UNIT 2 SOLUTION OF NON-LINEAR EQUATIONS

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2.0 INTRODUCTION

In this unit we will discuss one of the most basic problems in numerical analysis. The problem is called a root-finding problem and consists of finding values of the variable x (real) that satisfy the equation f(x) = 0, for a given function f. Let f be a real-value function of a real variable. Any real number α for which $f(\alpha) = 0$ is called a root of that equation or a zero of f. We shall confine our discussion to locating only the real roots of f(x), that is, locating non-real complex roots of f(x) = 0 will not be discussed. This is one of the oldest numerical approximation problems. The procedures we will discuss range from the classical Newton-Raphson method developed primarily by Isaac Newton over 300 years ago to methods that were established in the recent past.

Myriads of methods are available for locating zeros of functions and in first section we discuss bisection methods and fixed point method. In the second section, Chord Method for finding roots will be discussed. More specifically, we will take up regula-falsi method (or method of false position), Newton-Raphson method, and secant method. In section 3, we will discuss error analysis for iterative methods or convergence analysis of iterative method.

We shall consider the problem of numerical computation of the real roots of a given equation

$$f(x) = 0$$

which may be algebraic or transcendental. It will be assumed that the function f(x) is continuously differentiable a sufficient number of times. Mostly, we shall confine to simple roots and indicate the iteration function for multiple roots in case of Newton Raphson method.

All the methods for numerical solution of equations discussed here will consist of two steps. First step is about the location of the roots, that is, rough approximate value of the roots are obtained as initial approximation to a root. Second step consists of methods, which improve the rough value of each root.

A method for improvement of the value of a root at a second step usually involves a process of successive approximation of iteration. In such a process of successive approximation a sequence $\{X_n\}$ n = 0, 1, 2, … is generated by the method used starting with the initial approximation x_o of the root α obtained in the first step such that the sequence $\{X_n\}$ converges to α as $n \to \infty$. This x_n is called the nth approximation of nth iterate and it gives a sufficiently accurate value of the root α.

For the first step we need the following theorem:

Theorem 1: If f(x) is continuous in the closed internal [a, b] and f(a) are of opposite signs, then there is at least one real root α of the equation f(x) = 0 such that $a < \alpha < b$.

If further f(x) is differentiable in the open interval (a, b) and either $f'(x) < 0$ or $f'(x) > 0$ in (a, b) then f(x) is strictly monotonic in [a, b] and the root α is unique.

We shall not discuss the case of complex roots, roots of simultaneous equations nor shall we take up cases when all roots are targeted at the same time, in this unit.

2.1 OBJECTIVES

After going through this unit, you should be able to:

- find an approximate real root of the equation f(x) = 0 by various methods;
- know the conditions under which the particular iterative process converges;
- define 'order of convergence' of an iterative method; and
- know how fast an iterative method converges.

2.2 ITERATIVE METHODS FOR LOCATING ROOTS

One of the most frequently occurring problem in scientific work is to find the roots of equations of the form:

$$f(x) = 0$$

In other words, we want to locate zeros of the function f(x). The function f(x) may be a polynomial in x or a transcendental function. Rarely it may be possible to obtain the exact roots of f(x) = 0. In general, we aim to obtain only approximate solutions using some computational techniques. However, it should be borne in mind that the roots can be computed as close to the exact roots as we wish through these methods. We say x^* satisfies f(x) = 0 approximately when $|f(x^*)|$ is small or a point x^* which is close to a solution of f(x) = 0 in some sense like $|x^* - \alpha| < \varepsilon$ where α is a root of f(x) = 0.

To find an initial approximation of the root, we use tabulation method or graphical method which gives an interval containing the root. In this section, we discuss two iterative methods (i) bisection method and (ii) fixed-point method. In a later section we shall discuss about the rate of convergence of these methods.

2.2.1 Bisection Method

Suppose a continuous function f, defined on the interval [a, b], is given, with f(a) and f(b) of opposite signs, i.e. f(a) f(b) < 0, then by Intermediate Value Theorem stated below, there exists a number α on the real line such that $a < \alpha < b$, for which $f(\alpha) = 0$.

Theorem 2 (Intermediate-value Theorem): If the function f is continuous on the closed interval [a, b], and if $f(a) \leq y \leq f(a)$, then there exists a point c such that $a \leq c \leq b$ and $f(c) = y$.

The method calls for a repeated halving of subintervals of [a, b] and, at each step, locating the "half" containing α. To start with, $a_1 = a$ and $b_1 = b$, and let α_1 be the mid point of [a, b], that is $\alpha_1 = \frac{1}{2}(a_1 + b_1)$. If $f(\alpha_1) = 0$, then $\alpha = \alpha_1$. If not, then $f(\alpha_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$. If $f(a_1)\, f(\alpha_1) < 0$, then root lies in (a_1, α_1). Otherwise the root lies in (α_1, b_1). In the first case we set $a_2 = a_1$ and $b_2 = \alpha_1$ and in the later case we set $a_2 = \alpha_1$ and $b_2 = b_1$. Now we reapply the process to the interval (a_2, b_2). Repeat the procedure until the interval width is as small as we desire. At each step, bisection halves the length of the preceding interval. After n steps, the original interval length will be reduced by a factor $\frac{1}{2^n}$ enclosing the root.

Figure 1: Bisection Method

We now mention some stopping procedures that could be applied to terminate the algorithm. Select a tolerance $\varepsilon > 0$ and generate $\alpha_1, \alpha_2, \ldots \alpha_n$ until one of the following conditions is met:

(i) $\quad |\alpha_n - \alpha_{n-1}| < \varepsilon,$ (2.2.1)

(ii) $\quad \dfrac{|\alpha_n - \alpha_{n-1}|}{|\alpha_n|} < \varepsilon,\ \alpha_n \neq 0,$ or (2.2.2)

(iii) $\quad |f(\alpha_n)| < \epsilon$ (2.2.3)

While applying bisection method we repeatedly apply a fixed sequence of steps. Such a method is called an Iteration method.

However, it is pertinent to mention that difficulties can arise using any of these stopping criteria. For example, there exist sequence $\{\alpha_n\}$ with the property that the differences $\alpha_n - \alpha_{n-1}$ converge to zero while the sequence itself diverges. Also it is possible for $f(\alpha_n)$ to be close to zero while α_n differs significantly from α. The criteria given by (2.2.2) is the best stopping criterion to apply since it tests relative error.

Though bisection algorithm is conceptually clear, it has significant drawbacks. It is very slow in converging. But, the method will always converge to a solution and for this reason it is often used to obtain a first approximation for more efficient methods that are going to the discussed.

Theorem 3: Let $f \in C[a, b]$ and suppose $f(a).f(b) < 0$. The bisection procedure generates a sequence $\{\alpha_n\}$ approximating α with the property,

$$|\alpha_n - \alpha| \leq \frac{b-a}{2^n}, n \geq 1.$$

Now we illustrate the procedure with the help of an example.

Example 1

Find the least positive root of equation.

$$f(x) = x^3 + 4x^2 - 10 = 0$$

Check that f(x) has only one root in the interval in which this least positive root lies. Find α_4 by bisection algorithm.

