UNIT 3 DIFFERENTIATION

Structure

3.1 INTRODUCTION

It was the seventeenth century. Some European mathematicians were working on two basic problems :

i) Is it possible to find the tangent to a given curve at a given point of the curve?

ii) Is it possible to find the area under a given curve?

Two mathematical giants, Newton and Leibniz, independent of each other, solved these problems. The theory that they inventned in the process was **Calculus.**

In this first unit on differentiation, we propose to introduce the concept of a derivative which is a basic tool of calculus. Leibniz was motivated directly by the first problem given above — a problem which was of great significance for scientific applications. He recognised the derivative as the slope of the tangent to the curve at the given point. Newton, on the other hand, arrived at it by considering some physical problems such as determination of the velocity or the acceleration of a particle at a particle instant. He recognised the derivative as a rate of change of physical quantities. We shall now show that both these considerations lead to the concept of derivative as the limit of a ratio. Of course, to understand what a derivative is, you should have gone through Sec. 2 thoroughly.

Newton (1642-1727)

We shall first differentiate some standard functions using the definition of the derivative. The algebra of derivatives can then be effectively used to write down the derivatives of several functions which are algebraic combinations of these functions. We shall also discuss the chain rule of differentiation which offers an unbelievable simplification in the process of finding derivatives. We shall also establish a relationship between differentiable functions and continuous functions which you have studied in Unit 2.

Objectives

Leibniż (1646-1716)

After studying this unit you should be able to :

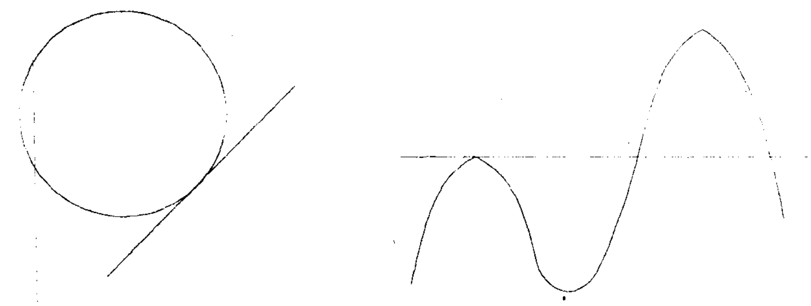
- draw a tangent to a given curve at a given point
- determine the rate of change of a given quantity with respect to another
- obtain the derivatives of some simple functions such as x^n, $|x|$, $\sqrt{x}$ etc. from the first principles
- find the derivatives of functions which can be written as the sum, difference, product, quotient of functions whose derivatives you already know
- derive and use the chain rule of differentiation for writing down the derivatives of a composite of functions
- discuss the relationship between continuity and derivability of a function.

3.2 THE DERIVATIVE OF A FUNCTION

Before defining a derivative, let us consider two problems in the next two subsections. The first is to find the slope of a tangent and the second is to find the rate of change of a given quantity in terms of another.

3.2.1 Slope of a Tangent

Let us consider the problem of finding a tangent to a given curve at a given point. But, what do we mean by the tangent to a curve? Euclid (300 B.C.) thought of a tangent as a line touching the curve at one point. This definition works fine in the case of a circle Fig. 1 (a), but it fails in the case of many other curves (see Fig. 1 (b)).

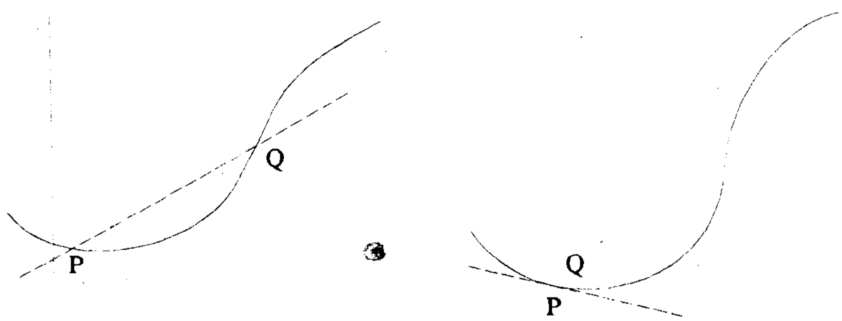


(a) (b)

Fig. 1

We may define a tangent to a curve at P to be a line which best approximates the curve near P. But this definition is still too vague. Then how can we define a tangent precisely? The concept of limit which you have studied in Unit 2 comes to our aid here.

Let P be a fixed point on the curve in Fig. 2 (a), and let Q be a nearby point on the curve. The line throughout P and Q is called a secant. We define the tangent line at P to be the limiting position (if it exists) of the secant PQ as Q moves towards P along the curve (Fig. 2 (b)).



(a) (b)

Fig. 2

It may not be always possible to find the limiting position of the secant. As we shall see later, there are curves which do not have tangents at some points. In fact, there are curves which do not have a tangent at any point!

There is another question which we can ask here. Suppose we know that a tangent to a curve exists at a point, how do we go about actually drawing the tangent?

The tangent of the angle which a line makes with the positive direction of the x-axis is called the slope of the line.

We have said earlier that the tangent at P is the limiting position of the secant PQ. With reference to a system of coordinate axes OX and OY (Fig. 3), we can also say that the tangent at P is a line through P whose slope is the limiting value of the slope of PQ as Q approaches P along the curve. The problem of determining the tangent is, then, the problem of finding the slope of the tangent line.

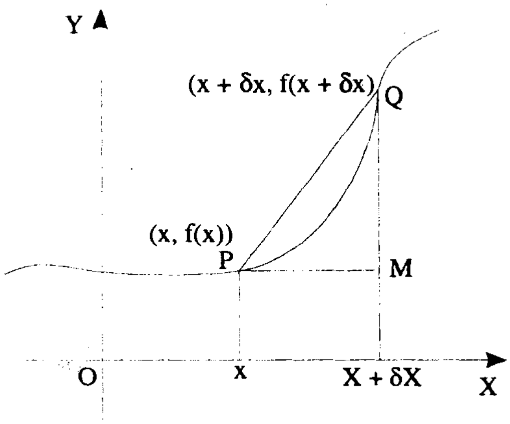


Fig. 3

Suppose the curve in Fig. 3 is given by $y = f(x)$. Let P $(x, f(x))$ be the point P and let Q $(x + \delta x, f(x + \delta x))$ be any other point on the curve. The prefix δ before a variable quantity means a small change in the quantity. Thus, δx means a small change in the variable x. **(Caution: δx is one inseparable quantity. It is not $\delta \times x$).** The coordinates $(x + \delta x, f(x + \delta x))$ indicate that Q is generally near P. If θ is the angle which PQ makes with the x-axis, then the slope of PQ = $\tan \theta$ = QM/PM

$$= \frac{f(x + \delta x) - f(x)}{\delta x}$$

Then limiting value of $\tan \theta$, as Q tends to P, (and hence $\delta x \to 0$) gives us the slope of the tangent at P. Thus,

$$\boxed{\text{The slope of the tangent line at } (x, f(x)) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}}$$

This indicates that the tangent line will exist at $(x, f(x))$ only if the limit of $\frac{f(x + \delta x) - f(x)}{\delta x}$ exists as $\delta x \to 0$.

