MINIMIZATION

aka. Simplification

Claude Elwood Shannon

Mathematician
Electrical Engineer
Cryptographer

M.Sc. Thesis (1937)

A Symbolic Analysis of Relay and Switching Circuits

Switching Algebra!

2-valued Boolean algebra

SWITCHING ALGEBRA

- Given S={0,1}
- Given § = × (AND), † = + (OR)
- S is closed, commutative, and distributive w.r.t × , +
- $e_\times$ = 1 and e_+ = 0

- Complement: for any x ∈S, there is y∈S such that
  - $0 \times 1 = e_+ = 0$
  - $0 + 1 = e_\times = 1$

We denote 0=1′, 1=0′

SWITCHING ALGEBRA
IS-A
BOOLEAN ALGEBRA

It satisfies all conditions of Boolean algebra!

Prove → Book: 2.3 axiomatic definition of Boolean algebra

Another sample of algebra in CS: Relational Algebra (SQL)

Is relational algebra a Boolean algebra? Check this when you take COMP-3150: Database Management Systems!

BASIC THEOREMS

Prove by postulates

$$X + X = X$$

$$X + X + X + \ldots + X = X$$

$$X + X = X$$

$$X + X + X + \ldots + X = X$$

$X + X \quad =$

$= (X + X)\, 1$ using identity $e_{\times} = 1$

$$X + X = X$$

$$X + X + X + \ldots + X = X$$

$X + X =$

$= (X + X)\ 1$ using identity $e_{\times}=1$

$= (X + X)\ (X+X')$ using complement property

$$X + X = X$$

$$X + X + X + \ldots + X = X$$

$X + X =$

$= (X + X)\, 1$ using identity $e_\times = 1$

$= (X + X)\,(X + X')$ using complement property

$= X + (XX')$ using distributive property of $+$ over $\times$

$$X + X = X$$

$$X + X + X + \ldots + X = X$$

$X + X =$

$= (X + X)\,1$ using identity $e_\times = 1$

$= (X + X)\,(X + X')$ using complement property

$= X + (XX')$ using distributive property of $+$ over $\times$

$= X + 0$ using complement property

$$X + X = X$$

$$X + X + X + \ldots + X = X$$

$X + X =$

$= (X + X)\,1$ using identity $e_{\times}=1$

$= (X + X)\,(X+X')$ using complement property

$= X + (XX')$ using distributive property of $+$ over $\times$

$= X + 0$ using complement property

$= X$ using identity property of $e_{+}=0$

$$X + 1 = 1$$

$$X + Y + Z + \ldots + 1 = 1$$

$$X + 1 = 1$$

$$X + Y + Z + \ldots + 1 = 1$$

$X + 1 \quad =$

$\quad\quad = (X + 1)\, 1$ using identity $e_{\times} = 1$

$$X + 1 = 1$$

$$X + Y + Z + \ldots + 1 = 1$$

$X + 1 \quad =$

$= (X + 1)\,1$ using identity $e_\times = 1$

$= (X + 1)\,(X+X')$ using complement property

$$X + 1 = 1$$

$$X + Y + Z + \ldots + 1 = 1$$

$X + 1 \quad =$

$= (X + 1)\, 1$ using identity $e_{\times}=1$

$= (X + 1)\, (X+X')$ using complement property

$= X + (1X')$ using distributive property of + over $\times$

$$X + 1 = 1$$
$$X + Y + Z + \ldots + 1 = 1$$

$X + 1 =$

$= (X + 1)\, 1$ using identity $e_\times = 1$

