UNIT 1 PRELIMINARIES IN PLANE GEOMETRY

Structure

1.1 INTRODUCTION

In this short unit, our aim is to re-acquaint you with some essential elements of two-dimensional geometry. We will briefly touch upon the distance formula and various ways of representing a straight line algebraically. Then we shall look at the polar representation of a point in the plane. Next, we will talk about symmetry with respect to origin or a coordinate axis is. Finally, we shall consider some ways in which a coordinate system can be transformed.

This collection of topics may seem random to you. But we have picked them according to our need. We will be using whatever we cover here, in the rest of the block. So, in later units we will often refer to a section, an equation or a formula from this unit.

You are probably familiar with the material covered in this unit. But please make sure to go through the following list of objectives and the exercises covered in the unit. Otherwise you may have some trouble in later units.

Objectives

After studying this unit you should be able to :

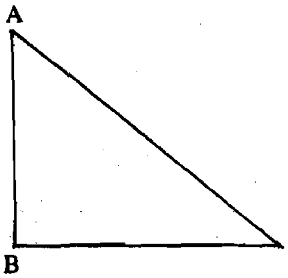
- find the distance between any two points, or a point and a line, in two-dimensional space;
- obtain the equation of a line in slope-intercept form, point-slope form, two-point form, intercept form or normal form;
- check if a curve is symmetric with respect to either coordinate axis or the origin;
- effect a change of coordinates with a shift in origin, or with a rotation of the axes;
- relate the polar coordinates and Cartesian coordinates of a point;
- obtain the polar form of an equation.

1.2 EQUATIONS OF A STRAIGHT LINE

In this section we aim to refresh your memory about the ways of representing points and lines algebraically in two-dimensional space. Since we expect you to be familiar with the matter, we shall cover the ground quickly.

Firstly, as you know, two-dimensional space can be represented by the Cartesian coordinate system. This is because there is a 1-1 correspondence between the points in a plane and ordered pairs of real numbers. If a point P is represented by (x, y) under this correspondence, then x is called the **abscissa** (or **x-coordinate**) of P and y is called the **ordinate** (or **y-coordinate**) of P.

According to the pythagoras theorem, in the right-angled triangle ABC,



$(AB)^2 + (BC)^2 = (AC)^2$

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane, then the distance between them is

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \qquad ...(1)$$

From Fig. 1, and by applying the Pythagoras theorem, you can see how we get (1).

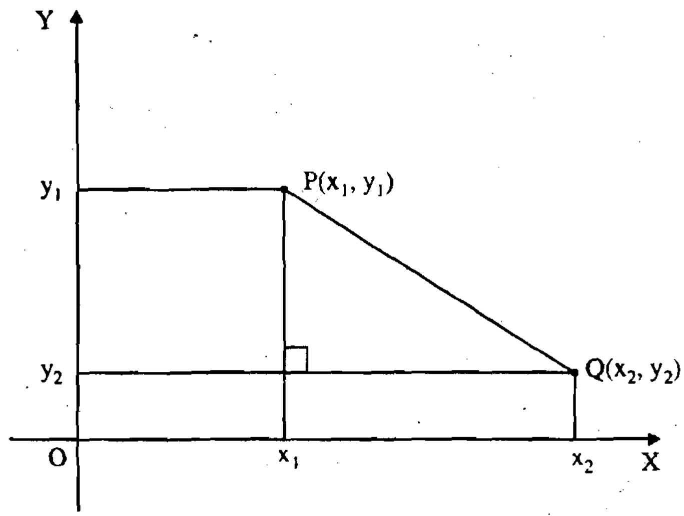


Fig. 1: Distance between two points.

(1) is called the **distance formula.**

Another formula that you must be familiar with is the following:

if the point R(x, y) divides the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio m:n (see Fig. 2), then

$$x = \frac{nx_1 + mx_2}{m+n} \text{ and } y = \frac{ny_1 + my_2}{m+n} \qquad ...(2)$$

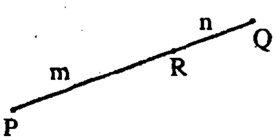


Fig. 2 : R divides the segment PQ in the ratio m:n

(2) is called the **section formula.**

To regain practice in using (1) and (2), you can try the following exercises.

E 1) What are the coordinates if the midpoint of the line segment with endpoints
a) A(5, –4) and B(–3, 2) ?
b) $A(a_1, a_2)$ and $B(b_2, b_2)$?

E 2) Check if the triangle PQR, where P, Q and R are represented by (1, 0), (–2, 3) and (1, 3), is an equilateral triangle.

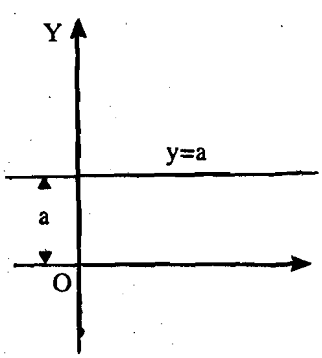


Fig. 3 : y = a is parallel to the x-axis.

The plural of 'axis' is 'axes'

Let us now write down the various ways of representing a straight line algebraically. We start with lines parallel to either of the axes. A line parallel to the x-axis is given by the equation

$$y = a, \qquad ...(3)$$

where a is some constant. This is because any point on the line will have the same ordinate (see Fig. 3).

What do you expect the equation of a line parallel to the y-axis to be ? It will be x = b, for some constant b.

Now let us obtain four forms of the equation of a line which is not parallel to either of the axes. Firstly, suppose we know that the line makes an angle α with the positive direction of the x-axis, and cuts the y-axis in (0, c). Then its equation will be

$$y = mx + c. \qquad \dots (5)$$

where $m = \tan\alpha$, and m is called its **slope** and c is its **intercept** on the y-axis. From Fig. 4 you should be able to derive (5), which is called the **slope-intercept form** of the equation of a line.

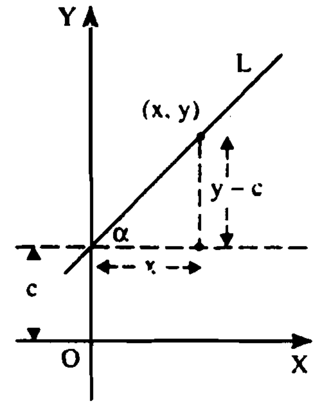


Fig. 4 : L is given by $y = x\tan\alpha + c$.

