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In this lecture we will discuss about Roll’s Theorem, Cauchy mean value theorem, Lagrange’s Mean Value Theorem and power series

Roll's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b) = 0,$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

Proof:

There are two cases arise

Case: I

$If\ f(x) = k$ (1)

$f(a) = k$

$f(b) = k$

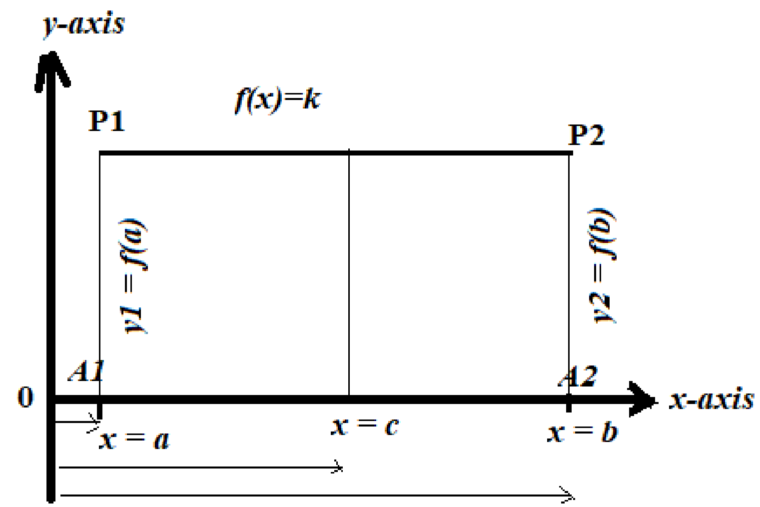
$\Rightarrow\ f(a) = f(b)$

From (1)

$f'(x) = 0$

$at\ x = c$

$f'(c) = 0$ *Hence proved*



Case: II

Consider a function which is continuous on [a, b] and differentiable (a, b) where $f(a) = f(b)$.

From Fig.

$f(c-h) - f(c) \leq 0$ (2)

Dividing both sides by –h and taking limit as $h \to 0$ *we have*

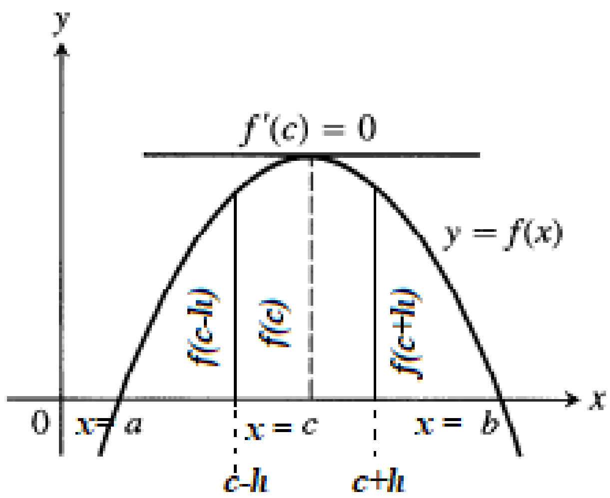
$$\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h} \geq 0$$
$\Rightarrow f'(c) \geq 0$

$\Rightarrow f'(c) > 0, f'(c) = 0 \qquad (A)$

Again from Fig.

$f(c+h) - f(c) \leq 0 \qquad (3)$

Dividing both sides by h and taking limit as $h \to 0$ *we have*



$$\lim_{h\to 0}\frac{f(c-h)-f(c)}{h} \leq 0$$
$i.e. f'(c) < 0, f'(c) = 0 \qquad (B)$

From (A) and (B) $f'(c) > 0$ and $f'(c) < 0$ at a time it is not possible i.e. only possible is

$$f'(c) = 0$$

Exercise

Discuss the validity of Roll's theorem. Find value of c (whenever possible).

