UNIT 2 COMPLEX NUMBERS

Structure

2.1 INTRODUCTION

In your studies so far you must have dealt with numbers, integers, rational numbers and real numbers. You also know that a shortcoming in N led mathematicians of several centuries ago to define negative numbers. Hence, the set Z was born. For similar reasons Z was extended to Q and Q to R at various stages in history. Then came a point when mathematicians looked for solutions of equations like $x^2 + 1 = 0$. Since $x^2 + 1 = 0$ has no solution in R, for a long time it was accepted that this equation has no solution. The Indian mathematicians Mahavira (in 850 A.D.) and Bhaskara (in 1150 A. D.) clearly stated that the square root of a negative quantity does not exist. Then, in the 16th century the Italian mathematician Cardano tried to solve the quadıatic equation $x^2 - 10x + 40 = 0$. He found that $x_1 = 5 + \sqrt{-15}$ and $x_2 = 5 + \sqrt{-15}$ satisfied the equation. But then, what is $\sqrt{-15}$? He, and other mathematicians, tried to give this expression some meaning. Even while making mathematical models of real life solutions, the mathematicians of the 17th and 18th centuries were coming across more and more examples of equations which had no real roots. To overcome this shortcoming the concept of a complex number slowly came into being. It was the famous mathematician Gauss (1777-1855) who used and popularised the name 'complex number' for numbers of the type $5 + \sqrt{-15}$.

In the early 1800s, a geometric representation of complex numbers was developed. This representation finally made complex numbers acceptable to all mathematicians. Since then complex numbers have seeped into all branches of mathematics. In fact, they have even been necessary for developing several areas in modern physics and engineering.

In this unit we aim to familiarise you with complex numbers and the different ways of representing them. We shall also discuss the basic algebraic operations on complex numbers. Finally, we shall acquaint you with a very useful result, namely, De Moivre's theorem. It has several applications. We shall discuss only two of them in some detail.

We would like to reiterate that whatever mathematics course you study, you will need the knowledge of the subject matter covered in this unit. So please go through it carefully and ensure that you have achieved the following objectives.

Objectives

After studying this unit you should be able to :

- define a complex number ;
- describe the geometrical, polar and exponential representations of a complex number ;
- apply the various algebraic operations on complex numbers ;
- prove and use de Moivre s theorem.

2.2 WHAT A COMPLEX NUMBERS IS

When you consider the linear equation $2x + 3 = 0$, you know that it has a solution, namely $x = \frac{-3}{2}$. But, can you always find a real solution of the equation $ax + b = 0$, where $a, b \in \mathbf{R}$ and $a \neq 0$? Is the required solution $x = \frac{-b}{a}$? It is, since $\left(\frac{-b}{a}\right) + b = 0$

Now, what happens if we try to look for real solutions of any quadratic equation over **R** ? Consider one such equation namely $x^2 + 1 = 0$, that is $x^2 = -1$. This equation has no solution in R since the square of any real number must be non-negative.

From about 250 A. D. onwards, nathematicians have been coming across quadratic equations, arising from real life situations, which did not have any real solutions. It was in the 16th century that the Italian mathematicians Cardano and Bombelli started a serious discussion on extending the number system to include square roots of nagative numbers. In the next two hundred years more and more instances were discovered in which the use of square roots of negative numbers helped in finding the solutions of real problems.

In 1777 the Swiss mathematician Euler introduced the "imaginary unit", which he denotes by the Greek letter iota, that is i. He defined $i = \sqrt{-1}$. Soon after, the great mathematician Carl Friedrich Gauss introduced the term complex numbers for numbers such as

$1 + i\ (=1 + \sqrt{-1})$ or $-2 + i\sqrt{5}\ (=-2 + \sqrt{-5}\ )$.

Nowadays these numbers are accepted and used in every field of mathematics.

Let us define a complex number now.

Definition : A complex number is a number of the form $x + iy$, where x and y are real numbers and $i^2 = -1$.

We say that x is the **real part** and y is the **imaginary part** of the complex number $x + iy$.

We write $x = \text{Re}(x + iy)$ and $y = \text{Im}(x + iy)$.

Caution : i) i is **not** a real number.

ii) $\text{Im}(x + iy)$ is the real number y, and not iy.

We denote the set of all complex numbers by C.

So, $C = \{x + iy \mid x, y \in \mathbf{R}\}$.

By convention, we will usually denote an element of C by z. So, whenever we will talk of a complex number z, we will mean $z = x + iy$ for some $x, y \in \mathbf{R}$. In fact, $z = \text{Re}\, z + i\, \text{Im}\, z$.

There is another convention that we follow while writing complex numbers, which we give in the following remark.

Remark 1: When you go through Sec. 2.4. 2, you will see that $iy = yi\ \forall\ y \in \mathbf{R}$. That is why we can write the complex number $x + iy$ as $x + y\,i$ also.

By convention, we write any complex number $x + iy$ for which $y \in \mathbf{Q}$, as $x + yi$. For example, we prefer to write $2 + i$, $2 + \frac{3}{2}i$ and $2 + \frac{5}{9}i$ for $2 + i1$, $2 + i\frac{3}{2}$ and $2 + i\frac{5}{9}$, respectively.

But if $Z \in C$ is the form $z = a + i\sqrt{b}$, $b \in \mathbf{R}$, then we prefer to write z in this form and not as $z = a + \sqrt{b}\, i$.

Now that you know what a complex number is, would you agree that the following numbers belong to C ?

$5 + \sqrt{-15}$, $3i$, $\sqrt{2}$, $\sqrt{-2}$

Each of them is a complex number because

$5 + \sqrt{-15} = 5 + i\sqrt{15}$

$3i = 0 + i3$

$\sqrt{2} = \sqrt{2} + i0$

$\sqrt{-2} = 0 + i\sqrt{2}$

From these examples you may have realised that some complex numbers can have their real part or their imaginary part equal to zero. We have names for such numbers.

$\sqrt{-a} = i\sqrt{a}\ \forall\ a \geq 0.$

Definition : Consider a complex number $z = x + iy$.

If $y = 0$ we say z is purely real.

If $x = 0$, we say z is purely imaginary.

We usually write the purely real number $x + oi$ as x only, and write the purely imaginary number $0 + iy$ as iy only.

Try these exercises now.

E1) Complete the following table :

z	Re z	Im z
$\frac{1 + \sqrt{-23}}{2}$		
i	0	0
$\frac{-1 + \sqrt{3}}{5}$		

E2) Is $\mathbf{R} \subseteq C$? Why

Now, given any complex number, we can define a related complex number in a very natural way, as follows.

Definition : Let $z = x + iy \in C$. We define the **complex conjugative** (or simply the conjugate) of z to be the complex number

$\bar{z} = x - iy.$

Thus, $\text{Re}\ \bar{z} = \text{Re}\ z$ and $\text{Im}\ \bar{z} = -\text{Im}\ z.$

For example, if $z = 15 + i$ then $\bar{z} = 15 - i$.

Try this simple exercise now.

E3) Obtain the conjugates of
$2 + 3i, 2 - 3i, 2, 3i$

In section 2.4.2 you will see one important use of the complex conjugate.
So far we have shown you an algebraic method of representing complex numbers. Now let us consider a geometrical way of doing so.

2.3 GEOMETRICAL REPRESENTATION

You know that we can geometrically represent real numbers on the number line. In fact there is a one - one correspondence between real numbers and points on the number line. You have also seen that a complex number is determined by two real numbers, namely, its real and imaginary parts. This observation led the mathematicians Wessel and Gauss to think of representing complex numbers as points in a plane. This geometric representation was given in the early 1800. It is called an **Argand diagram,** after the Swiss mathematician J. R. Argand, who propagated this idea.

Let us see what an Argand diagram is.

Take a rectangular set of axes OX and OY in the XOY plane. Any point in the plane is determined by its Cartesian coordinates. Now we consider any complex number x + iy. We represent it by the point in the plane with Cartesian coordinates (x, y). This representation is an Argand diagram. For example in Figure 1, P represents the complex number 2 + 3i, whose real part is 2 and imaginary part is 3. And what number does P′ represent? P′ corresponds to 2 – 3i.

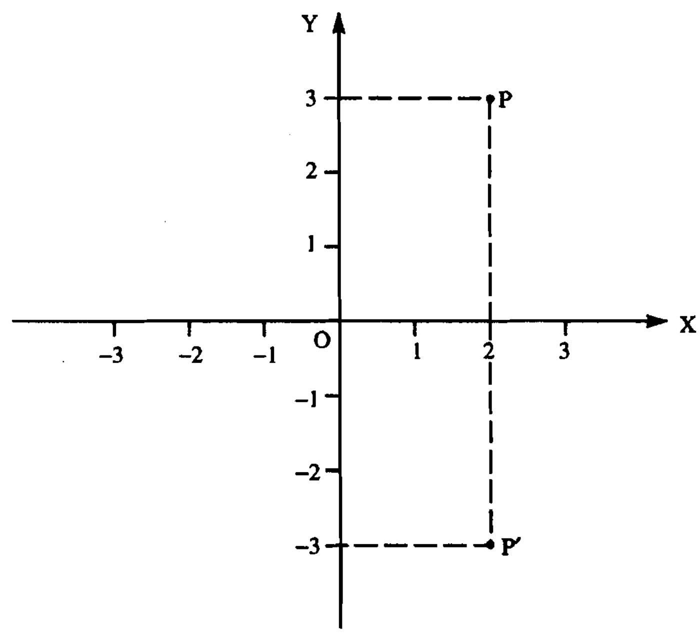

Fig. 1 : An Argand diagram

You may have realised that in an Argand diagram the purely real numbers lie along the x-axis and the purely imaginary numbers lie along the y-axis.

