# UNIT 4 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

## Structure



## 4.1 INTRODUCTION

In Unit 3 we have introduced the concept of derivatives. We have also talked about the algebra of derivatives and the chain rule which help us in calculating the derivatives of some complex functions. This unit will take you a step further in your study of differential calculus.

In this unit we shall first find the derivatives of standard trigonometric functions. We shall then go on to study the inverse function theorem and its applications in finding the derivatives of inverses of some standard functions. Finally, we shall see how the use of transformations can simplify the problem of differentiating some functions.

### Objectives

After studying this unit you should be able to :

- find the derivatives of trigonometric functions
- state and prove the inverse function theorem
- use the inverse function theorem to find the derivatives of inverse trigonometric functions
- use suitable transformations to differentiate given functions.

## 4.2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

In this section we shall calculate the derivatives of the six trigonometric functions; sin x, cos x, tan x, cot x, sec x and cosec x. You already know that these six functions are related to each other. For example, we have:

i) $\sin^2 x + \cos^2 x = 1$   ii) tan x = sin x/cos x, and many more identities which express the relationships between these functions. As you will soon see, our job of finding the derivatives of all trigonometric functions becomes a lot easier because of these identities. But let us first evaluate some important limits which will prove to be very useful later.

### 4.2.1 Some Useful Limits

In the next subsection we shall come across $\lim_{t \to 0} \frac{\sin t}{t}$ and $\lim_{t \to 0} \sin$ . So let's try to calculate these limits. For this, we first assume that $0 < t < \pi/2$ and consider a circle with radius 1 unit, given by $x^2 + y^2 = 1$ as shown in Fig. 1.

The line OT passes through the origin and has slope = tan t. Therefore, we can write its equation as $y = x \tan t$. This means that the y-coordinate of the point T is tan t, since its x-coordinate is 1.

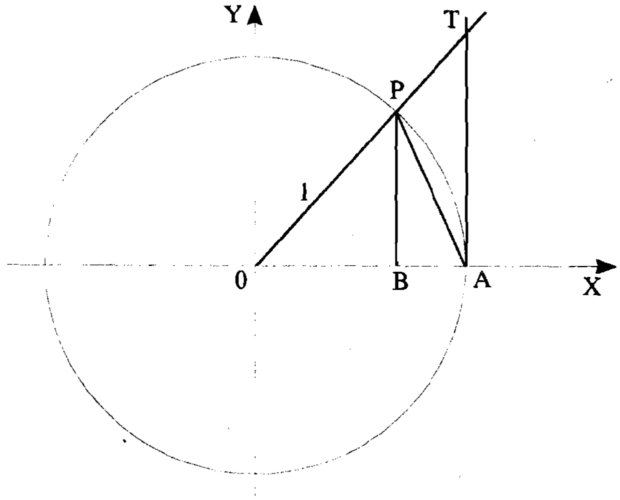



**Fig. 1**

From the figure we can see that

If the sectorial angle is $\theta$, the area of a sector of a circle of radius r is $(1/2)r^2\theta$

area of $\Delta$OPA < area of sector OPA < area of $\Delta$OTA ...(1)

Now, the area of $\Delta OPA = \frac{1}{2} \times 1 \times PB = \frac{1}{2} \sin t$,

The area of sector $OPA = \frac{1}{2} \times 1 \times t = \frac{1}{2} t$,

The area of $OTA = \frac{1}{2} \times 1 \times$ tant t

Thus, inequality (1) can be written as:

$\sin t < t < \tan t$ ... (2)

Since $0 < t < \pi/2$, $\sin t > 0$, therefore, from the left-hand inequality in (2) we get $0 < \sin t < t$ ...(3)

Now, if $-\pi/2 < t < 0$, then $0 < -t < \pi/2$, and applying (3) to $-t$, $0 < \sin(-t) < -t$ or $0 < -\sin t < -t$ since $\sin(-t) = -\sin t$. This means that if $-\pi/2 < t < 0$, then $t < \sin t < 0$ ...(4)

Here we are using various results about the order relation from Unit 1.


**Fig. 2**

In Fig. 2(a) and (b) you can see the representation of (3) and (4), respectively.

We can prove that $\lim_{t\to 0} \sin^2 t/2 = 0$ by using Theorem 3 of Unit 2 and by noting that $t \to 0 \Leftrightarrow t/2 \to 0$

We can combine (3) and (4) and write

$-|t| < \sin t < |t|$ for $-\pi/2 < t < \pi/2$, $t \neq 0$.

You have seen in Unit 2 that $\lim_{t\to 0} |t| = 0$. From this we can also say that $\lim_{t\to 0} -|t| = 0$.

