UNIT 3 THE HIGHER ARITHMETIC -11

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3.0 INTRODUCTION

One of the more intriguing mysteries of our times has been the fact that some of this century's most exceptional scientists and mathematicians - Theodor von Karman, George de Hevesy, Michael Polanyi, Leo Szilard, Eugene Wigner, John von Neumann, Edward Teller, Fejer, Haar, Riesz, Konig, Rado, to name a few came from one small country of Central Europe, Hungary. All these men made major contributions to mathematics, science or technology and two - de Hevesy and Wigner - won Nobel Prizes. When historians of science will speculate upon the reasons of this efflorescence of genius in Hungary, they will surely consider the remarkable role played by the Eotvos Competitions in, the development of its mathematical school. Named in honour of the distinguished experimental physicist Baron Lorand Eotvos, the competitions were held from 1894 onwards, and except for short breaks due to wars and similar exigencies, have continued until the present day, though under a different name.

The competitions were open to high school graduates preparing to enter the University: and the publication of the problems, and the names of the winners were events that excited much public interest. Although no calculus was needed, the problems were difficult, requiring for their solution ingenuity, insight and creative ability (but books or notes were permitted to be brought into the examinations). Side by side with the competitions a mathematics magazine called the Journal was organised, and ably edited for many years by the high school teacher Laszlo Racz, whose name will live in history for another reason: he was J. von Neumann's teacher. (Given that von Neumann's at age six could joke with his father in classical Greek, and had besides a photographic memory, teaching him would doubtless have been a daunting task! A readable account of the Eotvos competitions and the Journal which captures the spirit of the times was published by Tibor Rado, "On mathematical life in Hungary", *American Mathematical Monthly,* vol. 39, (1932) pp. 85 - 90).

G. Szego, later professor of mathematics at Stanford and winner of the competitions for 1912, in his introduction to the Hungarian Problem Book (comprising problems from the Eotvos competitions for 1894 - 1928, two vols., pub. The L. W. Singer Company, 1963) wrote: "I remember vividly the time when I participated in this phase of the Journal (in the years between 1908 and 1912); I would wait eagerly for the arrival of the monthly issue and my first concern was to look at the problem section, almost breathlessly, and to start grappling with the problems without delay. The names of the others who were in the same business were quickly known to me and frequently I read with considerable envy how they had succeeded with some problems which I could not handle with complete success, or how they had found a better solution (that is, simpler, more elegant or wittier) than the one I had sent in."

In this Unit we present some problems chosen from the Eotvos competitions: and some elegant geometrical theorems of antiquity as well as of more recent times, which may not be widely known. Geometry is no longer considered a "fashionable" branch of mathematics, though for many centuries it was held to be the exemplar of the deductive method. But Euclid has had his day. This is a pity, because many of the ancient and (comparatively recent theorems of geometry possess great aesthetic appeal. Morley's Theorem, that the trisectors of the angles of any triangle intersect at the vertices of an equilateral triangle, or Jung's Theorem that the spanning circle of a finite set of points of span d has a radius no greater d $3^{1/2}$ / 3 than are two instances.

3.1 OBJECTIVES

After studying through this unit, you would:

appreciate the need to THINK

be able to make use of clues provided to get a good understanding of problem solving

be able to solve problems involving applications of the theorem of Pythagoras

be able to get a flavour of some recurrence relations and their use

3.2 HUNGARIAN PROBLEMS

Hints or solutions to the twelve problems that follow below are provided at the end of this section. But you are strongly urged to grapple with the problems *before* turning to the solutions. Read one, shut the book, and follow the motto which IBM apparently inscribe even upon bathroom walls in their buildings: THINK. Keep a pencil and a sheet of paper handy, write the givens, experiment with simple cases, ruminate - for days if need he - and success will follow, and in its wake an immense feeling of achievement, and the confidence to tackle the other problems.