So, you have seen that, given $x + iy \in C$ we associate with it the unique point $(x,y) \in \mathbf{R}^2$. The converse is also true. That is given $(x, y) \in \mathbf{R}^2$, we can associate with it the unique complex number x + iy. This means that the following definition of a complex number is equivalent to our previous definition.

Definition : A complex number is an ordered pair of real numbers. In the language of sets, we can say that $\mathbf{C} = \mathbf{R} \times \mathbf{R}$.

With the help of this definition can you say when two complex numbers are equal ?

Definition : We say that two complex numbers (x_1, y_1) and (x_2, y_2) are **equal** iff $x_1 = x_2$ and $y_1 = y_2$.

In other words, $x_1 + iy_1 = x_2 + iy_2$ iff $x_1 = x_2$ and $y_1 = y_2$.

Thus, two elements of C are equal iff their real parts are equal and their imaginary parts are equal.

So, for example, $\frac{-1+\sqrt{-3}}{2} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, but

$\frac{-1+\sqrt{-3}}{2} \neq \frac{-1}{2} + i\frac{1}{2}$.

Try these exercises now.

E4) a) Plot the following elements of C in an Argand diagram :

$3, -1+i, \overline{-1+i}, i$

b) Plot the sets $S_1 = \{2+iy \mid y \in \mathbf{R}\}$, $S_2 = \{x+3i \mid x \in \mathbf{R}\}$ and $S_3 = \{x+ix \mid \in \mathbf{R}\}$ in an Argand diagram.

E5) Write down the elements of C represented by the points $\left(\frac{-1}{2}, \frac{1}{3}\right)$, $(2, 0)$ $(0, 2)$in the plane.

E6) For what values of k and m is $k+3i = \frac{1}{2} + im$?

While solving E4 you may have observed that **in an Argand diagram the point that represents $\bar{z}$ is the reflection in the x-axis of the point that represent z, for any $z \in C$.**

Here are two more exercises about complex conjugates.

E7) For which $z \in C$ will $z = \bar{z}$?

E8) For any $z \in C$, show that $\bar{\bar{z}} = z$, that is, the conjugate of the conjugate of z is z.

Now consider any non-zero complex number $z = x + iy$. We represent it by P in the Argand diagram in Figure 2. We call the distance OP the modulus of z, and denote it by $|z|$.

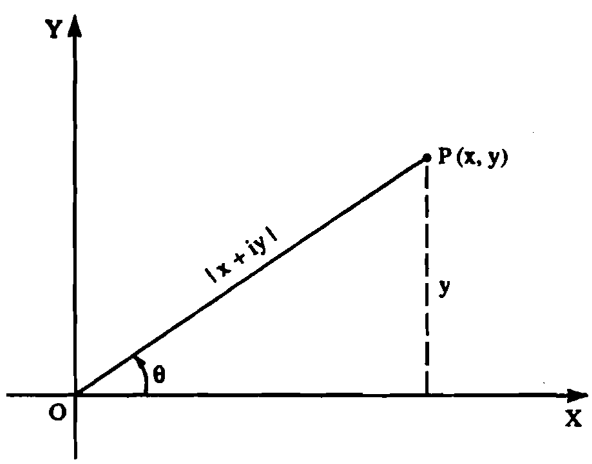

Fig. 2: Modulus and argument

Using the pythagoras theorem, we see that

$|z| = \sqrt{x^2 + y^2}$.

If z is real, what is $|z|$? It is just the absolute value of z.

Here's another important remark on the modulus.

Remark 2 : $Z \in C$, but $|z| \in R$

$z = 0$ iff $|z| = 0$

Now, if you see Figure 2 again, you will see that $\angle XOP = \theta$. We call θ an argument of $z = x + iy$.

For $z = 0$, $|z| = 0$ and its **argument is not defined.**

Now if $z \in C, z \neq 0$, will it have a unique argument ? If θ is an argument so are $2\pi + \theta, 4\pi + \theta$, etc. If we insist the θ lie in the range $-\pi < \theta \leq \pi$, then we get a unique argument. We call this value of θ **the argument** of z, and denote it by **Arg z.**

If, in Figure 2, we write $|z| = r$. Then you can see that $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$.

$\therefore x = r\cos\theta, y = r\sin\theta$. ......(1)

So, we can also write z as

$z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \text{Arg } z$.

This is called the **polar form** of z.

Note that, give $z = x + iy$ we can use (1) to obtain $\text{Arg } z = \tan^{-1}\left(\frac{y}{x}\right)$. However, as more than one angle between $-\pi$ and π have the same tan value, we must draw in argand diagram to find the right value of Arg z.

Let us look at an example.

Example 1 : (a) Obtain the modulus and argument of $1 + i$.

(b) Obtain z, if $|z| = 2$ and $\text{Arg } z = = \frac{\pi}{3}$.

Solution : (a) Let $z = 1 + i$.

Then Re $z = 1$, Im $z = 1$. Thus, $1 + i$ corresponds to $(1, 1)$, which lies in the first quadrant. We find that

$$|z| = \sqrt{(\text{Re } z)^2 + (\text{Im } z)^2} = \sqrt{1^2 + 1^2} = \sqrt{2}, \text{ and}$$

$$\text{Arg } z = \tan^{-1}\left(\frac{\text{Im } z}{\text{Re } z}\right) = \tan^{-1}(1) = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

Since z lies in the first quadrant, Arg z must be between 0 and $\frac{\pi}{2}$.

Thus, $\text{Arg } z = \frac{\pi}{4}$.

b) $z = |z|\,(\cos(\text{Arg } z) + i\sin(\text{Arg } z))$

$$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i.\frac{\sqrt{3}}{2}\right)$$

$$= 1 + i\sqrt{3}$$

Try the following exercises now.

E9) Write down the polar forms of the complex numbers listed in E4 (a).

E10) Show that $\{z \in C \mid |z| = 1\}$ is the equation of the circle $x^2 + y^2 = 1$ in R^2, and vice versa.

There is yet another way of representing a complex number. In fact this method is closely related to the polar representation. It uses the expression e^z, where $z \in C$. Let us define this expression.

Definition : For any $z = x + iy \in C$, we define

$e^z = e^x (\cos y + i \sin y)$.

In particular, if $z = iy$, a purely imaginary number, then we get

$$\boxed{\text{Euler's formula : } e^{iy} = \cos y + i \sin y \;\; \forall \in \mathbf{R}}$$

This formula is due to the famous Swiss mathematician Leonhard Euler. You will be using it quite often while dealing with complex numbers.

Now consider any $z \in C$. We write it in its polar form,

$z = r(\cos\theta + i \sin\theta)$.

Now, using Euler's formula we find that

$z = re^{i\theta}$.

This is the exponential form of the complex number z.

Fig. 3: Euler (1707-1783)

For example, the exponential from of $z = \frac{3\sqrt{3}}{2} + \frac{3i}{2}$

is $3\, e^{i\pi/6}$, since $|z| = 3$ and $\text{Arg } z = \frac{\pi}{2}$.

Try this exercise now.

E 11) Write the following complex numbers in polar form and exponential from:

$\sqrt{\frac{5}{2}} + i\sqrt{\frac{5}{2}}$, $1 + i$, -1, i.

By now you must be thoroughly familiar with the various ways of representing a complex number. Let us now discuss some operations on complex numbers.

2.4 ALGEBRAIC OPERATIONS

In this section we will discuss the addition, subtraction, multiplication and division of complex numbers. Let us first consider '+ ' and '–' in C.

2.4.1 Addition and Subtration

We will now define addition in C using the definition of addition in R.

Definition: The sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is the complex number $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, that is

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

Let us look at an example.

Example 2 : Find the sum of

i) $3 + i$ and $-2 + 4i$,

ii) -5 and $5 - i$.

Solution : i) $(3 + i) + (-2 + 4i) = (3 + (-2)) + i(1 + 4)$

$= 1 + 5i$

ii) $(-5)+(5-i) = (-5+0i)+(5-i) = (-5+5)+i(0-1)$

$= 0+i(C-1)$

$= -i.$

Have you observed that any complex number is the sum of a purely real and a purely imaginary number? This is because $x+iy = (x+0i) + (0+iy)$.

In the following exercises we ask you to verify some very important properties of addition in C.

E 12) a) Find the sum of $2+3i$ and $\overline{2+3i}$.

b) Show that $z + \bar{z} = 2 \operatorname{Re} z$ for any $z \in C$.

Addition in C is commutative and associative.

E 13) Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \ \forall \ z_1, z_2 \in C$.

E 14) a) Show that $z_1 + z_2 = z_2 + z_1$ for any $z_1, z_2 \in C$.

b) Show that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for any $z_1, z_2, z_3 \in C$.

E 15) Find the element $a + ib \in C$ such that

$z + (a+ib) = z \ \forall \ z \in C$.

If you have solved these exercises, you must have realised that the addition in C satisfies most of the properties that addition in **R** satisfies. Also, because of what you proved in E15, we say that $0 + i0$ (= 0) is the additive identity in **C**.

Now, can you define subtraction in C? We give you a very natural definition.

Definition : The **difference** $z_1 - z_2$ of two complex numbers $z_1 = x_1 + iy_1$ and

$z_2 = x_2 + iy_2$ is $z_1 + (-z_2)$,

where $-z_2 = (-x_2) + i(-y_2)$.

Thus, $z_1 - z_2 = z_1 + (-z_2)$

$= (x_1 - iy_1) + [(-x_2) + i(-y_2)]$

$= (x_1 - x_2) + i(y_1 - y_2)$.

So, what do you think **z − z** is, for any **z ∈ C**?

Let's see. Take $z = x + iy$. Then

$z - z = (x - x) + i(y - y) = 0$, the additive identity in C.

