UNIT 1 PRELIMINARIES IN THREE-DIMENSIONAL GEOMETRY

Structure

1.1 INTRODUCTION

With this unit we start our discussion of analytical geometry in three-dimensional space, or 3-space. The aim of this unit is to acquaint you with some basic facts about points, lines and planes in 3-space. We start with a short introduction to the cartesian coordinate system. Then we discuss various ways of representing a line and a plane algebraically. We also discuss angles between lines, between planes and between a line and a plane.

The facts covered in this unit will be used constantly in the rest of the course. Therefore, we suggest that you do all the exercises in the unit as you come to them. Further, please do not go to the next unit till you are sure that you have achieved the following objectives.

Objectives

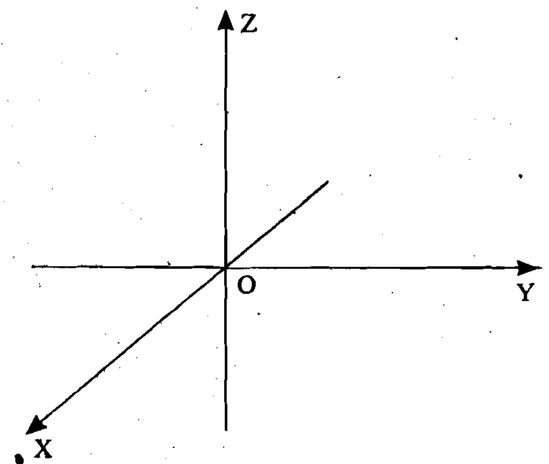
After studying this unit you should be able to :

- find the distance between any two points in 3-space;
- obtain the direction cosines and direction ratios of a line;
- obtain the equations of a line in canonical form or in two-point form;
- obtain the equation of a plane in three-point form, in intercept form or in normal form;
- find the distance between a point and a plane;
- find the angle between two lines, or between two planes, or between a line and a plane;
- find the point (or points) of intersection of two lines or of a line and a plane.

Now let us start discussion on points in 3-space.

1.2 POINTS

Let us start by generalising the two-dimensional coordinate system to three dimensions. You know that any point in two-dimensional space is given by two real numbers. To locate the position of a point in three-dimensional space, we have to give three numbers. To do this, we take three mutually perpendicular lines (axes) in space which intersect in a point O (see Fig. 1 (a)). O is called the **origin**. The positive directions OX, OY and OZ on these lines are so chosen that if a right-handed screw (Fig. 1 (b)) placed at O is rotated from OX to OY, it moves in the direction of OZ.



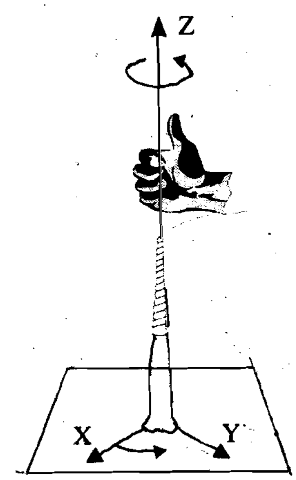


Fig. 1 : The Cartesian coordinate axes in three dimensions.

To find the coordinates of any point P in space, we take the foot of the perpendicular from P on the plane XOY (see Fig. 2). Call it M. Let the coordinates of M in the plane XOY be (x, y) and the length of MP be | z |. Then the coordinates of P are (x, y, z). z is positive or negative according as MP is in the positive direction OZ or not.

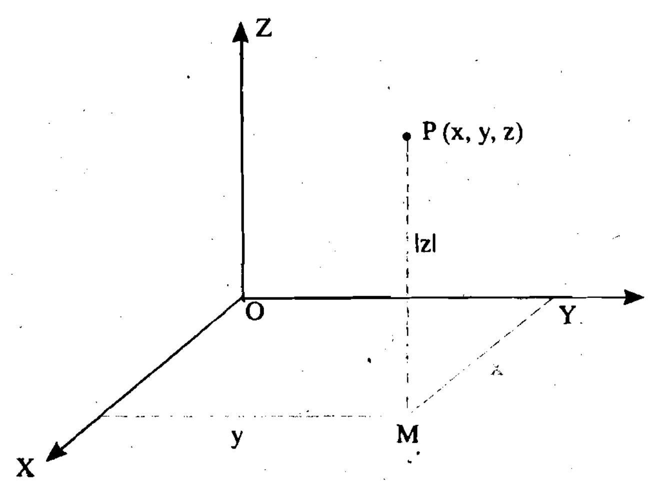


Fig. 2

So, for each point P in space, there is an ordered triple (x, y, z) of real numbers, i.e., an element of $\mathbf{R}^3$. Conversely, given an ordered triple of real numbers, we can easily find a point P in space whose coordinates are the given triple. So there is a one-one correspondence between the space and the set $\mathbf{R}^3$. For this reason, three-dimensional space is often denoted by the symbol $\mathbf{R}^3$. For a similar reason a plane is denoted by $\mathbf{R}^2$, and a line by $\mathbf{R}$.

Now, in two-dimensional space, the distance of any point P (x, y) from the origin is $\sqrt{x^2 + y^2}$. Using Fig. 2, can you extend this expression to three dimensions? By Pythagoras's theorem, we see that

$$\begin{aligned} OP^2 &= OM^2 + MP^2 \\ &= (x^2 + y^2) + z^2 \end{aligned}$$

$\therefore OP = \sqrt{x^2 + y^2 + z^2}$

Thus, the distance of P (x, y, z) from the origin is $\sqrt{x^2 + y^2 + z^2}$.

And then, what will the distance between any two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be? This is the **distance formula**

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, \quad ...(1)$$

as you may expect from Equation (1) of Unit 1.

Using (1), we can obtain the coordinates of a point R (x, y, z) that divides the join of P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) in the ratio m:n. They are

$$x = \frac{nx_1 + mx_2}{m+n}, y = \frac{ny_1 + my_2}{m+n}, z = \frac{nz_1 + mz_2}{m+n} \quad ...(2)$$

For example, to obtain a point A that trisects the join of P (1, 0, 0) and Q (1, 1, 1), we take

m = 1 and n = 2 (2). Then the coordinates of A are $\left(1, \frac{1}{3}, \frac{1}{3}\right)$.

Note that if we had taken m = 2, n = 1 in (2), we would have the other point that trisects PQ,

namely, $\left(1, \frac{2}{3}, \frac{2}{3}\right)$.