We cannot do better than quote Szego again: "We should not forget that the solution of any worthwhile problem very rarely comes to us easily and without hard work; it is rather the result of intellectual effort of days or weeks or months. Why should the young mind be willing to make this supreme effort? The explanation is probably the instinctive preference for certain values, that is, the attitude which rates intellectual effort and spiritual achievement higher than material advantage."

1. Prove that the expressions:

$$3x + 7y \text{ and } 9x + 4y$$

are divisible by 17 for the same set of integral values of x and y.

2. Prove that the expressions $21n + 4$ and $14n + 3$ have no factors (other than 1) in common for any value of n.

3. Determine all positive integers n for which $2^n + 1$ is divisible by 3.

4. Prove that for any natural number n, the expression

$$A = 2903^n - 803^n - 464^n + 261^n$$

is divisible by 1897.

5. Let a_l., a_2, a_3, ..., a_n, be an arbitrary arrangement of the numbers 1, 2, 3, n. Prove that, if n is odd, the product

$$(a_l - 1) \times (a_2 - 2) \times (a_3 - 3) \ldots (a_n - n)$$

is an even number.

6. Prove that for every integer n greater than 2

$$(n!)^2 > n^n$$

7. An arithmetic progression is a sequence of equally spaced terms:

$$a, a + d, a + 2d, a + 3d, \ldots..$$

where a is the first term of the progression, and d is the common difference. Prove that in an infinite arithmetic progression of natural numbers in which the common difference is not zero, not all the terms can be primes.

8. Prove that the product of four consecutive natural numbers cannot be the square of an integer.

9. Let a, b, c and d be four integers. Prove that the product of the six differences:

$$b-a, c-a, d-a, d-c, d-b, c-b$$

is divisible by 12.

10. For any integer n greater than 2, prove that the n^{th} power of the length of the hypotenuse of a right triangle is greater than the sum of the nth powers of the lengths of the smaller sides.

11. For any three consecutive integers, the cube of the largest cannot be the sum of the cubes of the other two.

12. Seventeen people correspond by mail with one another - each one with all the rest. In theft letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

Solutions

1. Let $u = 3x + 7y$, $v = 9x + 4y$, then $3u - v = 17y$,and $v = 3u + 7y$. Therefore, if u is divisible by 17, then v is also divisible by it. Similarly, if v is divisible by 17, then so must u be, since 3 and 17 have no factors in common.

2. Let f be a common factor between $21n + 4 = A$, say, and $14n + 3 - B$, say. Then f mast divide $pA + qB$, for all integer values of p and q. Choose $p = - 2, q = 3$. Then f must divide $3B - 2A = 1$; therefore f divides 1.

3. $2^n + 1$ is divisible by 3 for n = 1, 3, 5; thus it appears plausible that 3 will be a factor of $2^n + 1$ for odd values of n. For odd n it is possible to rewrite the given expression as $2^n - (-1)^n$, which is always factorable, one factor being 2 - (- 1), i.e. 3

4. A is expressible in the alternate forms:

$$(2903^n - 803^n) - (464^n - 261^n),$$

and

$$(2903^n - 464^n) - (803^n - 261^n)$$

Each parenthetical expression in the first form is divisible by 7, and in the second form by 271 (why?). Therefore the entire expression is divisible by both 271 and 7 (why?); therefore it is divisible by the product 7 x 271 (why?).

5. The number of factors in the product is odd (given that n is odd). By inspection, the sum of all the factors is 0. This is impossible if each of the factors was odd, because an odd number of odd numbers cannot sum to an even number, zero. Hence at least one factor in the product is even. Hence the product is even.

6. The expansion for $(n!)^2$ may be written in the form:

$$1 \times n \times 2 \times (n-1) \times 3 \times (n-2) \times \ldots \times (n-1) \times 2 \times n \times 1$$

In the set of products: $1 \times n$, $2 \times (n - 1)$, $3 \times (n - 2)$, etc, each term is greater than or equal to n. The first (and the last) product - $n \times 1$ - is smaller than the others; while each of the other terms, being of the form $(k + 1) \times (n-k)$ exceeds n, because

$$(k + 1) \times (n - k) = k \times (n - k) + (n - k) > k + (n - k) = n$$

There are n such products: therefore $(n!)^2 > n^n$.