Now applying the sandwich theorem (Theorem 2 of Unit 2) to the functions $-|t|$, sin t and $|t|$,

we get that $\lim \sin t = 0$

We shall use this result to calculate $\lim_{t\to 0} \cos t$. As you know, $\cos t = 1 - 2\sin^2 t/2$. This means

$$\lim_{t\to 0} \cos t = \lim_{t\to 0} (1 - 2\sin^2 t/2)$$
$$= 1 - 2\lim_{t\to 0} \sin^2 t/2 = 1$$

Thus, we get $\lim_{t\to 0} \cos t = 1$.

Now, let's get back to inequality (2): $\sin t < t < \tan t$. for $0 < t < \pi/2$. Since $0 < t < \pi/2$, $\sin t > 0$, and therefore, after dividing by $\sin t$, (2) becomes;

$1 < t/\sin t < 1/\cos t$

or $\cos t < \sin t/t < 1$ ...(5), $0 < t < \pi/2$

Now, since $\sin(-t) = -\sin t$, we see that $\sin(-t)/(-t) = \sin t/t$. This alongwith the result $\cos(-t) = \cos t$, shows that the inequality (5) holds even when $-\pi/2 < t < 0$. Thus,
$\cos t < \sin t/t < 1$, $-\pi/2 < t < \pi/2$, $t \neq 0$

Now, let us apply the sandwich theorem to the functions $\cos t$, $\sin t/t$ and 1, and take the limits at $t \to 0$. This give us:

$$\lim_{t\to 0} \sin t\, t/t = 1$$

**Example 1.** Suppose we want to find out

$$\lim_{t\to 0} \frac{\sin 3x}{x} \text{ and } \lim_{t\to 0} \frac{\sin 3x}{\sin 7x}$$

Let us first calculate $\lim_{t\to 0} \frac{\sin 3x}{x}$. For this we shall write $\frac{\sin 3x}{x} = \frac{\sin 3x}{x} \times 3$. If we replace 3x by t in the right hand side, and take the limit as $x \to 0$, we find that $t = 3x$ also tends to zero, and $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{t\to 0} \frac{\sin t}{t} \times 3$

$$= 3 \lim_{t\to 0} \frac{\sin t}{t} \qquad \text{(See Theorem 3 of Unit 2)}$$
$$= 3$$

To calculate $\lim_{x\to 0} \frac{\sin 5x}{\sin 7x}$ we start by writing

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 7x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \times \frac{7x}{\sin 7x} \times \frac{5}{7}$$
$$= \frac{5}{7} \lim_{x\to 0} \frac{\sin 5x}{5x} \lim_{x\to 0} \frac{7x}{\sin 7x}$$
$$= \frac{5}{7} \left( \text{since } \lim_{x\to 0} \frac{7x}{\sin 7x} = \frac{1}{\lim_{x\to 0} (\sin 7x / 7x)} = 1 \text{ by Theorem 3 of Unit 2} \right)$$

**Remark 1** in $\lim_{t\to 0} \frac{\sin t}{t} = 1$ or $\lim_{t\to 0} \cos t = 1$, the angle t is measured in radians. If in a particular problem, the angles are measured in degrees, we have to first convert these into radians before using these formulas. Thus,

$$\lim_{t\to 0} \frac{\sin t^\circ}{t} = \lim_{t\to 0} \frac{\sin(\pi t/180)}{t} = \frac{\pi}{180} \lim_{t\to 0} \frac{\sin(\pi t/180)}{\pi t/180} = \frac{\pi}{180}$$

See if you can solve this exercise now.

**E** E 1) Prove that a) $\lim_{x\to 0} \cos(a + x) = \cos a$

b) $\lim_{x\to 0} \sin(a + x) = \sin a$

### 4.2.2 Derivatives of Sin x and Cos x

We shall now find out the derivative of sin x from the first principles. If $y = f(x) = \sin x$, then by definition

Remember the formula

$\sin A - \sin B = 2 \sin \frac{(A-B)}{2} \cos \frac{(A+B)}{2}$ ?

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\sin(h/2)\cos(x+h/2)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \lim_{h\to 0} \cos(x + h/2)$$

$$= 1 \times \cos x = \cos x$$

Thus, we get

$$\mathbf{\frac{d}{dx}(\sin x) = \cos x}$$

Now, let us consider the function $y = f(x) = \cos x$ and find derivative. In this case,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin(h/2)\sin(x+h/2)}{h}$$

$$= -\lim_{h\to 0} \frac{\sin(h/2)}{h/2} \lim_{h\to 0} \sin(x + h/2)$$

$$= -\sin x$$

Thus, we have shown that

$$\mathbf{\frac{d}{dx}(\cos x) = -\sin x}$$

Actually, having first calculated $\frac{d}{dx}(\sin x)$, we could found out the derivative of cos x by using the formula :

$\cos x = \sin(x + \pi/2)$. This gives us,

$\frac{d}{dx}(\sin(x + \pi/2)) = \cos(x + \pi/2)$ can be proved by using the chain rule.

