UNIT 2 LIMITS AND CONTINUITY

Structure

2.1 INTRODUCTION

The last unit has helped you in recalling some fundamentals that will be needed in this course. We will now begin the study of calculus, starting with the concept of 'limit'. As you read the later units, you will realise that the seeds of calculus were sown as early as the third century B.C. But it was only in the nineteenth century that a rigorous definition of a limit was given by Weierstrass. Before him, Newton, d' Alembert and CAuchy had a clear idea abort limits, but none of them had given a formal and precise definition. They had depended, more or less, on intuition or geometry.

The introduction of limits revolutionised the study of calculus. The cumbersome proofs which were used by the Greek mathematicians have given way to neat, simpler ones.

You may already have an intuitive idea of limits. In Sec. 2 of this unit, we shall give you a precise definition of this concept. This will lead to the study of continuous functions in Sec.3. Most of the functions that you will come across in this course will be continuous. We shall also give you some examples of discontinuous functions.

Objectives

After reading this unit you should be able to :

- calculate the limits of functions whenever they exist,
- identify points of continuity and discontinuity of a function.

2.2 BASIC PROPERTIES OF R

In this section we will introduce you to the notion of 'limit'. We start with considering a situation which a lot of us are familiar with, such as train travel. Suppose we are travelling from Delhi to Agra by a train which will reach Agra at 10.00 a.m. As the time gets closer and closer to 10.00 a.m., the distance of the train from Agra gets closer and closer to zero (assuming that the train is running on time!). Here, if we consider time as out independent variable, denoted by t and distance as a function of time, say f(t), then we see that f(t) approaches zero as t approaches 10. In this case we say that the limit of f(t) is zero as t tends to 10.

Now consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2 + 1$. Let us consider Tables 1(a) and 1(b) in which we give the values of f(x) as x takes values nearer and nearer to 1.

In Table 1(a) we see values of x which are greater than 1. We can also express this by saying that x approaches 1 from the right. Similarly, we can say that x approaches 1 from the left in Table 1(b).

Table 1 (a)

x	1.2	1.1	1.01	1.001
f(x)	2.44	2.42	2.02	2.002

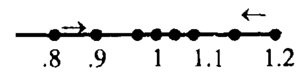


Table 1 (b)

x	0.8	0.9	0.99	0.999
f(x)	1.64	1.81	1,9801	1.9989

We find that, as x gets closer and closer to 1, f(x) gets closer and closer to 2. Alternatively, we express this by saying that as x approaches 1 (or tends to 1), the limit of f(x) is 2. Let us now give a precise meaning of 'limit'.

Definition 1 Let f be a function defined at all points near p (except possibly at p). Let L be a real number. We say that f approaches the limit L as x approaches p if, for each real number $\varepsilon > 0$, ε can find a real number $\delta > 0$ such that

This definition of limit was first stated by Karl Weierstrass, around 1850.

$0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon.$

As you know from Unit 1, $|x - p| < \delta$ means that $x \in \,]p - \delta, p + \delta[$ and $0 < |x - p|$ means that $x \neq p$. That is, $0 < |x - p| < \delta$ means that x can take any value lying between $p - \delta$ and $p + \delta$ except p.

(ε epsilon) and δ (delta) are Greek letters used to denote real numbers.

The limit L is denoted by $\lim\limits_{x \to p} f(x)$. We also write $f(x) \to L$ as $x \to p$.