For any $z \in C$, $(-z)$ is the additive inverse of z.

Try the following exercise now.

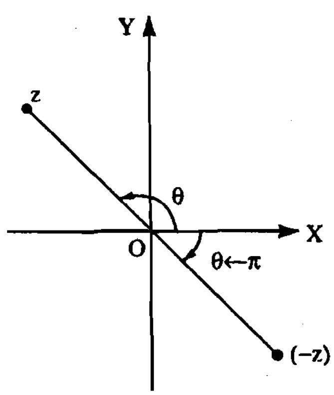

Fig. 4 : z and − z

E 16) Find $(-6+3i) - (-3-2i)$.

E 17) Find $z - \bar{z}$, for any $z \in C$

E 18) Find the relationship between

a) $|z|$ and $|-z|$,

b) Arg z and Arg $(-z)$,

for any $z \in C$. (see Figure 4.)

We will now make a brief remark on the graphical representation of the sum of complex numbers.

Remark 3 : The addition of two complex numbers has an important geometrical representation. Consider an Argand diagram (Figure 5) in which we represent two complex numbers (x_1, y_1) and (x_2, y_2) by the points P and Q.

If we complete the parallelogram whose adjacent sides are OP and OQ, the fourth vertex R represents the sum $(x_1, y_1) + (x_2, y_2)$.

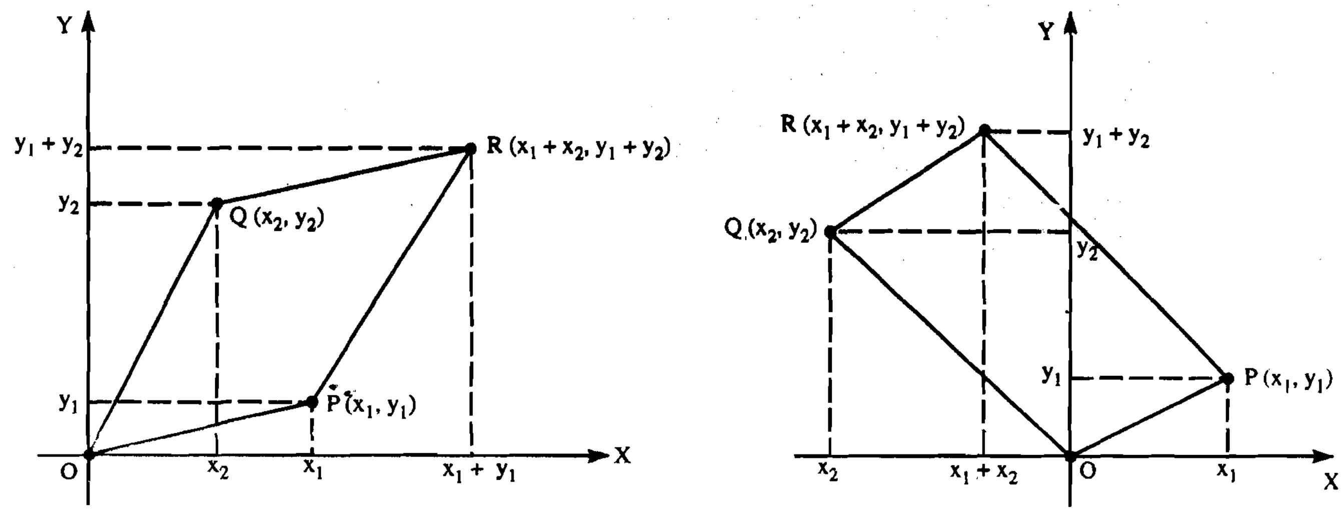

Fig. 5: Addition in C.

In vector algebra you will come across a similar parallelogram law of addition.

So far you have seen how naturally we have defined addition (and subtraction) in C by using addition (and subtraction) in **R**. Let us see if we can do the same for multiplication.

2.4.2 Multiplication And Division

We will now use multiplication in **R** to define multiplication in C. But the route is slightly circuitous. Consider the following product of two linear polynomials a + bx and c + dx, where a, b, c, d $\in$ **R**.

$(a + bx)\ (c + dx) = ac + (ad + bc)\, x + bd\, x^2.$

Now, if we put x = i in this, we get

$(a + ib)\,(c + id) = (ac - bd) + i\,(ad + bc)$, since $i^2 = -1$.

This is the way we shall define a product in C.

Definition : The product $z_1 z_2$ of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is the complex number

$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1).$

Or, in the language of ordered pairs.

$(x_1, y_1).\,(x_2, y_2) = (x_1 x_2 - y_1 y_2\,, x_1 y_2 + x_2 y_1).$

For example,

$(1, 2)\,(-3\,, 2) = [1.\,(-3) - 2.\,2,\ 1.2 + (-3).\,2].$

$= (-7, -4).$

Let us check and see what i^2 is according to this definition.

$i^2 = i.i = (0 + i)\,(0 + i) = (\,0 - 1) + i\,(0 - 0) = -1$, which is as it should be!

Multiplication has several properties, which you will discover if you try the following exercise.

E 19) Obtain $(x, y)\,(1, 0)$, $(x, y)\,(0, 1)$, $(x, y)\,(0, 0)$, $(x, 0)\,(y, 0)$ and $(x, y)\,(1, 1)\ \forall\,(x, y) \in C$.

E20) Prove that

a) $z_1 z_2 = z_2 z_1 \ \forall \ z_1, z_2 \in C$.

b) $(z_1 z_2) z_3 = z_1 (z_2 z_3) \ \forall \ z_1, z_2, z_3 \in C$.

c) $(x, y)\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = (1,0) \ \forall \ (x,y) \in C \setminus \{0\}$

Note that $x^2 + y^2 \neq 0$, since $(x, y) \neq (0, 0)$.)

d) $z\,\bar{z} = |z|^2 \ \forall \ z \in C$.

If you've solved these exercises, you must have realised that $z \,.\, 1 = z \ \forall \ z \in C$. This means that 1 is the **multiplicative identity** of C. Exercise E19 also says that
$z.0 = 0 \ \forall z \in C$,
$i(x+iy) = -y+ix \ \forall x, y \in \mathbf{R}$,
and that in case z_1 and z_2 are purely real numbers, our definition of multiplication coincides with the usual one for **R**

Also, from E20 (a) you can see that multiplication is commutative, and from E20 (b) you can see that multiplication is associative.

And, what does E20 (c) say? It says that for any non-zero element z of C, $\exists z' \in C$, such that $zz' = 1$. In this case we say that z' is the **multiplicative inverse** of z. So $z' = \frac{1}{z}$.

Using E20 let us see how to obtain the standard form of the quotient of a complex number by non-zero complex number. We will use a process similar to the one you must have used for rationalising the denominator in expressions like $\frac{a+b\sqrt{3}}{c+d\sqrt{2}}$. Consider an example.

Example 3 : Obtain $\frac{2+3i}{1-i}$ in the form $a+ib$, $a, b \in \mathbf{R}$.

Solution : E20 (d) gives us a clue to a method for making the denominator a real number. Let us multiply and divide $\frac{2+3i}{1-i}$ by $\overline{1-i}$. What do we get ?

$$\left(\frac{2+3i}{1-i}\right)\left(\frac{1+i}{1+i}\right) = \frac{(2+3i)(1+i)}{(1-i)(1+i)} = \frac{-1+5i}{1+1} = \frac{-1}{2} + \frac{5}{2}i$$

$z\,\bar{z} \in R \ \forall \ z \in C$.

So, $\frac{2+3i}{1-i} = \frac{-1}{2} + \frac{5}{2}i$.

If you've understood the way we have solved the example, you will have no problem in doing the following exercises.

E21) Obtain $\frac{-2+i}{\sqrt{-3}+i\sqrt{-4}}$

E22) For $a, b, c, d \in \mathbf{R}$ and $c^2 + d^2 \neq 0$, write $\frac{a+ib}{c+id}$ as an element of C.

E23) a) Show that $\frac{1}{1 \quad 6}$

b) Show that $\frac{1}{z} = \frac{1}{|z|^2}\bar{z}, \forall \ z \in C \setminus \{0\}$.

If you have done E22, then you know how to write the quotient of one complex number by a non-zero complex number in standard form.

Now, in Section 2.3 we introduced you to the polar form of a complex number. This form is very handy when it comes to multiplying or dividing complex numbers. Let us see why.

Suppose we know $z_1, z_2 \in C$ in their polar forms, say

$z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 \{(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)\}$$

$$= r_1 r_1 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \quad \text{....(2)}$$

So, $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ and

$\mathrm{Arg}(z_1 z_2) = (\theta_1 + \theta_2) + 2k\pi$, where we choose $k \in \mathbf{Z}$ so that

$-\pi < (\theta_1 + \theta_2) + 2k\pi \le \pi$.