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}(\sin(x + \pi/2))$$
$$= \cos(x + \pi/2) = -\sin x$$

In the next subsection we shall find the derivatives of the other four trigonometric functions by using similar formulas. But before that it is time to do some exercises.

**E** E 2) Find the derivatives of the following:

a) $\sin 2x$ b) $\cos^2 x$ c) $5 \sin^7 x \sin 3x$ d) $x^3 \cos 9x$

e) $\cos (\sin x)$

### 4.2.3 Derivatives of other Trigonometric Functions

We shall now find the derivatives of i) tan x ii) cot x iii) sec x iv) cosec x.

i) Suppose $y = f(x) = \tan x$. We know that $\tan x = \dfrac{\sin x}{\cos x}$

Hence, $$\frac{dy}{dx} = \frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \cos x \frac{d (\sin x)}{dx} - \sin x \frac{d (\cos x)}{dx}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\frac{d}{dx}(u / v) = \frac{v\, du / dx - u\, dv / dx}{v^2}$$

ii) Now, suppose $y = f(x) = \cot x$. Since $\cot x = 1/\tan x$, we get

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right)$$

$$= \frac{\tan x\, d(1) / dx - 1\, d (\tan x) / dx}{\tan^2 x}$$

$$= \frac{-\sec^2 x}{\tan^2 x} = - \operatorname{cosec}^2 x$$

iii) Now, let $y = f(x) = \sec x$. Since we know that $\sec x = 1/\cos x$, proceeding as in ii), we get

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

If you have followed i), ii) and iii) above, you sould not have any difficulty in finding the derivative of cosec x by using $\operatorname{cosec} x = 1/\sin x$.

E 3) Show that $\frac{d}{dx}$ (cosec x) = –cosec x cot x.

Let us summarise our results.

**Table 1**

| Function | Derivative |
|---|---|
| sin x | cos x |
| cos x | – sin x |
| tan x | $\sec^2 x$ |
| cot x | $-\operatorname{cosec}^2 x$ |
| sec x | sec x tan x |
| cosec x | – cosec x cot x |

**Remark 2** Here again we note that the angle is measured in radians. Thus

$$\frac{d}{dx}(\sin x^o) = \frac{d}{dx}\left(\sin \frac{\pi x}{180}\right) = \frac{\pi}{180} \cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos x^o$$

We shall now see how we can use these results to find the derivatives of some more complicated functions. The chain rule and the algebra of derivatives with which you must have become quite familiar by now, will come in handy again.

**Example 2** Let us differentiate i) $\sec^3 x$ ii) sec x tan x + cot x

i) Let $y = \sec^3 x$. If we write u = sec x, we get $y = u^3$. Thus,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2 \text{ sex } x \tan x$$
$$= 3 \sec^3 x \tan x$$

ii) If y = sec x tan x + cot x, then,

$$\frac{dy}{dx} = \frac{d}{dx}(\sec x \tan x) + \frac{d}{dx}(\cot x)$$
$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) - \operatorname{cosec}^2 x$$
$$= \sec x (\sec^2 x + \tan^2 x) - \operatorname{cosec}^2 x$$

**Remark 3.** sin x, cos x, sec x, cosec x are periodic functions with period $2\pi$. Their derivatives are also periodic with period $2\pi$. tan x and cot x are periodic with period $\pi$. Their derivatives are also periodic with period $\pi$.

We have been considering variables which are dimensionless. Actually, in practice, we may have to consider variables having dimensions of mass, length, time etc., and we have to be careful in interpreting their derivatives. Thus, we may be given that the distance x travelled by a particle in time t is x = a cos bt. Here, since bt has to be dimensionless (being an angle), b must have the dimension $\frac{1}{T}$. Similarly, x/a = cos bt has to be dimensionless. This means that a must have the same dimension as x. That is dimension of a is L.

Now $dx/dt = -ab \sin bt$ has the dimension as $ab = L \times 1/T = L/T$, which is not unexpected, since dx/dt is nothing but the velocity of that particle.

See if you can do these exercises now.

