Discrete Structures
(Discrete Mathematical Structures)
Fall 2022

Instructor: Mr. Muhammad Zafarullah

Lecture-1

Introduction
Propositional Logic

Instructor Introduction

Rules

- **Notebook/Register**
- **I encourage class participation**
- **Mobile phones – Silent or switch off**
- **80% Attendance (No Relaxation)**
- **Arrive on time in class**
- **If you do not understand a point, raise your hand and ask me to explain or contact during office hours**
- **No disturbance!!!! No Misconduct!!!!**
- **REMEMBER: Your first priority must be your studies**

Course Objectives

- Deep understanding of **discrete structures** used in **Computer Science**
- Developing **problem solving** and **analytical skills**
- Developing **algorithmic** and **computational skills**
  - Ability to understand **mathematical arguments** and their **design**
  - **Understanding of logic**
  - **Proofing techniques**

Course Outline

Discrete Structures			
Credit Hours:	3+0	**Prerequisites:**	None

Course Learning Outcomes (CLOs):		
At the end of the course the students will be able to:	**Domain**	**BT Level***
1. **Understand** the key concepts of Discrete Structures such as Sets, Permutations, Relations, Graphs, and Trees etc.	C	2
2. **Apply** formal logic proofs and/or informal, but rigorous, logical reasoning to real problems, such as predicting the behavior of software or solving problems such as puzzles.	C	3
3. **Apply** discrete structures into other computing problems such as formal specification, verification, databases, artificial intelligence, and cryptography.	C	3
4. **Differentiate** various discrete structures and their relevance within the context of computer science, in the areas of data structures and algorithms, in particular.	C	4
* BT= Bloom's Taxonomy, C=Cognitive domain, P=Psychomotor domain, A= Affective domain		

Course Outline

Course Content:
Mathematical reasoning, propositional and predicate logic, rules of inference, proof by induction, proof by contraposition, proof by contradiction, proof by implication, set theory, relations, equivalence relations and partitions, partial orderings, recurrence relations, functions, mappings, function composition, inverse functions, recursive functions, Number Theory, sequences, series, counting, inclusion and exclusion principle, pigeonhole principle, permutations and combinations, elements of graph theory, planar graphs, graph coloring, euler graph, Hamiltonian path, rooted trees, traversals.
Teaching Methodology:
Lectures, Written Assignments, Practical labs, Semester Project, Presentations
Course Assessment:
Sessional Exam, Home Assignments, Quizzes, Project, Presentations, Final Exam

Course Outline

Reference Materials:
1. Discrete Mathematics and Its Applications, 7th edition by Kenneth H. Rosen 2. Discrete Mathematics with Applications, 4th Edition by Susanna S. Epp 3. Discrete Mathematics, 7th edition by Richard Johnson Baugh 4. Discrete Mathematical Structures, 4th edition by Kolman, Busby & Ross 5. Discrete and Combinatorial Mathematics: An Applied Introduction by Ralph P. Grimaldi 6. Logic and Discrete Mathematics: A Computer Science Perspective by Winifred Grassman

Course Outline

- Formal Logic
- Quantifiers and Predicates
- Proof Techniques
- Number Theory
- Sequence and Summations
- Induction and Recursion
- Basic Set Theory
- Relations
- Functions
- Graphs
- Trees

Course Assessment/Grading

Component	Weightage
Sessional 1	10%
Sessional 2	15%
Terminal	50%
Quizzes	15%
Assignments	10%

- For all assignments, do follow the formatting guidelines given in course handbook.
- Submit all assignments in hard copy.
- No credit for copied or late submissions.
- No relaxation for students found cheating in any quiz or exam.
- To get good grade you must attend all lectures and perform good in all course assessments.

Reasons to Study Discrete Structures

- Proof

  Ability to understand and create mathematical argument

- Gateway to more advanced CS courses

  Data structures, algorithms, automata theory, formal languages, Database, networks, operating system, security etc.

Reasons to Study Discrete Structures

- It is the **mathematics underlying almost all of computer science**:
- Program verification
  - Analyzing algorithms for **correctness** and **efficiency**
- Finding **efficient algorithms**
  - (for sorting, searching, etc.)
- **Formalizing** security requirements
- Designing **cryptographic protocols** for enhanced security
- **Graph Theory** (Networks – both physical & social)

Problems solved using Discrete Math's

- How many secure passwords (using a specific number of characters or digits)?
- Probability of winning?
- How can I encrypt a message?
- Shortest paths between two cities using public transportation?
- How many steps required to sort 10,000 numbers? Is this algorithm correct?
- How to design a circuit that multiply two integers?

Logic

- Logic is the study of the principles and methods that distinguishes between a valid and an invalid argument.

- Logic deals with general reasoning laws, which you can trust.

Applications

- Applied in proving **program correctness** and **verification**
- Databases (Relational Algebra and calculus)
- Artificial Intelligence

Propositional Logic

- Proposition
  - A proposition is a declarative statement that is either TRUE or FALSE, but not both.
- Example 1
  - 2 + 2 = 4.
  - Lahore is the capital of Pakistan.
  - It is Sunday today.
  - Ali is student of this class.
- Example 2
  - What time is it?
  - X + 1 = 2.
  - Close the door.
  - Read this carefully.

Propositional Logic

- Letter are used to denote propositional variables, to symbolically represent propositions.
    - Letters used for this purpose are p, q, r, s,………
    - A propositional can have one of two values: true (T) or false (F).
- Example
    - ***p* =** “Islamabad is the capital of Pakistan”
    - ***q* =** “17 is divisible by 3”

Propositional Logic

- The area of logic that deals with propositions is called the *Propositional Calculus* or *Propositional Logic.*

- *Compound Propositions* are constructed by combining one or more propositions using logical operators (connectives).

- Examples
  - “3 + 2 = 5” **and** “Lahore is a city in Pakistan”
  - “The grass is green” **or** “ It is hot today”

Symbols for Logical Operators

Symbol	Meaning
$\neg$	Negation
$\wedge$	And, Conjunction
$\vee$	Or, Disjunction
$\rightarrow$	Implication
$\leftrightarrow$	Bi-Conditional

Logical Operators (Logical connectives)

- Negation

  - This just turns a false proposition to true and the opposite for a true proposition.
  - Symbol: $\neg$
  - Let *p* is a proposition. The statement

  “It is not the case that p.”

  is another proposition, called the negation of p.

  - The negation of *p* is written $\neg p$ and read as “not p”.

Logical Operator - Negation

- Logical operators are defined by **truth tables** –tables which give the output of the operator in the right-most column.
- Here is the truth table for negation:

p	**¬p**
T	F
F	T

Logical Operator - Negation

- Example

Let p = "Today is Friday."

The negation of p is

$\neg p$ = "It is not the case that today is Friday."

$\neg p$ = "Today is not Friday."

$\neg p$ = "It is not Friday today."

- What is negation of following proposition: "My PC runs Linux"

Logical Operator - Conjunction

- Conjunction is a *binary* operator in that it operates on two propositions when creating compound proposition. On the other hand, negation is a *unary* operator.
- Conjunction corresponds to English “and.”
- Symbol: ∧
- Let p and q be propositions. The conjunction of p and q, denoted by p ∧q, is the proposition “p and q”. The conjunction p ∧qis true when both p and q are true. If one of these is false, than the compound statement is false as well.

Logical Operator - Conjunction

- Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical Operator - Conjunction

- Example

Let p = “Today is Friday.”

and q = “It is raining today.”

$p \wedge q$ = “Today is Friday and it is raining today.”

Logical Operator - Conjunction

- Hamza’s PC has more than 16 GB free hard disk space, and the processor in Hamza’s PC runs faster than 1 GHz.

- It is cold but sunny.

Logical Operator - Disjunction

- Disjunction is also a binary operator.
- Disjunction corresponds to English “or.”
- Symbol: $\vee$

- Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition “p or q”. The conjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Logical Operator - Disjunction

- Truth Table

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Logical Operator - Disjunction

- Example

Let p = "Today is Friday."

and q = "It is raining today."

$p \vee q$ = "Today is Friday or it is raining today."

Example

Let p = “it is hot”,

q = “it is sunny”

- It is hot and sunny $p \land q$
- It is not hot but sunny $\neg p \land q$
- It is neither hot nor sunny $\neg p \land \neg q$

Logical Operator – Exclusive Or

- Symbol: $\oplus$
- Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false, and false otherwise.
- Truth Table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Logical Operator – Exclusive Or

- Example

Let p = “Students who have taken calculus can take this class.”

and q = “Students who have taken computer science can take this class.”

p ∨ q = “Students who have taken calculus or computer science can take this class.”

p ⊕ q = “Students who have taken calculus or computer science, but not both, can enroll in this class.”

Exclusive or Versus Inclusive or (Disjunction)

- Coffee or tea comes with dinner. Exclusive or
- A password must have at least three digits or be at least five characters long. Inclusive or
- Lunch includes soup or salad. Exclusive or
- Experience with C++ or Java is required. Inclusive or

Logical Operator – Implication

- $p \to q$ corresponds to English "if *p* then *q*," or "*p* implies *q*."
- Symbol: $\to$
- The implication p $\to$ q is the proposition that is false when p is true and q is false, and true otherwise.

$$p \to q$$

Hypothesis Conclusion

- Examples
  - If it is raining then it is cloudy.
  - If you get 100% on the final, then you will get an A.
  - If *p* then 2+2 = 4.

Logical Operator – Implication

- Truth Table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Logical Operator – Implication

- Alternate ways of stating an implication
  - p implies q
  - If p, q
  - p only if q
  - p is sufficient for q
  - q if p
  - q whenever p
  - q is necessary for p

Implication - Example

p: you get 100% on the final
q: you will get an A

- **p implies that q.**
  you get 100% on the final implies that you will get an A.
- **If p, then q.**
  If you get 100% on the final, then that you will get an A.
- **If p, q.**
  If you get 100% on the final, that you will get an A.
- **p is sufficient for q.**
  Get 100% on the final is sufficient for getting an A.
- **q if p.**
  you will get an A if you get 100% on the final.
- **q unless ¬ p.**
  you will get an A unless you don't get 100% on final.

Logical Operator – Implication

- Converse

  The proposition $q \to p$ is **converse** of $p \to q$.

- Contrapositive

  The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$.

- Inverse

  The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$.

Logical Operator – Implication

- Example

“The home team wins whenever it is raining?”

Because “q whenever p”, so $p \rightarrow q$, the original statement can be rewritten as “If it is raining, then the home team wins.”

- Contrapositive

  “If the home team does not win, then it is not raining.”
- Converse

  “If the home team wins, then it is raining.”
- Inverse

  “If it is not raining, then the home team does not win.”

Logical Operator – Bi-conditional

- $p \leftrightarrow q$ corresponds to English "*p* if and only if *q*."
- Symbol: $\leftrightarrow$
- The bi-conditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.
- Bi-conditional statements are also called *bi-implications*.
- Alternatively, it means "(if p then q) and (if q then p)"

- Example
  - "You can take the flight if and only if you buy a ticket."

Logical Operator – Bi-conditional

- Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Operator – Bi-conditional

p: You can take flight

q: You buy a ticket

$p \leftrightarrow q$

You can take flight if and only if you buy a ticket

What is the truth value when:

- you buy a ticket and you can take the flight ??
- $T \leftrightarrow T \equiv T$
- you don't buy a ticket and you can't take the flight ??
- $F \leftrightarrow F \equiv T$
- you buy a ticket but you can't take the flight ??
- $T \leftrightarrow F \equiv F$
- you can't buy a ticket but can take the flight ??
- $F \leftrightarrow T \equiv F$

Logical Operator – Bi-conditional

- Other English equivalents:
  - “p if and only if q”
  - “p is equivalent to q”
  - “p is necessary and sufficient for q”
  - “p iff q”
  - “If p then q, and conversely”

Bi-conditional -Example

p: "You can take the flight"

q: "You buy a ticket"

$p \leftrightarrow q$:

You can take the flight if and only if you buy a ticket

You can take the flight iff you buy a ticket

The fact that you can take the flight is necessary and sufficient for buying a ticket

Logical Operators Summary

		Not	Not	And	Or	Xor	Implication	Bi-conditional
p	**q**	**$\neg$p**	**$\neg$q**	**p$\wedge$ q**	**p $\vee$ q**	**p $\oplus$ q**	**p $\rightarrow$ q**	**p $\leftrightarrow$q**
T	T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F	F
F	T	T	F	F	T	T	T	F
F	F	T	T	F	F	F	T	T

Truth Table of Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the every propositions.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
  - This includes the atomic propositions

Truth Table of Compound Propositions

- $(p \vee \neg q) \rightarrow (p \wedge q)$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Truth Table of Compound Propositions

- $p \rightarrow (\neg q \wedge r)$

p	q	r	$\neg q$	$\neg q \wedge r$	$p \rightarrow (\neg q \wedge r)$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	F	T

Precedence of Logical Operators

- Just as in algebra, operators have precedence

$$4+3*2 = 4+(3*2), \qquad \text{not } (4+3)*2$$

- Example

This means that

$p \vee q \wedge \neg r \rightarrow s \leftrightarrow t$

yields: $(p \vee (q \wedge (\neg r)) \rightarrow s) \leftrightarrow (t)$

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Truth Tables

- Construct the truth table of following compound propositions
  - $p \rightarrow \neg p$
  - $p \oplus p$
  - $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

Chapter Reading

- ***Chapter 1***, Kenneth H. Rosen, Discrete Mathematics and Its Applications

Chapter Exercise (For Practice)

- Question # 1, 2, 3, 4, 8, 9, 13, 24, 27, 28, 31, 32

Discrete Structures
(Discrete Mathematics)
Fall 2202

Week-2 and Lecture-1

Applications of Propositional Logic

Logical Equivalence

Applications of Propositional Logic

- Translating English sentences (Formalization)
- System Specifications
- Boolean Searches
- Logic circuits

Translating English Sentences

- Steps to convert an English sentence to a statement in propositional logic
  - Identify propositions and represent using propositional variables.
  - Determine appropriate logical connectives

Translating English Sentences

- "I have neither given nor received help on this exam"

  Let p = I have given help on this exam

  q = I have received help on this exam

  $\neg p \wedge \neg q$

- Rephrase: It is not the case that either I have given or received help on this exams

  $\neg(p \vee q)$

Translating English Sentences

- "If I go to Harry's or to the country, I will not go shopping."
  - Let p = I go to Harry's
  - q = I go to the country.
  - r = I will go shopping.
- If p or q then not r

$$(p \lor q) \rightarrow \neg r$$

Translating English Sentences

- ***Page #: 14 Question #: 13***
- Let p = It is below freezing, q = It is snowing

a) It is below freezing and it is snowing
b) It is below freezing but not snowing
c) It is not below freezing and it is not snowing
d) It is either snowing or below freezing (or both)
e) If it is below freezing, it is also snowing
f) It is either below freezing or it is snowing (not both), but it is not snowing if it is below freezing
g) That it is below freezing is necessary and sufficient for it to be snowing

Translating English Sentences

- ***Page #: 14 Question #: 13***
- Let p = It is below freezing, q = It is snowing

a) It is below freezing and it is snowing $\mathbf{p \wedge q}$

b) It is below freezing but not snowing $\mathbf{p \wedge \neg q}$

c) It is not below freezing and it is not snowing $\mathbf{\neg p \wedge \neg q}$

d) It is either snowing or below freezing (or both) $\mathbf{p \vee q}$

e) If it is below freezing, it is also snowing $\mathbf{p \rightarrow q}$

f) It is either below freezing or it is snowing (not both), but it is not snowing if it is below freezing $\mathbf{(p \vee q) \wedge (p \rightarrow \neg q)}$

g) That it is below freezing is necessary and sufficient for it to be snowing $\mathbf{q \leftrightarrow p}$

Translating English Sentences

- ***Page #: 18 Example #: 1***
- "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

- Let a = You can access the Internet from campus

  c = You are a computer science major

  and

  f = You are a freshman" respectively
- a only if c or not f

$$a \rightarrow (c \lor \neg f).$$

Exercise

- ***Page #: 14 Question #: 14***
- Let *p* and *q* be the propositions "The election is decided" and "The votes have been counted," respectively. Express each of these compound propositions as an English sentence.

1. $\neg p$
2. $p \vee q$
3. $\neg p \wedge q$
4. $q \rightarrow p$
5. $\neg q \rightarrow \neg p$
6. $\neg p \rightarrow \neg q$
7. $p \leftrightarrow q$
8. $\neg q \vee (p \wedge q)$

System Specifications

1.2.3 System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Page #: 18 Example #: 3

The automated reply cannot be sent when the file system is full

p = The automated reply can be sent

q = The system is full

$q \longrightarrow \neg p$

Consistency

- System specifications should be **consistent**, They should not contain conflicting requirements that could be used to derive a contradiction

- When specifications are not consistent, there would be no way to develop a system that satisfies all specifications

- A list of propositions is **consistent** if it is possible to assign truth values to the proposition variables so that each proposition is true.

Determine whether these system specifications are **consistent**:

1. The diagnostic message is stored in the buffer or it is retransmitted.
2. The diagnostic message is not stored in the buffer.
3. If the diagnostic message is stored in the buffer, then it is retransmitted.

- Determine whether these system specifications are consistent:

1. The diagnostic message is stored in the buffer or it is retransmitted.
2. The diagnostic message is not stored in the buffer.
3. If the diagnostic message is stored in the buffer, then it is retransmitted.

- p = The diagnostic message is stored in the buffer
- q = The diagnostic message is retransmitted
- 1. $p \vee q$ 2. $\neg p$ 3. $p \to q$

1. $p \vee q$ 2. $\neg p$ 3. $p \rightarrow q$

Reasoning

- An assignment of truth values that makes all three specifications true must have *p* false to make $\neg p$ true.
- Because we want $p \vee q$ to be true but p must be false, *q* must be true.
- Because $p \rightarrow q$ is true when p is false and q is true
- we conclude that these specifications are **consistent**
- Let us do it with **truth table** now

- Is it remain consistent if the specification

“The diagnostic message is not retransmitted” is added?

p: The diagnostic message is stored in the buffer

q: The diagnostic message is retransmitted

1. $\boldsymbol{p} \vee \boldsymbol{q}$2. $\neg\boldsymbol{p}$3. $\boldsymbol{p} \rightarrow \boldsymbol{q}$

- Is it remain consistent if the specification

"The diagnostic message is not retransmitted" is added?

p: The diagnostic message is stored in the buffer

q: The diagnostic message is retransmitted

1. $p \vee q$2. $\neg p$3. $p \rightarrow q$

4. $\neg q$

Inconsistent

Discrete Structures
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Week -2, Lecture-2

Propositional Equivalence

Propositional Equivalence

- An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value

- Propositional Equivalence is extensively used in the construction of mathematical arguments.

Tautology and Contradiction

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology. A* compound proposition that is always false is called a *contradiction.*

p	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

- Show that $(p \land q) \rightarrow p$ is a tautology.

Logical Equivalence

- Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.

- The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology.

- The notation $p \equiv q$ denotes that p and q are logically equivalent.

Logical Equivalence

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg p$	$q \vee \neg p$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Logical Equivalence

- Converse

  The proposition $q \to p$ is **converse** of $p \to q$.

- Contrapositive

  The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$.

- Inverse

  The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$.

Logical Equivalence

				Implication	Inverse	Converse	Contrapositive
p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$q \to p$	$\neg q \to \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Logical Equivalence

Equivalence	Name
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distribution Laws

Logical Equivalence

- Distributive: $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

p	q	r	$q \land r$	$p \lor (q \land r)$	$(p \lor q)$	$(p \lor r)$	$(p \lor q) \land (p \lor r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Logical Equivalence

Equivalence	Name
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalence involving Implication

Logical Equivalence involving Implication
$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalence involving Bi-conditional

Logical Equivalence involving Bi-conditional
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Proof using Logical Equivalence

- Show that $\neg(p \wedge \neg q) \vee q \equiv \neg p \vee q$ is logically equivalent.

$\neg(p \wedge \neg q) \vee q$	
$\equiv (\neg p \vee \neg\neg q) \vee q$	De Morgan's
$\equiv (\neg p \vee q) \vee q$	Double negation
$\equiv \neg p \vee (q \vee q)$	Associative
$\equiv \neg p \vee q$	Idempotent

Proof using Logical Equivalence

Show that $(p \wedge q) \rightarrow q$ is a Tautology.

Proof:

$(p \wedge q) \rightarrow q$	
$\equiv \neg(p \wedge q) \vee q$	Implication
$\equiv (\neg p \vee \neg q) \vee q$	De Morgan
$\equiv \neg p \vee (\neg q \vee q)$	Associative
$\equiv \neg p \vee T$	Negation
$\equiv T$	Dominations

Proof using Logical Equivalence

- Show that $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.

$[p \wedge (p \rightarrow q)] \rightarrow q$

$\equiv [p \wedge (\neg p \vee q)] \rightarrow q$	Substitution for $\rightarrow$
$\equiv [(p \wedge \neg p) \vee (p \wedge q)] \rightarrow q$	Distributive
$\equiv [\,F \vee (p \wedge q)] \rightarrow q$	Negation
$\equiv (p \wedge q) \rightarrow q$	Identity
$\equiv \neg(p \wedge q) \vee q$	Substitution for $\rightarrow$
$\equiv (\neg p \vee \neg q) \vee q$	De Morgan's
$\equiv \neg p \vee (\neg q \vee q\,)$	Associative
$\equiv \neg p \vee T$	Negation
$\equiv T$	Domination

Proof using Logical Equivalence

Show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$ is logically equivalent.

$L.H.S = \neg(p \vee (\neg p \wedge q))$

$\equiv \neg p \wedge \neg(\neg p \wedge q)$	DeMorgan's Law
$\equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$	DeMorgan's Law
$\equiv \neg p \wedge (p \vee \neg q)$	Double Negation Law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive Law
$\equiv F \vee (\neg p \wedge \neg q)$	Negation Law
$\equiv (\neg p \wedge \neg q) \vee F$	Commutative Law
$\equiv \neg p \wedge \neg q$	Identity Law

$= R.H.S$

Chapter Reading

- ***Book: Kenneth H. Rosen, Discrete Mathematics and its Application, (8th Edition), McGraw Hill Education.***
- ***1.1 Propositional Logic***
- ***1.2: Applications of Propositional Logic***
- ***1.3: Propositional Equivalences***

Assignment

Chapter 1: Foundations: Logic and Proofs

- ***1.1 Propositional Logic*:** ***Page #: 13 to 17***
  Question #: 7, 8, 9, 10, 11, 12, 13, 18, 19,
  20, 38, 39, 40, 47, 48

- ***1.2: Applications of Propositional Logic*:** ***Page #: 23 to 26***
  Question #: 5, 6, 7, 8, 9, 10, 11, 12, 44, 45, 46, 47

- ***1.3: Propositional Equivalences*:** ***Page #: 38 to 40***
  Question #: 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,
  33, 34, 35, 36, 37, 38, 39

- ***Book: Kenneth H. Rosen, Discrete Mathematics and its Application, (8th Edition), McGraw Hill Education.***

- **Due Date: Within 15 days, after that marks will be deducted**

MAT401 - Discrete Structures (Discrete Mathematics) Fall 2022

Week -3 and Lecture – 1 & 2

Predicates and Quantifiers

SINDH MADRESSATUL ISLAM UNIVERSITY KARACHI
Chartered by Government of Sindh, Recognized by HEC.

Bitwise Operations

- Computer represents information using bits. A bit is a symbol with two possible values, namely, 0 (zero) and 1 (one).
- 1 represents T (true) and 0 represents (false).
- A variable is called a Boolean Variable if its value is either true or false.
- A Boolean Variable can be represented by a bit.
- A bit string is a series of Boolean values. Length of the string is the number of bits.
- 101010011 is nine Boolean values in one string

Bitwise Operations

- We can then do operations on these Boolean strings

- Each column is its own Boolean operation

Operations	Operator	Bit String1	Bit String 2	Result
Bitwise XOR	⊕	0101 1010	1011 0100	1110 1110
Bitwise OR	∨	0101 1010	1011 0100	1111 1110
Bitwise AND	∧	0101 1010	1011 0100	00010000

Predicate

- A predicate is a declarative statement with at least one variable (i.e. unknown value).

- A predicate, or propositional function, is a function that takes some variable(s) as arguments and returns True or False.

Predicate

- Suppose Q(x,y) = “x > y”

Proposition, YES or NO?		Predicate, YES or NO?	
Q(x,y)	No	Q(x,y)	Yes
Q(3,4)	Yes	Q(3,4)	No
Q(x,9)	No	Q(x,9)	Yes

Quantification

- Quantification expresses the extent to which a predicate is true over a range of elements.

- In English, the words *all*, *some*, *many*, *none*, and *few* are used in quantifications.

- The area of logic that deals with predicates and quantifiers is called the **predicate calculus.**

Types of Quantification

- A quantifier is “an operator that limits the variables of a proposition”.

- There are two types of Quantifications:
  - Universal Quantification
  - Existential Quantification

Universal Quantification

- What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement $x^2 < 10$ and the domain consists of the positive integers not exceeding 4?

Universal Quantification

- What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement $x^2 < 10$ and the domain consists of the positive integers not exceeding 4?

Solution:

- The statement $\forall x P(x)$ is the same as the conjunction

$P(1) \wedge P(2) \wedge P(3) \wedge\ P(4)$,

- Because $\boldsymbol{P(4) \equiv 4^2 < 10}$**, is false**,
- it follows that $\forall \boldsymbol{x}\ \boldsymbol{P(x)}$ **is false**.

Existential Quantification

- Represented by a backwards E: ∃
  - It means "there exists", there is", "for some", etc.
  - Let P(x) = x+1 > x

- We can state the following:
  - ∃x P(x)
  - English translation: "there exists (a value of) x such that P(x) is true"
  - English translation: "for at least one value of x, x+1>x is true"
  - English translation: "for some x, P(x)"

Existential Quantification

- The existential quantifier of P(x) is the proposition:
- "P(x) is true for some x in the universe of discourse."
- $\exists x\, P(x)$ is TRUE if there is an x for which P(x) is true.
- $\exists x\, P(x)$ is FALSE if P(x) is false for every single x.

Existential Quantification

- Note that you still must specify your universe
- Let P(x) = x+1 < x
    - There is no numerical value x for which x+1<x
    - Thus, ∃x P(x) is false

Existential Quantification

- Let $P(x) = x+1 > x$
  - There is a numerical value for which $x+1>x$
  - In fact, it's true for all of the values of x. Thus, $\exists x\ P(x)$ is true

- In order to show an existential quantification is true, you only have to find ONE value

- In order to show an existential quantification is false, you have to show it's false for ALL values

Existential Quantification

- **Example:** Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

- **Solution:** Because "$x > 3$" is sometimes true—for instance, when $x = 4$ the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.

Existential Quantification

- **Example:** Let Q(x) denote the statement "x ==x + 1."What is the truth value of the quantification ∃xQ(x), where the domain consists of all real numbers?

- **Solution:** Because Q(x) is false for every real number x, the existential quantification of Q(x), which is ∃xQ(x), is false.

Universal Quantification

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier**. We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$." An element for which $P(x)$ is false is called a **counterexample** to $\forall x P(x)$.

TABLE 1 Quantifiers.

Statement	***When True?***	***When False?***
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Existential Quantification

The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

Logically Equivalent

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

$$\neg \forall x P(x) \equiv \exists x \neg P(x).$$

$$\neg \exists x Q(x) \equiv \forall x \neg Q(x).$$

De Morgan's Laws for Quantifier.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

1.4 Predicates and Quantifiers
Exercise Page #: 56

1. Let $P(x)$ denote the statement "$x \leq 4$." What are these truth values?

Solution:	$P(x)$: $x \leq 4$.
a) $P(0)$	True
b) $P(4)$	True
c) $P(6)$	False

1.4 Predicates and Quantifiers
Exercise Page #: 56

2. Let $P(x)$ be the statement "The word x contains the letter a." What are these truth values?

Solution: $P(x)$: "The word x contains the letter a."

a) P(orange)	False
b) P(lemon)	False
c) P(true)	False
d) P(false)	True

1.4 Predicates and Quantifiers
Exercise Page #: 56

3. Let $Q(x, y)$ denote the statement "x is the capital of y." What are these truth values?

Solution: $Q(x, y)$: "x is the capital of y."

a) *Q*(Denver, Colorado)	True
b) *Q*(Detroit, Michigan)	False
c) *Q*(Massachusetts, Boston)	False
d) *Q*(New York, New York)	False

1.4 Predicates and Quantifiers Exercise Page #: 56

4. State the value of x after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement "$x > 1$," if the value of x when this statement is reached is

a) $x = 0$. ***value of x is unchanged, so x=0***
b) $x = 1$. ***value of x is unchanged, so x=1***
c) $x = 2$. ***value of x is changed, so x=1***

1.4 Predicates and Quantifiers Exercise Page #: 56

5. Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.

Solution:

$P(x)$: "x spends more than five hours every weekday in class,"
Domain: set of all students.

a) $\exists x P(x)$: There is a student who spends more than 5 hours every weekday in class.

1.4 Predicates and Quantifiers
Exercise Page #: 56

5. Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.

Solution:

$P(x)$: "x spends more than five hours every weekday in class,"
Domain: set of all students.

b) $\forall x P(x)$: Every student spends more than 5 hours every weekday in class.

1.4 Predicates and Quantifiers
Exercise Page #: 56

5. Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.

Solution:

$P(x)$: "x spends more than five hours every weekday in class,"

Domain: set of all students.

c) $\exists x \neg P(x)$: There is a student who does not spend more than 5 hours every weekday in class.

1.4 Predicates and Quantifiers
Exercise Page #: 56

5. Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.

Solution:

$P(x)$: "x spends more than five hours every weekday in class,"

Domain: set of all students.

d) $\forall x \neg P(x)$: No student spends more than 5 hours every weekday in class.

1.4 Predicates and Quantifiers Exercise Page #: 56

6. Let $N(x)$ be the statement "x has visited North Dakota," where the domain consists of the students in your school. Express each of these quantifications in English.