7. If the n^{th} term of the progression be denoted T_n, then it is clear that $T_n = a + (n - 1)d$. Thus the a^{th} term after a must be $a + (a - 1)d$, and the following term must be, $a + (a - 1)d + d$, i.e. $a(1 + d)$, which is a composite number.

8. Examine the product of the consecutive numbers 1, 2, 3, 4 (product = 24); of 2, 3, 4, 5 (product = 120); of 3, 4, 5, 6 (product = 360); in each case we find that the product is one less than a perfect square: thus, 24 is $5^2 -1$, 120 is $11^2 - 1$, 360 is $19^2 - 1$, and so on. Now consider the product of four consecutive integers:

$$n (n + 1) (n + 2) (n + 3)$$

where n is arbitrary. We will have proved the result if we could rewrite this product in the form $f(n) - 1$, where $f(n)$ is a perfect square expression in n. We first note that the term $n(n + 2)$ in the product looks "almost" like a perfect square:

$$n (n + 2) = n^2 + 2n = n^2 + 2n + 1 - 1 = (n + 1)^2 - 1.$$

Similarly,

$$(n + 1) (n + 3) = n^2 + 4n + 3 = n^2 + 4n + 4 - 1 = (n + 2)^2 - 1,$$

and therefore:

$$n(n + 1)(n + 2)(n + 3) = [(n + 1)^2 - 1] [(n + 2)^2 - 1]$$

which is an expression not quite the kind we're looking for!

Let's therefore experiment with a different order of expansion:

$$[n (n + 3)] [(n + 1) (n + 2)] = (n^2 + 3n) (n^2 + 3n + 2)$$
$$= (n^2 + 3n)^2 + 2(n^2 + 3n).$$
$$[n (n + 3)] [(n + 1) (n + 2)] = (n^2 + 3n)^2 + 2(n^2 + 3n) + 1 - 1$$
$$= (n^2 + 3n + 1)^2 - 1$$

i.e. the product of four consecutive integers is always 1 less than a perfect square. Since no two squares differ by 1 , the result is proved.

9. The product of the six differences will be divisible by 12 if it is divisible by 3 as well as by 4.

If an integer is divided by 3 the only possible remainders that it can yield are 0, 1 and 2. Now at least two of *a, b, c* and *d* must yield the same remainder when divided by 3 - if there are 4 persons on an elevator, and only three buttons have been pressed, at least two must want to get off on the same floor. However, if two integers leave the same remainder on division by 3, then their difference must be divisible by 3. (Because each integer must then be of the form $3k + r$ and $3l + r$; their difference is $3 (k - 1)$).Therefore one of $b - a$, $d - a$, $c - a$, $d - c$, $d - b$, $c - b$ must certainly be divisible by 3.

Similarly, when any number is divided by 4, the only remainders that we may expect are 0, 1, 2, and 3. If two of *a*, *b*, *c* and *d* yield the same remainder, then one of the 6 differences above must be divisible by 4, and hence the product of the six differences must also be divisible by 4.

If no two of a, b c and yield the same remainder on division by 4, then they must belong to one of the types $4k$, $4l + 1$, $4m + 2$, and $4n + 3$ for some *k, 1, m* and *n*. Thus two of *a, b, c* and *d* are even, and two are odd. Now. the product under consideration is built from the six differences between *a, b, c* and *d*. The difference of the even numbers from these four is even. The difference of the odd numbers is also even. Therefore the product of the six differences must be divisible by 4. Therefore it must also be divisible by 12.