In Figure 6 we give a graphic illustration of what we have just said.

cos (A + B) = cos A cos B– sin A sin B sin (A + B) = sin A cos B + cos A sin B

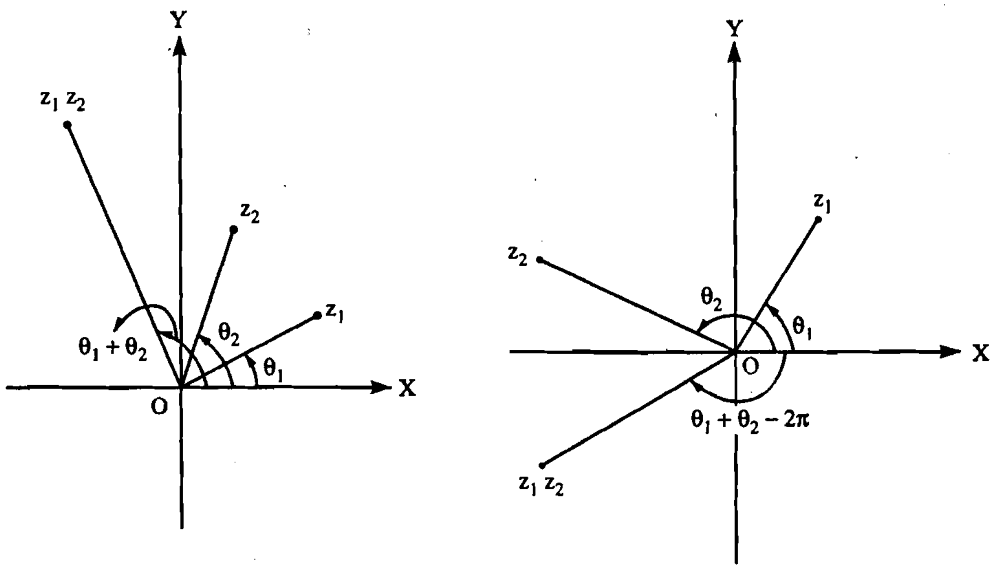

Figure 6: Production polar form.

Let us consider an example.

Example 4 : Obtain the product of $z_1 = 2(\cos 1 + i\sin 1)$ and $z_2 = \cos 3 + i\sin 3$.

Solution : Here $|z_1| = 2$ $\mathrm{Arg}\, z_1 = 1$, $|z_2| = 1$, $\mathrm{Arg}\, z_2 = 3$.

Therefore, $z_1 z_2 = 2(\cos(1+3) + i\sin(1+3))$

$$= 2(\cos 4 + i\sin 4).$$

Note the $\mathrm{Arg}(z_1 z_2) = 4$, since $4 > \pi$. We need to choose an integer k such that

$-\pi < 4 + 2k\pi \le \pi$, $k = -1$ serves the purpose. thus,

$\mathrm{Arg}(z_1 z_2) = 4 - 2\pi$.

Hence $(z_1 z_2) = 2(\cos(4 - 2\pi) + i\sin(4 - 2\pi))$ is the polar form of $z_1 z_2$.

We have a very nice method of finding the multiplicative inverse of a non zero complex number in an Argand diagram. Let us see what it is.

Let $z \in C \setminus \{0\}$ be represented by a point P (see Figure 7). Let Q represent the real number $|z|^2$. Let R be the reflection of P in the x-axis, so that R represents $\bar{z}$.

Now, through (1, 0) draw a line parallel to QR. Let it intersect the line OR in S. Then S represents $\frac{1}{z}$.

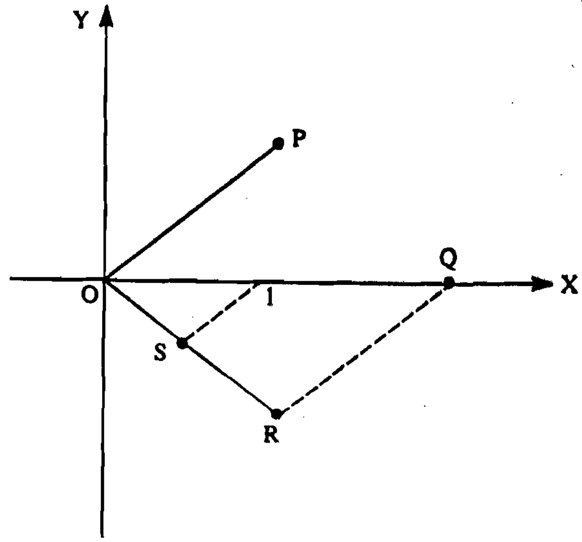

Figure 7: Finding the multiplication inverse.

Try the following exercises now.

E 24) Find the polar forms of z_1 and z_2 where $z_1 = -6$ and $z_2 = 1 + i$. Hence obtain $|z_1 z_2|$ and $\text{Arg}(z_1 z_2)$.

E25) Knowing the polar forms of $z_1, z_2 \in C, z_2 \neq 0$, obtain the polar form of $\frac{z_1}{z_2}$.

E26) Obtain $\frac{z_1}{z_2}$ in the polar form, where z_1 and z_2 are as in E 24. Represent $z_1, z_2, \bar{z}_2, \frac{1}{z_2}$ and $\frac{z_1}{z_2}$ in an Argand diagram.

We will use multiplication and division in the polar form a great deal in the next section. Before going to it, let us give you a rule that relates '+' and '×' in C. Do you know of such a law in **R**? You must have used the distributive law often enough. It says that
$a(b + c) = ab + ac \ \forall \ a, b, c \in$ **R**. The same law holds of C. Why don't you try and prove it?

Multiplicatin distributes over addition.

E27) a) Check that

$$(1 + i) \{(\sqrt{2} - 3i) + (5 + i)\}$$

$$= (1 + i)(\sqrt{2} - 3i) + (1 + i)(5 + i)$$

b) Prove that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \ \forall \ z_1 z_2 z_3 \in C$.

Now let us discuss a very useful theorem.

2.5 DE MOIVRE'S THEOREM

In the previous section we proved that if

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then $z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$.

In particular, if $z_1 = z_2$, then $r_1 = r_2$, $\theta_1 = \theta_2$, and we find that

$z_1^2 = r_1^2 (\cos 2\theta_1 + i\sin 2\theta_1)$.

In fact this is a particular case of a very nice fourmula, namely, that if $z = r(\cos\theta + i\sin\theta)$,

then $z^n = r^n(\cos n\theta + i\sin n\theta)$ for any integer n. To prove this result we need **De moivre's**

theorem, named after the French mathematician Abraham De Moivre (1667 - 1754). It may amuse you to know that De Moivre never explicity stated this result. But he seems to have known it and used it in his writings of 1730. It was Euler who explicitly stated and proved this result in 1748.

You can study induction in greater detail in the cours "Absiract Algebra".

Theorem 1 (De Moivre s theorem): $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for any $n \in \mathbf{Z}$ and any angle θ.

Proof : Let us first prove it for $n > 0$. We will prove this by using the following important principle.

Principle of Induction : Let P (n) be a set statements of one statement for each a positive integer n, such that

i) P (1) is true, and

ii) if P (m) is true for some $m \in \mathbf{N}$, then P (m+1) is true.

Then, P (n) is true $\forall\, n \in \mathbf{N}$.

How will we use this principle ? For any $n \in \mathbf{N}$, we will take P (n) to be the statement "$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$". We will first prove that it holds for n = 1 that is, P (1) is true. Then, we will assume that it holds for n = m for some $m \in N$, and prove that it is true for
n = m + 1. This will show that if P (m) is true, then so is P (m +1).

Now, for n = 1,

$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta = \cos 1.\theta + i\sin 1\theta$.

So the result is true for n =1.

Assume that it is true for n = m, that is,

$(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$. ....(3)

Now, $(\cos\theta + i\sin\theta)^{m+1}$

$= (\cos\theta + i\sin\theta)^m(\cos\theta + i\sin\theta)$

$= (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$ by (3)

$= \cos(m\theta + \theta) + i\sin(m\theta + \theta)$, by the formula (2) for products.

$= \cos(m+1)\theta + i\sin(m+1)\theta$

Hence, the result is true for n = m + 1.

Thus, by the principle of induction, the result is true $\forall\, n \in N$.

Now let us see what happens if n = 0.

We **define** $z^0 = 1$, for any $z \in C\setminus\{0\}$. (As in the case of **R**, 0^0 is not defined.)

Therefore, $(\cos\theta + i\sin\theta)^0 = 1$.

Also, $\cos 0.\theta + i\sin 0.\theta. = \cos 0 + i\sin 0 = 1$.

Thus, the result is also true for n = 0.

Now, what happens if n < 0? May be, you can prove this case. You can do the following exercises, which will lead you to the result.

E 28) Prove that $\dfrac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$, for any angle θ.

E 29) Let $n < 0$, say $n = -m$, where $m > 0$. Then

$$(\cos\theta + i\sin\theta)^n = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

Use this fact and De Moivre's theorem for positive integers to prove that

$(\cos\theta + i\sin\theta)^n = \cos n + i\sin\theta$.

So, De Moivre s theorem is true $\forall\ n \in \mathbf{Z}$.

Now, if $z = r(\cos\theta + i\sin\theta) \in C$, then $\forall\ n \in \mathbf{Z}$

$z^n = r^n(\cos\theta + i\sin\theta)^n$

$= r^n(\cos n\theta + i\sin n\theta)$, using De Moivre's theorem.

What we have shown is that

$$\boxed{[r(\cos\theta + \sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) \text{ for } r \in \mathbf{R}, \theta \in \mathbf{R}, n \in \mathbf{Z}.}$$

This result has several applications in mathematics and physics. We shall discuss two of them here.

2.5.1 Trigonometric Identities

One of the most useful applications of Theorem 1 is in proving identities that involve trigonometric functions like $\sin\theta$, $\cos\theta$, etc. Let us look at an example.

Example 5 : Find a formula for $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Solution : by De Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = \cos 4\theta + i\sin\theta 4 \qquad \text{.........(4)}$$

We can also expand the left hand side of (4) by using the binomial expansion. Then

$(\cos\theta + i\sin\theta)^4 = (\cos\theta)^4 + {}^4C_1(\cos\theta)^3(i\sin\theta) + {}^4C_2(\cos\theta)^2(i\sin\theta)^2$

$+ {}^4C_3\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$

$$= \cos^4\theta + 4i\sin\theta\cos^3\theta - 6\sin^2\theta\cos^2\theta - 4i\sin^3\theta\cos\theta + \sin^4\theta \qquad \text{......(5)}$$

Thus, comparing the real parts in (4) and (5), we get

$\cos 4\theta = \cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta$.

