UNIT 1 REAL NUMBERS AND FUNCTIONS

Structure

1.1 INTRODUCTION

This is the first unit of the course on Calculus. We thought it would be a good idea to acquaint you with some basic results about the real number system and functions, before you actually start your study of Calculus. Perhaps, you are already familiar with these results. But, a quick look through the pages will help you in refreshing your memory, and you will be ready to tackle the course.

In the next three sections of this unit, we shall present some results about the real number system. You will find a number of examples of various types of functions in Sections 1.5 to 1.7. You should also study the graphs of these functions carefully, in order to be able to visualise given functions. In fact, try to draw a graph whenever you encounter a new function. We shall systematically study the tracing of curves in Block 2 Unit 4.

Objectives

After reading this unit you should be able to :

- recall the basic properties of real numbers
- derive other properties with the help of the basic ones
- identify various types of bounded and unbounded intervals
- define a function and examine whether a given function is one-one/onto
- investigate whether a given function has an inverse or not
- define the scalar multiple, absolute value, sum, difference, product, quotient of the given functions and
- determine whether a given function is even odd, monotonic or periodic.

1.2 BASIC PROPERTIES OF R

In the next three sections, we are going to tell you about the set **R** of real numbers, which is all-pervading in mathematics. The real number system is the foundation on which a large part of mathematics, including calculus, rests. Thus, before we actually start learning calculus, it is necessary to understand the structure of the real number system.

You are already familiar with the operations of addition, subtraction, multiplication and division of real numbers, and also with inequalities. Here we shall quickly recall some of their properties. We start with the operation of addition:

A1 R is closed under addition.
If x and y are real numbers, then $x + y$ is a unique real number.

A2 Addition is associative.
$x + (y + z) = (x + y) + z$ holds for all x, y, z in **R**.

A3 Zero exists.
There is a real number 0 such that
$x + 0 = 0 + x = x$ for all x in **R**.

A4 Negatives exist.
For each real number x, there exists a real number y (called a negative or an additive inverse of x, and denoted by $-x$) such that $x + y = y + x = 0$.

A5 Addition is commutative.
$x + y = y + x$ holds for all x, y in **R**.

Similar to these properties of addition, we can also list some properties of the operation of multiplication:

M1 R is closed under multiplication.
If x and y are real numbers, then x.y is a unique real number.

M2 Multiplication is associative.
$x.(y.z) = (x.y).z$ holds for all x, y, z in **R**.

M3 Unit element exists.
There exists a real number 1 such that
$x.1 = 1.x = x$ for every x in **R**.

M4 Inverses exist.
For each real number x other than 0, there exists a real number y (called a multiplicative inverse of x and denoted by x^{-1}, or by $1/x$) such that
$x.y = y.x = 1$

M5 Multiplication is commutative.
$x.y = y.x$ holds for all x, y in **R**.
The next property involves addition as well as multiplication.

D Multiplication is distributive over addition.
$x.(y + z) = x.y + x.z$ holds for all, x, y, z in **R**

You may have come across a "field", in the course on Linear Algebra.

Remark 1: The fact that the above eleven properties are satisfied is often expressed by saying that the real numbers form a *field* with respect to the usual addition and multiplication operations.

Remark 1 (a): Usually the operator '.' is dropped in expressions, e.g., x.y may be denoted as xy.

In addition to the above mentioned properties, we can also define an order relation on **R** with the help of which we can compare any two real numbers. We write $x > y$ to mean that x is greater than y. The order relation '>' has the following properties:

01 Law of Trichotomy holds.
For any two real numbers a, b, one and only one of the following holds:
$a > b$, $a = b$, $b > a$.

02 '>' is transitive.
If $a > b$ and $b > c$, then $a > c$, $\forall$ a, b, c $\in$ **R**

03 Addition is monotone.
If a, b, c in **R** are such that $a > b$, then $a + c > b + c$.

04 Multiplication is monotone in the following sense.
If a, b, c in **R** are such that $a > b$ and $c > 0$, then $ac > bc$.
Caution: $a > b$ and $c < 0 \Rightarrow ac < bc$.

Remark 2: Any field together with a relation > satisfying 01 to 04 is called an **ordered field**. Thus **R** with the usual > is an example of an ordered field.

Notations: We write $x < y$ (and read x is less than y) to mean $y > x$. We write $x \leq y$ (and read x is less than or equal to y) to mean either $x < y$ or $x = y$. We write $x \geq y$ (and read x is greater than or equal to y) if either $x > y$ or $x = y$.

A number x is said to be positive or negative according as $x > 0$ or $x < 0$. If $x \geq 0$, we say that x is non-negative.

Now, you know that given any number $x \in \mathbf{R}$, we can always find a number $y \in \mathbf{R}$ such that $y \geq x$. (In fact, there are infinitely many such real numbers y). Let us see what happens when we take any sub-set of **R** instead of a single real number x. Do you think that, given a set $S \subseteq \mathbf{R}$, it is possible to find $u \in \mathbf{R}$ such that $u \geq x$ for **all** $x \in S$? Discuss the special case when S is empty.

Before we try to answer this question, let us look at a definition.

Definition 1: Let S be a subset of **R**. An element u in **R** is said to be an **upper bound** of S if $u \geq x$ holds for every x in S. We say that **S is bounded above,** if there is an upper bound of S.

Now we can reword our earlier questions as follows : Is it possible to find an upper bound for a given set ?

Let us consider the set $\mathbf{Z}^- = \{-1, -2, -3, -4, ...\}$

Now, each $x \in \mathbf{Z}^-$ is negative. Or, in other words, $x < 0$ for all $x \in \mathbf{Z}$. It is easily seen that, in this case, we are able to find an upper bound, namely zero, for our set $\mathbf{Z}^-$

On the other hand, if we consider the set of natural numbers, $\mathbf{N} = \{1, 2, 3 ....\}$, obviously we will not be able to find an upper bound. Thus **N** is not bounded above.

You will, of course, realise that if u is an upper bound **for a set S then u + 1, u + 2, u + 3,** ....., (in fact, u + r, where r is any positive number) are all upper bounds of S. For example, we have seen that 0 is an upper bound for $\mathbf{Z}^-$. Check that 1, 2, 3, 4, 8, ... are all upper bounds of $\mathbf{Z}^-$.

From among all the upper bounds of a set S, which is bounded above, we can choose an upper bound u such that u is less than or equal to every upper bound of S. It is easily seen that, if such a u exists, then it is unique. We call this u the **least upper bound** or **the supremum of** S. For example, consider the set

$T = \{x \in \mathbf{R} : x^2 \leq 4\} = \{x \in \mathbf{R} : -2 \leq x \leq 2\}$

Now 2, 3, 3.5, 4, $4 + \pi$ are all upper bounds for this set.

But you will see that 2 is less than any other upper bound.

Hence 2 is the supremum or the least upper bound of T.

You will agree that -1 is the l.u.b. (least upper bound) of $\mathbf{Z}^-$.

Note that for both the sets **T** and $\mathbf{Z}^-$, the l.u.b. belongs to the set. This may not be true in general. Consider the set of all negative real numbers $\mathbf{R}^- = \{x : x < 0\}$. The l.u.b. of this set is 0. But $0 \notin \mathbf{R}^-$.

Working on similar lines we can also define a **lower bound** for a given set S to be a real number v such that $v \leq x$ for all $x \in S$. We shall say that a set is bounded below, if we can find a lower bound for it. Further, from among all the lower bounds of a set S, which is bounded below, we can choose a lower bound v such that v is greater than or equal to every lower bound of S. It is easily seen that, if such a v exists, then it is unique. We call this v the **greatest lower bound** or the **infimum** of S.

As in the case of l.u.b., remember that the g.l.b. of a set may or may not belong to the set.

We shall say that a set $S \subset R$ is bounded if it has both an upper bound and a lower bound.

Based on this discussion you will be able to solve the following exercise.

E E1) Give examples to illustrate the following :

a) A set of real numbers having a lower bound,
b) A set of real numbers without any lower bound,
c) A set of real numbers whose g.l.b. does not belong to it,

d) A bounded set of real numbers.

Now we are ready to state an important property of **R**.

C The order is complete

Every non-empty subset S of **R** that is bounded above, has a supremum. (We shall use this property in Unit 10).

Many more properties are either restatements or consequences of these sixteen properties. Here is a list of some of them.

1 Zero is unique, i.e.,
If $x + 0' = x$ for all x in **R**, then $0' = 0$.

2 Additive inverse is unique, i.e.,
For each x in **R**, there is a unique y in **R** such that $x + y = y + x = 0$.

3 Addition is cancellative, i.e.,
If $x + y = x + z$, then $y = z$.

4 Unity is unique, i.e.,
If $x.1' = x$ for all x in **R**, then $1' = 1$.

5 Multiplicative inverse is unique, i.e.,
For each non-zero real number x, there is a unique y in **R** such that $xy = yx = 1$.

6 Multiplication is cancellative, i.e,
If $xy = xz$ and $x \neq 0$, then $y = z$.

Definition 2: If x and y are any two real numbers, the result of subtraction of y from x is denoted by $x - y$ and is defined as $x + (-y)$. Similarly, the division $x \div y$ (also denoted by x/y) is defined as xy^{-1}, provided $y \neq 0$.

Now we are ready to list a few more properties. You are already aware of these. But let us quickly recall them.

7 $-(x + y) = (-x) + (-y)$ for all x, y in **R**.

8 If $xy = 0$, then either $x = 0$ or $y = 0$.

9 $(x^{-1})^{-1} = x$ for all $x \neq 0$ in **R**.

10 If x and y are non zero numbers such that $x^{-1} = y^{-1}$, then $x = y$.

11 If $a < b$ and $c > 0$, then $ac < bc$.

12 a is positive if and only if $-a$ is negative.

13 If $a < b$ and $c < d$, then $a + c < b + d$.

14 If $a > b$ and $c < 0$, then $ac < bc$.

15 a^2 is non-negative for all a in **R**.