Remark 1 In Fig. 3 we have taken δx to be positive. But our discussion is valid even for negative values of δx.

Let us take an example.

Example 1 Suppose we want to determine the tangent to the parabola $y = x^2$ at the point P(2, 4).

In Fig. 4 we give a portion of the parabola in the vicinity of P (2,4).

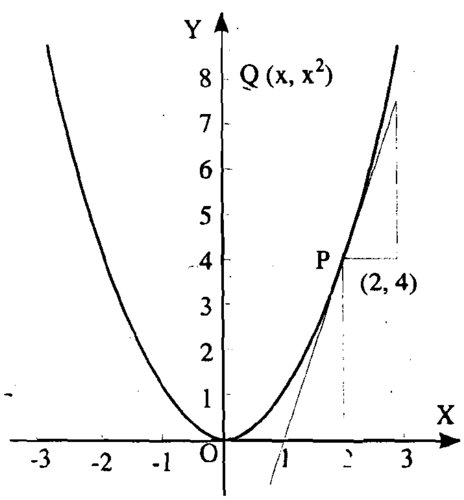


Fig. 4

Let Q (x, x^2) be any other point on the parabola. The slope of

$$PQ = \frac{\text{y coordinate of Q} - \text{y coordinate of P}}{\text{x coordinate of Q} - \text{x coordinate of P}}$$

$$= \frac{x^2 - 4}{x - 2}$$

Equation of a line passing through a point (x, y) and having space m is $y - y_1 = m(x - x_1)$

The tangent at P (2, 4) is the limiting position of PQ as $x \to 2$. Therefore, the slope of the tangent at P is

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x + 2) - (x - 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

The equation of the tangent line will be $(y - 4) = 4(x - 2)$

Now, how do we draw this tangent? Just mark the point P′ by moving a distance 1 unit from P, parallel to the x-axis to the right, and then, moving a distance equal to 4 units parallel to the y-axis upward. Join P to P′ as shown in Fig. 4. The coordinates of p′ are (2 + 1, 4 + 4). The resulting line will touch the parabola at P, and the slope of the tangent at P = tan θ = 4.

E E 1) Find the equation of the tangent to the following curves at the given points.

a) $y = 1/x$ at (2, 1/2)

b) $y = x^3$ at (1, 2)

In this subsection we have given a precise definition of a tangent to a curve. We have also seen how to draw the tangent to a given curve at a given point. Now let us consider the second problem mentioned at the beginning of this section.

3.2.2 Rate of Change

Suppose a particle is moving along a straight line, and covers a distance s in time t. The distance covered depends on the time t. That is s = f(t), a function of time. When the time changes to $t + \delta t$, the distance covered changes from $f(t) = s$ to $f(t + \delta t) = s + \delta s$. Therefore we can say that δs is the distance covered in the time δt. We want to know the average velocity of the particle during the time interval t to $t + \delta t$ (or $t + \delta t$ to t, according as $t > 0$ or $t < 0$).

Now, the average velocity = $\dfrac{\text{Total distance travelled}}{\text{Total time taken}}$

Therefore, the average velocity in the time interval $[t, t + \delta t]$ (or $[t - \delta t, t]$).

$= \frac{f(t+\delta t)-f(t)}{(t+\delta t)-t} = \frac{(s+\delta s)-s}{(t+\delta t)-t} = \frac{\delta s}{\delta t}$. where $\delta s = f(t+\delta t) - f(t)$.

But this does not give us the velocity of the particle at a **particular instant** t, which is called **the instantaneous** velocity. Rather it is the average velocity over the internal δf. How do we calculate this?

To find the velocity at a particular time t, we proceed to find the average velocity in the time interval $[t, t+\delta t]$ (or $[t+t, t]$) for smaller and smaller values of δt.

If δt is very small, then $t + \delta t$ is very near t and so the average velocity during the time interval δt would be very near the velocity at t. It seems reasonable, therefore, to define the instantaneous velocity at time t to be $\lim\limits_{\delta t \to 0} \frac{f(t+\delta t)-f(t)}{\delta t}$

Thus, we have $\boxed{v = \lim\limits_{\delta t \to 0} \frac{f(t+\delta t)-f(t)}{\delta t}}$

where $s = f(t)$ is the distance travelled in time t. Comparing this box with the one given at the end of the last subsection, we find that the concepts of the slope of a tangent and the instantaneous velocity are identical. Further, velocity can be considered as the rate of change of distance with respect to time. So, extending our definition of velocity to other rates of change, we can say that if a quantity y depends on x according to the rule $y = f(x)$, then the rate of change of y with respect to x can be defined as

$$\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$$

Example 2 Suppose we want to find the rate of change of the function f defined by $f(x) = x + 5$, $\forall x \in R$, at $x = 0$.

We shall first calculate the average rate of change of f in an interval $[0, \delta x]$.

This average rate of change of f in $[0, \delta x]$ is

$$\frac{f(0+\delta x)-f(0)}{(0+\delta x)-0} = \frac{f(\delta x)-f(0)}{\delta x}$$

$$= \frac{\delta x + 5 - 5}{\delta x} = \frac{\delta x}{\delta x} = 1$$

Hence, the rate of change of f at 0, which is the limiting value of this average rate as $\delta x \to 0$,

$$= \lim_{\delta t \to 0} \frac{f(0+\delta x)-f(0)}{\delta x} = \lim_{\delta t \to 0} 1 = 1.$$

Example 3 Suppose a particle is moving along a straight line and the distance s covered in time t is given by the equation $s = (1/2)t^2$. Let us draw the curve represented by the function $s = (1/2)t^2$, measuring time along x-axis and distance along y-axis. Let P and Q be points on the curve which correspond to $t_1 = 2$ and $t_2 = 4$.

We shall show that the average velocity of the particle in the time interval [2, 4] is the slope of the line PQ and the velocity at time $t_1 = 2$ is the slope of the tangent to the curve at $t_1 = 2$.

The curve represented by $s = (1/2)t^2$ is a parabola, as shown in Fig. 5. P and Q correspond to the values $t_1 = 2$ and $t_2 = 4$ of t. Now $s_1 = (1/2)t_1^2 = 2$ and $S_2 = (1/2)t_2^2 = 8$. Therefore, the coordinates of the points P and Q are (2, 2) and (4, 8), respectively.

The slope of PQ $= \frac{8-2}{4-2} = 6/2 = 3$.

Also, the distance travelled by the particle in the time $(t_2 - t_1)$ is $s_2 - s_1 = 8 - 2 = 6$. Therefore, the average velocity of the particle in the time $(t_2 - t_1)$ is

$$\frac{\text{distance travelled}}{\text{time taken}} = 6/2 = 3.$$

δt may be positive or negative

Hence, the slope of PQ is the same as the average velocity of the particle in the time $(t_2 - t_1)$.