You can try the following exercise on similar lines.

E 30) Find formulae for $\cos 3\theta$ in terms of $\cos\theta$ and $\sin 3\theta$ in terms of $\sin\theta$.

Now, for any $m \in N$ let us look at z^m, where $z \in C$ such that $|z| = 1$. Then, by De Moivre's theorem

$z^m = \cos m\theta + i\sin m\theta$

and $z^{-m} = \cos m\theta - i\sin m\theta$, since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ for any angle θ.

Thus $z^m + z^{-m} = 2\cos\theta\, m$, and

$$z^m - z^{-m} = 2i\sin m\theta. \qquad \text{.....(6)}$$

We can use these relations to express $\cos^m\theta$ and $\sin^m\theta$ in terms of $\cos m\theta$ and $\sin m\theta$ for $m = \pm 1, \pm 2, \ldots\ldots$ Let us consider an example.

Example 6 : Expand $2^{4n-2}(\cos^{4n}\theta + \sin^{4n}\theta)$ in terms of the cosines or sines of multiples of θ.

Solution: Putting $m = 1$ in the equations (6), we get

$2\cos\theta = z + \frac{1}{z}$ and $2\sin\theta = z - \frac{1}{z}$

$$\therefore\ 2^{4n}\cos^{4n}\theta = \left(z + \frac{1}{z}\right)^{4n}$$

$$= z^{4n} + 4n\, z^{4n-1}\,\frac{1}{z} + {}^{4n}C_2 z^{4n-2}\,\frac{1}{z^2} + \ldots + {}^{4n}C_{2n} z^{2n}\,\frac{1}{z^{2n}} + \ldots + 4n z\,\frac{1}{z^{4n-1}} + \frac{1}{z^{4n}},$$

by the binomial expansion.

$$= \left(z^{4n} + \frac{1}{z^{4n}}\right) + 4n\left(z^{4n-2} + \frac{1}{z^{4n-2}}\right) + \ldots + {}^{4n}C_{2n}.$$

Also, $2^{4n}\sin^{4n}\theta = \left(z - \frac{1}{z}\right)^{4n}$, since $i^{4n} = (i^4)^n = 1$

$$= \left(z^{4n} + \frac{1}{z^{4n}}\right) - 4n\left(z^{4n-2} + \frac{1}{z^{4n-2}}\right) + \ldots + {}^{4n}C_{2n}.$$

$$\therefore\ 2^{4n}(\cos^{4n}\theta + \sin^{4n}\theta) = 2\left(z^{4n} + \frac{1}{z^{4n}}\right) + 2\,({}^{4n}C_2)\left(z^{4n-4} + \frac{1}{z^{4n-4}}\right) + \ldots + 2\,({}^{4n}C_{2n})$$

$$= 2\{2\cos 4n\theta + 2\,({}^{4n}C_2)\cos(4n-4)\theta + \ldots\} + 2({}^{4n}C_{2n}),\ \text{using (5).}$$

$$\therefore\ 2^{4n-2}(\cos^{4n}\theta + \sin^{4n}\theta) = \cos 4n\theta + {}^{4n}C_2\cos(4n-4)\theta + \ldots + \frac{1}{2}\,{}^{4n}C_{2n}.$$

The procedure we have shown in Example 6 is very useful for solving differential equations involving trigonometic functions. It is also useful for finding the Laplace transform of such functions.

Why don't you try this exercise now ?

E31) Apply De Moivre's formula to prove that
   i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$
   ii) $\sin 2\theta = 2\sin\theta\cos\theta$

E32) Expand $\cos^6\theta - \sin^6\theta$ in terms of the cosines of multiples of θ.

Let us now look at another area in which we can apply De moivre's theorem with great success.

2.5.2 Roots of A Complex Number

In Section 2.2 we told you that the whole subject of complex numbers first arose in an attempt to find the square roots of -1. By now you know that we can always find two distinct complex square roots of any non - zero real number.

That is, given $a \in \mathbf{R} \setminus \{0\}$, $\exists$ distinct $z_1, z_2 \in \mathbf{C}$ such that $z_1^2 = a$, $z_2^2 = a$.

In fact, the set of complex numbers has a much stronger property, which is a major reason for its importance in mathematics. This property is :

given any $n \in \mathbf{N}$ and $z \in \mathbf{C}$, $z \neq 0$, we can find distinct $z_1, \ldots, z_n \in \mathbf{C}$ such that $z_k^n = z$ $\forall\, k = 1, \ldots, n$.

Each of these z_k's is called an **nth root of z.**

To extract all the nth roots of a complex number, we need De Moivre's theorem as well as the following result that we ask you to prove.

E33) Let x be a positive real number and $n \in N$. Show that there is one and only one positive real number b such that $b^n = x$.
(**Hint** : Let $r,s > 0$ be such that $r^n = x = s^n$. Suppose $r \neq s$. Then $r^n - s^n = 0$ and $r - s \neq 0$. Then you should reach a contradiction.)

We denote the unique positive nth root obtained in E 33 by $x^{1/n}$.

Now let us consider an example of **extraction of roots of** a complex number.

Example 7 : Obtain all the fifth roots of i in C.

Solution : Let $z = r(\cos\theta + i\sin\theta)$ be any 5th root of i. Then $z^5 = i$. The polar form of i is

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$. Therefore

$z^5 = i$.

$$\Rightarrow r^5(\cos\theta + i\sin\theta)^5 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$\Rightarrow r^5(\cos 5\theta + i\sin 5\theta) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \qquad \text{........ (7)}$$

by De Moivre's thorem.

Comparing the moduli (plural 'modulus') and arguments of the complex numbers on both sides of (7), we get

$$r^5 = 1 \text{ and } 5\theta = \frac{\pi}{2} + 2k\pi \text{ where } k = 0, \pm 1, \pm 2, \ldots\ldots$$

r is the unique positive real fifth root of 1 (see E33). Since $1 \in \mathbf{R}$, $r = 1$, that is $|z| = 1$.

The possible values of θ are

$$\theta = \frac{1}{5}\left(\frac{\pi}{2} + 2k\pi\right), k = 0, \pm 1, \pm 2\ldots.$$

Thus, the possibe 5th roots of i are

$$z = \cos\left(\frac{\pi}{10} + 2k\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{10} + 2k\frac{\pi}{5}\right), k = 0, \pm 1, \pm 2, \ldots$$

From this it seems that i has infinitely many 5th roots, one for each $k \in \mathbf{Z}$. But this is not true. There are only 5 distinct ones among these. They will be the values of z for

$K = -2, -1, 0, 1, 2$. Let us see why.

When $k = -2$, $= \cos\left(\frac{\pi}{10} - \frac{4\pi}{5}\right) + i\sin\left(\frac{\pi}{10} - \frac{4\pi}{5}\right)$

$= \cos\frac{7\pi}{10} - i\sin\frac{3\pi}{10} = z_{-2}$, say.

When $k = -1$, $z = \cos\frac{\pi}{10} - i\sin\frac{3\pi}{10} = z_{-1}$ say.

When $k = 0$, $z = \cos\frac{\pi}{10} + i\sin\frac{\pi}{10} = z_0$ say.

When $k = 1$, $z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = z_1$ say.

When $k = 2$, $z = \cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10} = z_2$, say.

$\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$

When $k = 3$, $z = \cos\frac{13\pi}{10} + i\sin\frac{13\pi}{10} = \cos\left(2\pi - \frac{7\pi}{10}\right) + i\sin\left(2\pi - \frac{7\pi}{10}\right) = z_{-2}$.

when $k = 4, z = \cos\frac{17\pi}{10} + i\sin\frac{17\pi}{10} = \cos\left(2\pi - \frac{3\pi}{10}\right) + i\sin\left(2\pi + \frac{3\pi}{10}\right) = z_{-1}$.

Similarly, when k = 5, you will get z , and so on.

Thus, k = 5, 6, 7,..... don't give us new values of z.

Now, if we put k = –3, we get $z = \cos\left(\frac{-11\pi}{10}\right) + \sin\left(\frac{-11\pi}{10}\right) = z_2$.

Simlarly, k = –4, –5.... will not give us new values of z.

Therefote, the only 5th roots of i are

$$\cos\left(\frac{\pi}{10} + 2k\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{10} + 2k\frac{\pi}{5}\right) \text{ for } k = 0, \pm 1, \pm 2.$$

Remark 4: We also get the 5th roots of i by taking k = 0, 1, 2. 3, 4 in

$\cos\left(\frac{\pi}{10} + \frac{2k\pi}{5}\right) + \sin\left(\frac{\pi}{10} + \frac{2k\pi}{5}\right)$, as you have seen. Only note that for k = 3 and k = 4, the angles θ will not lie in the range $-\pi < \theta \le \pi$. That's why we had taken k = 0, ± 1, ± 2.

Now, look at all the fifth roots of i.. How are their moduli related ? They have the same modulus, namely, $|i|^{1/5}$ (=1). Thus they all lie on the circle with centre (0, 0) and radius 1. These points will be equally spaced on the circle, since the arguments of consecutive points differ by $\frac{2\pi}{5}$, a constant. We plot them in the Argand diagram in Figure 8.