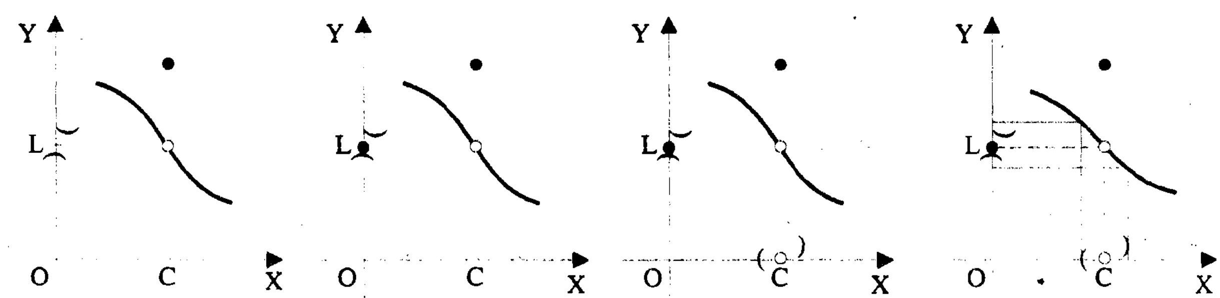
'$\to$' denotes 'tents to'

Note that, in the above definition, we take any real number $\varepsilon > 0$ and then choose some $\delta > 0$, so that $L - \varepsilon < f(x) < L + \varepsilon$, whenever $|x - p| < \delta$, that is, $p - \delta < x < p + \delta$.

In unit 1 we have also mentioned that $|x - p|$ can be thought of as the distance between x and p. In the light of this the definition of the limit of a function can also be interpreted as:

Given $\varepsilon > 0$, we can choose $\delta > 0$ such that if we choose x whose distance from p is less than δ, then the distance of its image from L must be less than ε. The pictures in Fig. 1 may help you absorb the definition.

The $\varepsilon - \delta$ definition does not give us the value of L. It just helpts u check whether a given number L is the limit of f(x).



for each $\varepsilon > 0$ $\quad$ $\exists$ a $\delta > 0$ s.t. $\quad$ $0 < |x - c| < \delta$ $\quad \Rightarrow \quad$ $|f(x) - L| < \varepsilon$

Fig. 1

Remember, the number ε is given first and the number δ is to be produced.

An important point to note here is that while taking the limit of f(x) as $x \to p$, we are concerned only with the values of f(x) as x takes values closer and closer to p, but not when x = p. For example, consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$. This function is not defined for x = 1, but is defined for all other $x \in \mathbf{R}$. However, we can still about is limit as $x \to 1$. This is because for taking the limit we will have to look at the values of f(x) as x tends to 1, but not when x = 1.

Nowlet us take the following examples:

Example 1 Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$. How can we find

$\lim\limits_{x \to 0} = f(x)$?

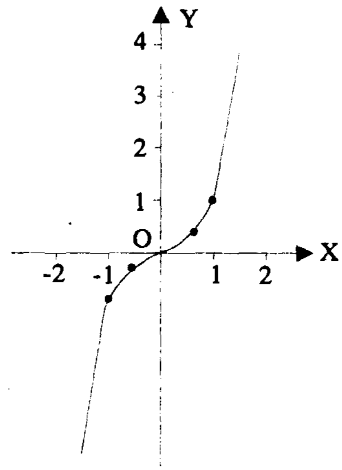


Fig. 2

Look at the graph of f in Fig. 2. You will see that when x is small, x^3 is also small. As x comes closer and closer to 0, x^3 also comes closer and closer to zero. It is reasonable to expect that $\lim f(x) = 0$ as $x \to 0$.

Let us prove that this is what happens. Take any real number $\varepsilon > 0$. Then, $|f(x) - 0| < \varepsilon \Leftrightarrow |x^3| < \varepsilon \Leftrightarrow |x| < \varepsilon^{1/3}$. Therefore, if we choose $\delta = \varepsilon^{1/3}$ we get $|f(x) - 0| < \varepsilon$ whenever $0 < |x - 0| < \delta$. This gives us $\lim\limits_{x\to 0} f(x) = 0$.

A useful general rule to prove $\lim\limits_{x\to a} f(x) = L$ is to write down $f(x) - L$ and then express it in terms of $(x - a)$ as much as possible.

Let us now see how to use this rule to calculate the limit in the following examples.