Solution: $N(x)$: "x has visited North Dakota,"
Domain: A set of all students in your school.

a) $\exists x N(x)$: There is a student who has visited North Dakota.

b) $\forall x N(x)$: Every student has visited North Dakota.

1.4 Predicates and Quantifiers
Exercise Page #: 56

6. Let $N(x)$ be the statement "x has visited North Dakota," where the domain consists of the students in your school. Express each of these quantifications in English.

Solution: $N(x)$: "x has visited North Dakota,"
Domain: A set of all students in your school.

c) $\neg\exists x N(x)$: Every student has not visited North Dakota.

d) $\exists x\, \neg N(x)$: There is a student who has not visited North Dakota.

1.4 Predicates and Quantifiers
Exercise Page #: 56

6. Let $N(x)$ be the statement "x has visited North Dakota," where the domain consists of the students in your school. Express each of these quantifications in English.

Solution: $N(x)$: "x has visited North Dakota,"
Domain: A set of all students in your school.

e) $\neg\forall x N(x)$: Some students has not visited North Dakota.

f) $\forall x\ \neg N(x)$: No student has visited North Dakota.

1.4 Predicates and Quantifiers
Exercise Page #: 56

7. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

Solution: $C(x)$: "x is a comedian" and $F(x)$: "x is funny"
Domain: A set of all people.

a) $\forall x(C(x) \rightarrow F(x))$: Every comedian is funny.

b) $\forall x(C(x) \wedge F(x))$: Every person is a funny comedian.

1.4 Predicates and Quantifiers
Exercise Page #: 56

7. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

Solution: $C(x)$: "x is a comedian" and $F(x)$: "x is funny"
Domain: A set of all people.

c) $\exists x(C(x) \rightarrow F(x))$: There exists a person such that if she or he is a comedian, then she or he is funny.

d) $\exists x(C(x) \land F(x))$: Some comedians are funny.

1.4 Predicates and Quantifiers
Exercise Page #: 56

8. Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.

Solution: $R(x)$: "x is a rabbit" and $H(x)$: "x hops"
Domain: A set of all animal.

a) $\forall x(R(x) \rightarrow H(x))$: Every rabbit hops.

b) $\forall x(R(x) \wedge H(x))$: Every animal is a rabbit and it hops.

1.4 Predicates and Quantifiers
Exercise Page #: 56

8. Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.

Solution: $R(x)$: "x is a rabbit" and $H(x)$: "x hops"
Domain: A set of all animal.

c) $\exists x(R(x) \rightarrow H(x))$: There is an animal, if it is rabbit then it hops.

d) $\exists x(R(x) \wedge H(x))$: Some animal are rabbit and they hop.

1.4 Predicates and Quantifiers
Exercise Page #: 56-57

9. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

Solution: $P(x)$: "x can speak Russian."

$Q(x)$: "x knows the computer language C++."

a) There is a student at your school who can speak Russian and who knows C++. $\exists x(P(x) \land Q(x))$

1.4 Predicates and Quantifiers
Exercise Page #: 56-57

9. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

Solution: $P(x)$: "x can speak Russian."
$Q(x)$: "x knows the computer language C++."

b) There is a student at your school who can speak Russian but who doesn't know C++. $\exists x(P(x) \land \neg Q(x))$

1.4 Predicates and Quantifiers
Exercise Page #: 56-57

9. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

Solution: $P(x)$: "x can speak Russian."

$Q(x)$: "x knows the computer language C++."

c) Every student at your school either can speak Russian or knows C++. $\forall x(P(x) \vee Q(x))$

1.4 Predicates and Quantifiers
Exercise Page #: 56-57

9. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

Solution: $P(x)$: "x can speak Russian."
$Q(x)$: "x knows the computer language C++."

d) No student at your school can speak Russian or knows C++. $\forall x \neg (P(x) \vee Q(x))$

1.4 Predicates and Quantifiers
Exercise Page #: 57

10. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

Solution:

$C(x)$: "x has a cat," $D(x)$: "x has a dog," $F(x)$: "x has a ferret."

Domain: A set of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.

$$\exists x(C(x) \land D(x) \land F(x))$$

1.4 Predicates and Quantifiers
Exercise Page #: 57

10. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

Solution:

$C(x)$: "x has a cat," $D(x)$: "x has a dog," $F(x)$: "x has a ferret."

Domain: A set of all students in your class.

b) All students in your class have a cat, a dog, or a ferret.

$$\forall x(C(x) \lor D(x) \lor F(x))$$

$$\forall x((C(x) \land D(x)) \lor F(x))$$

1.4 Predicates and Quantifiers
Exercise Page #: 57

10. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

Solution:

$C(x)$: "x has a cat," $D(x)$: "x has a dog," $F(x)$: "x has a ferret."

Domain: A set of all students in your class.

c) Some student in your class has a cat and a ferret, but not a dog.

$$\exists x(C(x) \land D(x) \land \neg F(x))$$

1.4 Predicates and Quantifiers
Exercise Page #: 57

10. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

Solution:

$C(x)$: "x has a cat," $D(x)$: "x has a dog," $F(x)$: "x has a ferret."

Domain: A set of all students in your class.

d) No student in your class has a cat, a dog, and a ferret.

$$\forall x\, \neg(C(x) \wedge D(x) \wedge F(x))$$

1.4 Predicates and Quantifiers
Exercise Page #: 57

10. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

Solution:

$C(x)$: "x has a cat," $D(x)$: "x has a dog," $F(x)$: "x has a ferret."

Domain: A set of all students in your class.

e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

$$\exists x\,(C(x) \land D(x) \land F(x))$$

1.4 Predicates and Quantifiers
Exercise Page #: 57

11. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

Solution: $P(x)$: "$x = x^2$"

Domain = A set of the integers

a) $P(0)$	=	T	**b)** $P(1)$	=	T
c) $P(2)$	=	F	**d)** $P(-1)$	=	F
e) $\exists x P(x)$	=	T	**f)** $\forall x P(x)$	=	F

1.4 Predicates and Quantifiers
Exercise Page #: 57

12. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

Solution: $Q(x)$: "$x + 1 > 2x$"
Domain = A set of all integers

a) $Q(0)$	=	T	**b)** $Q(-1)$	=	T
c) $Q(1)$	=	F	**d)** $\exists x Q(x)$	=	T
e) $\forall x Q(x)$	=	F	**f)** $\exists x \neg Q(x)$	=	T
g) $\forall x \neg Q(x)$	=	F			

1.4 Predicates and Quantifiers
Exercise Page #: 57

13. Determine the truth value of each of these statements if the domain consists of all integers.

Solution: Domain = A set of all integers.

a) $\forall n(n + 1 > n) = \mathrm{T}$

b) $\exists n(2n = 3n) = \mathrm{T}$

c) $\exists n(n = -n) = \mathrm{T}$

d) $\forall n(3n \leq 4n) = \mathrm{F}$

1.4 Predicates and Quantifiers
Exercise Page #: 57

14. Determine the truth value of each of these statements if the domain consists of all real numbers.

Solution: Domain = A set of all real numbers.

a) $\exists x(x^3 = -1) = T$

b) $\exists x(x^4 < x^2) = F$

c) $\forall x((-x)^2 = x^2) = T$

d) $\forall x(2x > x) = F$

1.4 Predicates and Quantifiers
Exercise Page #: 57

15. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

Solution: Domain = A set of all integers.

a) $\forall n(n^2 \geq 0) = \text{T}$

b) $\exists n(n^2 = 2) = \text{F}$

c) $\forall n(n^2 \geq n) = \text{T}$

d) $\exists n(n^2 < 0) = \text{F}$

1.4 Predicates and Quantifiers
Exercise Page #: 57

16. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

Solution: Domain = A set of all real numbers.

a) $\exists x(x^2 = 2) = \mathrm{T}$

b) $\exists x(x^2 = -1) = \mathrm{F}$

c) $\forall x(x^2 + 2 \geq 1) = \mathrm{T}$

d) $\forall x(x^2 \neq x) = \mathrm{F}$

1.4 Predicates and Quantifiers
Exercise Page #: 57

17. Suppose that the domain of the propositional function $P(x)$ consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.

Solution: Domain = {0, 1, 2, 3, 4}.

a) $\exists x P(x) = P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4)$

b) $\forall x P(x) = P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

c) $\exists x \neg P(x) = \neg P(0) \vee \neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$

1.4 Predicates and Quantifiers
Exercise Page #: 57

17. Suppose that the domain of the propositional function $P(x)$ consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.

Solution: Domain = {0, 1, 2, 3, 4}.

d) $\forall x \neg P(x) = \neg P(0) \wedge \neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4)$

e) $\neg \exists x P(x) = \neg(P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4))$

f) $\neg \forall x P(x) = \neg(P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4))$

1.4 Predicates and Quantifiers Exercise Page #: 57

18. Suppose that the domain of the propositional function $P(x)$ consists of the integers −2, −1, 0, 1, and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.

Solution: Domain = {-2 , -1, 0, 1, 2}.

a) $\exists x P(x) = P(-2) \lor P(-1) \lor P(0) \lor P(1) \lor P(2)$

b) $\forall x P(x) = P(-2) \land P(-1) \land P(0) \land P(1) \land P(2)$

1.4 Predicates and Quantifiers
Exercise Page #: 57

18. Suppose that the domain of the propositional function $P(x)$ consists of the integers −2, −1, 0, 1, and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.

Solution: Domain = {-2 , -1, 0, 1, 2}.

c) $\exists x \neg P(x) = \neg P(-2) \lor \neg P(-1) \lor \neg P(0) \lor \neg P(1) \lor \neg P(2)$

d) $\forall x \neg P(x) = \neg P(-2) \land \neg P(-1) \land \neg P(0) \land \neg P(1) \land \neg P(2)$

1.4 Predicates and Quantifiers
Exercise Page #: 57

18. Suppose that the domain of the propositional function $P(x)$ consists of the integers −2, −1, 0, 1, and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.

Solution: Domain = {-2 , -1, 0, 1, 2}.

e) $\neg\exists x P(x) = \neg(P(-2) \vee P(-1) \vee P(0) \vee P(1) \vee P(2))$

f) $\neg\forall x P(x) = \neg(P(-2) \wedge P(-1) \wedge P(0) \wedge P(1) \wedge P(2))$

1.4 Predicates and Quantifiers
Exercise Page #: 57

19. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

Solution: Domain = {1, 2, 3, 4, 5}.

a) $\exists x P(x) = P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$

b) $\forall x P(x) = P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$

1.4 Predicates and Quantifiers
Exercise Page #: 57

19. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

Solution: Domain = {1, 2, 3, 4, 5}.

c) $\neg\exists x P(x) = \neg(P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5))$

d) $\neg\forall x P(x) = \neg(P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5))$

1.4 Predicates and Quantifiers
Exercise Page #: 57

19. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

Solution: Domain = {1, 2, 3, 4, 5}.

e) $\forall x((x \neq 3) \rightarrow P(x)) \vee \exists x \neg P(x)$
$= (P(1) \wedge P(2) \wedge P(4) \wedge P(5)) \vee (\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5))$

1.4 Predicates and Quantifiers
Exercise Page #: 57

20. Suppose that the domain of the propositional function $P(x)$ consists of -5, -3, -1, 1, 3, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

Solution: Domain = {-5, -4, -3, -2, -1, 1, 2, 3, 4, 5}.

a) $\exists x P(x)$

$= P(-5) \vee P(-4) \vee P(-3) \vee P(-2) \vee P(-1) \vee P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$

1.4 Predicates and Quantifiers
Exercise Page #: 57

20. Suppose that the domain of the propositional function $P(x)$ consists of −5, −3, −1, 1, 3, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

Solution: Domain = {-5, -4, -3, -2, -1, 1, 2, 3, 4, 5}.

b) $\forall x P(x) = P(-5) \land P(-4) \land P(-3) \land P(-2) \land P(-1) \land P(1) \land P(2) \land P(3) \land P(4) \land P(5)$

1.4 Predicates and Quantifiers
Exercise Page #: 57

20. Suppose that the domain of the propositional function $P(x)$ consists of -5, -3, -1, 1, 3, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

Solution: Domain = {-5, -4, -3, -2, -1, 1, 2, 3, 4, 5}.

c) $\forall x((x \neq 1) \rightarrow P(x)) = P(-5) \wedge P(-4) \wedge P(-3) \wedge P(-2) \wedge P(-1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$

1.4 Predicates and Quantifiers Exercise Page #: 57

20. Suppose that the domain of the propositional function $P(x)$ consists of -5, -3, -1, 1, 3, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

Solution: Domain = {-5, -4, -3, -2, -1, 1, 2, 3, 4, 5}.

d) $\exists x((x \geq 0) \wedge P(x)) = P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$

1.4 Predicates and Quantifiers
Exercise Page #: 57

20. Suppose that the domain of the propositional function $P(x)$ consists of -5, -3, -1, 1, 3, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

Solution: Domain = {-5, -4, -3, -2, -1, 1, 2, 3, 4, 5}.

e) $\exists x(\neg P(x)) \land \forall x((x < 0) \rightarrow P(x))$ = $P(-5) \land P(-4) \land P(-3) \land P(-2) \land P(-1)$

Chapter Reading

Book: Kenneth H. Rosen,

Discrete Mathematics and Its Applications (Eight Edition)

Chapter 1: The Foundations: Logic and Proofs

Section 1.4: Predicates and Quantifiers

Exercise for Practice

Question 21 to 40, 59 and 60

MAT401 - Discrete Structures (Discrete Mathematics) Fall 2022

Week -4 and Lecture – 1 & 2

Introduction to Proofs

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Proof

A demonstration that a theorem is true.

OR

A proof is a valid argument that establishes the truth of a mathematical statement.

Applications of Proof

- Proving mathematical theorems,
- Designing algorithms and proving they meet their specifications,
- Verifying computer programs,
- Establishing operating systems are secure,
- Making inferences in artificial intelligence,
- Showing system specifications are consistent.

Terminologies of Proofs

Theorem: A statement that can be shown true.

Axiom: a statement that is assumed to be true and that can be used as a basis for proving theorems.

Lemma: a theorem used to prove other theorems

Corollary: a proposition that can be proved as a consequence of a theorem that has just been proved.

Terminologies of Proofs

Exhaustive proof: a proof that establishes a result by checking a list of all possible cases.

Proof by cases: a proof broken into separate cases, where these cases cover all possibilities.

Counterexample: an element x such that P(x) is false.

Terminologies of Proofs

Constructive existence proof: a proof that an element with a specified property exists that explicitly finds such an element.

Nonconstructive existence proof: a proof that an element with a specified property exists that does not explicitly find such an element.

Uniqueness proof: a proof that there is exactly one element satisfying a specified property.

Type of Proofs

Proof by

- implication,
- contraposition,
- contradiction, and
- induction,

Type of Proofs – Proof by Implication

Direct proof: a proof that $p \rightarrow q$ is true that proceeds by showing that q must be true when p is true.

Vacuous proof: a proof that $p \rightarrow q$ is true based on the fact that p is false.

Trivial proof: a proof that $p \rightarrow q$ is true based on the fact that q is true.

Proofs by Contraposition and Contradiction

Proof by contraposition: a proof that $p \to q$ is true that proceeds by showing that p must be false when q is false.

Proof by contradiction: a proof that p is true based on the truth of the conditional statement $\neg p \to q$, where q is a contradiction.

Proof by Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

Basis Step: We verify that $P(1)$ is true.

Inductive Step: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Type of Proofs – Proof by Implication

Direct proof: a proof that $p \rightarrow q$ is true that proceeds by showing that q must be true when p is true.

Proof by Implication – Direct Proof

Use a direct proof to show that the sum of two odd integers is even.

Solution: Let $n = 2k + 1$ and $m = 2l + 1$ be odd integers.

Then $n + m = 2(k + l + 1)$ is even because $(k + l + 1)$ is multiple of 2.

This is a direct proof.

Proof by Implication – Direct Proof

Let $P(n)$ be the proposition,

"If a and b are positive real numbers, then $(a + b)^n \geq a^n + b^n$."

Prove that $P(1)$ is true. What kind of proof did you use?

Solution: $P(1)$ is true because

$$(a + b)^1 = a + b \geq a^1 + b^1 = a + b$$

This is the direct proof.

Proof by Implication – Direct Proof

Use a direct proof to show that every odd integer is the difference of two squares.

Solution: Let n is odd,

we can write $n = 2k + 1$ for some integer k. Then

$$(k+1)^2 - k^2$$
$$= k^2 + 2k + 1 - k^2 = 2k + 1 = n$$

This is a direct proof.

Type of Proofs – Proof by Implication

Vacuous proof: a proof that $p \to q$ is true based on the fact that p is false.

Proof by Implication – Vacuous Proof

Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution: Note that,

$P(0)$ is "If $0 > 1$, then $0^2 > 0$."

We can show $P(0)$ using a vacuous proof.

Indeed, the hypothesis $0 > 1$ is false.

This tells us that $P(0)$ is automatically true.

Proof by Implication – Vacuous Proof

Prove that if n is an integer with $10 \leq n \leq 15$ which is a perfect square, then n is also a perfect cube.

Solution: Note that, there are no perfect squares n with $10 \leq n \leq 15$, because $3^2 = 9$ and $4^2 = 16$.

Hence, the statement that n is an integer with $10 \leq n \leq 15$ which is a perfect square is false for all integers n.

Consequently, the statement to be proved is true for all integers n.

Proof by Implication – Vacuous Proof

Let $x \in \mathbb{R}$. If $x^2 + 1 < 0$, then $x^5 \geq 4$.

Solution: Note that,

$$x^2 + 1 > x^2 \geq 0$$

Thus $x^2 + 1 < 0$ is false for all $x \in \mathbb{R}$, and so the implication is true.

This is a vacuous proof.

Type of Proofs – Proof by Implication

Trivial proof: a proof that $p \rightarrow q$ is true based on the fact that q is true.

Proof by Implication – Trivial Proof

Assume $P(n)$ is "if $ab > 0$, then $(ab)^n > 0$". Show that $P(0)$ is true.

Proof: $P(0)$ is "if $ab > 0$, then $(ab)^0 > 0$".

$$(ab)^0 = 1 > 0$$

Since the conclusion of $P(0)$ is true.

This is a trivial proof $p \to q$ is true when q is true.

Proof by Implication – Trivial Proof

Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$," where the domain consists of all nonnegative integers. Show that $P(0)$ is true.

Solution: The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$." Because $a^0 = b^0 = 1$, the conclusion of the conditional statement "If $a \geq b$, then $a^0 \geq b^0$" is true.

Hence, this conditional statement, which is $P(0)$, is true.

This is a trivial proof.

Note that the hypothesis, which is the statement "$a \geq b$," was not needed in this proof.

Proof by Implication – Trivial Proof

Let $x \in \mathbb{R}$. If $x > 0$, then $x^2 + 5 > 0$.

Solution: Since $x^2 \geq 0$, for all $x \in \mathbb{R}$,

it follows that $x^2 + 5 > x^2 \geq 0$.

Hence $x^2 + 5 \geq 0$.

Note that we didn't use the fact that $x > 0$,

since it holds for all $x \in \mathbb{R}$.

This is a trivial proof.

Type of Proofs – Proof by Contraposition

A proof that $p \rightarrow q$ is true that proceeds by showing that p must be false when q is false.

Proof by Contraposition

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution: Assume that n is even. Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1).$$

This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd.

Our proof by contraposition succeeded;

"If $3n + 2$ is odd, then n is odd."

Proof by Contraposition

Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

Solution:

Assume that it is not true that $x \geq 1$ or $y \geq 1$.

Then $x < 1$ and $y < 1$.

Adding these two inequalities,

we obtain $x + y < 2$,

which is the negation of $x + y \geq 2$.

Our proof by contraposition succeeded.

Proof by Contraposition

Show that if n is an integer and $n^3 + 5$ is odd, then n is even using a proof by contraposition.

Solution:

Assume that n is odd,

so $n = 2k + 1$ for some integer k.

Then $n^3 + 5 = 2(4k^3 + 6k^2 + 3k + 3)$.

Because $n^3 + 5$ is two times some integer,

it is even.

Type of Proofs – Proof by Contradiction

A proof that p is true based on the truth of the conditional statement $\neg p \rightarrow q$, where q is a contradiction.

Proof by Contradiction

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution: Let p be the proposition $\sqrt{2}$ is irrational.

For contradiction, let $\neg p$: $\sqrt{2}$ is ration.

So, $\sqrt{2} = \frac{a}{b}$, $b \neq 0$ and a and b have no common faction.

It mean, the fraction $\frac{a}{b}$ is in simplest form.

Now squaring both sides, $2 = \frac{a^2}{b^2} \Longrightarrow 2b^2 = a^2$

It means, a must be even, and b must be odd.

Let $a = 2c$, so $2b^2 = 4c^2 \Longrightarrow b^2 = 2c^2$, so b is even.

Contradiction: (i). b^2 is even, so b is even. But we let b is odd.

(ii). If a and b both are even, then fraction is not simplest form.

Proof by Contradiction

Give a proof by contradiction of the theorem, "If $3n + 2$ is odd, then n is odd."

Solution: Let p be "$3n + 2$ is odd" and q be "n is odd."

For proof by contradiction,

Let both p: 3n + 2 is odd and $\neg q$: n is not odd are true.

$\therefore n \in \mathbb{E}$, and $k \in \mathbb{Z}$ such that $n = 2k$.

or $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.

It means, "3n + 2 is even" is equivalent to the statement$\neg p$.

We have a contradiction.

So, proving that if 3n + 2 is odd, then n is odd.

Proof by Contradiction

Show that if n is an integer and $n^3 + 5$ is odd, then n is even using a proof by contradiction.

Solution: **Let $n^3 + 5$ is odd, and n is odd.**

$\because$ n is odd and the product of two odd is odd,

$\therefore$ n^2 is odd and then that n^3 is odd.

But then $5 = (n^3 + 5) - n^3$ would have to be even because it is the difference of two odd numbers.

$\therefore$ the supposition that $n^3 + 5$ and n were both odd is wrong.

Mistake in Proof

What is wrong with this famous supposed "proof" that 1 = 2?

Proof:

Step	Reason
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Factor both sides of (3)
5. $a + b = b$	Divide both sides of (4) by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2 = 1$	Divide both sides of (6) by b

Proof by Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

Basis Step: We verify that $P(1)$ is true.

Inductive Step: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

Proof by Principle of Mathematical Induction

Prove by Mathematical Induction.

Q. 1. $1 + 2 + 3 + \ \dots + n = \frac{1}{2}n(n + 1)$

Q. 2. $1^2 + 2^2 + 3^2 + \ \dots + n^2 = \frac{1}{6}n(n + 1)\ (2n + 1)$

Q. 3. $1^3 + 2^3 + 3^3 + \ \dots + n^3 = \left[\frac{1}{2}n(n + 1)\right]^2$

Q. 4. $2 + 4 + 6 + \ \dots + 2n = 2n(n + 1)$

Proof by Principle of Mathematical Induction

Prove by Mathematical Induction.

5. Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3$ whenever n is a nonnegative integer.

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ whenever n is a positive integer.

7. Prove that $3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4$ whenever n is a nonnegative integer.

8. Prove that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$ whenever n is a nonnegative integer.

Chapter Reading

Book: Kenneth H. Rosen,

Discrete Mathematics and Its Applications (Eight Edition)

Exercise for Practice

Section 1.7: Introduction to Proofs

Section 5.1: Mathematical Induction

MAT401 - Discrete Structures (Discrete Mathematics) Fall 2022

Week -5 and Lecture – 1 & 2

Introduction to Sets and Set Operations

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MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 1

Introduction to Sets

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Applications of Set

- Databases
- Data-type or type in computer programming
- Constructing discrete structures
- Finite state machine
- Modeling computing machine
- Representing computational complexity of algorithms

Set

A **set** is an unordered collection of well-defined distinct objects.

The term 'well defined' means a set must have some specific property so, that it can easily be identified whether an object belongs to the given set or not.

Whereas the word 'distinct' means different objects.

Examples of a Set

Q. Tick (✓) which of the following are sets?

i. Collection of the names of Presidents of Pakistan.

ii. Collection of names of captains of hockey teams of Pakistan.

iii. Collection of delicious dishes.

iv. Collection of intelligent students in your class.

v. Collection of greater numbers.

vi. Collection of English teachers in your school.

Examples of a Set

Q. Tick (✓) which of the following are sets?

i. Collection of the names of Presidents of Pakistan. ✓

ii. Collection of names of captains of hockey teams of Pakistan. ✓

iii. Collection of delicious dishes. χ

iv. Collection of intelligent students in your class. χ

v. Collection of greater numbers. χ

vi. Collection of English teachers in your school. ✓

Elements of a Set

The distinct objects are called **elements or members** of the set.

We write $a \in A$ to denote that **a is an element of the set A**.

The notation $a \notin A$ denotes that **a is not an element of the set A**.

Introduction to Sets

How to describe a set?

Tabular Form:

O = {1, 3, 5, 7, 9}.

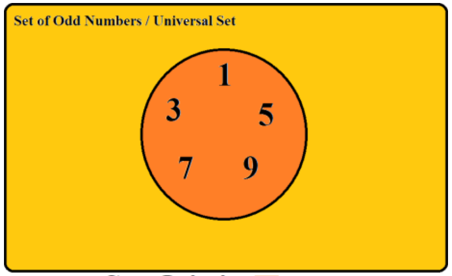
Descriptive Form:

The set **O** of odd positive integers less than 10

Set Builder Form:

$\mathbf{O} = \{x \in Z^{+} \mid x \text{ is odd and } x < 10\}$

Pictorial Form / Venn Diagram:



Set O is in

Introduction to Sets

Type of some number sets

N = {0, 1, 2, 3,…}, the set of all **natural numbers**

Z = {…,−2,−1, 0, 1, 2,…}, the set of all **integers**

$\mathbf{Z}^+$ = {1, 2, 3,…}, the set of all **positive integers**

Q = $\{\frac{p}{q} \mid p \in \boldsymbol{Z}, q \in \boldsymbol{Z}, \text{and } q \neq 0\}$, the set of all **rational numbers**

R, the set of all **real numbers**

$\mathbf{R}^+$, the set of all **positive real numbers**

C, the set of all **complex numbers**.

Real Numbers between a and b

Among the sets studied in calculus and other subjects are **intervals**, sets of all the real numbers between two numbers a and b, with or without a and b.

Real Numbers between a and b

If a and b are real numbers with $a \le b$, we denote these **intervals** by

$[a, b] = \{x \mid a \le x \le b\}$ **Closed interval**

$[a, b) = \{x \mid a \le x < b\}$ **Half closed interval**

$(a, b] = \{x \mid a < x \le b\}$ **Half open interval**

$(a, b) = \{x \mid a < x < b\}$ **Open interval**

Introduction to Sets

Real Numbers between a and b

$[a, b] = \{x \mid a \le x \le b\}$ **Closed interval**

$[a, b) = \{x \mid a \le x < b\}$ **Half closed interval**

$(a, b] = \{x \mid a < x \le b\}$ **Half open interval**

$(a, b) = \{x \mid a < x < b\}$ **Open interval**

Each of the intervals $[a, b]$, $[a, b)$, $(a, b]$, and (a, b) contains all the real numbers strictly between a and b.

The first two of these contain a and the first and third contain b.

Equal Sets

Two sets are ***equal*** if and only if they have the same elements.

Therefore, if A and B are sets, then A and B are equal if and only if

$$\forall x(x \in A \leftrightarrow x \in B).$$

We write $A = B$ if A and B are equal sets.

Empty Set

There is a special set that has no elements.

This set is called the **empty set**, or **null set**, and is denoted by $\emptyset$.

The empty set can also be denoted by $\{\ \}$

Singleton Set

A set with one element is called a **singleton set**.

A common error is to confuse the empty set $\emptyset$ with the set $\{\emptyset\}$, which is a singleton set.

The single element of the set $\{\emptyset\}$ is the empty set itself!

Empty Set vs Singleton Set

A useful analogy for remembering this difference is to think of folders in a computer file system.

The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

Introduction to Sets

Subsets vs Superset

The set A is a ***subset*** of B, and B is a ***superset*** of A, if and only if every element of A is also an element of B.

Subsets vs Superset

A is a Subset of B

$A \subseteq B$,

show that if $x \in A$ then $x \in B$.

A is Not a Subset of B

$A \nsubseteq B$,

find a single $x \in A$ such that $x \notin B$.

Subsets vs Superset

The notations indicate

$A \subseteq B$ A is a ***subset*** of the set B and

$B \supseteq A$ B is a ***superset*** of A.

So, $A \subseteq B$ and $B \supseteq A$ are equivalent statements.

Properties of Subsets

We see that $A \subseteq B$ if and only if the quantification

$$\forall x (x \in A \rightarrow x \in B)$$

Properties of Subsets

For every set S,

$\emptyset \subseteq \boldsymbol{S}$ Proper Subset

$\boldsymbol{S} \subseteq \boldsymbol{S}$ Improper Subset

Properties of Subsets

Proper subset

$A \subset B$

A is proper subset of B.

if and only if

$$\forall \boldsymbol{x}(\boldsymbol{x} \in \boldsymbol{A} \rightarrow \boldsymbol{x} \in \boldsymbol{B}) \wedge \exists \boldsymbol{x}(\boldsymbol{x} \in \boldsymbol{B} \wedge \boldsymbol{x} \notin \boldsymbol{A})$$

Improper subset

$A \subseteq B$ and $B \subseteq A$, then $A = B$

A and B are improper subsets of each other

$$\forall \boldsymbol{x}(\boldsymbol{x} \in \boldsymbol{A} \rightarrow \boldsymbol{x} \in \boldsymbol{B}) \text{ and } \forall \boldsymbol{x}(\boldsymbol{x} \in \boldsymbol{B} \rightarrow \boldsymbol{x} \in \boldsymbol{A})$$

$$\forall \boldsymbol{x}(\boldsymbol{x} \in \boldsymbol{A} \leftrightarrow \boldsymbol{x} \in \boldsymbol{B})$$

Properties of Subsets

Sets are Equal

Two sets A and B are equal,

if and only if

$A \subseteq B$ and $B \subseteq A$

then

$A = B$

Properties of Subsets

Sets are Equal

we have two equal sets

$$\boldsymbol{A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}}$$

and

$$\boldsymbol{B = \{x \mid x \textbf{ is a subset of the set } \{a, b\}\}.}$$

Properties of Subsets

The Size of a Set

Let ***S*** be a set.