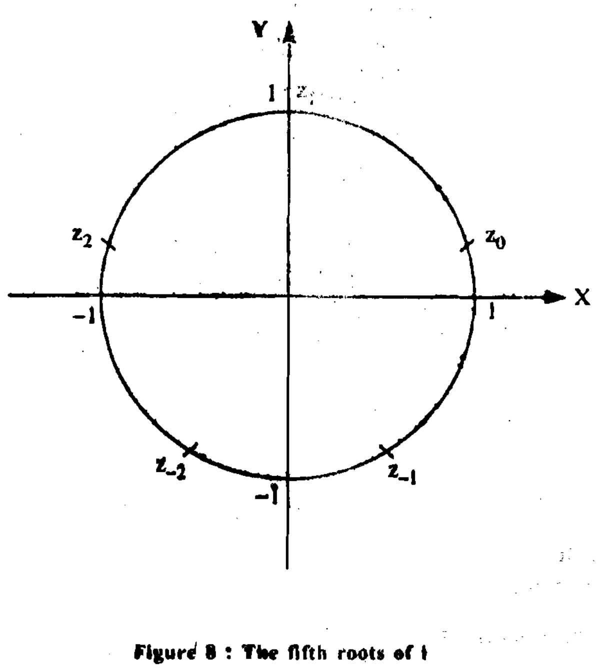

Figure 8 : The fifth roots of i

Using the same procedure as above we can obtain the distinct nth roots of any non-zero complex number, for any $n \in N$. Thus given, any non-zero complex number w, we write it in its polar form

$w = a(\cos\alpha + i\sin\alpha)$, where $a = |w|$ and $\theta = \text{Arg } w$.

By E33, there is a unique $r \in \mathbf{R}$, $r > 0$, such that $r^n = a$, that is, $r = a^{1/n}$. Then the distinct nth roots of w are

$$z_k = a^{1/n}\left(\cos\frac{\alpha + 2k\kappa}{n} + i\sin\frac{\alpha + 2k\kappa}{n}\right), \text{ for } k = 0, 1, \ldots\ldots, n-1.$$

Geometrically, they lie on a circle of radius $a^{1/n}$ and are equally spaced on it.

Note that

> a non - zero complex number has exactly n distinct nth roots for any $n \in N$. If z is one root, then the others are $z\alpha_1, \ldots, z\alpha_2$ where $\alpha_1 \ldots \alpha_{n-1}$ are the nth roots of unity.

Now you can do some exercises.

E 34) Find the complex cube roots of unity, that is, those $z \in C$ such that $z^3 = 1$. Plot them in an Argand diagram.

E35) Solve the equation $z^4 - 4z^2 + 4 - 2i = 0$.

(Hint : The equation can be rewritten as $(z^2 - 2)^2 = (1 + i)^2$.)

The cube roots of unity that you obtained in E34 are very important. We usually denote the cube root $\frac{-1+i\sqrt{3}}{2}$ by the Greek letter ω (omega).

Note that $\omega^2 = \left(\frac{-1+i\sqrt{3}}{2}\right)^2 = \frac{-1-i\sqrt{3}}{2}$, the other non - real cube root of unity. Thus,

> the three cube roots of unity are 1, ω, ω^2, where $\frac{-1+i\sqrt{3}}{2}$.

Also note that

$1 + \omega + \omega^2 = 0$. .....(8)

We will often use ω and (8) in Unit 3.

We will equally often use the following results, that we ask you to prove.

E 36) a) Let $a \in R$. Show that a has a real cube root r, and the cube roots of a are r, $r\omega$, $r\omega^2$.

b) Show that if $a \in \mathbf{R}$, $a < 0$ and n is an even positive integer, then a will not have a real nth root.

c) Let $z \in C \setminus \mathbf{R}$. Then show that z has three cube roots, and if any one of them is γ, the other two are $\gamma\omega$, $\gamma\omega^2$.

With this we come to the end of our discussion on complex numbers. Thus does't mean that you won't be **dealing** with them any more . In fact, you will often use whatever we have covered in this unit, while studying this course as well as other mathematics courses.

Let us take a brief look at the points covered in this unit.

2.6 SUMMARY

In this unit on complex numbers you have studied the following points.

1) The definition of a complex number:

   A complex number is a number of the form $x + iy$ where $x, y \in \mathbf{R}$ and $i = \sqrt{-1}$.

   Equivalently, it is a pair $(x,y) \in \mathbf{R} \times \mathbf{R}$.

2) x is the real part and y is the imaginary part of $x + iy$.

3) $x_1 + iy_1 = x_2 + iy_2$ iff $x_1 = x_2$ and $y_1 = y_2$.

4) The conjugate of $z = x + iy$ is $\bar{z} = x - iy$.

5) The geometric representation of complex numbers in Argand diagrams.

6) The polar form of $z = x + iy$ is $z = r(\cos\theta + i\sin\theta)$, where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \text{Arg } z = \tan^{-1}\left(\frac{y}{x}\right)$, where we choose the θ that corresponds to the position of z in an Argand diagram.

7) Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta \ \forall\ \theta \in \mathbf{R}$.

8) The exponential form of $z = x + iy$ is $z = re^{i\theta}$, where $r = |z|$ and $\theta = \text{Arg } z$

9) Addition, subtraction, multiplication and division in C: $\forall$ a, b, c, d, $\in$ **R**

$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$,

$(a + ib).(c + id) = (ac - bd) + i(ad + bc)$,

$$\frac{1}{a+ib} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}, \text{ when } a+ib \neq 0,$$

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{c^2+d^2}, \text{ for } c+id \neq 0.$$

10) For $z_1, z_2 \in C$,

$|z_1 z_2| = |z_1||z_2|$, $\text{Arg}(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2k\pi$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z^2}\right|, \text{ Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg } z_1 - \text{Arg } z_2 + 2m\pi, (\text{for } z_2 \neq 0)$$

where k, m $\in$ **Z** are chosen so that

$$-\pi < \text{Arg}(z_1 z_2) \leq \pi \text{ and } -\pi < \text{Arg}\left(\frac{z_1}{z_2}\right) \leq \pi$$

11) De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \ \forall\ n \in Z$ and any angle θ.

12) The use of De Moivre's theorem in proving trigonometric identities and for obtaining nth roots of complex numbers, where $n \in \mathbf{N}$.

13) The cube roots of unity are 1, ω, ω^2, where $\omega = \frac{-1 + i\sqrt{3}}{2}$.

Now that you have gone through this unit, please **go back to the objectives** listed in Section 2.1. Do you think you have achieved them? One way of finding out is to solve all the exercises that we have given you in this unit. If you would like to verify your solution or answers, you can see what we have written in the following section.

2.7 SOLUTION / ANSWERS

E1)

z	Re z	Im z
$\frac{1+\sqrt{-23}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{23}}{2}$
i	0	1
0	0	0
$\frac{-1+\sqrt{3}}{5}$	$\frac{-1+\sqrt{3}}{5}$	0

E2) Yes, because every real number x is the complex number x + 0i.

E3) 2 − 3i, 2 + 3i, 2, −3i.

E4) a)

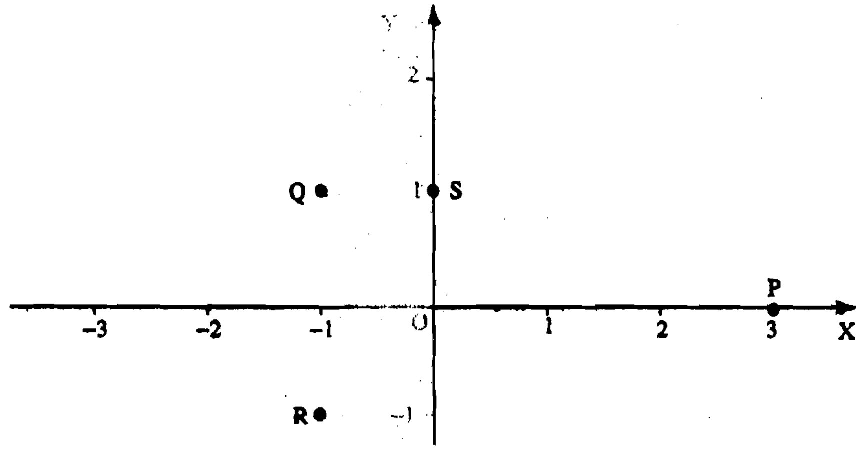

Fig. 9

P, Q, R and S represent 3, −1 + i, $\overline{-1+i}$ and i, respectively.

b)

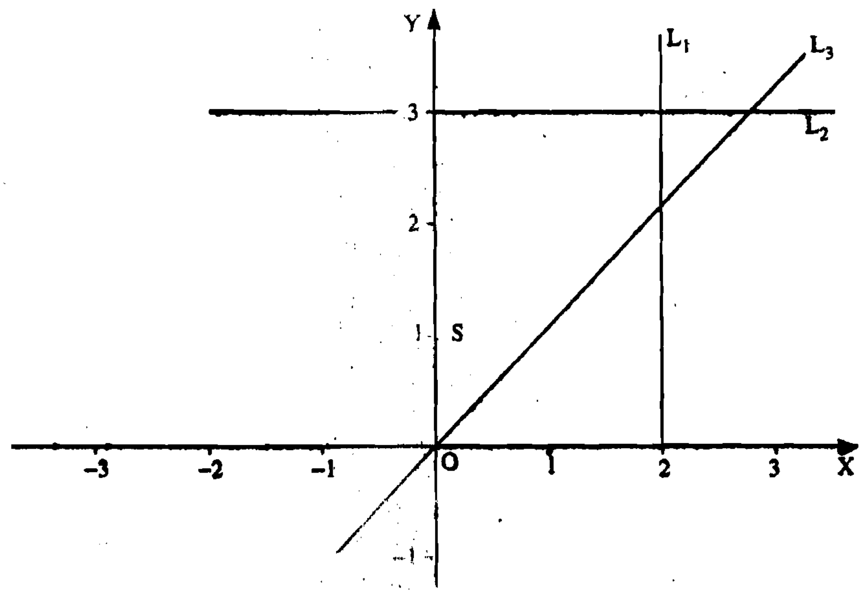

Fig. 10

L_1, L_2 and L_3 represent the sets S_1, S_2 and S_3, respectively.