Why don't you try some exercises now?

E 1) Find the distance between P (1, 1, −1) and Q (−1, 1, 1). What are the coordinates of the point R that divides PQ in the ratio 3:4?

E 2) Find the midpoint of the join of P (a, b, c) and Q (r, s, t).

Now let us shift our attention to lines.

1.3 LINES

In Unit 1 we took a quick look at lines in 2-space. In this section we will show you how to represent lines in 3-space algebraically. You will see that in this case a line is determined by a set of two linear equations, and not one linear equation, as in 2-space.

Let us start by looking at a triplet of angles which uniquely determine the direction of a line in 3-space.

1.3.1 Direction Cosines

Let us consider a Cartesian coordinate system with O as the origin and OX, OY, OZ as the axes. Now take a directed line L in space, which passes through O (see Fig. 3). Let L make angle α, β and γ with the positive directions of the x, y and z-axes, respectively. Then we define cos α, cos β and cos γ to be the **direction cosines** of L.

For example, the direction cosines of the x-axis are cos 0, cos $\pi/2$, cos $\pi/2$, that is, 1, 0, 0.

Note that the direction cosines depend on the frame of reference, or coordinate system, that we choose.

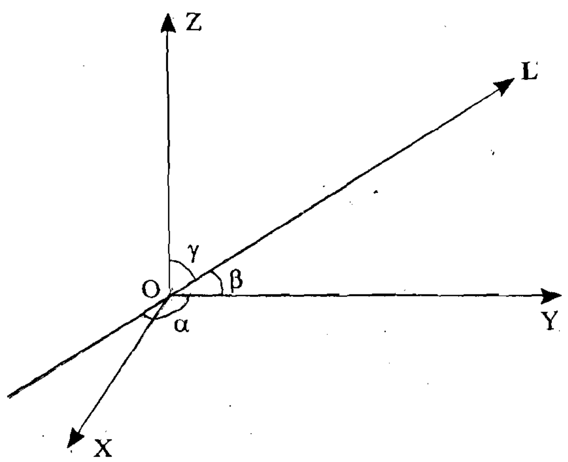


Fig. 3 : Cos α, cos β and cos γ are the direction cosines of the line L.

Now take any directed line L in space. How can we find its direction cosines with respect to a given coordinate system? They will clearly be the direction cosines of the line through O which has the same direction as L. For example, the direction cosines of the line through (1, 1, 1) and parallel to the x-axis are 1, 0, 0.

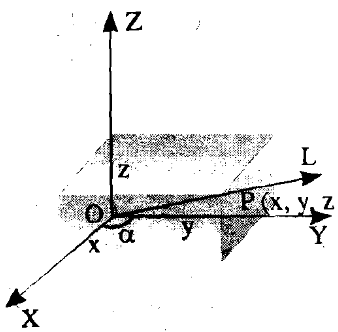


Fig. 4

Now let us consider some simple properties satisfied by the direction cosines of a line. Let the direction cosines of a line L be cos α, cos β and cos γ with respect to a given coordinate system. We can assume that the origin O lies on L. Let P (x, y, z) be any point on L. Then you can see from Fig. 4 that

$x = OP \cos \alpha$, $y = OP \cos \beta$ and $z = OP \cos \gamma$.

Since $OP^2 = x^2 + y^2 + z^2 = OP^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$, we find that

$$\boxed{\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \qquad \text{....(3)}}$$

This simple property of the direction cosines of a line is useful in several ways, as you'll see later in the course. Let us consider an example of its use.

Example 1: If a line makes angles $\frac{\pi}{4}$ and $\frac{\pi}{3}$ with the x and y axes, respectively, then what is the angle that it makes with the z-axis?

Solution : Put $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$ in (3). Then, if γ is the angle that the line makes with the z-axis, we get

$$\frac{1}{2} + \frac{1}{4} + \cos^2 \gamma = 1 \Rightarrow \cos \gamma = \pm \frac{1}{2} \Rightarrow \gamma = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}.$$

Thus, there will be two lines that satisfy our hypothesis. (Don't be surprised ! See Fig.

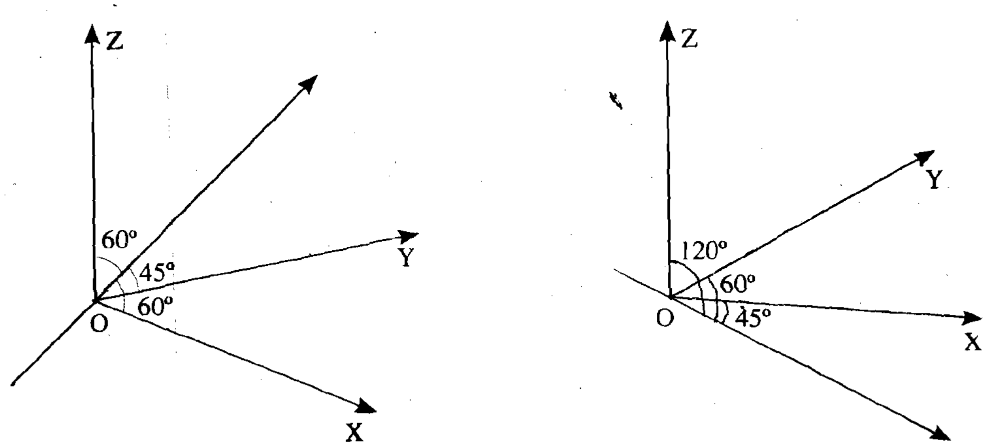


Fig. 5

They will make angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, respectively, with the z-axis.

There is another number triple that is related to the direction cosines of a line.

Definition : Three numbers a, b, c are called **direction ratios** of a line with direction cosines l, m and n, if $a = kl$, $b = km$, $c = kn$, for some $k \in \mathbf{R}$.

Thus, any triple that is proportional to the direction cosines of a line are its direction ratios.

For example, $\sqrt{2}$, 1, 1 are direction ratios of a line with direction cosines $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$.

You can try these exercises now.

E3) If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of a line, show that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

E4) Find the direction cosines of

a) the y and z axes,

b) the line $y = mx + c$ in the XY-plane.

E5) Let L be a line passing through the origin, and let P (a, b, c) be a point on it. Show that a, b, c are direction ratios of L.