Example 2 Let us calculate $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{x - 1}$

We know that division by zero is not defined. Thus, the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ is not defined at $x = 1$. But, as we have mentioned earlier, when we calculate the limit as x approaches 1, we do not take the value of the function at $x = 1$. Now, to obtain $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{x - 1}$, we first note that $x^2 - 1 = (x - 1)(x + 1)$, so that, $\dfrac{x^2 - 1}{x - 1} = x + 1$ for $x \neq 1$. Therefore $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{x - 1} = \lim\limits_{x\to 1} (x + 1)$.

As x approaches 1, we can intuitively see that this limit approaches 2. To prove that the limit is 2, we first write $f(x) - L = x + 1 - 2 = x - 1$, which is itself in the form $x - a$, since $a = 1$ in this case. Let us take any number $\varepsilon > 0$. Now,

$|(x + 1) - 2| < \varepsilon \Leftrightarrow |x - 1| < \varepsilon$

Thus, if we choose $\delta = \varepsilon$, in our definiton of limit, we see that

$|x - 1| < \delta = \varepsilon \Rightarrow |f(x) - L| = |x - 1| < \varepsilon$. This shows that $\lim\limits_{x\to 1} (x + 1) = 2$. Hence,

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Example 3 Let us prove that $\lim\limits_{x\to 3} (x^2 + 4) = 13$.

That is, we shall prove that $\forall\ \varepsilon > 0$. $\exists \delta > 0$ such that $|x^2 + 4 - 13| < \varepsilon$ whenver $|x - 3| < \delta$.

Here, $f(x) - L = (x^2 + 4) - 13 = x^2 - 9$, and $x - a = x - 3$.

Now we write $|x^2 - 9|$ in trms of $|x - 3|$:

$|x^2 - 9| = |x + 3||x - 3|$

Thus, apart from $|x - 3|$, we have a factor, namely $|x + 3|$ of $[x^2 - 9]$. To decide the limits of $|x + 3|$, let us put a restriction on δ. Remember, we have to choose δ. So let us say we choose a $\delta \leq 1$. What does this imply?

$|x - 3| < \delta \quad \Rightarrow |x - 3| < 1 \Rightarrow 3 - 1 < x < 3 + 1$

$\Rightarrow 2 < x < 4 \Rightarrow 5 < x + 3 < 7$. Recall Sec. 4, Unit 1.

Thus, we have $|x^2 - 9| < 7|x - 3| < \varepsilon$. Now when will this be true? It will be true when $|x - 3| < \varepsilon/7$. So this $\varepsilon/7$ is the value of δ we were looking for. But we have already chosen $\delta \leq 1$. This means that given $\varepsilon > 0$, the δ we choose should satisfy

$\delta \leq 1$ and also $\delta \leq \varepsilon/7$.

In other words, $\delta = \min\{1, \varepsilon/7\}$, should serve our purpose. Let us verify this :

$|x - 3| < \delta \Rightarrow |x - 3| < 1$ and $|x - 3| < \varepsilon/7 \Rightarrow |x^2 - 9| = |x + 3||x - 3| < 7 . \varepsilon/7 = \varepsilon$.

Remark 1 : If f is a constant function on **R**, that is, if $f(x) = k \ \forall\ x \in \mathbf{R}$, where k is some fixed real number, then $\lim_{x \to p} f(x) = k$.

Now please try the following exercises.

E E1) Show that

a) $\lim_{x \to 1} \frac{1}{x} = 1$

b) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3.$

Before we go further, let us ask, 'Can a function f(x) tend to two different limits as x tends to p'?

The answer is NO, as you can see from the following :

Theorem 1 If $\lim\limits_{x \to p} f(x) = L$ and $\lim\limits_{x \to p} f(x) = M$, then $L = M$.

Proof : Suppose $L \neq M$, then $|L - M| > 0$. Since $\lim\limits_{x \to p} f(x) = L$. If we take $\varepsilon = \frac{|L - M|}{2}$ then $\exists\ \delta_1 > 0$ such that

$|x - p| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$

Similarly, since $\lim\limits_{x \to p} f(x) = M, \exists\ \delta_2 > 0$ such that

$|x - p| < \delta_2 \Rightarrow |f(x) - M| < \varepsilon$

If we choose $\delta = \min|\delta_1, \delta_2|$, then $\delta > 0$ and $|x - p| < \delta$ will mean that $|x - p| < \delta_1$ and $|x - p| < \delta_2$.