E5) $\frac{-1}{2}+\frac{1}{3}$, 2, − 2i

E6) $k = \frac{1}{2}$, m = 3.

E7) Let z = x + iy. Then $\bar{z} = x - iy$.

$\therefore z = \bar{z} \Rightarrow x + iy = x - iy \Rightarrow y = -y \Rightarrow y = 0.$

$\therefore \forall z \in \mathbb{R}, z = \bar{z}$

E8) Let z = x + iy. Then $\bar{z} = x - iy$.

$\therefore \bar{\bar{z}} = \overline{x - iy} = x + iy = z$

E9) $3 = 3(\cos\theta + i\sin\theta)$

Now $|-1+i| = \sqrt{1+1} = \sqrt{2}$, and

Arg $(-1+i) = \tan^{-1}(-1) = -\pi/4$ or $3\pi/4$,

Since $-1+i$ corresponds to $(-1, 1)$, which lies in the 2nd quadrant, Arg $(-1+i) = \frac{3\pi}{4}$

$$\therefore (-1+i) = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).$$

$$\overline{-1+i} = -1 - i = \sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right).$$

$$= \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right)$$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}.$$

E10) For any $z = x + iy \in C$, $|z| = 1 \Leftrightarrow \sqrt{x^2+y^2} = 1 \Leftrightarrow x^2 + y^2 = 1$.

E11) $\sqrt{\frac{5}{2}} + i\sqrt{\frac{5}{2}} = \sqrt{5}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ (polar form)

$= \sqrt{5}\; e^{i\pi/4}$ (exponential form)

$1 + i = \sqrt{2}\;(\cos\frac{\pi}{4} = i\sin\frac{\pi}{4})$ (polar form)

$= \sqrt{2}\; e^{i\pi/4}$ (exponential form)

$-1 = \cos\pi + i\sin\pi$ (polar form)

$= e^{i\pi/2}$ (exponential from)

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (polar form)

$= e^{i/\pi}$ (exponential from)

E 12) a) $2 + 3i + \overline{2+3i} = 2 + 3i + 2 - 3i = 4 + 0i = 4$.

b) Let $z = x + iy$. Then

$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\,\text{Re}\,z$.

E 13) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$

$\therefore \overline{z_1 + z_2} = (x_1 + x_2) - (y_1 + y_2)$

$= (x_1 + x_2, -y_1 - y_2)$

$= (x_1, -y_1) + (x_1 - y_2)$

$= \bar{z}_1 + \bar{z}_2$

E 14) a) Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

Then $z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

$= (x_2 + x_1, y_2 + y_1)$, since $a + b = b + a\ \forall a, b \in \mathbf{R}$.

$= (x_2, y_2) + (x_1, y_1)$

$= z_2 + z_1$

b) Let $z_1 = (x_1, y_1)$, $z_2 = (x_2\, y_2)$, $z_3 = (x_3, y_3)$.

Then, use the fact that $(a + b) + c = a + (b + c)\ \forall a, b, c \in \mathbf{R}$, to prove the result.

E 15) Let $z = x + iy$.

Then $z + (a + ib) = z$

$\Rightarrow (x + iy) + (a + ib) = x + iy$

$\Rightarrow (x + a) + i(y + b) = x + iy$

$\Rightarrow x + a = x$ and $y + b = y$

$\Rightarrow a = 0, b = 0$

$\therefore a + ib = 0 + i0 = 0.$

E 16) $(-6-(-3)) + i(3-(-2)) = -3+5i.$

E 17) Let $z = x + iy$. Then

$$z - \bar{z} = (x+iy) - (x-iy) = (x-x) + i(y+y) = 2iy.$$
$$= i(2 \operatorname{Im} z).$$

E 18) Let $z = x + iy$. Then $-z = (-x) + i(-y)$. Thus,

a) $|z| = \sqrt{x^2+y^2}$, and

$$|-z| = \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2+y^2} = |z|.$$

b) $\operatorname{Arg} z = \tan^{-1}\left(\frac{y}{x}\right).$

$\operatorname{Arg}(-z) = \tan^{-1}\left(\frac{-y}{-x}\right) = \tan^{-1}\left(\frac{y}{x}\right) = \operatorname{Arg} z - \pi$, because $(-z)$ is the reflection of z in the origin.

E 19) $(x,y)(1,0) = (x,y)$

$Z.1 = \ \forall z \in C$

$(x,y)(0,1) = (-y,x)$

$Z.0 = 0 \ \forall z \in C$

$(x,y)(0,0) = (0,0)$

$(x,0)(y,0) = (xy,0)$

$(x,y)(1,1) = (x-y, x+y).$

E 20) a) Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Then

$$z_1 z_2 = (x_1, y_1)(x_2, y_2)$$
$$= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$
$$= (x_2 x_1 - y_2 y_1, x_2 y_1 + x_1 y_2), \text{ since } ab = ba \ \forall a, b, c \in \mathbf{R}.$$
$$= (x_2, y_2)(x_1, y_1)$$
$$= z_2 z_1.$$

b) If $z_1 = (x_1, y_1)\ z_2 = (x_2, y_2)\ z_3 = (x_3\ y_3)$ then

$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$ and

$z_2 z_3 = (x_2 x_3 - y_2 y_3, x_2 y_3 + x_3 y_2)$

Therefore, $(z_1 z_2) z_3$

$$= ((x_1 x_2 - y_1 y_2) x_3 - (x_1 y_2 + x_2 y_1) y_3, (x_1 x_2 - y_1 y_2) y_3 + x_3 (x_1 y_2 + x_2 y_1))$$
$$= (x_1 (x_2 x_3 - y_2 y_3) - y_1 (x_2 y_3 + x_3 y_2), x_1 (x_2 y_3 + x_3 y_2) + (x_2 x_3 - y_2 y_3) y_1)$$
$$= z_1 (z_2 z_3)$$

c) $$(x, y)\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = \left(\frac{x^2+y^2}{x^2+y^2}, \frac{-xy + xy}{x^2+y^2}\right) = (1,0).$$

Let $z = x + iy \in C$. Then

$$z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = \left(\sqrt{x^2+y^2}\right)^2 = |z|^2.$$

E 21) $$\frac{-2+i}{i\sqrt{3}+i(2i)} = \frac{-2+i}{-2+i\sqrt{3}}, \text{ since } i^2 = -1.$$

$$= \frac{(-2+i)(-2-i\sqrt{3})}{(-2)^2+(\sqrt{3})^2}$$

$$= \frac{4+\sqrt{3}}{7} + \frac{2}{7}(\sqrt{3}-1)i$$

E 22) $c^2 + d^2 \neq 0$ means that $c \neq 0$ or $d \neq 0$. Thus, $c + id \neq 0$. Hence $\frac{a+ib}{c+id}$ is meaningful.

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd) + i(bc-ad)}{c^2+d^2} = \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right).$$

E23) a) Let $z = x + iy \neq 0$. Then, from E20 (d) we know that $z\,\bar{z} = |z|^2$.

Therefore, $z\left(\frac{1}{|z|^2}\bar{z}\right) = 1$. Thus, $\frac{1}{|z|^2}\bar{z}$ is the multiplicative inverse of z.

b) For $z \neq 0$, $z.\frac{1}{z} = 1$, $\therefore |z|.\left|\frac{1}{z}\right| = |1| = 1$. $\therefore \left|\frac{1}{z}\right| = \frac{1}{|z|}$.

E24) $z_1 = 6(\cos\pi + i\sin\pi)$, $z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\therefore |z_1 z_2| = 6\sqrt{2}$ and

$\text{Arg}(z_1 z_2) = \left(\pi + \frac{\pi}{4}\right) + 2k\pi$, where $k \in \mathbf{Z}$ such that $-\pi < \text{Arg}(z_1 z_2) \le \pi$.

$\therefore \text{Arg}(z_1 z_2) = \frac{-3\pi}{4}$.

E25) If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta 2)}$

$= \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)$, multiplying and dividing by $(\cos\theta_2 - i\sin\theta_2)$.

$= \frac{r_1}{r_2}\cos(\theta_1 + \theta_2) + i\sin(\theta_1 - \theta_2))$

$= \frac{r_1}{r_2}(\cos\theta_1 - \theta_2 + 2k\pi) + i\sin(\theta_1 - \theta_2 + 2k\pi))$, where $k \in \mathbf{R}$ such that

$-\pi < \theta_1 - \theta_2 + 2k\pi \le \pi$.

E26) $\frac{z_1}{z_2} = \frac{6}{\sqrt{2}}\left(\cos\left(\pi - \frac{\pi}{4}\right) + i\sin\left(\pi - \frac{\pi}{4}\right)\right)$

$= \frac{6}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

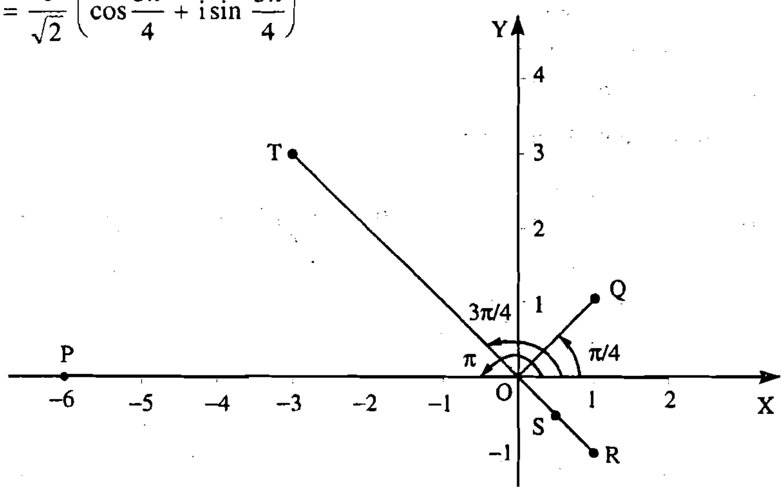

Fig. 11

The points P, Q, R, S and T in Figure 11 represent $z_1, z_2, \bar{z}_2, \frac{1}{z_2}$ and $\frac{z_1}{z_2}$, respectively.