16 If a and b are positive, then

i) $a^2 = b^2 \Leftrightarrow a = b$.
(The symbol $\Leftrightarrow$ is read as 'if and only if')

ii) $a^2 > b^2 \Leftrightarrow a > b$

iii) $a^2 < b^2 \Leftrightarrow a < b$

17 If $b > 0$, then $a^2 < b^2 \Leftrightarrow -b < a < b$.

You are also familiar with the following subsets of **R**:

1) The set N of natural numbers. Note that it is the smallest subset of **R** possessing the following properties:

i) $1 \in N$

ii) $k \in \mathbf{N} \Rightarrow k+1 \in \mathbf{N}$

2) The set **Z** of integers. It is the smallest subset of **R** possessing the following properties:

i) $\mathbf{Z} \supset \mathbf{N}$

ii) If $x, y \in \mathbf{Z}$, then $x - y \in \mathbf{Z}$.

3) The set **Q** of rational numbers. We observe that it is the smallest subset of **R** possessing the following properties:

i) $\mathbf{Q} \supset \mathbf{Z}$

ii) If $x, y \in \mathbf{Q}$ and $y \neq 0$, then $xy^{-1} \in \mathbf{Q}$.

You must have also studied the following properties of these sets.

1) $k \in \mathbf{N}$ if and only if k is a positive integer, that is, $k \in \mathbf{Z}$ and $k > 0$.

2) The operations of addition and multiplication on **N** satisfy A1, A2, A5, M1, M2, M3, M5 and D. They do not, however, satisfy A3, A4 and M4.

3) The operations on **Q** satisfy A1 to A5, M1 to M5 and 01 to 04. Therefore **Q** is an ordered field. But C is not satisfied, that is, **Q** is not order-complete.

We list here some more properties of these sets which you will find useful in our study of calculus:

4) **Archimedean Property:** If a and b are any real numbers and if $b > 0$, then there is a positive integer n such that $nb > a$.

5) If a is any real number, there is a positive integer n such that $n > a$ (Archimedean property applied to a and 1).

6) A real number s is the supremum of a set $S \subset \mathbf{R}$ if and only if the following conditions are satisfied.

i) $s \geq x$ for all x in S.

ii) For each $\varepsilon > 0$, there is a y in S such that $y > s - \varepsilon$.

ε (epsilon) is a Greek letter used to denote small real numbers.

For example, consider the set $A = \{x \in \mathbf{R} : 8 \leq x < 10\}$. 10 is the supremum of this set. Now, if we are given any ε, say, $\varepsilon = 0.01$, we should be able to find some $y \in A$ such that $y > 10 - 0.01 = 9.99$. As you can see, $y = 9.999$ serves our purpose.

Now 10.01 is also an upper bound for A. But 10.01 is not the supremum of A. For $\epsilon = 0.001$,
we cannot find any $y \in A$ such that $y > 10.01 - 0.001 = 10.009$.

7) Every nonempty set of real numbers that is bounded below, has an infimum.

The exercise below can now be done easily.

E E2) a) Show that the set of positive real numbers is bounded below. What is its infimum?

b) Write the characterisation of the infimum of a subset of **R**, which corresponds to 6) above. Give an example.

1.3 ABSOLUTE VALUE

In this section we shall define the absolute value of a real number. You will realise the importance of this simple concept as you study the later units.

Definition 3 : If x is a real number, its **absolute value**, denoted by $|x|$ (read as **modulus of** x, or **mod** x), is defined by the following rules:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

For example, we get

$|5| = 5, |-5| = 5,$

$|1.7| = 1.7, |-2| = 2, |0| = 0$

It is obvious that $|x|$ is defined for all $x \in \mathbf{R}$. The following theorem gives some of the important properties of $|x|$.

Theorem 1: If x and y be any real numbers, then

a) $|x| = \max\{-x, x\}$

b) $|x| = |-x|$

c) $|x|^2 = x^2 = |-x|^2$

d) $|x + y| \leq |x| + |y|$ **(the triangle inequality)**

e) $|x - y| \geq \big||x| - |y|\big|$

Proof:

a) By the law of trichotomy (01) applied to the real numbers x and 0, exactly one of the following holds:
i) $x > 0$, ii) $x = 0$, or iii) $x < 0$.

Let us consider these one by one.

i) If $x > 0$, then $|x| = x$ and $x > -x$, so that $\max\{-x, x\} = x$ and hence $|x| = \max\{-x, x\}$

ii) If $x = 0$, then $x = 0 = -x$, and therefore, $\max\{-x, x\} = 0$. Also $|x| = 0$, so that $|x| = \max\{-x, x\}$.

iii) If $x < 0$, then $|x| = -x$, and $-x > x$, so that $\max\{-x, x\} = -x$. Thus, again, $|x| = \max\{-x, x\}$.

From this it follows that $x \leq |x|$

b) $|-x| = \max\{-(-x), -x\} = \max\{x, -x\} = \max\{-x, x\} = |x|$.

c) If $x \geq 0$, then $|x| = x$, so that $|x|^2 = x^2$.

If $x < 0$, then $|x| = -x$, so that $|x|^2 = (-x)^2 = x^2$.

Therefore, for all $x \in \mathbf{R}$, $|x|^2 = x^2$.

Also $|-x|^2 = |x|^2$, because $|-x| = |x|$ by (b). Thus, we have $|x|^2 = x^2$

d) We shall consider two different cases according as

i) $x + y \geq 0$ or ii) $x + y < 0$.

Let $x + y \geq 0$. Then $|x + y| = x + y$. Now $x \leq |x|$ and $y \leq |y|$ by (a). Therefore.

$|x + y| = x + y \leq |x| + |y|$.

Let $x + y < 0$. Then $-(x + y) > 0$, that is,

$(-x) + (-y) > 0$ and we can use the result of (i)

for $-x$ and $-y$. Now $|x + y| = |-(x + y)|$ by (b).

Thus $|x + y| = |(-x) + (-y)| \leq |-x| + |-y|$, by (i).

$= |x| + |y|$, by (b).

Therefore, we get $|x + y| \leq |x| + |y|$.

Thus we find that for all $x, y \in \mathbf{R}$, $|x + y| \leq |x| + |y|$.

e) By writing $x = (x - y) + y$ and applying the triangle inequality to the numbers $x - y$ and y, we have

$|x| = |(x - y) + y| \leq |x - y| + |y|$,

so that $|x| - |y| \leq |x - y|$. ...(1)

Since (1) holds for all x and y in **R**,

therefore, by interchanging x and y in (1) we have

$|y| - |x| \le |y-x| = |-(x-y)| = |x-y|$.

So that $-(|x| - |y|) \le |x-y|$. ...(2)

From (1) and (2) we find that $|x| - |y|$ and its negative $-(|x| - |y|)$ are both less than or at the most equal to $|x-y|$. Therefore, max $\{|x|-|y|, -(|x|-|y|)\} \le |x-y|$.

But the left hand side of the above inequality is simply $\big||x|-|y|\big|$. Therefore, we have

$\big||x| - |y|\big| \le |x-y|$

That is, $|x-y| \ge \big||x|-|y|\big|$ for all $x, y \in \mathbf{R}$.

Now you should be able to prove some easy consequences of this theorem. The following exercise will also give you some practice in manipulating absolute values. This practice will come in handy when you study Unit 2.

E E3) Prove the following :

a) $x = 0 \Leftrightarrow |x| = c$

b) $|xy| = |x| . |y|$

c) $|1/x| = 1/|x|$, if $x \neq 0$

d) $|x-y| \le |x| + |y|$

e) $|x+y+z\} \le |x| + |y| + |z|$

f) $|xyz| = |x| . |y| . |z|$

e) and f) can be extended to any number of reals. Now if $a \in \mathbf{R}$ and $\delta > 0$, then $|x-a| < \delta \Rightarrow x - a < \delta$, and $-(x-a) < \delta$.

$x - a < \delta$, this means that $x < a + \delta$

$-(x-a) < \delta$, this means that $a - \delta < x$.

Thus, we get that $|x-a| < \delta \Rightarrow a - \delta < x < a + \delta$.

This means that the difference between x and a is not more than δ.

In the next section, we shall see how the set $\{x : |x-a| < \delta\}$ can be represented geometrically.

1.4 INTERVALS ON THE REAL LINE

Before we define an interval let us see what is meant by a number line. The real numbers in the set **R** can be put into one-to-one correspondence with the points on a straight line L. In other words, we shall associate a unique point on L to each real number and vice versa.

Consider a straight line L [see Fig. 1 (a)]. Mark a point O on it. The point O divides the straight line into two parts. We shall use the part to the left of O for representing negative real numbers and the part to the right of O for representing positive real numbers. We choose a point A on L which is to the right of O. We shall represent the number 0 by O and 1 by A. OA can now serve as a unit. To each positive real number x we can associate exactly one point P lying to the right of O on L, so that OP = | x | units (= x units). A negative real number y will be represented by a point Q lying to the left of O on the straight line L, so that OQ = | y | = – y units ($\because$ y is negative). We thus find that to each real number we can associate a point on the line. Also, each point S on the line reprsents a unique real number z, such that | z | = OS. Further, z is positive if S is to the right of O, and z is negative if S is to the left of O.

Distance is always non-negative.

This representation of real numbers by points on a straight line is often very useful. Because of this one-to-one correspondence between real numbers and the points of a straight line, we often call a real number "a point of R". Similarly L is called a "number line". Note that the absolute value or the modulus of any number x is nothing but its distance from the point O on the number line. In the same way, | x – y | denotes the distance between the two numbers x and y [see Fig. 1 (b)].

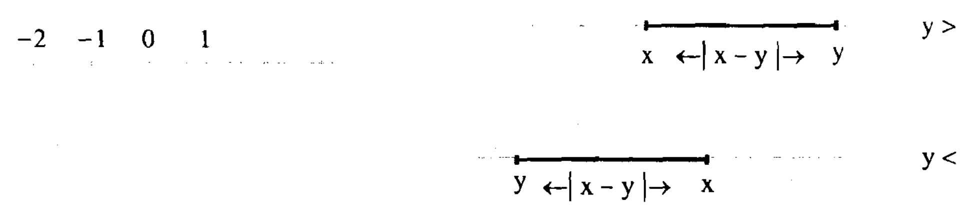


Figure 1 : (a) Number line **b) Distance between x and y is | x – y |**

Now let us consider the set of the real numbers which lie between two given real numbers a and b, where a ≤ b. Actually, there will be four different sets satisfying this loose condition.