Now, suppose we know the slope m of a line and that the point (x_1, y_1) lies on the line. Then, can we obtain the line's equation ? We can use (5) to get the **point-slope form**.

$$y - y_1 = m(x - x_1) \qquad \dots (6)$$

of the equation of the line.

We can also find the equation of a line that is not parallel to either axis if we know two distinct points lying on it. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the points on the line (see Fig. 5), then its equation in the **two-point form** will be

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad \dots (7)$$

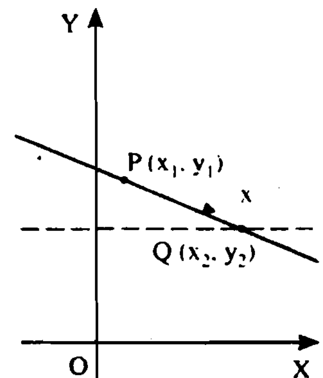


Fig. 5: The slope of PQ is $\tan\alpha$

Note that both the terms in the equation are well-defined since the denominators are not zero.

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + \left\{y_1 - x_2\left(\frac{y_2 - y_1}{x_2 - x_1}\right)\right\},$$

you can see that its slope is $\frac{y_2 - y_1}{x_2 - x_1}$, and its intercept on the y-axis is the constant term.

viz $\left\{y_1 - x_2\left(\frac{y_2 - y_1}{x_2 - x_1}\right)\right\}$

Why don't you try some exercises now ?

E 3) What are the equations of the coordinate axes ?

E 4) Find the equation of the line that cuts off an intercept of 1 from the negative direction of the y-axis, and is inclined at 120° to the x-axis.

E 5) What is the equation of a line passing through the origin and making an angle θ with the x-axis?

E 6) a) Suppose we know that the intercept of a line on the x-axis is 2 and on the y-axis is −3. Then show that its equation is

$$\frac{x}{2} - \frac{y}{3} = 1.$$

(**Hint:** See if you can use (7).)

b) More generally, if a line L cuts off an intercept a ($\neq 0$) on the x-axis and b($\neq 0$) on the y-axis (see Fig. 6), then show that its equation is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \dots (8)$$

(8) is called the **intercept form** of the equation of L.

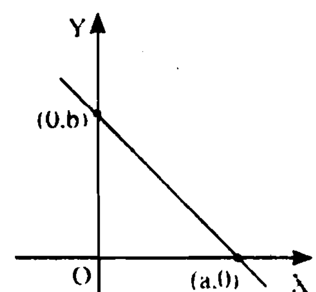


Fig. 6: L is given by $\frac{x}{a} + \frac{y}{b} = 1$

We can obtain the equation of a line in yet another form. Suppose we know the length p of the **perpendicular (or the normal)** from the origin to a line L, and the angle α that the perpendicular makes with the x-axis (see Fig. 7).

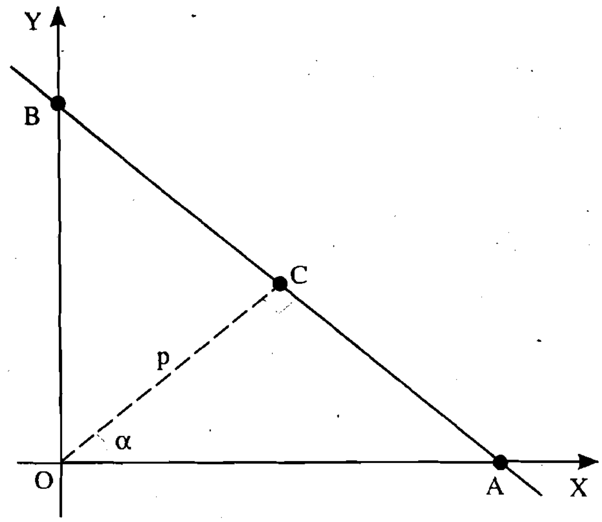


Fig. 7 : x cos α + y sin α = p is the normal form of AB.

Then, using (8) we can obtain the equation of L in the **normal form**

$x \cos \alpha + y \sin \alpha = p$ ...(9)

where p is the length of perpendicular from origin to the line and α is the angle which the perpendicular makes with the positive direction of x-axis.

Further, from Fig. 7.

A = OA = p sec α and

b = OB = p cosec α

using in (8), we get

For example, the line which is at a distance of 4 units from (0, 0), and for which $\alpha = 135°$, has equation $-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = 4$, that is, $x - y + 4\sqrt{2} = 0$.

Here's a small remark about the form (9).

The **distance of a line from a point** is the length of the perpendicular from the point to the line.

Remark 1 : In (9) p is positive and the coefficients of x and y are "normalised", that is, the sum of their squares is 1. Using these facts we can easily find the distance of any line from the origin.

For example, let us find the distance of the origin from the line you got in E4. We rewrite its equation as $-\sqrt{3}\,x - y = 1$. Then we divide throughout by $\sqrt{\sqrt{(3)}^2 + 1}$, to get $\frac{-\sqrt{3}}{2}x - \frac{1}{2}y = \frac{1}{2}$. This is in the form ax + by = c, where $a^2 + b^2 = 1$ and $c \geq 0$. Thus, the required distance is c, which is $\frac{1}{2}$.

Now, have you noticed a characteristics that is common to the equations (3) to (8)? They are all linear in two variables, that is, are of the form $ax + by + c = 0$, where $a, b, c \in$ **R** and at least one of a and b is non-zero. This is not a coincidence, as the following theorem tells us.

Theorem 1: A linear equation in two variables represents a straight line in two-dimensional space. Conversely, the equation of a straight line in the plane is a linear equation in two variables.

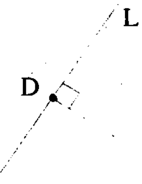


Fig. 8: PD is the distance from P to the line L.

So, for example, $2x + 3y - 1 = 0$ represents a line. What is its slope? We rewrite it as $y = -\frac{2}{3}x + \frac{1}{3}$, to find that its slope is $-\frac{2}{3}$. Do you agree that its intercepts on the x and y axes are $\frac{1}{2}$ and $\frac{1}{3}$, respectively? And what is its distance from the origin? To find this, we "normalise" the coefficients of x and y, that is, we divide the equation throughout by $\sqrt{2^2 + 3^2} = \sqrt{13}$. We get

$\frac{2}{\sqrt{13}}x + \frac{3}{\sqrt{13}}y = \frac{1}{\sqrt{13}}$, which is in the form (9). Thus, the required distance is $\frac{1}{\sqrt{13}}$.