Further, to calculate the slope of the tangent at P, we choose a point R $\left(2 + \delta t, \frac{1}{2}(2 + \delta t)^2\right)$

on the curve, near P. Then the required slope is $\lim_{\delta t \to 0}$ (slope of PR).

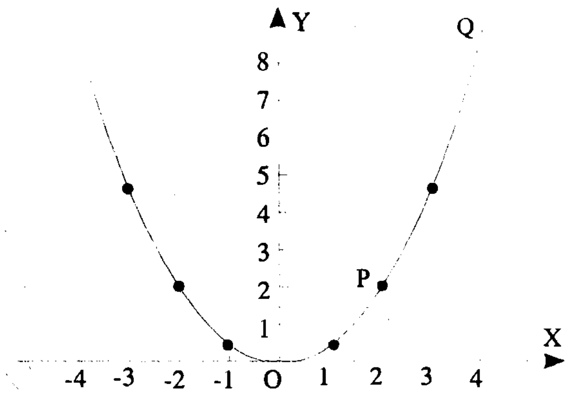


Fig. 5

$$\lim_{\delta t \to 0} \frac{\frac{1}{2}(2+\delta t)^2 - 2}{(2+\delta t)-2} = \lim_{\delta t \to 0} \frac{(4+\delta t)-\delta t}{2\delta t} = \lim_{\delta t \to 0} \frac{4+\delta t}{2} = 2$$

Ans, what is the velocity at t_1? It is $\lim_{\delta t \to 0} \delta s/\delta t$,, which is again equal to 2. Thus the velocity at t_1 is the same as the slope of the tangent at P.

Remark 2 i) Example 3 is a particular case of the general result: If the path of a particle moving according to s = f(t) is shown in the ts-plane and if P and Q are points on the path which correspond to $t = t_1$ and $t = t_2$, then the average velocity of the particle in time $(t_2 - t_1)$ is given by the slope of PQ and the velocity at time t_1 is given by the slope of the tangent at P.

ii) Distance is always measured in units of length (metres, centimetres, feet) and so velocity v really means v units of distance per unit of time. The slope of the tangent is a dimensionless number, while the velocity has the dimension of length/time.

Now you can try some exercises on your own.

E E 2) A particle is thrown vertically upwards in the air. The distance it covers in time t is given by $s(t) = ut - (1/2)\, gt^2$ where u is the initial velocity and g denotes the acceleration due to gravity. Find the velocity of the particle at any time t.

E E 3) Find the rate of change of the area of a circle with respect to its radius when the radius is 2 cm. (Hint: Express the area of a circle as a function of its radius first).

E E 4) Find the average rate of change of the function f defined by $f(x) = 2x^2 + 1$, $\forall x \in \mathbf{R}$ in the interval $[1, 1 + h]$ and hence evaluate the rate of change of f at $x = 1$.

3.2.3 The Derivative

We have seen that the slope of a tangent and the rate of growth have the same basic concept behind them. Won't it be better, then, to give a separate name to this basic concept, and study it independently of its diverse applications? We give it the name "derivative".

Definition 1 Let $y = f(x)$ be a real-valued function whose domain is a subset D on **R**. Let $x \in D$. If

$$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$ exists, then it is called the **derivative of f at x.**

The notation dy/dx is due to Leibniz and $f'(x)$ is due to Lagrange (1736-1813)

Now, if we write $f(x + \delta x) = y + \delta y$, then derivative of $f = \lim_{\delta x \to 0} \delta y/\delta x$. Here δy denotes the change in y caused by a change δx in x.

The derivative is denoted variously by $f'(x)$, dy/dx or Df. The value of $f'(x)$ at a point x_0 is denoted by $f'(x_0)$. Thus,

$$f(x_0) = \lim_{\delta x \to 0} \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}$$

If, in the expression $f(x_0) = \lim_{\delta x \to 0} \frac{f(x_0 + \delta x) - (f(x_0)}{\delta x}$ we write

$x_0 + \delta x = x$, we get $\delta x = x - x_0$, and $\lim_{\delta x_0 \to 0} \Leftrightarrow \lim_{x \to x_0}$

$$f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

This is an alternative expression for the derivative of f at the point x_0.

Remark 3 In this definition x and y are real numbers and are two dimensionless numbers. If x and y are dimensional quantities (length, time, distance, velocity, area, volume) then the derivative will also have a dimension. For convenience, we shall always treat x and y as dimensionless real numbers. The appropriate dimensions can be added later.

Caution: 'dy' and 'dx' in the expression dy/dx are **not separate entities**. You cannot cancel 'd' from dy/dx to get y/x. The notation only suggests the fact that the derivative is obtained as a ratio.

When $f'(x)$ exists, we say that f is **differentiable (or derivable)** at x. When f is differentiable at each point of its domain D, then f is said to be a **differentiable function**. The process of obtaining the derivative is called **differentiation**. The function f' which associates to each point x of D, the derivative $f'(x)$ at x, is called the **derived function** of f. Thus, the domain of the derived function is {x D: $f'(x)$ exists}.

The process of finding the derivative of a function by actually calculating the limit of the ratio

$\frac{f(x + \delta x) - f(x)}{\delta x}$ is called **differentiating from first principles.**

As we shall see later, it is not always necessary to find a derivative from the first principles. We shall develop certain rules which can be used to write down the derivatives of some functions without actually finding the limit. Some such rules are contained in the next section.

3.3 DERIVATIVES OF SOME SIMPLE FUNCTIONS

In this section we shall find the derivatives of some simple functions like the constant function, the power function and the absolute value function. We shall illustrate the method of finding the derivative by the first principle method through some examples.

Example 4 Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a constant function, that is, $f(x) = c$ for all $x \in \mathbf{R}$, c being a real number. We shall show that f is differentiable, and its derivative is zero.

Now, $$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{c - c}{\delta x} = \lim_{\delta x \to 0} 0 = 0$$

Hence, a constant function is differentiable and its derivatives is equal to zero at any point of its domain.

The result of the above example can be seen more easily geometrically. The constant function $f(x) = c \ \forall \ x \in \mathbf{R}$ represents the straight line $y = c$ which is parallel to the x - axis (Fig. 6). If we join any two points, P and Q, on it, the line PQ is parallel to the x-axis. Hence, the angle made by PQ with the x-axis is zero. This means the slope of PQ is $\tan 0 = 0$. Since $f'(x)$ is the limit of this slope as $Q \rightarrow P$, we get $f'(x) = 0$ for all x in the domain of f.