**E** E 4) Find the derivatives of :

a) cosec 2x     b) $\cot x + \sqrt{\operatorname{cosec} x}$     c) 5cot 9x

## 4.3 DERIVATIVES OF INVERSE FUNCTIONS

We have seen in Unit 1 that the graphs of a function and its inverse are very closely related to each other. If we are given the graph of a function, we have only to take its reflection in the line $y = x$, to obtain the graph of its inverse. In this section we shall establish a relation between the derivatives of a function and its inverse.

### The Inverse Function Theorem

Let us take two functions f and g, which are inverses of each other. We have already seen in Unit 1 that in this case, $gof(x) = g(f(x) = x$, for all x for which f is defined, $fog(y) = f(g(y)) = y$ for all y for which g is defined. Now, suppose that both f and g are differentiable. Then, by applying the chain rule to differentiate $g(f(x)) = x$, we get $g'(f(x)).f'(x) = 1$ or $g'(y).f'(x) = 1$, where $g = f(x)$ where $g'(f(x))$ is derivative w.r.t f(x).

This means that if $f'(x) \neq 0$, we can write $g'(y) = 1/f'(x)$. So we have been able to find some relation between the derivatives of these inverse functions. Let us state our results more precisely.

The strict monotonicity condition in this theorem implies that f is one-one and thus ensures the existence of $f^{-1}$.

**Theorem 1 (The Inverse Function Theorem)**
Let f be differentiable and strictly monotonic on an interval I. If $f'(x) \neq 0$ at a certain x in I, then $f^{-1}$ is differentiable at $y = f(x)$ and

$(f^{-1})'(y) = 1/f'(x)$.

Thus, we have the inverse function rule:

$$(f^{-1})'(y) = \frac{1}{f'(x)} \quad \text{or} \quad \frac{dx}{dy} = \frac{1}{dy/dx}$$

The derivative of the inverse function is the reciprocal of the derivative of the given function.

Soon we shall see that this rule is very useful if we want to find the derivative of a function when the derivative of its inverse function is already known. This will become clear when we consider the derivatives of the inverses of some standard functions. But first, let us use this rule to find the derivative of $f(x) = x^r$, where r is a rational number. In unit 3 we have already proved that $\frac{d}{dx}(x^n) = nx^{n-1}$ when n is an integer. We shall use this fact in proving the general case.

$x^r$ may not be always defined. For example, if $x = -1$ and $r = 1/2$, $x^r = \sqrt{-1}$ is not defined in **R**.

**Theorem 2** If $y = f(x) = x^r$, where r is a rational number for which $x^r$ and $x^{r-1}$ are both defined, then $\frac{d}{dx}(x^r) = rx^{r-1}$.

**Proof:** Let us first consider the case when $r = 1/q$, q being any non-zero integer. In this case, $y = f(x) = x^{\frac{1}{q}}$. Its inverse function g will be given by $x = g(y) = y^q$. This means

$$\frac{dx}{dy} = g'(y) = qy^{q-1}$$

Thus, by the inverse function rule, we get

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{qy^{q-1}}$$

$$= \frac{1}{q\left(x^{1/q}\right)^{q-1}}$$

$$= \frac{1}{qx(q^{1/q})/q} = \frac{1}{q}\ x^{-(q-1)/q}$$

$$= \frac{1}{q}\ x^{(1/q)-1} = rx^{r-1}$$

So far, we have seen that the theorem is true when r is of the form 1/q, where q is an integer. Now, having proved this, let us take the general case when $r = p/q$, $p, q \in \mathbf{Z}$ (q is, of course, non-zero). Here,

$$y = f(x) = x^r = x^{p/q}$$

So, $\frac{dy}{dx} = \frac{d}{dx}(x^{p/q}) = \frac{d}{dx}(x^{1/q})^p$

Now, $\frac{d}{dx}(x^{1/q})^p = p(x^{1/q})^{p-1}\frac{d}{dx}(x^{1/q})$, by chain rule

$= p(x^{1/q})^{p-1}\ (-1/q)\ x^{(1/q)-1}$

$= (p/q)\ x^{(p/q)-1}$

Thus, $\frac{dy}{dx} = \frac{d}{dx}(x^r) = (p/q)x^{(p/q)-1} = rx^{r-1}$

This completes the proof of the theorem.

The usefulness of this theorem is quite clear from the following example.

**Example 3.** Suppose we want to differentiate

$$y = (x^{5/6} + \sqrt{x})^{1/11}$$

We write $u = x^{5/6} + \sqrt{x}$. This gives us, $y = u^{1/11}$.
By chain rule, we get

$$\frac{dy}{dx} = \frac{1}{11}(x^{5/6} + \sqrt{x})^{(1/11)-1}\left(\frac{5}{6}x^{(5/6)-1} + \frac{1}{2}x^{-1/2}\right)$$

Thus, $\frac{dy}{dx} = \frac{1}{66}(x^{5/6} + \sqrt{x})^{-10/11}(5x^{-1/6} + 3x^{-1/2})$

Why don't you try these exercises now ?