10. Let the hypotenuse be *c*, and the smaller sides be *a* and *b*. Then

$$c^n = c^2 c^{(n-2)} = (a^2 + b^2) c^{(n-2)} = a^2 c^{(n-2)} + b^2 c^{(n-2)}$$

But $$c^{(n-2)} > a^{(n-2)}, \text{ and } c^{(n-2)} > b^{(n-2)}$$

Therefore:
$$c^n > a^2 a^{(n-2)} + b^2 b^{(n-2)} = a^n + b^n$$

11. Let the three integers be $(n-1)$, n, and $(n + 1)$. If possible, let

$$(n+1)^3 = (n-1)^3 + n^3$$

so that,

$$n^2 (n-6) = 2$$

The left hand side is positive only if $n > 6$, in which case it is greater than 2. Thus there is no *n* for which the above equation can be true.

12 Select any person ,say *A*. He corresponds with 16 other people. He has only three topics to discuss say the topics *I, II* and *III*. Therefore he must write to at least six of them on the same topic. Assume A write to six persons on topic *I*. If any one of these six writes to another of them on the same topic, then there are three writers corresponding on topics *I*, and we have nothing to prove.

Assume therefore these six write to each other only on topics *II* and *III*. Suppose B is one of these six. He writes to the other five only on topics *II* and *III*. Therefore, he must write to at least three of these five on the same topic, say topic *II*.

Now there are two possibilities for these last three people. If one writes to another on topic *II*, then we have found three corresponding on topic *II*. If none of them writes on topic *II*, then all three must write on topic *III*, proving the assertion.

Check Your Progress 1

1. In a contest of intelligence, three problems *A, B,* and *C* were posed. Among the contestants there were 25 who solved at least one problem each. Of all the contestants who did not solve problem *A*, the number who solved *B* was twice the number who solved *C*. The number of participants who solved only problem *A* was one more than the number who solved problem *A* and at least one other problem. Of all students who solved just one problem, half did not solve problem *A*. How many students solved only problem *B*?

2. In a sports contest there were m medals awarded on n successive days ($n > 1$). On the first day 1 medal and 1/7 of the remaining $m - 1$ medals were awarded. On the second day 2 medals and in of the now remaining medals was awarded; and so on. On the n^{th} and last day, the remaining *n* medals were awarded. How many days did the contest last, and how many medals were awarded altogether?

3. Prove that the fraction $(21n + 4) / (14n + 3)$ cannot be reduced for any n.

4. Determine all three-digit numbers *N* such that *N* is divisible by 11, and *N*/11 equals the sum of the squares of the digits of *N*.

5. Find the smallest natural number *n* which ends in 6, and which is such that if its last digit is removed and placed in front of the remaining digits, the new number is four times as large as the original number.

3.3 AN ARCHIMEDEAN RESULT

Archimedes, the first practical mathematician whose record we have, lived in Alexandria in the third century B. C., where he studied under the disciples of Euclid. His great achievement lay in demonstrating that mathematics could be applied to gain an understanding of the world. He created the sciences of statics (in which are studied the equilibria of bodies at rest), and hydrostatics, the study of forces on bodies immersed in fluids, and stated the laws of levers. He invented machines such as the compound pulley, and a screw for raising water. But he disparaged these inventions, regarding, in the words of Plutarch, as "ignoble and sordid the business of mechanics and every sort of art which is directed to use and profit; he placed his whole ambition in those speculations the beauty and subtlety of which are untainted by any admixture of the common needs of life".

He computed a creditable value for *pi*, and derived formulae for the area of a circle and the volume of a sphere; but a less well-known result answered the question: what is the volume common to the orthogonal (i.e. at right angles) intersection of two cylinders of diameter *d*? Though his original proof is lost, it is speculated that he might have reasoned as follows:

Suppose the axes of the cylinders to lie along the axes of X and Y, so that half the cylinders lie above the X-Y plane, and half below, as illustrated in Fig. 1.

