Logic

Logic

- It relates to Knowledge Representation (KR) and inferencing in AI programs
- Typically a reference to systematic method of reasoning

Ex.

F1: If it is hot and humid, then it will rain.

F2: If it is humid, then it is hot.

F3: It is humid now.

F1, F2, F3 are facts.

The question is will it rain?

R: will it rain

It is hot, it is humid and it will rain are assertions which are

- Simple in the sense that they can not be further decomposed.
- Declarative with the characteristic that these may be true or false but never both,.i.e., excluded middle.

In symbolic logic, the method of reasoning which is followed is first to identify simple declarative facts called <u>Atoms</u>

An atom is a symbol representing a simple proposition which has the following characteristics:

- not decomposable
- has T or F values but not both.

Let

P: It is hot

Q: It is humid

R: It will rain

Then F1 can be written as $P \wedge Q \rightarrow R$

F2 can be written as $Q \rightarrow P$

F3 can be written as Q

Given F1, F2, F3, can one conclude R? Or

Does R logically follow from $P \wedge Q \rightarrow R, Q \rightarrow P, Q$?

On propositional calculus, we have

- Simple declarative statements identified as atoms.
- Calculus – basically refers to a set of rules for calculating with symbols. So we shall have a set of rules to combine propositions and compound propositions often called formulas.
- A set of theorems which when systematically employed will help us establish new propositions.

Given G and H as propositions (atoms), this table gives us truth values of formulas

G	H	~G	G∧H	G∨H	G→H	G↔H
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Given this table and hierarchical description of operators, it is possible to evaluate any other formula.

An Interpretation of a formula

- Let $G \triangleq (P \wedge Q) \rightarrow (R \leftrightarrow (\sim S))$

For a particular assignment of P, Q, R, S, we can obtain an interpretation of the formula as follows:

Let P,Q,R,S be respectively T, F, T, T.

This assignment has the interpretation for G as

$$(T \wedge F) \rightarrow \big(T \leftrightarrow (\sim T)\big)$$
$$F \quad \rightarrow (T \leftrightarrow F)$$
$$F \quad \rightarrow F$$
$$T$$

Interpretation of G

Definition: Given a propositional formula G, Let A_1, A_2,… , A_n be the atoms occurring in the formula G, then an interpretation of G is an assignment of truth values of A_1, A_2,… , A_n in which every A_i is assigned T or F but not both.

Definition: A formula G is said to be true under (or in) an interpretation if and only if G is evaluated to T in the interpretation, otherwise G is said to be false under the interpretation.

Clearly for n distinct propositions, there shall be 2^n interpretations.

Extreme cases are always interesting

$$G \triangleq ((P \rightarrow Q) \wedge P) \rightarrow Q$$

P	Q	$P \rightarrow Q$		$(P \rightarrow Q) \wedge P$	$((P \rightarrow Q) \wedge P) \rightarrow Q$
T	T	T	T	T	T
T	F	F	F	F	T
F	T	T	F	F	T
F	F	T	F	F	T

Extreme cases are always interesting

$G \triangleq (P \rightarrow Q) \wedge (P \wedge \sim Q)$

P	Q	$\sim Q$	$P \rightarrow Q$	$P \wedge \sim Q$	$(P \rightarrow Q) \wedge (P \wedge \sim Q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	T	F	F
F	F	T	T	F	F

Definition: A formula is said to be valid if and only if it is true under all its interpretation.

A formula is said to be invalid if and only if it is not valid.

Definition: A formula is said to be inconsistent (unsatisfiable) if and only if it is false under all its interpretations.

A formula is said to be consistent (satisfiable) if and only if it is not inconsistent.

- A formula is valid if and only if its negation is inconsistent.
- A formula is inconsistent if and only if its negation is valid.
- A formula is invalid if and only if there is at least one interpretation under which the formula is false.
- A formula is consistent if and only if there is one interpretation under which the formula is true.
- If a formula is valid then it is consistent but not vice versa.
- If a formula is inconsistent then it is invalid but not vice versa.

Examples

- $P \wedge \sim P$ is inconsistent and therefore also invalid.

- $P \vee \sim P$ is valid and therefore consistent.

- $P \rightarrow \sim P$ is invalid yet it is consistent.

Inference Rules in Propositional Logic

Given a set of propositions – regarded as premises, the attempt is to find the truth value of a new proposition from the premises.

This is also considered as – inferencing from premises.

There are two classic rules of inferencing

- Modus ponen
- Chain rule

Modus ponen

From P and $P \to Q$ infer Q

This is sometimes written as

$$
\begin{array}{r}
P \\
P \to Q \\
\hline
Q \\
\hline
\end{array}
$$

Given P is true, the condition that Q is false is false, establishing Q is true.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example

F1: If it is hot and humid then it will rain.