$= (X + 1)(X + X')$ using complement property

$= X + (1X')$ using distributive property of $+$ over $\times$

$= X + X'$ using identity $e_\times = 1$

$$X + 1 = 1$$

$$X + Y + Z + \ldots + 1 = 1$$

$X + 1 \quad =$

$= (X + 1)\,1$ using identity $e_\times = 1$

$= (X + 1)\,(X + X')$ using complement property

$= X + (1X')$ using distributive property of + over $\times$

$= X + X'$ using identity $e_\times = 1$

$= 1$ using complement property

$$X + XY = X$$

$$X + XY + XZW + \ldots + XWAD = X$$

Absorption

$$X + XY = X$$

$$X + XY + XZW + \ldots + XWAD = X$$

$X + XY =$

$= X1+XY$ using identity $e_{\times}=1$

$= X(1 + Y)$ using distributive property of $\times$ over $+$

$= X1$ using previous theorem $x+1=1$

$= X$ using identity $e_{\times}=1$

Absorption

DUALITY

Dual(F) = OR $\rightleftarrows$ AND, 1 $\rightleftarrows$ 0

Dual(F) may or may not equal to F!

DUALITY

$$X+1 \rightleftarrows X0$$

$$X+X' \rightleftarrows XX'$$

$$(X+Y)' \rightleftarrows (XY)'$$

DUALITY

A postulate or a proved theorem for F
Also a postulate or a proved theorem for Dual(F)

DUALITY

$$\{X+1=1\} \rightleftarrows \{X0 = 0\}$$
$$\{X+X'=1\} \rightleftarrows \{XX'=0\}$$
$$\{(X+Y)'=X'Y'\} \rightleftarrows \{(XY)'=X'+Y'\}$$

$$X(X+Y) = X$$

$$X(X+Y)(X+Z) \ldots (X+W) = X$$

$\{X(X+Y)= X\}$ ⇄ $\{X+XY=X\}$

⇄ We proved the dual version

⇄ Using the duality property, this is also true!

Absorption

BASIC THEOREMS

prove by truth table

For equality proof, $F_1=F_2$, for all possibility in the input variables (all rows), both side of equation must have equal value for same input variables.

For inequality proof, $F_1 \neq F_2$, find at least one possibility (a row) that have different values.

MINIMIZATION

I) Boolean Algebra (algebraically)

aka. Algebraic Manipulation

$$F = ZY'X' + ZYX + ZYX' + ZY'X$$

$4 \times$ 3-input-AND

$1 \times$ 4-input-OR

$$F = ZY'X' + ZYX + ZYX' + ZY'X$$

$$F = Z\,(Y'X' + YX + YX' + Y'X)$$

$$F = Z\,(Y'X' + YX + YX' + Y'X)$$

$$F = Z\,(Y'(X'+X) + YX + YX')$$

$$F = Z\,(Y'(X' + X) + YX + YX')$$

$$F = Z\,(Y'1 + YX + YX')$$

$$F = Z(Y'1 + YX + YX')$$

$$F = Z(Y' + YX + YX')$$

$$F = Z(Y' + YX + YX')$$

$$F = Z\,(Y' + Y\,(X+X'))$$

$$F = Z(Y' + Y(X + X'))$$

$$F = Z(Y' + Y1)$$

$$F = Z\,(Y' + Y)$$

$$F = Z\,(Y' + Y)$$

$$F = Z1$$

$$F = Z$$

0 gates!

$$F = ZY'X' + ZYX + ZYX' + ZY'X$$

4 × 3-input-AND
1 × 4-input-OR

$$F = ZY'X' + ZYX + ZYX' + ZY'X$$

$$F = ZY' (X'+X) + ZYX + ZYX'$$

$$F = ZY'(X'+X) + ZYX + ZYX'$$

$$F = ZY' (X'+X) + ZY (X+X')$$

$$F = ZY'(X'+X) + ZY(X+X')$$

$$F = ZY'1 + ZY1$$

$$F = ZY' + ZY$$

$$F = Z\,(Y' + Y)$$

$$F = Z1$$

$$F = Z$$

0 gates!