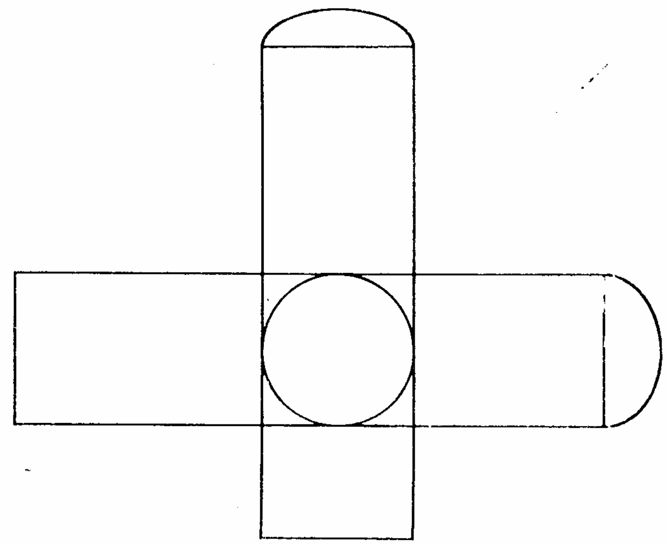


Figure 1 : Two orthogonally intersecting cylinders

The area intercepted in the X -Y plane is in the shape of a square of side *d*, and is d^2. A circle inscribed within this square (radius = d/2) has the area *pi* $*d^2$ / 4, so that the ration of the area of the circle to that of the circle to that of the square is *pi* / 4.

Consider now the another plan parallel to the X-Y plan, but infinitesimally above it. This plan too intersects the volume common to both cylinders in a square of side infinitesimally less than *d*; and a circle inscribed within this square bears likewise a proportion of area equal to *pi / 4* of the square.

Consider now an infinite sequence of such planes above the X -Y plan. Each plan interests the volume common to both cylinders in a square; and each has an area infinitesimally less than the area of the plane beneath it. But each square encloses a circle to which its sides are tangent, whose diameter

decreases as one proceeds upwards from the X-Y plane, and whose area is *pi* / 4 of the circumscribing square. The stack of squares of steadily diminishing sides forms the upper half of the volume of the intersection - the volume sought. And the stack of enclosed circles forms a hemisphere inscribed within this volume.

Similarly consider an infinite number of planes lying below the X-Y plane, each intersecting the cylinders in area of square cross-section. As before, the intersection of each plane with the cylinders results in a square, whose sides are tangent to an inscribed circle. The set of squares in the planes beneath the X-Y plane fill the lower half of die volume sought. And the set of inscribed circles within these squares are cumulatively equivalent to a hemisphere of diameter *d*.

The stack of squares above and below the X-Y plane completes the volume created by the intersecting cylinders; and this volume includes a sphere of diameter *d*.

Inasmuch as the ratio of the area of each circle to its circumscribing square is *pi* / 4, the ratio of the
volume of the enclosed sphere to the volume sought is also *pi* / 4.

But the volume of a sphere of diameter d is known to be pi x d^3 / 6.

Hence the volume we seek is $(4 / pi) \times (pi \times d^3 / 6) = (2 / 3) \times d^3$.

This, incidentally, is one of the few instances in which a volume bounded by curved surfaces is determinable without recourse to the integral calculus; but the glimmerings of the calculus are of course visible in the Archimedean method. (And this is also one of the few instances where such a curved volume does not involve pi. Strange!)

Check Your Progress 2

1. The ancient Greeks needed to dig a tunnel through a large mountain. To save time they decided to have two teams dig simultaneously from opposite sides of the mountain. How could they ensure that both would dig in the same straight line (which may be assumed to be horizontal)?

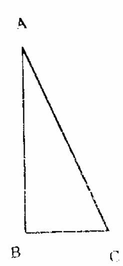
2. Prove that the area of a regular dodecagon - a 12-sided polygon - inscribed in a circle of unit radius is precisely three units. (Another *pi*-less area.)