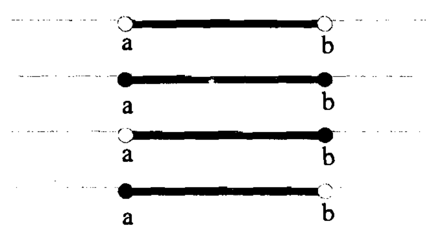
These are :

i) $]a, b[= \{ x : a < x < b \}$

ii) $[a, b] = \{ x : a \leq x \leq b\}$

iii) $]a, b] = \{ x : a < x \leq b\}$

iv) $[a, b[= \{ x : a \leq x < b\}$



The representation of each of these sets is given alongside. Each of these sets is called an **interval**, and a and b are called the end points of the interval. The interval]a, b[, in which the end points are not included, is called an **open interval**. Note that in this case we have drawn a hollow circle around a and b to indicate that they are not included in the graph. The set [a, b] contains both its end points and is called a **closed interval**. In the representation of this closed interval, we have put thick black dots at a and b to indicate that they are included in the set.

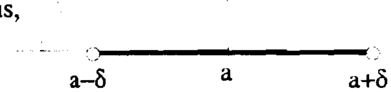
The sets [a, b[and]a, b] are called **half-open (or half-closed) intervals** or **semi-open (or semi-closed) intervals**, as they contain only one end point. This fact is also indicated in their geometrical representation.

If a = b,]a, a[=]a, a] = [a, a[= ϕ and [a, a] = a.

Each of these intervals is bounded above by b and bounded below by a.

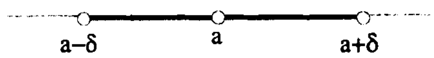
Can we represent the set $I = \{x : | x - a | < \delta \}$ on the number line? Yes, we can. We know that | x – a | can be thought of as the distance between x and a. This means I is the set of all numbers x, whose distance from a is less than δ. Thus,

$I = \{x : | x - a | < \delta\}$



is the open interval $]a-\delta, a+\delta[$. Similarly, $I_1 = \{x : |x-a| \le \delta\}$ is the closed interval $[a-\delta, a+\delta]$. Sometimes we also come across sets like $I_2 = \{x : 0 < |x-a| < \delta\}$. This means if $x \in I_2$, then the distance between x and a is less than δ, but is not zero. We can also say that the distance between x and a is less than δ, but $x \neq a$. Thus,

$$I_2 =]a-\delta, a+\delta[\setminus \{a\}$$
$$=]a-\delta, a[\cup]a, a+\delta[.$$



Apart from the four types of intervals listed above, there are a few more types. These are:

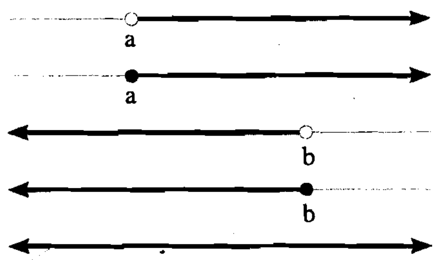
$]a, \infty[= \{x : a < x\}$ (open right ray)

$[a, \infty[= \{x : a \le x\}$ (closed right ray)

$]-\infty, b[= \{x : x < b\}$ (open left ray)

$]-\infty, b] = \{x : x \le b\}$ (closed left ray)

$]-\infty, \infty[= \mathbf{R}$ (open interval)



As you can see easily, none of these sets are bounded. For instance, $]a, \infty[$ is bounded below, but is not bounded above, $]-\infty, b]$ is bounded above, but is not bounded below. Note that ∞ and $-\infty$ does not denote a real number, it merely indicates that an interval extends without limits.

We note further that if S is any interval (bounded or unbounded) and if c and d are two elements of S, then all numbers lying between c and d are also elements of S.

E E4) State whether the following are true or false.

a) $0 \in [1, 8]$, b) $-1 \in]-\infty, 2[$

c) $1 \in [1, 2]$ d) $5 \in]5, \infty[$

E E5) Represent the intervals in E 4) geometrically.

1.5 FUNCTIONS

Now let us move over to present some basic facts about functions which will help you refresh your knowledge. We shall look at various examples of functions and shall also define inverse functions. Let us start with the **definition of a function.**

1.5.1 Definition and Examples

Definition 4: If X and Y are two sets, a **function** f from X to Y, is a rule or a correspondence which connects every member of X to a **unique** member of Y. We write f: $X \to Y$ (read as "f is a function from X to Y) X is called the **domain** and Y is called the **co-domain** of f. We shall denote by f(x) that unique element of Y which is associated to $x \in X$.

The following examples will help you in understanding this definition better.

Example 1: $f : \mathbf{N} \to \mathbf{R}$, defined by $f(x) = -x$. is a function since the rule $f(x) = -x$ associates a unique member $(-x)$ of **R** to every member x of **N**. The domain here is **N** and the co-domain is **R**.

Example 2: The rule $f(x) = x/2$ does not define a function from $\mathbf{N} \to \mathbf{Z}$ as odd natural numbers like 1, 3, 5 ...... from **N** cannot be connected to any member of **Z**.

Example 3: Every natural number can be written as a product of some prime numbers. Consider the rule f(x) = a prime factor of x, which connects elements of **N**. Here since $6 = 2 \times 3$, f(6) has two values : f(6) = 2 and f(6) = 3. This rule does not associate a unique number with 6 and hence does not give a function from **N** to **N**.

Thus, you see, to describe a function completely we have to specify the following three things:

a) the domain

b) the co-domain, and

c) the rule which associates a unique member of the co-domain to each member of the domain.

The rule which defines a function need not always be in the form of a formula. But it should clearly specify (perhaps by actual listing) the correspondence between X and Y.

If $f: X \to Y$, then $y = f(x)$ is called the **image of x under f** or the **f-image** of x. The set of f-images of all members of X, i.e., $\{f(x) : x \in X\}$ is called the **range** of f and is denoted by f(X). It is easy to see that $f(X) \subseteq Y$.

Remark 3 a) Throughout this course we shall consider functions for each of which whose domain and co-domain are both subsets of **R**. Such functions are often called **real functions** or real-valued functions of a real variable. We shall, however, simply use the word 'function' to mean a real function.

b) The variable x used in describing a function is often called a dummy variable because it can be replaced by any other letter. Thus, for example, the rule $f(x) = -x$, $x \in$ **N** can as well be written in the form $f(t) = -t$, $t \in$ **N** or as $f(u) = -u$, $u \in$ **N**. The variable x (or t or u) is also called an **independent variable**, and f(x) is **dependent** on this independent variable.

Graph of a function: A convenient and useful method for studying a function is to study it through its graph. To draw the graph of a function $f : X \to Y$, we choose a system of coordinate axes in the plane. For each $x \in X$, the ordered pair (x, f(x)) determines a point in the plane (see Fig. 2). The set of all the points obtained by considering all possible values of x (remember that the domain of f is X) is the graph of the function f. The role that the graph of a function plays in the study of the function will become clear as we proceed further. In the meantime let us consider some more examples of functions and their graphs.

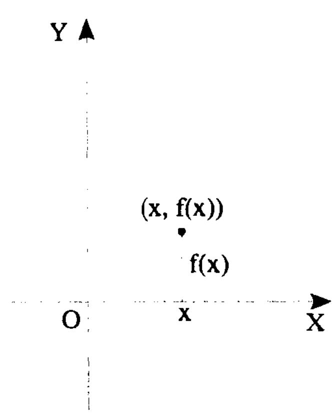


Fig. 2

1) **A constant function:** The simplest example of a function is a constant function. A constant function sends all the elements of the domain to just one element of the co-domain.

For example, let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 1$.

Alternatively, we may write

$f: x \to 1 \ldots \forall x \in \mathbf{R}$

The graph of f is as shown in Fig. 3.

It is the line y = 1.

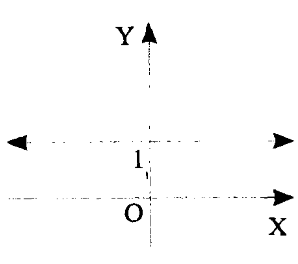


Fig. 3

In general, the graph of a constant function $f : x \to c$ is straight line which is parallel to the x-axis at a distance of $|c|$ units from it.

2) **The identity function:** Another simple but important example of a function is a function which sends every element of the domain to itself.

Let X be any non-empty set, and let f be the function on X defined by setting $f(x) = x \; \forall \, x \in X$.

This function is known as the identity function on X and is denoted by i_X.

The graph of i_R the identity function on **R**, is shown in Fig. 4. It is the line $y = x$.

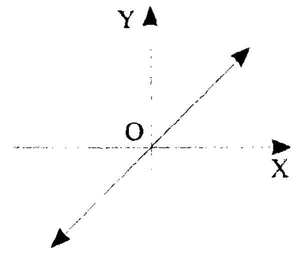


Fig. 4

3) **Absolute value Function:** Another interesting function is the absolute value function (or modulus function) which can be defined by using the concept of the absolute value of a real number as:

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

The graph of this function is shown in Fig. 5. It consists of two rays, both starting at the origin and making angles $\pi/4$ and $3\pi/4$, respectively, with the positive direction of the x-axis.

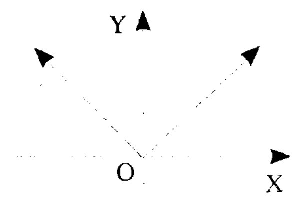
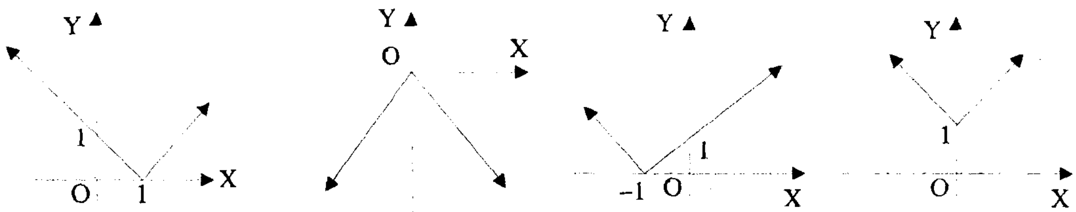


Fig. 5

E 6) Given below are the graphs of four functions depending on the notion of absolute value. The functions are $x \to -|x|$, $x \to |x| + 1$, $x \to |x + 1|$, $x \to |x - 1|$, though not necessarily in this order. (The domain in each case is **R**). Can you identify them?