E 5) Differentiate

a) $5(x^3 + x^{1/3})$ b) $(\sqrt[5]{x} - \sqrt[9]{x})x^2$

## 4.4 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

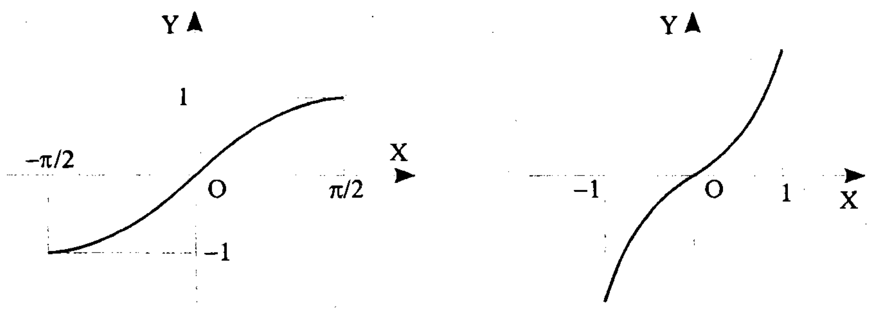
In the last section we have seen how the inverse function theorem helps us in finding the derivative of $x^n$ where n is a rational number. We shall now use that theorem to find the derivatives of inverse trigonometric functions.

We have noted in Unit 1, Section 5, that sometimes when a given function is not one-one, we can still talk about its inverse, provided we restrict its domain suitably. Now, sin x is neither a one-one, nor an onto function from **R** to **R**. But if we restrict its domain $[-\pi/2, \pi/2]$, and co-domain to $[-1, 1]$, then it becomes a one-one and onto function, and hence the existence of its inverse is assured. In a similar manner we can talk about the inverses of the remaining trigonometric functions if we place suitable restrictions on their domains and co-domains.

Now that we are sure of the existence of inverse trigonometric functions, let's go ahead and find their derivatives.

### 4.4.1 Derivatives of $Sin^{-1}$ x and $Cos^{-1}$ x

Let us consider the function $y = f(x) = \sin x$ on the domain $[-\pi/2, \pi/2]$. Fig. 3(a) shows the graph of this function. Its inverse is given by $g(y) = \sin^{-1}(y) = x$. We can see clearly that sin x is strictly increasing on $[-\pi/2, \pi2]$.




(a) (b)

Fig. 3

We also know that the derivative $\frac{d}{dx}$ (sin x) = cos x exists and is non-zero for all $x \in ]-\pi/2, \pi/2[$.

This means that sin x satisfies the conditions of the inverse function theorem. We can, therefore, conclude that $\sin^{-1} y$ is differentiable on. ]–1, 1[, and

Since sin x = y, cos x = $\sqrt{1-y^2}$ for $-\pi/2 < x < \pi/2$.

$$\frac{d}{dy}(\sin^{-1}(y)) = \frac{1}{f'(x)} = \frac{1}{\cos x}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

Thus, we have the result

$$\frac{d}{dt}(\sin^{-1} t) = \frac{1}{\sqrt{1-t^2}}$$

Remember, $\sin^{-1}x$ is not the same as $(\sin x)^{-1} = 1/\sin x$ or $\sin x^{-1} = \sin 1/x$.

Fig. 3(b) shows the graph of $\sin^{-1}$ x.

We shall follow exactly the same steps to find out the derivative of the inverse cosine function.

Let's start with the funciton y = f(x) = cos x, and restrict its domain to [0, π] and its co-domain to [–1, 1]. Its inverse function g(y) = $\cos^{-1}$ y exists and the graphs of cos x and $\cos^{-1}$ x are shown in Fig. 4 (a) and 4(b), respectively.