Solution

Consider the following tables of values.

X	0	1	2
f(x)	–10	–5	14

Take $a_1 = 1$, $b_1 = 2$, since $f(a_1) f(b_1) < 0$.

We give the four iterations in the following tabular form.

N	**a_n**	**b_n**	**α_n**	**$f(\alpha_n)$**
1	1	2	1.5	2.375
2	1	1.5	1.25	– 1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	– 0.84839
5	1.3125	1.375	1.34375	– 0.35098

After four iterations, we have $\alpha_4 = 1.3125$ approximating the root α with an error $|\alpha - \alpha_4| \leq |1.375 - 1.325| = .050$ and since $1.3125 < \alpha$.

$$\frac{|\alpha - \alpha_4|}{|\alpha|} < \frac{|b_5 - a_5|}{|a_5|} < \frac{.050}{1.3125} \leq \frac{.5}{10} = \frac{1}{2} 10^{-1} = \frac{1}{2} 10^{1-2}$$

That is, the approximation is correct to at least 2 significant digits.

Remarks 1: Generally the first stage methods for location of the roots of f(x) = 0 are (i) Tabulation method and (ii) Graphical method. The method of tabulation is very crude and labourious and we have used it in the above example to some extent in locating the least positive root of f(x) = 0. In graphical method we plot the graph of the curve y = f(x) on the graph paper and the points where the curve crosses the x-axis gives approximate values of the roots.

2.2.2 Fixed-point Method (or Method of Interation)

This method is also known as Method of Successive Approximations or Method of Iteration. In this method, we write the equation f(x) = 0.

For example $x^3 - x - 1 = 0$ can be written as,

$$x = (1+x)^{\frac{1}{3}}$$

or $$x = \frac{1+x}{x^2}$$

or $$x = \sqrt{\frac{1+x}{x}}$$

Now solving f(x) = 0 is equivalent to solving x = g(x).

Each such g(x) given above is called an iteration function. In fact, these are infinite number of ways in which the original equation f(x) = 0 can be written as x = g(x). Out of all these functions where one is to be selected, will be discussed in the following analysis.

Definition 1: A number ξ is called a fixed point of g(x) if $g(\xi) = \xi$ and g is called the iteration function.

Our problem is now to find out fixed point(s) of g(x). Graphically x = g(x) is equivalent to solving y = x and y = g(x).

Figure 2: Fixed Point Method

Once an iteration function is chosen, to solve x = g(x), we start with some suitable value x_0 close to the root (how to choose this will be explained) and calculate $x_1 = g(x_0)$ (the first approximation), then $x_2 = g(x_1)$ (second approximation) and so on.

In general

$$x_{n+1} = g(x_n),\ n = 0, 1, 2 \ldots$$

The sequence $\{x_n\}$ converges (under some suitable conditions on g) to a number ξ (say). If g is continuous then this gives $\xi = g(\xi)$, that is, ξ is a fixed point of g(x).

Concerning the existence, uniqueness of a fixed point and convergence of the sequence, we state a theorem below:

Theorem 4 (Fixed Point Theorem): Let iteration function g(x) be defined and continuous on a closed interval I = [a, b]. suppose further that g(x) satisfies the following:

(i) $g(x) \in I$ for all $\in I$

(ii) g(x) is differentiable on I = [a, b]
and there exists a non-negative number k < 1 such that for all $x \in I$, $|g'(x)| \leq k < 1$.

Then

(a) $g(x)$ has a fixed point ξ,

(b) the fixed point is unique, and

(c) the sequence $\{x_1\}$ generated from the rule $x_{n+1} = g(x_n)$ converges to ξ, the fixed point of $g(x)$, when $x_0 \in [a, b]$

Proof: (a) **Existence:** Suppose $g(a) = a$ or $g(b) = b$, then there is nothing to be proved. So, suppose $g(a) \neq a$ and $g(b) \neq b$. Then $g(a) > a$ and $g(b) < b$ since $g(x) \in I$ for all $x \in I$.

Consider $h(x) = g(x) - x$

Then $h(a) = g(a) - a > 0$ and
$h(b) = g(b) - b < 0$

Also $h(x)$ is continuous on I since $g(x)$ is so. Hence by Intermediate Value Theorem, there exists a number ξ, $a < \xi < b$ such that $h(\xi) = 0 \Rightarrow$

$$g(\xi) - \xi = 0, \text{ i.e., } g(\xi) = \xi$$

Hence $g(x)$ has a fixed point in I.

(b) **Uniqueness**: From (2.2.4)

$$h'(x) = g'(x) - 1, \text{ but } |g'(x)| \leq k < 1$$

Hence $h'(x) < 0$.

Therefore, $h(x)$ is a decreasing function and it crosses x-axis only once, i.e. $h(x)$ vanishes only once in I.

Therefore $g(x) - x = 0$ only for unique value of x in (a, b). Hence uniqueness.

(c) **Convergence**: Let ξ be the fixed point of $g(x)$. We have
$\xi = g(\xi)$ and $x_{n+1} = g(x_n)$.

Let $e_{n+1} = \xi - x_{n+1} = g(\xi) - g(x_n) = g'(\eta_n)\ (\xi - x_n)$, where η_n lies between x_n and ξ, that is, $e_{n+1} = g'(\eta_n)e_n$.

Thus, we have $|e_{n+1}| \leq k\ |e_n|$. Using this repeatedly

$$|e_n| \leq k^n\ |e_0|$$

$$\lim_{m\to\infty} |e_n| = 0 \text{ since } k < 1,$$

$$\text{i.e. } \lim_{m\to\infty} |\xi - x_n| = 0 \Rightarrow \{x_n\} \to \xi$$

(The sequence $\{x_n\}$ converges to the number ξ)

Hence proved.

Thus, it may be noted that the iterative scheme $x_{n+1} = g(x_n)$ converges under the condition $|g'(x)| < 1$, $x \in [a, b]$.

Example 2

For $x^3 - x - 1 = 0$, find a positive root by the fixed point method. Find minimum number of iterations so that nth approximate x_n is correct to 4 decimal places.

Solution

Write $x = (1+x)^{\frac{1}{3}} = g(x)$.

The root lies between 1 and 2 since f(1) = –1 and f(2) = 3.

Also $g(1) - 1 = 2^{\frac{1}{3}} - 1 = +ve$

$g(2) - 2 = 3^{\frac{1}{3}} - 2 = -ve$

$I = [a, b] = [1, 2]$

$g'(x) = \dfrac{1}{3(1+x)^{\frac{2}{3}}}$ is decreasing function and

$$\max_{x \in I} |g'(x)| = g'(1) = \frac{1}{3 \times 2^{\frac{2}{3}}} = k < 1.$$

Since $g'(x)$ = +ve, therefore g(x) is increasing.

$$\max_{x \in I} g(x) = g(2) = 3^{\frac{1}{3}} = 1.442 < 2$$

$$\min_{x \in I} g(x) = g(1) = 2^{\frac{1}{3}} > 1.$$

Hence, $g(x) \, \varepsilon \, I$ for all $x \in I$.

Therefore, $x_{n+1} = (1 + x_n)^{\frac{1}{3}}$ generates a sequence of numbers which converges to a fixed point of g(x), (starting with $x_o \in I$).