(a) (b) (c) (d)

4) **The Exponential Function:** If a is a positive real number other than 1, we can define a function f as:

$f: \mathbf{R} \to \mathbf{R}$

$f(x) = a^x \qquad \{a > 0, a \neq 1\}$

This function is known as the **exponential function**. A special case of this function, where $a = e$, is often found useful. Fig. 6 shows the graph of the function $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) = e^x$. This function is also called the **natural exponential function**. Its range is the set $\mathbf{R}^+$ of positive real numbers.

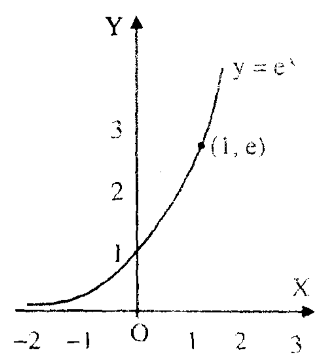


Fig. 6

5) **The Natural Logarithmic Function:** This function is defined on the set $\mathbf{R}^+$ of positive real numbers, with $f: \mathbf{R}^+ \to \mathbf{R}$ such that $f(x) = \ln(x)$. The range of this function is **R**. Its graph is shown in Fig. 7.

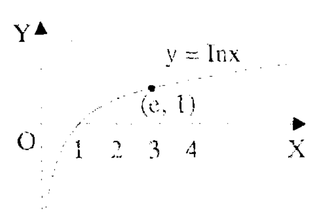


Fig. 7

6) **The Greatest Integer Function:** Take a real number x. Either it is an integer, say n (so that $x = n$) or it is not an integer. If it is not an integer, we can find (by the Archimedean property of real numbers) an integer n, such that $n < x < n + 1$. Therefore, for each real number x we can find an integer n, such that $n \leq x < n + 1$. Further, for a given real number x, we can find only one such integer n. We say that n is the greatest integer not exceeding x, and denote it by $[x]$. For example, $[3] = 3$ and $[3.5] = 3$, $[-3.5] = -4$. Let us consider the function defined on R by setting $f(x) = [x]$.

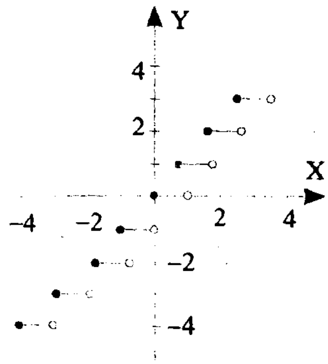


Fig. 8

This function is called the **greatest integer function**. The graph of the function is as shown in Fig. 8. (It resembles the steps on an infinite staircase).

Notice that the graph consists of infinitely many line segments of unit length, all parallel to the x-axis.

7) **Other Functions** The following are some important classes of functions.

a) **Polynomial Functions** $f(x) = a_0x^n + a_1x^{n-1} + \text{-------------} + a_n$ where $a_0, a_1, \text{-------------}, a_0$ are given real numbers (constants) and n is a positive integer.

b) **Rational Functions** $f(x) = g(x)/k(x)$, where $g(x)$ and $k(x)$ are polynomial functions of degree n and m. This is defined for all real x, for which $k(x) \neq 0$.

c) **Trigonometric or Circular Functions** $f(x) = \sin x$, $f(x) = \cos x$, $f(x) = \tan x$, $f(x) = \cos x$, $f(x) = \sec x$, $f(x) = \operatorname{cosec} x$.

d) **Hyperbolic Funcions** $f(x) = \cosh x = \dfrac{(e^x + e^{-x})}{2}$, $f(x) = \sinh x = \dfrac{(e^x - e^{-x})}{2}$. We shall study these in detail in Unit 5.

1.5.2 Inverse Functions

In this sub-section we shall see what is meant by the inverse of a function. But before talking about the inverse, let us look at some special categories of functions. These special types of functions will then lead us to the definition of the inverse of a function.

One-one and Onto Functions

Consider the function $h : x \to x^2$, defined on the set **R**. Here $h(2) = h(-2) = 4$. Thus 2 and -2 are distinct members of the domain **R**, but their h-images are the same. (Can you find some more numbers whose h-images are equal?) This may be expressed by saying that $\exists x, y$ such that $x \neq y$ but $h(x) = h(y)$.

Now, consider the function $g: x \to 2x + 3$

Here you will be able to see that if x_1 and x_2 are two distinct real numbers, then $g(x_1)$ and $g(x_2)$ are also distinct.

For, $x_1 \neq x_2 \Rightarrow 2x_1 \neq 2x_2 \Rightarrow 2x_1 + 3 \neq 2x_2 + 3 \Rightarrow g(x_1) \neq g(x_2)$

We have considered two functions here. While one of them, namely g, sends distinct members of the domain to distinct members of the co-domain, the other, namely h, does not always do so. We give a special name to functions like g above.

Definition 5: A function $f: x \to Y$ is said to be a **one-one function** (a $(1-1)$ function or an injective function) if the images of distinct members of X are distinct members of Y.

Thus the function g above is one-one, whereas h is not one-one.

Remark 4: ˜he condition "the images of distinct members of X are distinct members of Y" in the above de.inition can be replaced by either of the following equivalent conditions:

a) For every pair of members x, y of X, $x \neq y \Rightarrow f(x) \neq f(y)$

b) For every pair of members x, y of X, $f(x) = f(y) \Rightarrow x = y$.

We have observed earlier that for a function $f: X \to Y$.

$f(X) \subseteq Y$. This opens two possibilities:

i) $f(X) = Y$, or ii) $f(X) \subsetneq Y$, that is, $f(X)$ is a proper subset of Y.

The function $h : x \to x^2 \ \forall x \in \mathbf{R}$ falls in the second category. Since the square of any real number is always non-negative, $h(\mathbf{R}) = \mathbf{R}^+ \cup \{0\}$, the set of non-negative real numbers. Thus $h(\mathbf{R}) \subsetneq \mathbf{R}$.

On the other hand, the function $g : x \to 2x + 3$ belongs to the first category. Given any $y \in \mathbf{R}$ (co-domain) if we take $x = (1/2)y - 3/2$, we find that $g(x) = y$. This shows that every member of the co-domain is a g-image of some member of the domain and thus, is in the range $g(\mathbf{R})$. From this we get that $g(\mathbf{R}) = \mathbf{R}$. The following definition characterises this property of the function.

Definition 6 A function $f: X \to Y$ is said to be an **onto function** (or a surjective function) if every member of Y is the image of some member of X. If f is a function from X onto Y, we often write: $f : x \xrightarrow{\text{onto}} Y$ (or $f : X \to\to Y$).

Thus, h is not an onto function, whereas g is an onto function. Functions which are both one-one and onto are of special importance in mathematics. Let us see what makes them special.

Consider a function f: $X \rightarrow Y$ which is both one-one and onto. Since f is an onto function, each $y \in Y$ is the image of some $x \in X$. Also, since f is one-one, y cannot be the image of two distinct members of X. Thus, we find that to each $y \in Y$ there corresponds a **unique** $x \in X$ such that $f(x) = y$. Consequently, f sets up a one-to-one correspondence between the members of X and Y. It is this one-to-one correspondence between members of X and Y which makes a one-one and onto function so special, as we shall soon see.

Consider the function $f : \mathbf{N} \rightarrow E$ defined $f(x) = 2x$, where E is the set of even natural numbers. We can see that f is one-one as well as onto. In fact, to each $y \in E$ there exists $y/2 \in \mathbf{N}$, such that $f(y/2) = y$. The correspondence $y \rightarrow y/2$ defines a function, say g, from E to **N** such that $g(y) = y/2$.

The function g so defined is called an **inverse of f**. Since, to each $y \in E$ there corresponds, a **unique** $x \in \mathbf{N}$ such that $f(x) = y$, only one such function g can be defined corresponding to a given function f. For this reason g is called **the** inverse of f.

As you will notice, the function g is also one-one and onto and therefore it will also have an inverse. You must have already guessed that the inverse of g is the function f.

From this discussion we have the following :

If f is one-one and onto function from X to Y, then there exists a unique function $g : Y \rightarrow X$ such that for each $y \in Y$, $g(y) = x \Leftrightarrow y = f(X)$. The function g so defined is called the **inverse** of f. Further, if g is the inverse of f, then f is the inverse of g, and the two function f and g are said to be the inverses of each other. The inverse of a function f is usually denoted by f^{-1}.

To find the inverse of a given function f, we proceed as follows:

Solve the equation $f(x) = y$ for x. The resulting expression for x (in terms of y) defines the inverse function.

Thus, if $f(x) = \frac{x^5}{5} + 2$, we solve $\frac{x^5}{5} + 2 = y$ for x.

This gives us $x = \{5(y - 2)\}^{\frac{1}{5}}$. Hence f^{-1} is the function defined by $f^{-1}(y) = \{5(y - 2)\}^{\frac{1}{5}}$.

1.5.3 Graphs of Inverse Functions

There is an interesting relation between the graphs of a pair of inverse functions because of which, if the graph of one of them is known, the graph of the other can be obtained easily.

Let f: $X \rightarrow Y$ be a one-one and onto function, and let $g : Y \rightarrow X$ be the inverse of f. A point (p, q) lies on the graph of f $\Leftrightarrow q = f(p) \Leftrightarrow p = g(q) \Leftrightarrow (q, p)$ lies on the graph of g. Now the points (p, q) and (q, p) are reflections of each other with respect to (w.r.t.) the line $y = x$. Therefore, we can say that the graphs of f and g are reflections of each other w.r.t. the line $y=x$.

Therefore, it follow that, if the graph of one of the functions f and g is given, that of the other can be obtained by reflecting it w.r.t. the line $y = x$. As an illustration, the graphs of the functions $y = x^3$ and $y = x^{1/3}$ are given in Fig. 9.

Do you agree that these two functions are inverses of each other? If the sheet of paper on which the graphs have been drawn is folded along the line $y = x$, the two graphs will exactly coincide.

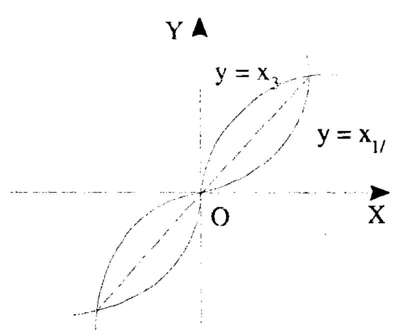


Fig. 9

E E 7) Compare the graphs of ln x and e^x given in Figs. 6 and 7 and verify that they are inverses of each other.