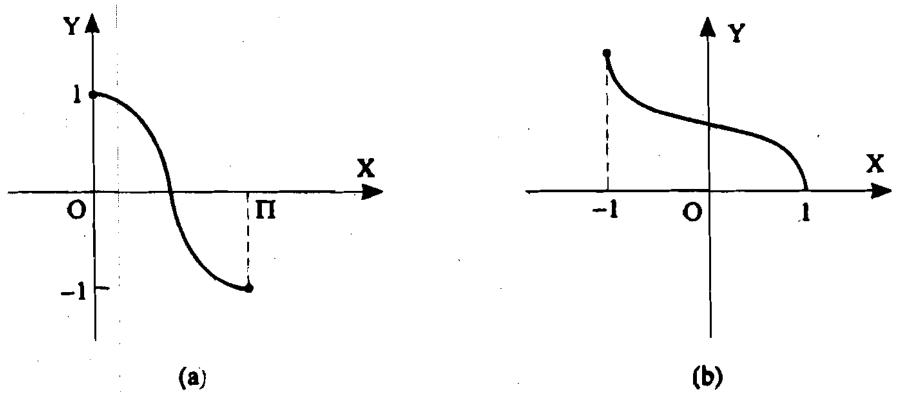



Fig. 4

As in the earlier case, we can now check that the conditions of the inverse function theorem are satisfied and conclude that $\cos^{-1}$ y is differentiable in ]–1, 1[. Further

$$\frac{d}{dy}(g(y)) = \frac{d}{dy}(\cos^{-1} y) \frac{1}{f'(x)} = \frac{1}{-\sin x}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

Since cos x = y, sin x = $\sqrt{1-y^2}$ for $0 < x < \pi$.

This gives us the result

$$\frac{d}{dt}(\cos^{-1} t) = \frac{-1}{\sqrt{1-t^2}}$$

You can apply these two results to get the derivatives in the following exercises.

E 6) Differentiate

a) $\sin^{-1}(5x)$

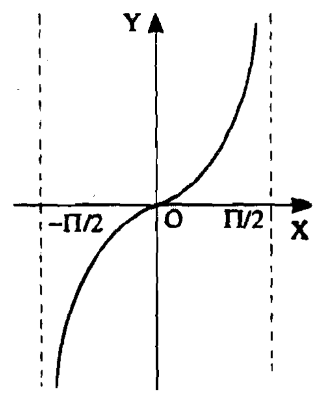
b) $\cos^{-1}\sqrt{x}$

c) $\sin x \cos^{-1}(x^3 + 2)$

E 7) a) By looking at the graph of tan x given alongside, indicate the interval to which the domain of tan x should be restricted so that the existence of its inverse is guaranteed.

b) What will be the domain for $\tan^{-1} x$?

c) Prove that $\frac{d}{dx}(\tan^{-1} x) = 1/(1 + x^2)$ in its domain.




In this section we have calculated the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ and if you have done E 7), you will have calculated the derivative of $\tan^{-1} x$ also. Proceeding along exactly similar lines, we shall be able to see that

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}$$

### 4.4.2 Derivatives of $Sec^{-1}$ x and $Cosec^{-1}$ x

Let's tackle the inverses of the remaining two trigonometric functions now.

To find $\sec^{-1} x$, we proceed as follows :

If $y = \sec^{-1} x$, then $x = \sec y$ or $1/\cos y = x$, which means that $1/x = \cos y$. This gives us $y = \cos^{-1}(1/x)$, where, $|x| \geq 1$.

Remember, we have seen that $\cos^{-1} t$ is defined in the interval $[-1, 1]$.

Thus, $y = \sec^{-1} x = \cos^{-1}(1/x)$, $|x| \geq 1$

From this we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1}(1/x))$$

$$= \frac{-1}{\sqrt{1-1/x^2}} \cdot \frac{d}{dx}(1/x)$$

$$= \frac{-|x|}{\sqrt{x^2-1}}(-1/x^2)$$

$$= \frac{1}{|x|\sqrt{x^2-1}}, \; |x| > 1$$

Note that although $\sec^{-1} x$ is defined for $|x| \geq 1$, the derivative of $\sec^{-1} x$ does not exist when $x = 1$.

Thus, we have

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \; |x| > 1.$$

**E** E 8) Following exactly similar steps, show that

$$\frac{d}{dx}(\mathrm{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, \; |x| > 1.$$

**Example 4** Suppose we want to find the derivative of $y = \sec^{-1} 2\sqrt{x}$.
By chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sec^{-1} 2\sqrt{x})$$

$$= \frac{1}{2\sqrt{x}\sqrt{4x-1}} \frac{d}{dx}(2\sqrt{x})$$

$$= \frac{1}{2\sqrt{x}\sqrt{4x-1}} \times \frac{1}{\sqrt{x}}$$

$$= \frac{1}{2x\sqrt{4x-1}}$$

Now, you will be able to solve these exercises using the results about the derivatives of inverse trigonometric functions.

