima and minima of the function graphed here. Identify the interval at which the function concave up and concave down.

1. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



2. $f(x) = x^4 - 2x^2 + 4$



Q#3 Identify and locate the relative extrema and point of inflection of the curve:

$$f(x) = x^3 - 12x - 5$$

Solution:

$f(x) = x^3 - 12x - 5 \qquad (1)$

$f'(x) = 3x^2 - 12 \qquad (2)$

$f''(x) = 6x \qquad (3)$

<u>At critical Point:</u> $\boldsymbol{f'(x) = 0}$

$\boldsymbol{Eq.(2)} \Rightarrow 3x^2 - 12 = 0$

$\Rightarrow 3(x^2 - 4) = 0$

$\Rightarrow (x - 2)(x + 2) = 0$

$\Rightarrow x - 2 = 0 \; or \; x + 2 = 0$

$\Rightarrow x = -2, \qquad 2$

<u>For local maxima and minima:</u>

$At\ x = -2 \qquad Eq.(3) \Rightarrow f''(-2) = 6(-2) = -12 < 0 \Rightarrow maximum\ point$

$Therefore, f(x) has\ maximum\ value\ at\ x = -2.$

$At\ x = 2 \qquad Eq.(3) \Rightarrow f''(2) = 6(2) = 12 > 0$

$Therefore, f(x) has\ minimum\ value\ at\ x =$1

For Point of Inflection: $f''(x) = 0$

$Eq. (3) \Rightarrow 6x = 0$

$\Rightarrow x = 0$

$Put\ x = 0, -2\ and\ 2\ to\ find\ inflection\ point,$
$maximum\ poin\ and\ minimum\ point.$

$At\ x = 0\ (1)$

$\Rightarrow f(0) = 0 - 0 - 5 = -5$

$Inflection\ point\ is\ (0, -5)$

$Now, put\ x = -2\ in\ (1)$

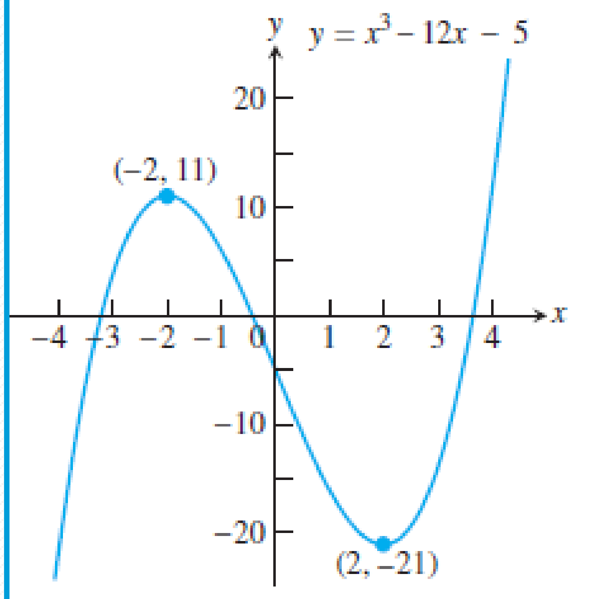
$(1) \Rightarrow f(-2) = (-2)^3 - 12(-2) - 5 = 11$

$Maximum\ point\ is\ (-2, 11)$

$Put\ x = 2\ in\ (1)$

$(1) \Rightarrow f(2) = (2)^3 - 12(2) - 5 = -21$

$Maximum\ point\ is\ (2, -21).$



Error Determination

If x is a quantity to be measured and if Δx is the error in x, then

(i) Δx is known as absolute error in x.

(ii) $\frac{\Delta x}{x}$ *is known as Rlative (or average)error in x.*

$(iii)\ \dfrac{\Delta x}{x} \times 100$ *is known as Percentage error in x*

Example:1 The percentage error in measuring the edge of a cube is 2%. Find the percentage error in computing the volume of the cube.

Solution:

If V is the volume of the cube with edge **a**, then

$$V = a^3$$

We approximate the error $\frac{\Delta V}{V}$ in V by use of differentials

$$\frac{\Delta V}{V} = \frac{\mathrm{d}v}{V} = \frac{3a^2 da}{a^3} = 3\frac{da}{a} = 3(0.02) = 0.06$$

$$Percentage\ error\ in\ V = \frac{\Delta V}{V} \times 100 = 0.06 \times 100 = 6\%$$

Example:2 The powe dissipated in a resistor is given by $P = \frac{E^2}{R}$.
Using applications of differential calculus, find the approximate percentage change in P when E is increased 3% and R is decreased by 2%.

Solution:

$$P = \frac{E^2}{R}$$

$$\Rightarrow lnP = 2lnE - lnR$$

On differentiating we get

$$\frac{\Delta P}{P} = 2\frac{\Delta E}{E} - \frac{\Delta R}{R}$$

Percentage change:

$$\frac{\Delta P}{P} \times 100 = 2\,\frac{\Delta E}{E} \times 100 - \frac{\Delta R}{R} \times 100$$

$$= 2\,(3\%) - (-2\%)$$

$$= 6\% + 2\%$$

$$= 8\%$$

Absolute Maximum and Minimum

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

V EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \leqslant x \leqslant 4$$

SOLUTION Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $\left(-\frac{1}{2}, 4\right)$. The values of f at these critical numbers are

$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the endpoints of the interval are

$$f\left(-\tfrac{1}{2}\right) = \tfrac{1}{8} \qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.