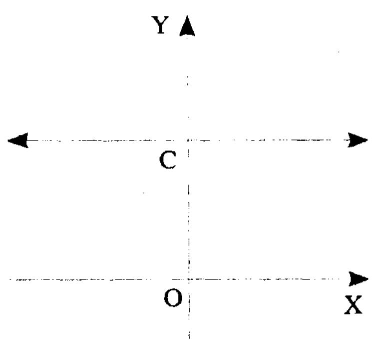


Fig. 6

Example 5 We now show that, if n is a positive integer, then $D(x^n) = nx^{n-1}$.

In order to obtain $D(x^n)$, in case it exists, we have to determine

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Notice that we have used the letter h (instead of our usual δx) to denote the small change in the variable x. We are, in fact, free to use any notation; but δx and h are the more commonly used ones.

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h},$$

$$= \lim_{h \to 0} \frac{(x^n + nhx^{n-1} + \ldots + h^n) - x^n}{h} \quad \text{(by bionomial theorem)}$$

$$= \lim_{h \to 0} \left\{ nx^{n-1} + \frac{n(n-1)}{2} hx^{n-2} + \ldots + h^{n-1} \right\}$$

$$= nx^{n-1}$$

The result of the above example is very useful. We shall show later, that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all non-zero $x \in \mathbf{R}$ even when n is a negative integer. The result also holds for all $x > 0$ if n is any non-zero real number. Of course, if $n = 0$, then $x^n = 1 \;\forall\; x$ and hence, $Dx^n = 0$ for all $x \in \mathbf{R}$. This means that the result is trivially true for $n = 0$. Nevertheless, right now we are in a position to prove this result for $n = 1/2$. That is,

$\frac{d}{dx}(\sqrt{x}) = \left(\frac{1}{2}\right) x^{-1/2}$, and this we do in the following example.

Example 6 We shall show that the function f defined by $f(x) = \sqrt{x}$, $x > 0$ is differentiable.

$\sqrt{a}$ is defined for $x < 0$.

We have, $$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{(x+y)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-1/2}$$

The result of our next example is of great significance. Recall that, in Sec. 2 we mentioned that there are functions that have no tangents at some point (or equivalently, have no derivative there). This example will illustrate this fact. Before giving the example we give some definitions.

$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, if it exists, is called the **right hand derivative** of f(x) at x = a and is written as $Rf'(a)$. Likewise, $\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$ is called the **left hand derivative** of f(x) at x = a and is written as $Lf'(a)$. If $f'(a)$ exists, we must have $Rf(a) = Lf'(a) = f'(a)$ (See Unit 2, Theorem 4).

Example 7 The function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x|$ is not derivable at x = 0 but is derivable at every other point of its domain.

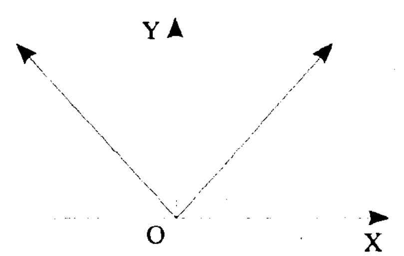


Fig. 7

Fig. 7 shows the graph of this function.

To prove that the given function is not derivable at x = 0, we have to show that

$\lim_{h \to 0} \frac{|0+h|-|0|}{h}$ does not exist. In fact as we shall see, RF′(0) and Lf′(0) both exist, but they are not equal.

Now $Rf'(0) = \lim_{h \to 0} \frac{|0+h|-|0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \frac{h}{h} = 1$ (since $|h| = h$ for $h > 0$)

And $Lf'(0) = \lim_{h \to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h \to 0} \frac{|h|}{h} = \frac{-h}{h} = -1$ (since $|h| = -h$ for $h < 0$)

Therefore, $RF'(0) = 1 \neq -1 = Lf'(0)$. Hence $f'(0)$ does not exist. We shall now show that the function is derivable at every other point.

First, let $x > 0$. Choose h so that $|h| < x$. This will ensure that $x + h > 0$ whether $h > 0$ or $h < 0$. Now,

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{|x+h|-|x|}{h}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h}$$

$$= \lim_{h \to 0} h/h = 1$$

Thus f is derivable at x, and $f'(x) = 1$ for all $x > 0$.

You can now complete the solution by solving E 5).

E E 5) Show that $y = |x|$ is derivable and $f'(x) = -1$ at all points $x < 0$.

E E 6) Show that each of the following functions is derivable at $x = 2$. Find $f'(2)$ in each case.

a) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x$

b) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = ax + b$ where a and b are fixed real numbers.

E E7) Find dy/dx, wherever it exists, for each of the following functions:

a) $y = x^3$ b) $y = |x+1|$ c) $y = \sqrt{2x+1}$, $x \geq -\frac{1}{2}$

So far we have obtained derivatives of certain functions by differentiating from the first principles. That is, each time we have calculated $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$.

But the process of taking limits is a very lengthy and complicated affair. In the section we shall see how to simplify the process of differentiation for some functions.

3.4 ALGEBRA OF DERIVATIVES

Consider the function $f(x) = \frac{2x^3 + 3x^2}{x^4 - 1}$. If we try to find the derivative of this function from the first principles, we will have to do lengthy, complicated calculations. However, a close look at this function reveals that it is composed of several functions: constant functions like 2, 3 and -1, and power functions like x^3, x^2 and x^4. We already know the derivatives of these functions. Can we use this knowledge to find the derivative of f(x)? In this section we shall state and prove some theorems which help us do just that.

3.4.1 Derivative of a Scalar Multiple of a Function

Let $f: \mathbf{R} \to \mathbf{R}$ be a differentiable function and let $c \in \mathbf{R}$. Then, consider the function $y = cf(x)$. We call this function a scalar multiple of f by c (see Unit 1). The derivative of y with respect to x is

$$\lim_{h \to 0} \frac{(cf)(x+h) - cf(x)}{h} = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= \lim_{h\to 0} c\, \frac{f(x+h)-f(x)}{h}$$

$$= c \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{(by Theorem 3 of Unit 2)}$$

$$= cf'(x)$$

Thus, we have just proved the following theorem.

Theorem 1 If f is a differentiable function and $c \in \mathbf{R}$ then cf is differentiable and $(cf)'(x) = cf'(x)$.

Example 8 To differentiate $y = 7|x|$ we apply the scalar multiple rule obtained in Theorem 1 at all points where the function $|x|$ is differentiable and get

$$\frac{d}{dx}(7|x|) = 7\frac{d}{dx}(|x|)$$

But, in view of Example 7, when $x = 0$, $\frac{d}{dx}(|x|)$ does not exist. When $x > 0$, $\frac{d}{dx}(|x|) = 1$ and

when $x < 0$, $\frac{d}{dx}(|x|) = -1$.

$$\text{Therefore, } \frac{d}{dx}(7|x|) = 7\frac{d}{dx}(|x|)$$

$$= \begin{cases} 7 \text{ when } x > 0 \\ -7 \text{ when } x < 0 \end{cases}$$

and $\frac{d}{dx}(7|x|)$ does not exist at $x = 0$.

Note: In example 8 we have used the fact that if $f'(x)$ does not exist at a point then $(cf).(x)$ also does not exist at that point.