**E** E 9) Differentiate,

a) $\cot^{-1}(x/2)$

b) $\dfrac{\cot^{-1}(x+1)}{\tan^{-1}(x+1)}$

c) $\cos^{-1}(5x+4)$

d) $\sec^{-1}\left(\dfrac{x \sin\theta}{1 - x\cos\theta}\right)$

e) $\mathrm{cosec}^{-1}(x+1) + \sec^{-1}(x-1)$

## 4.5 USE OF TRANSFORMATIONS

Sometimes the process of finding derivatives is simplified to a large extent by making use of some suitable transformations. In this section we shall see some examples which will illustrate this fact.

**Example 5** Suppose we want to find the derivative of

$y = \cos^{-1} (4x^3 - 3x)$

As you know, we can differentiate this function by using the formula for the derivative of $\cos^{-1} x$ and the chain rule. But suppose we put $x = \cos \theta$, then we get

$$\begin{aligned} y &= \cos^{-1} (4 \cos^3 \theta - 3 \cos \theta) \\ &= \cos^{-1} (\cos 3\theta) \qquad (\cos 3\theta = 4 \cos^3\theta - 3 \cos \theta) \\ &= 3\theta \\ &= 3 \cos^{-1} x. \end{aligned}$$

Now this is a much simpler expression, and can be differentiated easily as:

$$\frac{dy}{dx} = \frac{-3}{\sqrt{1 - x^2}}$$

**Example 6** To differentiate $y = \tan^{-1} \left( \frac{\sqrt{1 + x^2} - 1}{x} \right)$, we use the transformation $x = \tan \theta$.

This gives us,

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left[\frac{1-(1-2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right]$$

$$= \tan^{-1}(\tan\theta/2)$$

$$= \theta/2 = \frac{\tan^{-1}x}{2}.$$

Now, we can write $\dfrac{dy}{dx} = \dfrac{1}{2(1+x^2)}$

Let's tackle another problem.

**Example 7** Suppose we want to differentiate $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

For this, let $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ and $z = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$. Our aim is to find dy/dz. We shall use the transformation $x = \tan\theta$. This gives us

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta, \text{ and}$$

$$z = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta.$$

Now if we differentiate y and z with respect to $\theta$, we get $dy/d\theta = 2$ and $dz/d\theta = 2$.

Therefore, $\dfrac{dy}{dz} = \dfrac{dy/d\theta}{dz/d\theta} = 1.$

Alternatively, we have $y = z$. Hence, $dy/dz = 1$.

So, you see, a variety of complex problems can be solved easily by using transformations. The key to a successful solution is, however, the choice of a suitable transformation. We are giving some exercises below, which will give you the necessary practice in choosing the right transformation.

**E** E 10) Find the derivatives of the following functions using suitable transformation :

a) $\sin^{-1}(3x - 4x^3)$

b) $\cos^{-1}(1 - 2x^2)$

c) $\sin^{-1}\left(\dfrac{2x}{1+x}\right)$

d) $\tan^{-1}\left(\dfrac{3x - x^3}{1+3x^2}\right)$

e) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.

Now let us summarise the points covered in this unit.

## 4.6 SUMMARY

In this unit we have

1. calculated the derivatives of trigonometric functions:

| Function | Derivative |
|---|---|
| sin x | cos x |
| cos x | – sin x |
| tan x | $\sec^2 x$ |
| cot x | $-\operatorname{cosec}^2 x$ |
| sec x | sec x tan x |
| cosec x | –cosec x cot x |

2. discussed the inverse function theorem and used the rule

   $\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(y)}$ to prove that $\frac{d(x^r)}{dx} = rx^{r-1}$, where r is a rational number.

3. used the inverse function theorem to find the derivatives of inverse trigonometric functions:

| Function | Derivative |
|---|---|
| $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$ |
| $\cos^{-1} x$ | $\frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$ |
| $\tan^{-1} x$ | $\frac{1}{1+x^2}, x \in \mathbf{R}$ |
| $\cot^{-1} x$ | $\frac{-1}{1+x^2}, x \in \mathbf{R}$ |
| $\sec^{-1} x$ | $\frac{1}{\lvert x\rvert\sqrt{x^2-1}}, \lvert x \rvert > 1$ |
| $\operatorname{cosec}^{-1} x$ | $\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}, \lvert x \rvert > 1$ |

4. used transformations to simplify the problems of finding the derivatives of some functions.