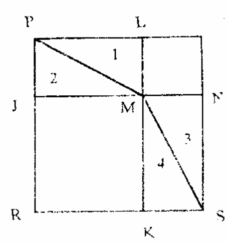
3. Assume for a moment that the earth is a perfectly uniform sphere of radius 6400 km. Suppose a thread equal to the length of the circumference of the earth was placed along the equator, and drawn to a tight fit. Now suppose that the length of the thread is increased by 12 cm, and that it is pulled away uniformly in all directions. By how many cm. will the thread be separated from the earth's surface?

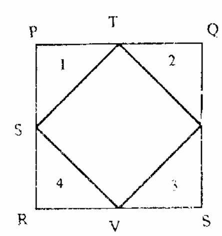
4. A cylindrical hole of length three inches is drilled through a sphere. Find the volume of material removed.

3.4 THE THEOREM OF PYTHAGORAS AND IRRATIONAL NUMBERS

The Theorem of Pythagoras is of course the most important in all of geometry, for it crystallises the concept of distance between two points in the plane. (Even a vegetarian may forgive Pythagoras for sacrificing fifty buffaloes in celebration of his discovery, as legend has it!) Here we present a proof possibly the proof that is discover himself enunciated.







Construction:

Given the triangle ABC right-angled at B, construct a square PQR with side AB + BC. Place four copies of the triangle inside the square in the two configurations shown in Figs. 3 and 4.

Proof:

In Fig 3, the area remaining is that of the square on the sides AB and BC; in Fig 4 the area remaining is that of the square on the side AC of the original triangle. In each case the area remaining is:

Area (PQRS) - 4 x Area (ABC)

Hence the square on AC equal the sum of the square on AB and BC.

An *irrational* number is a number that cannot be expressed in me form p/q, where p and q are intergers. The square root of 2, by Pythagoras' Theorem the diagonal of a square of side one unit, is such a number. Suppose that you had two pieces of string, one of length 1 unit, and the other of length $2^{1/2}$ units, and I gave you the following task: "Cut up these two strings into pieces of equal length, as small as you like, but they must be the same length for both strings." Then you would not be able to accomplish my task, ever! For $2^{1/2}$ is *incommensurate* with 1. This discovery perplexed Pythagoras and his students so greatly - because it meant that the world was not as simple and harmonious as they had imagined it to be - that they swore to keep it a secret. Legend has it that one of their number revealed it , and was therefore drowned in a shipwreck - by act of divine retribution!

To prove the irrationality of $2^{1/2}$ we follow Euclid again and argue by reduction ad absurdum:

Assume that $2^{1/2} = p / q$, where p and q are positive integers that have been expressed in their lowest terms. Then $p^2 = 2q^2$, which implies that p^2 is even, i.e. p is even. If p is even, then $p = 2k$, for some k. Therefore $4k^2 = 2q^2$, i.e. $q^2 = 2k^2$, which implies that q^2 is even, i.e. that q is even. If p and q are both
even, they could not have been in their lowest terms, invalidating our original assumption. hence $2^{1/2}$ cannot be expressed as a rational fraction. It follows from Pythagoras Theorem that the diagonal of a square is incommensrable with its side, i.e. that there is no unit of which both side and diagonal are intergral multiples.

Another way of establishing the same result is detailed in the following steps :

$$p^2 = 2q^2;$$

Subtract pq;

$$p^2 - pq = 2q^2 - pq,$$
$$p(p - q) = q(2q - p),$$

so that

$$p/q = (2q - p) / (p - q)$$

But the right hand side is a representation of p/q with a smaller numerator than p, and a smaller denominator than q, which is impossible.

A third way of proving the same result is to rewrite the equation $p^2 = 2q^2$ in the form $p^2/q = 2q$, and to observe that the left hand side of the new equation is a fraction, while its right hand side is an integer, which is impossible.

Many people disparage *reductio ad absurdum* because they feel that it yields indirect proofs. However, G. H. Hardy, one of the foremost mathematicians of our time, wrote, "Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit dm any chess gambit: a chess player may offer the sacrifice of a pawn, or even a piece, but a mathematician offers the game.