We have $k = \dfrac{1}{3 \times 2^{\frac{2}{3}}} < 1$ and

$|e_n| \leq k^n |e_n|$ and $|e_o| < 1$. Hence for the desired accuracy we have

$$|e_n| \leq \left(\frac{1}{3 \times 2^{\frac{2}{3}}} \right)^n < \frac{1}{2} \, 10^{-4} \Rightarrow n = 7.$$

Remark 2: In the following figures we observe the importance of $g'(x)$ in the neighbourhood of a fixed point ξ.

Figure 3

In the neighbourhood of ξ, $|g'(x)|>1$ (the sequences converge in these cases Fig. 3).
In the neighbourhood of ξ, $|g'(x)| < 1$ (the sequences converge in these two cases Fig. 4).

Figure 4

Remark 3: In numerical problems, one may follow the following procedure to find an interval [a, b].

In order to use this method one needs only to see if $|g'(x)|<1$ at a point in the neighbourhood of the root. Therefore, determining an interval I is not necessary.

Choose an interval [a, b] by some trial and check for the following:

(i) $a - g(a)$ and $b - g(b)$ must be of opposite sign (with $b - g(b) > 0$).
(ii) $|g'(x)| \le k < 1$ for $x \in [a, b]$
(iii) $g'(x)$ is continuous on [a, b].

If above conditions are not satisfied try for a smaller interval and so on.

Example 3

Find the smallest positive root of $e^{-x} - \cos x = 0$ by the fixed point method.

Solution

To locate the smallest positive root, we draw the figures of
$y = e^{-x}$ and $y = \cos x$

Figure 5 **Figure 6**

Figure shows that the desired root lies between 0 and $\frac{\pi}{2}$ i.e. in $(0, \frac{\pi}{2})$.

Now let us try x = $\cos^{-1}(e^{-x}) = g(x)$

$$g'(x) = \frac{1}{\sqrt{e^{2x}-1}}$$

To make this less than 1, we must choose $e^{2x} - 1 > 1$, that is, $e^{2x} > 2$. This gives $x > \frac{1}{2}\ln 2$ This suggest that we should take the suggested interval

$(\frac{1}{2} \ln 2, \frac{\pi}{2})$, but to take a closed interval, we consider $I = [\ln 2, \frac{\pi}{2}]$.

Derivative of g(x) implies that g(x) is an increasing function.

$$\max_{x\in I} g(x) = g(\ln 2) = \cos^{-1}(e^{-\ln 2})$$

$$= \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} = \frac{22}{21} > \ln 2$$

$$\max_{x\in I} g(x) = g(\frac{\pi}{2}) = \cos^{-1}(e^{-\frac{\pi}{2}}) < \frac{\pi}{2}$$

since $e^{-\frac{\pi}{2}}$ is positive. Hence g(x) $\in$ I for all x $\in$ I.

$$\max_{x\in I} |g'(x)| = \frac{1}{\sqrt{e^{2\ln 2}-1}} = \frac{1}{\sqrt{e^{\ln^4}-1}} = \frac{1}{\sqrt{3}} = k < 1$$

since $g'(x)$ is a decreasing function.

Hence all the sufficient conditions of the Theorem 4 are satisfied.

Now further, suppose that we want to find minimum number of iteration required to get 4 decimal place accuracy. Let n be the minimum number of iterations required for the desired accuracy.

$$|e_n| \le k^n |e_o| \le \frac{1}{2} 10^{-4}$$

$|e_o| \le \left|\frac{\pi}{2} - \ln 2\right| \le 1.$ Thus the given condition is satisfied if

$\left(\frac{1}{\sqrt{3}}\right)^n \le \frac{1}{2}10^{-4}$. That is,

$-n \log_{10}\sqrt{3} \le -4 - \log_{10} 2$ i.e.

$n \ge \frac{4.301}{0.238} = 18.07$ i.e., n =19

Example 4: Find the iteration function and interval I = [a, b] which satisfy the conditions of the theorem of fixed point to find the smallest positive root of x = tan x.

Solution:

We rewrite the equation x = tan x as $x = n\pi + \tan^{-1} x, n = 0, \pm 1, \pm 2, ..$

We know that $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$, so for desired root we take n = 1, that is,

We consider x = $\pi + \tan^{-1} x = g(x)$ and

consider I = $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

For x $\in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, we have $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$, hence $\frac{\pi}{2} < g(x) < \frac{3\pi}{2}$

Also $\max_{x\in I} |g'(x)| = \max_{x\in I} \frac{1}{1+x^2} = \frac{1}{1+\frac{\pi^2}{4}} = \frac{4}{4+\pi^2} < 1$

(Since $g'(x)$ is a decreasing function).

Hence for any $x_o \in I = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, the sequence generated by the fixed-point iteration method will converge.

Remark 5: If ξ is a fixed point of g(x) lying in the open interval (c, d) on which $|\varphi'(x)|$ then the sequence $\{x_n\}$ generated with g(x) as iteration function will not converge to ξ, however close x_o to ξ is taken except accidentally. (Consider the root $\xi = 2$ of $f(x) = x^2 - x - 2 = 0$ with $g(x) = x^2 - 2$).

Remark 6: If ξ is a fixed point of g(x) such that $|g'(\xi\xi| = 1$, then the iteration function with g(x) may or may not converge to ξ. However, if $|g'(\xi\xi| < 1$, in some deleted neighbourhood of ξ, then it will converge to ξ, with x_o taken sufficiently close to ξ. If $|g'(\xi\xi)| > 1$, in some deleted neighbourhood of ξ, then sequence will not converge to ξ.

Remark 7: The conditions mentioned in fixed-point theorem are sufficient but not necessary.

Now we discuss one example, which is very simple but conveys the fact that if a function f(x) has more zeros i.e. f(x) = 0 has more than one real root, then we may have to consider different g(x) for different roots.

Example 5: Find the iteration function g(x) and corresponding interval to get the two roots 1 and 2 by fixed point iteration method for the equations

$$x^2 - 3x + 2 = 0$$

Solution:

(a) For the root x = 1 if we consider $x = \sqrt{3x-2} = g(x)$, then

$g'(x) = \frac{3}{2\sqrt{3x-2}}$ and

$g'(1) = 1$. Hence we choose $g(x) = \frac{x^2+2}{3}$, $I_1 = \left[\frac{1}{2}, \frac{5}{4}\right]$

$g'(x) = \frac{2x}{3} > 0$ for $x \in I_1$. Hence g(x) is increasing. Also $\max_{x\in I_1} |g'(x)| = \frac{5}{6} < 1$.

$$\max_{x\in I_1} g(x) = \frac{\frac{25}{16}+2}{3} = \frac{57}{48} < \frac{5}{4}$$

$$\max_{x\in I_1} g(x) = \frac{\frac{1}{4}+2}{3} = \frac{3}{2} < \frac{1}{2}$$

Hence all the conditions of the theorem are satisfied.