If there are exactly ***n*** distinct elements in ***S*** where ***n*** is a nonnegative integer,

we say that ***S*** is a finite set,

and that ***n*** is the cardinality of ***S***.

The cardinality of ***S*** is denoted by |***S***|.

Properties of Subsets

The Size of a Set

Let A be the set of odd positive integers less than 10. Then $|A| = 5$.

Let S be the set of letters in the English alphabet. Then $|S| = 26$.

The null set has no elements, it follows that $|\emptyset| = 0$.

Properties of Subsets

The Size of a Set

A set is said to be ***infinite*** if it is not finite.

The set of positive integers is infinite.

Properties of Subsets

Power set of a Set

Given a set S,

the power set of S is the set of all subsets of the set S.

The power set of S is denoted by P(S).

Properties of Subsets

Power set of a Set

What is the power set of the set $\{0, 1, 2\}$?

The power set $P(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$.

Hence,

$$P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Properties of Subsets

Power set of a Set

What is the power set of the empty set {∅}?

The empty set has exactly one subset, namely, itself.

Consequently, P(∅) = {∅}.

The set {∅} has exactly two subsets,

namely, ∅ and the set {∅} itself.

Therefore, P({∅}) = {∅, {∅}}.

Properties of Subsets

Cartesian Products

Let A and B be sets.

The Cartesian product of A and B,

denoted by $A \times B$,

is the set of all ordered pairs (a, b),

where $a \in A$ and $b \in B$.

Hence,

$$\boldsymbol{A \times B = \{(a, b) \mid a \in A \land b \in B\}}.$$

Properties of Subsets

Cartesian Products

Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where

A = {1, 2} and B = {a, b, c}

The Cartesian product A × B is

A × B = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}.

The Cartesian product B × A is

B × A = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}.

This is not equal to $A \times B \neq B \times A$

Properties of Subsets

Cartesian Products

What is the Cartesian product A × B × C,

where A = {0, 1}, B = {1, 2}, and C = {0, 1, 2}?

The Cartesian product A × B × C consists of all ordered triples (a, b, c), where a ∈ A, b ∈ B, and c ∈ C.

Hence,

$$\boldsymbol{A \times B \times C}$$
$$\boldsymbol{= \{(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),}$$
$$\boldsymbol{(0,2,2),(1,1,0),(1,1,1),(1,1,2),(1,2,0),}$$
$$\boldsymbol{(1,2,1),(1,2,2)\}.}$$

2.1 Exercise, Page 131, 132 & 133

1. List the members of these sets.

a) $\{x \mid x \text{ is a real number such that } x^2 = 1\}$

b) $\{x \mid x \text{ is a positive integer less than } 12\}$

c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$

d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

Introduction to Sets

2.1 Exercise, Page 131, 132 & 133

2. Use set builder notation to give a description of each of these sets.

a) $\{0, 3, 6, 9, 12\}$

b) $\{-3, -2, -1, 0, 1, 2, 3\}$

c) $\{m, n, o, p\}$

2.1 Exercise, Page 131, 132 & 133

3. Which of the intervals

$$(\mathbf{0},\mathbf{5}),(\mathbf{0},\mathbf{5}],[\mathbf{0},\mathbf{5}),[\mathbf{0},\mathbf{5}],(\mathbf{1},\mathbf{4}],[\mathbf{2},\mathbf{3}],(\mathbf{2},\mathbf{3})$$

contain

a) 0? **b)** 1?

c) 2? **d)** 3?

e) 4? **f)** 5?

2.1 Exercise, Page 131, 132 & 133

4. For each of these intervals, list all its elements or explain why it is empty.

a) $[a, a]$

b) $[a, a)$

c) $(a, a]$

d) (a, a)

e) (a, b), where $a > b$

f) $[a, b]$, where $a > b$

2.1 Exercise, Page 131, 132 & 133

5. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other..

a) the set of airline flights from New York to New Delhi, the set of nonstop airline flights from New York to New Delhi.

b) the set of people who speak English, the set of people who speak Chinese.

c) the set of flying squirrels, the set of living creatures that can fly.

2.1 Exercise, Page 131, 132 & 133

6. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

a) the set of people who speak English, the set of people who speak English with an Australian accent

b) the set of fruits, the set of citrus fruits

c) the set of students studying discrete mathematics, the set of students studying data structures

2.1 Exercise, Page 131, 132 & 133

7. Determine whether each of these pairs of sets are equal.

a) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}, \{5, 3, 1\}$

b) $\{\{1\}\}, \{1, \{1\}\}$

c) $\emptyset, \{\emptyset\}$

8. Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other of these sets.

2.1 Exercise, Page 131, 132 & 133

9. For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in \mathbf{R} \mid x$ is an integer greater than $1\}$

b) $\{x \in \mathbf{R} \mid x$ is the square of an integer$\}$

c) $\{2,\{2\}\}$

d) $\{\{2\},\{\{2\}\}\}$

e) $\{\{2\},\{2,\{2\}\}\}$

f) $\{\{\{2\}\}\}$

2.1 Exercise, Page 131, 132 & 133

10. For each of the sets in Exercise 9, determine whether {2} is an element of that set.

a) $\{x \in \mathbf{R} \mid x$ is an integer greater than $1\}$

b) $\{x \in \mathbf{R} \mid x$ is the square of an integer$\}$

c) $\{2,\{2\}\}$

d) $\{\{2\},\{\{2\}\}\}$

e) $\{\{2\},\{2,\{2\}\}\}$

f) $\{\{\{2\}\}\}$

2.1 Exercise, Page 131, 132 & 133

11. Determine whether each of these statements is true or false.

a) $0 \in \emptyset$

b) $\emptyset \in \{0\}$

c) $\{0\} \subset \emptyset$

d) $\emptyset \subset \{0\}$

e) $\{0\} \in \{0\}$

f) $\{0\} \subset \{0\}$

g) $\{\emptyset\} \subseteq \{\emptyset\}$

Introduction to Sets

2.1 Exercise, Page 131, 132 & 133

12. Determine whether these statements are true or false.

a) $\emptyset \in \{\emptyset\}$

b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$

c) $\{\emptyset\} \in \{\emptyset\}$

d) $\{\emptyset\} \in \{\{\emptyset\}\}$

e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

2.1 Exercise, Page 131, 132 & 133

13. Determine whether each of these statements is true or false.

a) $x \in \{x\}$

b) $\{x\} \subseteq \{x\}$

c) $\{x\} \in \{x\}$

d) $\{x\} \in \{\{x\}\}$

e) $\emptyset \subseteq \{x\}$

f) $\emptyset \in \{x\}$

2.1 Exercise, Page 131, 132 & 133

14. Use a Venn diagram to illustrate the subset of odd integers in the set of all positive integers not exceeding 10.

15. Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter *R* in the set of all months of the year.

16. Use a Venn diagram to illustrate the relationship $A \subseteq B$ and $B \subseteq C$.

Introduction to Sets

2.1 Exercise, Page 131, 132 & 133

21. What is the cardinality of each of these sets?

a) $\{a\}$

b) $\{\{a\}\}$

c) $\{a, \{a\}\}$

d) $\{a, \{a\}, \{a, \{a\}\}\}$

22. What is the cardinality of each of these sets?

a) $\emptyset$

b) $\{\emptyset\}$

c) $\{\emptyset, \{\emptyset\}\}$

d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

2.1 Exercise, Page 131, 132 & 133

23. Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$ **b)** $\{a, b\}$ **c)** $\{\emptyset, \{\emptyset\}\}$

25. How many elements does each of these sets have where a and b are distinct elements?

a) $P(\{a, b, \{a, b\}\})$

b) $P(\{\emptyset, a, \{a\}, \{\{a\}\}\})$

c) $P(P(\emptyset))$

2.1 Exercise, Page 131, 132 & 133

26. Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

29. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find

a) $A \times B$.

b) $B \times A$.

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 2

Set Operations

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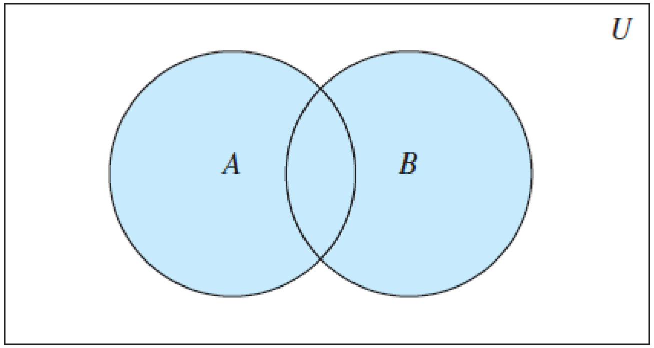
Sets Operations

Union of Sets

Let A and B be sets. The ***union*** of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

Venn Diagram



$A \cup B$ is shaded.

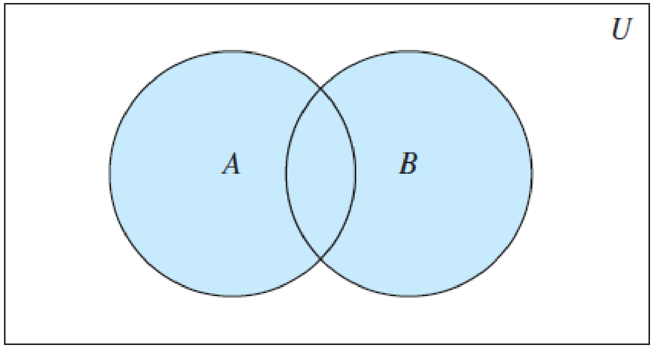
Union of Sets

Example

The union of the sets

{1, 3, 5} and {1, 2, 3} is the set

$\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.



$A \cup B$ is shaded.

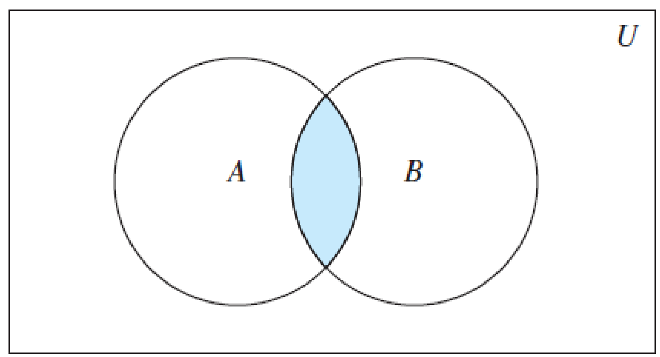
Sets Operations

Intersection of Sets

Let A and B be sets. The ***intersection*** of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x \mid x \in A \land x \in B\}.$$

Venn Diagram



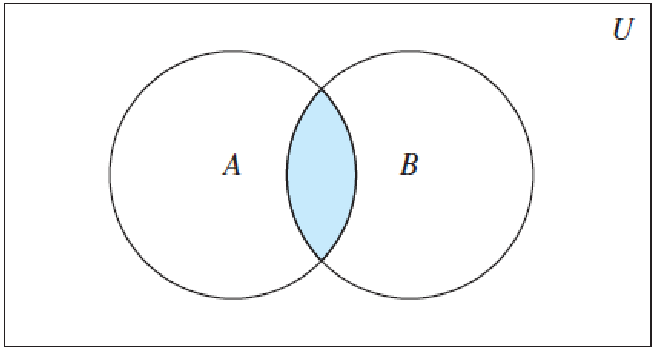
$A \cap B$ is shaded.

Intersection of Sets

Example

The intersection of the sets {1, 3, 5} and {1, 2, 3} is the set

$\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$.

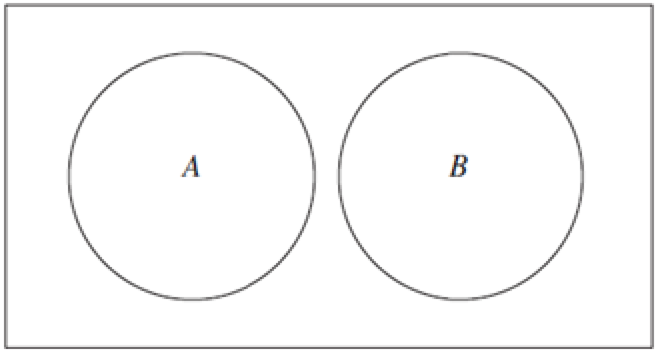


$A \cap B$ is shaded.

Sets Operations

Disjoint Sets

Two sets are called ***disjoint*** if their intersection is the empty set.



A and B are Disjoint Sets

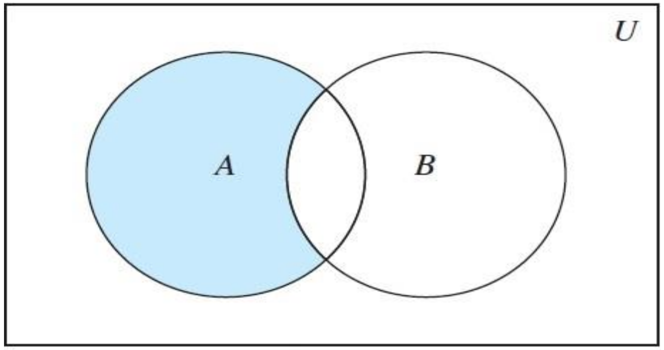
Sets Operations

Difference of Sets

Let A and B be sets. The ***difference*** of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B.

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

Venn Diagram



$A - B$ is shaded.

Sets Operations

Difference of Sets

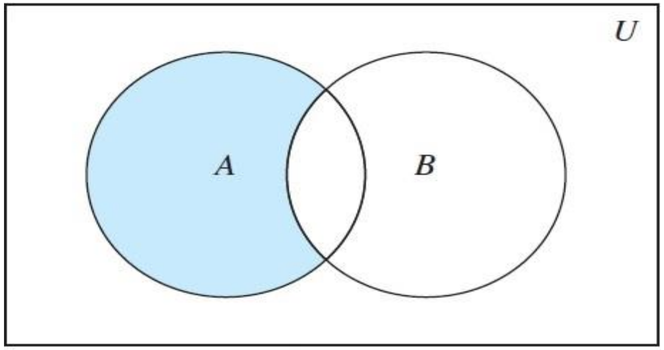
Example

The difference of

{1, 3, 5} and {1, 2, 3}

is the set,

$\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$.



$A - B$ is shaded.

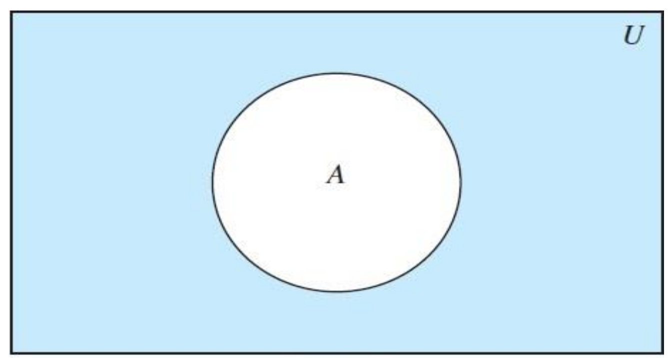
Complement of a Sets

Let U be the universal set. The ***complement*** of the set A, denoted by A, is the complement of A with respect to U.

The complement of the set A is $U - A$.

$$A = \{x \in U \mid x \notin A\}.$$

Venn Diagram



$\overline{A}$ is shaded.

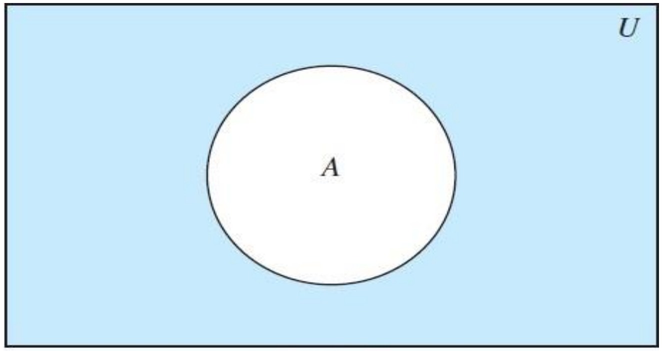
Sets Operations

Complement of a Sets

Let *U* be the universal set.

The ***universal set***, containing all objects under consideration.

Venn Diagram



$\overline{A}$ is shaded.

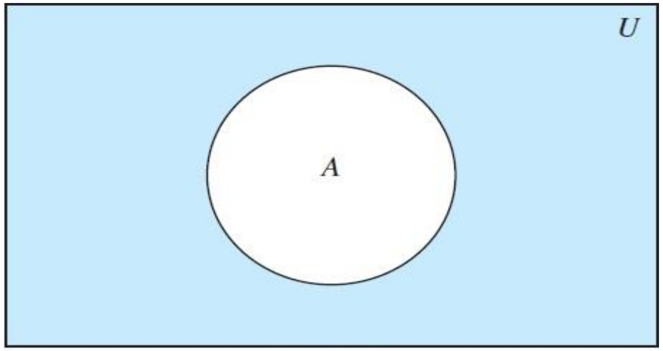
Complement of a Sets

Example

Let A be the set of positive integers greater than 10 with universal set the set of all positive integers. Then

$$\bar{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Venn Diagram



$\bar{A}$ is shaded.

Sets Operations

Set Identities

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set Identities

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: We can prove this identity with the following steps.

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}$$

Set Identities

Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

Sets Operations

2.2 Exercise, Page 144 & 145

1. Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.

a) $A \cap B$

b) $A \cup B$

c) $A - B$

d) $B - A$

Sets Operations

2.2 Exercise, Page 144 & 145

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in your school

b) the set of sophomores at your school who are not taking discrete mathematics

c) the set of students at your school who either are sophomores or are taking discrete mathematics

d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

Sets Operations

2.2 Exercise, Page 144 & 145

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$. **b)** $A \cap B$.

c) $A - B$. **d)** $B - A$.

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$. **b)** $A \cap B$.

c) $A - B$. **d)** $B - A$.

2.2 Exercise, Page 144 & 145

14. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

15. Prove the second De Morgan law by showing that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$

a) by showing each side is a subset of the other side.

b) using a membership table.

Sets Operations

2.2 Exercise, Page 144 & 145

16. Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$.

b) $A \subseteq (A \cup B)$.

c) $A - B \subseteq A$.

d) $A \cap (B - A) = \emptyset$.

e) $A \cup (B - A) = A \cup B$.

18. Given sets A and B in a universe U, draw the Venn diagrams of each of these sets.

a) $A \rightarrow B = \{x \in U \mid x \in A \rightarrow x \in B\}$

b) $A \leftrightarrow B = \{x \in U \mid x \in A \leftrightarrow x \in B\}$

Sets Operations

2.2 Exercise, Page 144 & 145

27. Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find

a) $A \cap B \cap C$. **b)** $A \cup B \cup C$.

c) $(A \cup B) \cap C$. **d)** $(A \cap B) \cup C$.

28. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B \cup C)$ **b)** $\overline{A} \cap \overline{B} \cap \overline{C}$

c) $(A - B) \cup (A - C) \cup (B - C)$

2.2 Exercise, Page 144 & 145

29. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B - C)$ **b)** $(A \cap B) \cup (A \cap C)$

c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

30. Draw the Venn diagrams for each of these combinations of the sets A, B, C, and D.

a) $(A \cap B) \cup (C \cap D)$

b) $\overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$

c) $A - (B \cap C \cap D)$

Sets Operations

2.2 Exercise, Page 144 & 145

31. What can you say about the sets A and B if we know that

a) $A \cup B = A$? **b)** $A \cap B = A$?

c) $A - B = A$? **d)** $A \cap B = B \cap A$?

e) $A - B = B - A$?

32. Can you conclude that $A = B$ if A, B, and C are sets such that

a) $A \cup C = B \cup C$? **b)** $A \cap C = B \cap C$?

c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$?

Sets Operations

2.2 Exercise, Page 144 & 145

The ***symmetric difference*** of A and B,

denoted by $A \oplus B$,

is the set containing those elements in either A or B, but not in both A and B.

38. Find the symmetric difference of

$\{1, 3, 5\}$ and $\{1, 2, 3\}$.

2.2 Exercise, Page 144 & 145

39. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.

40. Draw a Venn diagram for the symmetric difference of the sets A and B.

Sets Operations

2.2 Exercise, Page 144 & 145

Fuzzy sets are used in artificial intelligence.

Each element in the universal set U has a ***degree of membership***,

which is a real number between 0 and 1 (including 0 and 1), in a fuzzy set S.

2.2 Exercise, Page 144 & 145

The ***fuzzy set*** S is denoted by listing the elements with their degrees of membership (elements with 0 degree of membership are not listed).

Sets Operations

2.2 Exercise, Page 144 & 145

The ***complement of a fuzzy set*** *S* is the set $\bar{S}$, with the degree of the membership of an element in $\bar{S}$ equal to 1 minus the degree of membership of this element in *S*.

The ***union of two fuzzy sets*** *S* and *T* is the fuzzy set $S \cup T$, where the degree of membership of an element in $S \cup T$ is the maximum of the degrees of membership of this element in *S* and in *T*.

The ***intersection of two fuzzy*** sets *S* and *T* is the fuzzy set $S \cap T$, where the degree of membership of an element in $S \cap T$ is the minimum of the degrees of membership of this element in *S* and in *T*.

Sets Operations

2.2 Exercise, Page 144 & 145

we write

{0.6 Alice, 0.9 Brian, 0.4 Fred, 0.1 Oscar, 0.5 Rita}

for the set F (of famous people) to indicate that

Alice has a 0.6 degree of membership in F,

Brian has a 0.9 degree of membership in F,

Fred has a 0.4 degree of membership in F,

Oscar has a 0.1 degree of membership in F, and

Rita has a 0.5 degree of membership in F

Also suppose that R is the set of rich people with R = {0.4 Alice, 0.8 Brian, 0.2 Fred, 0.9 Oscar, 0.7 Rita}.

2.2 Exercise, Page 144 & 145

73. Find $\overline{F}$ (the fuzzy set of people who are not famous) and $\overline{R}$ (the fuzzy set of people who are not rich).

74. Find the fuzzy set $F \cup R$ of rich or famous people.

75. Find the fuzzy set $F \cap R$ of rich and famous people.

Chapter Reading

Book: Kenneth H. Rosen,

Discrete Mathematics and Its Applications (Eight Edition)

Exercise for Practice

Section 2.1: Sets

Section 2.2: Set Operations

MAT401 - Discrete Structures (Discrete Mathematics) Fall 2022

Week - 6 and Lecture – 1 & 2

Functions

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MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 1

Functions

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Applications of Function

- Define discrete structures such as sequences and strings.

- Represent the time that a computer takes to solve problems of a given size.

- Represent the complexity of algorithms.

Functions

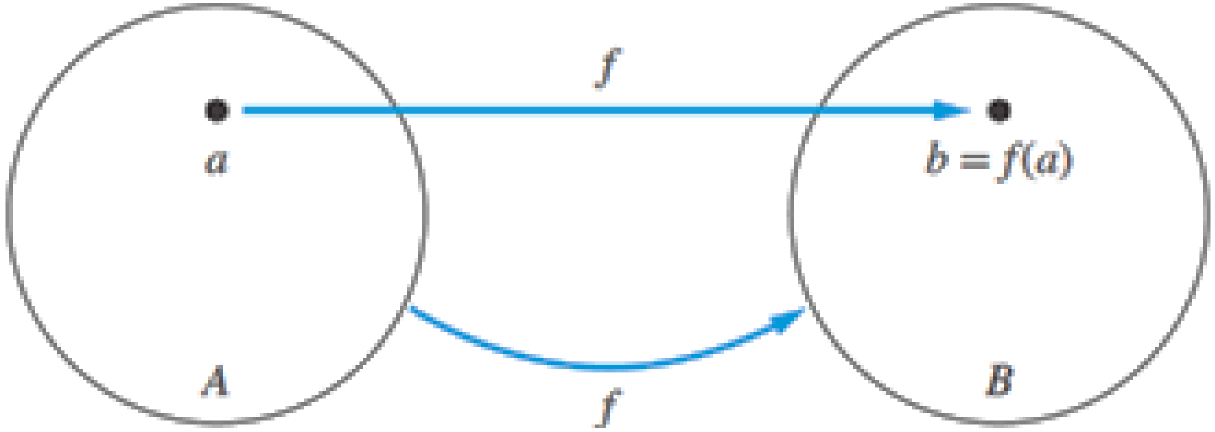
Let A and B be nonempty sets. A ***function*** f from A to B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ if ***b* is the unique element of *B*** assigned by the function f to the element a of A.

Functions

If f is a function from A to B, we write $f : A \to B$.

Functions are sometimes also called ***mappings*** or ***transformations***.



Functions

Domain & *Codomain*
Image & *Preimage*

If f is a function from A to B, we say that ***A* is the *domain*** of f and ***B* is the *codomain*** of f.

If $\boldsymbol{f(a) = b}$, we say that ***b* is the *image*** of a and ***a* is a *preimage*** of b.

Functions

Domain *&* *Codomain*

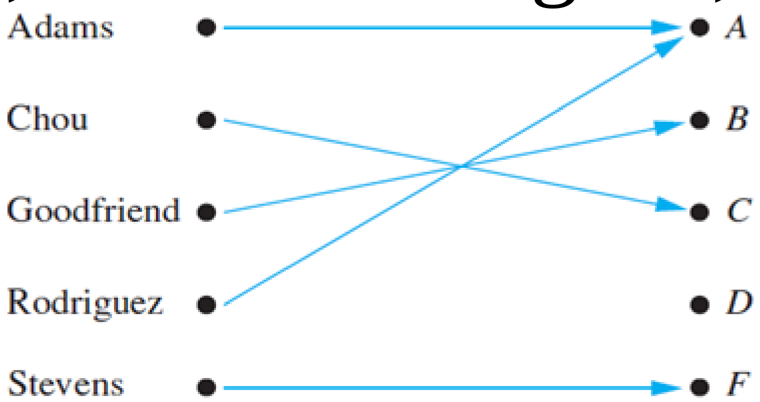
Image *&* *Preimage*

The ***range***, or ***image***, of f is the set of all images of elements of A.

Also, if f is a function from A to B, we say that ***f maps A to B***.

Examples of a Function

Suppose that each student in a discrete mathematics class is assigned a letter grade from the set {A, B, C, D, F}. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens.



Functions

Examples of a Function

Let G be the function that assigns a grade to a student in our discrete mathematics class.

Note that G(Adams) = A, for instance.

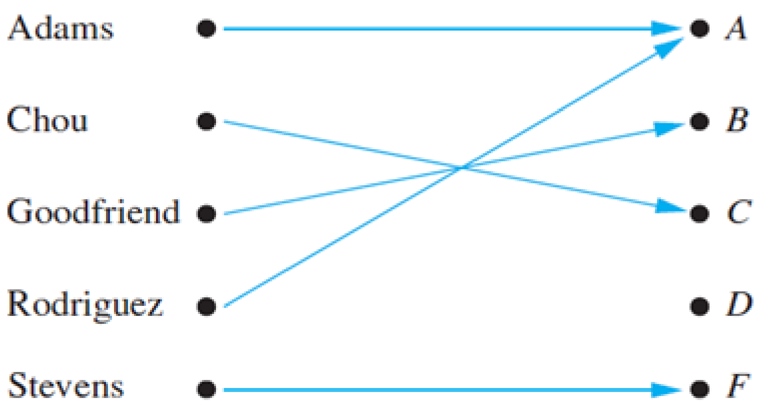
The domain of G is the set

{Adams, Chou, Goodfriend, Rodriguez, Stevens}, and

the codomain is the set $\{A, B, C, D, F\}$.

The range of G is the set

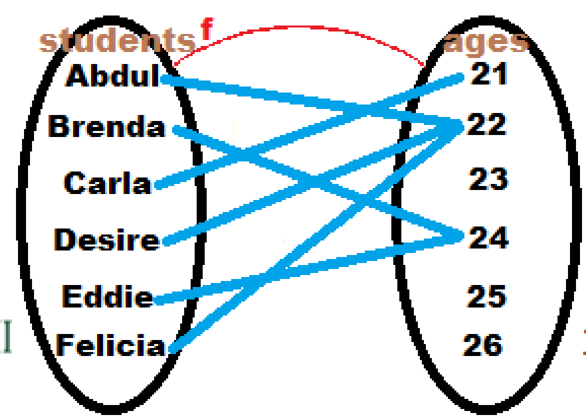
$\{A, B, C, F\}$, because each grade except D is assigned to some student.



Examples of a Function

Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

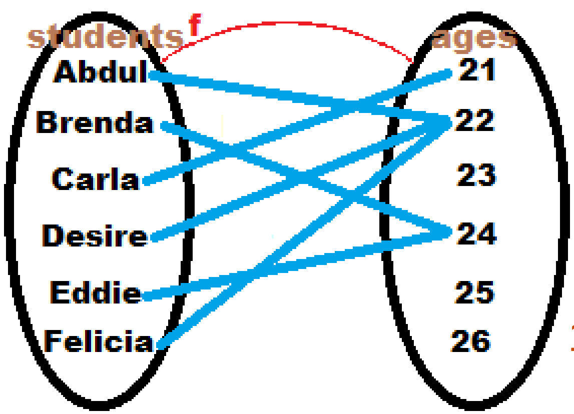
Solution:



Examples of a Function

Solution: For the **domain**, we take the set **{Abdul, Brenda, Carla, Desire, Eddie, Felicia}**. We also need to specify a **codomain**, which needs to contain all possible ages of students. Because it is highly likely that all students are less than 26 years old, we can take **the set of positive integers less than 27** as the codomain.

The **range** of the function is the set **{21, 22, 24}.**



Examples of a Function

Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ assign the square of an integer to this integer.

Then, $f(x) = x^2$, where

the domain of f is the set of all integers,

the codomain of f is the set of all integers, and

the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$.

Functions

Real-Valued Function & Integer-Valued Function

A function is called ***real-valued*** if its codomain is the set of real numbers, and

it is called ***integer-valued*** if its codomain is the set of integers.

Two real-valued functions or two integer valued functions with the same domain can be added, as well as multiplied.

