PROJECTIVE REPRESENTATION OF FINITE GROUPS

CÉSAR GALINDO

Abstract. In this note we collect basic definitions and results on projective representations.

1. Basic definition on group cohomology

We will denote by $\mu \subset \mathbb{C}^*$ the subgroup of all roots of unity. By the universal coefficient theorem we know that for every finite group $H^n(G, \mathbb{C}^*) = H^n(G, \mu)$, that mean that we can assume that every cocycle has values in roots of unity.

Recall that the universal coefficient theorem say that for any finite abelian group there is an split exact sequence

$$0 \to Ext(H_{n-1}(G, \mathbb{Z}), A) \to H^n(G, A) \to \mathrm{Hom}(H_n(G, \mathbb{Z}), A) \to 0.$$

If $A = \mathbb{C}^*$, we have that $H^n(G, \mathbb{C}^*) \cong \mathrm{Hom}(H_n(G, \mathbb{Z}), \mathbb{C}^*) \cong \mathrm{Hom}(H_n(G, \mathbb{Z}), \mu_m)$, where m is the exponent of $H_n(G, \mathbb{Z})$, in other words we have the exact sequence

$$0 \to Ext(H_{n-1}(G, \mathbb{Z}), \mu_m) \to H^n(G, \mu_m) \to H^n(G, \mathbb{C}^*) \to 0.$$

The imporant part is that every cohomology class in $H^n(G, \mathbb{C}^*)$ is cohomologous to a cocycle with values in μ_m.

Given $\alpha \in Z^2(G, \mathbb{C}^*)$ a α-projective representation of G is a pair (ρ, V), where V is a complex vector space and $\rho : G \to GL(V)$ ia mapping such that

- $\rho(1) = id_V$,
- $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$, for all $x, y \in G$.

If (ρ_V, V) and (ρ_W, W) are α-projective representation a linear isomorphism $f : V \to W$ is a linear isomorphism of α-projective representation if $= \rho_W = f\rho_V f^{-1}$. There is other equivalence relation called projective equivalence, but we are only interested in linear equivalence.

We will denote by $\mathbb{C}_\alpha[G]$ the group algebra of G twisted by α. As vector space $\mathbb{C}_\alpha[G]$ has basis $\{\overline{g} : g \in G\}$ and product $\overline{x}\overline{y} = \alpha(x, y)\overline{xy}$ for all $x, y \in G$.

Proposition 1.1. *Let $\alpha Z^2(G, \mathbb{C}^*)$ be of order n (in particular take values in μ_n) and let ζ be a primite n-th root of unity, and $G_\alpha^* = \langle \alpha(x, y)\overline{g} : g, x, y \in G\rangle$.*

- *The group G_α^* is a central extension of μ_n by G with associated 2-cocycle α.*
- $\mathbb{C}[G_\alpha^*] \cong \prod_{i=1}^n \mathbb{C}_{\alpha^i}[G]$.

- *If $\rho : G \to GL(V)$ is an α-projective representation, then $\rho^* : G^*_\alpha \to GL(V)$ defined by $\rho^*(\zeta^i \overline{g}) = \zeta^i \rho(g)$ is a linear representation of G.*
- *Conversely, given a linear representation $\rho^* : G^*_\alpha \to GL(V)$,such that $\rho^*(\zeta) = \zeta$, then $\rho : G \to GL(V)$ defined by $\rho(g) = \rho(\overline{g})$ is an α-projective representation.*

2. How to compute all projective characters.

Let G be a finite group and $\alpha \in Z^2(G, \mathbb{C}^*)$ a 2-cocycle. We assume that α take values in root of unity, in fact in our case we know α takes values in root of unit of order the exponent of $H_3(G, \mathbb{Z})$.

- First, we need to compute the order of α, that is the less n such that α^n is the constant function 1.
- Second, assume that the order of α is n then we need to construct the group G^*_α, this is the extension of μ_n by G associated with the 2-cocycle α.
- Assume that ζ is primitive root of unity. Search all characters $\chi : G^*_\alpha \to \mathbb{C}$ such that $\chi(\zeta) = \zeta\chi(1)$. The function $\chi_* : G \to \mathbb{C}$, defined by $\chi_*(g) = \chi(\overline{g})$ is an α-projective character.

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Departamento de Matemáticas, Universidad de los Andes, Carrera 1 N. 18A - 10, Bogotá, Colombia

E-mail address: `cn.galindo1116@uniandes.edu.co`