Try the following exercise now.

E E8) Differentiate the following, using Theorem 1.

a) $(5/3)x^3$ b) $8\sqrt{x}$

3.4.2 Derivative of the Sum of Two Functions

We know the sum of two functions f and g defined on R, denoted by f + g, is defined as $(f+g)(x) = f(x) + g(x)\ \forall\, x \in R$

Let f and g be differentiable from **R** to **R**. Let us examine whether f + g, the sum of the functions f and g, is differentiable. Now,

$$\lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{\{f(x+h)+g(x+h)-f(x)-g(x)\}}{h}$$

$$= \lim_{h\to 0} \left\{\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right\}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x)$$

Thus, we have proved the following :

Theorem 2 The sum of two differentiable functions f and g is a differentiable function and each of $(f+g)'(x) = f'(x) + g'(x) \ \forall \ x \in \mathbf{R}$.

The above result can be easily extended to a finite sum, that is,

$$\frac{d}{dx}(f_1 + f_2 + ... + f_n) = \frac{df_1}{dx} + \frac{df_2}{dx} + ... + \frac{df_n}{dx}$$

where $f_1, ..., f_n$ are differentiable functions.

Remark 4 From Theorems 1 and 2 it follows that if f and g are differentiable functions, then $f - g$ is also a differentiable function (since $f - g = f + (-g)$), and $(f-g)'(x) = f'(x) - g'(x)$.

Let us see how Theorem 2 is useful in the following example.

Example 9 To differentiate $3x^2 + 41x - 9$, we apply Theorem 2, and get,

$$\frac{d}{dx}(3x^2 + 41x - 9) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(41x) + \frac{d}{dx}(-9)$$

$$\text{Now, } \frac{d}{dx}(3x^2) = 3\frac{dx^2}{dx} \text{ (in view of the theorem)}$$

$$= 3 \times 2x = 6x$$

$$\frac{d}{dx}(41x) = 41\frac{dx}{dx} = 41$$

$$\text{and } \frac{d}{dx}(-9) = 0 \text{ (see Example 4).}$$

$$\text{Thus, } \frac{d}{dx}(3x^2 + 41x - 9) = 6x + 41$$

You are now in a position to solve this exercise.

E E9) Differentiate the following :

a) $5x^3 + 2$ (b) $a_n + a_1x + a_2x_2 + ... + a_nx^n$, where $a_1 \in \mathbf{R}$ for $i = 1, 2, ..., n$.

3.4.3 Derivative of the Product of Two Functions

Let f and g be two differentiable functions on **R**. We want to find out whether their product fg is also differentiable.

Now, $\lim_{h\to 0} \frac{fg(x+h)-fg(x)}{h}$

$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h)-f(x)\,g(x)}{h}$ (We have added and subtracted f(x) g(x + h))

$= \lim_{h\to 0} \frac{\{f(x+h)-f(x)\}g(x+h)+\{g(x+h)-g(x)\}\,f(x)\}}{h}$

$= \lim_{h\to 0} \left\{\frac{f(x+h)-f(x)}{h}\,g(x+h)\right\} + \lim_{h\to 0}\left\{\frac{g(x+h)-g(x)}{h}\,f(x)\right\}$

$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \lim_{h\to 0} g(x+h) + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \lim_{h\to 0} f(x)$

(Ref. Unit 2, Theorem 3)

$= f'(x)\,g(x) + g'(x)\,f(x)$

Thus, we get the following

Theorem 3 The product of two differentiable functions is again a differentiable function and its derivative at any point x is given by the formula.

$(fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

We can extend this result to the product of three differentiable functions. This gives us $(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

You see, you have to **differentiate only** one function at a time. This result can also be extended to the product of any finite number of differentiable functions. Thus, if $f_1, \ldots f_n$ are differentiable functions, then,

$(f_1 f_2 \ldots f_n)'(x) = f_1'(x)f_2(x) \ldots f_n(x) + f_1(x)f_2'(x)f_3(x) \ldots f_n(x) + \ldots + f_1(x)f_2(x) \ldots f_n'(x).$

Theorem 3 is very useful in simplifying calculations, as you can see in the following example.

Example 10 The differentiate $f(x) = x^2(x+4)$, we take $g(x) = x^2$, $h(x) = x + 4$. We have, $f(x) = x^2(x+4) = g(x)h(x)$

Now, $g'(x) = \frac{d}{dx}(x^2) = 2x$ and $h'(x) = \frac{d}{dx}(x+4) = 1.$

$$\begin{aligned} \text{Thus, } f'(x) &= g'(x)h(x) + h'(x)g(x) \\ &= 2x\,(x+4) + 1 \times x^2 \\ &= 2x^2 + 8x + x^2 = 3x^2 + 8x \end{aligned}$$

Remark 5 You could also have differentiated $x^2(x+4)$ without using Theorem 3, as follows :

$x^2(x+4) = x^3 + 4x^2$

$$\begin{aligned} \text{Therefore, } \frac{d}{dx}(x^2(x+4)) &= \frac{d}{dx}(x^3 + 4x^2) \\ &= \frac{d}{dx}(x^3) + \frac{d}{dx}(4x^2) \text{ (By Theorem 2)} \\ &= 3x^2 + 4(2x) = 3x^2 + 8x \end{aligned}$$

This shows that the same function can be differentiated by using different methods. You may use any method that you find convenient. This observation should also help you to check the correctness of your result. (We assume that you would not make the same mistake while using two different methods !)

E 10) Using Theorem 3, differentiate the following functions. Also, differentiate these functions without using Theorem 3, and compare the results.

a) $x\sqrt{x}$ b) $(x^5+2x^3\,5)^2$ c) $(x+1)(x+2)(x+3)$

3.4.4 Derivative of the Quotient of Two Functions

Let $\phi = f/g$, where f and g are differentiable functions on **R**, and $g(x) \neq 0$ for any x. Then,

$$\lim_{h\to 0} \frac{\phi(x+h) - \phi(x)}{h} = \lim_{h\to 0} \frac{(f/g)(x+h) - (f/g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \lim_{h\to 0} \frac{g(x)\,f(x+h) - f(x)\,g(x+h)}{h\,g(x)g(x+h)}$$

$$= \lim_{h\to 0} \frac{g(x)\dfrac{f(x+h) - f(x)}{h} - f(x)\dfrac{g(x+h) - g(x)}{h}}{g(x+h)\,g(x)}$$

(by adding and subtracting f(x)g(x) from the numerator)

$$= \frac{\lim\limits_{h\to 0}\left\{g(x)\dfrac{f(x+h) - f(x)}{h} - f(x)\dfrac{g(x+h) - g(x)}{h}\right\}}{\lim\limits_{h\to 0} g(x+h) - g(x)}$$

$$= \frac{\lim\limits_{h\to 0}\left[g(x)\left\{\dfrac{f(x+h) - f(x)}{h}\right\}\right] - \lim\limits_{h\to 0}\left[f(x)\left\{\dfrac{g(x+h) - g(x)}{h}\right\}\right]}{\lim\limits_{h\to 0} g(x+h)\ \lim\limits_{h\to 0} g(x)}$$

$$= \frac{g(x)\,f'(x) - f(x)g'(x)}{(g(x))^2}$$

Thus, we get the following.