1. $f(x) = x^2 - 3x + 2$ on $[1, 2]$
2. $f(x) = \sin^2 x$ on $[0, \pi]$
3. $f(x) = x^2 - 7x + 12$ on $[3, 4]$
4. $f(x) = x^2 + 2x - 1$ on $[0, 1]$

1. Solution:

$f(x) = x^2 - 3x + 2 \qquad (1)$

$Now, at\ x = 1$

$(1) \Rightarrow f(1) = 1^2 - 3(1) + 2 = 3 - 3 = 0$

$at\ x = 2$

$(1) \Rightarrow f(2) = 2^2\ - 3(2) + 2 = 4 - 6 + 2 = 6 - 6 = 0$

$\Rightarrow f(1) = f(2)$

$Hence, Roll's\ theorem\ is\ valid. Therefore, by\ Roll's\ theorem$

$f'(c) = 0 \qquad\qquad (2)$

$From\ (1) \quad f'(x) = 2x - 3$

$at\ x = c, \quad f'(c) = 2c - 3 \qquad (3)$

$Comapring\ (2) and\ (3)$

$2c - 3 = 0$

$\Rightarrow 2c = 3 \quad \Rightarrow c = \frac{3}{2} = 1.5$

2. Solution:

$f(x) = sin^2 x$ (1)

$Now, at\ x = 0$

$(1) \Rightarrow f(0) = Sin^2(0) = 0$

$at\ x = \pi$

$(1) \Rightarrow f(\pi) = sin^2(\pi) = 0$

$\Rightarrow f(0) = f(\pi)$

$Hence, Roll's\ theorem\ is\ valid. Therefore, by\ Roll's\ theorem$

$f'(c) = 0$ (2)

$From\ (1)$ $f'(x) = 2sinxcosx = \sin(2x)$

$at\ x = c,$ $f'(c) = \sin(2c)$ (3)

$Comapring\ (2) and\ (3)$

$\sin(2c) = 0$

$\Rightarrow 2c = 0, \pi \ \Rightarrow c = \frac{\pi}{2}$

3. Solution:

$f(x) = x^2 - 7x + 12 \qquad (1)$

$Now, at\ x = 3$

$(1) \Rightarrow f(3) = 3^2 - 7(3) + 12 = 9 - 21 + 12 = 0$

$at\ x = 4$

$(1) \Rightarrow f(4) = 4^2 \ - 7(4) + 12 = 16 - 28 + 12 = 0$

$\Rightarrow f(3) = f(4)$

$Hence, Roll's\ theorem\ is\ valid. Therefore, by\ Roll's\ theorem$

$f'(c) = 0 \qquad (2)$

$From\ (1) \quad f'(x) = 2x - 7$

$at\ x = c, \quad f'(c) = 2c - 7 \qquad (3)$

$Comapring\ (2) and\ (3)$

$2c - 7 = 0$

$\Rightarrow 2c = 7 \quad \Rightarrow c = \frac{7}{2} = 3.5$

4. Solution:

$f(x) = x^2 + 2x - 1 \qquad (1)$

$Now, at\ x = 0$

$(1) \Rightarrow f(0) = 0^2 + 2(0) - 1 = -1$

$at\ x = 1$

$(1) \Rightarrow f(1) = 1^2 + 2(1) - 1 = 3 - 1 = 2$

$\Rightarrow f(0) \neq f(1)$

$Hence, Roll's\ theorem\ is\ not\ valid.$

Lagrange's Mean Value Theorem

Statement: *Let $f(x)$ be a function such that*

(i) $f(x)$ *is continuous on* $[a, b], b > a$

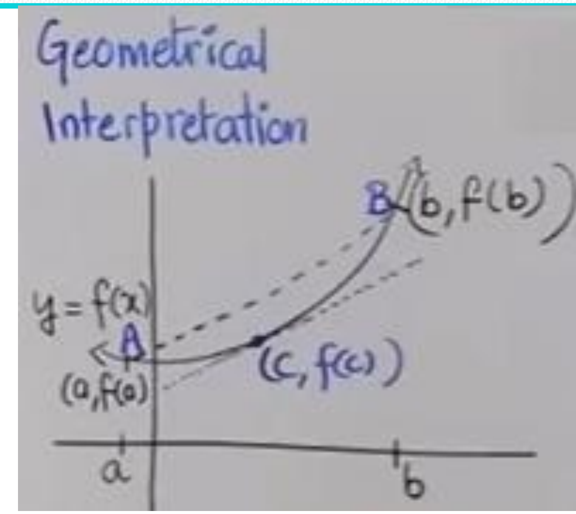
(ii) $f(x)$ *is differentiable on* $]a, b[$