E6) Suppose we change the direction of the line L in Fig. 3 to the opposite direction. What will the direction cosines of L be now?

Let us now see how the direction cosines or ratios can be used to find the equation of a line.

1.3.2 Equations of a Straight Line

We will now find the equations of a line in different forms. Let us assume that the direction cosines of a line are l, m and n, and that the point P (a, b, c) lies on it.

Then, if Q (x, y, z) is any other point on it, let us complete the cuboid with PQ as one of its diagonals (see Fig. 6).

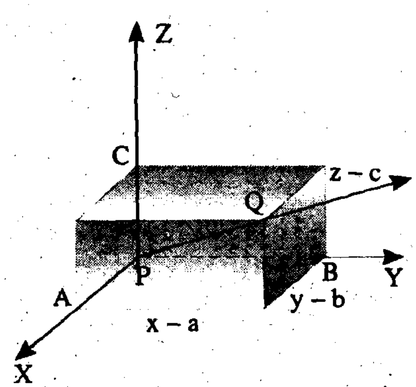


Fig. 6 : (x – a), (y – b) and (z – c) are direction ratios of PQ.

Then $PA = x - a$, $PB = y - b$ and $PC = z - c$. Now, if $PQ = r$, you can see that

$\cos\alpha = \frac{x-a}{r}$, that is,

$l = \frac{x-a}{r}$. Similarly, $m = \frac{y-b}{r}$, $n = \frac{z-c}{r}$.

Thus, any point on the line satisfies the equations.

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}. \qquad ...(4)$$

Note that (4) consists of pairs of equations,

$\frac{x-a}{l}=\frac{y-b}{m}$ and $\frac{y-b}{m}=\frac{z-c}{n}$, or $\frac{x-a}{l}=\frac{z-c}{n}$ and $\frac{y-b}{m}=\frac{z-c}{n}$ or

$\frac{x-a}{l}=\frac{y-b}{m}$ and $\frac{x-a}{l}=\frac{z-c}{n}$

Conversely, and pair of equations of the form (4) represent a straight line passing through (a, b, c) and having direction ratios l, m and n.

(4) is called the **canonical form of the equations of a straight line.**

For example, the equations of the straight line passing through (1, 1, 1) and having direction cosines $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are

$\frac{x-1}{1/\sqrt{3}}=\frac{y-1}{-1/\sqrt{3}}=\frac{z-1}{1/\sqrt{3}}$, that is,

$\frac{x-1}{1}=\frac{y-1}{(-1)}=\frac{z-1}{1}$.

Note that this is in the form (4), but 1, –1, 1 are **direction ratios** of this line, and not its direction cosines.

Remark 1 : By (4) we can see that the equations of the line passing through (a, b, c) and having direction cosines l, m, n are

$$x=a+lr,\ y=b+mr,\ z=c+nr, \text{ where } r\in \mathbf{R}. \qquad ...(5)$$

This is a **one-parameter form** of the equations of a line, in terms of the parameter r.

Let us now use (4) to find another form of the equations of a line. Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) lie on a line L. Then, if l, m and n are its directions cosines, (4) tells us that the equations of L are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \qquad ...(6)$$

Since Q lies on L, we get

$$\frac{x_2-x_1}{l}=\frac{y_2-y_1}{m}=\frac{z_2-z_1}{n} \qquad ...(7)$$

Then (6) and (7) give us

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \qquad ...(8)$$

(8) is the generalisation of Equation (7) of Unit 1, and is called the **two-point form** of the equations of a line in 3-space.

For example, the equations of the line passing through (1, 2, 3) and (0, 1, 4) are $x-1=y-2=-(z-3)$.

Note that, while obtaining (8) we have also shown that

> if P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) lie on a line L, then x_2-x_1, y_2-y_1 and z_2-z_1 are direction ratios of L.

Now you can try some exercises.

E 7) Find the equations to the line joining (–1, 0, 1) and (1, 2, 3).

E 8) Show that the equations of a line through (2, 4, 3) and (–3, 5, 3) are $x + 5y = 22, z = 3$.

Now let us see when two lines are perpendicular.

1.3.3 Angle Between Two Lines

In Unit 1 you saw that the angle between two lines in a plane can be obtained in terms of their slopes. Now we will find the angle between two lines in 3-space in terms of their direction cosines.

Let the lines L_1 and L_2 have direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, respectively. Let θ be the angle between L_1 and L_2. Now let us draw straight lines L_1' and L_2' through the origin with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, respectively. Then choose P and Q on L_1' and L_2', respectively, such that OP = OQ = r. Then the coordinates of P are (l_1r, m_1r, n_1r), and of Q are (l_2r, m_2r, n_2r). Also, θ is the angle between OP and OQ (see Fig. 7). Now

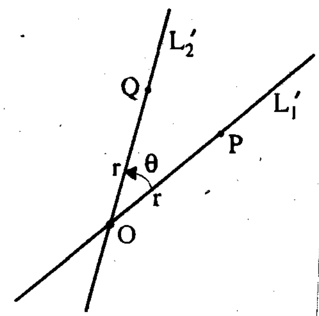


Fig. 7

$$PQ^2 = (l_1 - l_2)^2 r^2 + (m_1 - m_2)^2 r^2 + (n_1 - n_2)^2 r^2$$
$$= 2(1 - l_1 l_2 - m_1m_2 - n_1 n_2) r_2,$$

using $l_1^2 + m_1^2 + n_1^2 = 1, l_2^2 + m_2^2 + n_2^2 = 1$.

Also, from Fig. 7 and elementary trigonometry, we know that

$$PQ^2 = OP^2 + OQ^2 - 2OP \,.\, OQ \cos\theta$$
$$= 2(1 - \cos\theta) r^2.$$

Therefore, we find that θ is given by the relation

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \quad \text{...(9)}$$

Using (9) can you say when two lines are perpendicular ? They will be perpendicular iff $\theta = \pi/2$, that is iff

$$l_1l_2 + m_1m_2 + n_1n_2 = 0. \quad \text{...(10)}$$

Now, suppose we consider direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 of L_1 and L_2, instead of their direction cosines. Then, is it true that L_1 and L_2 are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$?

If you just apply the definition of direction ratios, you will see that this is so.