In general the distance of a point $P(x_1, y_1)$ from a line $ax + by + c = 0$ (see Fig. 8) is given by

$$\left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right| \quad \text{... (10)}$$

You may like to try some exercises now.

E 7) Find the distance of (1, 1) from the line which has slope –1 and intercept $\frac{1}{2}$ on the y-axis.

E 8) What is the distance of

a) $y = mx + c$ from (0, 0)?

b) $x = 5$ from (1, 1)?

c) $x \cos \alpha + y \sin \alpha = p$ form $(\cos \alpha, \sin \alpha)$?

d) (0, 0) from $2x + 3y = 0$?

E 9) Prove the equation (9).

Let us now see what the angle between two lines is. Suppose the slope-intercept forms of the lines are $y = m_1x + c_1$ and $y = m_2x + c_2$ (see Fig. 9).

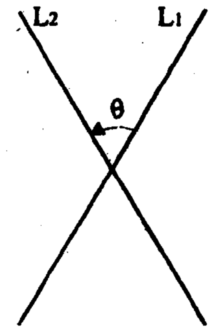


Fig. 9: θ is the angle between the lines L_1 and L_2.

Then the angle θ between them is given by

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}. \quad \text{...(11)}$$

Tan θ can be positive or negative. If it is positive, θ is acute. If $\tan \theta < 0$, then θ is the obtuse angle between the lines (which would be $\pi - \theta$ in Fig. 9).

Note that the constant terms in the equations of the lines play no role in finding the angle between them.

Now, from (11) can you say when two lines are parallel or perpendicular? The conditions follow immediately if you remember what tan 0 and tan $\frac{\pi}{2}$ are. Thus, the lines $y = m_1x + c_1$ and $y = m_2x + c_2$.

i) are parallel if $m_1 = m_2$, and ...(12)

ii) are perpendicular if $m_1m_2 = -1$. ...(13)

For example, $y = 2x + 3$ and $x + 2y = 5$ are perpendicular to each other, and $y = 2x + 3$ is parallel to $y = 2x + c \ \forall \ c \in \mathbf{R}$.

Why not try an exercise now?

E 10) a) Find the equation of the line parallel to $y + x + 1 = 0$ and passing through (0, 0).

b) What is the equation of the line perpendicular to the line obtained in (a), and passing through (2, 1)?

c) What is the angle between the line obtained in (b) and $2x = y$?

Let us now stop our discussion on lines, and move on to more general equations. We shall discuss a concept that will help us to trace the conics in the next unit.

1.3 SYMMETRY

While studying this block you will come across several equations in x and y. Their geometric representations are called **curves**. For example, a line is represented by the equation $ax + by + c = 0$, and a circle with radius a and centre (0, 0) is represented by the equation $x^2 + y^2 - a^2 = 0$.

Note that these equations are of the form F(x, y) = 0, where F(x, y) denotes their left hand sides.

Now suppose the curve C, represented by an equation F(x, y) = 0, is such that when (x, y) lies on it, then so does (x, –y).
Then $F(x, y) = 0 \Rightarrow F(x, -y) = 0$.
(For example, $x^2 + y^2 = a^2 \Rightarrow x^2 + (-y^2) = a^2$).

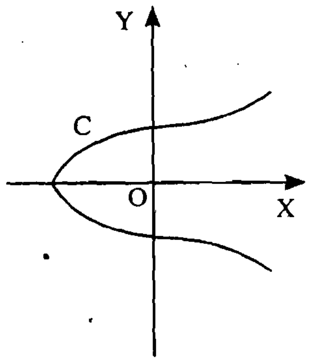


Fig. 10: The curve C is symmetric about the x-axis.

In this situation we say that C is **symmetric about the x-axis**. Similarly, C will be **symmetric about the y-axis** if $F(x, y) = 0 \Rightarrow F(-x, y) = 0$.

We say that C is **symmetric about the origin** (0, 0) if
$F(x, y) = 0 \Rightarrow F(-x, -y) = 0$.

Let us look at an example. The circle $x^2 + y^2 = 9$ is symmetric about both the axes and the origin. On the other hand, the line y = x is not symmetric about any of the axes, but it is symmetric about the origin.

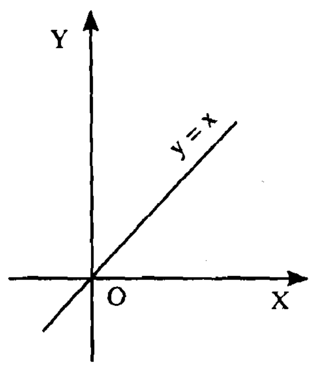


Fig. 11: The line y = x is symmetric about the origin.

Geometrically, if a curve is symmetric about the x-axis, it means that the portion of the curve below the x-axis is the mirror image of the portion above the x-axis (see Fig. 10). A similar visual interpretation is true for symmetry about the y-axis. And what does symmetry about the origin mean geometrically? It means that the mirror image of the portion of the curve in the first quadrant is the portion in the third quadrant, and the mirror image of the portion in the second quadrant is the portion in the fourth quadrant (see Fig. 11).

Why don't you try some exercises on symmetry to see if you have grasped the concept?

E 11) Which axis is the curve $y^2 = 2x$ symmetric about? Is it symmetric about the origin?