Such that there must be a number $c \in (a, b)$ *such that*

$$\boldsymbol{f(b) - f(a) = f'(c)(b - a)}$$

Or

$$\boldsymbol{\frac{f(b)-f(a)}{b-a} = f'(c)}$$



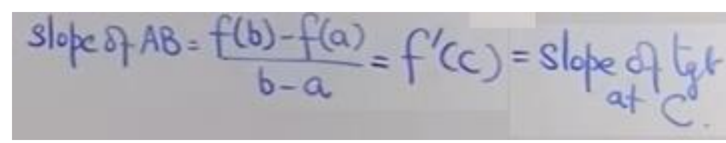
Proof:

Let $f(x)$ is a continuous function on $[a, b]$ and differentiable on (a, b) then $\emptyset(x)$ be a new function which is also continuous on $[a, b]$ and differentiable on (a, b) and this can be written as

$$\emptyset(x) = Mf(x) + Nx \qquad (1)$$

Where M and N are constants

By Roll's theorem



$$\boldsymbol{\emptyset(a) = \emptyset(b)}$$

$$\boldsymbol{\Rightarrow Mf(a) + Na = Mf(b) + Nb}$$

$$\Rightarrow Mf(a) - Mf(b) = Nb - Na$$
$$\Rightarrow M[f(a) - f(b)] = (b - a)N$$
$$\Rightarrow -M[f(b) - f(a)] = (b - a)N$$
$$\Rightarrow \frac{f(b) - f(a)}{b - a} = -\frac{N}{M} \qquad (2)$$

Again by Roll's theorem

$$\emptyset'(c) = 0$$

From (1)

$$\emptyset'(x) = Mf'(x) + N$$
$$put\ x = c$$
$$\Rightarrow \emptyset'(c) = Mf'(c) + N\ but\ \emptyset'(c) = 0$$
$$\Rightarrow Mf'(c) + N = 0$$
$$\Rightarrow f'(c) = -\frac{N}{M} \qquad (3) Comparing\ (2) and\ (3)$$
$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$Example$: 1 $If\ f(x) = x^3\ - 3x - 1$ on $\left[-\frac{11}{7}, \frac{13}{7}\right]$. $Find\ the\ value\ of\ C\ by\ LMVT$.

Solution:

$f(x) = x^3\ - 3x - 1 \qquad\qquad (1)$

Put $x = -\frac{11}{7}\ \ in\ (1)$

(1)=> $f\left(-\frac{11}{7}\right) = \left(-\frac{11}{7}\right)^3 - 3\left(-\frac{11}{7}\right) - 1 = -0.17$

Put $x = \frac{13}{7}\ in(1)$

(1)=>$f\left(\frac{13}{7}\right) = \left(\frac{13}{7}\right)^3 - 3\left(\frac{13}{7}\right) - 1 = -0.17$

Now, differentiate (1) w.r.t. x

$f'(x) = 3x^2\ - 3$

$at\ x = c, f'(c) = 3c^2\ - 3,$

By LMVT

$$\boldsymbol{f'(c)} = \frac{\boldsymbol{f(b) - f(a)}}{\boldsymbol{b - a}} = \frac{(-0.17) - (-0.17)}{\frac{13}{7} - \left(-\frac{11}{7}\right)} = \frac{0}{\frac{13}{7} + \frac{11}{7}} = 0$$