And when are two lines parallel? Clearly, they are parallel if they have the same or opposite directions. Thus, the lines L_1 and L_2 (given above) will be parallel iff $l_1 = l_2, m_1 = m_2, n_1 = n_2$ or $l_1 = -l_2, m_1 = -m_2, n_1 = -n_2$. In particular, this means that if a, b, c and a′, b′, c′ are direction ratios of L_1 and L_2 respectively, then

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

Let us just summarise what we have said.

> Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are
>
> (i) perpendicular iff $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$;
>
> (ii) parallel iff $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

For example, the line $\frac{x}{2} = y = \frac{z}{3}$ is not parallel to the x-axis, since 2, 1, 3 are not proportional to 1, 0, 0. Further, x = y = z is perpendicular to x = – y, z = 0, since 1, 1, 1 and 1, – 1, 0 are the direction ratio of these two lines, and 1 (1) + 1 (–1) + 1 (0) = 0.

Why don't you try some exercises now?

E 9) Find the angle between the lines with direction ratios 1, 1, 2 and $\sqrt{3}$, $-\sqrt{6}$, 4, respectively.

E 10) If 3 lines have direction ratios 1, 2, 3; 1, –2, 1 and 4, 1, –2, respectively, show that they are mutually perpendicular.

In this section you saw that a line in 3-space is represented by a pair of linear equations. In the next section you will see that this means that a line is the intersection of two planes.

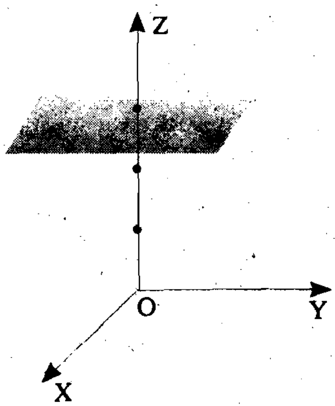


Fig. 8: The plane z = 3

1.4 PLANES

In this section you will see that a linear equation represents a plane in 3-space. We will also discuss some aspects of intersecting planes, as well as the intersection of a line and a plane.

Let us first look at some algebraic representations of a plane.

1.4.1 Equations of a Plane

Consider the XY-plane in Fig. 1 (a). The z-coordinate of every point in this plane is 0. Conversely, any point whose z-coordinate is zero will be in the XY-plane. Thus, the equation z = 0 describes the XY-plane.

Similarly, z = 3 describes the plane which is parallel to the XY-plane and which is placed 3 units above it (Fig. 8).

And what is the equationof the YZ-plane? Do you agree that it is x = 0?

Note that each of these planes satisfies the property that if any two points lie on it, then the line joining them also lies on it. This property is the defining property of a plane.

Definition : A **Plane** is a set of points such that whenver P and Q belong to it then so does every point on the line joining P and Q.

Another point that you may have noticed about the planes mentioned above is that their equations are linear in x, y and z. This fact is true of any plane, according to the following theorem.

Theorem 1 : The general linear equation Ax + By + Cz + D = 0, where at least one of A, B, C is non-zero, represents a plane in three-dimensional space.

Further, the converse is also true.

We will not prove this result here, but will always use the fact that a plane is synonymous with a linear equation in 3 variables. Thus, for example, because of Theorem 1 we know that 2x + 5z = y represents a plane.

At this point we would like to make an important remark.

Remark 2 : In 2-space a linear equation represents a line, while in 3-space a linear equation represents a plane. For example, y = 1 is the line of Fig. 9 (a), as well as the plane of Fig. 9 (b).

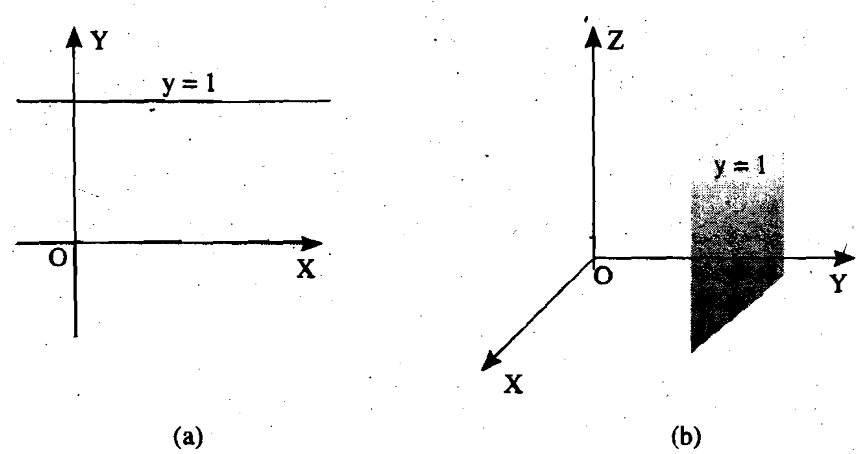


Fig. 9 : The same equation represents a line in 2-space and a plane in 3-space.

Let us now obtain the equation of a plane in different forms. To start with, we have the following results.

Theorem 2 : Three non-collinear points determine a plane. In fact, the unique plane passing through the non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by the determinant equation.

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \qquad (11)$$

We will not prove this result here, but we shall use it quite a bit.

As an example, let us find the equation of the plane which passes through the points (1, 1, 0), (−2, 2, −1) and (1, 2, 1). It will be

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 1 & 2 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix} - y \begin{vmatrix} 1 & 0 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + z \begin{vmatrix} 1 & 1 & 1 \\ -2 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$- \begin{vmatrix} 1 & 1 & 0 \\ -2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 0.$$

$\Rightarrow 2x + 3y - 3z = 5.$

Why don't you try some exercises now ?

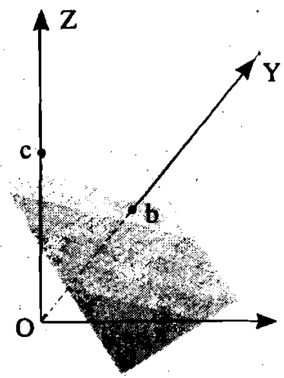


Fig. 10 : The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

E 11) Show that the four points $(0,-1,-1)$, $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ are coplanar, that is, lie on the same plane.
(**Hint :** Obtain the equation of the plane passing through any three of the points, and see if the fourth point lies on it.)

E 12) Show that the equation of the plane which makes intercepts $2, -1, 5$ on the three axes is $\frac{x}{2}+\frac{y}{(-1)}+\frac{z}{5}=1$.
(**Hint :** The plane makes an intercept 2 on the x-axis means that it intersects the x-axis at $(2, 0, 0.)$.

