Predicate Logic

Predicate Logic

- Used for representation of knowledge expressible in natural language
- We will be limiting ourselves to first order predicate logic
- This requires introduction of
  - A notion of constants
  - A notion of variables
  - A notion of quantifiers $\forall$ and $\exists$
  - A notion of function
  - A notion of function name

- Constants – basically identifiable individual symbols from a given domain, e.g., 3, Red
- Variables – which take values from a given domain, e.g., x,y
- Function symbols – which take variables/constants as parameters and evaluate to a value in a domain, e.g.,
  - father(kamal)
  - half-of(x)
  - dist(x,y,u,v)   n parameters means n-place function
- Predicate symbols – which takes a set of constants as parameters and evaluate/map to a value true or false. (n parameters means n-place function)

Definitions

- Term
  - A constant is a term
  - A variable is a term
  - If f is a n-place function and $t_1, t_2, \ldots, t_n$ are terms, then $f(t_1, t_2, \ldots, t_n)$ is a term
  - All terms are generated using the above rules

- EX:

plus(plus-one(x),x) is a 2-place function with each parameter in it as term. Hence it is a term.

Definitions

- Atom
  - Like in propositional logic where we had a notion of atom as a declarative statement
    - Which was not decomposable any further
    - Which has a value T or F but not both

We have a notion of atom in predicate logic also.

An <u>atom</u> is any n-place predicate symbol. If t_1, t_2, …, t_n are terms, then $P(t_1, t_2, \ldots, t_n)$ is a atom

Definitions

- Quantifiers - ∀ and ∃ are symbols for universal and existential quantifiers.

$\forall x(Q(x) \rightarrow P(x))$ For all x Q implies P

notion of interpretation, i.e., for every

assignment of variable x from the given domain

Note that the scope of the variable x extends over to P also, i.e., scope of x extends over to $Q(x) \rightarrow P(x)$

Bound and free variables

- An occurrence of a variable in formula is bound if and only if its occurrence is within the scope of a quantifier
- An occurrence of a variable in a formula is free if and only if there is at least one occurrence of it which is not bound in the formula.

- $\forall x(P(x, y))$ x is bound, y is free
- $\forall x(P(x, y) \land \forall y Q(x, y))$ x is bound, y is both bound and free

Well formed formula (WFF)

- Well formed formula in First Order Predicate Logic (FOPL) is recursively defined as follows:
  - An atom is a formula
  - If F and G are formulas, then $\sim (F)$, $(F \vee G)$, $(F \wedge G)$, $F \longrightarrow G$, $F \leftrightarrow G$ are formulas
  - If F is a formula and x is a free variable in F, the $\forall x F$ and $\exists x F$ are formulas
  - Formulas are generated by repeated application of the above rules.

Example: Axioms of natural numbers

A1: For every number, there is one and only one immediate successor.

A2: There is no number for which 0 is successor.

A3: For every number other than 0, there is one and only one immediate predecessor.

Let

$f(x)$ represents successor of x

$g(x)$ represents predecessor of x

$e(x, y)$ represents equals predicate

Axioms of natural numbers in FOPL

A1: For every number, there is one and only one immediate successor.

$$\forall x(\exists y\left(e\big(y,f(x)\big)\wedge\forall z\left(e\big(z,f(x)\big)\rightarrow e(y,z)\right)\right))$$

A2: There is no number for which 0 is successor.

$$\sim(\exists x\ e\big(0,f(x)\big))$$

A3: For every number other than 0, there is one and only one immediate predecessor.

$$\forall x(\sim e(x,0)\rightarrow\left(\left(\exists y\ e\big(y,g(x)\big)\wedge\forall z(e\big(z,g(x)\big)\rightarrow e(y,z))\right)\right))$$

Interpretation of a formula

An interpretation of a formula F in FOPL consists of a non-empty domain D, and an assignment of values to each variable, function symbol and predicate symbol occurring as follows:

- To each variable, we assign an element in D.
- To each n-place function symbol, we assign a mapping from D^n to D.
- To each n-place predicate symbol, we assign a mapping from D^n to (T,F)

We say that the formula is interpreted over the domain D.

Formulas are evaluated to T or F according to the following rule.

- If the truth values of formulas G and H are evaluated, then the truth values of ~G, G∧H, G∨H, G→H and G↔H are evaluated.
- $\forall x\ G$ evaluates to T if the truth value of G evaluate to T for every x in D otherwise it evaluates to false.
- $\exists x\ G$ evaluates to T if the truth value of G evaluate to T for at least one x in D otherwise it evaluates to false.

Ex: Evaluate the formula $\forall x \; \exists y \; P(x, y)$

Given : D={1,2} and

P(1,1)=T P(1,2)=F

P(2,1)=F P(2,2)=T

If x=1, G evaluates to T because P(1,1)=T

If x=2, G evaluates to T because P(2,2)=T

Hence $\forall x \; \exists y \; P(x, y)$ evaluates to T.