3.5 THE DIVISION OF A PLANE BY STRAIGHT LINES

What is the maximum number of regions Rn into which a plane can be divided by n infinite straight lines no two of which are parallel, and no three of which go through the same point?

We start by considering the numbers of regions formed by zero, one line, two lines, three lines, etc. in the attempt to arrive at a general formula. It's always advantageous to look at the simple cases to begin with. When there are no lines, the plane consists of one region.

Let's state this fact mathematically:

$$R_0 = 1$$

One line divides the plane into two regions:

$$R_1 = 2$$

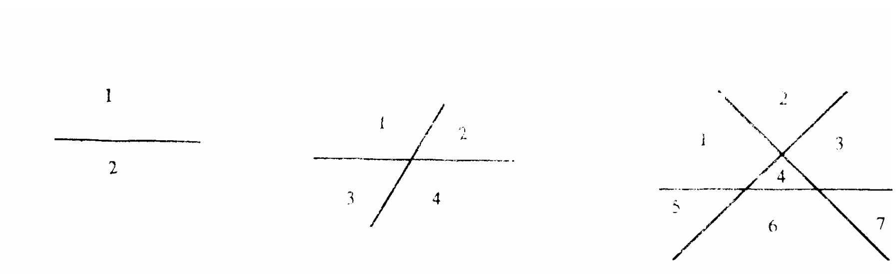


Figure 5

Two lines divide the plane into four regions, and three into seven (Fig. 5). How about four lines? Check that 11 regions are obtained. We can now guess a formula:

$$R_n = R_{n-1} + n$$

Our experience with small values of n has helped us derive a "recurrence relation", which seems to be valid for the cases considered. And it's not difficult to give the formula a geometrical interpretation: The *nth* line increases the number of regions by n because it cuts the previous $n - 1$ lines in $n - 1$ places. Since R_0 is 1 (this may be thought of as a "boundary condition"), we can work backwards from R. to find a general formula for it in terms of n alone:

$$\begin{aligned} R_n &= R_{n-1} + n \\ &= R_{n-2} + (n-1) + n \\ &= R_{n-3} + (n-2) + (n-1) + n \end{aligned}$$

$$\begin{aligned} &= R_0 + 1 + 2 + \dots\dots + (n-2) + (n-1) + n \\ &= 1 + S_n, \end{aligned}$$

where S_n is the sum of the first n positive integers.

To evaluate S_n we'll adopt a method that Carl Friedrich Gauss (possibly the greatest mathematician of all time) discovered in 1786, when he was nine years old. As Gauss used to tell the story years after the event, his school master, exasperated by the rowdy youngsters in his charge, assigned the class the following problem: find the sum of all the integers from 1 through 100. Gauss wrote the correct answer (5050) within a few moments on his slate, placed it face downwards on his desk, and waited.

His teacher was appropriately astounded on seeing his tablet, blank but for the answer. Every other boy had attempted the solution by laboriously performing 99 separate additions but had it wrong. Gauss solved the problem by observing that when the sum is arranged as below: ,

$$1 + 2 + 3 + 4 + 5 + \ldots + 96 + 97 + 98 + 99 + 100$$

and the first term paired with the last:

$$1 + 100 = 101$$

the second with the second from the last:

$$2 + 99 = 101$$

the third with the third from the last:

$$3 + 98 = 101$$

each pair adds to 101. There are fifty such pairs. The sum of numbers from 1 through 100 must therefore be 5050!

Using Gauss' trick we can write:

$$S_n = 1 + 2 + 3 + \ldots\ldots + (n - 2) + (n - 1) + n$$
$$S_n = n + (n - 1) + (n - 2) + \ldots\ldots + 3 + 2 + 1$$

$$2S_n = (n + 1) + (n + 1) + (n + 1) + \quad + (n + 1)$$
$$= n\,(n + 1)$$

Thus

$$S_n = n(n + 1)/2,$$

and

$$R_n = 1 + n\,(n + 1)\,/\,2.$$

Check Your Progress 3

1. You are given three integers x_l, x_2 and x_3. Prove that the average of two of these must be an integer.

2. A lattice point is one which has integer co-ordinates. Given five lattice points (x_1, y_l), ... (x_5, y_5) in the plane, the mid-point of one of the ten lines between pairs of points is necessarily a lattice point.