F2: If it is humid then it is hot.

F3: It is humid now.

Will it Rain?

Let

P: It is hot

Q: It is humid

R: It will rain

Then F1 can be written as $P \wedge Q \rightarrow R$

F2 can be written as $Q \rightarrow P$

F3 can be written as Q

Given F1, F2, F3, we have to establish R.

Using modus ponen on F2, F3

$$Q$$

$$Q \rightarrow P$$

$$P$$

Using P =true and Q = true means $P \wedge Q$ is true.

LHS in F1 is true, so RHS must be true. Hence R is established.

(F1 uses implication operator which suggests that the condition that if LHS is true and RHS is false   is false).

Chain Rule

From $P \to Q$ and $Q \to R$, infer $P \to R$

i.e. $P \to Q$

$Q \to R$

$P \to R$

Examples

- Ex1:

Given (Programmer likes LISP) → (Programmer hates COBOL)

And (Programmer hates COBOL) → (Programmer likes recursion)

Conclude: (Programmer likes LISP) → (Programmer likes recursion)

- Ex2:

i. $(P \rightarrow Q) \rightarrow ((P\ or\ Q) \rightarrow (R \rightarrow Q))$

ii. $(R \rightarrow Q) \rightarrow (R\ or\ S)$

iii. $P \rightarrow Q$

Establish $(P\ or\ Q) \rightarrow (R\ or\ S)$

Examples

i and iii by modus ponen establish

$(P \ or \ Q) \to (R \to Q)$ iv

iv and ii by chain rule establish

$(P \ or \ Q) \to (R \ or \ S)$

In establishing the truth values, it is useful to use theorems of logic. In fact, one can regard them as tautologies (inferencing tools)

Some Useful Tautologies

- $P \wedge (P \rightarrow Q) \rightarrow Q$ modus ponen
- $\sim Q \wedge (P \rightarrow Q) \rightarrow \sim P$ modus ponen
- $(P \vee Q) \wedge \sim P \rightarrow Q$ disjunctive syllogism
- $(P \rightarrow \mathrm{Q}) \wedge (Q \rightarrow \mathrm{R}) \rightarrow (P \rightarrow R)$ chain rule
- $(P \rightarrow Q) \rightarrow ((P \vee R) \rightarrow (Q \vee R))$

<u>Method1</u>

These tautologies can be obtained using truth table.

Use of rule of substitution: It states that if C(A) is a tautology and we substitute formula B for any occurrence of A in C, the C(B) is a tautology.

Method2

Simplify the formula and see if it reduces to $P \vee \sim P$ which is a tautology.

Ex:

$(P \wedge (P \longrightarrow Q)) \longrightarrow Q$

To show this we take

$\sim (P \wedge (P \longrightarrow Q)) \vee Q$ and show that this is a tautology.

Disjunctive normal form: $P \longrightarrow Q = \sim P \vee Q$

Example

contd

$$\sim (P \wedge (P \longrightarrow Q)) \vee Q$$
$$= (\sim P \vee \sim (P \longrightarrow Q)) \vee Q$$
$$= (\sim P \vee \sim (\sim P \vee Q)) \vee Q$$
$$= (\sim P \vee (P \wedge \sim Q)) \vee Q$$
$$= ((\sim P \vee P) \wedge (\sim P \vee \sim Q)) \vee Q$$
$$= (T \wedge (\sim P \vee \sim Q)) \vee Q$$
$$= (\sim P \vee \sim Q) \vee Q$$
$$= \sim P \vee \sim Q \vee Q$$
$$= \sim P \vee T$$
$$= T$$

Example

F1: If it is summer time, it is warm.

F2: If it is winter time, it is cold.

F3: It is summer time or winter time.

Establish it is warm or cold.

Let

P: It is summer time

Q: It is warm

R: It is winter time

S: It is cold

Then F1 : $P \longrightarrow Q$

F2 : $R \rightarrow S$

F3 : $P \vee R$

F4 : $Q \vee S$ (to establish)

Example

Contd…

F1 and $(P \to Q) \to ((P \vee R) \to (Q \vee R))$ gives $(P \vee R) \to (Q \vee R)$

F2 and$(R \to S) \to ((R \vee Q) \to (S \vee Q))$ gives $(R \vee Q) \to (S \vee Q)$

Chain rule applies to the above two gives $(P \vee R) \to (S \vee Q)$

Using this and F3 and employing modus ponen establishes $(S \vee Q)$ which is same as $(Q \vee S)$

Proof by resolution principle in propositional logic

Given a set of axioms (premises), to prove a consequent.

The general principle is proof by contradiction and effort is to show that negation of consequent leads to a contradiction.