$\Rightarrow\ 3c^2\ -3 = 0,$

$\Rightarrow 3c^2 = 3,$

$\Rightarrow c^2 = 1,$

$\Rightarrow c = \pm 1$

Practice Problem: $If\ f(x) = x^3\ - 5x^2 + 4x - 2$ on $[2,4]$. $Find\ the\ value\ of\ C\ by\ LMVT$

Practice Problem: $If\ f(x) = x^3 - 5x^2 + 4x - 2$ on $[2,4]$. $Find\ the\ value\ of\ C\ by\ LMVT$

Solution:

$f(x) = x^3 - 5x^2 + 4x - 2 \qquad (1)$

Put $x = 2\ in\ (1)$

(1)=> $f(2) = (2)^3 - 5(2)^2 + 4(2) - 2 = 8 - 20 + 8 - 2 = -6$

Put $x = 4\ in (1)$

(1)=>$f(4) = (4)^3 - 5(4)^2 + 4(4) - 2 = 64 - 80 + 16 - 2 = -2$

Now, differentiate (1) w.r.t. x

$f'(x) = 3x^2 - 10\mathrm{x} + 4$

$at\ x = c, f'(c) = 3c^2 - 10\mathrm{c} + 4,$

By LMVT

$$\boldsymbol{f'(c) = \frac{f(b) - f(a)}{b - a}} = \frac{(-2) - (-6)}{4 - 2} = \frac{4}{2} = 2$$

$\Rightarrow\ 3c^2\ -10c + 4 = 2,$

$\Rightarrow 3c^2 - 10c + 2 = 0,$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{(4)^2 - 4(3)(2)}}{2(3)}$$

$$c = \frac{-4 \pm \sqrt{16 - 24}}{6} = \frac{-4 \pm \sqrt{-8}}{6} == \frac{-4 \pm 2\sqrt{2}i}{6} = \frac{-2 \pm \sqrt{2}i}{3}$$

Practice Problem: $If\ f(x) = x^2 - 7x^2 + 12$ on $[3,4]$. $Find\ the\ value\ of\ C$.

Cauchy Mean Value Theorem

Statement:

$\boldsymbol{Let\ f(x)\ and\ g(x) be\ two\ function\ that\ satisfying}$
$\boldsymbol{following\ conditions.}$

$\boldsymbol{(i) f(x)\ and\ g(x) both\ are\ \ continuous\ on\ [a, b],}$

$\boldsymbol{(ii) f(x)\ and\ g(x)\ both\ are\ differentiable\ on\ ]a, b[}$

$\boldsymbol{(iii) g'^{(x)} \neq 0, \quad \forall\ x\ then}$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Example:1 $If\ f(x) = e^x\ and\ g(x) = e^{2x}\ on\ [0, ln2].$

$Find\ value\ of\ C\ by\ Cauchy\ mean\ value\ theorem.$

Solution:

$f(x) = e^x \quad (1)$

$g(x) = e^{2x} \quad (2)$

Since, by Cauchy mean value theorem

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \qquad (3)$$

$at\ x = a = 0,$

$(1) \Rightarrow f(0) = e^0 = 1, \qquad (2) \Rightarrow g(0) = e^0 = 1$

$at\ x = b = ln2,$

$(1) \Rightarrow f(ln2) = e^{ln2} = 2, \qquad (2) \Rightarrow g(ln2) = e^{2ln2} = e^{ln4} = 4$

Again from equation (1) and (2)

$(1) \Rightarrow f'(x) = e^x, \qquad (2) \Rightarrow g'(x) = 2e^{2x}$

At $x = c,$

$f'(c) = e^c, \qquad g'(x) = 2e^{2c}$

Putting all above values in (3)

$$(3) \Rightarrow \frac{e^c}{2e^{2c}} = \frac{2-1}{4-1} = \frac{1}{3} \Rightarrow \frac{1}{2e^c} = \frac{1}{3} \Rightarrow 2e^c = 3 \Rightarrow c = ln\left(\frac{3}{2}\right)$$

Example:2 $If\ f(x) = x^2\ and\ g(x) = x^3.$

$Verify, Cauchy\ mean\ value\ theorem\ on\ [1, 2].$

$Also\ find\ value\ of\ C.$

Solution:

$f(x) = x^2 \quad (1)$

$g(x) = x^3 \quad (2)$

Since, by Cauchy mean value theorem

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \qquad (3)$$

$at\ x = a = 1,$

$(1) \Rightarrow f(1) = (1)^2 = 1,$ $\qquad (2) \Rightarrow g(1) = (1)^3 = 1$

$at\ x = b = 2,$

$(1) \Rightarrow f(2) = (2)^2 = 4,$ $\qquad (2) \Rightarrow g(2) = (2)^3 = 8$

Again from equation (1) and (2)

$(1) \Rightarrow f'(x) = 2x,$ $\qquad (2) \Rightarrow g'(x) = 3x^2$

At $x = c,$

$f'(c) = 2c,$ $\qquad g'(x) = 3c^2$

Putting all above values in (3)

$$(3) \Rightarrow \frac{2c}{3c^3} = \frac{4-1}{8-1} = \frac{3}{7} \Rightarrow \frac{1}{c^2} = \frac{9}{14} \Rightarrow c^2 = \frac{14}{9} \Rightarrow c = \sqrt{\left(\frac{14}{9}\right)}$$

Power Series :-

A series $\sum_{n=0}^{\infty} C_n (x-\alpha)^n = C_0 + C_1(x-\alpha) + C_2(x-\alpha)^2 + C_3(x-\alpha)^3 + \cdots\cdots$ is a power series around α and C_n's are called coefficients.

The series $\sum_{n=0}^{\infty} C_n x^n$ is a power series around '0'.

e.g. $\sum_{n=0}^{\infty} n^2 x^n$, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=0}^{\infty} n^3 (x-1)^n$ etc.

Power Series

The equation of the form

$$\sum_{n=0}^{n} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots,$$

is called a power series.

Consider,

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots \qquad (1)$$

then

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots.$$

$$f''(x) = 2\cdot 1a_2 + 3\cdot 2a_3(x-a) + 4.3a_4(x-a)^2 + \cdots.$$

$$f'''(x) = 3!\,a_3 + 4!\,a_4(x-a) + \cdots.$$

$\vdots$

Put $x = a$,

$$f(a) = a_0$$

$$f'(a) = a_1$$

$$f''(a) = 2!\,a_2 \;\; or \; a_2 = \frac{f''(a)}{2!}$$

$$f'''(a) = 3!\,a_3 \;\; or \; a_3 = \frac{f'''(a)}{3!}$$

$\vdots$

$Putting\ these\ values\ of\ a_0, a_1, a_2, \dots in\ eq.(1)$

$$(1) \Rightarrow\ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \qquad (2)$$

$This\ equation\ is\ called\ Taylor\ series\ for\ f(x)\ at\ x = a.$

$Put\ a = 0\ in\ equation\ (2) then$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \qquad (3)$$

$The\ equation\ (3) is\ called\ \boldsymbol{Machaurin\ series}\ for\ f(x).$

$Example: 3\ Find\ Machaurian's\ series\ for\ f(x) = e^x.$

$Solution:$

$The\ Machaurin's\ series\ f(x)\ is$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \qquad (1)$$

$Here, \qquad f(x) = e^x$

$Now,$

$f(x) = e^x \qquad \Rightarrow f(0) = 1$

$f'(x) = e^x \qquad \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \qquad \Rightarrow f''(0) = e^0 = 1$

$\vdots$

$$(1) \Rightarrow e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$e^x = \sum_{n=0}^{n} \frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Practice Problems: Find Manchurian series for