$$F = ZY'X' + ZYX + ZYX' + ZY'X$$

$4 \times$ 3-input-AND
$1 \times$ 4-input-OR

$$F = ZY'X' + ZYX' + Z'Y'X$$

$3 \times$ 3-input-AND

$1 \times$ 3-input-OR

$$F = ZY'X' + ZYX' + Z'Y'X$$

$$F = ZX'(Y'+Y) + Z'Y'X$$

$$F = ZX' \, (Y' + Y) + Z'Y'X$$

$$F = ZX'1 + Z'Y'X$$

$$F = ZX' + Z'Y'X$$

1 × 2-input-AND
1 × 3-input-AND
1 × 2-input-OR

$$F = ZYX' + ZYX' + Z'Y'X$$

3 × 3-input-AND
1 × 3-input-OR

MINIMIZATION

I) Boolean Algebra (algebraically)

- Needs to be smart. It is hard due to guesswork (which rules to apply?)
- If the number of variables (ABCDEF...) and/or number of minterms (MAXTERMS) grows
- No Algorithm
- Is the result minimal?!

MINIMIZATION

II) Map (Karnaugh map, K-map)

aka. Graphical Manipulation

MINIMIZATION

II) Map (Karnaugh map, K-map)

aka. Graphical Manipulation

Algorithm; Straightforward, up to six variables

Result is always minimal

TRUTH TABLE

Y	X	F
0	0	?
0	1	?
1	0	?
1	1	?