(b) Now for the other root 2, consider

If $g(x) = \frac{x^2+2}{3}$, then $g'(2) = \frac{4}{3} > 1$. Hence we choose $g(x) = \sqrt{3x-2}$ with

$I_2 = \left[\frac{3}{2}, \frac{5}{2}\right]$

$g'(x) = \frac{3}{2\sqrt{3x-2}} > 0$ for all $x \in I_2$, so g(x) is increasing.

$\max_{x \in I_2} g(x) = \sqrt{\frac{11}{2}} < \frac{5}{2}$

$\min_{x \in I_2} g(x) = \sqrt{\frac{5}{2}} > \frac{3}{2}$ so g(x) : maps I_2 into itself.

Also $\max_{x \in I_2} |g'(x)| = \frac{3}{2\sqrt{\frac{9}{2}-2}} = \frac{3}{\sqrt{10}} < 1$ (since $g'(x)$ is a decreasing function).

Hence all the conditions for the fixed point theorem are satisfied.

In the following two examples, we use the corollary to the fixed point theorem (Theorem 4).

Example 6: The equation $f(x) = x^4 - x - 10 = 0$ has a root in the interval [1, 2]. Derive a suitable iteration function φ(x) such that the sequence of iterates obtained from the method $x_{k+1} = \varphi(x_k)$, k = 0, 1, 2, ... converges to the root of f(x)=0. Using this method and the initial approximation $x_0 = 1.8$, iterate thrice.

Solution: Choose $\varphi(x) = (x+10)^{1/4}$, I = [1, 2].

Then $\varphi'(x) = \frac{1}{4}(x+10)^{-\frac{3}{4}} = \frac{1}{4} \cdot \frac{1}{(x+10)^{\frac{3}{4}}}$

$\max_{x \in I_2} \varphi'(x) = \frac{1}{4} \cdot \frac{1}{(11)^{\frac{3}{4}}}$

i.e., $\varphi'(x) < 1$ for $x \in [1, 2]$

Given $x_0 = 1.8$

$x_1 = (1.8+10)^{1/4} = 1.8534 = 1.86$

$x_2 = (1.8+10)^{1/4} = 1.8534 = 1.86$

$x_3 = (1.8+10)^{1/4} = 1.8534 = 1.86$

Example 7: The equation $f(x) = x^3 - 5x + 1 = 0$ has a root in the interval [0, 1]. Derive a suitable iteration function φ(x), such that the sequence of iterates obtained from the formula $x_{k+1} = \varphi(x_k)$, k = 0, 1, 2, ... converge to the root of f(x)=0. Using this formula and the initial approximation $x_0 = 0.5$, iterate thrice.

Solution: $\varphi(x) = \frac{x^3+1}{5}$ is chosen since $\varphi'(x) = \frac{3x^2}{5}$ and $\max_{0 \in x \in I} \varphi'(x) < 1$.

With $x_0 = 0.5$, $x_1 = 0.225 = 0.23$, $x_2 = 0.202$.

What about choosing $\varphi(x) = (5x-1)^{\frac{1}{3}}$?

What is $\max_{0 \in x \in I} \varphi'(x)$ in this case?

2.3 CHORD METHODS FOR FINDING ROOTS

In the previous section we have introduced you to two iterative methods for finding the roots of an equation f(x) = 0, namely bisection method and fixed point method. In this section we shall discuss three iterative methods: regula-falsi, Newton-Raphson, and Secant methods. In the next section we shall discuss the efficiency of these methods.

2.3.1 Regula-falsi Method

This method attempts to speed up bisection method retaining its guaranteed convergence. Suppose we want to find a root of the equation f(x) = 0 where f(x) is a continuous function. We start this procedure also by locating two points x_o and x_1 such that $f(x_o)f(x_1) < 0$.

Let us consider the line joining $(x_o, f(x_o))$ and $(x_1, f(x_1))$. This line cuts the x-axis at some point, say x_2. We find $f(x_2)$. If $f(x_2)f(x_0) < 0$, then we replace x_1 by x_2 and draw a straight line connecting $(x_2, f(x_2))$ and $\{x_o, f(x_o)\}$. If $f(x_2)$ and $f(x_o)$ are such that $f(x_2)f(x_0) > 0$, then x_0 is replaced by x_2 and draw a straight line connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Where the straight line crosses x – axis. that print gives x_3. In both the cases, the new interval obtained is smaller then the initial interval. We repeat the above procedure. Ultimately the sequence is guaranteed to converge to the desired root.

Figure 7

The equation of the chord PQ is $y - f(x_0) = \dfrac{f(x_1) - f(x_o)}{x_1 - x_o}(x - x_o)$

This cuts x-axis at the point x_2 given by

$$0 - f(x_o) = \frac{f(x_1) - f(x_o)}{x_1 - x_o}(x_2 - x_o)$$

i.e. $x_2 = \dfrac{x_o f(x_1) - x_1 f(x_o)}{f(x_1) - f(x_o)}$.

In general, $x_{r+1} = \dfrac{x_{r-1} f(x_r) - x_r f(x_{r-1})}{f(x_r) - f(x_{r-1})}$, r = 1, 2, 3,....

If $f(x_2) = 0$, then x_2 is the required root. If $f(x_2) \neq 0$ and $f(x_o)f(x_2) < 0$, then the next approximation lies in (x_o, x_2). Otherwise it lies in (x_2, x_1). Repeat the process till $|x_{i+1} - x_i| < \varepsilon$.

Example 8: The equation $2x^3 + 5x^2 + 5x + 3 = 0$ has a root in the interval [–2, –1]. Starting with $x_o = -2.0$ and $x_1 = -1.0$ as initial approximations, perform three iteration of the Regula-falsi method.

Solution:

$f(-2) = -16 + 20 - 10 + 3 = -3$

$f(-1) = -2 + 5 - 5 + 3 = 1$, and $f(-2)\, f(-1) < 0$

$$x_2 = \frac{x_o f(x_1) - x_1 f(x_o)}{f(x_1) - f(x_o)} = \frac{-2\times 1-(-1)(-3)}{1-(-3)} = \frac{-5}{4}\text{ , i.e.,}$$

$x_2 = -1.25$ (First iteration)

$$f(x_2) = \frac{-125}{32} + 5\times\frac{25}{16} + 5\times\frac{-5}{4} + 3 = \frac{21}{32}$$

The root lies in (x_o, x_2)

$x_3 = -1.384$ or 1.39 (Second iteration) since

$$x_3 = \frac{-2\times\frac{21}{32} - \left(-\frac{5}{4}\right)\times(-3)}{\frac{21}{32}-(-3)} = \frac{-\frac{42}{32}-\frac{5}{4}}{\frac{21}{32}+3} = \frac{-\frac{42-120}{32}}{\frac{21+96}{32}}$$

$$= \frac{-162}{117} = -1.384 \approx -1.39.$$

For next iteration find $f(x_3)$ and proceed in similar fashion.

2.3.2 Newton-Raphson Method

Newton-Raphson method or N–R method in short.