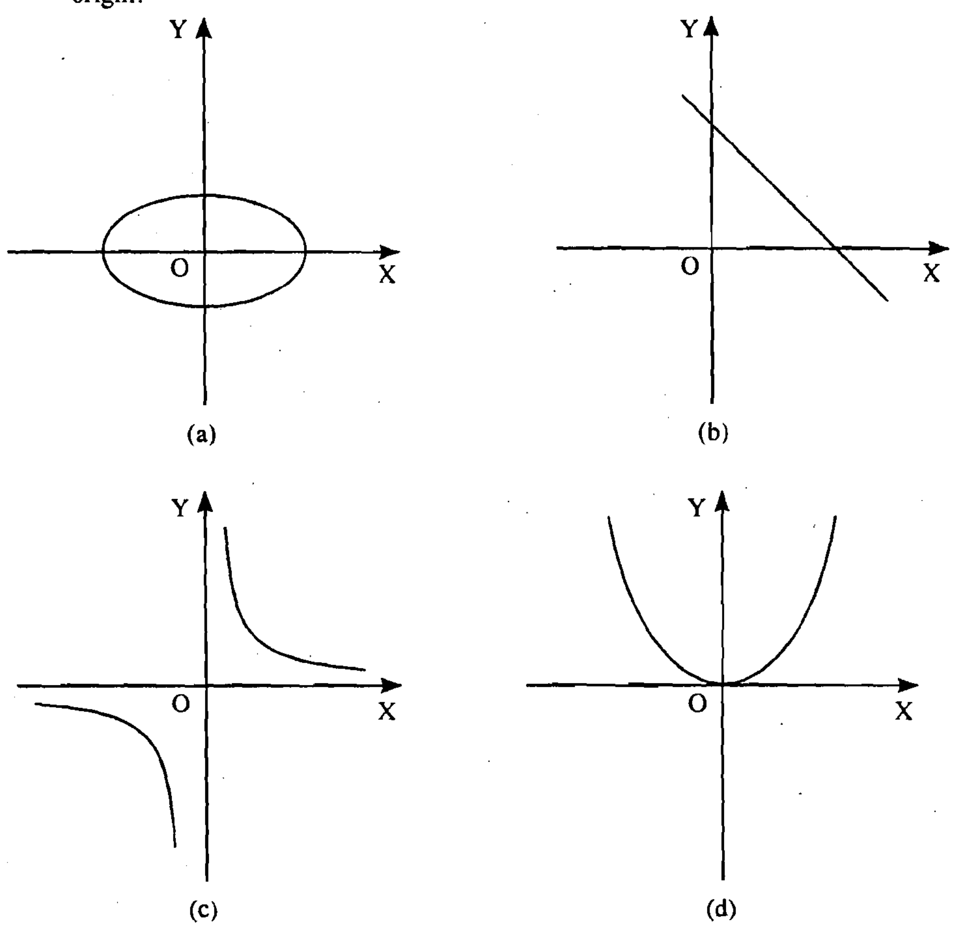


Fig. 12

E 12) Discuss the symmetries of the line $y = 2$.

E 13) Which of the curves in Fig. 12 are symmetric about the x-axis? And which ones are symmetric with respect to the origin?

E14) a) Show that if $F(x, y) = 0$ is symmetric about the x-axis, then $F(x, y) = 0$ iff $F(x, -y) = 0$.

b) Show that if $f(x, y) = 0$ is symmetric about both the axes, then it is symmetric about the origin. Is the converse true?

There is another concept that you will need while studying Units 2 and 3, which we shall now take up.

1.4 CHANGE OF AXES

In the next unit you will see that the general equation of a circle is $x^2 + y^2 + 2ux + 2vy + c = 0$. But we can always choose a coordinate system in which the equation simplifies to $x^2 + y^2 = r^2$, where r is the radius of the circle. To see why this happens, we need to see how to choose an appropriate set of coordinate axes. We also need to know how the coordinates of a point get affected by the transformations to a new set of axes. This is what we will discuss in this section.

There are several ways in which axes can be changed. We shall see how the coordinates of a point in a rectangular Cartesian coordinate system are affected by two types of changes, namely, translation and rotation.

1.4.1 Translating the Axes

The first type of change of axes that we consider is a shift in the origin without changing the direction of the axes.

Let XOY be a rectangular Cartesian coordinate system. Suppose a point O′ has the coordinates (a, b) in this system, what happens if we shift the origin to O′?
Let O′X′, parallel to OX, be the new x-axis. Similarly let O′Y′, parallel to OY, be the new y-axis (see Fig. 13). Now, suppose a point P has the coordinates (x, y) and (x′, y′) with respect to the old and the new coordinate systems, respectively.

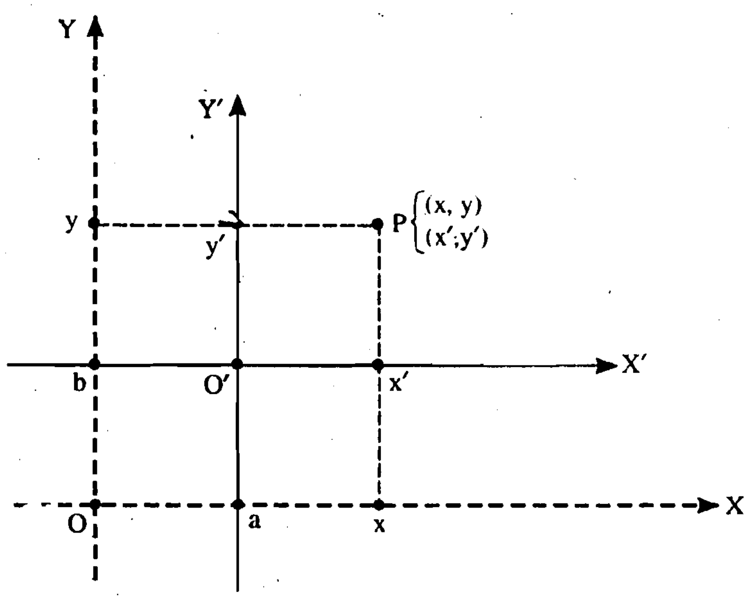


Fig. 13: Translation of axes through (a, b)

How are they related? From Fig. 13 you can see that
$x = x' + a$ and $y = y' + b$. ...(14)

Thus, the new coordinates are given by
$x' = x - a$ and $y' = y - b$. ...(15)

For example, if we shift the origin to (–1, 2), the new (or current) coordinates (x', y') of a point $P(x, y)$ will be given by $x' = x + 1$, $y' = y - 2$.

When we shift the origin, keeping the axes parallel, we say that we are **translating the axes**. So, whenever we translate the axes to a point (a, b), we are transforming the coordinate system to a system with parallel axes through (a, b). We can write this briefly as **transforming to parallel axes through (a, b).**

Now, if we translate the axes to a point (a, b), what will the resultant change in any equation be? Just replace x by $x' + a$ and y by $y' + b$ in the equation and you get the new equation. For example, the straight line $x + 2y = 1$ becomes $(x' + a) + 2(y' + b) = 1$, that is, $x' + 2y' + a + 2b = 1$ in the new system.

Now for some exercises!

E15) If we translate the axes to (–1, 3), what are the new coordinates of the origin of the previous system? Check your answer with the help of a diagram.

E16) Transform the quadratic equation $5x^2 + 3y^2 + 20x - 12y + 17 = 0$ to parallel axes
a) through the point (–2, 2), and
b) through the point (1, 1).