In E12 did you notice the relationship between the intercepts and the coefficient of the equation?

In general, you can check that the equation of the plane making intercepts a, b and c on the coordinate axes (see Fig. 10) is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad \ldots(12)$$

This is because $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ lie on it.

(12) is called the **intercept form** of the equation of a plane.

Let us see how we can use this form.

Example 2 : Find the intercepts on the coordinate axes by the plane $2x-3y+5z=4$.

Solution : Rewriting the equation, we get

$$\frac{x}{2}-\frac{y}{\frac{4}{3}}+\frac{z}{\frac{4}{5}}=1$$

Thus, the intercepts on the axes are 2, $\frac{4}{3}$ and $\frac{4}{5}$.

Now here is an exercise on the use of (12).

E13) Show that the planes $ax+by+cz+d=0$ and $Ax+By+Cz+D=0$ are the same iff a, b, c, d and A, B, C, D are proportional.

(**Hint :** Rewrite the equations in intercept form. The two planes will be the same iff their intercepts on the axes are equal).

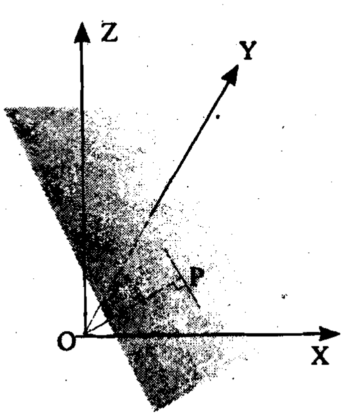


Fig. 11: Obtaining the normal form of the equation of the plane.

Let us now consider another form of the equation of a plane. For this, let us drop a perpendicular from the origin O onto the given plane (see Fig. 11). Let it meet the plane in the point P. Let $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ be the direction cosines of OP and $p = |OP|$. Further, let the plane make intercepts a, b and c on the x, y, and z axes, respectively. Then

$$\cos\alpha=\frac{p}{a},\ \cos\beta=\frac{p}{b} \text{ and } \cos\gamma=\frac{p}{c}. \qquad \ldots(13)$$

Now, from (12) we know that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

The, using (13), this equation becomes

$$x\cos\alpha+y\cos\beta+z\cos\gamma= \qquad \ldots(14)$$

This is called the **normal form** of the equation of the plane

For example, let us find the normal form of the equation of the plane in Fig. 9 (b). The perpendicular from the origin onto it is of length 1 and lies along the x-axis. Thus, its direction cosines are 1, 0, 0. Thus, from (14) we get its equation as $x = 1$.

Note that (14) is of the form $Ax + By + Cz = D$, where $|A| \le 1$, $|B| \le 1$, $|C| \le 1$ and $D \ge 0$.

Now, suppose we are given a plane $Ax + By + Cz + D = 0$. From its equation, can we find the length of the normal from the origin to it? We will use E13 to do so.

Suppose the equation of the plane in the normal form is

$x \cos\alpha + y \cos\beta + z \cos\gamma - p = 0.$

Then this is the same as the given equation of the plane. So, by E13 we see that there is a constant k such that
$\cos\alpha = kA$, $\cos\beta = kB$, $\cos\gamma = kC$, $p = -kD$.

Then, by (3) we get

$k^2(A^2 + B^2 + C^2) = 1$, that is, $k = \dfrac{\pm 1}{\sqrt{A^2 + B^2 + C^2}}$.

So, $p = -kD \dfrac{|D|}{\sqrt{A^2 + B^2 + C^2}}$, where we take the absolute value of D since $p \ge 0$.

$+ D$

Thus, the length of the perpendicular from the origin onto the plane $Ax + By + Cz = 0$ is

$$\frac{|D|}{\sqrt{A^2 + B^2 + C^2}}. \qquad ...(15)$$

For example, the length of the perpendicular from the origin onto $x + y + z = 1$ is $\dfrac{1}{\sqrt{3}}$.

Now let us go one step further. Let us find the distance between the point (a, b, c) and the plane $AX + By + Cz + D = 0$, that is, the length of the perpendicular from the point to the plane. To obtain it we simply shift the origin to (a, b, c), without changing the direction of the axes. Then, just as in Sec. 1.4.1, if X, Y, Z are the current coordinates, we get $x = X + a$. $y = Y + b$, $z = Z + c$.

We have discussed the translation of axes in detail in Unit 7.

So the equation of the plane in current coordinates is $A(X + a) + B(Y + b) + C(Z + c) + D = 0$.

Thus, the length of the normal from (a, b, c) to the plane $Ax + By + Cz + D = 0$ is the same as the length of the normal from the current origin to

$A(X + a) + B(Y + b) + C(Z + C) + D = 0$, that is,

$$\frac{|Aa + Bb + Cc + D|}{\sqrt{A^2 + B^2 + C^2}}$$

For example, the length of the normal from (4, 3, 1) to $3x - 4y + 12z + 14 = 0$ is

$$P = \frac{|12 - 12 + 12 + 14|}{\sqrt{3^2 + 4^2 + 12^2}} = \frac{26}{13} = 2.$$

Now you may like to do the following exercises.

E 14) Find the distance of (2, 3, –5) from each of the coordinate planes, as well as from $x + y + z = 1$.

Show that if the sum of the squares of the distance of (a, b, c) from the planes $x + y + z = 0$, $x = z$ and $x + z = 2y$ is 9, then $a^2 + b^2 + c^2 = 9$.

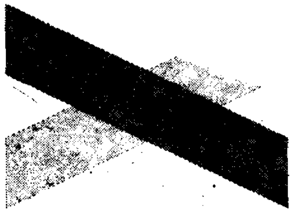


Fig. 12 : A straight line is the intersection of two planes.

Now that you are familiar with the various equations of a plane let us talk of the intersection of planes.

1.4.2 Intersecting Planes and Lines

In Sec. 4.3.2, you saw that a line is represented by two linear equations. Thus, it is the intersection of two planes represented by these equations (see Fig. 12).

In general, we have the following remark.

Remark 3 : A straight line is represented by a linear system of the form $ax + by + cz + d = 0$, $Ax + By + Cz + D = 0$. We write this in short as $ax + by + cz + d = 0 = Ax + By + Cz + D$. For example, $3x + 5y + z - 1 = 0 = 2x + 1$ represents the line obtained on intersecting the planes $3x + 5y + z = 1$ and $2x + 1 = 0$

Now, suppose we are given a line.