Properties of Real-Valued Function

Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 \,.\, f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$

$$(f_1 \,.\, f_2)(x) = f_1(x)\, f_2(x).$$

Examples of a Function

Let f_1 and f_1 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution: From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and $(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4$.

Examples of a Function

Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$.

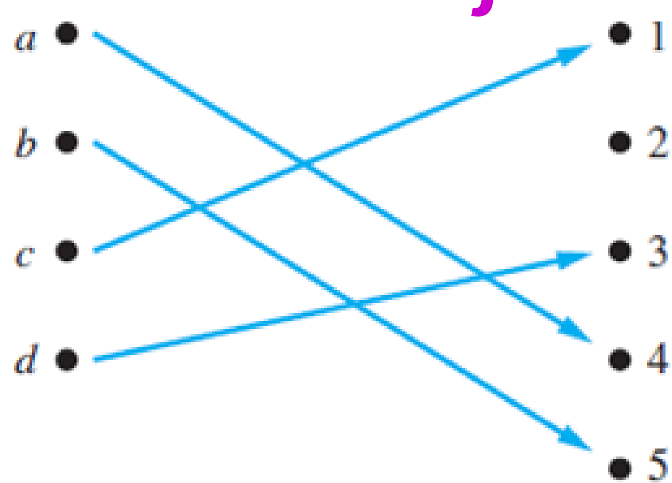
The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

Functions

One-to-One Function or Injective Function

A function f is said to be ***one-to-one***, or an ***injection***, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

A function is said to be ***injective*** if it is one-to-one.



One-to-One Function or Injective Function

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution: The function $f(x) = x^2$ is not one-to-one because, for instance,

$f(1) = f(-1) = 1$, but $1 \neq -1$.

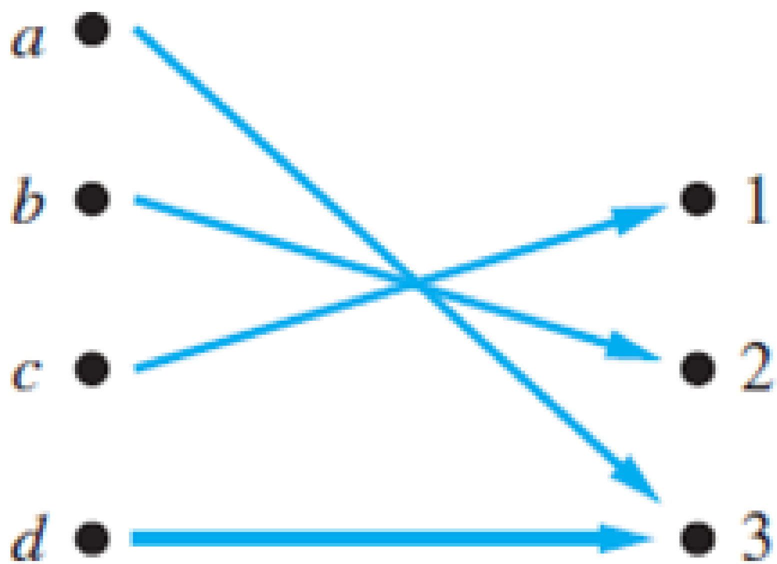
Increasing Function vs Decreasing Function

A function f whose domain and codomain are subsets of the set of real numbers is called ***increasing*** if $f(x) \le f(y)$, and ***strictly increasing*** if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called ***decreasing*** if $f(x) \ge f(y)$, and ***strictly decreasing*** if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.

Functions

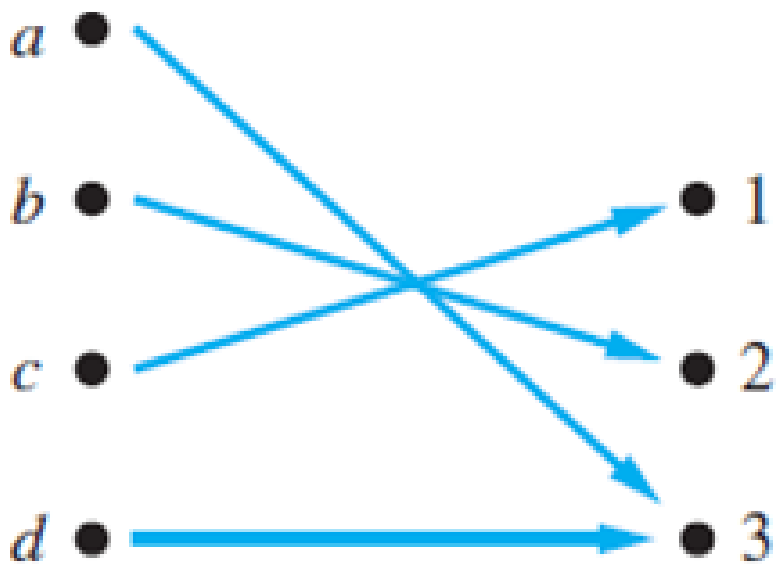
Onto Function or Surjective Function

A function f from A to B is called ***onto***, or a ***surjection***, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called ***surjective*** if it is onto.



Onto Function or Surjective Function

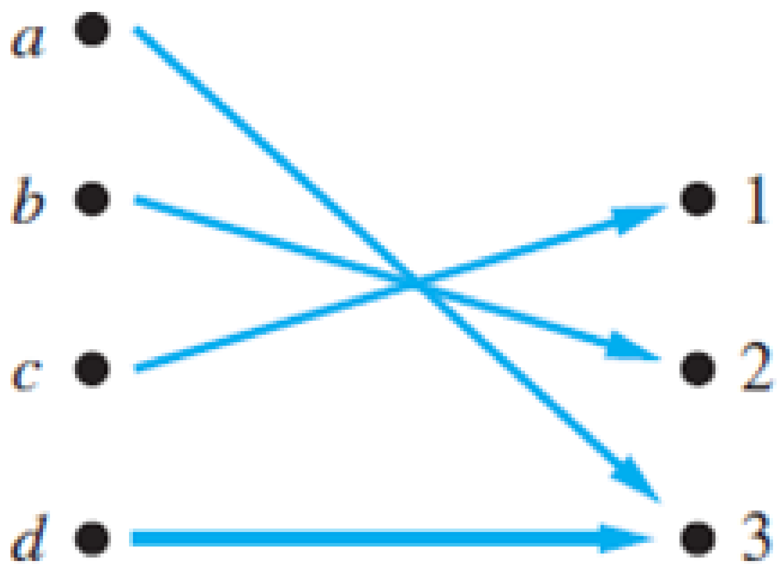
A function f is onto if $\forall y \exists x(f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.



Functions

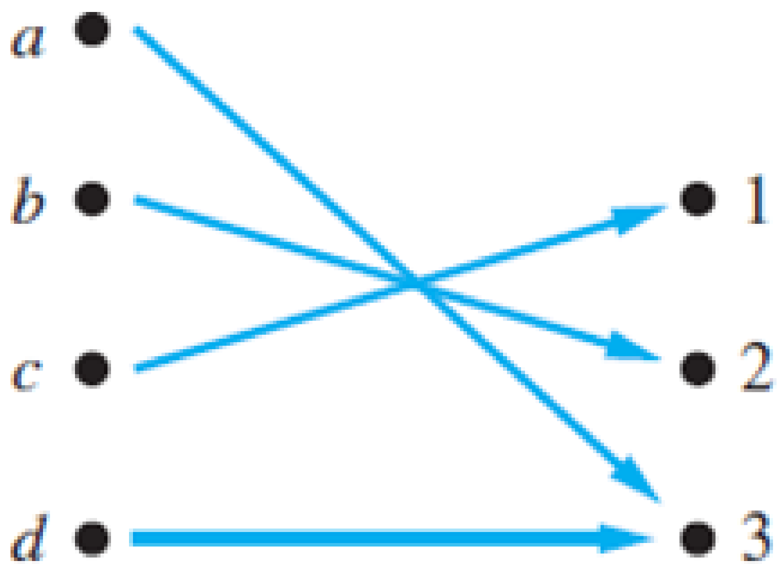
Onto Function or Surjective Function

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?



Onto Function or Surjective Function

Solution: Because all three elements of the codomain are images of elements in the domain, we see that f is onto.



Onto Function or Surjective Function

Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: The function f is not onto because there is no integer x with $x^2 = -1$, for instance.

Onto Function or Surjective Function

Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

Solution: This function is onto, because for every integer y there is an integer x such that $f(x) = y$.

To see this, note that $f(x) = y$ if and only if $x + 1 = y$, which holds if and only if $x = y - 1$. **(Note that $y - 1$ is also an integer, and so, is in the domain of f.)**

One-to-One Onto or Bijective Function

The function f is a ***one-to-one correspondence***, or a ***bijection***, if it is both one-to-one and onto. We also say that such a function is ***bijective***.

One-to-One Onto or Bijective Function

Suppose that each worker in a group of employees is assigned a job from a set of possible jobs, each to be done by a single worker.

In this situation, the function f that assigns a job to each worker is one-to-one. To see this, note that if x and y are two different workers, then $f(x) \neq f(y)$ because the two workers x and y must be assigned different jobs.

Consider the function f that assigns jobs to workers. The function f is onto if for every job there is a worker assigned this job. The function f is not onto when there is at least one job that has no worker assigned it.

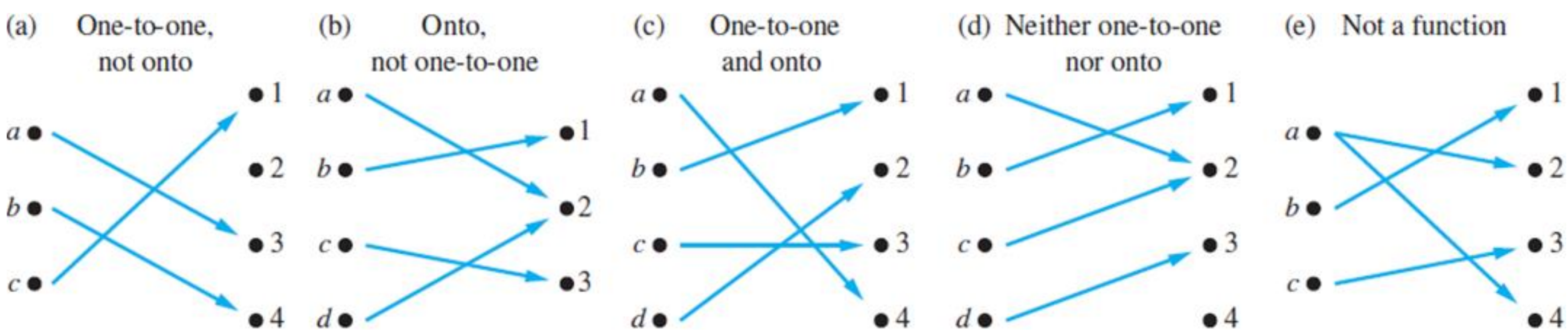
One-to-One Onto or Bijective Function

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?

Solution: The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.

Functions and their Types

Examples of different types of correspondence



Functions

Identity Function

Let A be a set. The ***identity function*** on A is the function $\iota_A : A \to A$, where $\iota_A(x) = x$ for all $x \in A$.

In other words, the identity function ιA is the function that assigns each element to itself.

The function ι_A is one-to-one and onto, so it is a bijection. (Note that ι is the Greek letter iota.)

Identification of a Function

Suppose that $f : A \longrightarrow B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

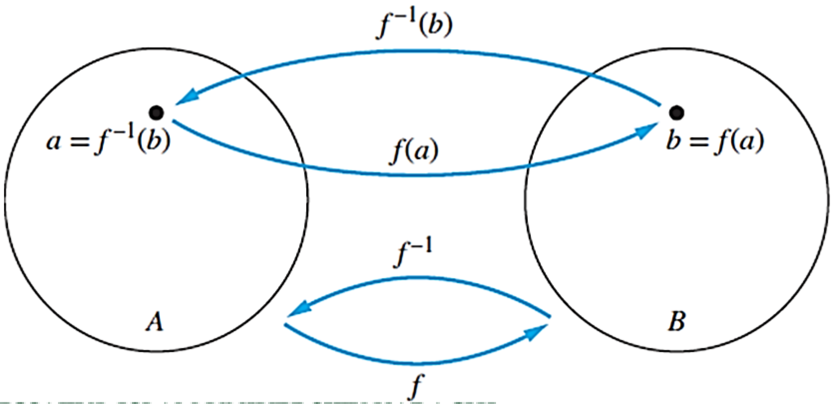
To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Inverse Function of a Function

Let f be a one-to-one correspondence from the set A to the set B. The ***inverse function*** of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.



Inverse Function of a Function

Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Inverse Function of a Function

Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

Inverse Function of a Function

Let f be the function from **R** to **R** with $f(x) = x^2$. Is f invertible?

Solution: Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible.

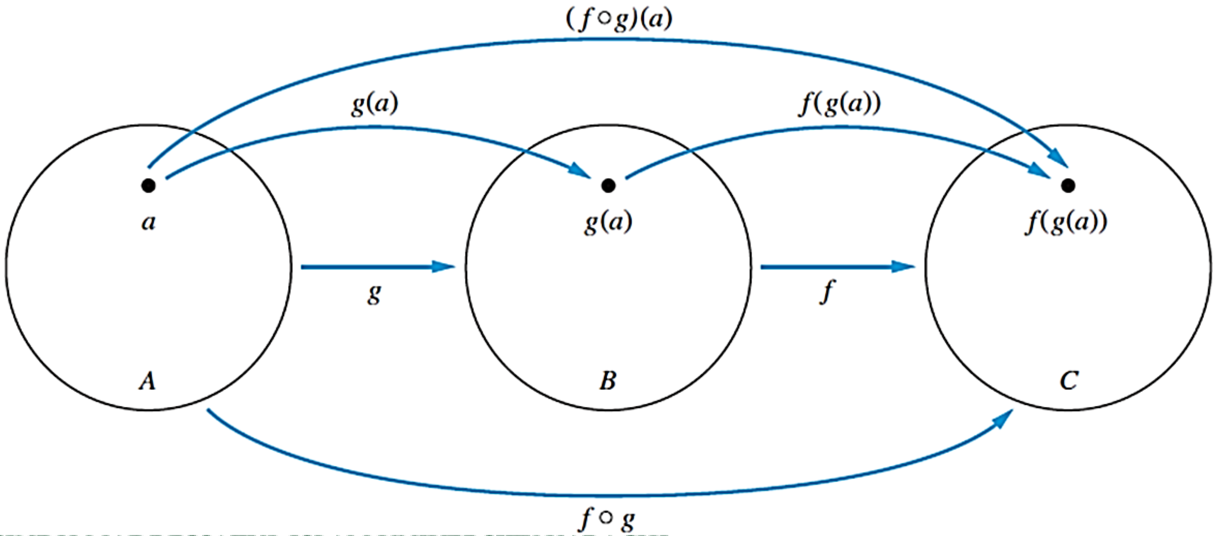
(Note we can also show that f is not invertible because it is not onto.)

Composition of the functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C.

The ***composition*** of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is the function from A to C defined by

$$(f \circ g)(a) = f(g(a)).$$



Composition of the functions

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

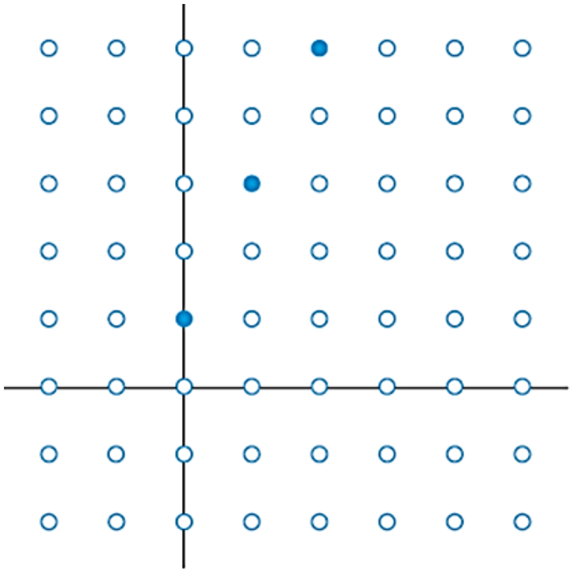
and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

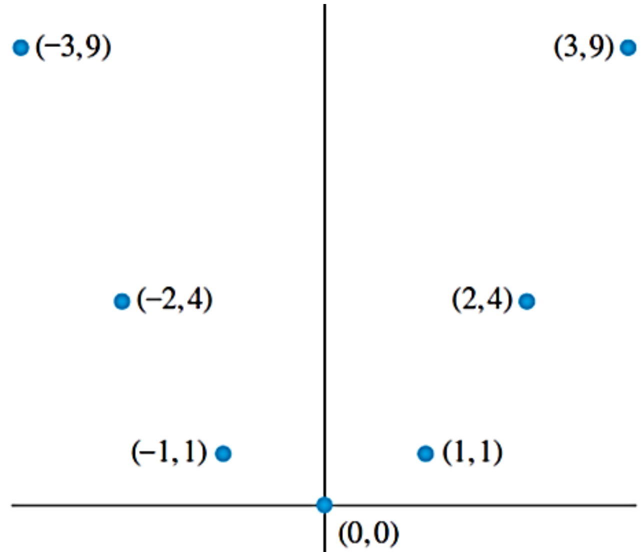
Functions

Graph of a Function

Let f be a function from the set A to the set B. The ***graph*** of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.



The graph of $f(n) = 2n + 1$ from $\mathbf{Z}$ to $\mathbf{Z}$.



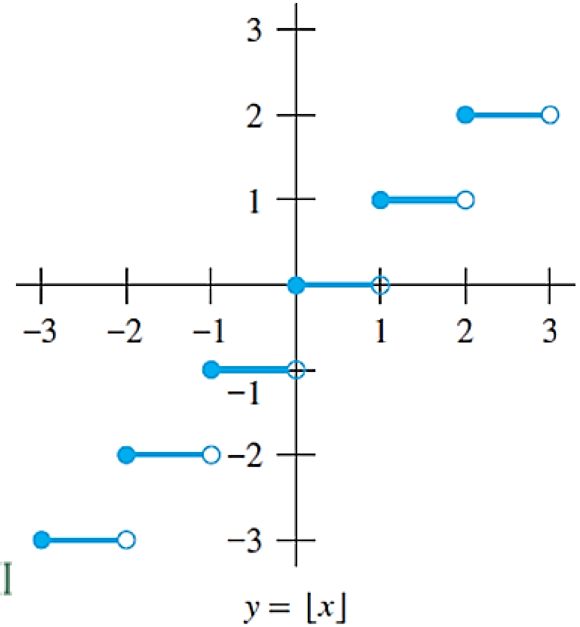
The graph of $f(x) = x^2$ from $\mathbf{Z}$ to $\mathbf{Z}$.

Functions

Some Important Functions

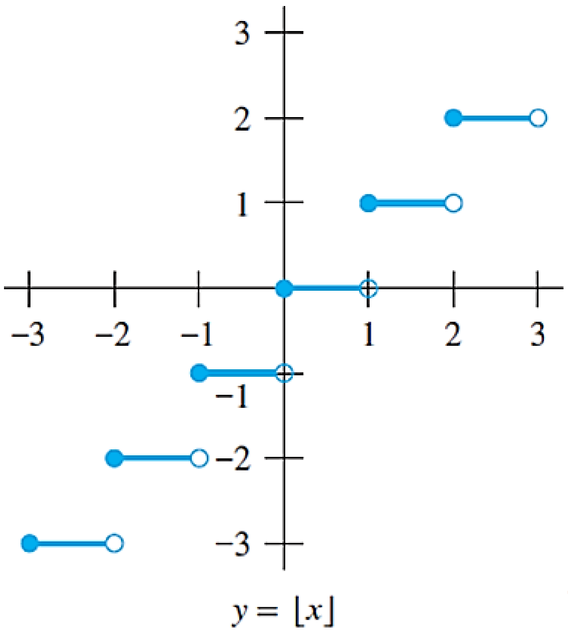
The ***floor function*** assigns to the real number x the largest integer that is less than or equal to x.

The value of the floor function at x is denoted by $\lfloor x \rfloor$.



Some Important Functions

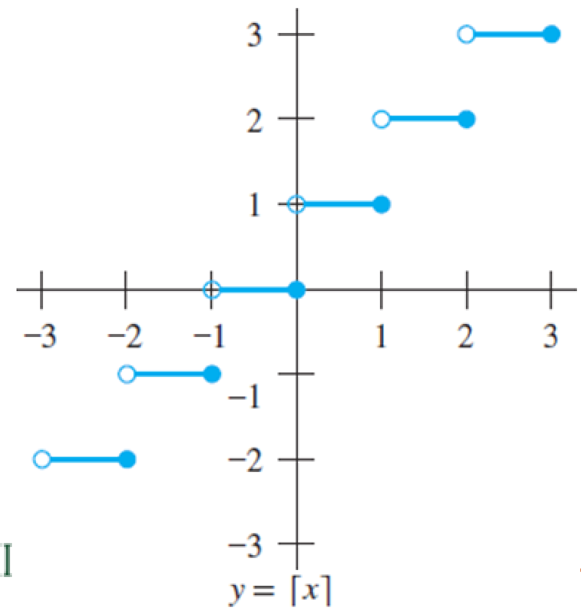
Remark: The ***floor function*** is often also called the greatest integer function. It is often denoted by [x].



Functions

Some Important Functions

The ***ceiling function*** assigns to the real number *x* the smallest integer that is greater than or equal to *x*. The value of the ceiling function at *x* is denoted by $\lceil x \rceil$.



Functions

Properties of Floor and Ceiling Functions

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

n is an integer and ***x*** is a real number

Examples of a Floor and Ceiling Functions

These are some values of the floor and ceiling functions:

$$\left\lfloor \tfrac{1}{2} \right\rfloor = 0, \qquad \left\lceil \tfrac{1}{2} \right\rceil = 1,$$

$$\left\lfloor -\tfrac{1}{2} \right\rfloor = -1, \qquad \left\lceil -\tfrac{1}{2} \right\rceil = 0,$$

$$\lfloor 3.1 \rfloor = 3, \qquad \lceil 3.1 \rceil = 4,$$

$$\lfloor 7 \rfloor = 7, \qquad \lceil 7 \rceil = 7.$$

Examples of a Floor and Ceiling Functions

Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

Solution:

Number of bits to encode = 100

Number of bits in one byte = 8

Examples of a Floor and Ceiling Functions

Solution:

Number of bits to encode = 100

Number of bits in one byte = 8

To determine the number of bytes needed, we determine the smallest integer that is at least as large as the quotient when 100 is divided by 8, the number of bits in a byte.

Consequently, $\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$ bytes are required.

Examples of a Floor and Ceiling Functions

In Asynchronous Transfer Mode (ATM) ***(a communications protocol used on backbone networks)***, data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 1 minute over a connection that transmits data at the rate of 500 kilobits per second?

***Solution*: Data transmission,**

in 1 second = 500 kilobits = 500,000 bites

in 1 minute = $500{,}000 \times 60 = 30{,}000{,}000$ bits.

Each ATM cell = 53 bytes = $53 \times 8 = 424$ bits long.

Examples of a Floor and Ceiling Functions

Solution: Data transmission,

in 1 second = 500 kilobits = 500,000 bites

in 1 minute = $500{,}000 \times 60 = 30{,}000{,}000$ bits.

Each ATM cell = 53 bytes = $53 \times 8 = 424$ bits long.

To determine the number of cells that can be transmitted in 1 minute, we determine the largest integer not exceeding the quotient when 30,000,000 is divided by 424.

Consequently, $\lfloor 30{,}000{,}000/424 \rfloor = 70{,}754$ ATM cells can be transmitted in 1 minute over a 500 kilobit per second connection

Factorial Function

Factorial Function $f: N \rightarrow Z^+$, denoted by $f(n) = n!$.

The value of $f(n) = n!$ is the product of the first n positive integers, so

$$\boldsymbol{f(n) = 1 \cdot 2 \cdots (n-1) \cdot n}$$

$$[\boldsymbol{and\ f(0) = 0! = 1}]$$

Factorial Function

We have

$$f(1) = 1! = 1,$$

$$f(2) = 2! = 1 \cdot 2 = 2,$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720, \text{ and}$$

$$f(20) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20 =$$

$$2{,}432{,}902{,}008{,}176{,}640{,}000.$$

Partial and Total Functions

A ***partial function*** *f* from a set *A* to a set *B* is an assignment to each element *a* in a subset of *A*, called the ***domain of definition of f***, of a unique element *b* in *B*.

The sets *A* and *B* are called the *domain* and *codomain* of *f*, respectively.

We say that *f* is *undefined* for elements in *A* that are not in the domain of definition of *f* .

When the domain of definition of *f* equals *A*, we say that *f* is a ***total function***.

Partial and Total Functions

The function $f : \mathbf{Z} \rightarrow \mathbf{R}$,

where $f(n) = \sqrt{n}$ is a ***partial function*** from $\mathbf{Z}$ to $\mathbf{R}$,

where the ***domain of definition*** is the set of nonnegative integers.

Note that f is undefined for negative integers.

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 2

Functions - Exercise

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Functions - Exercise

2.3 Exercise – Page #: 161 to 164

1. Why is f not a function from $\boldsymbol{R}$ to $\boldsymbol{R}$ if

a) $f(x) = 1/x$?

***f* (0) is not defined.**

b) $f(x) = \sqrt{x}$?

***f* (*x*) is not defined for *x* < 0.**

c) $f(x) = \pm\sqrt{x^2 + 1}$?

***f* (*x*) is not well defined because there are two distinct values assigned to each *x*.**

2.3 Exercise – Page #: 161 to 164

2. Determine whether f is a function from $\boldsymbol{Z}$ to $\boldsymbol{R}$ if

a) $f(n) = \pm n$.

f **(*x*) is not well defined because there are two distinct values assigned to each *x*.**

b) $f(n) = \sqrt{n^2 + 1}$.

It is a function.

c) $f(n) = 1/(n^2 - 4)$.

f **(2) is not defined.**

2.3 Exercise – Page #: 161 to 164

4. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) the function that assigns to each nonnegative integer its last digit.

Domain = Set of nonnegative integers.

Range = {0, 1, 2, 3, …, 9}

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

4. b) the function that assigns the next largest integer to a positive integer

Domain = {1, 2, 3, …}

Range = {2, 3, 4, …}

c) the function that assigns to a bit string the number of one bits in the string

Domain = Set of bitstrings = {0, 1}

Range = Set the natural numbers = {1, 2, 3, …}.

2.3 Exercise – Page #: 161 to 164

4.

d) the function that assigns to a bit string the number of bits in the string

Domain = Set of bitstrings = {0, 1}

Range = Set the natural numbers = {1, 2, 3, …}.

2.3 Exercise – Page #: 161 to 164

5. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) the function that assigns to each bit string the number of ones in the string minus the number of zeros in the string

Domain = The set of bit strings;

Range = The set of integers

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

b) the function that assigns to each bit string twice the number of zeros in that string

Domain = The set of bit strings;

Range = The set of even nonnegative integers

c) the function that assigns the number of bits left over when a bit string is split into bytes (which are blocks of 8 bits)

Domain = The set of bit strings;

Range = The set of nonnegative integers not exceeding

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

d) the function that assigns to each positive integer the largest perfect square not exceeding this integer

Domain = The set of integers;

Range = The set of set of perfect square

2.3 Exercise – Page #: 161 to 164

6. Find the domain and range of these functions.

a) the function that assigns to each pair of positive integers the first integer of the pair

Domain = The set of ordered pair of integers;

Range = The set of integers

2.3 Exercise – Page #: 161 to 164

b) the function that assigns to each positive integer its largest decimal digit

Domain = The set of all positive integers;

Range = {1, 2, 3, …, 9}

c) the function that assigns to a bit string the number of ones minus the number of zeros in the string

Domain = The set of bitstrings;

Range = The set of natural numbers

2.3 Exercise – Page #: 161 to 164

d) the function that assigns to each positive integer the largest integer not exceeding the square root of the integer

Domain = The set of all positive integers;

Range = The set of square roots of all positive integers minus one.

e) the function that assigns to a bit string the longest string of ones in the string

Domain = The set of bitstrings;

Range = The set of natural numbers

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

7. Find the domain and range of these functions.

a) the function that assigns to each pair of positive integers the maximum of these two integers

Domain = The set of ordered pair of integers;

Range = The set of integers

b) the function that assigns to each positive integer the number of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 that do not appear as decimal digits of the integer

Domain = The set of all positive integers;

Range = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

c) the function that assigns to a bit string the number of times the block 11 appears

Domain = The set of bit strings;

Range = The set of natural numbers

d) the function that assigns to a bit string the numerical position of the first 1 in the string and that assigns the value 0 to a bit string consisting of all 0s

Domain = The set of bit strings;

Range = The set of natural numbers

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

8. Find these values.

a) $\lfloor 1.1 \rfloor = \mathbf{1}$

b) $\lceil 1.1 \rceil = \mathbf{2}$

c) $\lfloor -0.1 \rfloor = \mathbf{-1}$

d) $\lceil -0.1 \rceil = \mathbf{0}$

e) $\lceil 2.99 \rceil = \mathbf{3}$

f) $\lceil -2.99 \rceil = \mathbf{-2}$

g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor = \lfloor \frac{1}{2} + \mathbf{1} \rfloor = \lfloor \mathbf{\frac{3}{2}} \rfloor = \mathbf{1}$

h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil = \lceil \mathbf{0} + \mathbf{1} + \frac{1}{2} \rceil = \lceil \mathbf{\frac{3}{2}} \rceil = \mathbf{2}$

2.3 Exercise – Page #: 161 to 164

9. Find these values.

a) $\left\lceil \frac{3}{4} \right\rceil = \mathbf{1}$

b) $\left\lfloor \frac{7}{8} \right\rfloor = \mathbf{0}$

c) $\left\lceil -\frac{3}{4} \right\rceil = \mathbf{0}$

d) $\left\lfloor -\frac{7}{8} \right\rfloor = \mathbf{-1}$

e) $\lceil 3 \rceil = \mathbf{3}$

f) $\lfloor -1 \rfloor = \mathbf{-1}$

g) $\left\lfloor \frac{1}{2} + \left\lceil \frac{3}{2} \right\rceil \right\rfloor = \left\lfloor \frac{1}{2} + \mathbf{2} \right\rfloor = \left\lfloor \mathbf{\frac{5}{2}} \right\rfloor = \mathbf{2}$

h) $\left\lfloor \frac{1}{2} \cdot \left\lfloor \frac{5}{2} \right\rfloor \right\rfloor = \left\lfloor \frac{1}{2} \cdot \mathbf{2} \right\rfloor = \lfloor \mathbf{1} \rfloor = \mathbf{1}$

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

10. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

11. Which functions in Exercise 10 are onto?

a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$

One-to-one onto

b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$

Neither one to one nor onto

c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

Neither one to one nor onto

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

12. Determine whether each of these functions from **Z** to **Z** is one-to-one.

13. Which functions in Exercise 12 are onto?

a) $f(n) = n - 1$ **One-to-one onto**

b) $f(n) = n^2 + 1$ **Neither one-to-one nor onto**

c) $f(n) = n^3$ **One-to-one**

d) $f(n) = \lceil n/2 \rceil$ **Onto**

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

14. Determine whether $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = 2m - n$. **Onto**

b) $f(m, n) = m^2 - n^2$. **Not onto**

c) $f(m, n) = m + n + 1$. **Onto**

d) $f(m, n) = |m| - |n|$. **Not onto**

e) $f(m, n) = m^2 - 4$. **Not onto**

2.3 Exercise – Page #: 161 to 164

15. Determine whether the function $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = m + n.$ **Onto**

b) $f(m, n) = m^2 + n^2.$ **Not onto**

c) $f(m, n) = m.$ **Onto**

d) $f(m, n) = |n|.$ **Not onto**

e) $f(m, n) = m - n.$ **Onto**

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

16. Consider these functions from the set of students in a discrete mathematics class. Under what conditions is the function one-to-one if it assigns to a student his or her

18. Specify a codomain for each of the functions in Exercise 16. Under what conditions is each of these functions with the codomain you specified onto?