If a given function is not one-one on its domain, we can choose a subset of the domain on which it is one-one, and then define its inverse function. For example, consider the function $f : x \to \sin x$.

Since we know that $\sin(x + 2\pi) = \sin x$, obviously this function is not one-one on **R**. But if we restrict it to the interval $[-\pi/2, \pi/2]$, we find that it is one-one. Thus, if $f(x) = \sin x \;\forall\; x \in [\pi/2, \pi/2]$, then we can define

$f^{-1}(x) = \sin^{-1}(x) = y$ if $\sin y = x$.

Similarly, we can define $\cos^{-1}$ and $\tan^{-1}$ functions as inverse of cosine and tangent functions if we restrict the co-domain to $[0, \pi]$ and $]-\pi/2, \pi/2[$, respectively.

E E 8) Which of the following functions are one-one?

a) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x|$
b) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x - 1$.
c) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$
d) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1$

E E 9) Which of the following functions are onto?

a) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x + 7$
b) $f : \mathbf{R}^+ \to \mathbf{R}$ defined by $f(x) = \sqrt{x}$
c) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2 + 1$
d) $f : \mathbf{X} \to \mathbf{R}$ defined by $f(x) = 1/x$

where X stands for the set of non-zero real numbers.

E E 10) Show that the function $f : X \to X$ such that $f(x) = \dfrac{x+1}{x-1}$, where X is the set of all real numbers except 1, is one-one and onto. Find its inverse.

E E 11) Give one example of each of the following :

a) a one-one function which is not onto.
b) onto function which is not one-one.
c) a function which is neither one-one nor onto.

1.6 NEW FUNCTIONS FROM OLD

In this section we shall see how we can construct new functions from some given functions. This can be done operating upon the given functions in a variety of ways. We give a few such ways here.

1.6.1 Operations on Functions

Scalar Multiple of a Function

Consider the function $f : x \rightarrow 3x^2 + 1 \ \forall\, x \in \mathbf{R}$. The function $g : x \rightarrow 2\,(3x^2 + 1) \ \forall\, x \in \mathbf{R}$ is such that $g\,(x) = 2f\,(x) \ \forall\, x \in \mathbf{R}$. We say that $g = 2f$, and that g is a scalar multiple of f by 2. In the above example there is nothing special about the number 2. We could have taken any real number to construct a new function from f. Also, there is nothing special about the particular function that we have considered. We could as well have taken any other function. This suggests the following definition: Let f be a function with domain D and let k be any real number. The scalar multiple of f by k is a function with domain D. It is denoted by kf and is defined by setting $(kf)\,(x) = kf(x)$.

Two special cases of the above definition are important.

i) Given any function f, if $k = 0$, the function kf turns out to be the zero function. That is, $0.f = 0$.

ii) If $k = -1$, the function kf is called the negative of f and is denoted simply by $-f$ instead of the clumsy $-1f$.

Absolute Value Function (or modulus function) of a given function

Let f be a function with domain D. The absolute value function of f, denoted by $|\,f\,|$ and read as mod f is defined by setting.

$(|\,f\,|)\,(x) = |\,f(x)\,|$, for all $x \in D$.

Since $|\,f(x)\,| = f(x)$, if $f(x) \geq 0$, f and $|\,f\,|$ have the same graph for those value of x for which $f(x) \geq 0$.

Now let us consider those values of x for which $f\,(x) < 0$.

Here $|\,f\,(x)\,| = -\,f\,(x)$. Therefore, the graphs of f and $|\,f\,|$ are reflections of each other w.r.t. the x-axis for those values of x for which $f\,(x) < 0$.

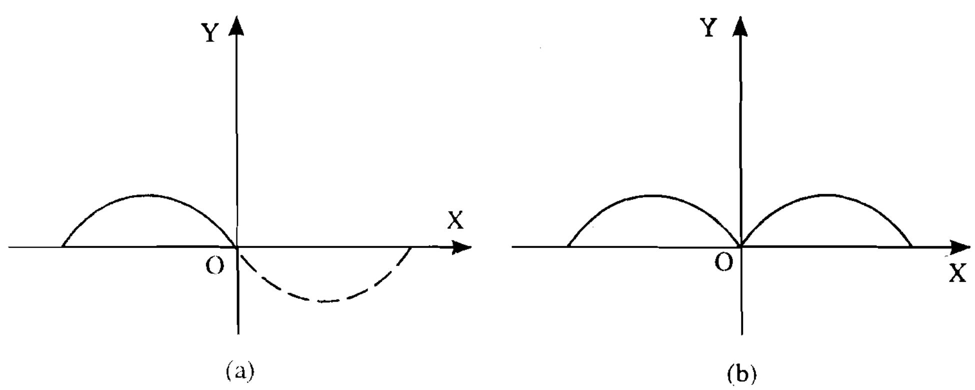


Fig. 10

As an example, consider the graph in Fig. 10 (a). The portion of the graph below the x-axis (that is, the portion for which $f(x) < 0$) has been shown by a dotted line.

To draw the graph of $|f|$ we retain the undotted portion in Fig. 10 (a) as it is, and replace the dotted portion by its reflection w.r.t. the x-axis (see Fig. 10b).

Sum, difference, Product and Quotient of two functions
If we are given two functions with a common domain, we can form-several new functions by applying the four fundamental operations of addition, subtraction, multiplication and division on them.

i) Define a functions s on D by setting
$s(x) = f(x) + g(x)$.
The function s is called the sum of the functions f and g, and is denoted by $f + g$. Thus,
$(f+g)(x) = f(x) + g(x)$

ii) Define a function d on D by setting
$d(x) = f(x) - g(x)$.
The function d is the function obtained by subtracting g from f, and is denoted by $f-g$. Thus, for all $x \in D$
$(f-g)(x) = f(x) - g(x)$.

iii) Define a function p on D by setting
$p(x) = f(x)\,g(x)$.

The function p, called the product of the functions f and g, is denoted by fg. Thus, for all $x \in D$

$(fg)(x) = f(x)\,g(x)$

iv) Define a function q on D by setting $q(x) = f(x)/g(x)$, provided $g(x) \neq 0$ for $x \in D$. The function q is called the quotient of f by g and is denoted by f/g. Thus,

$(f/g)(x) = f(x)/g(x) \quad (g(x) \neq 0$ for any $x \in D)$.

Remark 5: In case $g(x) = 0$ for some $x \in D$. We can consider the set, say D, of all those values of x for which $g(x) \neq 0$, and define f/g on D by setting $(f/g)(x) = f(x)/g(x)\ \forall\, x \in D$.

Example 4: Consider the functions $f : x \to x^2$ and $g : x \to x^3$. Then the functions $f + g$, $f - g$, fg are defined as

$(f+g)(x) = x^2 + x^3$,

$(f-g)(x) = x^2 - x^3$.

$(fg)(x) = x^5$

Now, $g(x) = 0 \Leftrightarrow x^3 = 0 \Leftrightarrow x = 0$. Therefore, in order to define the function f/g, we shall consider only non-zero values of x. If $x \neq 0$, $f(x)/g(x) = x^2/x^3 = 1/x$. Therefore f/g is the function.

$f/g : x \to 1/x$, whenever $x \neq 0$.

All the operations defined on functions till now, were similar to the corresponding operations on real numbers. In the next subsection we are going to introduce an operation which has no parallel in **R**. Composite functions play a very important role in calculus. You will realise this as you read this course further.

1.6.2 Composite of Functions

We shall now describe a method of combining two functions which is somewhat different from the ones studied so far. Uptill now we have considered functions with the same domain. We shall now consider a pair of functions such that the co-domain of one is the domain of the other.

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. We define a function $h : X \to Z$ by setting $h(x) = g(f(x))$.

To obtain $h(x)$, we first take the f-image, $f(x)$, of an element x of X. This $f(x) \in Y$, which is the domain of g. We then take the g image of $f(x)$, that is, $g(f(x))$, which is an element of Z. This scheme has been shown in Fig. 11.

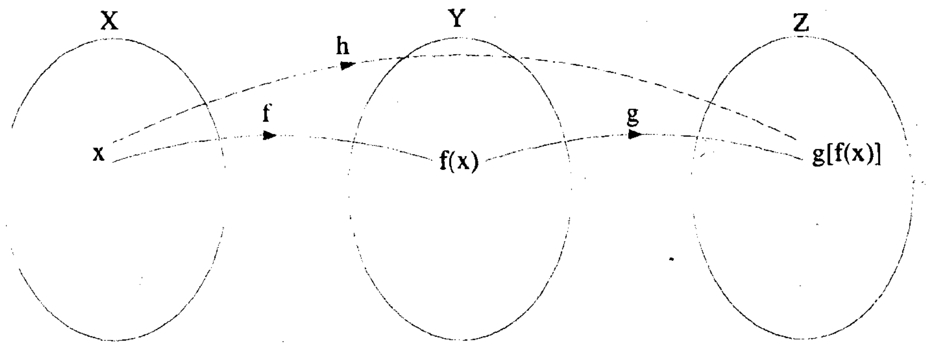


Fig. 11

The function h, defined above, is called the composite of f and g and is written as g_of. Note the order. We first find the f-image and then its g-image. Try to distinguish it form f_og, which will be defined only when Z is a subset of X. Also, in that case, fog is a function from y to y.

Example 5: Consider the functions $f : x \rightarrow x^2 \ \forall \ x \in \mathbf{R}$ and $g : x \rightarrow 8x + 1 \ \forall \ x \in \mathbf{R}$. g_of is a function from **R** to itself, defined by $(g_of)(x) = g(f(x)) = g(x^2) = 8x^2 + 1 \ \forall \ x \in \mathbf{R}$. f_og is a function from **R** to itself defined by $(f_og)(x) = f(g(x)) = f(8x + 1) = (8x + 1)^2$. Thus g_of and f_og are both defined, but are different from each other.

The concept of composite function is used not only to combine functions, but also to look upon a given function as made up of two simpler functions. For example, consider the function.