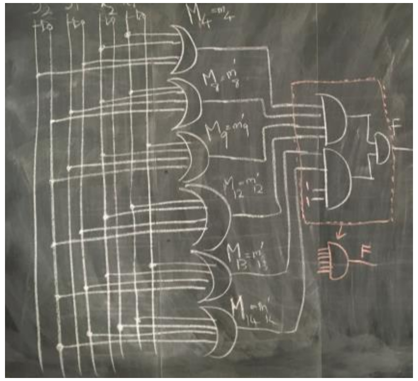
BOOLEAN FUNCTION

Algebraic Expression

$$F_{SoP}(Y,X) = \sum \text{minterms}$$

$$F_{PoS}(Y,X) = \prod \text{MAXTERMs}$$

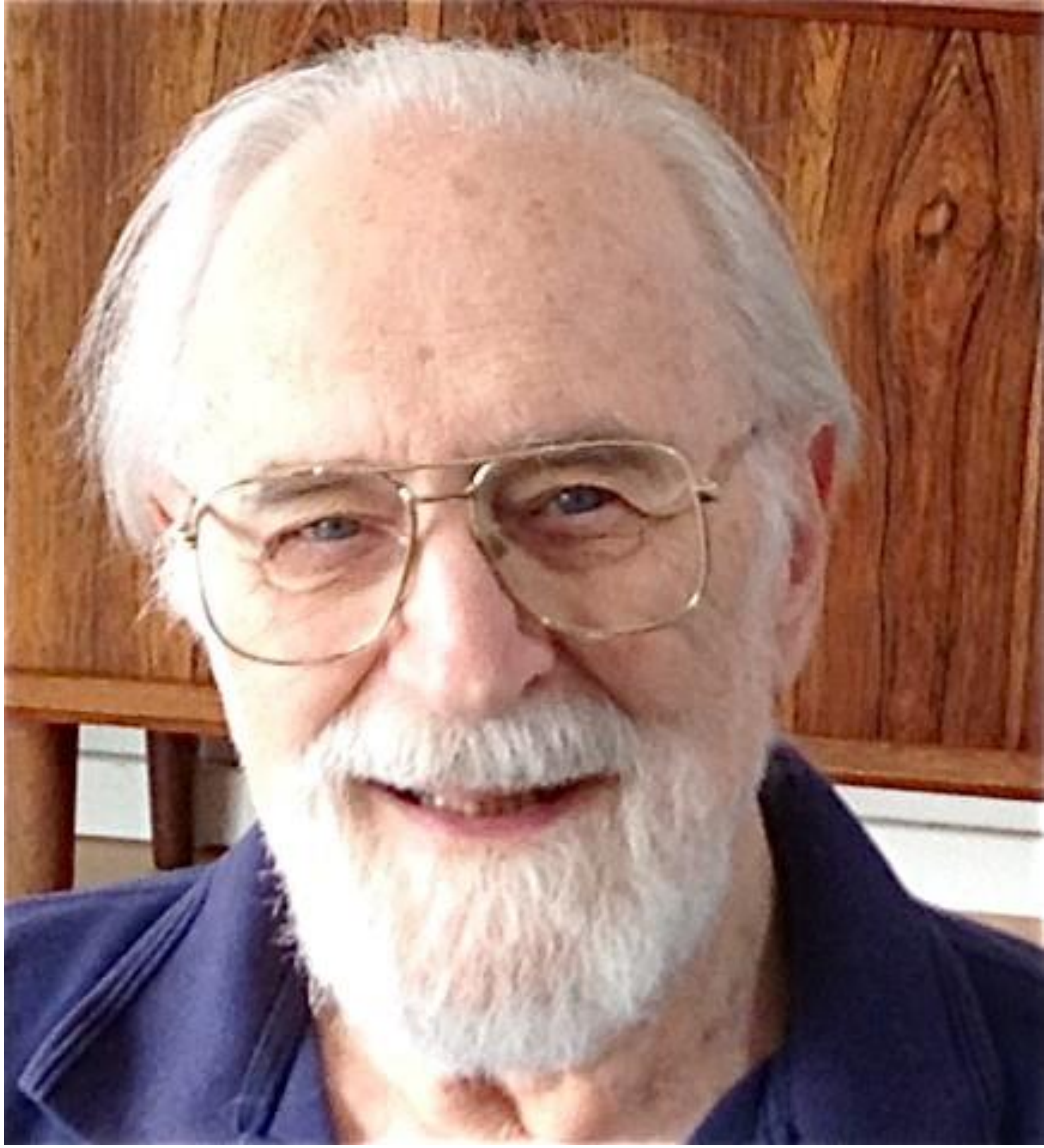
LOGIC CIRCUIT



Maurice Karnaugh

Physicist
Mathematician
Inventor

Bell Labs (1954)
"The Map Method for Synthesis of Combinational Logic Circuits”