If you've done E16, you would have realised how much simplification can be achieved by an appropriate shift of the origin.

Over here we would like to make an important observation.

Note: When you apply a translation of axes to a curve, the shape of the curve doesn't change. For example, a line remains a line and a circle remains a circle of the same radius. Such a transformation is called a **rigid body motion**.

Now let us consider another kind of change of axes.

1.4.2 Rotating the Axes

Let us now see what happens if we change the direction of the coordinate axes without shifting the origin. That is, we shall consider the transformation of coordinates when the rectangular Cartesian system is rotated about the origin through an angle θ. Let the coordinate system XOY be rotated through an angle θ in the anticlockwise direction

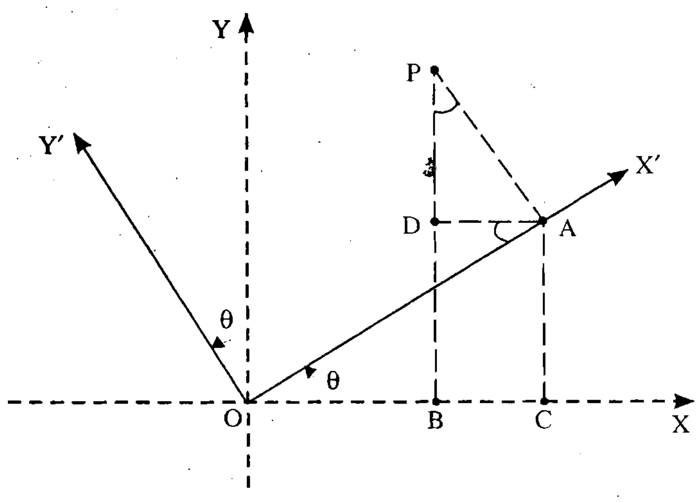


Fig. 14: The axes OX′ and OY′ are obtained by rotating the axes OX and OY through an angle θ.

about O in the XOY plane. Let OX′ and OY′ be the new axes (see Fig. 14). Let P be a point with coordinates (x, y) in the XOY system, and (x′, y′) in the X′OY′ system. Drop perpendiculars PA and PB from P to OX′ and OX, respectively. Also draw AC perpendicular to OX, and AD perpendicular to PB. Then $x = OB$, $y = PB$, $x' = OA$, $y' = PA$.

Also $\angle DAO = \angle AOC = \theta$. Therefore, $\angle DPA = \theta$.

$$\begin{aligned} \text{Thus, } x &= OB = OC - AD \\ &= OA \cos\theta - PA \sin\theta \\ &= x' \cos\theta - y' \sin\theta \qquad \dots (16) \\ \text{and } y &= PB = AC + PD \\ &= x' \sin\theta + y' \cos\theta \qquad \dots (17) \end{aligned}$$

(16) and (17) give us x and y in terms of the new coordinate x′ and y′.

Now, how can we get x′ and y′ in terms of x and y?

Note that the xy-system can be obtained from the x′y′-system by rotating through angle $(-\theta)$. Thus, if we substitute $-\theta$ for θ, x′ by x and y′ by y in (16) and (17), we get x′ and y′ in terms of x and y.

Thus, $x' = x \cos\theta + y \sin\theta$

and $y' = -x \sin\theta + y \cos\theta$ ... (18)

For example, the coordinates of a point P with current co-ordinates (x, y), when the rectangular axes are rotated in the anticlockwise direction through 45°, become

$$x' = x \cos 45^\circ + y \sin 45^\circ = \frac{1}{\sqrt{2}}(x + y)$$

$$y' = -x \sin 45^\circ + y \cos 45^\circ = \frac{1}{\sqrt{2}}(y - x)$$

Now, what happens if we shift the origin and rotate the axes? We will need to apply all the transformations (14) – (17) to get the current coordinates.

For example, suppose we transform to axes inclined at 30° to the original axes, the equation $11x^2 + 2\sqrt{3}\,xy + 9y^2 = 12(x\sqrt{3} + y + 1)$, and then translate the system through $\left(\frac{1}{2}, 0\right)$, what do we get? We first apply (16) and (17), to get

$11(x' - \sqrt{3} - y')^2 + 2\sqrt{3}(x'\sqrt{3} - y')(x' + y'\sqrt{3}) + 9(x' + y'\sqrt{3})^2$ we have multiplied both sides by 4

$= 12\{\sqrt{3}(x'\sqrt{3} - y') + (x' + y'\sqrt{3}) + 1\}$, that is,

$$6\left(x' - \frac{1}{2}\right)^2 + 4y'^2 = 3.$$

Now if we shift the origin to $\left(\frac{1}{2}, 0\right)$ and use (14), we find that the new coordinates (X, Y) are related to (x′, y′) by $x' = X + \frac{1}{2}$, $y' = Y + 0$.

Thus, the equation will become

$$6X^2 + 4Y^2 = 3.$$

Isn't this an easier equation to handle than the one we started with? In fact, both the translation and rotation have been carefully chosen so as to simplify the equation at each stage.

Note: The rotation of axes is a rigid body motion. Thus, when such a transformation is applied to a curve, its position may change but its shape remains the same.

Try these exercises now.

E 17) Write the equation of the straight line $x + y = 1$ when the axes are rotated through 60°.

E 18) a) Suppose the origin is shifted to (–2, 1) and the rectangular Cartesian axes are rotated through 45°. Find the resultant transformation of the equation $x^2 + y^2 + 4x - 2y + 4 = 0$.

b) Now, first rotate the axes through 45° and then shift the origin to (–2, 1). What is the resulting transformation of the equation in (a)?

c) From (a) and (b) what do you learn about interchanging the transformations of axes, (You can study more about this in our course 'Linear Algebra').

So far we have been working with Cartesian coordinates. But is there any other coordinate system that we can use? Let's see.

1.5 POLAR COORDINATES

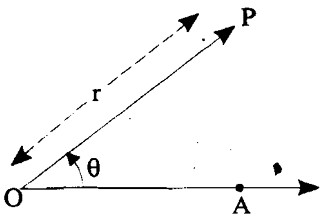


Fig. 15: Polar coordinates.

In the late 17th century the mathematician Bernoulli invented a coordinate system which is different from, but intimately related to, the Cartesian system. This is the **polar coordinate system**, and was used extensively by Newton. You will realise the utility of this system when you study conics in Unit 2. Now, let us see what polar coordinates are.