2.3 Exercise – Page #: 161 to 164

a) mobile phone number.

If there are no two students with a common phone, the function that assigns to a student his or her mobile phone number is one-to-one.

The set of all mobile phone numbers, there are some which didn't assign to any students, so it is not onto.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

b) student identification number.

Since the student identification number is unique, and hence there are no two students with the same identification number, the function that assigns to a student his or her identification number is one-to-one.

The set of all student identification numbers, there are some which didn't assign to any students, so it is not onto.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

c) final grade in the class.

Two student can have the same final grade in the class. Therefore, in general case this function is not one-to-one. It can be one-to-one if the number of students is less or equal to the number of final grades and different students have different final grades.

It may one-to-one onto but depend upon the number of students.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

d) home town.

This function in general case is not one-to-one because of two students can live in the same town. It can be injective if different students have different home town.

It is onto, if students belongs to all town of the city.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

17. Consider these functions from the set of teachers in a school. Under what conditions is the function one-to-one if it assigns to a teacher his or her

19. Specify a codomain for each of the functions in Exercise 17. Under what conditions is each of the functions with the codomain you specified onto?

a) office.

Depends on whether teachers share offices, it may not one-to-one. There is many offices at the school other than teachers' offices; probably not onto.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

b) assigned bus to chaperone in a group of buses taking students on a field trip.

One-to-one assuming only one teacher per bus

Set of buses going on the trip; onto, assuming every bus gets a teacher chaperone

c) salary.

Most likely not one-to-one, especially if salary is set by a collective bargaining agreement. Set of real numbers; not onto

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

d) social security number.

One-to-one because every teacher has unique social security number. Set of strings of nine digits with hyphens after third and fifth digits; not onto.

2.3 Exercise – Page #: 161 to 164

20. Give an example of a function from **N** to **N** that is

a) one-to-one but not onto.

A function from N to N that is $f(n) = 2n$ is one-to-one because of $f(a) = f(b)$ implies $2a = 2b$, and hence $a = b$,

but it is not onto because of the preimage of the odd number 11 is empty set.

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

b) onto but not one-to-one.

A function from N to N that is

$$\boldsymbol{f(n)=\begin{cases}1, & if\ n=1\\ n-1, & if\ n>1\end{cases}}$$

is not one-to-one onto because of the preimage of each natural number nn contains $n+1$, but it is not one-to-one as $f(1)=1=f(2)$.

2.3 Exercise – Page #: 161 to 164

c) both onto and one-to-one (but different from the identity function).

A function from N to N

$$f(n) = \begin{cases} 1, & if\ n = 2 \\ 2, & if\ n = 1 \\ n, & if\ n \geq 1 \end{cases}$$

obviously is onto and one-to-one, and it is different from the identity function.

2.3 Exercise – Page #: 161 to 164

d) neither one-to-one nor onto.

A function from N to N

$$\boldsymbol{f(n) = 2022}$$

is neither one-to-one nor onto.

Indeed, since $\boldsymbol{f(1) = f(2) = f(3) = 2021}$,
we conclude that it is not one-to-one.

2.3 Exercise – Page #: 161 to 164

21. Give an explicit formula for a function from the set of integers to the set of positive integers that is

a) one-to-one, but not onto.

The function $f(x)$ with

$$f(x) = \begin{cases} 3x + 1, & if\ x \geq 0 \\ -3x + 1, & if\ n < 0 \end{cases}$$

b) onto, but not one-to-one.

The function $f(x)$ with

$$f(x) = |x| + 1$$

2.3 Exercise – Page #: 161 to 164

c) one-to-one and onto.

The function $f\,(x)$ with

$$f(x) = \begin{cases} 2x + 1, & if\ x \geq 0 \\ -2x, & if\ n < 0 \end{cases}$$

d) neither one-to-one nor onto.

The function $f\,(x)$ with

$$f\,(x) = x^2 + 1$$

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

22. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = -3x + 4$ **Yes**

b) $f(x) = -3x^2 + 7$ **No**

c) $f(x) = \frac{(x+1)}{(x+2)}$ **No**

d) $f(x) = x^5 + 1$ **Yes**

2.3 Exercise – Page #: 161 to 164

23. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = 2x + 1$ **Yes**

b) $f(x) = x^2 + 1$ **No**

c) $f(x) = x^3$ **Yes**

d) $f(x) = \frac{(x^2 + 1)}{(x^2 + 2)}$ **No**

2.3 Exercise – Page #: 161 to 164

30. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

a) $f(x) = 1$. $\boldsymbol{f(S) = \{1\}}$

b) $f(x) = 2x + 1$. $\boldsymbol{f(S) = \{0, 1, 5, 9, 15\}}$

c) $f(x) = \lceil \frac{x}{5} \rceil$. $\boldsymbol{f(S) = \{0, 1, 2\}}$

d) $f(x) = \lfloor \frac{x^2+1}{3} \rfloor$. $\boldsymbol{f(S) = \{0, 1, 5, 16\}}$

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

31. Let $f(x) = \lfloor \frac{x^2}{3} \rfloor$. Find $f(S)$ if

a) $S = \{-2, -1, 0, 1, 2, 3\}$. $\quad f(S) = \{0, 1, 3\}$

b) $S = \{0, 1, 2, 3, 4, 5\}$. $\quad f(S) = \{0, 1, 3, 5, 8\}$

c) $S = \{1, 5, 7, 11\}$. $\quad f(S) = \{0, 8, 16, 40\}$

d) $S = \{2, 6, 10, 14\}$. $\quad f(S) = \{1, 12, 33, 65\}$

2.3 Exercise – Page #: 161 to 164

32. Let $f(x) = 2x$ where the domain is the set of real numbers. What is

a) $f(\mathbf{Z})$?

$$\{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$$

b) $f(\mathbf{N})$?

$$\{2, 4, 6, \dots\}$$

c) $f(\mathbf{R})$?

$$\mathbb{R}$$

2.3 Exercise – Page #: 161 to 164

38. Find $\boldsymbol{f} \circ \boldsymbol{g}$ and $\boldsymbol{g} \circ \boldsymbol{f}$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from **R** to **R**.

$$\boldsymbol{f \circ g = f\big(g(x)\big) = f(x+2)}$$

$\boldsymbol{f(x) = x^2 + 1}$ **then**

$$\boldsymbol{f(x+2) = (x+2)^2 + 1}$$

$$\boldsymbol{f(x+2) = x^2 + 4x + 4 + 1}$$

$$\boldsymbol{f(x+2) = x^2 + 4x + 5}$$

2.3 Exercise – Page #: 161 to 164

38. Find $\boldsymbol{f} \circ \boldsymbol{g}$ and $\boldsymbol{g} \circ \boldsymbol{f}$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from **R** to **R**.

$$\boldsymbol{g \circ f = g(f(x)) = g(x^2 + 1)}$$

$\boldsymbol{g(x) = x + 2}$ **then**

$$\boldsymbol{g(x^2 + 1) = (x^2 + 1) + 2}$$

$$\boldsymbol{g(x^2 + 1) = x^2 + 3}$$

2.3 Exercise – Page #: 161 to 164

40. Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $\boldsymbol{f \circ g = g \circ f}$.

$$\boldsymbol{f \circ g = f\big(g(x)\big) = f(cx + d)}$$

$$f(x) = ax + b$$

$$f(cx + d) = a(cx + d) + b = acx + ad + b$$

$$\boldsymbol{g \circ f = g\big(f(x)\big) = g(ax + b)}$$

$$g(x) = cx + d$$

$$g(ax + b) = c(ax + b) + d = acx + bc + d$$

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

$$\boldsymbol{f \circ g = f(g(x)) = f(cx + d)}$$

$$f(x) = ax + b$$

$$f(cx + d) = a(cx + d) + b = acx + ad + b$$

$$\boldsymbol{g \circ f = g(f(x)) = g(ax + b)}$$

$$g(x) = cx + d$$

$$g(ax + b) = c(ax + b) + d = acx + bc + d$$

We have $\boldsymbol{f \circ g = g \circ f}$

$$acx + ad + b = acx + bc + d$$

Therefore $\boldsymbol{b = d}$ and $\boldsymbol{a = c}$

2.3 Exercise – Page #: 161 to 164

44. Let f be the function from **R** to **R** defined by $f(x) = x^2$. Find

Therefore $\boldsymbol{f^{-1}(x) = \pm\sqrt{x}}$

a) $f^{-1}(\{1\}) = \boldsymbol{\{-1, 1\}}$

b) $f^{-1}(\{x|0<x<1\}) = \boldsymbol{\{x| -1 < x < 0 \vee 0 < x < 1\}}$.

c) $f^{-1}(\{x|x>4\}) = \boldsymbol{\{x|x > 2 \vee x < -2\}}$.

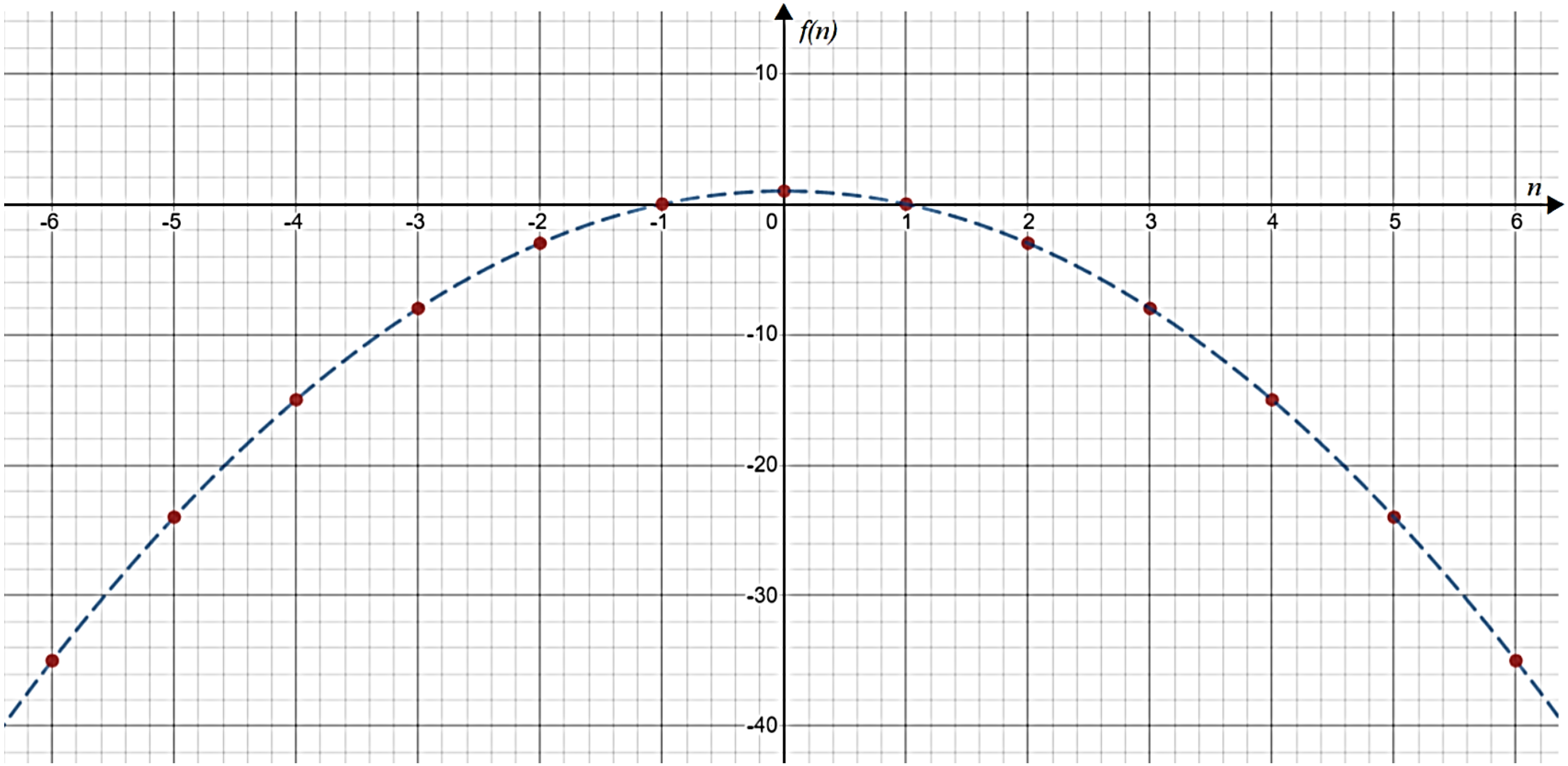
45. Let $g(x) = \lfloor x \rfloor$. Find

a) $g^{-1}(\{0\}) = \{x \mid 0 \leq x < 1\}$.

b) $g^{-1}(\{-1, 0, 1\}) = \{x \mid -1 \leq x < 2\}$.

c) $g^{-1}(\{x \mid 0 < x < 1\}) = \emptyset$.

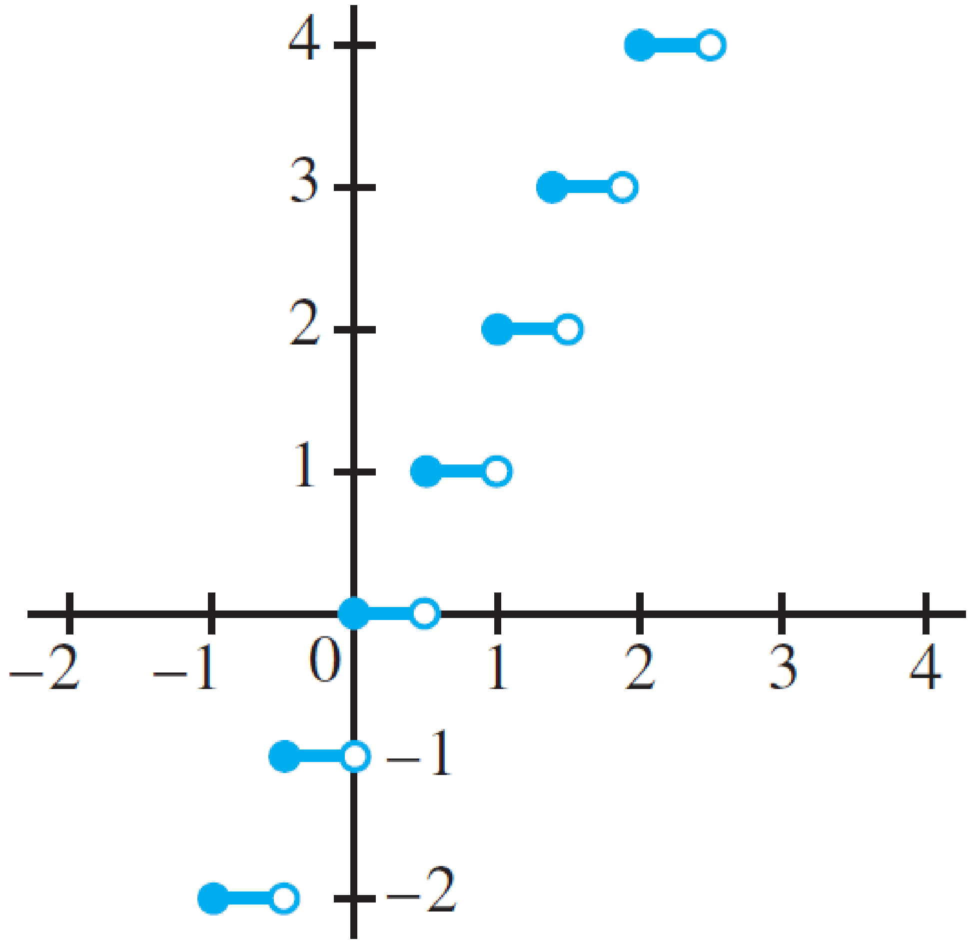
64. Draw the graph of the function $f(n) = 1 - n^2$ from **Z** to **Z**.



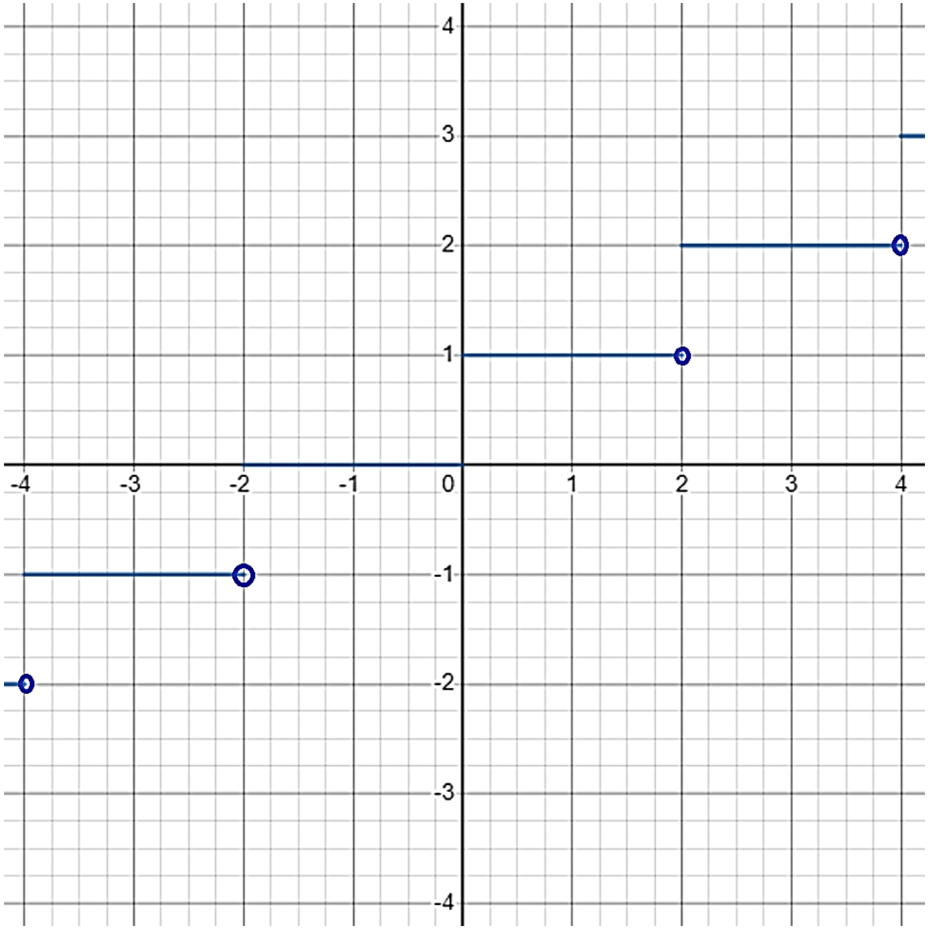
Functions - Exercise

2.3 Exercise – Page #: 161 to 164

65. Draw the graph of the function $f(x) = \lfloor 2x \rfloor$ from $\mathbf{R}$ to $\mathbf{R}$.



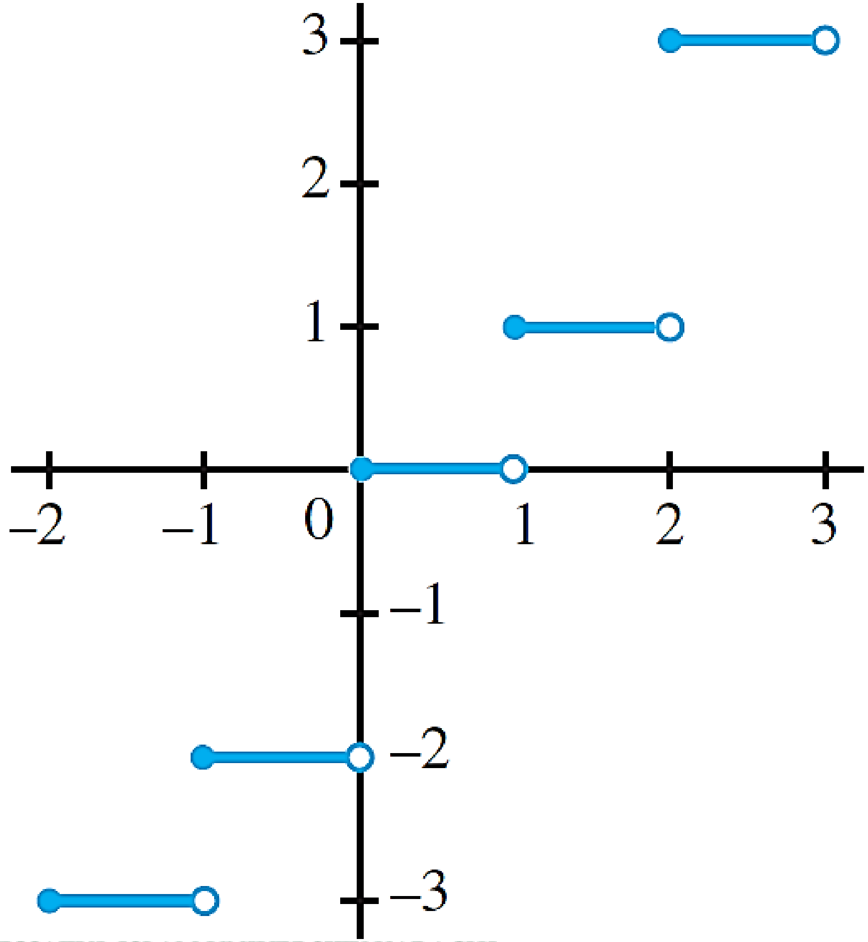
66. Draw the graph of the function $f(x) = \lfloor x/2 \rfloor$ from **R** to **R**.



Functions - Exercise

2.3 Exercise – Page #: 161 to 164

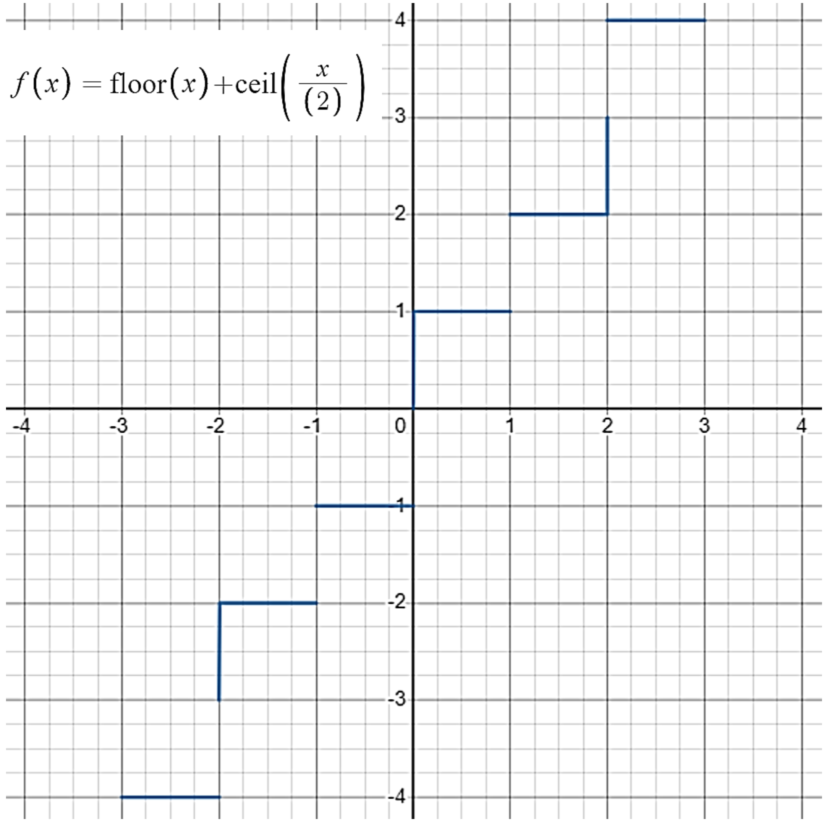
67. Draw the graph of the function $f(x) = \lfloor x \rfloor + \lfloor x/2 \rfloor$ from **R** to **R**.



Functions - Exercise

2.3 Exercise – Page #: 161 to 164

68. Draw the graph of the function $f(x) = \lceil x \rceil + \lfloor x/2 \rfloor$ from **R** to **R**.



https://www.desmos.com/

Functions - Exercise

2.3 Exercise – Page #: 161 to 164

69. Draw graphs of each of these functions.

a) $f(x) = \lfloor x + 12 \rfloor$

b) $f(x) = \lfloor 2x + 1 \rfloor$

c) $f(x) = \lceil x/3 \rceil$

d) $f(x) = \lceil 1/x \rceil$

e) $f(x) = \lceil x - 2 \rceil + \lfloor x + 2 \rfloor$

f) $f(x) = \lfloor 2x \rfloor \lceil x/2 \rceil$

g) $f(x) = \lceil \lfloor x - 12 \rfloor + 12 \rceil$

2.3 Exercise – Page #: 161 to 164

70. Draw graphs of each of these functions.

a) $f(x) = \lceil 3x - 2 \rceil$

b) $f(x) = \lceil 0.2x \rceil$

c) $f(x) = \lfloor -1/x \rfloor$

d) $f(x) = \lfloor x2 \rfloor$

e) $f(x) = \lceil x/2 \rceil \lfloor x/2 \rfloor$

f) $f(x) = \lfloor x/2 \rfloor + \lceil x/2 \rceil$

g) $f(x) = \lfloor 2 \lceil x/2 \rceil + 12 \rfloor$

2.3 Exercise – Page #: 161 to 164

71. Find the inverse function of $f(x) = x^3 + 1$.

$$f(x) = x^3 + 1$$

$$y = x^3 + 1$$

$$y - 1 = x^3$$

$$x^3 = y - 1$$

$$x = (y - 1)^{\frac{1}{3}}$$

$$x = \sqrt[3]{(y - 1)}$$

$$f^{-1}(x) = \sqrt[3]{(x - 1)}$$

Introduction to Functions

Chapter Reading

Book: Kenneth H. Rosen,

Discrete Mathematics and Its Applications (Eight Edition)

Exercise for Practice

Section 2.3: Functions

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Week - 7 and Lecture – 1 & 2

Inclusion–Exclusion

SINDH MADRESSATUL ISLAM UNIVERSITY KARACHI
Chartered by Government of Sindh, Recognized by HEC

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 1

Principle of Inclusion–Exclusion

1885 SMI UNIVERSITY SINDH MADRESSATUL ISLAM UNIVERSITY KARACHI
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Principle of Inclusion–Exclusion

APPLICATIONS OF INCLUSION - EXCLUSION

- To find the number of primes less than a positive integer.

- Counting the number of onto functions from one finite set to another.

- The well-known hatcheck problem can be solved using the principle of inclusion–exclusion.

Principle of Inclusion–Exclusion

INTRODUCTION

A discrete mathematics class contains 30 women and 50 sophomores.

How many students in the class are either women or sophomores?

This question cannot be answered unless more information is provided.

INTRODUCTION

Adding the number of women in the class and the number of sophomores probably does not give the correct answer, because women sophomores are counted twice.

This observation shows that the number of students in the class that are either sophomores or women is the sum of the number of women and the number of sophomores in the class minus the number of women sophomores.

Principle of Inclusion–Exclusion

DEFINITION

If a task can be done in either n_1 ways or n_2 ways,

then the number of ways to do the task is

$n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The is known as the ***subtraction rule*** or ***principle of inclusion–exclusion***, especially when it is used to count the number of elements in the union of two sets.

Principle of Inclusion–Exclusion

DEFINITION

Suppose that A_1 and A_2 are sets. Then,

$|A_1|$ ways to select an element from A_1 and

$|A_2|$ ways to select an element from A_2.

The number of ways to select an element *from A_1 or from A_2*, that is, the number of ways to select an element from their union $|A_1 \cup A_2|$,

is $|A_1| + |A_2|$, the sum of the number of ways to select an element from A_1 and the number of ways to select an element from A_2,

minus $|A_1 \cap A_2|$, the number of ways to select an element that is *in both A_1 and A_2*.