$h : x \rightarrow \sin(3x + 7)$

We can think of it as the composite (g_of) of the functions $f : x \rightarrow 3x + 7 \ \forall \ x \in \mathbf{R}$ and $g : u \rightarrow \sin u \ \forall \ u \in \mathbf{R}$.

Now let us try to find the composites f_og and g_of of the functions:

$f : x \rightarrow 2x + 3 \ \forall \ x \in \mathbf{R}$, and $g : x \rightarrow (1/2)x - 3/2 \ \forall \ x \in \mathbf{R}$

Note that f and g are inverses of each other. Now $g_of(x) = g(f(x)) = g(2x + 3)$

$$= \frac{1}{2}(2x + 3) - \frac{3}{2} = x.$$

Similarly, $f_og(x) = f(g(x)) = f(x/2 - 3/2) = 2(x/2 - 3/2) + 3 = x$. Thus, we see that $g_of(x) = x$ and $f_og(x) = x$ for all $x \in \mathbf{R}$. Or, in other words, each of g_of and f_og is the identity function on **R**. What we have observed here is true for any two functions f and g which are inverses of each other. Thus, **if $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are inverses of each other, then g_of and f_og are identity functions.** Since the domain of g_of is X and that of f_og is Y, we can write this as :

$g_of = i_X$, $f_og = i_Y$.

This fact is often used to test whether two given functions are inverses of each other.

1.7 TYPES OF FUNCTIONS

In this section we shall talk about various types of functions, namely, even, odd, increasing, decreasing and periodic functions. In each case we shall also try to explain the concept through graphs.

1.7.1 Even and Odd Functions

We shall first introduce two important classes of functions: even functions and odd functions. Consider the functions f defined on **R** by setting

$f(x) = x^2 \qquad \forall \ x \in \mathbf{R}$.

You will notice that $f(-x) = (-x)^2 = x^2 = f(x) \ \forall \ x \in \mathbf{R}$

This is an example of an even function. Let's take a look at the graph (Fig. 12) of this function. We find that the graph (a parabola) is symmetrical about the y-axis. If we fold the paper along the y-axis, we shall see that the parts of the graph on both sides of the y-axis completely coincide with each other. Such functions are called even functions. Thus, a function f, defined on **R** is even, if, for each $x \in \mathbf{R}$, $f(-x) = f(x)$.

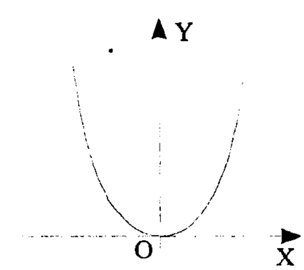
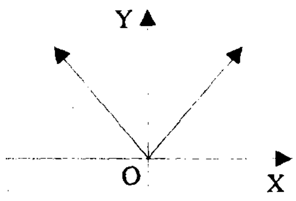


Fig. 12

The graph of an even function is symmetric with respect to the y-axis. We also note that if the graph of a function is symmetric with respect to the y-axis, the function must be an even function. Thus, if we are required to draw the graph of an even function, we can use this property to our advantage. We only need to draw that part of the graph which lies to the right of the y-axis and then just take its reflection w.r.t. the y-axis to obtain the part of the graph which lies to the left of the y-axis.



(a)

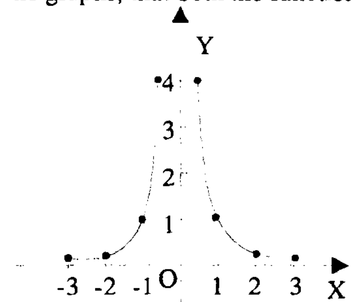
E E 12) Given below are two examples of even functions, alongwith their graphs. Try to convince yourself, by calculations as well as by looking at the graphs, that both the functions are, indeed, even functions.

a) The absolute value function on **R**

$f : x \to |x|$

The graph of f is shown alongside.

b) The function g defined on the set of non-zero real numbers by setting $g(x) = 1/x^2, x \neq 0$.
The graph of g is shown alongside.



(b)

Now let us consider the function f defined by setting $f(x) = x^3 \ \forall\, x \in \mathbf{R}$. We observe that $f(-x) = (-x) = (-x)^3 = -x^3 = -f(x) \ \forall\, x \in \mathbf{R}$. If we consider another function g given by $g(x) = \sin x$ we shall be able to note again that $g(-x) = \sin(-x) = -\sin x = -g(x)$.

The functions f and g above are similar in one respect: the image of $-x$ is the negative of the image of x. Such functions are called odd functions. Thus, a function f defined on **R** is said to be an **odd function** if $f(-x) = -f(x) \ \forall\, x \in \mathbf{R}$.

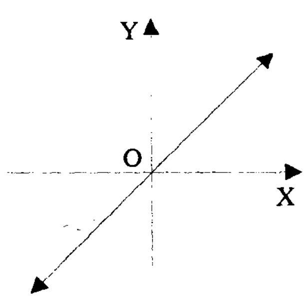
If $(x, f(x))$ is a point on the graph of an odd function f, then $(-x, -f(x))$ is also a point on it. This can be expressed by saying that the graph of an odd function is symmetric with respect to the origin. In other words, if you turn the graph of an odd function through 180° about the origin you will find that you get the original graph again. Conversely, if the graph of a function is symmetric with respect to the origin, the function must be an odd function. The above facts are often useful while handling odd functions.

E E 13) We are giving below two functions alongwith their graphs. By calculations as well as by looking at the graphs, find out for each whether it is even or odd.

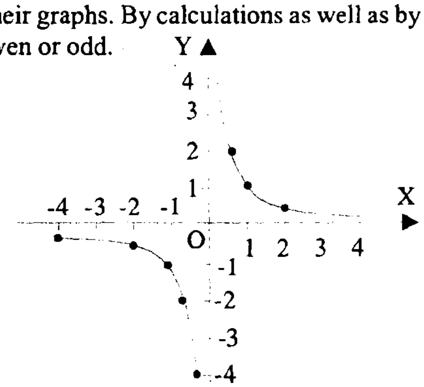
a) The identity function on **R**:

$f : x \to x$

b) The function g defined on the set of non-zero real numbers by setting $g(x) = 1/x, x \neq 0$



(a)



(b)

While many of the functions that you will come across in this course will turn out to be either even or odd, there will be many more which will be neither even nor odd. Consider, for example, the function

$f: x \to (x+1)^2$

Here $f(-x) = (-x+1)^2 = x^2 - 2x + 1$. Is $f(x) = f(-x)\ \forall\ x \in \mathbf{R}$?

The answer is 'no'. Therefore, f is not an even function. Is $f(x) = -f(-x)\ \forall\ x \in \mathbf{R}$? Again, the answer is 'no'. Therefore f is not an odd function. The same conclusion could have been drawn by considering the graph of f which is given in Fig. 13.

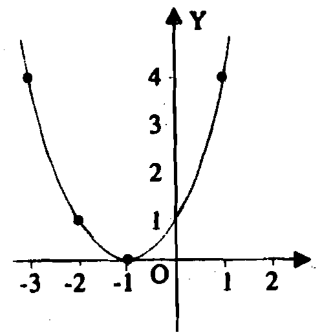


Fig. 13

You will observe that the graph is symmetric neither with respect to the y-axis, nor with respect to the origin.

Now there should be no difficulty in solving the exercise below.

E E 14) Which of the following functions are even, which are odd, and which are neither even nor odd?

a) $x \to x^2 + 1,\ \forall\ x \in \mathbf{R}$

b) $x \to x^3 - 1,\ \forall\ x \in \mathbf{R}$

c) $x \to \cos x,\ \forall\ x \in \mathbf{R}$

d) $x \to x\,|x|,\ \forall\ x \in \mathbf{R}$

e) $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

1.7.2 Monotone Functions

In this sub-section we shall consider two types of functions:

i) Increasing and ii) Decreasing

Any function which conforms to any one of these types is called a monotone function. Does the profit of a company increase with production? Does the volume of gas decrease with increase in pressure? Problems like these require the use of increasing or decreasing functions. Now let us see what we mean by an increasing function. Consider the function g and h defined by

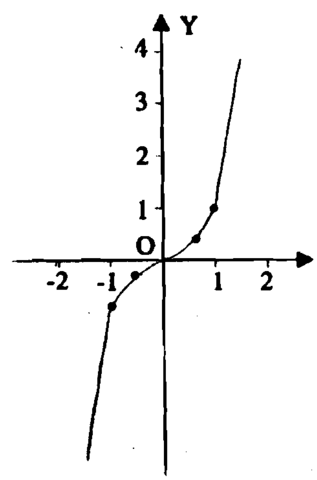


Fig. 14

$g(x) = x^3$ and $h(x) = \begin{cases} -x, & \text{if } x \le 0 \\ 1, & \text{if } x > 0 \end{cases}$

Note that whenever $x_2 > x_1$, we get $x_2^3 > x_1^3$, that is, $g(x_2) > g(x_1)$.

In other words, as x increases, g(x) also increases. This fact can also be seen from the graph of g shown in Fig. 14.

Let us find out how h(x) behaves as x increases. In this case we see that if $x_2 > x_1$, then $h(x_2) \ge h(x_1)$. (You can verify this by choosing any values for x_1 and x_2). Equivalently, we can say that h (x) increases (or does not decrease) as x increases. The same can be seen from the graph of h in Fig. 15.

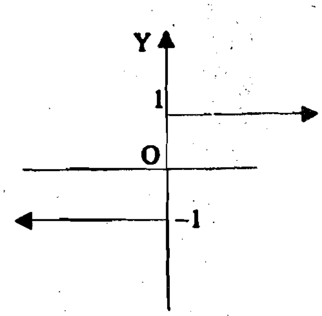


Fig. 15

Functions like g and h above are called increasing or non-decreasing functions. Thus, a function f defined on a domain D is said to be **increasing** (or **non-decreasing**) if, for every pair of elements $x_1, x_2 \in D$, $x_2 > x_1 \Rightarrow f(x_2) \ge f(x_1)$. Further, we say that f is **strictly increasing** if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ (strict inequality).

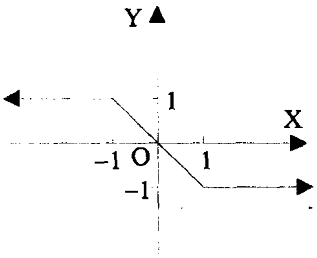
Clearly, the function $g: x \to x^3$ discussed above, is a strictly increasing function; while h is not a strictly increasing function.