To define them, we first fix a pole O and a polar axis OA, as shown in Fig. 15. Then we can locate any point P in the plane, if we know the distance OP, say r, and the angle AOP, say θ radians. (Does this remind you of the geometric representation of complex numbers?) Thus, given a point P in the plane, we can represent it by a pair (r, θ), where r is the "directed distance" of P from O and θ is ∠ AOP, measured in radians in the anticlockwise direction. We use the term "directed distance" because r can be negative also. For instance, the point P in Fig. 16 can be represented by $\left(5, \frac{5\pi}{4}\right)$ or $\left(-5, \frac{\pi}{4}\right)$.

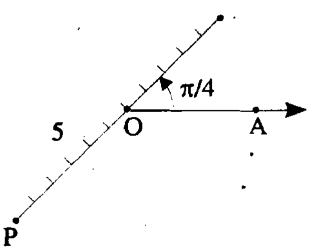


Fig. 16: P's polar coordinates are $\left(-5 - \frac{\pi}{4}\right)$

A point has many different polar coordinates.

Note that by this method the point O corresponds to (0, θ), for any angle θ.

Thus, for any point P, we have a pair of real numbers (r, θ) that corresponds to it. They are called the **polar coordinates**.

Now, if we keep θ fixed, say θ = α, and let r take on all real values, we get the line OP (see Fig. 17), where ∠ AOP = α. Similarly, keeping r fixed, say r = a, and allowing θ to take all real values, the point P(r, θ) traces a circle of radius a, with centre at the pole (Fig. 18). Here note that a negative value of θ means that the angle has magnitude | θ |, but is taken in the clockwise direction. Thus, for example the point $\left(2, -\frac{\pi}{2}\right)$ is also represented by $\left(2, \frac{3\pi}{2}\right)$.

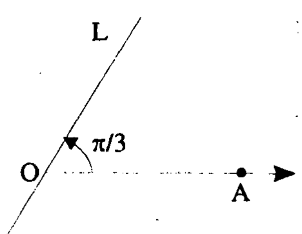


Fig. 17: The line L is given by θ = -/3.

As you have probably guessed, the Cartesian and polar coordinates are very closely related. Can you find the relationship? From Fig. 19 you would agree that the relationship is

$$\left.\begin{aligned} & x = r\cos\theta,\ y = r\sin\theta, \text{ or} \\ & r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\frac{y}{x} \end{aligned}\right\} \qquad \ldots (19)$$

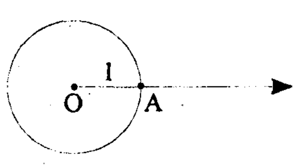


Fig. 18: circle r = 1.

Note that the origin and the pole are coinciding here. This is usually the situation.

We use this relationship often while dealing with equations. For example, the Cartesian equation of the circle $x^2 + y^2 = 25$, reduces to the simple polar form r = 5. So we may prefer to use this simpler form rather than the Cartesian one. As θ is not mentioned, this means θ varies from 0 to 2π to 4π and so on.

Doing the following exercises will help you get used to polar coordinates.

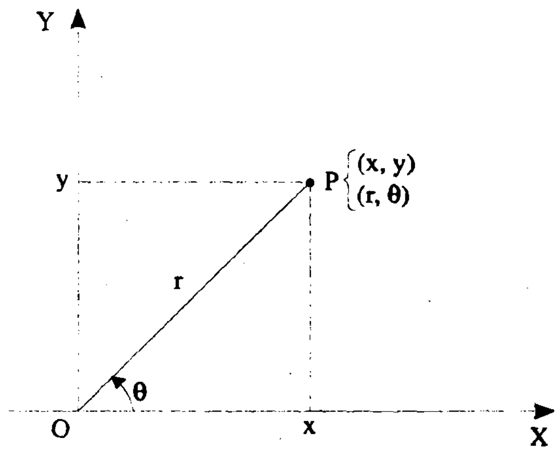


Fig. 19: Polar and Cartesian coordinates.

E 19) From (9) and (19), show that the polar equation of the line AB in Fig. 7 is $r \cos(\theta - \alpha) = p$.

Also see Unit 9 of MTE-01 (Calculus) for more about tracing of curves.

E 20) Draw the graph of the curve $r \cos\left(\theta - \frac{\pi}{4}\right) = 0$, as r and θ vary.

E 21) Find the Cartesian forms of the equations
a) $r^2 = 3r \sin\theta$
b) $r = a(1 - \cos\theta)$, where a is a constant.

Apart from the polar coordinate system, we have another method of representing points on a curve. This is the representation in terms of a parameter. You will come across this simple method in the next unit, when we discuss each conic separately.

Let us now summarise that we have done in this unit

1.6 SUMMARY

In this unit we have briefly run through certain elementary concepts of two-dimensional analytical geometry. In particular, we have covered the following points :

1) The distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

2) The distance between (x_1, y_1) and the line $ax + by + c = 0$ is
$$\left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

3) Any line parallel to the x-axis is $y = a$, and parallel to the y-axis is $x = b$, for some constants a and b.

4) The equation of a line in
i) slope-intercept form is $y = mx + c$,
ii) point-slope form is $y - y_1 = m(x - x_1)$,
iii) two-point form is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$,
iv) intercept form is $\frac{x}{a} + \frac{y}{b} = 1$,
v) normal form is $x \cos\alpha + y \sin\alpha = p$.

5) The angle between two lines with slopes m_1 and m_2 is
$$\tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right).$$

The lines are parallel if $m_1 = m_2$, and perpendicular if $m_1 m_2 = -1$.

6) Symmetry about the coordinate axes and the origin.

7) i) If we translate the axes to (a, b), keeping the directions of the axes unchanged, the new coordinates x′ and y′ are given by $x' = x - a$ and $y' = y - b$.
ii) If we rotate the axes through an angle θ, keeping the origin unchanged, the new coordinates x′ and y′ are given by
$x' = x \cos \theta + y \sin \theta$
$y' = -x \sin \theta + y \cos \theta$.

8) A point P in a plane can be represented by a pair of real numbers (r, θ), where r is the directed distance of P from the pole O and θ is the angle that OP makes with the polar axis, measured in radians in the anticlockwise direction. These are the polar coordinates of P. They are related to the Cartesian coordinates (x, y) of P by $r^2 = x^2 + y^2$ and
$\theta = \tan^{-1} \frac{y}{x}$

In the next unit we shall start our study of ellipses and other conics. But before going to it, **please make sure that you have achieved the unit objectives listed in Sec. 1.1.** One way of checking is to ensure that you have done all the exercises in the unit. Our solutions to these exercises are given in the following section.