$(i) cosx$

$(ii) \frac{1}{1-x}$

Q1 Apply maclaurin's series to prove That

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$

Sol $f(x) = \ln(1+x)$, $f(0) = \ln(1+0) = \ln 1 = 0$

$\boxed{f(0)=0}$

Diff w.r.t x

$f'(x) = \frac{1}{1+x}\frac{d}{dx}(1+x)$

$f'(x) = \frac{1}{1+x}$, $\boxed{f'(0)=1}$

$f'(x) = (1+x)^{-1}$ Diff w.r.t. x

$f''(x) = (-1)(1+x)^{-2}(0+1)$

$f''(x) = (-1)(1+x)^{-2}$, $f''(0) = -1(1+0)^{-2} = -1$

$\boxed{f''(0)=-1}$

Diff again w.r.t. x

$f'''(x) = (-1)(-2)(1+x)^{-3}(0+1)$, $f'''(0) = 2(1+0)^{-3}$

$= 2(1+x)^{-3}$

$\boxed{f'''(0)=2}$

Diff aga w.r.t. x

$f^{(4)}(x) = 2(-3)(1+x)^{-4}$

$= -6(1+x)^{-4}$, $f^{(4)}(0) = -6(1+0)^{-4}$

$\boxed{f^{(4)}(0) = -6}$

As we Know By Maclaurin's Series

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots$

But $f(x) = \ln(1+x)$

$\ln(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots\cdots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$

$\frac{2}{3!} = \frac{\cancel{2}}{3\cdot\cancel{2}\cdot 1}$

$\frac{6}{4!} = \frac{\cancel{6}}{4\cdot\cancel{3}\cdot\cancel{2}\cdot 1}$

Apply maclaurin's series to prove That

(i) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots\cdots$

Sol $f(x) = \ln(1-x)$, $f(0) = \ln(1-0) = \ln 1 = 0$

$\boxed{f(0)=0}$

Diff w.r.t x

$f'(x) = \frac{1}{1-x}\frac{d}{dx}(1-x)$

$f'(x) = \frac{-1}{1-x}$, $\boxed{f'(0) = -1}$

$f'(x) = -(1-x)^{-1}$ Diff w.r.t. x

$f''(x) = (+1)(1-x)^{-2}(0-1)$

$f''(x) = (-1)(1-x)^{-2}$, $f''(0) = -1(1-0)^{-2} = -1$

$\boxed{f''(0) = -1}$

Diff again w.r.t. x

$f'''(x) = (-1)(-2)(1-x)^{-3}(0-1)$

$= -2(1-x)^{-3}$

$f'''(0) = -2(1-0)^{-3}$

$\boxed{f'''(0) = -2}$

Diff again w.r.t. x

$f^{(4)}(x) = -2(-3)(1-x)^{-4}(0-1)$

$= -6(1-x)^{-4}$, $f^{(4)}(0) = -6(1-0)^{-4}$

$\boxed{f^{(4)}(0) = -6}$

As we Know By Maclaurin's Series

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \cdots$

But $f(x) = \ln(1-x)$

$\ln(1-x) = 0 + x(-1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(-6) + \cdots$

$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \cdots$$

$$\cos x = \cdots$$

$$\ln(1+x) = \cdots$$

$$\ln(1-x) = \cdots$$

$Example: 1\ Find\ Taylor's\ series\ for\ sinx\ at\ x = \frac{\pi}{2}.$

$Solution:$

$The\ Talor's\ series\ at\ x = a\ is$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \quad (1)$$

$Here, \quad f(x) = sinx \quad and\ a = \frac{\pi}{2}$

$Now,$

$f(x) = sinx \qquad \Rightarrow f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = cosx \qquad \Rightarrow f'\left(\frac{\pi}{2}\right) = 0$

$f''^{(x)} = -sinx \qquad \Rightarrow f''^{\left(\frac{\pi}{2}\right)} = -1$

$f'''(x) = -cosx \qquad \Rightarrow f'''\left(\frac{\pi}{2}\right) = 0$

$$(1) \Rightarrow sinx = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \cdots$$

$Example: 2\ Find\ Taylor's\ series\ for\ f(x) = \dfrac{1}{\sqrt{x}}\ at\ x = 9.$

$Solution:$

$The\ Talor's\ series\ at\ x = a\ is$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \quad (1)$$