KARNAUGH MAP

/ˈkɑːrnɔː/

1-Variable KARNAUGH MAP

X	F
0	m_0
1	m_1

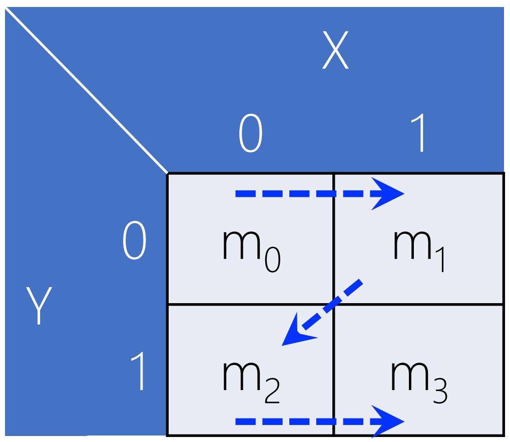
X	
0	1
m_0	m_1

2-Variable KARNAUGH MAP

Y	X	F
0	0	m_0
0	1	m_1
1	0	m_2
1	1	m_3

		X	
		0	1
Y	0	m_0	m_1
	1	m_2	m_3

Y	X	F
0	0	m_0
0	1	m_1
1	0	m_2
1	1	m_3



Y	X	F
0	0	1
0	1	0
1	0	0
1	1	0

$F(Y,X) = m_0 = Y'X'$

		X	
		0	1
Y	0	1	0
	1	0	0

$F(Y,X) = Y'X'$

Y	X	F
0	0	0
0	1	1
1	0	0
1	1	0

$F(Y,X) = m_1 = Y'X$

Y \ X	0	1
0	0	1
1	0	0

$F(Y,X) = Y'X$

Y	X	F
0	0	0
0	1	0
1	0	1
1	1	0

$F(Y,X) = m_2 = YX'$

Y \ X	0	1
0	0	0
1	1	0

$F(Y,X) = YX'$

Y	X	F
0	0	0
0	1	0
1	0	0
1	1	1

$F(Y,X) = m_3 = YX$

		X	
		0	1
Y	0	0	0
	1	0	1

$F(Y,X) = YX$

Y	X	F
0	0	1
0	1	1
1	0	0
1	1	0

$$
\begin{aligned}
F(Y,X) &= m_0 + m_1 \\
&= Y'X' + Y'X \\
&= Y'(X' + X) \\
&= Y'
\end{aligned}
$$

		X	
		0	1
Y	0	1	1
	1	0	0

$$F(Y,X) = Y'$$

Y	X	F
0	0	0
0	1	0
1	0	1
1	1	1

$$
\begin{aligned}
F(Y,X) &= m_2 + m_3 \\
&= YX' + YX \\
&= Y(X' + X) \\
&= Y
\end{aligned}
$$

		X	
		0	1
Y	0	0	0
	1	1	1

$F(Y,X) = Y$

Y	X	F
0	0	0
0	1	1
1	0	0
1	1	1

$$\begin{aligned} F(Y,X) &= m_1 + m_3 \\ &= Y'X + YX \\ &= X(Y' + Y) \\ &= X \end{aligned}$$

		X	
		0	1
Y	0	0	1
	1	0	1

$$F(Y,X) = X$$

Y	X	F
0	0	1
0	1	0
1	0	1
1	1	0

$$
\begin{aligned}
F(Y,X) &= m_0 + m_2 \\
&= Y'X' + YX' \\
&= X'(Y' + Y) \\
&= X'
\end{aligned}
$$

		X	
		0	1
Y	0	1	0
	1	1	0

$$F(Y,X) = X'$$

Y	X	F
0	0	1
0	1	1
1	0	1
1	1	0

$$
\begin{aligned}
F(Y,X) &= m_0 + m_1 + m_2 \\
&= Y'X' + Y'X + YX' \\
&= Y'(X' + X) + YX' \\
&= Y' + YX'
\end{aligned}
$$

Y \ X	0	1
0	1	1
1	1	0

$$F(Y,X) = Y' + YX'$$

Y	X	F
0	0	1
0	1	1
1	0	1
1	1	0

$$
\begin{aligned}
F(Y,X) &= m_0 + m_1 + m_2 \\
&= Y'X' + Y'X + YX' \\
&= X'(Y' + Y) + Y'X \\
&= X' + Y'X
\end{aligned}
$$

Y \ X	0	1
0	1	1
1	1	0

$F(Y,X) = X' + Y'X$

Y	X	F
0	0	1
0	1	1
1	0	1
1	1	0

$$
\begin{aligned}
F(Y,X) &= m_0 + m_1 + m_2 \\
&= Y'X' + Y'X + YX' \\
&= Y'X' + Y'X' + Y'X + YX' \\
&= Y'(X' + X) + Y'X' + YX' \\
&= Y' + Y'X' + YX' \\
&= Y' + X'(Y' + Y) \\
&= Y' + X'
\end{aligned}
$$

Y \ X	0	1
0	1	1
1	1	0

$$F(Y,X) = Y' + X'$$

Y	X	F
0	0	1
0	1	1
1	0	1
1	1	1

$$
\begin{aligned}
F(Y,X) &= m_0 + m_1 + m_2 + m_3 \\
&= Y'X' + Y'X + YX' + YX \\
&= Y'(X' + X) + Y(X' + X) \\
&= Y' + Y \\
&= 1
\end{aligned}
$$