Ex: Evaluate $\exists x(P(f(x)) \wedge Q(x, f(a)))$

Given: D={1,2}   a=1

	f(1)=2	f(2)=1
and	P(1)=F	P(2)=T
	Q(1,1)=T	Q(1,2)=T
	Q(2,1)=F	Q(2,2)=T

If x=1

$(P(f(1)) \wedge Q(1, f(1)))$

$(P(2) \wedge Q(1,2))$

$(T \wedge T)$

T

If x=2

$(P(f(2)) \wedge Q(2, f(1)))$

$(P(1) \wedge Q(2,1))$

$(F \wedge T)$

F

Hence the formula evaluates to T over the given domain.

Definitions

- A formula G is valid if and only if every interpretation of G evaluates to True.
- A formula G is inconsistent (unsatisfiable) if and only if there exist no interpretation of G that evaluates to true.
- A formula is a logical consequence of formulas F_1, F_2, …, F_n if and only if for every interpretation I, If $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ is true in I, G is also true in I.

Note: In propositional logic, we can not obtain interpretation over infinite domains. Clearly FOPL goes far beyond the propositional logic as the interpretations may be infinite in number.

Prenex Normal Form

A formula F in FOPL is said to be in a Prenex normal form if and only if formula F is in the form $(Q_1x_1\ Q_2x_2\ \ldots\ Q_nx_n)M$

where

every Q_ix_i, i=1,…,n is either $\forall xi$ or $\exists x_i$

and M is a formula containing no quantifiers.

$Q_1x_1\ Q_2x_2\ \ldots\ Q_nx_n$ are called the prefix

and M is called the matrix of the formula.

Examples:

$\forall x \forall y (P(x,y) \land Q(y))$

$\forall x \forall y \exists z (q(x,y) \to R(z))$

Now clearly one would like to explore if every formula in FOPL is expressible in prenex normal form.

To begin with, this requires establishing equivalence between formulas.

Definition:

Two formulas F and G are equivalent if and only if the truth value of F and G are same under every interpretation.

Some equivalent formulas which are also laws

$Qx\ F(x) \vee G = Qx(F(x) \vee G)$

$Qx\ F(x) \wedge G = Qx(F(x) \wedge G)$

$\sim (\forall x\ F(x)) = \exists x(\sim F(x))$

$\sim (\exists x\ F(x)) = \forall x(\sim F(x))$
$\forall x F(x) \wedge \forall x G(x) = \forall x(F(x) \wedge G(x))$
$\exists x F(x) \vee \exists x G(x) = \exists x(F(x) \vee G(x))$

Note that

(i) $\forall x F(x) \lor \forall x G(x) \neq \forall x(F(x) \lor G(x))$

(ii) $\exists x F(x) \land \exists x G(x) \neq \exists x(F(x) \land G(x))$

However by renaming of variables (rule of substitution) (i) may be rewritten as

$\forall x F(x) \lor \forall y G(y)$ which can be represented as $\forall x \forall y(F(x) \lor G(y))$

In general, we have the following laws

$\mathrm{Q}_1 x F(x) \lor Q_2 x G(x) = Q_1 x Q_2 y(F(x) \lor G(y))$

$\mathrm{Q}_1 x F(x) \land Q_2 x G(x) = Q_1 x Q_2 y(F(x) \land G(y))$

Transforming Formulas to Prenex Normal Form

1. Use the laws

   $F \leftrightarrow G = (F \rightarrow G) \wedge (G \rightarrow F)$

   $F \rightarrow G = \sim F \vee G$

2. Repeatedly use the following to bring the negation symbol in front of an atom.

   $\sim(\sim F) = F$

   $\sim(F \vee G) = \sim F \wedge \sim G$

   $\sim(F \wedge G) = \sim F \vee \sim G$

   $\sim \forall x F(x) = \exists x(\sim F(x))$

   $\sim \exists x F(x) = \forall x(\sim F(x))$

3. Rename bound variables to bring Q symbols in front.
4. Bring the Q symbols in front.

Example: Convert to Prenex normal form

$\forall x P(x) \rightarrow \exists x Q(x)$
$= \sim(\forall x P(x)) \lor \exists x Q(x)$
$= \exists x(\sim P(x)) \lor \exists x Q(x)$
$= \exists x(\sim P(x) \lor Q(x))$

which is in Prenex normal form

Inferencing in Predicate Logic

F1: Some patients like all doctors.

F2: No patient like any quack.

C: Therefore, no doctor is a quack.

These are statements in natural language.

The first two represent knowledge in the domain.

Third one is a conclusion to be inferred from the knowledge.

Inferencing in Predicate Logic

Let l(x,y) denote x likes y

D(y) denote y is a doctor

Q(y) denote y is a quack

P(x) denote x is a patient

We can write F1 and F2 in FOPL as follow:

$\exists x(P(x) \land \forall y(D(y) \rightarrow l(x,y)))$

$\forall x(P(x) \land \forall y(Q(y) \rightarrow \sim l(x,y)))$

Let I be an arbitrary interpretation over the domain.

Suppose F1 and F2 are true for the interpretation, then C must be true for this interpretation.