$ax + by + cz + d = 0 = Ax + By + Cz + D$.

This line clearly lies in both the planes $ax + by + cz + d = 0$ and $Ax + By + Cz + D = 0$. In fact, it lies in infinitely many planes given by

$$(ax + by + cz + d) + k(Ax + By + Cz + D) = 0, \qquad ...(17)$$

where $k \in \mathbf{R}$. This is because any point (x, y, z) lies on the line iff it lies on $ax + by + cz + d = 0$ as well as $Ax + By + Cz + D = 0$.

Let us see an example of the use of (17).

Example 3 : Find the equation of the plane passing through the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and the point $(0, 7, -7)$.

Solution : The line is the intersection of $2(x + 1) = -3(y - 3)$ and $x + 1 = -3(z + 2)$, that is, $2x + 3y - 7 = 0 = x + 3z + 7$.

Thus, by (17), any plane passing through it is of the form

$(2x + 3y - 7) + k(x + 3z + 7) = 0$ for some $k \in \mathbf{R}$.

Since $(0, 7, -7)$ lies on it, we get

$21 - 7 + k(-21 + 7) = 0$, that is, $k = 1$.

Thus, the required plane is

$3x + 3y + 3z = 0$, that is, $x + y + z = 0$.

You can do the following exercise on the same lines.

E 16). Find the equation of the plane passing through $(1, 2, 0)$ and the line $x \cos\alpha + y \cos\beta + z \cos\gamma = 1$, $x + y = z$.

Now, given a line and a plane, will they always intersect? And, if so, what will their intersection look like? In Fig. 13 we show you the three possibilities.

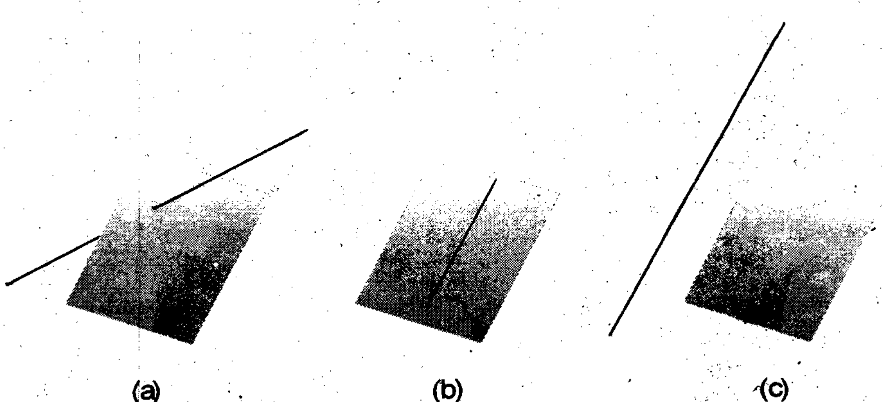


Fig. 1.3 : (a) A line and a plane can either intersect in a point, or (b) the line can lie in the plane, or (c) they may not intersect at all.

Let us see some example.

Example 4 : Check whether the plane $x+y+z=1$ and the straight line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ intersect. If they do, then find their point (or points) of intersection.

Solution : By Remark 1 you know that any point on the line can be given by $x=t$, $y=1+2t$ and $z=2+3t$, in terms of a parameter t. So, if the line and plane intersect, then $(t, 1+2t, 2+3t)$ must lie on the plane $x+y+z=1$ for some t. Let us substitute these values in the equation. We get $t+1+2t+2+3t=1$, that is, $6t=-2$, that is, $t=-\frac{1}{3}$. Thus, the line and plane intersect in a point, and the point of intersection is $\left(-\frac{1}{3}, \frac{1}{3}, 1\right)$.

Example 5 : Find the point (or points) of intersection of

(a) $\frac{x+2}{2}=\frac{y+3}{3}=\frac{z-4}{-2}$ and $3x+2y+6z=12$.

(b) $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{-2}$ and $x+z=1$.

Solution : a) Any point on the line is of the form $(2k-2, 3k-3, -2k+4)$, where $k \in \mathbf{R}$.

Thus, if there is any point of intersection, it will be given by substituting this triple in $3x+2y+6z=12$.

So, we have

$3(2k-2)+2(3k-3)+6(-2k+4)=12$

$\Rightarrow 0=0.$

This is true $\forall\, k \in \mathbf{R}$. Thus, for every $k \in \mathbf{R}$, the triple $(2k-2, 3k-3, -2k+4)$ lies in the plane. This means that the whole line lies in the plane.

b) Any point on the line is of the form $(2t+1, t+2, -2t+3)$, where $t \in \mathbf{R}$. This lies on $x+z=1$ if, some t, $(2t+1)+(-2t+3)=1$, that is, if $4=1$, which is false. Thus, the line and plane do not intersect.

You can use the same method for finding the point (or points) of intersection of two lines. In following exercises you can check if you've undestood the method.

E 17) Find the point of intersection of the line $x=y=z$ and the plane $x+2y+3z=3$.

E 18) Show that the line $x-1=\frac{1}{2}(y-3)=\frac{1}{3}(z-5)$ meets the line $\frac{1}{3}(x+1)=\frac{1}{5}(y-4)=\frac{1}{7}(z-9)$.

Now, consider any two planes. Can we find the angle between them? We can, once we have the following definition.

Definition : The **angle between two planes** is the angle between the normals to them from the origin.

So, now let us find the angle between two planes. Let the equations of the planes, in the normal form, be $l_1x+m_1y+n_1z=p_1$ and $l_2x+m_2y+n_2z=p_2$. Then the angle between the normals is $\cos^{-1}(l_1l_2+m_1m_2+n_1n_2)$.

Thus, the angle between two planes $l_1x+m_1y+n_1z=p_1$ and $l_2x+m_2y+n_2z=p_2$ is

$$\cos^{-1}(l_1l_2+m_1m_2+n_1n_2). \qquad \ldots(18)$$

$$\cos\theta = \frac{aA + bB + cC}{\sqrt{a^2+b^2+c^2}\,.\sqrt{A^2+B^2+C^2}} \qquad \text{...(19)}$$

This is because a, b, c and A, B, C are direction ratios of the normals to the two planes, so that $\frac{a}{\sqrt{a^2+b^2+c^2}}$, $\frac{b}{\sqrt{a^2+b^2+c^2}}$, $\frac{c}{\sqrt{a^2+b^2+c^2}}$ and $\frac{A}{\sqrt{A^2+B^2+C^2}}$, $\frac{B}{\sqrt{A^2+B^2+C^2}}$, $\frac{C}{\sqrt{A^2+B^2+C^2}}$ are their direction cosines.