Y \ X	0	1
0	1	1
1	1	1

$$
\begin{aligned}
F(Y,X) &= m_0+m_1+m_2+m_3 \\
&= 1
\end{aligned}
$$

Y	X	F
0	0	0
0	1	1
1	0	1
1	1	0

$$F(Y,X) = m_1 + m_2$$
$$= Y'X + YX'$$

Y \ X	0	1
0	0	1
1	1	0

$$F(Y,X) = m_1 + m_2$$
$$= Y'X + YX'$$

Y	X	F
0	0	1
0	1	0
1	0	0
1	1	1

$$F(Y,X) = m_0 + m_2$$
$$= Y'X' + YX$$

Y \ X	0	1
0	1	0
1	0	1

$$F(Y,X) = m_0 + m_2$$
$$= Y'X' + YX$$

3-Variable KARNAUGH MAP

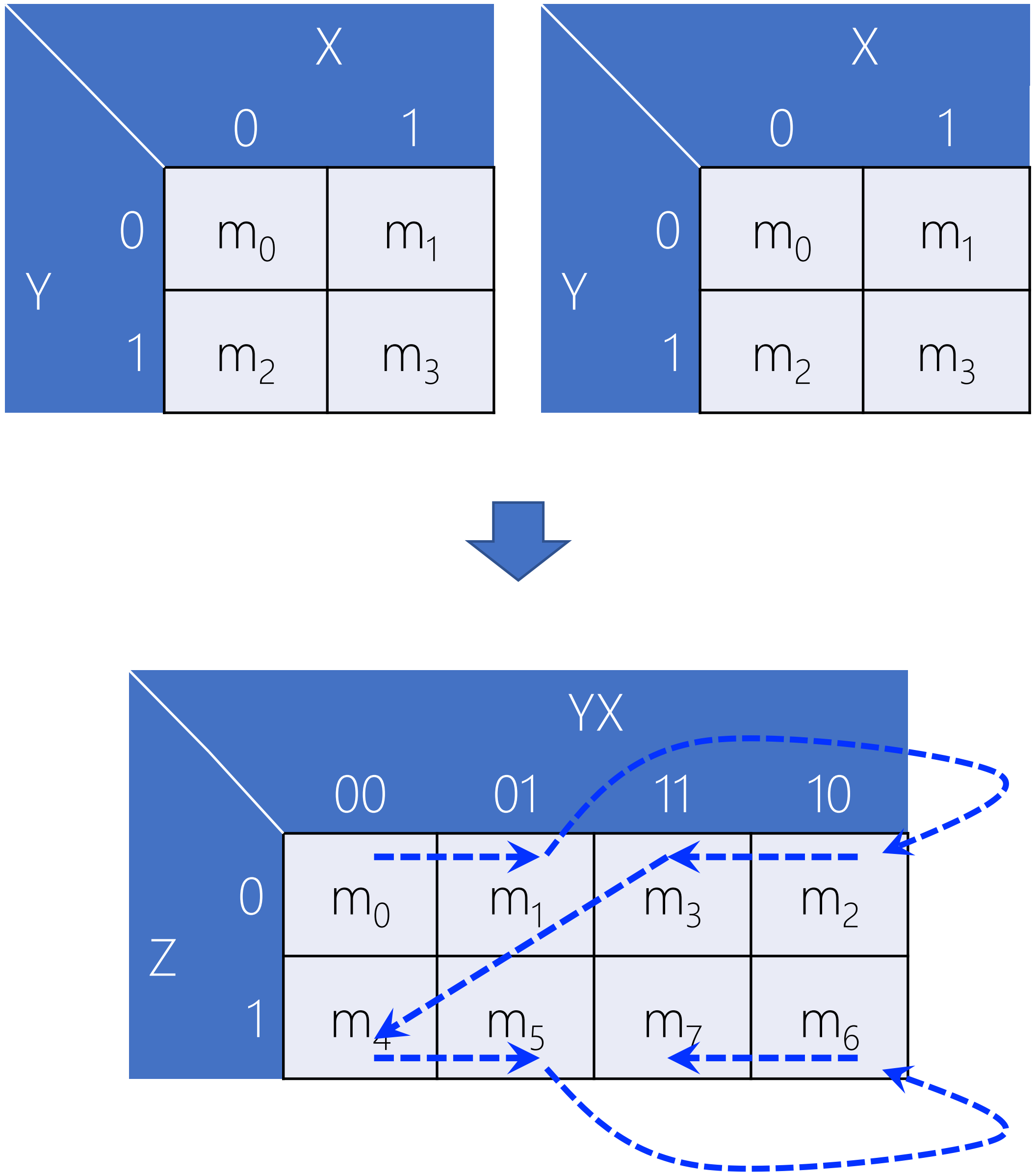
Z	Y	X	F
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