Here $OT = \frac{OP}{OQ}$ and $\angle XOT = \angle XOP - \angle XOQ$.

E27) a) $\text{LHS} = (1+i)[(\sqrt{2}+5)-2i] = (7+\sqrt{2}) + i(3+\sqrt{2})$

$\text{RHS} = [(\sqrt{2}+3) + i(\sqrt{2}-3)] + (4+6i) = (\sqrt{2}+7) + i(\sqrt{2}+3)$.

Thus, LHS = RHS.

LHS stands for left hand side and RHS stands for right hand side

b) Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + iy_3$.

Then $z_1(z_1 + z_3) = (x_1 + iy_1)[(x_2 + x_3) + i(y_2 + y_3)]$

$= [x_1(x_2 + x_3) - y_1(y_2 + y_3)] + i[x_1(y_2 + y_3) + y_1(x_2 + x_3)]$

$= (x_1 x_2 - y_1 y_2) + (x_1 x_3 - y_1 y_3) + i[x_1 y_2 + x_2 y_1) + (x_1 y_3 + x_3 y_1)]$

$= [(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)] + [(x_1 x_3 - y_1 y_3) + i(x_1 y_3 + x_3 y_1)]$

$= z_1 z_2 + z_1 z_3$.

You can also solve this by writing z_1, z_2 and z_3 in polar form, If you do, you must remember to be careful about $z_i = 0$ for any i.

E28) $\dfrac{1}{\cos\theta + \sin\theta} = \dfrac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta.$

E29) For $n < 0$, say $n = -m$, $m > 0$,

$$(\cos - i\ \sin\theta)^n = \frac{1}{(\cos\theta + i\sin\theta)^m} = \left(\frac{1\ .}{\cos\theta + i\sin\theta}\right)^m$$

$= (\cos\theta - i\sin\theta)^m$

$= [\cos(-\theta) + i\sin(-\theta)]^m$

$= \cos(-m\theta) + i\sin(-m\theta)$, since $m > 0$.

$= \cos n\theta + i\sin n\theta$.

E30) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$.

Also, $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta)$.

Thus, comparing real parts of the two equalities, we get

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$.

Similarly, comparing the imaginary parts we get

$\sin 3\theta = 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$.

E31) $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$, and

$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$.

$\therefore \cos 2\theta = \cos^2\theta - \sin^2\theta$, and

$\sin 2\theta = 2\sin\theta\cos\theta$.

E32) Let $z = \cos\theta + i\sin\theta$. Then, using (6)

$$(2\cos\theta)^6 = \left(z + \frac{1}{z}\right)^6 = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20,$$

$$(2i\sin\theta)^6 = \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$\therefore 2^6\left(\cos^6\theta - \sin^6\theta\right) = 2\left(z^6 + \frac{1}{z^6}\right) + 30\left(z^2 + \frac{1}{z^2}\right)$$

$= 4\cos 6\theta + 60\cos 2\theta$

$\Rightarrow \text{Cos}^6\theta - \sin^6\theta = \dfrac{1}{16}(\cos 6\theta + 15\cos 2\theta)$

E33) Let $r, s \in \mathbf{R}$, $r, s > 0$ and $r^n = x = s^n$. We will prove the result by contradiction (sec appendix to this block). Suppose $r \neq s$. Then

$r^n - s^n = (r - s)(r^{n-1} + r^{n-2}s + \ldots + rs^{n-2} + s^{n-1}) = 0$.

Since $r > 0$, $s > 0$, $r^{n-1} + r^{n-2}s + \ldots + rs^{n-2} + s^{n-1} > 0$.

Also, $r - s \neq 0$.

But then how can the product of two non - zero numbers be zero ? So we reach a contradiction. Therefore, our assumption must be false. Thus, r = s.

E 34) Let z = (cosθ + θsinθ) be a cube root of 1 = cos 0 + i sin 0,

Then $r = 1^{1/3} = 1,\ \theta = \frac{0+2k\pi}{3} = \frac{2k\pi}{3}$ for k = 0, 1, –1.

∴ The roots are $1.\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right)$ for k = 0, 1, – 1

Thus, the roots are 1, $\frac{-1}{2} + i\frac{\sqrt{3}}{2}$ and $\frac{-1}{2} - i\frac{\sqrt{3}}{2}$.

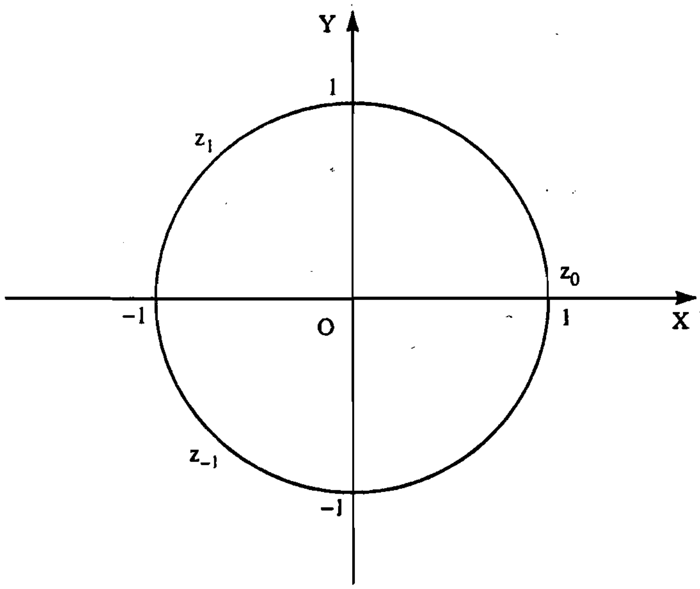

Figure 12: Cube roots of unity.

E 35) We want to obtain those z ∈ C for which

$(z^2 - 2) = \pm(1 + i)$, that is

$z^2 - 2 = 1 + i$ and $z^2 - 2 = -(1 + i)$, that is,

$z^2 = 3 + i$ and $z^2 = 1 - i$.

Thus, we want to find the square roots of 3 + i and 1 – i.

Now, $3 + i = \sqrt{10}\left\{\cos\left(\tan^{-1}\frac{1}{3}\right) + i\sin\left(\tan^{-1}\frac{1}{3}\right)\right\}$.

Thus, the square roots of 3 + i are

$10^{1/4}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ and $10^{1/4}\left\{\cos\left(\frac{\theta}{2} + \pi\right) + i\sin\left(\frac{\theta}{2} + \pi\right)\right\}$,

where $\theta = \tan^{-1}\frac{1}{3}$.

Also $1 - i = \sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + \sin\left(\frac{-\pi}{4}\right)\right)$, so that the square roots of 1 – i are

$2^{1/4}\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$ and $2^{1/4}\left(\cos\frac{7\pi}{8} - i\sin\frac{7\pi}{8}\right)$.

These 4 square roots are the 4 roots of the given equation.

E 36) a) If a ≥ 0, then by E33, a has a real cube root, $a^{1/3}$. Now, a = a (cos 0 + i sin 0).

Thus, the cube roots of a are

$a^{1/3}\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right)$, k – 0, 1, 2,

that is, $a^{1/3}, a^{1/3}\omega^{1/3}, a^{1/3}\omega^2$.

If a < 0, then – a > 0. Thus, – a has a real cube root, say b. Then r = – b is a real cube root of a. And $|r| = |a|^{1/3}$ that is, $r = -|a|^{1/3}$ (since r is negative).

Now a = | a | (cosπ + i sin π). Therefore, the cube roots of a are

$$|a|^{1/3}\left(\cos\frac{(2k+1)\pi}{3}+i\sin\frac{(2k+i)\pi}{3}\right),\ k=0,1,2.$$

$$= r(\cos\pi+i\sin\pi)\left(\cos\frac{(2k+1)\pi}{3}+\sin\frac{(2k+1)\pi}{3}\right),\ k=0,1,2.$$

(since $-1=\cos\pi+i\sin\pi$)

$$= r\left(\cos\frac{(2k+4)\pi}{3}+i\sin\frac{(2k+4)\pi}{3}\right),\ k=0,1,2.$$

Thus, the cube roots of a are r, rω rω^2.

b) Let n = 2m, m $\in$ **N**. Then, for any b $\in$ **R**,

$b^n = b^{2m} = (b^2)^m \geq 0.$

Thus, $b^n \neq a$ for any b $\in$ **R**. Hence, a can't have a real nth root.

c) Let $z = r(\cos\theta + i\sin\theta)$, in polar form.

Then its cube roots are $r^{1/3}\left(\cos\frac{\theta+2k\pi}{3}+i\sin\frac{\theta+2k\pi}{2}\right)$, k = 0, 1, 2

Thus, if $\gamma = r^{1/3}\left(\cos\frac{\theta}{3}+i\sin\frac{\theta}{3}\right)$, then the other roots are

$$r^{1/3}\left(\cos\frac{\theta+2\pi}{3}+i\sin\frac{\theta+2\pi}{3}\right)=\gamma\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=\gamma\,\omega,\text{and}$$

$$r^{1/3}\left(\cos\frac{\theta+4\pi}{3}+i\sin=\frac{\theta+4\pi}{3}\right)=\gamma\,\omega^2.$$