Thus,

> the planes $ax + by + cz + d = 0$ and $Ax + By + Cz + D = 0$
>
> i) are parallel iff $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$, and
>
> ii) are pependicular iff $aA + bB + cC = 0$.

Let us consider an example of the utility of these conditions.

Example 6 : Find the equation of a plane passing through the line of intersection of the planes $7x - 4y + 7z + 16 = 0$ and $4x + 3x - 2z + 13 = 0$, and which is pependicular to the plane $2x - y - 2z + 5 = 0$.

Solution : The general equation of the plane through the line of intersection is given by $7x - 4y + 7z + 16 + k(4x + 3y - 2z + 13) = 0$.

$\Rightarrow (7 + 4k)\,x + (3k - 4)\,y + (7 - 2k)\,z + 13k + 16 = 0.$

This will be perpendicular to $2x - y - 2z + 5 = 0$, if

$2(7 + 4k) - (3k - 4) - 2(7 - 2k) = 0$, that is, $k = -\frac{4}{9}$.

Thus, the required equation of the plane is $47x - 48y + 71z + 92 = 0$.

Try these exercises now.

E 19) Find the equation of the plane through (1, 2, 3) and parallel to $3x + 4y - 5z = 0$.

E 20) Find the angle between the planes $x + 2y + 2z = 5$ and $2x + 2y + 3 = 0$.

E 21) Show that the angle between the line $\frac{x - \overset{a}{\cancel{\alpha}}}{\alpha} = \frac{y - b}{\beta} = \frac{z - c}{\gamma}$ and the plane

$Ax + By + Cz + D = 0$ is $\sin^{-1}\left(\frac{A\alpha + B\beta + C\gamma}{\sqrt{A^2+B^2+C^2}\,.\sqrt{\alpha^2+\beta^2+\gamma^2}}\right)$.

(**Hint :** The required angle is the complement of the angle between the line and the normal to the plane).

And now let us end the unit by summarising what we have done in it.

1.5 SUMMARY

In this unit we have covered the following points.

1) Distance formula : The distance between the points (x, y, z) and (a, b, c) is

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}.$$

2) The coordinates of a point that divides the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio m : n are

$$\left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}, \frac{nz_1+mz_2}{m+n}\right)$$

3) If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of a line, then $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.

4) The canonical form of the equations of a line passing through the point (a, b, c) and having direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ is

$$\frac{x-a}{\cos\alpha}=\frac{y-b}{\cos\beta}=\frac{z-c}{\cos\gamma}$$

5) The two-point form of the equations of a line passing through (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

6) The angle between two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is

$$\cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,.\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

Thus, these lines are perpendicular iff $a_1a_2+b_1b_2+c_1c_2=0$, and parallel iff $a_1=ka_2$, $b_1=kb_2$, $c_1=kc_2$, for some $k\in\mathbf{R}$.

7) The equation of a plane is of the form $Ax+By+Cz+D=0$, where $A, B, C, D\in\mathbf{R}$ and not all of A, B, C are zero.

Conversely, such an equation always represents a plane.

8) The plane determined by the three points

(x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0.$$

9) The equation of the plane which makes intercepts a, b and c on the x, y and z-axes, respectively, is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

10) The normal form of the equation of a plane is $x\cos\alpha+y\cos\beta+z\cos\gamma=p$, where p is the length of the pependicular from the origin onto the plane and cos a, $\cos\beta$, $\cos\gamma$ are the direction cosines of the perpendicular.

11) The length of the pependicular from a point (a, b, c) onto the plane $Ax+By+Cz+D=0$ is

$$\frac{|Aa + Bb + Cc + D|}{\sqrt{A^2 + B^2 + C^2}}$$

12) A line is the intersection of two planes.

13) The general equation of a plane passing through the line ax + by + cz + d = 0 = Ax + b By + Cz + D is (ax + by + cz + d) + k (Ax + By + Cz + D) = 0, where $k \in \mathbf{R}$.

14) The angle between the planes ax + by + cz + d = 0 and Ax + By + Cz + D = 0 is

$$\cos^{-1}\left(\frac{aA + bB + cC}{\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{A^2 + B^2 + C^2}}\right).$$

And now you may like to go back to Sec. 4.1, and see if you've achieved the **unit objectives** listed there. As you know by now, one way of checking this is to ensure that you have done all the exercises in the unit. You may like to see our solutions to the exercises. We have given them in the following section.

1.6 SOLUTIONS/ANSWERS

E 1) $PQ = \sqrt{(1-(-1))^2 + (1-1)^2 + (-1-1)^2} = \sqrt{8}$

The coordinates of R are $\left(\frac{1}{7}, 1, -\frac{1}{7}\right)$.

E 2) $\left(\frac{a+r}{2}, \frac{b+s}{2}, \frac{c+t}{2}\right)$.

E 3) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2$.

E 4) a) 0, 1, 0 and 0, 0, 1, respectively.

b) Any line in the XY-plane makes the angle $\pi/2$ with the z-axis. Now, if $m = \tan\theta$, then $y = mx + c$ makes an angle θ with the x-axis, and $\pi/2 - \theta$ with the y-axis. Thus, its direction cosines are $\cos\theta$, $\sin\theta$, 0.

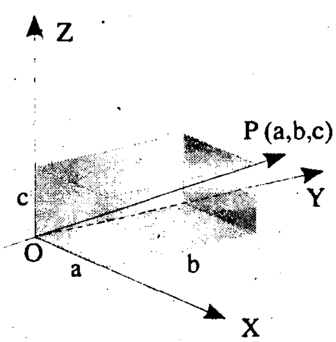


Fig. 14

E 5) In Fig. 14 we have depicted the situation.

Let OP = r. Then you can see that the direction cosines of L are $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$.

Thus, a, b, c are direction ratios of L.

E 6) Now, the line L makes angles of $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$ with the positive directions of the x, y and z-axes, respectively. Thus, its direction cosines are $-\cos\alpha$, $-\cos\beta$ and $-\cos\gamma$.