Theorem 4 The quotient f/g of two differentiable functions f and g such that $g(x) \neq 0$, for any x in its domain, is again a differentiable function and its derivative at any point x is given by the following formula :

$$(f/g)'(x) = \frac{g(x)\, f'(x) - f(x) g'(x)}{(g(x))^2}$$

This can also be written as

$$\frac{d}{dx}\left(\frac{\text{numerator}}{\text{denominator}}\right)$$

$$= \frac{(\text{denominator})\ (\text{derivative of numerator}) - (\text{numerator})\ (\text{derivative of denominator})}{(\text{denominator})}$$

We will obtain an important corollary to Theorem **4** now.

Corollary 1 If g is a function such that $g(x) \neq 0$ for any x in its domain, then

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right)' = \frac{-g'(x)}{(g(x))^2}$$

Proof: In the result of Theorem 4, take f to be the constant function 1. Then $f'(x) = 0$ for all x.

Therefore, $\left(\frac{1}{g(x)}\right) = \left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\, f'(x) - f(x) g'(x)}{(g(x))^2}$ where $f(x) = 1$.

$$= \frac{g(x) \times 0 - 1 \times g'(x)}{(g(x))^2} = \frac{-g'(x)}{(g(x))^2}$$

Example 11 We shall now show that $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a negative integer and $x \neq 0$.

We have already proved this result for a positive integer n in Example 5.

Consider the function : $\mathbf{R} \setminus \{0\} \to \mathbf{R}$ given by $f(x) = x^{-m}$, where $m \in \mathbf{N}$. Then $f(x) = 1/x^{-m}\ \forall\, x \in \mathbf{R}$. Thus, $f = 1/g$, where $g(x) = x^m$ for all $x \in \mathbf{R}$, $x \neq 0$. g is a differentiable function and $g(x) \neq 0$ as $x \neq 0$. So, except at $x = 0$, we find that

$$f'(x) = \frac{-g'(x)}{(g(x))^2} \text{ (from Corollary 1)}$$

$$= \frac{-mx^{m-1}}{(x^m)^2} \quad (g'(x) = mx^{m-1} \text{ by using Example 5})$$

$$= \frac{-mx^{m-1}}{(x^{2m})} = -ms^{-m-1}$$

Denoting $-m$ by n, we get $f(x) = x^n$, and $f'(x) = nx^{n-1}$

Example 12 Let us differentiate the function f given by $f(x) = (x^{-2} + 2) / (x^2 + 2x)$

We can write f as the quotient g/h where $g(x) = (x^{-2} + 2)$ and $h(x) = x^2 + 2x$.

Now, $g'(x) = \frac{d}{dx}(x^{-2}) + \frac{d}{dx}(2) = -2x^{-3} + 0 = \frac{-2}{x^3}$

Also $h'(x) = 2x + 2$.

$$\text{Therefore, } f'(x) = \frac{h(x)\, g'(x) - g(x) h'(x)}{(h(x))^2}$$

$$= \frac{(x^2 + 2x)\ (-2/x^3) - (x^{-2} + 2)\ (2x + 2)}{(x^2 + 2x)^2}$$

$$= \frac{-4x^{-1} - 6x^{-2} - 4x - 4}{(x^2 + 2x)^2}$$

$$= \frac{-(4x + 4 + 4x^{-1} - 6x^{-2})}{(x^2 + 2x)^2}$$

E E 11) Differentiate

a) $\frac{2x + 1}{x + 5}$ b) $\frac{1}{a + bx + cx^2 + dx^3}$, where a, b, c, d are fixed real numbers

c) $\frac{2x^3 + 3x^2}{x^4 - 1}$

E E 12) Obtain the derivative of $1/f(x)$ by differentiating from first principles, assuming that $f(x) \neq 0$ for any x.

E E 13) Differentiate $f(x) = \dfrac{2 + 5x + 7x^{-1}}{x^5}$ by three different methods.

3.4.5 The Chain Rule of Differentiation

The chain rule of differentiation is a rule for differentiating a composite of functions (Ref. Unit 1). It is a remarkable rule which helps us to differentiate complicated functions in an easy and elegant way.

We establish the rule in the following theorem.

Theorem 5 Let $y = g(u) = g(f(x)) = (gof)(x)$, so that y is the composite function gof. We are given that y, regarded as a function of u, is differentiable and u regarded as a function of x is differentiable with respect to x. We want to prove that y, regarded as a function of x, is also differentiable. To do this we must show that $\lim\limits_{\delta x \to 0} \delta y/\delta x$ exists where δy is the change in the variable y corresponding to a change δx in the variable x. Now, δu, be the change in the value of u corresponding to a change δx in the value of x, is given by $\delta u = f(x + \delta x) - f(x)$.

We have $\lim\limits_{\delta x \to 0} \delta u = \lim\limits_{\delta x \to 0} \left(\dfrac{\delta u}{\delta x}\, \delta x\right)$

$$= \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \times \lim_{\delta x \to 0} \delta x$$

$$= \frac{du}{dx} \times 0 = 0$$

This means that $\delta u \to 0$ as $\delta x \to \mathbf{0}$

We assume however that $\delta u \neq 0$.

This implies that $\lim_{\delta x \to 0} \frac{\delta y}{\delta u} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u}$ exists and is equal to $\frac{dy}{du}$.

Now, $\frac{\delta y}{\delta u} = \frac{\delta y}{\delta u} \frac{\delta y}{\delta x}$, and we know that

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta u} = \frac{dy}{du} \text{ and } \lim_{\delta x \to 0} \frac{\delta u}{\delta x} = \frac{du}{dx}$$

Hence, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \cdot \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

Hence dy/dx exists and is equal to $dy/du \times du/dx$.

You may find it more convenient to remember and use the rule in the following form:

If $h(x) = g(f(x))$ is the composite of two differentiable functions g and f, then h is differentiable and $h'(x) = g'(f(x))f'(x)$ where $g'(f(x))$ denotes derivative of $g(f(x))$ w.r.t $f(x)$

To clarify this rule let us look at the following example

Example 13 Here we shall differentiate $y = (2x + 1)^3$ with respect to x.

Let $u = 2x + 1$. Then $y (2x + 1)^3 = u^3$.

Now y is a differentiable function of u and u is a differentiable function of x. $dy/du = 3u^2$ and $du/dx = 2$. Hence we can use the chain rule to get $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= 3u^2 . 2 = 6u^2 = 6(2x + 1)^2.$$

You might be thinking that there was really no necessity of using the chain rule here. We could simply expand $(2x + 1)^3$ and then write the derivative. But the situation is not always as simple as in this example. You would appreciate the power of the chain rule after using it in the next example.