It can be introduced by basing it on the Taylor's expansion as explained below. Let x_o be an initial approximation and assume that x_o is close to the exact root α and $f'(x_o) \neq 0$. Let $\alpha = x_o + h$ where h is a small quantity in magnitude. Let f(x) satisfy all the conditions of Taylor's theorem. Then

$f(x_o + h) = f(x_o) + h f'(x_o) + ...$

The method is derived by assuming that the term involving h^2 is negligible and that $f(x_o) + h f'(x_o) = 0$ i.e. $f(x_o) + (\alpha - x_o)f'(x_o) = 0$

$$\text{i.e. } \alpha \approx x_o - \frac{f(x_o)}{f'(x_o)}$$

$$\text{i.e. } x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$$

Geometrically the next approximation, x_1, is the abscissa of the point of intersection of the tangent PT and the x-axis in Figure 8.

The iteration scheme is

$$.\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, ...$$

Figure 8 : Newton – Raphson Method

N-R method is an extremely powerful technique, but it has a major difficulty – the need to know the value of the derivative of f at each approximation or iteration. derivative evaluation, we discuss a slight variation, known as Secant Method next.

Example 9: Newton-Raphson method is used to find the pth root of a positive real number R. Set up the iteration scheme. Perform three iterations of the method for R=16.0, p =3, starting with the initial approximation 2.0.

Solution: Let us denote pth root of R by x i.e.

$x = R^{1/p}$ or $x^p - R = 0$.
$f'(x) = px^{p-1}$.

Newton-Raphson Iteration scheme is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$
$$= x_k - \frac{x_k^p - R}{px_k^{p-1}},$$

On simiplification we get $x_{k+1} = \left(1 - \frac{1}{p}\right)x_k + \frac{R}{p\,x_k^{p-1}}$, $k = 0, 1, 2, \ldots$

For R = 16, p = 3, x_o = 2, we get $x_1 = \frac{8}{3} = 2.67$, $x_2 = \frac{91}{36} = 2.53$,

Remark 8: If a root is repeated m times, the N–R method is modified as

$$x_{k+1} = x_k - in\frac{f(x_k)}{f'(x_k)}$$

Example 10 : The quadric equation $x^4 - 4x^2 + 4 = 0$ has a double root. Starting with x_o = 1.5, compute two iterations by Newton-Raphson method.

Solution: For m-repeated root of f(x) = 0, the iteration scheme in case of Newton-Raphson method is given by:

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}, \ k = 0, 1, 2, \ldots$$

In this case, we have

$$x_{k+1} = x_k - \frac{2(x_k^4 - 4x_k^2 + 4)}{4x_k^3 - 8x_k}, \ (\text{since } m = 2 \text{ and } f(x) = x^4 - 4x^2 + 4)$$

With x_o = 1.5, we have

$$x_1 = \frac{3}{2} - \frac{2 \times \frac{1}{16}}{\frac{3}{2}} = \frac{17}{12} = 1.41$$

Example 11: Perform two iterations of the Newton-Raphson method to find an approximate value of $\frac{1}{15}$ starting with the initial approximation x_o = 0.02

Solution: Suppose we want to find the reciprocal of the number N.

Let $f(x) = \frac{1}{x} - N$

Then $f'(x) = \frac{1}{x^2}$ and the iteration scheme is

$$x_{k+1} = x_k - \frac{\frac{1}{x_k} - N}{\frac{-1}{x_k^2}} = 2x_k - Nx_k^2, \; k = 0, 1, 2...$$

In this case we $x_{k+1} = 2x_k - 15\ x_k^2$, k = 0, 1,2. This gives $x_1 = 0.034$, $x_2 = 0.051$, $x_3 = 0.063$, etc.

2.3.3 Secant Method

This method is a modification of the regula-falsi method and retains the use of secants throughout, but dispenses with the bracketing of the root. Given a function f(x) and two given points x_o and x_1,

We compute,

$$x_2 = x_o - \frac{f(x_o)}{\frac{f(x_1) - f(x_o)}{x_1 - x_o}} = \frac{x_o f(x_1) - x_1 f(x_o)}{f(x_1) - f(x_o)}.$$

Figure 9

Above figure illustrates how x_{n+1} is obtained.

Example 12: Apply the Secant method to find a root of the equation

$2x^3 + 3x^2 + 3x + 1 = 0$. Take the initial approximations as $x_o = -0.2$ and $x_1 = -0.4$.

Solutions:

Let $f(x) = 2x^3 + 3x^2 + 3x + 1$

$f(-0.2) = 0.504$

$f(-0.4) = 0.152$

$$x_2 = \frac{x_o f(x_1) - x_1 f(x_o)}{f(x_1) - f(x_o)}, = \frac{.1712}{.352} = 00.48$$

You may now solve the following exercises:

E1) Using the Newton-Raphson method, find the square root of 10 with initial approximation $x_o = 3$.

E2) A fixed point iteration to find a root of $3x^3 + 2x^2 + 3x + 2 = 0$ close to $x_o = -0.5$ is written as $x_{k+1} = -\dfrac{2 + 3x_k + 2x_k^2}{3x_k^2}$

Does this iteration converge? If so, iterate twice. If not, write a suitable form of the iteration, show that it converges and iterate twice to find the root.

E3) Do three iterations of Secant method to find an approximate root of the equation.
$3x^3 - 4x^2 + 3x - 4 = 0$
Starting with initial approximations $x_o = 0$ and $x_1 = 1$.

E4) Do three iterations of fixed point iteration method to find the smallest positive roots of $x^2 - 3x + 1 = 0$, by choosing a suitable iteration function, that converges. Start with $x_o = 0.5$.

E5) Obtain the smallest positive root of the equation of $x^3 - 5x + 1 = 0$ by using 3 iterations of the bisection method.

E6) Starting with $x_o = 0$, perform two iterations to find an approximate root of the equation $x^3 - 4x + 1 = 0$, using Newton-Raphson method.

E7) Do three iterations of the Secant method to solve the equation
$x^3 + x - 6 = 0$,
starting with $x_o = 1$ and $x_2 = 2$.

E8) Apply bisection method to find an approximation to the positive root of the equation.
$2x - 3 \ \text{Sin}x - 5 = 0$
rounded off to three decimal places.

E9) It is known that the equation $x^3 + 7x^2 + 9 = 0$ has a root between -8 and -7. Use the regula-falsi method to obtain the root rounded off to 3 decimal places. Stop the iteration when $\left|x_{i+1} \quad x_i\right| < 10 \quad ^4$

E10) Determine an approximate root of the equation
$\text{Cos}x - xe^{x} = 0$
using Secant method with the two initial approximations as $x_o = 0$ and $x_1 = 1$. Do two iterations

2.4 ITERATIVE METHODS & CONVERGENCE CRITERIA

Let $\{x_n\}$ be a sequence of iterates of a required root α of the equation $f(x) = 0$, generated by a given method.

The error at the nth iteration, denoted by e_n is given by

$$e_n = \alpha - x_n$$

The sequence of iterates $\{x_n\}$ converge to α if and only if $e_n \to 0$ as $n \to \infty$ otherwise the sequence of iterates diverges.

For each method of iteration considered by us, we shall discuss conditions under which the iteration converges.

Let x_0, x_1, x_2, etc be a sequence generated by some iterative method.