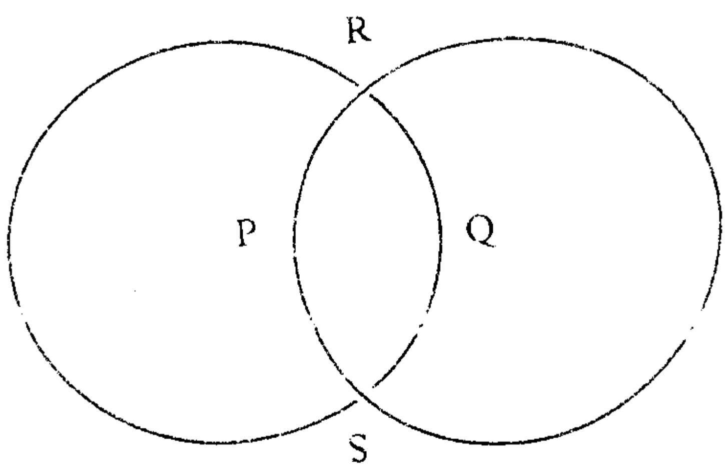
3. Given a set A consisting of 10 numbers between 1 and 100: prove that there exist two distinct subsets of A whose elements sum to the same number.

4. Given a set X containing n positive integers: prove that there is a subset of X whose elements add up to a multiple of n.

3.6 MINIMUM SPANNING CIRCLES

Consider a finite set of n points in the plane; then the set of distances between any two points is also a finite set, which has a largest member, say d, called the *span* of the set of points. Clearly, if a set of points has span d, then it is possible to draw a circle of radius d that encloses all the points.

But it is possible to draw a smaller circle that still encloses the entire set: first locate a pair of points, P and Q, whose distance from each other is d. (If there are several such pairs, we may choose any one.) Construct two circles of radius d with centres at P and Q (Fig. 6).



$PQ = d$, $OP = d/2$, $PS = d$, maximally

Figure 6

The circle centred at *P* passes through *Q*, while the circle centred at *Q* passes through *P*. These circles intersect at *R* and *S*. The entire set of *n* points lies within each circle. Therefore it must lie within their intersection, But the intersection itself is enclosed within die circle which has *RS* for its diameter. Since *PS* is maximally *d*, the radius of *this* spanning circle, by Pythagoras' Theorem, is $d\ 3^{1/2}/2$, (why?) a bound lower than the one we had started with. This bound lean be further improved, and this is *Jung's Theorem:* Every finite set of points of span *d* has a spanning circle no greater than $d\ 3^{1/2}/3$, a still better bound. Though smaller spanning circles are possible for certain finite sets of points, there are some which require the spanning circle to be at least this large.

Check Your Progress 4

1. If n lines are drawn in the plane, then it is possible to shade regions black or white so that contiguous regions have different colours.

2. Find the radius of the spanning circle of the vertices of an equilateral triangle of side *d*.

3. The span of a finite set of points is also called its *diameter*. Prove that a set of *n* points has at most *n* diameters.

3.7 SUMMARY

In this Unit you have learnt the application of techniques introduced in the last Unit to a number of problems. Techniques used in solving these problems rely solely upon logic, and not upon any sophisticated mathematics at all. Yet powerful results are derived. These same techniques can be applied to geometrical problems, for example computing the number of divisions that can be made of a plane by intersecting straight lines, or determining the radius of the spanning circle of a finite set of points in the plane. You have also seen how elegantly the greatest mathematician of antiquity determined the volume common to two orthogonally intersecting cylinders; and how the greatest mathematician of recent times summed the first hundred integers. These examples are important to us for another reason: they help introduce the subject of creativity in problem solving, dealt with in the next Unit.