Example 14 To differentiate $(x^3 + 2x^2 - 1)^{100}$,

let $y = (x^3 + 2x^2 - 1)^{100}$ and let $u = (x^3 + 2x^2 - 1)$. Then $y = u^{100}$
Since dy/du and du/dx both exist, and $dy/du = 100u^{99}$ and $du/dx = 3x^2 + 4x$, therefore, by chain rule, $dy/dx = dy/du = dy/du.du/dx$.

$= 100\, u^{99} . (3x^2 + 4x)$

$= 100\, (x^3 + 2x^2 - 1)^{99} (3x^2 + 4x)$

Can you really attempt to solve the above example without using the chain rule? Don't you think the rule has simplified matters a lot for you ?

Instead of introducing u explicitly each time while applying the chain rule, after a little practice you would find it more convenient to do away with u and arrange the working in the above example as follows :

$$\frac{dy}{dx} = \frac{d(x^3 + 2x^2 - 1)^{100}}{d(x^3 + 2x^2 - 1)} \frac{d}{dx} (x^3 + 2x^2 - 1)$$

$$= 100\, (x^3 + 2x^2 - 1)^{100-1} \quad (3x^2 + 4x)$$

$$= 100\, (x^3 + 2x^2 - 1)^{99} \, (3x^2 + 4x)$$

Our next example illustrates that this rule can be extended to three functions.

Example 15 To differentiate $[(5x+2)^2+3]^4$, we write $y=\{(5x+2)^2+3\}^4$, $u=(5x+2)^2+3$ and $v=5x+2$

Then $y=u^4$ and $u=v^2+3$. That is, y is a function of u, u is a function of v, and v is function of x. By extending the chain rule, we get

$$\frac{dy}{dx}=\frac{dy}{du}\frac{dy}{du}\frac{dv}{dx}$$

This gives,

$$\begin{aligned} dy/dx &= 4u^3 . 2v . 5 = 40\,u^3v \\ &= 40\,[(5x+2)^2+3]^3\,(5x+2) \end{aligned}$$

This example illustrates that there may be situations in which we may go on using chain rule for a function of a function ..., and so on. This perhaps justifies the name 'chain' rule. Thus, if $g_1 \ldots g_n$ and h are functions such that $h=(g_1 o g_2 o \ldots o g_n)(x)$, then

$$h'(x)=g'_1(g_2 o \ldots o g_n(x))\, g'_2(g_3 o \ldots o g_n)(x) \ldots g'_{n-1}(g_n)(x) . g'_n(x)$$

E 14) Find dy/dx for each of the following using the chain rule:

a) $\dfrac{5}{1+5x+7x^2}$ b) $\dfrac{(2x+3)^2}{1+(2x+3)^3}$ c) $\left\{(9x+5)^3+(9x+5)^{-3}\right\}^7$

3.5 CONTINUITY VERSUS DERIVABILITY

We end this unit with the relationship of differentiability with continuity, which we have studied in Unit 2. In Sec. 3 of Unit 2 we proved that the functon $y=|x|$ is continuous $\forall x \in \mathbf{R}$. We have also proved that this function is derivable at every point except at $x=0$ in Example 7. This means that the function $y=|x|$ is continuous at $x=0$, but is not derivable at that point. Thus, this shows that a function can be continuous at a point without being derivable at that point. However, we will now prove that if a function is derivable at a point, then it must be continuous at that point; or derivability $\Rightarrow$ continuity.

Recall that a function f is said to be continuous at a point x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.

Theorem 6 Let f be a function defined on an interval I. If f is derivable at a point $x_0 \in I$, then it is continuous at x_0.

Proof:

If $x \neq x_0$ then we may write $f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0} . (x - x_0)$

Since f is derivable at x_0, therefore, $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and equals $(f'(x_0)$.

Thus, taking limits as $x \to x_0$, we have,

$$\lim_{x \to x_0} [f(x) - f(x_0)]$$

$$= \lim_{x \to x_0} \left\{ \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right\}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0)$$

$$= f'(x_0) \times 0 = 0$$

Therefore $\lim_{x \to x_0} f(x) - \lim_{x \to x_0} f(x_0) = 0$

That is, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} f(x_0) = f(x_0)$

Consequently, f is continuous at x_0.

As we have seen, the function $y = | x |$ is continuous but not derivable at only one point, $x = 0$. But there are some continuous functions which are not derivable at infinitely many points. For instance, look at Fig. 8.

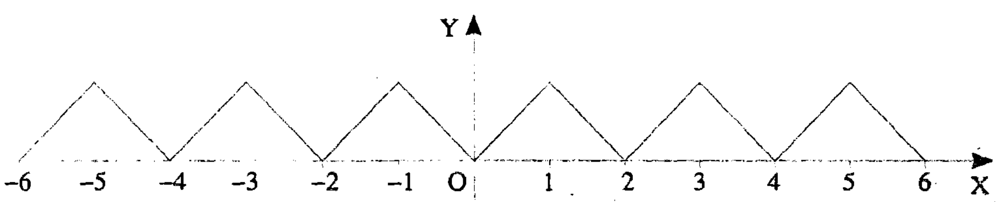


Fig. 8

It shows the graph of a continuous function which is not derivable at infinitely many points. Can you mark those infinitely many points at which this function is not derivable ? You can take your hint from the graph of the functon $y = | x |$.

The situation is, in fact, much worse. There are functions which are **continuous everywhere** but **differentiable nowhere.** The discovery came as a surprise to the nineteeth century mathematicians who believed, till then, that if a function is so bad that it is not derivable at any point, then it can't be so good that it is continuous at every point. The first such function was put forth by Weierstrass (although he is said to have attributed the discovery to Riemann) in 1812. He showed that the function f given by $f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$, where a is an odd integer and b is a positive constant between 0 and 1 such that $ab > 1 + 3\pi/2$, is a function which is continuous everywhere, but derivable nowhere. It will not be possible for us to prove this assertion at this stage.

Sometimes we use Theorem 6 to prove that a given function is continuous at a given point. We prove that its derivative exists at that point. By Theorem 6 then, the community automatically follows.

E E 15) Is the function $f : [0, 1] \to R : f(x) = \frac{(2x + 3)^{50}}{9x + 2}$ continuous at $x = 0.1$?

3.6 SUMMARY

We conclude this unit by summarising what we have covered in it.

1. For any function $y = f(x)$

   $\lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$ (if it exists) is called the **derivative** of f at x, denoted by $f'(x)$.

   The function f' is the derived function. The derivative $f'(x)$ is the slope of the tangent to the curve $y = f(x)$ at the point (x, y). The derivative also gives the rate of change of the function with respect to the independent variable.

2. The derivative of a constant function is 0.

   $\dfrac{d}{dx}(x^n) = nx^{n-1}$,

   where n is any integer (and $x \neq 0$ if $n \leq 0$).