2.4.1 Order of Convergence of Iterative Methods

Definition 2: If an iterative method converges, that is, if $\{x_n\}$ converges to the desired root α, and two constants p $\geq$ 1 and C > 0 exist such that

$$\lim_{n\to\infty} \left|\frac{e_{n+1}}{e_n^p}\right| = C \text{ (C does not depend on n)}$$

then p is called the order of convergence of the method and C is called the asymptotic error constant. An iterative method with higher order of convergence than 1 is expected to converge rapidly. If p = 1, 2, 3, …, then the convergence is called linear, quadratic, cubic… respectively.

(i) For the Fixed Point Iteration method the order of convergence is generally 1, that is, it is of first order (convergence is linear).

(ii) For the Newton-Raphson method, with x_o near the root, the order of convergence is 2, that is, of second order (convergence is quadratic).

(iii) For the Secant Method order of convergence is 1.618 $\approx$ 1.62 but it is not guaranteed to converge.

The bisection method is guaranteed to converge, but convergence is slow. Regula-falsi method is guaranteed to converge. However, it is slow and order of convergence is 1.

2.4.2 Convergence of a Fixed Point Method

Theorem 5:

If g ’(x) is continuous in some neighbourhood of the fixed point ξ of g, then the fixed point method converges linearly provided $g'(\xi) \neq 0$.

Proof: $e_{n+1} = \xi - x_{n+1}$

$$= g(\xi) - g(x_n)$$

$$= g'(\eta_n)\,(\xi - x_n)$$

for some η_n lying between x_n and ξ.

$e_{n+1} = e_n\, g'(\eta_n)$

Since g ’(x) is continuous in a neighbourhood of ξ, we can write

$g'(\eta_n) = g'(\xi) + h_n$ such that $\lim\limits_{n\to\infty} h_n = 0$.

$e_{n+1} = e_n\{g'(\xi) + h_n\}$

On taking $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = g'(\xi) = C \neq 0$$

Hence p = 1, since $\dfrac{|e_{n+1}|}{|e_n|} = |g'(\xi)|$

Therefore, fixed point method converges linearly.

Note: Smaller the value of $|g'(\xi)|$, faster would be the convergence.

Theorem 6: If g ’’(x) is continuous in some neighbourhood of the fixed point ξ of g, then the fixed point method converges quadratically, provided $g'(\xi) = 0$ and $g''(\xi) \neq 0$.

Proof: We have

$e_{n+1} = \xi - x_{n+1}$

$e_{n+1} = g(\xi) - g(x_n)$

$= g(\xi) - g(\xi - e_n)$

By using Taylor's series expansion, we have

$e_{n+1} = g(\xi) - \{g(\xi) - e_n g'(\xi) + \frac{e_n^2}{2} g''(\eta_n)\}$

for some η_n lying in the interval of x_n and ξ. That is,

$e_{n+1} = -\frac{e_n^2}{2}\left[g''(\xi) + h_n\right]$ since $g''(x)$ is continuous, $h_n \to 0.$ as $n \to \infty$.

Thus, $\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \frac{|g''(\xi)|}{|2|} = C \neq 0$

Here p = 2, hence convergence is quadratic.

Fixed-point iteration is effective when it converges quadratically, as in Newton-Raphson method discussed below.

N–R Method

We define for equation f(x) = 0 an iterative function g(x) as

$g(x) = x - \frac{f(x)}{f'(x)}$, then the method is called Newton's method. We state a theorem without proof which suggests an interval in which if x_o is taken then Newton's method converges. We generate the sequence $\{x_n\}$ as $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, n =1, 2, 3, …

2.4.3 Convergence of Newton's Method

Theorem 7:

Suppose we are to solve f(x) = 0. If $f'(x) \neq 0$ and $f''(x)$ is continuous on the closed finite interval [a, b] and let the following conditions be satisfied:

(i) $f(a)\,f(b) < 0$

(ii) $f''(x)$ is either ≥ 0 or ≤ 0 for all $x \in [a, b]$

(iii) At the end points a, b

$$\frac{|f(a)|}{|f'(a)|} < b - a \text{ and } \frac{|f(b)|}{|f'(b)|} < b - a.$$

Then Newton's method converges to the unique solution ξ of f(x) = 0 in [a, b] for any choice of $x_0 \in [a, b]$.

Theorem 8: Let f(x) be twice continuously differentiable in an open interval containing a simple root ξ of f(x) = 0. Further let f"(x) exists in neighbourhood of. Then the Newton's method converges quadratically.

Proof: $g(x) = x - \dfrac{f(x)}{f'(x)}$ is continuously differentiable in some open neighbourhood of ξ. On differentiating g(x), we get

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f'(x)}{[f'(x)]^2}$$

$$= \frac{f(x)f''(x)}{[f'(x)]^2}$$

$g''(\xi) = 0$, since $f(\xi) = 0$ $\quad$ ($f'(\xi) \neq 0$ for a simple root ξ)

$$\text{Also } g''(x) = \frac{-2f(x)[f'(x)][f''(x)]^2 + [f'(x)]^2\{f'(x)f''(x) + f(x)f'''(x)\}}{[f'(x)]^4}$$

$$= \frac{[f'(x)]^2 f''(x) + f(x)f'(x)f'''(x) - 2f(x)[f''(x)]^2}{[f'(x)]^3}$$

$$g''(\xi) = \frac{f''(\xi)}{f'(\xi)} \neq 0$$

By Taylor's formula, we have

$$e_{n+1} = \xi - x_{n+1} = g(\xi) - g(x_n)$$

$= -g'(\xi)(x_n - \xi) - \dfrac{1}{2}g''(\eta_n)(x_n - \xi)^2$ for some η_n between ξ and x_n. That is

$$e_{n+1} = g'(\xi)e_n - \frac{1}{2}g''(\eta_n)e^2_n$$

since $g'(\xi) = 0$, and $g''(x)$ is continuous, we have

$$e_{n+1} \cong -\frac{1}{2}g''(\xi)\, e_n^2 .$$

Hence the Newton's Method converges quadratically if of x_o is chosen sufficiently close to ξ, where ξ is a simple root of ξ.

2.4.4 Rate of Convergence of Secant Method

Suppose f(x) = 0 is to be solved. Consider the curve y = f(x).

Figure 10

Let the chord AB through the points $A(x_{n-1}, f(x_{n-1}))$ and $B(x_n, f(x_n))$ be drawn. Suppose this intersects x–axis at C. Denote this value of x by x_{n+1}. That is

$$y - f(x_{n-1}) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1})$$

$y = 0$, $x = x_{n+1}$, we get

$$x_{n+1} = x_{n-1} - \frac{f(x_{n-1})(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

This is known as secant method. The sequence $\{x_n\}$ is generated with starting points x_0, x_1. We get x_2, reject x_0, and use x_1, x_2 to get x_3 and so on.