Because there are $|A_1 \cup A_2|$ ways to select an element in either A_1 or in A_2, and $|A_1 \cap A_2|$ ways to select an element common to both sets,

we have $$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Principle of Inclusion–Exclusion

PROBLEM

Question: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: We have a bit string of length eight then,

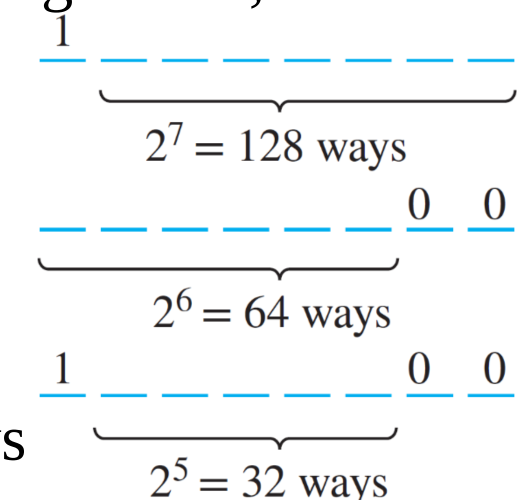
There are two choices for each place 0 or 1.

Number of ways,

begin with a 1 = 2^7 = **128** ways

ending with the two bits 00 = 2^6 = **64** ways

begin with 1 and end with a 00 = 2^5 = **32** ways



If either start with a 1 bit or end with the two bits 00

= **128** + **64** – **32** = **160** ways

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Q. 50. How many bit strings of length seven either begin with two 0s or end with three 1s?

$$\underline{0}\ \underline{0}\ \underbrace{\_\ \_\ \_\ \_\ \_}_{2^5 = 32\ Ways}$$

$$\underbrace{\_\ \_\ \_\ \_}_{2^4 = 16\ Ways}\ \underline{1}\ \underline{1}\ \underline{1}$$

$$\underline{0}\ \underline{0}\ \underbrace{\_\ \_}_{2^2 = 4\ Ways}\ \underline{1}\ \underline{1}\ \underline{1}$$

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Q. 51. How many bit strings of length 10 either begin with three 0s or end with two 0s?

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Q. 52. How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Solution: Let's break down the problem:

The answer for this is simply $\boldsymbol{2^{10}}$ because you have 10 slots of bits xxxxxxxxxx.

In each slot u can either put 0 or 1 so the total possible strings

$= 2\times2\times2\times2\times2\times2\times2\times2\times2\times2 = \boldsymbol{2^{10}}$

Principle of Inclusion–Exclusion

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Bit strings of length 10 with 5 consecutive 1's.

Total 6 possible positions exists. Note that extra 0 is added from the 2nd position to avoid counting duplicate bit strings twice.

11111_ _ _ _ _ (5 slots ⇨ 2^5) 011111_ _ _ _ (4 slots ⇨ 2^4)

_ 011111_ _ _ (4 slots ⇨ 2^4) _ _ 011111 _ _ (4 slots ⇨ 2^4)

_ _ _ 011111_ (4 slots ⇨ 2^4) _ _ _ _ 011111 (4 slots ⇨ 2^4)

Total bit strings of length 10 contain 5 consecutive ones

$= 2^5 + 5 \times 2^4 = 112$

Problem

Exercise 6.1 (Page #: 418)

Class Assignment

Total bit strings of length 10 contain 5 consecutive ones

$= 2^5 + 5 \times 2^4 = 112$

Similarly,

Total bit strings of length 10 contain 5 consecutive zeros

$= 2^5 + 5 \times 2^4 = 112$

A = bit strings of length 10 contain either five consecutive 1s

B = bit strings of length 10 contain either five consecutive 0s

"Or" means union

Principle of Inclusion–Exclusion

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

By set theory we know that A ∪ B = A + B - A ∩ B

Here A ∩ B means that the bit string contains 5 consecutive ones and 5 consecutive zeroes.

1111100000 and ***0000011111***

There are only two such possibilities = 112 + 112 - 2 = 222

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Q. 53. How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?

Solution: Let's break down the problem:

Note that extra 1 is added from the 2nd position to avoid counting duplicate bit strings twice.

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Bit strings of length 8 with three consecutive 0's. The possible positions are:

000_ _ _ _ _ (5 slots ⇨ 2^5)

_ 1000_ _ _ (4 slots ⇨ 2^4)

_ _ _ 1000_ (4 slots ⇨ 2^4)

1000_ _ _ _ (4 slots ⇨ 2^4)

_ _ 1000 _ _ (4 slots ⇨ 2^4)

_ _ _ _ 1000 (4 slots ⇨ 2^4)

Total bit strings of length 8 contain three consecutive zeroes

$= 2^5 + 5 \times 2^4 = 112$

Principle of Inclusion–Exclusion

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Similarly, bit strings of length 8 with four consecutive 1's. The possible positions are:

1111 _ _ _ _ (5 slots ⇨ 2^4)

01111 _ _ _ (4 slots ⇨ 2^3)

_ 01111 _ _ (4 slots ⇨ 2^3)

_ _ 01111 _ (4 slots ⇨ 2^3)

_ _ _ 01111 (4 slots ⇨ 2^3)

Total bit strings of length 8 contain four consecutive ones

$$= 2^4 + 4 \times 2^3 = 48$$

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

A = bit strings of length 8 contain 000, B = bit strings of length 8 contain 1111

"Or" means union, we know that **A ∪ B = A + B - A ∩ B**

Here A ∩ B means that the common bit strings, those are:

00001111, 11110000, 11111000, 00011111,

00011110, 10001111, 01111000, 11110001,

00011000, 00010001, 00010000, 00001000,

10001000. **A ∪ B = A + B - A ∩ B**

There are only two such possibilities = 112 + 48 - 13 = ***147***

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 2

Principle of Inclusion–Exclusion

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Principle of Inclusion–Exclusion

PROBLEM

Question: A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution: A computer company received applications for a job.

No. of application = $|U|$ = 350

major in computer science = $|A_1|$ = 220

major in business = $|A_2|$ = 147

major in both computer science **and** Business =$|A_1 \cap A_2|$= 51

Principle of Inclusion–Exclusion

PROBLEM

Solution: A computer company received applications for a job.

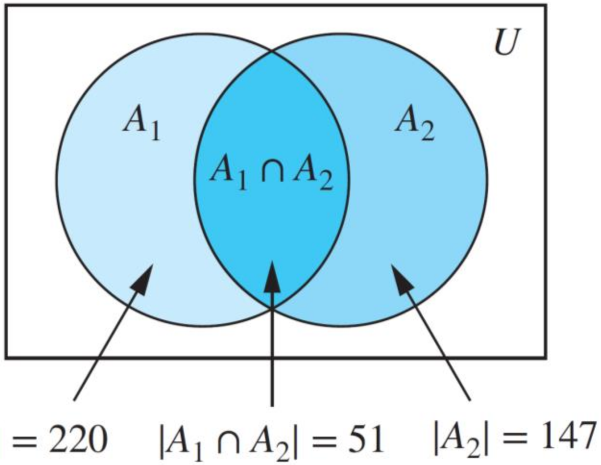
No. of application = $|U|$ = 350

major in computer science = $|A_1|$ = 220

major in business = $|A_2|$ = 147

major in both computer science **and** Business =$|A_1 \cap A_2|$= 51

major in both computer science **or** Business =$|A_1 \cup A_2|$= ?



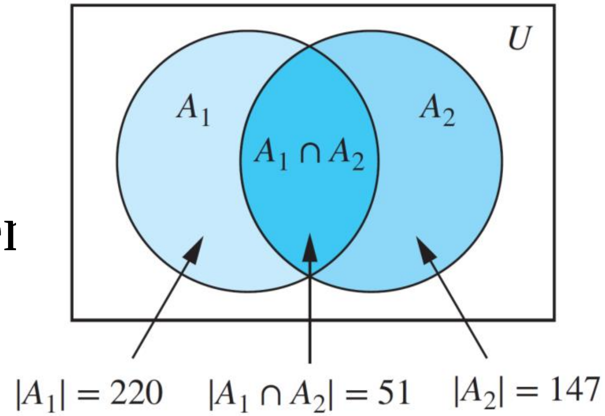
Principle of Inclusion–Exclusion

PROBLEM

major in both computer science **or** Business = $|A_1 \cup A_2|$ = ?

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

= 220 + 147 – 51 = 316



Applicants who majored in **neither** computer science **nor** business =

$$\begin{aligned} |\overline{A_1 \cup A_2}| &= |U| - |A_1 \cup A_2| \\ &= |U| - (|A_1| + |A_2| - |A_1 \cap A_2|) \\ &= 350 - (220 + 147 - 51) \\ &= 350 - 316 \\ &= 34 \end{aligned}$$

PROBLEM

Exercise 6.1 (Page #: 418)

Class Assignment

Q. 54. Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?

PROBLEM

Exercise 6.1 (Page #: 419)

Class Assignment

Q. 55. How many positive integers not exceeding 100 are divisible either by 4 or by 6?

PROBLEM

Exercise 6.1 (Page #: 419)

Class Assignment

Q. 56. How many different initials can someone have if a person has at least two, but no more than five, different initials?

Assume that each initial is one of the 26 uppercase letters of the English language.

Solution: $26^2 + 26^3 + 26^4 + 26^5$

$= 12,356,604$

PROBLEM

Exercise 6.1 (Page #: 419)

Class Assignment

Q. 57. Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters $*$, >, <, !, +, and =.

a) How many different passwords are available for this computer system?

PROBLEM

Exercise 6.1 (Page #: 419)

Class Assignment

Solution: We have the following characters; No. of;

lowercase letters = 26 uppercase letters = 26

digits = 10 special characters = 6

Total available characters = 26 + 26 + 10 + 6 = ***68***

Possible passwords $= 68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12}$

$$= 9{,}920{,}671{,}339{,}261{,}325{,}541{,}376 \approx 9.9 \times 10^{21}$$

PROBLEM

Exercise 6.1 (Page #: 419)

Class Assignment

b) How many of these passwords contain at least one occurrence of at least one of the six special characters?

Solution: The number of passwords that contain *at least* one of the six characters in the "special set" is the total minus those the contain **none** from that set. The total possibilities containing none from the special set can be calculated using a reduced character space of 62, that is

$$(68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12}) - (62^8 + 62^9 + 62^{10} + 62^{11} + 62^{12})$$

$$= 6{,}641{,}514{,}961{,}387{,}068{,}437{,}760 \approx 6.6 \times 10^{21}$$

PROBLEM

Exercise 6.1 (Page #: 419)

Class Assignment

c) Using your answer to part (a), determine how long it takes a hacker to try every possible password, assuming that it takes one nanosecond for a hacker to check each possible password.

Solution*:** We known that 1 sec = 1,000,000,000 ns. A hacker check 1,000,000,000 possible password in one second. So, Therefore, he can check ***9,920,671,339,261,325,541,376 possible password in

$$\frac{9{,}920{,}671{,}339{,}261{,}325{,}541{,}376}{1{,}000{,}000{,}000} = 9{,}920{,}671{,}339{,}261 \text{ second}$$

$$= \frac{9{,}920{,}671{,}339{,}261}{31{,}556{,}952} \text{ year} \approx \mathbf{314373.56\ years}$$

PROBLEM

Exercise 6.1 (Page #: 419)

Class Assignment

Q. 58. The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter, either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters.)

PROBLEM

Exercise 6.1 (Page #: 419)

Class Assignment

Q. 59. The name of a variable in the JAVA programming language is a string of between 1 and 65,535 characters, inclusive, where each character can be an uppercase or a lowercase letter, a dollar sign, an underscore, or a digit, except that the first character must not be a digit. Determine the number of different variable names in JAVA.

Principle of Inclusion–Exclusion

PROBLEM

Exercise 6.1 (Page #: 419)

Class Assignment

Q. 60. The International Telecommunications Union (ITU) specifies that a telephone number must consist of a country code with between 1 and 3 digits, except that the code 0 is not available for use as a country code, followed by a number with at most 15 digits. How many available possible telephone numbers are there that satisfy these restrictions?

Introduction to Functions

Chapter Reading

Book: Kenneth H. Rosen,

Discrete Mathematics and Its Applications (Eight Edition)

Exercise for Practice

Section 6.1.4 The Subtraction Rule

(Inclusion–Exclusion for Two Sets)

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Week - 8 and Lecture – 1 & 2

The Pigeonhole Principle

SINDH MADRESSATUL ISLAM UNIVERSITY KARACHI
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MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 1

The Pigeonhole Principle – An Introduction and Examples

SINDH MADRESSATUL ISLAM UNIVERSITY KARACHI
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The Pigeonhole Principle

APPLICATIONS

- Very useful in counting problems.
- Application of ceiling function.
- This principle is applicable in many fields like Number Theory, Probability, Algorithms, Geometry, etc.

The Pigeonhole Principle

INTRODUCTION

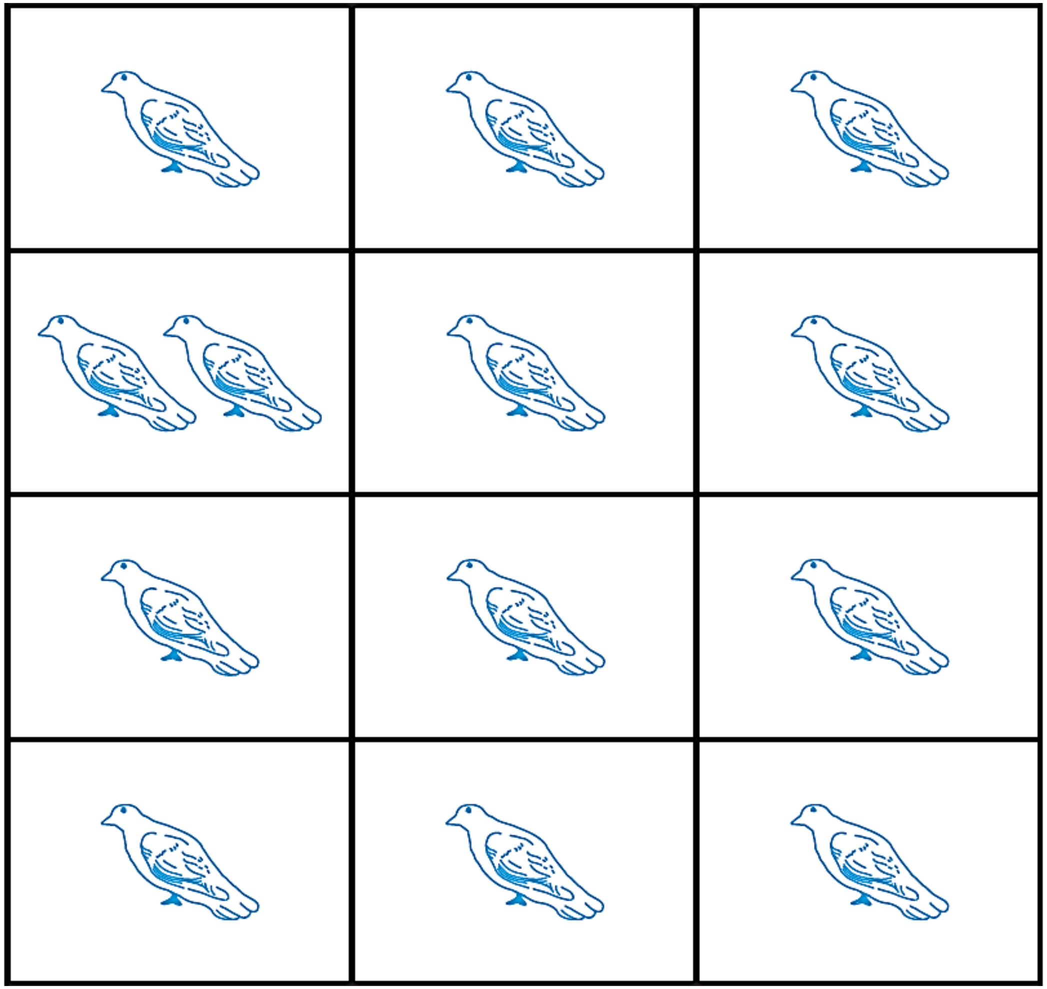
Suppose that a flock of 13 pigeons flies into a set of 12 pigeonholes to roost.

Because there are 13 pigeons but only 12 pigeonholes,

a least one of these 13 pigeonholes must have at least two pigeons in it.

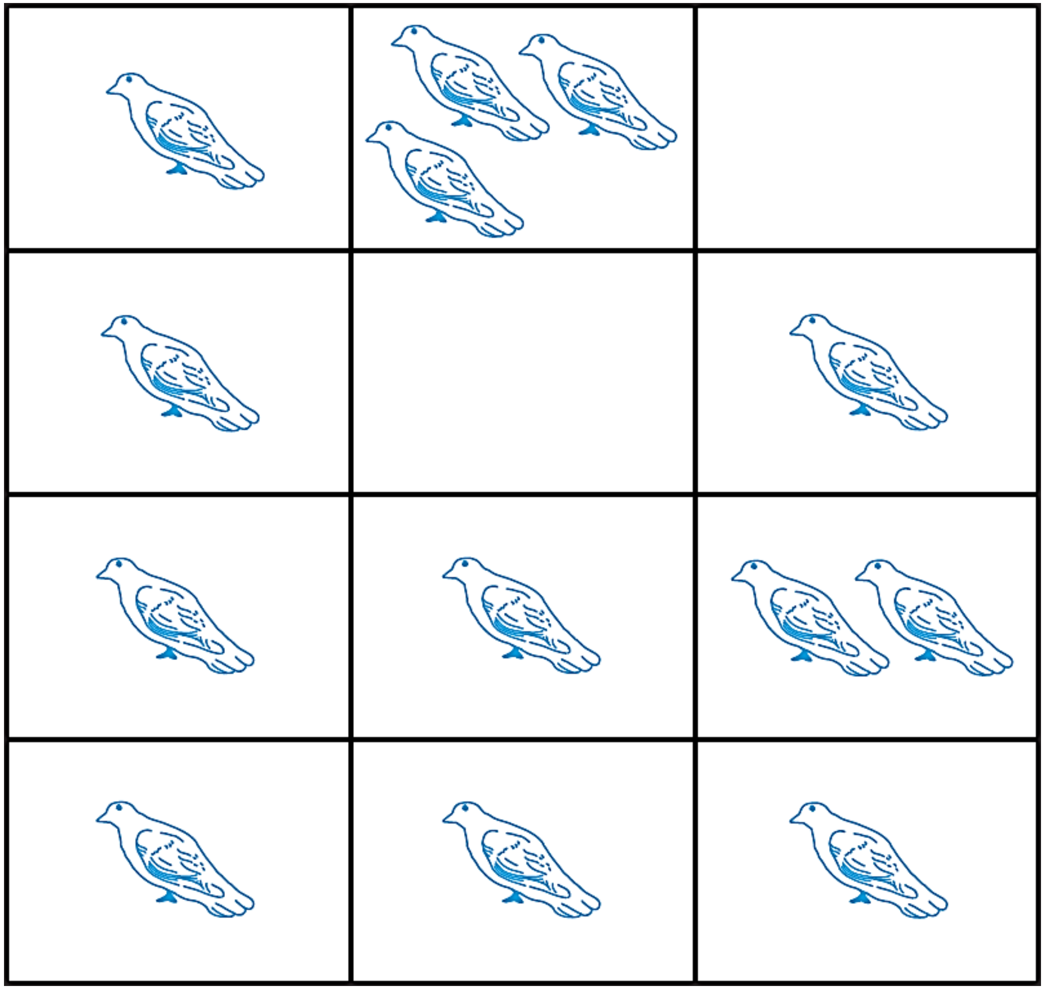
The Pigeonhole Principle

INTRODUCTION



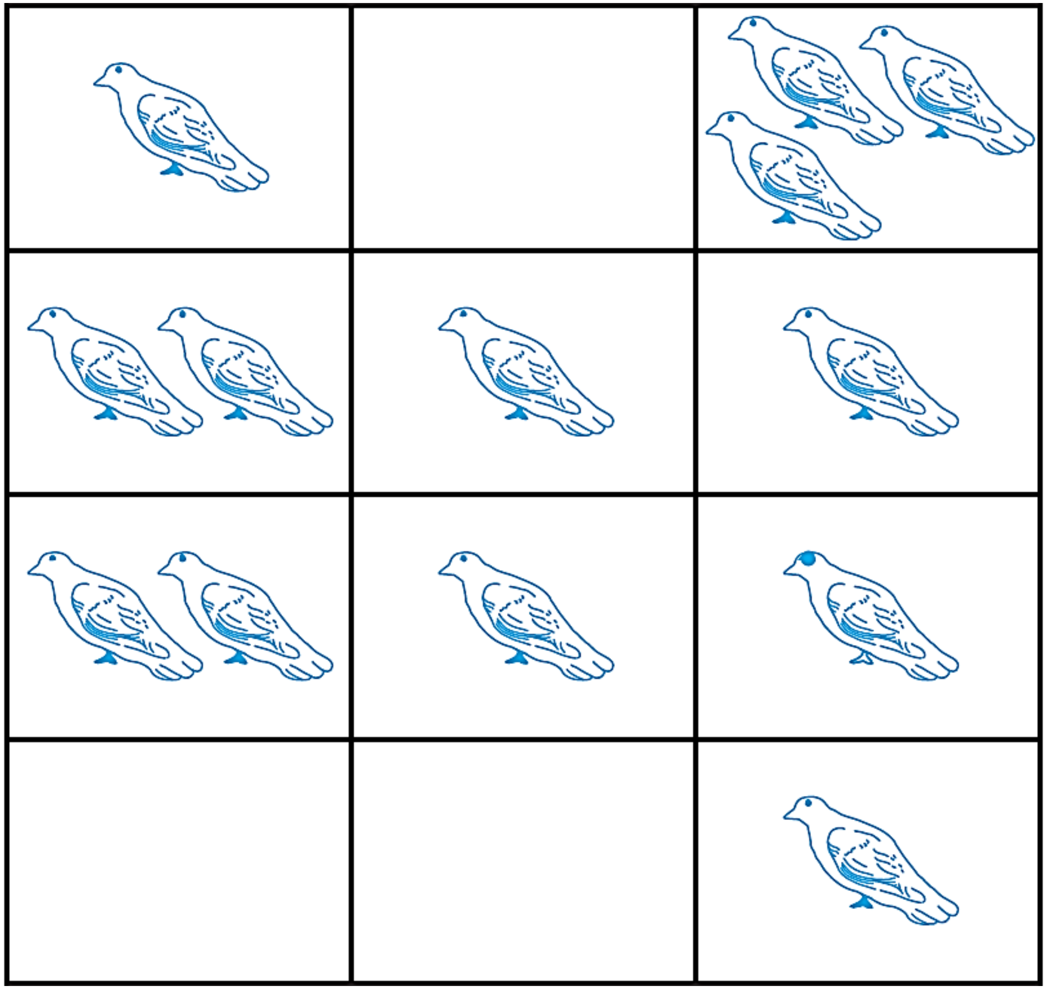
The Pigeonhole Principle

INTRODUCTION



The Pigeonhole Principle

INTRODUCTION



The Pigeonhole Principle

THEOREM – 1

If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

The ***pigeonhole principle*** is also called the ***Dirichlet drawer principle***, after the nineteenth century German mathematician G. Lejeune Dirichlet, who often used this principle in his work.

The Pigeonhole Principle

COROLLARY - 1

A function f from a set with k + 1 or more elements to a set with k elements is not one-to-one.

The Pigeonhole Principle

EXAMPLES – PAGE #: 421

Example – 1

Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example – 2

In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

EXAMPLES – PAGE #: 421

Example – 3

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

***Solution*:** There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

The Pigeonhole Principle

DECIMAL EXPANSION

The decimal expansion of a number is its representation in base-10 (i.e., in the decimal system).

The following table summarizes the decimal expansions of the first few unit fractions (numerator is always one).

fraction	decimal expansion	fraction	decimal expansion
1	1	$\frac{1}{11}$	$0.\overline{09}$
$\frac{1}{2}$	0.5	$\frac{1}{12}$	$0.08\overline{3}$
$\frac{1}{3}$	$0.\overline{3}$	$\frac{1}{13}$	$0.\overline{076923}$
$\frac{1}{4}$	0.25	$\frac{1}{14}$	$0.0\overline{714285}$
$\frac{1}{5}$	0.2	$\frac{1}{15}$	$0.0\overline{6}$
$\frac{1}{6}$	$0.1\overline{6}$	$\frac{1}{16}$	0.0625
$\frac{1}{7}$	$0.\overline{142857}$	$\frac{1}{17}$	$0.\overline{0588235294117647}$
$\frac{1}{8}$	0.125	$\frac{1}{18}$	$0.0\overline{5}$
$\frac{1}{9}$	$0.\overline{1}$	$\frac{1}{19}$	$0.\overline{052631578947368421}$
$\frac{1}{10}$	0.1	$\frac{1}{20}$	0.05

DECIMAL EXPANSION

Consider a sequence of eight numbers 1, 11, 111, 1111, 11111, 111111, 1111111, 11111111 is divided by 7 then prove that at least two remainders are same.

Decimal Expansion	
$\frac{1}{7}$ remainder is 1	$\frac{11111}{7}$ remainder is 2
$\frac{11}{7}$ remainder is 4	$\frac{111111}{7}$ remainder is 0
$\frac{111}{7}$ remainder is 6	$\frac{1111111}{7}$ remainder is 1
$\frac{1111}{7}$ remainder is 5	$\frac{11111111}{7}$ remainder is 4

Thus, remainders of $\frac{1}{7}$ and $\frac{1111111}{7}$ is same. **Hence Proved.**

The Pigeonhole Principle

DECIMAL EXPANSION

Consider a sequence of eight numbers 1, 11, 111, 1111, 11111, 111111, 1111111, 11111111 is divided by 7 then prove that difference of same remainder is multiple of 7.

Decimal Expansion	
$\frac{1}{7}$ remainder is 1	$\frac{11111}{7}$ remainder is 2
$\frac{11}{7}$ remainder is 4	$\frac{111111}{7}$ remainder is 0
$\frac{111}{7}$ remainder is 6	$\frac{1111111}{7}$ remainder is 1
$\frac{1111}{7}$ remainder is 5	$\frac{11111111}{7}$ remainder is 4

Thus, remainders of $\frac{1}{7}$ and $\frac{1111111}{7}$ is same,

then 1111111-1=1111110 is multiple of 7. **Hence Proved.**

The Pigeonhole Principle

EXAMPLES – PAGE #: 422

Example – 4

Show that for every integer *n* there is a multiple of *n* that has only 0s and 1s in its decimal expansion.

***Solution*:** Let *n* be a positive integer. Consider the *n* + 1 integers 1, 11, 111,…, 11…1 (where the last integer in this list is the integer with *n* + 1 1s in its decimal expansion). Note that there are *n* possible remainders when an integer is divided by *n*. Because there are *n* + 1 integers in this list, by the pigeonhole principle there must be two with the same remainder when divided by *n*. The larger of these integers less the smaller one is a multiple of *n*, which has a decimal expansion consisting entirely of 0s and 1s.

THE GENERALIZED PIGEONHOLE PRINCIPLE

Introduction

Among any set of 21 decimal digits there must be 3 that are the same.

This follows because when 21 objects are distributed into 10 boxes, one box must have more than 2 objects.

THE GENERALIZED PIGEONHOLE PRINCIPLE

Theorem – 2

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 5 (*PAGE #: 422*)

Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 6 (*PAGE #: 422*)

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 6 (*PAGE #: 422*)

Solution: The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

THE GENERALIZED PIGEONHOLE PRINCIPLE

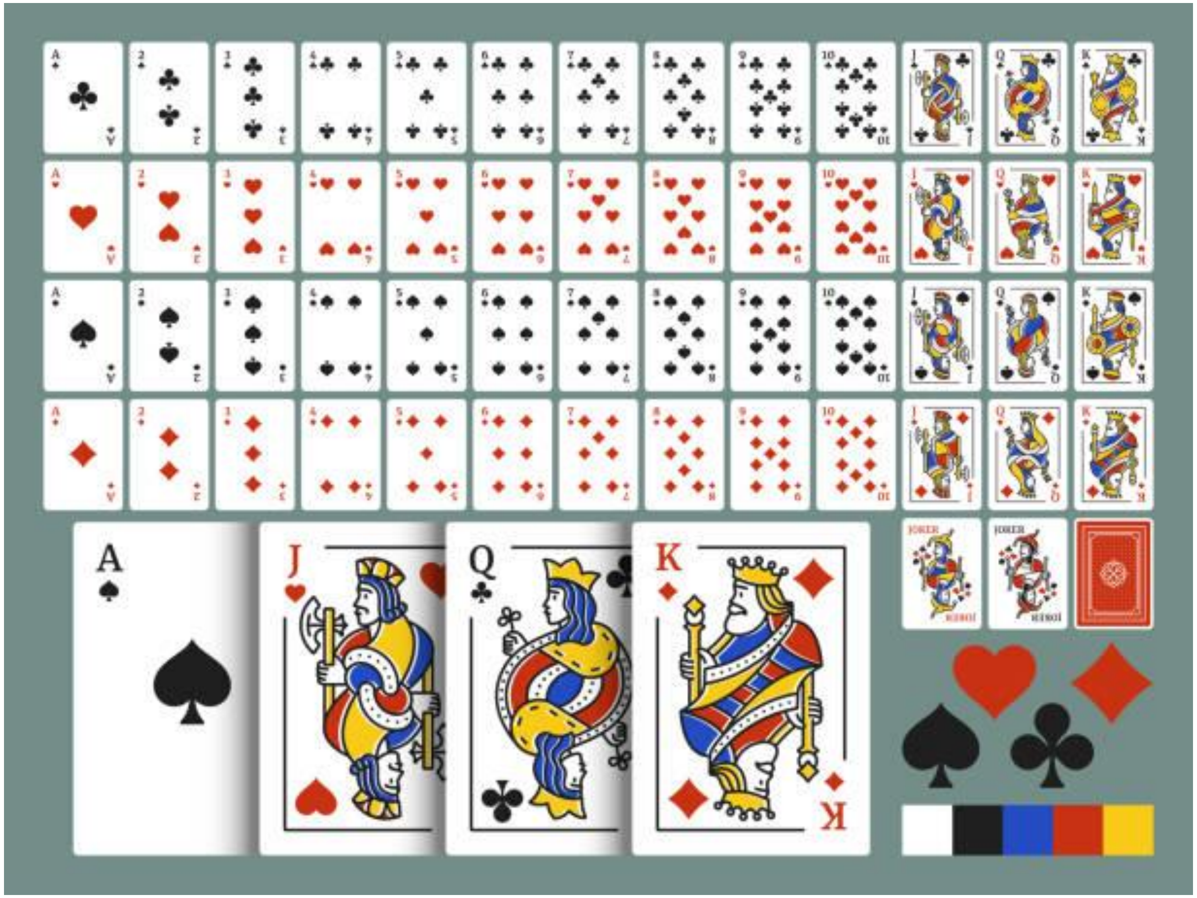
Example – 7 (*PAGE #*: *422*)

A standard deck of 52 cards has 13 kinds of cards, with four cards of each of kind, one in each of the four suits, hearts, diamonds, spades, and clubs.