## 4.7 SOLUTIONS AND ANSWERS

E 1) a) $\cos(a + x) = \cos a \cos x - \sin a \sin x$

$\lim_{x\to 0} \cos(a + x) = \cos a \lim_{x\to 0} \cos x - \sin a \lim_{x\to 0} \sin x$

b) similar $= \cos a$

E 2) a) $2 \cos 2x$ b) $2\cos x \frac{d}{dx}(\cos x) = -2 \sin x \cos x$

c) $5(3 \sin^7 x \cos 3x + 7 \sin^6 x \cos x \sin 3x)$
$= 5 \sin^6 x (3 \sin x \cos 3x + 7\cos x \sin 3x)$

d) $3x^2 \cos 9x - 9x^3 \sin 9x$

e) $-\sin(\sin x) \cos x$

E 3) $$\frac{d}{dx}(\text{cosec } x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\sin x \times 0 - 1 \cos x}{\sin^2 x}$$

$$= \frac{\cos x}{\sin^2 x} = -\text{cosec } x \cot x.$$

E 4) a) $-2 \text{ cosec } 2x \cot 2x$

b) $-\text{cosec}^2 x + \frac{1}{\sqrt{2 \text{cosec } x}}(-\text{cosec } x \cot x)$

c) $-45 \text{ cosec}^2 9x.$

E 5) a) $5(3x^2 + x^{-2/3})$

b) $2(\sqrt[5]{x} - \sqrt[9]{x})x + x^2(\frac{1}{5}x^{-4/5} - \frac{1}{9}x^{-8/9})$

E 6) a) $\frac{5}{\sqrt{1 - 25x^2}}$ b) $\frac{-1}{2\sqrt{x}\sqrt{1-x}}$

c) $\sin x \frac{-3x^2}{\sqrt{1-(x^3+2)^2}} + \cos x \cos^{-1}(x^3 + 2)$

E 7) a) $\tan x$ restricted to $]-\pi/2, \pi/2[$ is a strictly increasing one-one function of x. Thus, its inverse exists when restricted to $]-\pi/2, \pi/2[$.

b) The domain of $\tan^{-1} x$ is $]-\infty, \infty[$.

c) If $y = f(x) = \tan x$,

$$\frac{d}{dy}(\tan^{-1} y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{1 + y^2}$$

Hence $d/dx (\tan^{-1} x) = \frac{1}{1 + x^2}$.

E 8) $y = \text{cosec}^{-1} x \Rightarrow \text{cosec } y = x = \sin y = 1/x \Rightarrow$
$y = \sin^{-1}(1/x)$ where $|x| \geq 1$.

Thus, $$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}(1/x))$$

$$= \frac{1}{\sqrt{1 - 1/x^2}}(-1/x^2)$$

$$= \frac{|x|}{\sqrt{x^2 - 1}} (-1/x^2)$$

$$= \frac{-1}{|x|\sqrt{x^2 - 1}}, |x| > 1$$

E 9) a) $\dfrac{-1}{2(1 + x^2/4)}$

b) $\dfrac{-\tan^{-1}(x+1)\left(\dfrac{1}{1+(x+1)^2}\right) - \cot^{-1}(x+1)\left(\dfrac{1}{1+(x+1)^2}\right)}{(\tan^{-1}(x+1))^2}$

c) $\dfrac{5}{1 - (5x+4)^2}$

d) $\dfrac{1}{\left|\dfrac{x\sin\theta}{1 - x\cos\theta}\right|\sqrt{\dfrac{x^2\sin^2\theta}{(1 - x\cos\theta)} - 1}} \; \dfrac{\sin\theta\,(1 - x\cos\theta) + x\sin\theta\cos\theta}{(1 - x\cos\theta)^2}$

e) $\dfrac{-1}{|x+1|\sqrt{(x+1)^2 - 1}} + \dfrac{1}{|x-1|\sqrt{(x-1)^2 - 1}}$

E 10) a) Put $x = \sin\theta \Rightarrow y = \sin^{-1}(3x - 4x^3)$

$= \sin^{-1}(3\sin\theta - 4\sin^3\theta)$
$= \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1} x.$

$$= \frac{dy}{dx} = \frac{3}{\sqrt{1 - x^2}}$$

b) $x = \cos\theta/2 \Rightarrow y = \cos^{-1}(1 - 2x^2)$
$= \cos^{-1}(1 - 2\cos^2\theta/2)$
$\Rightarrow y = \cos^{-1}(-\cos\theta) = \cos^{-1}(\cos(\pi - \theta))$
$= \pi - \theta = \pi - 2\cos^{-1} x.$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1 - x^2}}$$

c) Put $x = \tan\theta \Rightarrow y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = 2\tan^{-1} x$

$$\frac{dy}{dx} = \frac{2}{1 + x^2}$$

d) Put $x = \tan\theta$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{dx - x^3}{1 - 3x^2}\right)\right) = \frac{3}{1+x^2}$$

e) Put $x = \tan\theta$

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)\right) = \frac{2}{1 + x^2}$$