Let $e_{n+1} = \xi - x_{n+1}$

$$= \xi - \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

writing $e_n = \xi - x_n$, $e_{n-1} = \xi - x_{n-1}$

and $x_n = \xi - e_n$, $x_{n-1} = \xi - e_{n-1}$

we get

$$e_{n+1} = \frac{e_{n-1} f(\xi - e_n) - e_n f(\xi - e_{n-1})}{f(\xi - e_n) - f(\xi - e_{n-1})}$$

$$= \frac{e_{n-1}\left\{ f(\xi) - e_n f'(\xi) - \frac{e_n^2}{2} f''(\xi) + \text{higher order terms} \right\} - e_n \left\{ f(\xi) - e_{n-1} f'(\xi) + \frac{e_n^2}{2} f''(\xi) + \text{higher order terms} \right\}}{f(\xi) - e_n f'(\xi) - f(\xi) + e_{n-1} f'(\xi)}$$

since $\because f(\xi) = 0$

$$e_{n+1} \cong \frac{\frac{1}{2}\left(e_n^2 e_{n-1} - e_n e_{n-1}^2\right) f''(\xi)}{-(e_n - e_{n-1}) f'(\xi)}$$

$$= -\frac{1}{2}\frac{f''(\xi)}{f'(\xi)} e_n e_{n-1}, \text{ for sufficiently large } n$$

Let $-\frac{1}{2}\frac{f''(\xi)}{f'(\xi)} = \alpha$ (fixed constant).

Then $e_{n+1} = \alpha\, e_n e_{n-1}$

Suppose $e_{n+1} = C\, e_n^p \Rightarrow e_n = C e_{n-1}^p$ substituting these in previous results

$$C e_n^p = \alpha\, e_n \left(\frac{e_n}{C}\right)^{\frac{1}{p}} = \frac{\alpha}{C^{1/p}} e_n^{1 + \frac{1}{p}}$$

Equating powers of p, on both sides we get

$$P = 1 + \frac{1}{p} \Rightarrow p^2 - p - 1 = 0$$

$$P = \frac{1 \pm \sqrt{5}}{2}, \qquad p = 1.618.$$

Now after comparing the rate of convergence of fixed point method, N–R Method and Secant method, we find that secant is faster than fixed point month and N–R method is faster than secant method. For further qualitative comparison refer the books mentioned.

Apart from the rate of convergence, the amount of computational effort required for iteration and the sensitivity of the method to the starting value and the intermediate values, are two main basis for comparison of various iterative methods discussed here. In the case of Newton's method, if $f'(x)$ is near zero anytime during the iterative cycle, it may diverge. Furthermore, the amount of computational effort to compute $f(x)$ and $f''(x)$ is considerable and time consuming. Whereas the fixed point method is easy to programme.

You may now solve the following exercises.

E11) Let M denote the length of the initial interval $[a_0, b_0]$. Let $(x_0, x_1, x_3 ...)$ represent the successive midpoints generated by the bisection method. Show that $|x_{i+1} - x_i| = \frac{M}{2^{i+2}}$

Also show that the number n of iterations required to generate an approximation to a root to an accuracy ε is given by

$$n > -2 - \frac{\log\left(\varepsilon/M\right)}{\log 2}$$

E12) If $x = \xi$ is a zero of f(x) of order 2, then $f(\xi) = 0$, $f'(\xi) = 0$ and $f''(\xi) \neq 0$. Show that in this case Newton-Raphson's method no longer converges quadratically. Also show that if $f'(\xi) = 0$, $f''(\xi) \neq 0$ and $f'''(x)$ is continuous in the neighbourhood of ξ, the iteration

$$x_{i+1} = x_i - \frac{2f(x_i)}{f(x_i)} = g(x_i)$$

does converge quadratically.

E13) The quadratic equation $x^4 - 4x^2 + 4 = 0$ has a double root at $\sqrt{2}$. Starting with $x_0 = 1.5$, compute three successive approximations to the root by Newton-Raphson method. Do this with $g_1(x) = x - \frac{f(x)}{f'(x)}$ and $g_2(x) = x - \frac{2f(x)}{f'(x)}$ and comment on the order of convergence from your results.

E14) The following are the five successive iterations obtained by the Secant method to find the real positive root of the equation $x^3 - x - 1 = 0$ starting with $x_0 = 1.0$ and $x_1 = 2.0$.

n	2	3	4	5	6	7
x_n	1.166667	1.2531120	1.3372064	1.3238501	1.3247079	1.3247180

Calculate $|e_n|$ and $|e_{n+1}|/|e_n e_{n-1}|$ for n = 2, 3, 4. Also compute the constant directly $\left(f''(\xi)/2f'(\xi)\right)$ assuming the value of ξ correct to eight decimal places as ξ= 1.324718.

E15) If $a_0 = 0$ and $b_0 = 1.0$, how many steps of the bisection method are needed to determine the root with an error of at most 10^{-5}?

2.5 SUMMARY

In this unit we have covered the following points:

The methods for finding an approximate solution of equation in one variable involve two steps:

(i) Find an initial approximation to a root.

(ii) Improve the initial approximation to get more accurate value of the root.

The following iterative methods have been discussed:

(i) Bisection method

(ii) Fixed point iteration method

(iii) Regula-falsi method

(iv) Newton-Raphson method

(v) Secant method

We have introduced the convergence criterion of an iteration process.
We have obtained the order/rate of convergence for the iterative methods discussed.
Finally we have given a comparative performance of these methods.

2.6 SOLUTIONS/ANSWERS

E1) $x = \sqrt{10}$, i.e. $x^2 = 10$. $f(x) = x^2 - 10$

$$x_{n+1} = x_n - \frac{x_n^2 - 10}{2x_n} = \frac{x_n^2 + 10}{2x_n}, \quad n = 0, 1, 2,$$

$$x_o = 3, x_1 = \frac{19}{6} = 3.167, \quad x_2 = \frac{(3.167)^2 + 10}{6.334} = 3.162$$

E2) Here $\phi(x) = -\frac{1}{3x^2}\left(2 + 3x + 2x^2\right)$

$$|\phi'(x)| = \left|\frac{1}{3}\frac{(4+3x)}{x^3}\right| > 1 \quad \text{at } x_o = -0.5$$

Hence iteration does not converge.

If $\phi(x) = -\frac{1}{3}\left(2 + 2x^2 + 3x^3\right)$, then

$$|\phi'(x)| = \left|-\frac{1}{3}\left(4x + 9x^2\right)\right| < 1 \quad \text{at } x_o = -0.5$$

Hence in this case iteration converges

First iteration $x_1 = -0.708$

Second iteration $x_2 = -0.646$

E3) $f(x) = 3x^3 - 4x^2 + 3x - 4$, $x_o = 0$, $x_1 = 1$.

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, 3$$

This gives $x_2 = 2$, $x_3 = 1.167$, $x_4 = 1.255$

E4) Root lies in [0, 1]. We take.

$$x = \frac{x^2+1}{3} = g(x)$$

$$g'(x) = \frac{2x}{3} \Rightarrow |g'(x)| < 1$$

Starting with $x_0 = 0.5$, we have

$$x_1 = \frac{5}{12} = 0.417,\ x_2 = \frac{169}{432} = 0.391 \text{ and } x_3 = 0.384$$

E5) f(0) > 0 and f(1) < 0. The smallest positive root lies in]0, 1[.

No. of bisection	Bisected value x_i	$f(x_i)$	Improved interval
1	0.5	– 1.375	]0,0.5[
2	0.25	– 0.09375	]0,0.25[
3	0.125	0.37895	]0.125, 0.25[

It is enough to check the sign of $f(x_0)$ – the value need not be calculated.