The Pigeonhole Principle

The Generalized Pigeonhole Principle

Example – 7 (Page #: 422)



THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 7 (*PAGE #*: *422*)

a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are selected?

Solution: Suppose there are four boxes, one for each suit, and as cards are selected, they are placed in the box reserved for cards of that suit.

The Pigeonhole Principle

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 7 (*PAGE #: 422*)

Solution: Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing at least $\lceil N/4 \rceil$ cards. Consequently, we know that at least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$.

The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is

$N = 2 \cdot 4 + 1 = 9$, so nine cards suffice.

The Pigeonhole Principle

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 7 (*PAGE #: 422*)

b) How many must be selected from a standard deck of 52 cards to guarantee that at least three hearts are selected?

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 7 (*PAGE #: 422*)

Solution: We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three hearts.

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 8 **(*PAGE #*: *422*)**

What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form *NXX*-*NXX*-*XXXX*, where the first three digits form the area code, *N* represents a digit from 2 to 9 inclusive, and *X* represents any digit.)

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 8 (*PAGE #: 422*)

Solution: There are eight million different phone numbers of the form *NXX-XXXX* (calculated in previous). Hence, by the generalized pigeonhole principle, among 25 million telephones, at least $\lceil 25{,}000{,}000/8{,}000{,}000 \rceil = 4$ of them must have identical phone numbers. Hence, at least four area codes are required to ensure that all 10-digit numbers are different.

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 9 (*PAGE #: 422*)

Example 9, although not an application of the generalized pigeonhole principle, makes use of similar principles.

The Pigeonhole Principle

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 9 (PAGE #: 422)

Suppose that a computer science laboratory has 15 workstations and 10 servers. A cable can be used to directly connect a workstation to a server. For each server, only one direct connection to that server can be active at any time. We want to guarantee that at any time any set of 10 or fewer workstations can simultaneously access different servers via direct connections. Although we could do this by connecting every workstation directly to every server (using 150 connections), what is the minimum number of direct connections needed to achieve this goal?

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 9 **(*PAGE #: 422*)**

Solution: Suppose that we label the workstations W_1, W_2, …, W_{15} and the servers S_1, S_2,…, S_{10}. First, we would like to find a way for there to be far fewer than 150 direct connections between workstations and servers to achieve our goal. One promising approach is to directly connect W_k to S_k for $k = 1, 2,\ldots, 10$ and then to connect each of $W_{11}, W_{12}, W_{13}, W_{14}$, and W_{15} to all 10 servers. This gives us a total of $10 + 5 \cdot 10 = 60$ direct connections.

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 9 (*PAGE #: 422*)

Solution: We need to determine whether with this configuration any set of 10 or fewer workstations can simultaneously access different servers. We note that if workstation W_j is included with $1 \leq j \leq 10$, it can access server S_j, and for each workstation W_k with $k \geq 11$ included, there must be a corresponding workstation W_j with $1 \leq j \leq 10$ not included, so W_k can access server S_j. (This follows because there are at least as many available servers S_j as there are workstations W_j with $1 \leq j \leq 10$ not included.)

THE GENERALIZED PIGEONHOLE PRINCIPLE

Example – 9 (PAGE #: 422)

Solution: So, any set of 10 or fewer workstations are able to simultaneously access different servers. But can we use fewer than 60 direct connections? Suppose there are fewer than 60 direct connections between workstations and servers. Then some server would be connected to at most $\lfloor 59/10 \rfloor = 5$ workstations. This means that the remaining nine servers are not enough for the other 10 or more workstations to simultaneously access different servers. Consequently, at least 60 direct connections are needed. It follows that 60 is the answer.

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 2

The Pigeonhole Principle - Exercise

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EXERCISE 6.2 (*PAGE #: 426*)

1. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.

***Solution*:** As we known that there are six classes, but only five weekdays, the pigeonhole principle shows that at least two classes must be held on the same day.

The Pigeonhole Principle

EXERCISE 6.2 (*PAGE #: 426*)

2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

***Solution*:** As we known that there are 26 letters, and 30 students so by using Pigeonhole Principle we have ceiling function $\lceil 30 / 26 \rceil = 2$.

The Pigeonhole Principle

EXERCISE 6.2 (PAGE #: 426)

3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

a) How many socks must he take out to be sure that he has at least two socks of the same color?

***Solution*:** By the pigeonhole principle, the answer is 3. If three socks are taken from the drawer, at least two must have the same color.

$$1(red)+1(black)+1(either\ red\ or\ black)=3$$

EXERCISE 6.2 (*PAGE #: 426*)

3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

b) How many socks must he take out to be sure that he has at least two black socks?

***Solution*:** He needs to take out 14 socks in order to insure at least two are black socks. If he does so, then at most 12 of them are brown, so at least two are black.

EXERCISE 6.2 (PAGE #: 426)

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?

***Solution*:** By the pigeonhole principle, the answer is 5, She is picking up alternating colors, and thus she can only be sure she holds 3 balls of the same color after 5 pickups.

The Pigeonhole Principle

EXERCISE 6.2 (*PAGE #: 426*)

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

b) How many balls must she select to be sure of having at least three blue balls?

***Solution*:** By the pigeonhole principle, the answer is 13, the first 10 balls are red. So, in order to be sure, she has 3 blue balls, she needs to pickup 13 times.

EXERCISE 6.2 (PAGE #: 426)

5. Undergraduate students at a college belong to one of four groups depending on the year in which they are expected to graduate. Each student must choose one of 21 different majors. How many students are needed to assure that there are two students expected to graduate in the same year who have the same major?

***Solution*:** By the pigeonhole principle, maximum possible different major/group=84. So, 85th student must be same as one of those 84 student. Therefore, answer is 85.

EXERCISE 6.2 (*PAGE #: 426*)

6. There are six professors teaching the introductory discrete mathematics class at a university. The same final exam is given by all six professors. If the lowest possible score on the final is 0 and the highest possible score is 100, how many students must there be to guarantee that there are two students with the same professor who earned the same final examination score?

***Solution*:** By the pigeonhole principle, If possible, scores are from 0 to 100 with both inclusive it means number of possible scores = 101.

The Pigeonhole Principle

EXERCISE 6.2 (PAGE #: 426)

Now, If there was only one professor grading the students, in order to ensure that that there are two students with the same professor who earned the same final examination score, The number of students would have to be = 101 + 1 = 102 students.

Meanwhile, for each student, since there are 6 professors, the possible combination for a score will be = 6 possible combinations.

Therefore, number of students that will guarantee that there are two students with the same professor who earned the same final examination score must be a minimum of: $(6 \times 101) + 1 = 607$

EXERCISE 6.2 (*PAGE #: 426*)

7. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

Solution: We known that there are four possible remainders when an integer is divided by 4, those are 0, 1, 2, 3.

Now the pigeonhole principle implies that given five integers, at least two have the same remainder.

EXERCISE 6.2 (PAGE #: 426)

8. Let d be a positive integer. Show that among any group of $d + 1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

***Solution*:** We known that there are only d possible remainders modulo d (namely, 0,1,...,d−1).

By the pigeonhole principle, at least two of the d +1 remainders would be equal.

The Pigeonhole Principle

EXERCISE 6.2 (*PAGE #: 426*)

9. Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

Solution: Let $a, a+1, \ldots, a+n-1$ be the integers in the sequence. The integers $(a+i) \bmod n$, $i = 0, 1, 2, \ldots, n-1$, are distinct, because $0 < (a+j)-(a+k) < n$ whenever $0 \leq k < j \leq n-1$. Because there are n possible values for $(a+i) \bmod n$ and there are n different integers in the set, each of these values is taken on exactly once. By the pigeonhole principle, it follows that there is exactly one integer in the sequence that is divisible by n.

EXERCISE 6.2 (PAGE #: 426)

10. Show that if f is a function from S to T, where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$, or in other words, f is not one-to-one.

***Solution*:** By the pigeonhole principle, think of elements of S as objects and elements of T as boxes. A function sends each element of S to an element of T (i.e. "puts an object into a box"). Since there are more elements of S (objects) than elements of T (boxes), there is at least one element of T that is the image of at least two elements of S.

EXERCISE 6.2 (PAGE #: 426)

11. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

***Solution*:** 4951

12. Let (xi, yi), i = 1, 2, 3, 4, 5, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

EXERCISE 6.2 (PAGE #: 426)

13. Let (x_i, y_i, z_i), $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$, be a set of nine distinct points with integer coordinates in *xyz* space. Show that the midpoint of at least one pair of these points has integer coordinates.

14. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1,b_1) and (a_2, b_2) such that $a_1 \, mod \, 5 = a_2 \, mod \, 5$ and $b_1 \, mod \, 5 = b_2 \, mod \, 5$?

Introduction to Functions

Chapter Reading

Book: Kenneth H. Rosen,

Discrete Mathematics and Its Applications (Eight Edition)

Exercise for Practice

Section 6.2 The Pigeonhole Principle

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Week - 8 and Lecture – 1 & 2

Permutation & Combination

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MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 1

Permutation & Combination

An Introduction & Examples

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APPLICATIONS

- Very useful in counting problems.
- Application of ceiling function.
- This principle is applicable in many fields like Number Theory, Probability, Algorithms, Geometry, etc.

INTRODUCTION

- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, ***where the order of these elements matters***.

- Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, ***where the order of the elements selected does not matter***.

PERMUTATION - DEFINITION

- A ***permutation*** of a set of distinct objects is an ordered arrangement of these objects.
- We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of *r* elements of a set is called an ***r-permutation***.

PERMUTATION - *DEFINITION*

If $\boldsymbol{n}$ is a positive integer and $\boldsymbol{r}$ is an integer with $\mathbf{1} \le \boldsymbol{r} \le \boldsymbol{n}$, then there are

$$\boldsymbol{P(n,r)} = \boldsymbol{n(n-1)(n-2)\cdots(n-r+1)}$$

r-permutations of a set with $\boldsymbol{n}$ distinct elements.

PERMUTATION - COROLLARY

If ***n*** and ***r*** are integers with $0 \le r \le n$, then

$$P(n, r) = \frac{n!}{(n-r)!}$$

PERMUTATION - EXAMPLE – 1 (PAGE #: 428)

In how many ways can we select three students from a group of five students to stand in line for a picture?

***Solution*:** First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After that, there are three ways to select the third student in the line. By the product rule, there are $\mathbf{5 \cdot 4 \cdot 3 = 60}$ ways to select three students from a group of five students to stand in line for a picture.

PERMUTATION - EXAMPLE – 1 (PAGE #: 428)

In how many ways can we arrange all five of these students in a line for a picture?

***Solution*:**

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange all five students in a line for a picture.

PERMUTATION - EXAMPLE – 2 (PAGE #: 429)

Let $S = \{1, 2, 3\}$.

The ordered arrangement 3, 1, 2 is a permutation of S.

The ordered arrangement 3, 2 is a 2-permutation of S.

PERMUTATION - EXAMPLE – 3 (PAGE #: 429)

Let ***S* = {*a*, *b*, *c*}**. The 2-permutations of *S* are the ordered arrangements ***a*, *b***; ***a*, *c***; ***b*, *a***; ***b*, *c***; ***c*, *a***; and ***c*, *b***. Consequently, there are ***six 2-permutations*** of this ***set with three elements***. There are always six 2-permutations of a set with three elements. There are three ways to choose the first element of the arrangement. There are two ways to choose the second element of the arrangement, because it must be different from the first element. Hence, by the product rule, we see that $P(3, 2) = 3 \cdot 2 = 6$. the first element. By the product rule, it follows that $P(3, 2) = 3 \cdot 2 = 6$.

PERMUTATION - EXAMPLE – 4 (PAGE #: 430)

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution: Since it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is

$$P(100, 3) = 100 \cdot 99 \cdot 98 = 970{,}200.$$

PERMUTATION - EXAMPLE – 5 (*PAGE #: 430*)

Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

***Solution*:** The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are $\boldsymbol{P(8, 3) = 8 \cdot 7 \cdot 6 = 336}$ possible ways to award the medals.

PERMUTATION - EXAMPLE – 6 (PAGE #: 430)

Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

***Solution*:** The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily. Consequently, there are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways for the saleswoman to choose her tour.

PERMUTATION - EXAMPLE – 7 (PAGE #: 431)

How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

***Solution*:**

Because the letters *ABC* must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block *ABC* and the individual letters *D*, *E*, *F*, *G*, and *H*. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters *ABCDEFGH* in which *ABC* occurs as a block.

COMBINATION - DEFINITION

- An ***r-combination*** of elements of a set is an unordered selection of ***r*** elements from the set. Thus, an ***r-combination*** is simply a subset of the set with ***r*** elements.

- The number of ***r-combinations*** of a set with ***n*** distinct elements is denoted by $\boldsymbol{C(n,r)}$. Note that $\boldsymbol{C(n,r)}$ is also denoted by $\boldsymbol{\binom{n}{r}}$ and is called a ***binomial coefficient***.

COMBINATION - DEFINITION

The number of ***r*-combinations** of a set with ***n*** elements, where ***n*** is a nonnegative integer and ***r*** is an integer with $\mathbf{0 \le r \le n}$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

COMBINATION - COROLLARY

Let n and r be nonnegative integers with $r \leq n$.

Then

$$C(n, r) = C(n, n - r)$$

COMBINATION - EXAMPLE – 8 (PAGE #: 431)

How many different committees of three students can be formed from a group of four students?

***Solution*:** We need only find the number of subsets with three elements from the set containing the four students. We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the four students to leave out of the group. This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen does not matter.

COMBINATION - EXAMPLE – 9 (*PAGE #: 431*)

Let S be the set $\{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a ***3-combination*** from *S*.

(Note that $\{4, 1, 3\}$ is the same ***3-combination*** as $\{1, 3, 4\}$, because the order in which the elements of a set are listed does not matter.)

COMBINATION - EXAMPLE – 10 (PAGE #: 431)

We see that $C(4,2) = 6$,

because the ***2-combinations*** of $\{a, b, c, d\}$ are

the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

COMBINATION - EXAMPLE – 11 (PAGE #: 432)

How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Solution: The order in which the five cards are dealt from a deck of 52 cards does not matter, there are

$$C(52,5)=\frac{52!}{5!(52-5)!}=\frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1}=2{,}598{,}960$$

different hands of five cards that can be dealt.

COMBINATION - EXAMPLE – 11 (PAGE #: 432)

Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Solution: The order in which the 47 cards are dealt from a deck of 52 cards does not matter, there are

$$C(52,47)=\frac{52!}{47!(52-47)!}=\frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1}=2{,}598{,}960$$

different hands of five cards that can be dealt.

COMBINATION - EXAMPLE – 11 (PAGE #: 432)

In ***Example 11*** we observed that

$$\boldsymbol{C(52, 5) = C(52, 47)}.$$

This is not surprising because selecting five cards out of 52 is the same as selecting the 47 that we leave out.

The identity $\boldsymbol{C(52, 5) = C(52, 47)}$ is a special case of the useful identity for the number of ***r-combinations*** of a set given in ***Corollary***.

COMBINATION - EXAMPLE – 12 (PAGE #: 434)

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

Solution: The answer is given by the number of 5-combinations of a set with 10 elements. By Theorem 2, the number of such combinations is

$$C(10,5)=\frac{10!}{5!(10-5)!}=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1}=525$$

Combination - Example – 13 (Page #: 434)

A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

COMBINATION - EXAMPLE – 13 (PAGE #: 434)

Solution: The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter. The number of such combinations is

$$C(30,6) = \frac{30!}{6!(30-6)!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 . 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775.$$

COMBINATION - EXAMPLE – 14 (PAGE #: 434)

How many bit strings of length n contain exactly r **1s**?

Solution: The positions of r ***1s*** in a bit string of length n form an r-combination of the set $\{1, 2, 3, \ldots, n\}$.

Hence, there are $C(n, r)$ bit strings of length n that contain exactly r ***1s***.

COMBINATION - EXAMPLE – 15 (PAGE #: 434)

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

COMBINATION - EXAMPLE – 15 (PAGE #: 434)

Solution: By the product rule, the answer is the product of the number of ***3-combinations*** of a set with nine elements and the number of ***4-combinations*** of a set with 11 elements. The number of ways to select the committee is

$$C(9,3)\,.\,C(11,4) = \frac{9!}{3!(9-3)!}\cdot\frac{11!}{11!(11-4)!} = 27,720.$$

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 2
Permutation & Combination

Exercise 6.3

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EXERCISE 6.3 (*PAGE #: 435*)

Q. 1. List all the permutations of $\{a, b, c\}$.

Solution: In permutation, order of these elements matters.

Number of elements = 3

Number of permutation = 3! = 3 . 2 . 1 = 6

So, the list of all permutation are

abc, acb, bac, bca, cab, cba

EXERCISE 6.3 (*PAGE #: 435*)

Q. 2. How many different permutations are there of the set $\{a, b, c, d, e, f, g\}$?

Solution: In permutation, order of these elements matters.

Number of elements = 7

Total number of permutation = 7!

= 7 . 6 . 5 . 4 . 3 . 2 . 1 = **5040**

EXERCISE 6.3 (PAGE #: 435)

Q. 3. How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?

Solution: In permutation, order of these elements matters. Number of elements = ***7***

The last position is fixed with latter ***a***, so there are **6** letters are available for places are vacant.

Total number of permutation = 6!

$= 6 \,.\, 5 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 1 =$ **720**

EXERCISE 6.3 (*PAGE #: 435*)

Q. 4. Let $S = \{1, 2, 3, 4, 5\}$.

a) List all the ***3-permutations*** of S.

Solution: In ***3-permutation***, order of the elements matters.

Numbers of permutation $= \dfrac{5!}{(5-3)!} = 60$

So, the list of all permutation are,

EXERCISE 6.3 (PAGE #: 435)

So, the list of all permutations are

$(1,2,3)$, $(1,2,4)$, $(1,2,5)$, $(1,3,2)$, $(1,3,4)$, $(1,3,5)$,
$(1,4,2)$, $(1,4,3)$, $(1,4,5)$, $(1,5,2)$, $(1,5,3)$, $(1,5,4)$,
$(2,1,3)$, $(2,1,4)$, $(2,1,5)$, $(2,3,1)$, $(2,3,4)$, $(2,3,5)$,
$(2,4,1)$, $(2,4,3)$, $(2,4,5)$, $(2,5,1)$, $(2,5,3)$, $(2,5,4)$,
$(3,1,2)$, $(3,1,4)$, $(3,1,5)$, $(3,2,1)$, $(3,2,4)$, $(3,2,5)$,
$(3,4,1)$, $(3,4,2)$, $(3,4,5)$, $(3,5,1)$, $(3,5,2)$, $(3,5,4)$,
$(4,1,2)$, $(4,1,3)$, $(4,1,5)$, $(4,2,1)$, $(4,2,3)$, $(4,2,5)$,
$(4,3,1)$, $(4,3,2)$, $(4,3,5)$, $(4,5,1)$, $(4,5,2)$, $(4,5,3)$,
$(5,1,2)$, $(5,1,3)$, $(5,1,4)$, $(5,2,1)$, $(5,2,3)$, $(5,2,4)$,
$(5,3,1)$, $(5,3,2)$, $(5,3,4)$, $(5,4,1)$, $(5,4,2)$, $(5,4,3)$.

EXERCISE 6.3 (*PAGE #: 435*)

Q. 4. Let $S = \{1, 2, 3, 4, 5\}$.

b) List all the ***3-combinations*** of S.

Solution: In ***3-combination***, order of the elements does not matter.

Numbers of combination $= \dfrac{5!}{3!(5-3)!} = 10$

So, the list of all combination are,

EXERCISE 6.3 (*PAGE #: 435*)

So, the list of all combination are

{1,2,3}, {1,2,4}, {1,2,5},
{1,3,4}, {1,3,5}, {1,4,5},
{2,3,4}, {2,3,5}, {2,4,5},
{3,4,5}.

EXERCISE 6.3 (PAGE #: 435)

Q. 5. Find the value of each of these quantities.

a) $P(6, 3) = 120$

b) $P(6, 5) = 720$

c) $P(8, 1) = 8$

d) $P(8, 5) = 6720$

e) $P(8, 8) = 40,320$

f) $P(10, 9) = 3,628,800$

EXERCISE 6.3 (PAGE #: 435)

Q. 6. Find the value of each of these quantities.

a) $C(5, 1) = 5$

b) $C(5, 3) = 10$

c) $C(8, 4) = 70$

d) $C(8, 8) = 1$

e) $C(8, 0) = 1$

f) $C(12, 6) = 924$

EXERCISE 6.3 (PAGE #: 435)

Q. 7. Find the number of 5-permutations of a set with nine elements.

Solution: In ***5-permutation***, order of the elements matters.

Numbers of permutation

$$= \frac{9!}{(9-5)!} = 15,120$$

EXERCISE 6.3 (PAGE #: 435)

Q. 8. In how many different orders can five runners finish a race if no ties are allowed?

Solution: In ***5-permutation***, order of the elements matters. Numbers of permutation

$$= \frac{5!}{(5-5)!} = 120$$

or numbers of permutation = $5! = 120$

EXERCISE 6.3 (PAGE #: 435)

Q. 9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?

Solution: The order of the elements matters. Numbers of permutation

$$= \frac{12!}{(12-3)!} = 1320$$

EXERCISE 6.3 (PAGE #: 435)

Q. 10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?

Solution: The order of the elements matters. Numbers of permutation / order

$$= 6! = 720$$

EXERCISE 6.3 (PAGE #: 435)

Q. 11. How many bit strings of length 10 contain

a) exactly four 1s?

Solution: The order of 1s in the bit strings does not matter. Numbers of bit strings

$$\mathbf{C(10,4) = \frac{10!}{4!(10-4)!} = 210}$$

EXERCISE 6.3 (PAGE #: 435)

Q. 11. How many bit strings of length 10 contain

b) at most four 1s?

Solution: The order of 1s in the bit strings does not matter. Numbers of bit strings

$$\mathbf{C(10,0)+C(10,1)+C(10,2)+C(10,3)+C(10,4)=386}$$

EXERCISE 6.3 (PAGE #: 435)

Q. 11. How many bit strings of length 10 contain

c) at least four 1s?

Solution: The order of 1s in the bit strings does not matter. Numbers of bit strings

$$\mathbf{C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10) = 848}$$

EXERCISE 6.3 (PAGE #: 435)

Q. 11. How many bit strings of length 10 contain

d) an equal number of 0s and 1s?

Solution: The order of 1s in the bit strings does not matter. Numbers of bit strings

$$\mathbf{C(10, 5) = 252}$$

EXERCISE 6.3 (PAGE #: 435)

Q. 12. How many bit strings of length 12 contain

a) exactly three 1s? $C(12,3)$

b) at most three 1s? $C(12,1)+C(12,2)+C(12,3)$

c) at least three 1s?

$$C(12,3)+C(12,4)+C(12,5)$$
$$+C(12,6)+C(12,7)+C(12,8)$$
$$+C(12,9)+C(12,10)+C(12,11)+C(12,12)$$

d) an equal number of 0s and 1s?

$$C(12,6)$$

EXERCISE 6.3 (*PAGE #: 435*)

Q. 13. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

Solution: We have, a group contains n men and n women. The order of the people is important (since we want to determine the number of different orders) So, we will use the permutations.

EXERCISE 6.3 (PAGE #: 435)

For women: we need to select $\boldsymbol{n}$ women from the $\boldsymbol{n}$ women.

So, $(\boldsymbol{n}, \boldsymbol{r}) = (\boldsymbol{n}, \boldsymbol{n})$, $n = n$ and $r = n$

$$\mathbf{P}(\boldsymbol{n}, \boldsymbol{n}) = \frac{\boldsymbol{n}!}{(\boldsymbol{n} - \boldsymbol{n})!} = \boldsymbol{n}!$$

For men: we apply the same rule.

$\mathbf{P}(\boldsymbol{n}, \boldsymbol{n}) = \boldsymbol{n}!$

EXERCISE 6.3 (*PAGE #: 435*)

Now, The possible arrangements alternation women and men are:

WMWMW...M or MWMWM...W

We use the product rule: $\boldsymbol{n}! \cdot \boldsymbol{n}! = (\boldsymbol{n}!)^2$ ways to arrange the women and the men in the pattern WMWMW...M.

Again, use product rule: $\boldsymbol{n!{\cdot}n!{=}(n!)^2}$ ways to arrange the women and the men in the pattern MWMWM...W.

EXERCISE 6.3 (PAGE #: 435)

Now, At last we use sum rule:

There are $(n!)^2 + (n!)^2 = 2(n!)^2$ ways to arrange the women and men.

EXERCISE 6.3 (PAGE #: 435)

Q. 14. In how many ways can a set of two positive integers less than 100 be chosen?

***Solution*:** There are 99 positive integers less than 100. We must choose set of two So the number of ways can be written as

$$C(99,2) = \frac{99!}{2!\,(99-2)!} = 4851$$

EXERCISE 6.3 (PAGE #: 435)

Q. 15. In how many ways can a set of five letters be selected from the English alphabet?

Solution: We apply the combination because in set the order does not matter.

We have to select 5 letters from 26 letters.

$$C(26,5) = \frac{26!}{5!\,(26-5)!} = 65,780$$

EXERCISE 6.3 (*PAGE #: 435*)

Q. 16. How many subsets with an odd number of elements does a set with 10 elements have?

Solution: Number of elements in a set = 10

Number of subsets with odd number of elements

$$C(10,1)+C(10,3)+C(10,5)+C(10,7)+C(10,9)=512$$

EXERCISE 6.3 (PAGE #: 435)

Q. 17. How many subsets with more than two elements does a set with 100 elements have?

Solution: Number of elements in a set = 100

Number of subsets of a set = $\mathbf{2^{100}}$

Number of subsets with almost two elements

$$\mathbf{C(100,0)+C(100,1)+C(100,2)=5051}$$

Number of subsets with more than two elements

$$\mathbf{= 2^{100} - 5051}$$

EXERCISE 6.3 (PAGE #: 435)

Q. 18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?

***Solution*:** Number of outcome a coin flipped one time = 2

Number of outcome a coin flipped eight times = $\mathbf{2^8}$

EXERCISE 6.3 (PAGE #: 435)

Q. 18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

b) contain exactly three heads?

Solution:

$$C(8,3) = 56$$

EXERCISE 6.3 (PAGE #: 435)

Q. 18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

c) contain at least three heads?

Solution:

$$C(8,3)+C(8,4)+C(8,5)$$
$$+C(8,6)+C(8,7)+C(8,8)=219$$

EXERCISE 6.3 (PAGE #: 435)

Q. 18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

d) contain the same number of heads and tails?

Solution:

$$\boldsymbol{C(8,4) = 70}$$

Permutation & Combination

Chapter Reading

Book: Kenneth H. Rosen,

Discrete Mathematics and Its Applications (Eight Edition)

Exercise for Practice

Section 6.3 Permutation & Combination

Question 19 to 27

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Week - 10 and Lecture – 1 & 2

Sequence & Summations

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MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 1

Sequence & Summations

An Introduction & Examples

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APPLICATIONS

- Very useful in counting problems.

- This is applicable in many fields like
  - ✓ Number Theory,
  - ✓ Probability,
  - ✓ Algorithms,
  - ✓ Geometry, etc.

SEQUENCE - DEFINITION

A ***sequence*** is a function from a subset of the set of integers (usually either the set {0, 1, 2,…} or the set {1, 2, 3,…}) to a set *S*.

We use the notation $\boldsymbol{a_n}$ to denote the image of the integer *n*.

We call $\boldsymbol{a_n}$ a *term* of the sequence.

SEQUENCE – EXAMPLE #: 1 (PAGE #: 165)

Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}.$$

The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots,$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

GEOMETRIC PROGRESSION - DEFINITION

A ***geometric progression*** is a sequence of the form $a, ar, ar^2, \ldots, ar^n, \ldots$

where the ***initial term a*** and the ***common ratio r*** are real numbers.

GEOMETRIC PROGRESSION – EXAMPLE #: 2 (PAGE #: 166)

The sequence

$\{b_n\}$ with $\boldsymbol{b_n = (-1)^n}$, with initial term and common ratio are 1 and -1 respectively is geometric progression if we start at $n = 0$.

The list of terms $b_0, b_1, b_2, b_3, b_4, \ldots$ begins with $1, -1, 1, -1, 1, \ldots;$

GEOMETRIC PROGRESSION – EXAMPLE #: 2 (PAGE #: 166)

The sequence

$\{c_n\}$ with $\boldsymbol{c_n = 2 \cdot 5^n}$, with initial term and common ratio are 2 and 5 respectively, is geometric progressions, if we start at $n = 0$.

The list of terms $c_0, c_1, c_2, c_3, c_4, \ldots$ begins with 2, 10, 50, 250, 1250,…;

GEOMETRIC PROGRESSION – EXAMPLE #: 2 (PAGE #: 166)

The sequence

$\{d_n\}$ with $\boldsymbol{d_n = 6 \cdot \left(\frac{1}{3}\right)^n}$ with initial term and common ratio are 6 and $\frac{1}{3}$ respectively is geometric progressions, if we start at $n = 0$.

The list of terms $d_0, d_1, d_2, d_3, d_4, \ldots$ begins with

$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$

ARITHMETIC PROGRESSION - DEFINITION

An ***arithmetic progression*** is a sequence of the form $a, a + d, a + 2d, \ldots, a + nd, \ldots$ where the ***initial term a*** and the ***common difference d*** are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function

$$f(x) = dx + a.$$

Arithmetic Progression – Example #: 3 (Page #: 166)

The sequence $\{s_n\}$ with $s_n = -1 + 4n$ is arithmetic progressions with initial terms and common differences equal to −1 and 4 respectively, if we start at $n = 0$.