1.7 SOLUTIONS/ANSWERS

E 1) a) $\left(\frac{5-3}{2}, \frac{-4+2}{2}\right) = (1, -1)$.

b) $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)$

E 2) PQ= $\sqrt{1-(-2)^2+(0-3)^2} = \sqrt{18}$

QR = $\sqrt{(-2-1)^2+(3-3)^2} = 3$

PR= $\sqrt{(1-1)^2+(0-3)^2} = 3$

Thus, the sides of the triangle are not equal in length.
Hence, ΔPQR is not equilateral.

E 3) The x and y-axis are $y = 0$ and $x = 0$, respectively.

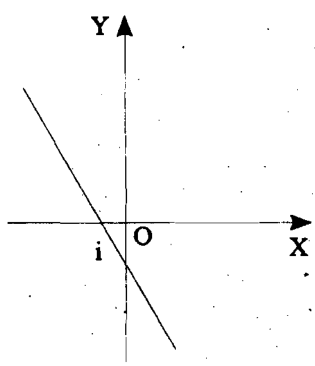


Fig. 20: $y = -(\sqrt{3}\,x + 1)$.

E 4) In Fig. 20 we have drawn the line. Its equation is $y = mx + c$, where $c = -1$ and $m = \tan 120° = -\sqrt{3}$.
Thus, the required equation is
$y = -(\sqrt{3}\,x + 1)$.

E 5) Here c = 0. Thus, the equation is $y = x \tan \theta$.

E 6) a) (2, 0) and (0, –3) lie on the line. Thus, its two-point form is
$\frac{y-0}{-3-0} = \frac{x-2}{0-2}$, that is, $2y = 3(x-2)$.

b) (a, 0) and (0, b) lie on the line. Thus, its equation is
$\frac{y-0}{b-0} = \frac{x-a}{0-a} \Rightarrow \frac{x}{a} + \frac{y}{b} = 1.$

E 7) The equation of the line is $y = -x + \frac{1}{2}$, that is, $2x + 2y - 1 = 0$.
The distance of (1, 1) from this line is
$\left|\frac{2.1+2.1-1}{\sqrt{4+4}}\right| = \frac{3}{\sqrt{8}}$

E 8) a) $\left|\dfrac{m.0-0+c}{\sqrt{m^2+1}}\right| = \dfrac{3}{\sqrt{m^2+1}}$.

b) $\left|\dfrac{1-5}{1}\right| = 4.$

c) $|1-p|$.

d) 0.

E 9) Using the intercept form (9) and Fig. 7, we see that the equation of the line is

$$\frac{x}{OA}+\frac{y}{OB}=1. \qquad \text{...(20)}$$

Now, $\angle OAC = \dfrac{\pi}{2} - \alpha$ and $\angle OBC = \alpha$. Thus,

$$OA = OC \operatorname{cosec}\left(\frac{\pi}{2}-\alpha\right) = p \sec\alpha = \left(\frac{p}{\cos\alpha}\right), \text{ and}$$

$$OB = OC \operatorname{cosec}\alpha = \frac{p}{\sin\alpha}.$$

Thus, (20) $\Rightarrow \dfrac{x\cos\alpha}{p}+\dfrac{y\sin\alpha}{p}=1$

$\Rightarrow x\cos\alpha + y\sin\alpha = p.$

E 10) a) Any line parallel to $y + x + 1 = 0$ is of the form $y + x + c = 0$. where $c \in \mathbf{R}$. Since (0, 0) lies on it, $0 + 0 + c = 0$, that is, $c = 0$. Thus, the required line is $y + x = 0$.

b) The slope of the line $y + x = 0$ is -1. Thus, the slope of any line perpendicular to it is 1, by (13). Thus, the equation of the required line is of the form $y = x + c$, where $c \in \mathbf{R}$. Since (2, 1) lies on it, $1 = 2 + c \Rightarrow c = -1$. Thus, the required line is $y = x - 1$.

c) In this case $m_1 = 1$, $m_2 = 2$. Thus, the angle between the lines is

$$\theta = \tan^{-1} = \left(\frac{1-2}{1+1\times 2}\right) = \tan^{-1}\left(-\frac{1}{3}\right) = -\tan^{-1}\frac{1}{3}.$$

Note that both $-\tan^{-1}\left(-\frac{1}{3}\right)$ and $\tan^{-1}\left(-\frac{1}{3}\right)$ are angles between the given lines.

E 11) If we substitute y by $(-y)$ in the given equation, it remains unchanged. Thus, the curve is symmetric about the x-axis. If we substitute x by $(-x)$, the curve changes to $y^2 = -2x$, Thus, it is not symmetric about the y-axis.

If we substitute $(-x)$ and $(-y)$ for x and y, respectively, in the equation, it changes to $y^2 = -2x$. Thus, it is not symmetric about the origin.

E 12) It is not symmetric about either axis or the origin.

E 13) (a) is symmetric with respect to the x-axis.
(a) and (d) are symmetric with respect to the y-axis.
(a) and (c) are symmetric with respect to the origin.

E 14) a) The curve is symmetric about the x-axis. Thus,
$f(x, y) = 0 \Rightarrow f(x, -y) = 0 \ \forall\ x, y \in \mathbf{R}$.
$\therefore F(x, -y) = 0 \Rightarrow F(x, -(-y)) = 0 \Rightarrow F(x, y) = 0 \ \forall\ x, y \in \mathbf{R}$.
Hence, the equivalence.

b) The curve is symmetric about both the axes.
Now, $F(x, y) = 0$
$\Rightarrow F(x, -y) = 0$, because of symmetry about the x-axis.
$\Rightarrow F(-x, -y) = 0$, because of symmetry about the y-axis.

$\Rightarrow$ F is symmetric about the origin.
The converse is clearly not true, as you can see from Fig. 11.

E15) In Fig. 21 we show the new and old systems.