We shall now study another concept which is, in some sense, complementary to that of an increasing function.

Consider the function f_1 defined on **R** by setting.

$$f_1(x) = \begin{cases} 1, & \text{if } x \le -1 \\ -x, & \text{if } -1 < x < 1 \\ -1, & \text{if } x \ge 1 \end{cases}$$

The graph of f_1 is as shown in Fig. 16.



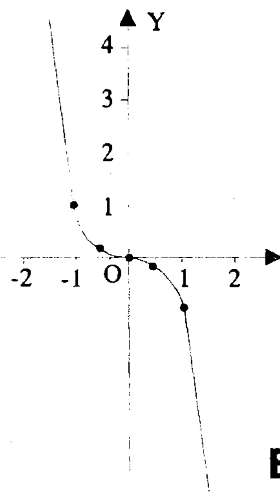
Graph of f_1
Fig. 16

From the graph we can easily see that as x increases f_1 does not increase.

That is, $x_2 > x_1 \Rightarrow f_1(x_2) \le f_1(x_1)$ or $f_1(x_2) \ge f_1(x_1)$

Now consider the function $f_2 : x \rightarrow -x^3$ $(x \in \mathbf{R})$

The graph of f_2 is shown in Fig. 17.



Graph of f_2
Fig. 17

Since $x_2 > x_1 \Rightarrow x_2^3 > x_1^3 \Rightarrow -x_2^3 < -x_1^3 \Rightarrow f_2(x_2) < f_2(x_1)$, we find that as x increases, $f_2(x)$ decreases. Functions like f_1 and f_2 are called decreasing or non-increasing functions. The above two examples suggest the follwing definition:

A function f defined on a domain D is said to be **decreasing** (or non-increasing) if for every pair of elements x_1, x_2, $x_2 > x_1 \Rightarrow f(x_2) \le f(x_1)$. Further, f is said to be strictly decreasing if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$.

We have seen that, of the two decreasing functions f_1 and f_2, f_2 is strictly decreasing, while f_1 is not strictly decreasing. A function f defined on a domain D is said to be a **monotone function** if it is either increasing or decreasing on D.

All the four functions (g, h, f_1, f_2) discussed above are monotone functions. The phrases 'monotonically increasing' and 'monotonically decreasing' are often used for 'increasing' and 'decreasing', respectively.

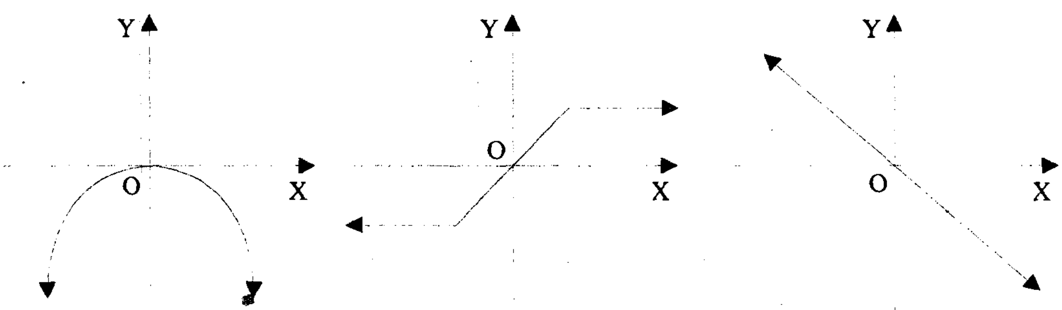
While many functions are monotone, there are many others which are not monotone. Consider, for example, the function.

$f : x \rightarrow x^2$ $(x \in \mathbf{R})$.

You have seen the graph of f in Fig. 12. This function is neither increasing nor decreasing.

If we find that a given function is not monotone, we can still determine some subsets of the domain on which the function is increasing or decreasing. For example, the function $f(x) = x^2$ is strictly decreasing in $]-\infty, 0]$ and is strictly increasing in $[0, \infty[$.

E E 15) Given below are the graphs of some functions. Classify them as non-decreasing, strictly decreasing, neither increasing nor decreasing:



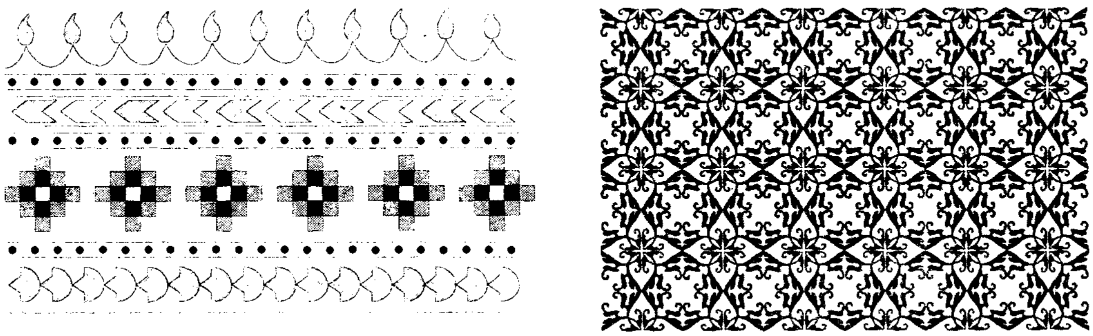
(a) (b) (c)

1.7.3 Periodic Functions

In this section we are going to tell you about yet another important class of functions, known as periodic functions.

Periodic functions occur very frequently in application of mathematics to various branches of science. Many phenomena in nature such as propagation of water waves, sound waves, light waves, electromagnetic waves etc. are periodic and we need periodic functions to describe them. Similarly, weather conditions and prices can also be described in terms of periodic functions.

Look at the following patterns :



(a) (b)

Fig. 18

You must have come across patterns similar to the ones shown in Fig. 18 on the borders of sarees, wall papers etc. In each of these patterns a design keeps on repeating itself. A similar situation occurs in the graphs of periodic functions. Look at the graphs in Fig. 19.

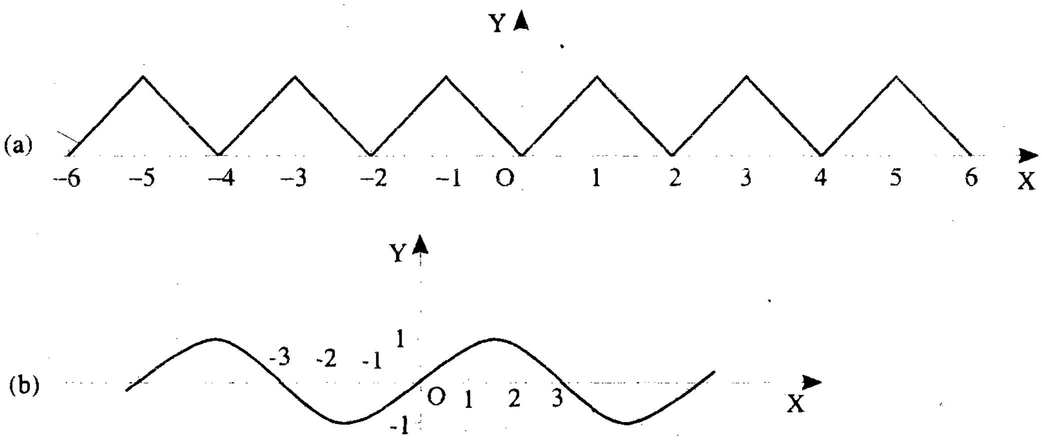


Fig. 19

In each of the figures shown above the graph consists of a certain pattern repeated infinitely many times. Both these graphs represent periodic functions. To understand the situation, let us examine these graphs closely.

Consider the graph in Fig. 19(a). The portion of the graph lying between $x = -1$ and $x = 1$ is the graph of the function $x \to |x|$ on the domain $-1 \leq x \leq 1$.

This portion is being repeated both to the left as well as to the right, by translating (pushing) the graph through two units along the x-axis. That is to say, if x is any point of $[-1, 1]$, then the ordinates at $x, x \pm 2, x \pm 4, x \pm 6, ...$ are all equal. The graph therefore represents the function f defined by

$f(x) = |x|$, if $-1 \leq x \leq 1$ and $f(x + 2) = f(x)$.

The graph in Fig. 19(b) is the graph of the sine function, $x \to \sin x$, $\forall x \in \mathbf{R}$. You will notice that the portion of the graph between 0 and 2π is repeated both to the right and to the left. You know already that $\sin(x + 2\pi) = \sin x$, $\forall x \in \mathbf{R}$. We now give a precise meaning to the term "a periodic function".

A function f defined on a domain D is said to be a **periodic function** if there exists a positive read number p such that $f(x + p) = f(x)$ for all $x \in D$. The number p is said to be a period of f.

The smallest positive ineger p with the property described above is called *the* period of f.

As you know, $\tan(x + n\pi) = \tan x$ $\forall n \in N$. This means that $n\pi$, $n \in N$ are all periods of the tangent function. The smallest of $n\pi$, that is π, is the period of the tangent function. See if you can do this exercise.

E 16) a) What is the period each of the functions given in Fig. 19(a) and (b)?

b) Can you give one other period of each of these functions?

As another example of a periodic function, consider the function f defined on **R** by setting $f(x) = x - [x]$

Let us recall that [x] stands for the greatest integer not exceeding x.

The graph of this function is as shown in Fig. 20.

From the graph (as also by calculation) we can easily see that

$f(x + n) = f(x) \ \forall \ x \in \mathbf{R}$. and for each positive integer n.

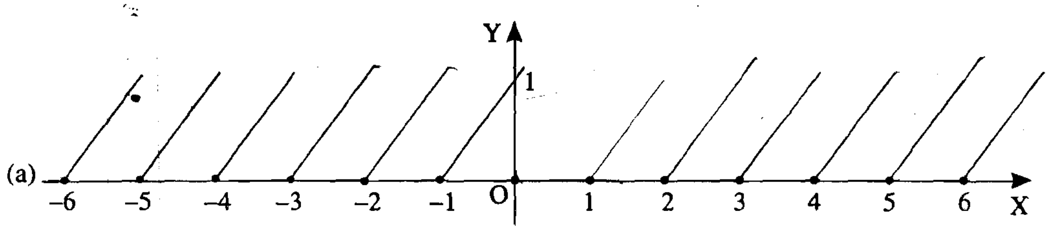


Fig. 20

The given function is therefore periodic, the numbers 1, 2, 3, 4 being all periods. The smallest of these, namely 1, is **the** period.