		X	
		0	1
Y	0	m_0	m_1
	1	m_2	m_3

		X	
		0	1
Y	0	m_0	m_1
	1	m_2	m_3

		YX			
		00	01	11	10
Z	0	m_0	m_1	m_3	m_2
	1	m_4	m_5	m_7	m_6

Z	Y	X	F
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7



Z	Y	X	F
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

Z \ YX	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

Y

Z	Y	X	F
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

Z \ YX	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

Y'

Z	Y	X	F
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

Z \ YX	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

X

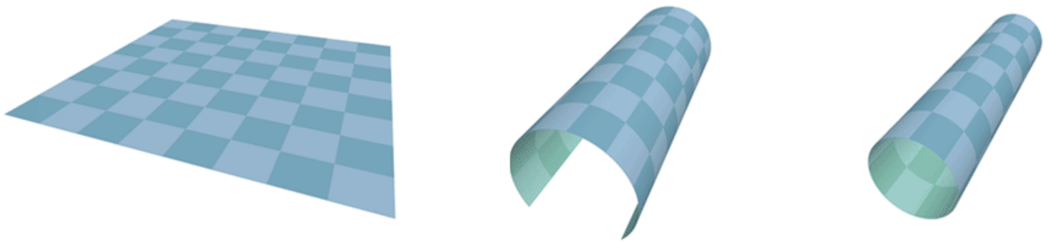
Z	Y	X	F
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

Z \ YX	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

X' ?

Z	Y	X	F
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

Z \ YX	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6



Z	Y	X	F
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

Z \ YX	00	01	11	10	Z
0	m_0	m_1	m_3	m_2	0
1	m_4	m_5	m_7	m_6	1

X' X'

Z	Y	X	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Z \ YX	00	01	11	10
0	1	1	1	1
1	0	0	1	1

$$
\begin{aligned}
F(Y,X) &= \sum m(0,1,2,3,6,7) \\
&= Z'Y'X' + Z'Y'X + Z'YX' + Z'YX + ZYX' + ZYX \\
&= ?
\end{aligned}
$$

Z	Y	X	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

$$F(Y,X) = \sum m(0,1,2,3,6,7)$$
$$= Z'Y'X' + Z'Y'X + Z'YX' + Z'YX + ZYX' + ZYX$$
$$= ?$$

Z \ YX	00	01	11	10
0	1	1	1	1
1	0	0	1	1

$$F(Y,X) = \sum m(0,1,2,3,6,7)$$
$$= Z' +$$

Z	Y	X	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

$$
\begin{aligned}
F(Y,X) &= \sum m(0,1,2,3,6,7) \\
&= Z'Y'X' + Z'Y'X + Z'YX' + Z'YX + ZYX' + ZYX \\
&= ?
\end{aligned}
$$