Step1: Convert all the given propositions to clausal form

Ex:

P	P	
$(P \wedge Q) \longrightarrow R$	$\sim P \vee \sim Q \vee R$	
$(S \vee T) \longrightarrow Q$	$(\sim S \wedge \sim T) \vee Q$	$\sim S \vee Q$
		$\sim T \vee Q$
T	T	

To prove R, we proceed as follows:

Step2: Negate the clause to be proved and add this to the axioms, i.e., $\sim R$

Step3: Repeat until either a contradiction is found or no progress can be made.

- Select two clauses. Let us call them parent clauses
- Resolve them
  - The clauses can be resolved whenever there are complementary literals.

$P \qquad \sim P \vee \sim Q \vee R \qquad \sim S \vee Q \qquad \sim T \vee Q \qquad T \qquad \sim R$

$\sim P \vee \sim Q$

$\sim Q$

$\sim T$

Contradiction

Hence R is established

Example

F1: $P \longrightarrow Q$

F2: $\mathrm{R} \longrightarrow S$

F3: $P \vee R$

Establish $\mathrm{Q} \vee S$

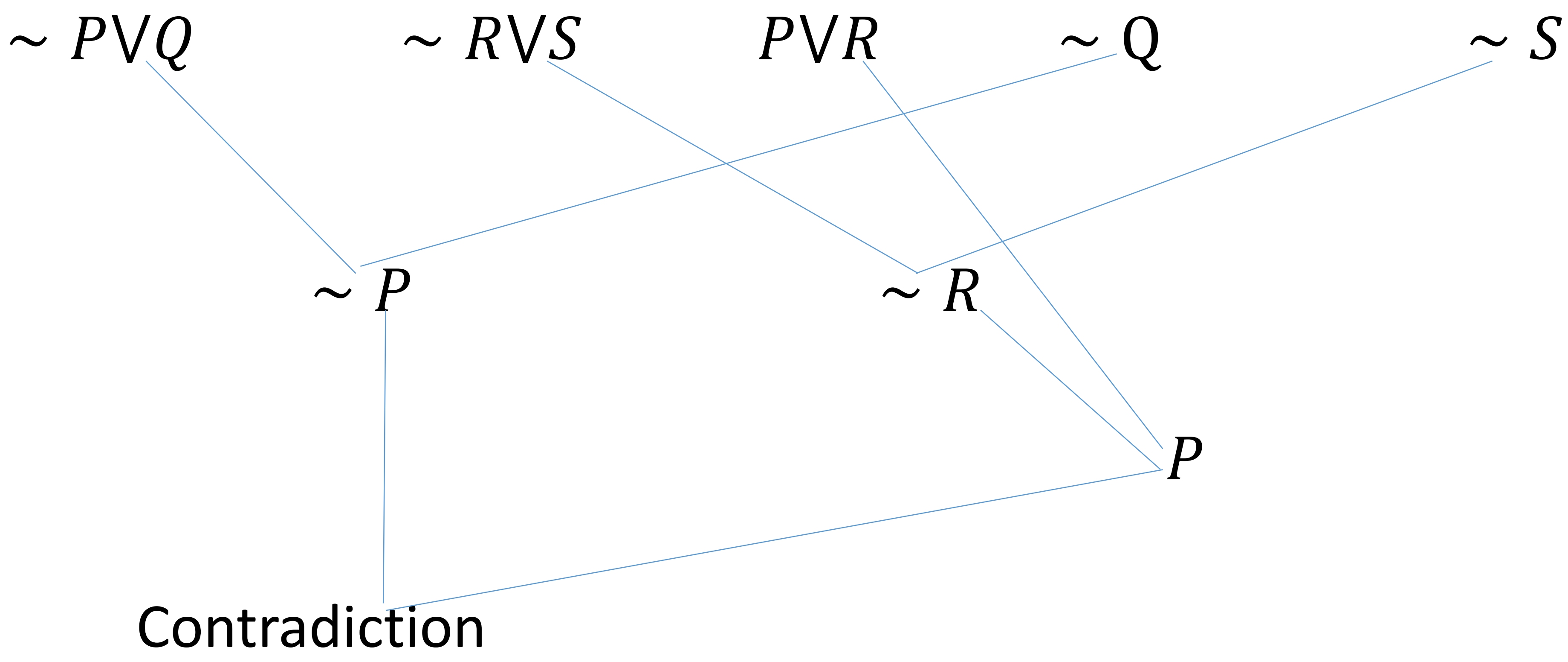
Step1: negate consequent $\sim (\mathrm{Q} \vee S)$

Convert to Clausal form

$\sim P \vee Q$ $\sim R \vee S$ $P \vee R$

$\sim \mathrm{Q}$ $\sim S$

Example



Hence $Q \vee S$ is established.(Note the way parallelism can be exercised)