$Here, \qquad f(x) = \dfrac{1}{\sqrt{x}} \quad and\ a = 9$

$Now,$

$$f(x) = \frac{1}{\sqrt{x}} \qquad \Rightarrow f(9) = \frac{1}{3}$$

$$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} \qquad \Rightarrow f'(9) = -\frac{1}{2}(9)^{-\frac{3}{2}} = \frac{1}{2\cdot 3^3}$$

$$f''(x) = \frac{3}{4}x^{-\frac{5}{2}} \qquad \Rightarrow f''(9) = \frac{3}{4}(9)^{-\frac{5}{2}} = \frac{3}{2^2\cdot 3^5}$$

$$f'''(x) = -\frac{15}{8}x^{-\frac{7}{2}} \qquad \Rightarrow f'''^{(9)} = -\frac{15}{8}(9)^{-\frac{7}{2}} = -\frac{15}{2^3\cdot 3^7}$$

$$(1) \Rightarrow \frac{1}{\sqrt{x}} = \frac{1}{3} - \frac{1}{2\cdot 3^3}(x-9) + \frac{3}{2^2\cdot 2!\cdot 3^5}(x-9)^2 - \frac{15}{2^3\cdot 3!\cdot 3^7}(x-9)^3 + \cdots$$

TAILOR THEOREM

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + \cdots$$

Q2 Show That $\cos(x+h) = \cos x - h \sin x - \frac{h^2}{2!} \cos x + \frac{h^3}{3!} \sin x + \cdots$ and Evaluate $\cos 61^\circ$

Sol

$f(x) = \cos x$, $f(x+h) = \cos(x+h)$

Diff w.r.t. x

$f'(x) = -\sin x$

Diff w.r.t. x

$f''(x) = -\cos x$

Diff again w.r.t x

$f'''(x) = -(-\sin x) = \sin x$

As we know $f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$

$$\cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{2!}(-\cos x) + \frac{h^3}{3!}(\sin x) + \cdots$$

$$\cos(x+h) = \cos x - h \sin x - \frac{h^2}{2!} \cos x + \frac{h^3}{3!} \sin x + \cdots$$

Let $x = 60^\circ$, $h = 1^\circ = 0.01745$

$$\cos(60^\circ + 1^\circ) = \cos 60^\circ - (0.01745) \sin 60^\circ - \frac{(0.01745)^2}{2} \cos 60^\circ + \cdots$$

$$\cos 61^\circ \approx .4848$$

$\cos 59^\circ$
$\cos 31^\circ$
$\cos 29^\circ$
$\cos 46^\circ$
$\cos 44^\circ$

EXERCISE NO 2.8

Q3 Show that: $2^{x+h} = 2^x\left[1 + h(\ln 2) + \frac{h^2}{2!}(\ln 2)^2 + \frac{h^3}{3!}(\ln 2)^3 + \cdots\right]$

Sol

$f(x) = 2^x$, $f(x+h) = 2^{x+h}$

Diff w.r.t. x

$f'(x) = \frac{d}{dx}(2^x)$ using $\frac{d}{dx}(a^x) = a^x \cdot \ln a$

$f'(x) = 2^x \cdot \ln 2$

Diff w.r.t. x

$f''(x) = \ln 2 \frac{d}{dx} 2^x$

$= \ln 2 \cdot (2^x) \cdot \ln 2$

$f''(x) = (\ln 2)^2 \cdot 2^x$

Diff w.r.t. x

$f'''(x) = (\ln 2)^2 \cdot 2^x \cdot \ln 2$

$= (\ln 2)^3 \cdot 2^x$

$\vdots$

By Taylor Theorem

$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$

$2^{x+h} = 2^x + h(2^x \ln 2) + \frac{h^2}{2!}(\ln 2)^2 2^x + \frac{h^3}{3!}(\ln 2)^3 2^x + \cdots$

$= 2^x\left[1 + h(\ln 2) + \frac{h^2}{2!}(\ln 2)^2 + \frac{h^3}{3!}(\ln 2)^3 + \cdots\right]$