   $\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$

3. The function $y = |x|$ is derivable at every point except at $x = 0$.

4. $(cf)' = cf'$, c a constant.

   $(f+g)' = f' - g'$

   $(f-g)' = f' - g'$

   $(fg)' = fg' + gf'$

   $(f/g)' = \dfrac{g'f - f'g}{g^2}$

   $(gof)' = g'(f).f'$

5. Every derivable function is continuous. The converse is not true, that is, there exist functions which are continuous but not differentiable.

3.7 SOLUTIONS AND ANSWERS

E1) a) $\left.\dfrac{dy}{dx}\right|_{x=1} = -1/4.$ Equation : $(y - 1/2) = (-1/4)(x - 2)$ or $x + 4y = 4$

b) $\left.\dfrac{dy}{dx}\right|_{x=1} = 3.$ Equation: $(y - 1) = 3(x - 1)$

E2) $v = ds/dt = u - gt$

E3) area $= A = \pi r^2$. $\left.\dfrac{dA}{dr}\right|_{r=1} = 4\pi$

E4) average rate of change of f in [1, 1 + h] $= \dfrac{f(1+h)-f(1)}{h}$

$$= \frac{2(1+h)^2 - 1 - (2 \times 1^2 + 1)}{h} = 4 + 2h$$

rate of change of f at x = 1 $= \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$, where h may be positive or negative.

$$= \lim_{h\to 0} (4 + 2h) = 4$$

E5) If $x < 0$. choose $0 < h < |x|$ then $x + h < 0$ and

$$\lim_{h\to 0} \frac{|x+h| - |x|}{h} = \frac{-(x+h)-(-x)}{h} = \frac{-h}{h} = -1$$

Thus $f'(x) = -1$ if $x < 0$. f is derivable for $x < 0$.

E6) a) $f'(2) = \lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0} \dfrac{2+h-2}{h} = 1$

b) $f'(2) = \lim\limits_{h\to 0} \dfrac{a(2+h)+b-(a \times 2 + b)}{h}$

$$= \lim_{h\to 0} \frac{ah}{h} = a.$$

E7) a) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = \lim\limits_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2$

b) If $x > -1$, $x + 1 > 0$, choose $h > 0$ s.t. $h < |x + 1|$.

then $x + h + 1 > 0$ and $\lim\limits_{h\to 0} \dfrac{|x+h+1| - |x+1|}{h}$

$$= \lim_{h\to 0} \frac{(x+h+1)-(x+1)}{h} = 1.$$

If $x < -1$, $\lim\limits_{h\to 0} \dfrac{|x+h+1| - |x+1|}{h} = -1$.

Thus dy/dx exists when $x > -1$. It does not exist when $x < -1$. It does not exist when $x = -1$ since $Rf'(-1) = 1$ and $Lf'(-1) = -1$.

c) $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \times \dfrac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}}$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2(x+h)+1} + \sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

E8) a) $\dfrac{d}{dx}\left(\dfrac{5}{3}x^3\right) = \dfrac{5}{3}\dfrac{d}{dx}(x^3) = \dfrac{5}{3} \times 3x^2 = 5x^2$

b) $\dfrac{d}{dx}\left(8\sqrt{x}\right) = 8\dfrac{d}{dx}(\sqrt{x}) = \dfrac{4}{\sqrt{x}}$

E9) a) $15x^2$ b) $a_1 + 2a_2x + \ldots + na_nx^{n-1}$

E10) a) $\dfrac{d}{dx}\left(x\sqrt{x}\right) = \left(\dfrac{1}{2}\right)x^{-1/2}.\,x + 1.\sqrt{x} = \dfrac{1}{2}x^{1/2} + x^{1/2} = \dfrac{3}{2}\sqrt{x}$

b) $\dfrac{d}{dx}\{(x^5 + 2x^3 + 5)(x^5 + 2x^3 + 5)\}$

$$= (x^5 + 2x^3 + 5)(5x^4 + 6x^2) + (5x^4 + 6x^2)(x^5 + 2x^3 + 5)$$

$$= 2(x^5+2x^3+5)(5x^4+6x^2)$$

c) $dy/dx = (x+2)(x+3)+(x+1)(x+3)+(x+1)(x+2)$

E11) a) $\dfrac{2(x+5)-(2x+1)}{(x+5)^2} = \dfrac{9}{(x+5)^2}$

b) $\dfrac{-(b+2cx+3dx^2)}{(a+bx+cx^2+dx^3)^2}$

c) $\dfrac{(x^4-1)(6x^2+6x)-(2x^3+3x^2)(4x^3)}{(x^4-1)^2}$

$$= \frac{6x(x+1)(x^4-1)-4x^5(2x+3)}{(X^4-1)^2}$$

E12) $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right) = \lim\limits_{h\to 0} \dfrac{\dfrac{1}{f(x+h)} - \dfrac{1}{f(x)}}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\,\frac{f(x)-f(x+h)}{f(x)\,f(x+h)}$$

$$= \frac{\lim\limits_{h\to 0} \dfrac{f(x)-f(x+h)}{h}}{f(x)\lim\limits_{h\to 0} f(x+h)}$$

$$= \frac{-f'(x)}{f(x)^2}$$

E13) a) $f(x) = 2x^{-5}+5x^{-4}+7x^{-6}$
$f'(x) = -10x^{-6}-20x^{-5}-42x^{-7}$

b) $f'(x) = \dfrac{x^5(5-7x^{-2})-5x^4(2+5x+7x^{-1})}{x^{10}}$

$= x^{-10}(-20x^5-42x^3-10x^4)$
$= -20x^{-5}-42x^{-7}-10x^{-6}$

c) $f(x) = x^{-5}(2+5x+7x^{-1})$
$f'(x) = x^{-5}(5-7x^{-2})-5x^{-6}(2+5x+7x^{-1})$
$= -20x^{-5}-42x^{-7}-10x^{-6}$

E14) a) $u = 1+5x+7x^2,\ y = 5/2$

$dy/dx = (-5/u^2)(5+14x) = \dfrac{-25-70x}{(1+5x+7x^2)}$

b) $u = 2x+3,\ y = u^2/(1+u^3)$

$$\frac{dy}{dx} = \frac{2u(1+u^3)-3u^4}{(1+u^3)^2} \times 2$$

$$= \frac{4(2x+3)[1+(2x+3)^3]-6(2x+3)^4}{[1+(2x+3)^2]^2}$$

c) $u = 9x+5,\ v = u^3+u^{-3}\ y = v^7$
$dy/dx = 7v^6(3u^2-3u^{-4}) \times 9$
$= 63[(9x+5)^3+(9x+5)^{-3}]^6[3(9x+5)^2-3(9x+5)^{-4}]$

E15) $f'(x) = \dfrac{50(2x+3)^{49}\ 2(9x+2)-9(2x+3)^{50}}{(9x+2)^2}$

exists at $x = 0.1$. Hence the function is continuous at $x = 0.1$.