The list of terms $s_0, s_1, s_2, s_3, \ldots$ begins with −1, 3, 7, 11,….

***ARITHMETIC PROGRESSION* – *EXAMPLE #: 3* (*PAGE #: 166*)**

The sequence $\{s_n\}$ with $\boldsymbol{t_n = 7 - 3n}$ is arithmetic progressions with initial terms and common differences equal to 7 and –3 respectively, if we start at $n = 0$.

The list of terms $t_0, t_1, t_2, t_3, \ldots$ begins with $7, 4, 1, -2, \ldots$.

STRINGS

Sequences of the form $a_1, a_2, \ldots, a_n$ are often used in computer science.

These finite sequences are also called ***strings***.

This string is also denoted by $a_1a_2\ldots a_n$.

STRINGS

The ***length*** of a string is the number of terms in this string.

The ***empty string***, denoted by λ, is the string that has no terms. The empty string has length zero.

STRINGS – *EXAMPLE #: 4* (*PAGE #: 166*)

The string ***abcd*** is a string of length four.

RECURRENCE RELATION

A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_0, $a_1, \ldots,$ a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a ***solution of a recurrence relation*** if its terms satisfy the recurrence relation.

Recurrence Relation – Example #: 5 (Page #: 167)

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$

for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$.

It then follows that $a_2 = 5 + 3 = 8$ and

$a_3 = 8 + 3 = 11$.

RECURRENCE RELATION – EXAMPLE #: 6 (PAGE #: 167)

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$

for $n = 2, 3, 4, \ldots,$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution: We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$ and

$a_3 = a_2 - a_1 = 2 - 5 = -3$. We can find a_4, a_5, and each successive term in a similar way.

Recurrence Relation – Initial Condition

The ***initial conditions*** for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect.

For instance, the initial condition in ***Example 5*** is $a_0 = 2$, and the initial conditions in ***Example 6*** are $a_0 = 3$ and $a_1 = 5$.

FIBONACCI SEQUENCE

We define a particularly useful sequence defined by a recurrence relation, known as the ***Fibonacci sequence***, after the Italian mathematician Fibonacci who was born in the 12th century.

FIBONACCI SEQUENCE

The ***Fibonacci sequence***, $f_0, f_1, f_2, \ldots,$ is defined by

the initial conditions $\boldsymbol{f_0 = 0}$, $\boldsymbol{f_1 = 1}$, and

the recurrence relation $\boldsymbol{f_n = f_{n-1} + f_{n-2}}$

for $\boldsymbol{n = 2, 3, 4, \ldots}$.

FIBONACCI SEQUENCE – EXAMPLE #: 7 (PAGE #: 168)

Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

Solution: The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previous two terms. Because the initial conditions tell us that $f_0 = 0$ and $f_1 = 1$, using the recurrence relation in the definition we find that

FIBONACCI SEQUENCE – EXAMPLE #: 7 (PAGE #: 168)

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

RECURRENCE RELATION – EXAMPLE #: 8 (PAGE #: 168)

Suppose that $\{a_n\}$ is the sequence of integers defined by $a_n = n!$, the value of the factorial function at the integer n, where $n = 1, 2, 3, \ldots$. Because

$$n! = n[(n-1)(n-2)\ldots 2 \cdot 1] = n(n-1)! = na_{n-1},$$

we see that the sequence of factorials satisfies the recurrence relation $a_n = na_{n-1}$, together with the initial condition $a_1 = 1$.

CLOSED FORMULA

We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a ***closed formula***, for the terms of the sequence.

CLOSED FORMULA – EXAMPLE #: 9 (PAGE #: 168)

Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$.

Solution: Suppose that $a_n = 3n$ for every nonnegative integer n. Then, for $n \geq 2$, we see that $2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n$. Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

CLOSED FORMULA – EXAMPLE #: 9 (PAGE #: 168)

Determine whether the sequence $\{a_n\}$, where $a_n = 2^n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$.

Solution: Suppose that $a_n = 2^n$ for every nonnegative integer n. Note that $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$. Because $2a_1 - a_0 = 2 \cdot 2^1 - 1 = 3 \neq a_2$, we see that $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

Closed Formula – Example #: 9 (Page #: 168)

Determine whether the sequence $\{a_n\}$, where $a_n = 5$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$.

Solution: Suppose that $a_n = 5$ for every nonnegative integer n. Then for $n \geq 2$, we see that $a_n = 2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$. Therefore, $\{a_n\}$, where $a_n = 5$, is a solution of the recurrence relation.

Closed Formula – Example #: 10 (Page #: 168)

Solve the recurrence relation and initial condition in recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$ and suppose that $a_0 = 2$.

Solution* (Method-I):** We can successively apply the recurrence relation, starting with the initial condition $a_1 = 2$, and working ***upward until we reach a_n to deduce a closed formula for the sequence. We see that

FORWARD SUBSTITUTION

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

$$\vdots$$

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1).$$

CLOSED FORMULA – EXAMPLE #: 10 (PAGE #: 168)

Solve the recurrence relation and initial condition in recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$ and suppose that $a_0 = 2$.

Solution* (Method-II):** We can also successively apply the recurrence relation, starting with the term a_n and working ***downward until we reach the initial condition $a_1 = 2$ to deduce this same formula. The steps are

BACKWARD SUBSTITUTION

$$a_n = a_{n-1} + 3$$
$$= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2$$
$$= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3$$
$$\vdots$$
$$= a_2 + 3(n - 2) = (a_1 + 3) + 3(n - 2) = 2 + 3(n - 1).$$

COMPOUND INTEREST – EXAMPLE #: 11 (*PAGE #: 169*)

Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Compound Interest – Example #: 11 (Page #: 169)

Solution: Let P_n denote the amount in the account after n years. we see that the sequence $\{P_n\}$ satisfies the recurrence relation. $\mathbf{P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}}$.

The initial condition is $P_0 = 10{,}000$.

We can use an iterative approach to find a formula for P_n. Note that

COMPOUND INTEREST – EXAMPLE #: 11 (PAGE #: 169)

$$P_1 = (1.11)P_0$$

$$P_2 = (1.11)P_1 = (1.11)^2 P_0$$

$$P_3 = (1.11)P_2 = (1.11)^3 P_0$$

$$\vdots$$

$$P_n = (1.11)P_{n-1} = (1.11)^n P_0.$$

Compound Interest – Example #: 11 (Page #: 169)

When we insert the initial condition $\mathbf{P_0=10,000}$, the formula $\boldsymbol{P_n = (1.11)^n 10,000}$ is obtained.

Inserting $n = 30$ into the formula

$P_n = (1.11)^n 10,000$ shows that after 30 years the account contains

$P_{30} = (1.11)^{30} 10,000 = \$228,922.97.$

Closed Formula – Example #: 12 (Page #: 170)

Find formulae for the sequences with the following first five terms:

(a) 1, 1⁄2, 1⁄4, 1⁄8, 1⁄16.

Solution: (a) We recognize that the denominators are powers of 2. The sequence with $a_n = 1/2^n$, $n = 0, 1, 2, \ldots$ is a possible match.

This proposed sequence is a geometric progression with $a = 1$ and $r = 1/2$.

Closed Formula – Example #: 12 (Page #: 170)

Find formulae for the sequences with the following first five terms:

(b) 1, 3, 5, 7, 9.

Solution: (b) We note that each term is obtained by adding 2 to the previous term. The sequence with $a_n = 2n + 1$, $n = 0, 1, 2, \ldots$ is a possible match.

This proposed sequence is an arithmetic progression with $a = 1$ and $d = 2$.

CLOSED FORMULA – EXAMPLE #: 12 (PAGE #: 170)

Find formulae for the sequences with the following first five terms:

(c) 1, −1, 1, −1, 1.

Solution: (c) The terms alternate between 1 and −1. The sequence with $a_n = (-1)^n$, $n = 0, 1, 2\ldots$ is a possible match.

This proposed sequence is a geometric progression with $a = 1$ and $r = -1$.

CLOSED FORMULA – EXAMPLE #: 13 (PAGE #: 170)

How can we produce the terms of a sequence if the first 10 terms are 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

***Solution*:** In this sequence, the integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and the integer 4 appears four times.

A reasonable rule for generating this sequence is that the integer n appears exactly ***n*** times, so the next five terms of the sequence would all be 5, the following six terms would all be 6, and so on. The sequence generated this way is a possible match.

Closed Formula – Example #: 14 (Page #: 170)

How can we produce the terms of a sequence if the first 10 terms are 5, 11, 17, 23, 29, 35, 41, 47, 53, 59?

***Solution*:** Note that each of the first 10 terms of this sequence after the first is obtained by adding 6 to the previous term.

Consequently, the nth term could be produced by starting with 5 and adding 6 a total of $n - 1$ times; that is, a reasonable guess is that the nth term is $5 + 6(n - 1) = 6n - 1$. (This is an arithmetic progression with $a = 5$ and $d = 6$.)

RECURRENCE RELATION

Some Useful Sequences

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, …
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, …
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, …

LUCAS SEQUENCE – EXAMPLE #: 15 (PAGE #: 171)

How can we produce the terms of a sequence if the first 10 terms are 1, 3, 4, 7, 11, 18, 29, 47,76, 123?

***Solution*:** Each successive term of this sequence, starting with the third term, is the sum of the two previous terms. That is, 4 = 3 + 1, 7 = 4 + 3, 11 = 7 + 4, and so on.

Consequently, if L_n is the nth term of this sequence, by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_1 = 1$ and $L_2 = 3$ (the same recurrence relation as the Fibonacci sequence, but with different initial conditions).

This sequence is known as the ***Lucas sequence***, after the French mathematician Francois Edouard Lucas.

RECURRENCE RELATION – EXAMPLE #: 16 (PAGE #: 172)

Conjecture a simple formula for *an* if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

Solution: It is reasonable to suspect that the terms of this sequence are generated by a formula involving 3^n. Comparing these terms with the corresponding terms of the sequence $\{3^n\}$, we notice that the nth term is 2 less than the corresponding power of 3.We see that $a_n = 3^n - 2$ for $1 \leq n \leq 10$ and conjecture that this formula holds for all n.

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Lecture – 2
Sequence & Summations

Exercise 2.4

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SUMMATIONS – EXAMPLE #: 17 (PAGE #: 173)

Use summation notation to express the sum of the first 100 terms of the sequence $\{aj\}$, where $aj = \frac{1}{j}$ for $j = 1, 2, 3, \ldots$.

Solution: The lower limit for the index of summation is 1, and the upper limit is 100. We write this sum as

$$\sum_{j=1}^{100} \frac{1}{j}$$

SUMMATIONS – EXAMPLE #: 18 (PAGE #: 173)

What is the value of

$$\sum_{j=1}^{100} j^2$$

Solution: We have

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$= 1 + 4 + 9 + 16 + 25 = 55$$

SUMMATIONS – EXAMPLE #: 19 (PAGE #: 173)

What is the value of

$$\sum_{j=4}^{8}(-1)^{k}$$

Solution: We have

$$\sum_{k=4}^{8}(-1)^{k} = (-1)^{4} + (-1)^{5} + (-1)^{6} + (-1)^{7} + (-1)^{8}$$

$$= 1 + (-1) + 1 + (-1) + 1 = 1$$

Sequence & Summations

SUMMATIONS – THEOREM (PAGE #: 174)

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

SUMMATIONS – EXAMPLE #: 21 (PAGE #: 175)

Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{j=1}^{4}\sum_{j=1}^{3} ij$$

SUMMATIONS – EXAMPLE #: 21 (PAGE #: 175)

Solution: To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\sum_{j=1}^{4}\sum_{j=1}^{3} ij = \sum_{j=1}^{4}(i + 2i + 3i) = \sum_{j=1}^{4} 6i$$

$$6 + 12 + 18 + 24 = 60$$

SUMMATIONS – EXAMPLE #: 22 (PAGE #: 175)

What is the value of

$$\sum_{s \in \{0,2,4\}} s$$

Solution: Because the above summation represents the sum of the values of *s* for all the members of the set {0, 2, 4}, it follows that

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6$$

Sequence & Summations

SOME USEFUL SUMMATION FORMULAE

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

SUMMATIONS – EXAMPLE #: 23 (PAGE #: 175)

Find

$$\sum_{k=50}^{100} k^2$$

Solution: First note that because

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$$

Sequence & Summations

SUMMATIONS – EXAMPLE #: 23 (PAGE #: 175)

$$\sum_{k=50}^{100} k^2 = \frac{1}{6}(100.101.201) - \frac{1}{6}(49.50.99)$$

$$= 338,350 - 40,425$$

$$= 297,925$$

EXERCISE 2.4 (PAGE #: 177)

1. Find these terms of the sequence $\{an\}$, where $an = 2 \cdot (-3)n + 5n$.

a) a_0 **b)** a_1 **c)** a_4 **d)** a_5

2. What is the term a_8 of the sequence $\{an\}$ if an equals

a) 2^{n-1}? **b)** 7?

c) $1 + (-1)^n$? **d)** $-(-2)^n$?

Sequence & Summations

EXERCISE 2.4 (PAGE #: 177)

3. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

a) $2^n + 1$? **b)** $(n + 1)^{n+1}$?

c) $\lfloor n/2 \rfloor$? **d)** $\lfloor n/2 \rfloor + \lceil n/2 \rceil$?

4. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

a) $(-2)^n$? **b)** 3?

c) $7 + 4^n$? **d)** $2^n + (-2)^n$?

EXERCISE 2.4 (PAGE #: 177)

5. List the first 10 terms of each of these sequences.

a) the sequence that begins with 2 and in which each successive term is 3 more than the preceding term

b) the sequence that lists each positive integer three times, in increasing order

EXERCISE 2.4 (PAGE #: 177)

c) the sequence that lists the odd positive integers in increasing order, listing each odd integer twice

d) the sequence whose nth term is $n! - 2n$

e) the sequence that begins with 3, where each succeeding term is twice the preceding term

EXERCISE 2.4 (PAGE #: 177)

f) the sequence whose first term is 2, second term is 4, and each succeeding term is the sum of the two preceding terms

g) the sequence whose *n*th term is the number of bits in the binary expansion of the number *n*

h) the sequence where the *n*th term is the number of letters in the English word for the index *n*

Exercise 2.4 (Page #: 177)

6. List the first 10 terms of each of these sequences.

a) the sequence obtained by starting with 10 and obtaining each term by subtracting 3 from the previous term

b) the sequence whose nth term is the sum of the first n positive integers

EXERCISE 2.4 (PAGE #: 177)

c) the sequence whose *n*th term is $3^n - 2^n$

d) the sequence whose *n*th term is $\lfloor \sqrt{n} \rfloor$

e) the sequence whose first two terms are 1 and 5 and each succeeding term is the sum of the two previous terms

f) the sequence whose *n*th term is the largest integer whose binary expansion has *n* bits

EXERCISE 2.4 (PAGE #: 177)

g) the sequence whose terms are constructed sequentially as follows: start with 1, then add 1, then multiply by 1, then add 2, then multiply by 2, and so on

h) the sequence whose *n*th term is the largest integer k such that $k! \leq n$

EXERCISE 2.4 (PAGE #: 177)

7. Find at least three different sequences beginning with the terms 1, 2, 4 whose terms are generated by a simple formula or rule.

8. Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

EXERCISE 2.4 (PAGE #: 177)

9. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}$, $a_0 = 2$

b) $a_n = a_{n-1}^2$, $a_1 = 2$

c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

d) $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$

e) $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

Sequence & Summations

EXERCISE 2.4 (PAGE #: 177)

10. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = -2a_{n-1}$, $a_0 = -1$

b) $a_n = a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$

c) $a_n = 3a_{n-1}^2$, $a_0 = 1$

d) $a_n = na_{n-1} + a_{n-2}^2$, $a_0 = -1$, $a_1 = 0$

e) $a_n = a_{n-1} - a_{n-2} + a_{n-3}$, $a_0 = 1$, $a_1 = 1$, $a_2 = 2$

EXERCISE 2.4 (PAGE #: 178)

18. A person deposits \$1000 in an account that yields 9% interest compounded annually.

a) Set up a recurrence relation for the amount in the account at the end of n years.

b) Find an explicit formula for the amount in the account at the end of n years.

c) How much money will the account contain after 100 years?

EXERCISE 2.4 (PAGE #: 178)

19. Suppose that the number of bacteria in a colony triples every hour.

a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

EXERCISE 2.4 (PAGE #: 178)

20. Assume that the population of the world in 2017 was 7.6 billion and is growing at the rate of 1.12% a year.

a) Set up a recurrence relation for the population of the world n years after 2017.

b) Find an explicit formula for the population of the world n years after 2017.

c) What will the population of the world be in 2050?

EXERCISE 2.4 (PAGE #: 178)

21. A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the nth month.

a) Set up a recurrence relation for the number of cars produced in the first n months by this factory.

b) How many cars are produced in the first year?

c) Find an explicit formula for the number of cars produced in the first n months by this factory.

EXERCISE 2.4 (PAGE #: 178)

22. An employee joined a company in 2017 with a starting salary of \$50,000. Every year this employee receives a raise of \$1000 plus 5% of the salary of the previous year.

a) Set up a recurrence relation for the salary of this employee n years after 2017.

b) What will the salary of this employee be in 2025?

c) Find an explicit formula for the salary of this employee n years after 2017.

Sequence & Summations

EXERCISE 2.4 (PAGE #: 179)

33. Compute each of these double sums.

a) $\sum_{i=1}^{2} \sum_{j=1}^{3} (i+j)$

b) $\sum_{i=0}^{2} \sum_{j=0}^{3} (2i+3j)$

c) $\sum_{i=1}^{3} \sum_{j=0}^{2} i$

d) $\sum_{i=0}^{2} \sum_{j=1}^{3} ij$

34. Compute each of these double sums.

a) $\sum_{i=1}^{3} \sum_{j=1}^{2} (i-j)$

b) $\sum_{i=0}^{3} \sum_{j=0}^{2} (3i+2j)$

c) $\sum_{i=1}^{3} \sum_{j=0}^{2} j$

d) $\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3$

Sequence & Summations

EXERCISE 2.4 (PAGE #: 179)

45. What are the values of the following products?

a) $\prod_{i=0}^{10} i$

b) $\prod_{i=5}^{8} i$

c) $\prod_{i=1}^{100} (-1)^i$

d) $\prod_{i=1}^{10} 2$

46. Express $n!$ using product notation.

47. Find $\sum_{j=0}^{4} j!$.

48. Find $\prod_{j=0}^{4} j!$.

Sequence & Summations

Chapter Reading

Book: Kenneth H. Rosen,

Discrete Mathematics and Its Applications (Eight Edition)

Exercise for Practice

Section 2.4 ***Sequence & Summations***

MAT401 - Discrete Structures
(Discrete Mathematics)
Fall 2022

Week - 11 and Lecture – 1 & 2

Algorithms

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ALGORITHMS

APPLICATIONS

- Very useful in Data Structures.

- This is applicable in many fields like
  - ✓ Programming
  - ✓ Solving a Problem

ALGORITHMS

APPLICATIONS

- In this lecture, we will introduce algorithms for two of the most important problems in computer science, searching for an element in a list and sorting a list so its elements are in some prescribed order, such as increasing, decreasing, or alphabetic.
    - ✓ searching for an element in a list and
    - ✓ sorting a list so its elements are in some prescribed order,

such as increasing, decreasing, or alphabetic.

ALGORITHMS - DEFINITION

An ***algorithm*** is a finite sequence of precise instructions for performing a computation or for solving a problem.

The term algorithm is a corruption of the name ***Al-Khowarizmi***, a mathematician of the ninth century

ALGORITHMS - DESCRIPTION

Algorithms are described using both ***English*** and an easily understood form of ***pseudocode***.

ALGORITHMS - PSEUDOCODE

Pseudocode is an intermediate step between an English language description of the steps of a procedure and a specification of this procedure using an actual programming language.

ALGORITHMS - PSEUDOCODE

Procedure Statements

The pseudocode for an algorithm begins with a procedure statement that gives the name of an algorithm, lists the input variables, and describes what kind of variable each input is.

ALGORITHMS - PSEUDOCODE

Procedure Statements

The statement

procedure *maximum (L: list of integers)*

is the first statement in the pseudocode description of the algorithm, which we have named maximum, that finds the maximum of a list ***L*** of integers.

ALGORITHMS - PSEUDOCODE

Assignments and Other Types of Statements

An assignment statement is used to assign values to variables. In an assignment statement the left-hand side is the name of the variable, and the right-hand side is an expression that involves constants, variables that have been assigned values, or functions defined by procedures. The right-hand side may contain any of the usual arithmetic operations.

ALGORITHMS - PSEUDOCODE

Assignments and Other Types of Statements

The symbol **:=** is used for assignments. Thus, an assignment statement has the form

variable := expression

For example, the statement

max := a

assigns the value of a to the variable max.

ALGORITHMS - PSEUDOCODE

Assignments and Other Types of Statements

A statement such as

x := largest integer in the list L

can also be used. This sets ***x*** equal to the largest integer in the list ***L***.

To translate this statement into an actual programming language would require more than one statement.

ALGORITHMS - PSEUDOCODE

Assignments and Other Types of Statements

Also, the instruction

interchange a and b

can be used to interchange a and b.

We could also express this one statement with several assignment statements, but for simplicity, we will often prefer this abbreviated form of pseudocode.

ALGORITHMS - PSEUDOCODE

Comments

The statements enclosed in curly **{ }** braces are not executed. Such statements serve as comments or reminders that help explain how the procedure works.

ALGORITHMS - PSEUDOCODE

Comments

For instance, the statement

{ x is the largest element in L }

can be used to remind the reader that at that point in the procedure the variable ***x*** equals the largest element in the list ***L***.

ALGORITHMS - PSEUDOCODE

Conditional Constructions

The simplest form of the conditional construction that we will use is

if *condition* ***then*** *statement*

or

if *condition* ***then***

block of statements

ALGORITHMS - PSEUDOCODE

Conditional Constructions

The condition is checked, and if it is true, then the statement or block of statements given is carried out.

ALGORITHMS - PSEUDOCODE

Conditional Constructions

The pseudocode ***if*** *condition* ***then***

statement 1

statement 2

statement 3

…

statement n

tells us that the statements in the block are executed sequentially if the condition is true.

ALGORITHMS - PSEUDOCODE

Conditional Constructions

We require the use of a more general type of construction. This is used when we wish to do one thing when the indicated condition is true, but another when it is false. We use the construction

if *condition* ***then*** *statement 1*

else *statement 2*

ALGORITHMS - PSEUDOCODE

Conditional Constructions

Sometimes, we require the use of an even more general form of a conditional. The general form of the conditional construction that we will use is

if *condition 1* ***then*** *statement 1*

else if *condition 2* ***then*** *statement 2*

else if *condition 3* ***then*** *statement 3*

...

else if *condition n* ***then*** *statement n*

else *statement n + 1*

ALGORITHMS - PSEUDOCODE

Loop Constructions

There are two types of loop construction in the pseudocode in this book. The first is the "for" construction, which has the form

for *variable := initial value* ***to*** *final value*

statement

or

for *variable := initial value* ***to*** *final value*

block of statements

where initial value and final value are integers.

ALGORITHMS - PSEUDOCODE

Loop Constructions

We can use the "for" loop construction to find the sum of the positive integers from 1 to n with the following pseudocode.

sum := 0

for *i := 1* ***to*** *n*

sum := sum +i

Also, the more general "for" statement, of the form

for *all elements with a certain property*

is used in this text.

ALGORITHMS - PSEUDOCODE

Loop Constructions

The second type of loop construction that we will use is the "while" construction. This has the form ***while*** *condition*

statement

or

while *condition*

block of statements

ALGORITHMS - PSEUDOCODE

Loop Constructions

When this construction is used, the condition given is checked, and if it is true, the statements that follow are carried out, which may change the values of the variables that are part of the condition.

ALGORITHMS - PSEUDOCODE

Loop Constructions

As an example, we can find the sum of the integers from 1 to n using the following block of pseudocode including a “while” construction.

$$sum := 0$$

$$\textbf{while } n > 0$$

$$sum := sum + n$$

$$n := n - 1$$

ALGORITHMS - PSEUDOCODE

Loops within Loops

Loops or conditional statements are often used within other loops or conditional statements.

ALGORITHMS - PSEUDOCODE

Using Procedures in Other Procedures

We can use a procedure from within another procedure (or within itself in a recursive program) simply by writing the name of this procedure followed by the inputs to this procedure. For instance,

max(L)

will carry out the procedure max with the input list L.

ALGORITHMS - PSEUDOCODE

Return Statements

We use a return statement to show where a procedure produces output. A return statement of the form

***return** x*

produces the current value of x as output. The output x can involve the value of one or more functions, including the same function under evaluation, but at a smaller value

ALGORITHMS - PSEUDOCODE

Return Statements

For instance, the statement

$$\textbf{return}\ f\,(n - 1)$$

is used to call the algorithm with input of $n - 1$.

This means that the algorithm is run again with input equal to $n - 1$.

ALGORITHMS - PSEUDOCODE

1. What is the difference between the following blocks of two assignment statements?

a* := *b	and	***b* := *c***
b* := *c		***a* := *b***

Explanation*:** After the first block is executed, ***a has been assigned the original value of ***b*** and ***b*** has been assigned the original value of ***c***,

whereas after the second block is executed, ***b*** is assigned the original value of ***c*** and ***a*** the original value of ***c*** as well.

ALGORITHMS - PSEUDOCODE

2. Give a procedure using assignment statements to interchange the values of the variables ***x*** and ***y***. What is the minimum number of assignment statements needed to do this?

ALGORITHMS - PSEUDOCODE

2. Answer:

Following is the algorithm to interchange the value of two variable ***x*** and ***y***.

step 1:Read the two integers ***x*** and ***y***.

step 2 :***t*** = ***x***

Step 3: ***x*** = ***y***

step 4: ***y*** = ***t***

The minimum number of assignment to do this is 3

ALGORITHMS - PSEUDOCODE

Explanation:

After reading two integer ***x*** and ***y***, create a variable "***t***" of integer type.

With the help of variable "***t***", we can swap the value of variable ***x*** and ***y***.

It requires 3 assignment to interchange the value.

ALGORITHMS - PSEUDOCODE

3. Show how a loop of the form

for *i* := *initial value* **to** *final value*

statement

can be written using the "**while**" construction.

***Alternative Method*:**

The following construction does the same thing.

i := *initial value*

while $i \leq$ *final value*

statement

ALGORITHMS – EXAMPLE #: 1 (PAGE #: 202)

Describe an algorithm for finding the maximum (largest) value in a finite sequence of integers.

SOLUTION: We perform the following steps.

1. Set the temporary maximum equal to the first integer in the sequence. ***(The temporary maximum will be the largest integer examined at any stage of the procedure.)***

ALGORITHMS – EXAMPLE #: 1 (PAGE #: 202)

2. Compare the next integer in the sequence to the temporary maximum, and if it is larger than the temporary maximum, set the temporary maximum equal to this integer.

3. Repeat the previous step if there are more integers in the sequence.

4. Stop when there are no integers left in the sequence. The temporary maximum at this point is the largest integer in the sequence.

ALGORITHMS – EXERCISE 3.1 (PAGE #: 213)

1. List all the steps used by algorithm to find the maximum of the list ***1, 8, 12, 9, 11, 2, 14, 5, 10, 4***.

Solution: $max := 1,$

$i := 2, max := 8,$

$i := 3, max := 12,$

$i := 4,$

$i := 5,$

$i := 6,$

ALGORITHMS – EXERCISE 3.1 (PAGE #: 213)

1. List all the steps used by algorithm to find the maximum of the list ***1, 8, 12, 9, 11, 2, 14, 5, 10, 4***.

Solution: $max := 1,$

$i := 2, max := 8,$

$i := 3, max := 12,$

$i := 4,$

$i := 5,$

$i := 6,$

$i := 7, max := 14,$

$i := 8,$

$i := 9,$

$i := 10,$

$i := 11$

ALGORITHMS – *EXERCISE 3.1* (*PAGE #: 213*)

2. Determine which characteristics of an algorithm described in the text the following procedures have and which they lack.

a) **procedure** *double*(n: positive integer)

$$\textbf{while } n > 0$$

$$n := 2n$$

Explanation: The pseudocode for doubling a positive integer: procedure double (n : positive integer) while $n > 0$ $n := 2n$. This is simply an infinite loop that has no output. It certainly is a finite set of precise instructions, but it's completely useless since it doesn't solve any problem.

Algorithms – Exercise 3.1 (Page #: 213)

b) **procedure** *divide*(n: positive integer)

$$\textbf{while } n \geq 0$$

$$m := 1/n$$

$$n := n - 1$$

***Explanation*:** The variable m isn't declared, but that's not a problem. We might take the final value of m to be the output, since that final value is 1. At least it's not an infinite loop.

ALGORITHMS – *EXERCISE 3.1* (*PAGE #: 213*)

c) **procedure** *sum*(*n*: positive integer)

$$sum := 0$$

while $i < 10$

$$sum := sum + i$$

***Explanation*:** More bugs. The variable *n* is never used, and the variable *i* is never initialized. Note that *i* is never incremented either, so this is also an infinite loop. We can assume sum was intended to be the output. Since it isn't clear what the problem to be solved is supposed to be, we can't tell to what degree this succeeds or fails to achieve the solution.

ALGORITHMS – *EXERCISE 3.1* **(*PAGE #: 213*)**

d) **procedure** *choose*(*a*, *b*: integers)

$x :=$ either a or b

Explanation: This fails to be definite because of the choice. As a "nondeterministic" algorithm, it is interesting. They all have inputs from specified sets. The question of correctness can't even be considered since we don't know what they're supposed to do. The question of generality also can't be considered as we don't know what the purposes are.

ALGORITHMS

Chapter Reading

Book: Kenneth H. Rosen,

Discrete Mathematics and Its Applications (Eight Edition)

Exercise for Practice

Section 3.1 ***Algorithms***