Z \ YX	00	01	11	10
0	1	1	1	1
1	0	0	1	1

$$
\begin{aligned}
F(Y,X) &= \sum m(0,1,2,3,6,7) \\
&= Z' + ZY
\end{aligned}
$$

Z	Y	X	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

$$
\begin{aligned}
F(Y,X) &= \sum m(0,1,2,3,6,7) \\
&= Z'Y'X'+Z'Y'X+Z'YX'+Z'YX+ZYX'+ZYX \\
&= ?
\end{aligned}
$$

Z \ YX	00	01	11	10
0	1	1	1	1
1	0	0	1	1

$$
\begin{aligned}
F(Y,X) &= \sum m(0,1,2,3,6,7) \\
&= Z' + Y
\end{aligned}
$$

Z	Y	X	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

$$F(Y,X) = \prod M(4,5)$$
$$= (Z'+Y+X)\ (Z'+Y+X')$$
$$= ?$$

Z \ YX	00	01	11	10
0	1	1	1	1
1	0	0	1	1

$$F(Y,X) = \prod M(4,5)$$
$$= ?$$

Z	Y	X	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

$$
\begin{aligned}
F(Y,X) &= \prod M(4,5) \\
&= (Z'+Y+X)\,(Z'+Y+X') \\
&= ?
\end{aligned}
$$

Z \ YX	00	01	11	10
0	0	0	0	0
1	1	1	0	0

$$
\begin{aligned}
F'(Y,X) &= \sum m(4,5) \\
&= ZY'
\end{aligned}
$$

Z	Y	X	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

$$
\begin{aligned}
F(Y,X) &= \prod M(4,5) \\
&= (Z'+Y+X)\,(Z'+Y+X') \\
&= ?
\end{aligned}
$$

Z \ YX	00	01	11	10
0	1	1	1	1
1	0	0	1	1

$$
\begin{aligned}
F(Y,X) &= \prod M(4,5) \\
&= (F')' \\
&= (ZY')' \\
&= Z'+Y
\end{aligned}
$$

Z	Y	X	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

$$F(Y,X) = \sum m(0,1,2,3,6,7)$$
$$= Z'Y'X'+Z'Y'X+Z'YX'+Z'YX+ZYX'+ZYX$$
$$= ?$$

$$F(Y,X) = \prod M(4,5)$$
$$= (Z'+Y+X)\,(Z'+Y+X')$$
$$= ?$$

Z \ YX	00	01	11	10
0	1	1	1	1
1	0	0	1	1

$$F(Y,X) = \sum m(0,1,2,3,6,7)$$
$$= Z' + Y$$

Z \ YX	00	01	11	10
0	1	1	1	1
1	0	0	1	1

$$F(Y,X) = \prod M(4,5)$$
$$= (F')'$$
$$= (ZY')'$$
$$= Z'+Y$$

Z	Y	X	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$$F(Y,X) = \sum m(0,2,4,\ 6)$$
$$= Z'Y'X' + Z'YX' + ZY'X' + ZYX'$$
$$= ?$$

		YX			
		00	01	11	10
Z	0	1	0	0	1
	1	1	0	0	1

Z	Y	X	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$$
\begin{aligned}
F(Y,X) &= \sum m(0,2,4,6) \\
&= Z'Y'X' + Z'YX' + ZY'X' + ZYX' \\
&= ?
\end{aligned}
$$

Z \ YX	00	01	11	10
0	1	0	0	1
1	1	0	0	1

$$
\begin{aligned}
F(Y,X) &= \sum m(0,2,4,6) \\
&= X'
\end{aligned}
$$

Z	Y	X	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$$
\begin{aligned}
F(Y,X) &= \prod M(1,3,5,7) \\
&= (Z+Y+X')(Z+Y'+X')(Z'+Y+X')(Z'+Y'+X') \\
&= ?
\end{aligned}
$$

Z \ YX	00	01	11	10
0	1	0	0	1
1	1	0	0	1

$$
\begin{aligned}
F(Y,X) &= \prod M(1,3,5,7) \\
&= (X)' \\
&= X'
\end{aligned}
$$

4-Variable KARNAUGH MAP