The approximate value of the desired root is 0.1875.

E6) Here $f(x) = x^3 - 4x + 1$, $x_0 = 0$.

The iteration formula is $x_{i+1} = x_2 - \dfrac{f(x_x)}{f^1(x_i)}$

i.e. $x_{i+1} = \dfrac{2x_i^3 - 1}{3x_i^2 - 4}$.

This gives

$x_1 = 0.25, x_1 = 0.254095 \approx 0.2541$

E7) $f(x) = x^3 + x = 6, x_0 = 1,\ x_1 = 2$

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \qquad n = 1, 2, 3, \ldots$$

This gives $x_2 = 1.5,\ x_3 = 1.609 \approx 1.61,\ x_4 = 1.64.$

E8) Here f(x) = 2x – 3 Sinx – 5

x	0	1	2	2.5	2.8	2.9
f(x)	– 5.0	– 5.51224	– 3.7278	– 1.7954	– 0.4049	0.0822

Thus a positive root lies in the interval [2.8, 2.9].

No. of bisection	Bisected value x_0	$f(x_0)$	Improved Interval
1	2.85	–0.1624	[2.85, 2.9]
2	2.875	–0.0403	[2.875, 2.9]
3	2.8875	–0.02089	[2.875, 2.8875]
4	2.88125		

E9) $$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(-8).9 - (-7)(-55)}{0+55} = 7.1406$$

Similarly $x_3 = -7.168174$

The iterated values are presented in tabular form below:

No. of intersections	Interval	Bisected value x_0	The function value $f(x_i)$
1	] –8, –7[	–7.1406	1.862856
2	] –8, –7.1406[	–7.168174	0.358767
3			
4			
5			
6			

Complete the above table. You can find that the difference between the 5th and 6th iterated values is $|7.1748226 - 7.1747855| = 0.0000371$ signaling a stop to the iteration. We conclude that –7.175 is an approximate root rounded to the decimal places.

E10) Here $f(x) = cosx - xe^2$, $x_0 = 0$ and $x_1 = 1$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 0.3146653378$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 0.4467281466$$

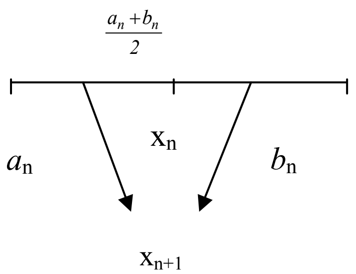
E11) Starting with bisection method with initial interval $[a_0, b_0]$ (recall that in each step the interval width is reduced by $\frac{1}{2}$ we have

$$b_1 - a_1 = \frac{b_0 - a_0}{2} = \frac{M}{2}$$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2}$$

and finally $b_n - a_n = \dfrac{b_0 - a_0}{2^n}$

Let $x_n = \dfrac{a_n - b_n}{2}$.



Then $x_{n+1} - x_n = \dfrac{a_{n+1} + b_{n+1}}{2} - \dfrac{a_n + b_n}{2}$

We have either $a_{n+1} = \dfrac{a_n + b_n}{2}$ and $b_{n+1} = b_n$

or $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n + b_n}{2}$.

In either case

$$|x_{n+1} - x_n| = \frac{b_n - a_n}{2^2} = \frac{b_0 - a_0}{2^{n+2}}$$

We want $|x_{n+1} - x_n| = \frac{M}{2^{n+2}} < \varepsilon$.

This is satisfied if

$$log\left(\frac{M}{2^{n+2}}\right) < log\varepsilon$$

log M – (n + 2) log 2 < log ε
nlog2 > – 2log2 + log M – log ε

$$n > -2 - \frac{log\left(\varepsilon/M\right)}{log2}$$

E12) In case we have for a simple root ξ

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g(x_n)$$

$g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$. Thus

$g'(\xi\xi = 0$ since f(ξ) = 0 and $g''(\xi\xi = \frac{f''(\xi\xi}{f'(\xi\xi}(f'(\xi\xi \neq 0)$

But given that f(ξ) = 0 = f'(ξ) and f''(ξ) ≠ 0.

In this case

$$\lim_{x\to\xi} \frac{f(x)f''(x)}{[f'(x)]^2} = \lim_{x\to\xi} \frac{f(x)}{[f'(x)]^2} \lim_{x\to\xi} f''(x)$$

$$= \lim_{x\to\xi} \frac{f'(x)}{2f(x)f''(x)} \lim_{x\to\xi} f''(x) = \frac{1}{2}$$ (by L'Hospital Rule)

That is $g'(\xi\xi = \frac{1}{2} \neq 0$

Hence it does not converge quadratically.

In case

$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$ where g(x) = $x - \frac{2f(x)}{f'(x)}$

$g'(x) = \frac{2f(x)f''(x) - [f'(x)]^2}{[f'(x)]^2}$ and $g'(\xi\xi = 0$.

Since

g'(x) $= \frac{2f(x)f''(x)}{[f'(x)]^2} - 1$ and

$$\lim_{x\to\xi} g'(x) = \lim_{x\to\xi} \frac{2f(x)f''(x)}{[f'(x)]^2} - 1$$

$$= \lim_{x\to\xi} \frac{2f(x)}{[f'(x)]^2} \cdot \lim_{x\to\xi} f''(x) - 1$$

$$= 2\times\frac{1}{2} - 1 = 0.$$

E13) $f(1.5) = 5.0625 - 9 + 4 = .0625$
$f'(1.5) = 13.5 - 12 = 1.5$

With $g_1(x)$

$x_1 = 1.5 - \frac{.0625}{1.5} = 1.5 - .04 = 1.46$

$f(1.46) = 4.543 - 8.52 + 4 = 0.02$
$f'(1.46) = 12.45 - 11.68 = 0.77$

$x_2 = 1.46 - \frac{0.02}{0.77} = 1.44$

With $g_2(x)$

$x_1 = 1.5 - \frac{2\times.0625}{1.5} = 1.5 - 0.08 = 1.42$

$f(1.42) = 4.065 - 8.065 + 4 = 0$
$x_2 = 1.42$

Actual root = 1.4142. Hence convergence is faster with $g_2(x)$ with two decimal digit arithmetic.

E14) We have the following results in tabular form:

n	1	2	3	4	5
e_n	0.1580513	0.0716060	0.012884	0.0008679	0.0000101
$\|e_{n+1}\|/\|e_n e_{n-1}\|$		1.1034669	0.9705400	0.9318475	

Also $f''(\xi)/2f'(\xi) = 0.93188$ $(\xi = 1.3247180)$ $|e_5|/|e_4 e_3| = 0.9318475.$ Hence agreement is good for n = 4.

E15) Here M = $b_0 - a_0 = 1 - 0 = 1$

$$n > -2 - \frac{\log(\varepsilon/M)}{\log 2}$$

Here $\varepsilon = 10^{-5}$

So $n > -2 - \frac{\log 10^{-5}}{\log 2}$

$$= -2 + \frac{5\log 10}{\log 2}$$

$$= -2 + \frac{5}{\log 2}$$

$$= -2 + 5\times 3.322$$
$$= -2 + 16.61$$
$$= 14.61$$

$n \geq 15$