Inferencing in Predicate Logic

Since F1 is true in some interpretation I, there is indeed a value e of x in the domain which satisfies F1

$P(e) \land \forall y \big(D(y) \rightarrow l(e, y)\big)$

For the same e, F2 becomes

$P(e) \land \forall y \big(Q(y) \rightarrow \sim l(e, y)\big)$

Since P(e) is true under the interpretation I, it follows

$\forall y \big(D(y) \rightarrow l(e, y)\big)$

$\forall y \big(Q(y) \rightarrow \sim l(e, y)\big)$

Clearly if D(y) is false, $D(y) \rightarrow \sim Q(y)$ holds

If D(y) is true, then $l(e, y)$ holds and if $l(e, y)$ holds then $\sim Q(y)$ is true because $(\mathrm{Q}(y) \rightarrow l(e, y)) \leftrightarrow (l(e, y) \rightarrow \sim Q(y))$

Hence $D(y) \rightarrow \sim Q(y)$ is established.

Mixing Universal and Existential Quantifiers

Let us consider

$\forall y(\exists x\ equals(plus(1, x), y))$ over the set of integers.

It states that for every integer y, there is at least one integer x, having the property that x+1=y. This is a valid statement.

Now consider $\exists x(\forall y\ equals(plus(1, x), y))$

It states that there exists an integer x having the property that for every y, (x+1)=y , which is clearly invalid.

So, in general, the existential and universal quantifiers can not be commuted.

Mixing Universal and Existential Quantifiers

If we have to move an existential quantifier in obtaining Prenex normal form, then it should be positioned at a point, to the left of all those universal quantifiers whose variable do not appear in the scope of the variables covered by the existential quantifier.

$Q_1 x_1 Q_2 x_2 \ldots . Qrxr Q_{r+1} x_{r+1} \ldots Qn x_n \exists z$
$Q_1 x_1 Q_2 x_2 \ldots . Qrxr \exists z Q_{r+1} x_{r+1} \ldots Qn x_n$
Where

$Q_1 Q_2 \ldots . Qr$ are quantifiers that cover the scope of $\exists z$ (value of z depends on them)

$Q_{r+1} \ldots Q_n$ are quantifiers with scope before $\exists z$ appears (value of z does not depend on them)

Skolmisation

Consider

$\exists z P(z) \land \forall y (\exists x\ equals(y, plus(x, 1)))$ …………. (1)

Now clearly, one can obtain a formula by replacing z with constant a.

$P(a) \land \forall y (\exists x\ equals(y, plus(x, 1)))$ …………… (2)

Note that by choosing the constant a, which is different from any other constant in the formula, we have been able to remove the existential quantifier.

Skolmisation

Clearly the formula (2) can be rewritten as

$\forall y(P(a) \land \exists x\ equals(y, plus(x, 1)))$ …………… (3)

Also

$\forall y(\exists x(P(a) \land \exists x\ equals(y, plus(x, 1))))$ …………… (4)

We notice that clearly the choice of x depends on our initial choice of y. Hence we should be able to replace all occurrences of x by some function g(y) and rewrite (4) as

$\forall y(P(a) \land equals(y, plus(g(y), 1)))$ …………… (5)

In general, a standard form from prenex normal form could be obtained with no existential quantifier,

Skolem Standard Form

Consider a formula F in Prenex normal form
$Q_1x_1Q_2x_2 \ldots .QnxnM$
where M is a matrix in conjunctive normal form

Suppose Q_r is an existential quantifier with associated variable x_r, $1 \leq r \leq n$ such that no universal quantifier appears to its left, i.e., $Q_1Q_2 \ldots .Qr_{-1}$ are not universal quantifiers, then one could choose a new constant c, different from those that have made appearance in M, and substitute it for all occurrences of x_r, and deleteQ_rx_r from the prefix. Such a constant c is called a skolem constant.

Skolem Standard Form

Similarly, if $Q_{s1}Q_{s2} \ldots . Q_{\mathrm{sm}}$ were the universal quantifiers appearing before Q_r , $1 \ll s1 < s2 \ldots \ldots < sm < r$, we choose a new m-place function f different from other function symbols, replace all occurrences of x_r in M by $\mathrm{f}(\mathrm{x}_{s1,}\, xs_2 \ldots . , xsm)$ and delete $Q_r x_r$ from the prefix.

The formula we obtain is in skolem standard form.

Examples

Ex1: Obtain a standard form of the formula
$\exists x \forall y \forall z \exists u \forall v \exists w\ P(x, y, z, u, v, w)$

Scanning the formula from left, it can be written as
$\forall y \forall z \exists u \forall v \exists w\ P(a, y, z, u, v, w)$
$\forall y \forall z \forall v \exists w\ P(a, y, z, g(y, z), v, w)$
$\forall y \forall z \forall v\ P(a, y, z, g(y, z), v, h(y, z, v))$

Ex2:
$\forall x \exists y \exists z (\sim P(x, y) \wedge Q(x, z) \vee R(x, y, z))$

Which can be written as
$\forall x (\sim P(x, g(x)) \wedge Q(x, h(x)) \vee R(x, g(x), h(x)))$