E 7) $\frac{x+1}{2} = \frac{y}{2} = \frac{z-1}{2}$.

E 8) The equations are

$\frac{x+3}{5} = \frac{y-5}{-1} = \frac{z-3}{0} = r$, say, that is,

$-(x+3) = 5(y-5)$ and $z = 3$, that is,

$x + 5y = 22, z = 3$.

E 9) The direction cosines of the line with direction ratios 1, 1, 2 are $\frac{1}{\sqrt{1^2+1^2+2^2}}$, $\frac{1}{\sqrt{1^2+1^2+2^2}}$, $\frac{2}{\sqrt{1^2+1^2+2^2}}$, that is, $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$.

Similarly, the direction cosines of the other line are $\frac{\sqrt{3}}{5}, -\frac{\sqrt{6}}{5}, \frac{4}{5}$,

Thus,. if θ is the angle between them,

$$\cos\theta = \left(\frac{1}{\sqrt{6}}\right)\left(\frac{\sqrt{3}}{5}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{-\sqrt{6}}{5}\right) + \left(\frac{2}{\sqrt{6}}\right)\left(\frac{4}{5}\right)$$

$$= \frac{1}{5\sqrt{6}}\left(8 + \sqrt{3} - \sqrt{6}\right).$$

E 10) Since $1(1) + 2(-2) + 3(1) = 0$.
$1(4) - 2(1) + 1(-2) = 0$, and
$1(4) + 2(1) + 3(-2) = 0$,
the lines are mutually perpendicular.

E 11) The equation of the plane passing through the first there points is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \end{vmatrix} = 0.$$

$\Rightarrow 5x - 7y + 11z + 4 = 0.$

Since $(-4, 4, 4)$ satisfies it, the 4 points are coplanar.

E 12) The points $(2, 0, 0)$, $(0, -1, 0)$, $(0, 0, 5)$ lie on the plane.

Thus, its equation is

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 5 & 1 \end{vmatrix} = 0 \Rightarrow 5x - 10y + 2z = 10$$

$$\Rightarrow \frac{x}{2} + \frac{y}{(-1)} + \frac{z}{5} = 1.$$

E 13) The intercepts of the two planes on the axes are $-\frac{d}{a}, -\frac{d}{b}, -\frac{d}{c}$

and $-\frac{D}{A}, -\frac{D}{B}, -\frac{D}{C}$, respectively. Thus, the planes coincide

iff $-\frac{d}{a} = -\frac{D}{A}, -\frac{d}{b} = -\frac{D}{B}, -\frac{d}{c} = -\frac{D}{C}$, that is,

iff $\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = \frac{d}{D}$, that is, iff a, b, c, d and A, B, C, D are proportional.

E 14) The distance of $(2, 3, -5)$ from the XY- plane, $z = 0$, is $\frac{|-5|}{\sqrt{1^2}} = 5$.

Similarly, the distance of $(2, 3, -5)$ from $x = 0$ and $y = 0$ is 2 and 3, respectively.
Its distance from $x + y + z = 1$ is

$$\frac{|2 + 3 - 5 - 1|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

E 15) We know that

$$\left(\frac{|a+b+c|}{\sqrt{}}\right)^2 + \left(\frac{|a-c|}{\sqrt{}}\right)^2 + \left(\frac{|a-2b+c|}{\sqrt{}}\right)^2 = 9.$$

$\Rightarrow a^2 + b^2 + c^2 = 9.$

E 16) The equation of the plane passing through the given line is
$(x \cos\alpha + y \cos\beta + z \cos\gamma - 1) + k(x + y - z) = 0,$ ...(20)
where $k \in \mathbf{R}$ is chosen so that $(1, 2, 0)$ lies on the plane.

$$\therefore \quad (\cos\alpha + 2\cos\beta - 1) + 3k = 0 \Rightarrow k = \frac{1}{3}(1 - \cos\alpha - 2\cos\beta).$$

Thus, the required equation is obtained by putting this value of k in (20).

E 17) Any point on the line is (t, t, t). The line and plane will intersect if, for some $t \in \mathbf{R}$.

$$t + 2t + 3t = 3 \Rightarrow t = \frac{1}{2}.$$

Thus, the plane and line intersect in only one point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

E 18) Any point on the first line is given by
$(t+1, 2t+3, 3t+5), t \in \mathbf{R}$.
Any point on the second line is given by
$(3k-1, 5k+4, 7k+9), k \in \mathbf{R}$.

The two lines will intersect if $t + 1 = 3k - 1$, $2t + 3 = 5k + 4$ and $3t + 5 = 7k + 9$ for some t and k in $\mathbf{R}$.

On solving these equations we find that they are consistent, and $k = 5$ gives us the common point. Thus, the point of intersection is $(14, 29, 44)$.

E 19) Any plane parallel to $3x + 4y - 5z = 0$ is of the form
$3x + 4y - 5z + k = 0$, where $k \in \mathbf{R}$.
Since $(1, 2, 3)$ lies on it, $3 + 8 - 15 + k = 0 \Rightarrow k = 4$.
Thus, the required plane is $3x + 4y - 5z + 4 = 0$.

E 20) If the angle is θ, then

$$\cos\theta = \frac{1(2) + 2(2) + (2(0)}{\sqrt{9}\ \sqrt{8}} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \theta = \pi/4.$

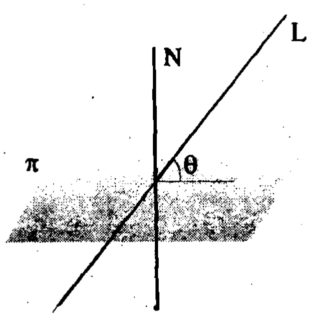


Fig 15 : The line L makes an angle θ with the plane π and $(\pi/2 - \theta)$ with the normal N to π.

E21) If θ is the angle between the line and the plane, then $\frac{\pi}{2} - \theta$ is the angle between the line and the normal to the plane (see Fig. 15). Now, A, B, C are the direction ratios of the normal. Thus,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{A\alpha + B\beta + C\gamma}{\sqrt{A^2 + B^2 + C^2} \cdot \sqrt{\alpha^2 + \beta^2 + \gamma^2}}$$

$$\Rightarrow \sin\theta = \frac{A\alpha + B\beta + C\gamma}{\sqrt{A^2 + B^2 + C^2} \cdot \sqrt{\alpha^2 + \beta^2 + \gamma^2}}$$