Thus the given function is periodic and has the period 1.

Remark 6 Monotonicity and periodicity are two properties of functions which cannot coexist. A monotone function can never be periodic, and a periodic function can never be monotone.

In general, it may not be easy to decide whether a given function is periodic or not. But sometimes it can be done in a straight forward manner. Suppose we want to find whether the function $f: x \to x^2 \ \forall \ x \in \mathbf{R}$ is periodic or not. We start by assuming that it is periodic with period p.

Then we must have $p > 0$ and $f(x + p) = f(x) \ \forall \ x$

$$\Rightarrow (x + p)^2 = x^2 \quad \forall x$$
$$\Rightarrow 2xp + p^2 = 0 \quad \forall x$$
$$\Rightarrow p(2x + p) = 0 \quad \forall x$$

Considering $x \neq -p/2$, we find that $2x + p \neq 0$. Thus, $p = 0$. This is a contradiction.

Therefore, there does not exist may positive number p such that $f(x + p) = f(x)$, $\forall \ x \in \mathbf{R}$ and, consequently, f is not periodic.

E 17) Examine whether the following functions are periodic or not. Write the periods of the periodic functions.

a) $x \to \cos x$ b) $x \to x + 2$

c) $x \to \sin 2x$ d) $x \to \tan 3x$

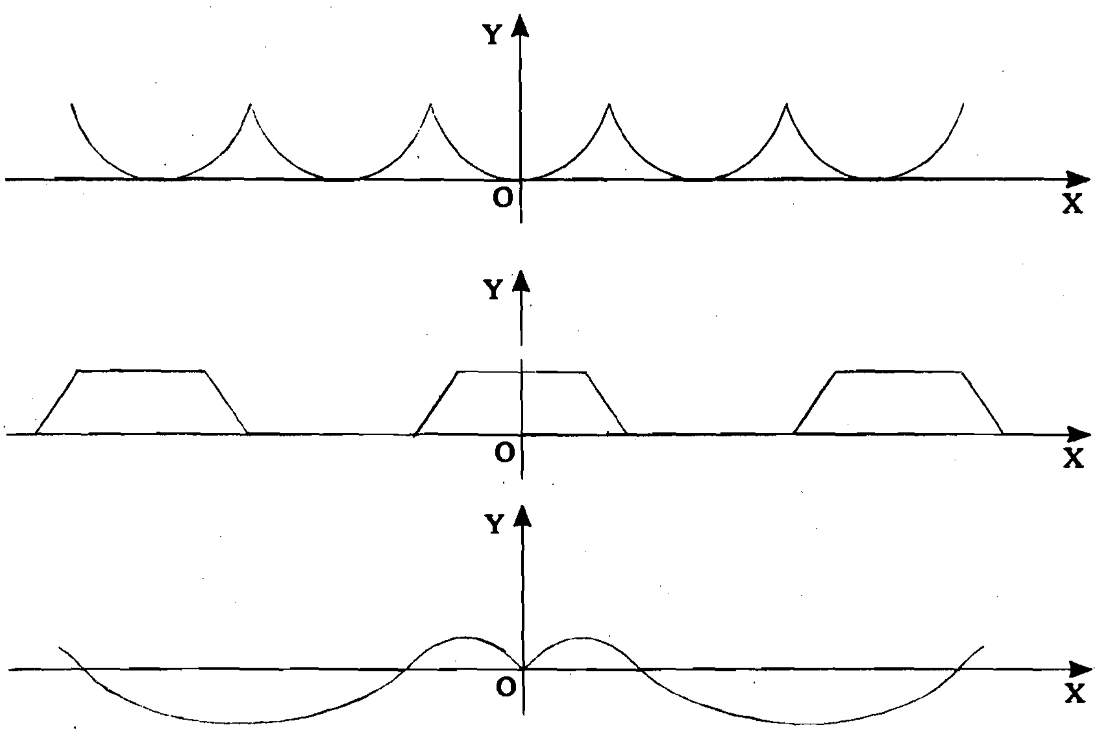
e) $x \to \cos(2x + 5)$ f) $x \to \sin x + \sin 2x$

E 18) The graphs of three functions are given below: classify the functions as periodic and non-periodic:

(a)

(b)

(c)



E 19) Is the sum of two periodic functions also periodic? Give reasons for you answer.

We end with summarising what we have discussed in this units.

1.8 SUMMARY

In this unit we have

1 briefly revised the basic properties of real numbers,

2. defined the absolute value of a real number x as

$$x = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

3 discussed various types of intervals in **R**

Open: $]a, b[\; = \{x \in \mathbf{R} : a < x < b\}$

closed: $[a, b] \; = \{x \in \mathbf{R} : a \leq x \leq b\}$

semi-open: $]a, b] \; = \{x \in \mathbf{R} : a < x \leq b\}$

or $[a, b[\; = \{x \in \mathbf{R} : a \leq x < b\}$,

where $a, b \in \mathbf{R}$,

4 defined a function and discussed various types of functions along with their graphs: one-one, onto, even, odd, monotone, periodic.

5 defined composite of functions and discussed the existence of the inverse of a function.

1.9 SOLUTIONS AND ANSWERS

E 1) a) The set $\{1, 2, 3, ...\}$ has a lower bound, e.g., 0.

b) The set $\{ ...., -3, -2, -1, 0, 1, 2, ...\}$ does not have a lower bound.

c) The g.l.b of the set $S = \{1, {}^1/_2, {}^1/_3 ...., {}^1/_n, ...\}$ is 0, and $0 \notin S$.

d) $\{x : x \in \mathbf{R} \text{ and } 1 \leq x \leq 2\}$ is a bounded set as it is bounded above by 2 and below by 1.

E 2) a) A real number p is positive if $p > 0$. Hence 0 is a lower bound for the set P of positive real numbers. Thus the set P is bounded below. Its infimum is 0.

b) A real number r is the infimum of a set $S \subset \mathbf{R}$ if and only if the following conditions are satisfied:

i) $r \leq x$ for all $x \in S$.

ii) For each $\varepsilon > 0$ there is $y \in S$ such that $y < r + \varepsilon$.

The set P in a) above has infimum 0, since

i) $0 < p$ for all $p \in P$ and

ii) For each $\varepsilon > 0$ there is $\varepsilon/2 \in P$ such that $\varepsilon/2 < 0 + \varepsilon = \varepsilon$

E3) a) $|x| = \max\{x, -x\}$. Hence $x = 0 \Rightarrow |x| = 0$ and

$|x| = 0 \Rightarrow \max. \{x, -x\} = 0 \Rightarrow x = 0$.

b) There are three cases 1) $x \geq 0, y \geq 0$

2) x and y have opposite signs

e) $x < 0, y < 0$.

We take 2). Suppose $x > 0, y < 0$, then,

$|x| = \max\{x, -x\} = x, |y| = \max\{y, -y\} = -y$.

$xy < 0 \Rightarrow |xy| = -xy = x \times (-y) = |x||y|$.

1) and 3) can be proved similarly.

c) If $x > 0$, $|x| = x$ and $|1/x| = 1/x = 1/|x|$

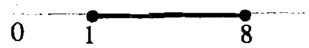
If $x < 0$, $|x| = -x$ and $|1/x| = -1/x = 1/|x|$

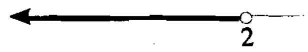
d) $|x - y| = |x + (-y)| \le |x| + |-y| = |x| + |y|$

e) $|x + y + z| = |(x + y) + z| \le |x + y| + |z| \le |x| + |y| + |z|$

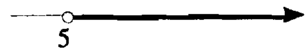
f) $|xyz| = |xy||z| = |x||y||z|$

E4) a) False b) True c) True d) False

E5) a) 

b) 

c) 

d) 

E6) a) $x \to |x - 1|$ b) $x \to -|x|$

c) $x \to |x + 1|$ d) $x \to |x| + 1$

E8) b) is one-one

E9) a) is onto

E10) $f(x) = \dfrac{x+1}{x-1} = 1 + \dfrac{2}{x-1}$

$$f(x) = f(y) \Rightarrow 1 + \frac{2}{x-1} = 1 + \frac{2}{y-1}$$

$$\Rightarrow \frac{2}{x-1} = \frac{2}{y-1} \Rightarrow x - 1 = y - 1 \Rightarrow x = y$$

Hence f is one-one.

If $y \in X$, put $x = \dfrac{y+1}{y-1}$. Then $x \in X$ and $y = f(x)$. Hence, f is onto.

$$f^{-1}(x) = \frac{x+1}{x-1}$$

E11) a) $f : \mathbf{R}^+ \to \mathbf{R} : f(x) = \sqrt{x}$

b) $f : \mathbf{R} \to \mathbf{R}^+ \cup \{0\} : f(x) = x^2$

c) $f : \mathbf{R} \to \mathbf{R} : f(x) = x^2$

E12) a) $f(x) = |x| \Rightarrow f(-x) = |-x| = |x| = f(x)$. Hence, f is even.

b) $g(x) = 1/x^2 \Rightarrow g(-x) = 1/(-x)^2 = g(x)$. Hence, g is even.

E13) a) $f(x) = x \Rightarrow f(-x) = -x = -f(x)$. Hence, f is odd.

b) $g(x) = 1/x \Rightarrow g(-x) = -1/x = -g(x)$. Hence, g is odd.

E14) a), c and e) are even

d) is odd

b) is neither even nor odd.

E15) a) neither increasing, nor decreasing

b) non-decreasing c) strictly decreasing

E16) The period of the function in Fig. 19 a) is 2. Other periods are 4, 6, 8, ...

The period of the function in Fig. 19 b) is 2π. Other periods are 4π, 6π, ...

E17) a) Periodic with period 2π

Since $\cos(x + 2\pi) = \cos x$ for all x.

b) not periodic

c) Periodic with period π.

d) Periodic with period $\pi/3$.

e) Periodic with period π.

f) Periodic with period 2π.

E18) a) and b) are periodic, c) is not.

E19) No. For example, $x - [x]$ and $|\sin x|$ are periodic, but their sum is not.