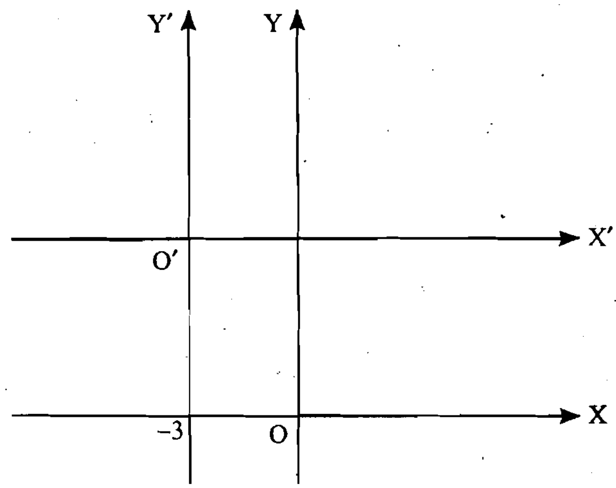


Fig. 21: The coordinates of O are (1, −3) with respect to X'O'Y'.

E16) a) If the new coordinates are x' and y', then x = x' − 2, y = y' + 2. Thus, the equation becomes

$$5(x' - 2)^2 + 3(y' + 2)^2 + 20(x' - 2) - 12(y' + 2) + 17 = 0$$

$$\Rightarrow 5x'^2 + 3y'^2 - 15 = 0$$

$$\Rightarrow \frac{x'^2}{3} + \frac{y'^2}{5} = 1.$$

b) The equation becomes

$$5(x' + 1)^2 + 3(y' + 1)^2 + 20(x' + 1) - 12(y' + 1) + 17 = 0$$

$$\Rightarrow 5x'^2 + 3y'^2 + 30x' - 6y' + 33 = 0.$$

E17) Here $x = \frac{x'}{2} + \frac{y'\sqrt{3}}{2}$ and $y = \frac{x'\sqrt{3}}{2} + \frac{y'}{2}$

Thus, x + y = 1, becomes

$$\left(\frac{x'}{2} - \frac{y'\sqrt{3}}{2}\right) + \left(\frac{x'\sqrt{3}}{2} + \frac{y'}{2}\right) = 1, \text{ that is,}$$

$$x'(1 + \sqrt{3}) + y'(1 - \sqrt{3}) = 2.$$

E18) a) By shifting the origin, the new coordinates x' and y' are related to x and y by x = x' − 2, y = y' + 1.
Thus, the equation becomes

$$(x' - 2)^2 + (y' + 1)^2 + 4(x' - 2) - 2(y' + 1) + 4 = 0$$

$$\Rightarrow x'^2 + y'^2 = 1 \qquad \text{...(21)}$$

Now, rotating the axes through 45°, we get coordinates x and y given by,

$$x' = \frac{X - Y}{\sqrt{2}} \text{ and } y' = \frac{X + Y}{\sqrt{2}}$$

Thus, (21) becomes

$$\left(\frac{X - Y}{\sqrt{2}}\right)^2 + \left(\frac{X + Y}{\sqrt{2}}\right)^2 = 1$$

$$\Rightarrow X^2 - 2XY + Y^2 + X^2 + 2XY + Y^2 = 2.$$

$$\Rightarrow X^2 + Y^2 = 1$$

b) If we first rotate the axes, the given equation becomes

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2 + \left(\frac{x'+y'}{\sqrt{2}}\right)^2 + 4\left(\frac{x'-y'}{\sqrt{2}}\right) - 2\left(\frac{x'+y'}{\sqrt{2}}\right) + 4 = 0$$

$$\Rightarrow x'^2 + y'^2 + 2\sqrt{x'} - 3\sqrt{2y'} + 4 = 0$$

Now, shift the origin to (–2, 1), the equation (2) becomes

$$X^2 + Y^2 + X(\sqrt{2} - 4) + Y(2 - 3\sqrt{2}) + 9 - 5\sqrt{2} = 0.$$

c) From (a) and (b) you can see that a change in the order of transformations makes a difference. That is, if T_1 and T_2 are two transformations, then T_1 followed by T_2 need not be the same as T_2 followed by T_1. Diagrammatically, the circles C_1 and C_2 in Fig. 22 correspond to the final equations in (a) and (b), respectively.

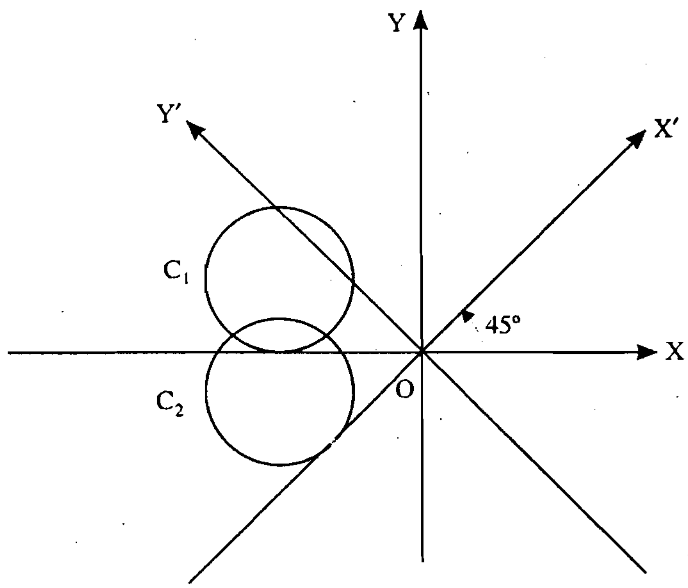


Fig. 22

E19) From (9), the equation of L is
$x \cos \alpha + y \sin \alpha = p.$
Using (19), this becomes
$r(\cos \theta \cos \alpha + \sin \theta \sin \alpha) = p.$
$\Rightarrow r \cos(\theta - \alpha) = p.$

E20)

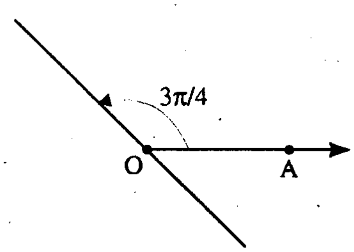


Fig. 23 : The line $r \cos\left(\theta - \frac{\pi}{4}\right) = 0$.

E21) a) Since $r^2 = x^2 + y^2$ and $y = 4 \sin \theta$, the equation becomes $x^2 + y^2 = 3y$.
b) The equation becomes

$$\sqrt{x^2 + y^2} = a\left(1 - \frac{x}{x^2 + y^2}\right)$$

$$\Rightarrow x^2 + y^2 + a(x + \sqrt{x^2 + y^2}) = 0.$$