$|a + b| \leq |a| + |b|$

In this case we will have both $|f(x) - L| < \varepsilon$, as well as, $|f(x) - M| < \varepsilon$.

So that $|L - M| = |L - f(x) + f(x) - M| \leq |f(x) - L| + |f(x) - M| < \varepsilon + \varepsilon = 2\varepsilon = |L - M|$.

That is, we get $|L - M| < |L - M|$, which is a contradiction. Therefore, our supposition is wrong. Hence $L = M$.

We now state and prove a theorem whose usefulness will be clear to you in Unit 4.

Theorem 2 Let f, g, and h be functions defined on an interval I containing a, except possibly at a, Suppose

i) $f(x) \leq g(x) \leq h(x)\ \forall\ x \in I \setminus |a|$

ii) $\lim\limits_{x \to a} f(x) = L = \lim\limits_{x \to a} h(x)$

Then $\lim\limits_{x \to a} g(x)$ exist and is equal to L.

Proof: By the definition of limit, given $\varepsilon > 0, \exists\ \delta_1 > 0$ and $\delta_2 > 0$ such that
$|f(x) - L| < \varepsilon$ for $0 |x - a| < \delta_1$ and
$|h(x) - L| < \varepsilon$ for $0 |x - a| < \delta_2$.
Let $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon \text{ and } |h(x) - L| < \varepsilon$$
$$\Rightarrow L - \varepsilon \leq f(x) \leq L + \varepsilon, \text{ and}$$
$$L - \varepsilon \leq h(x) \leq L + \varepsilon$$

We also have $f(x) \leq g(x) \leq h(x)\ \forall\ x \in I \setminus \{a\}$
Thus, we get $0 < |x - a| < \delta \Rightarrow L - \varepsilon \leq f(x) \leq g(x) \leq h(x) \leq L + \varepsilon$
In other words, $0 < |x - a| < \delta \Rightarrow |g(x) - L| < \varepsilon$

Therefore $\lim\limits_{x \to a} g(x) = L$.

Theorem 2 is also called the **sandwich theorem (or the squeeze theorem),** because g is being sandwiched between f and h. Let us see how this theorem can be used.

Example 4 Given that $|f(x) - 1| \leq 3(x + 1)^2\ \forall\ x \in \mathbf{R}$, can we calculate $\lim\limits_{x \to -1} f(x)$?

We know that $-3(x + 1)^2 \leq f(x) - 1 \leq 3(x + 1)^2\ \forall\ x$. This means that
$-3(x + 1)^2 + 1 \leq f(x) \leq 3(x + 1)^2 + 1 \leq \forall\ x$. Using the sandwich theorem and the fact that
$\lim\limits_{x \to -1} [-3(x + 1)^2 + 1] = 1 = \lim\limits_{x \to -1} [3(x + 1)^2 + 1]$, we get $\lim\limits_{x \to -1} f(x) = 1$.

In the next section we will look at the limits of the sum, product and quotient of functions.

2.2.1 Algebra of limits

Now that you are familiar with limits. Let us state some basic properties of limits. (Their proofs are beyond and scope of this course.)

Theorem 3 Let f and g be two functions such that

$\lim\limits_{x \to p} f(x)$ and $\lim\limits_{x \to p} g(x)$ exist. Then

i) $\lim\limits_{x \to p} [f(x) + g(x)] = \lim\limits_{x \to p} f(x) + \lim\limits_{x \to p} g(x)$ Sum rule

ii) $\lim\limits_{x \to p} [f(x)g(x)] = \left[\lim\limits_{x \to p} f(x)\right]\left[\lim\limits_{x \to p} g(x)\right]$ Product rule

iii) $\lim\limits_{x \to p} \frac{1}{g(x)} = \frac{1}{\lim\limits_{x \to p} g(x)}$, provided $\lim\limits_{x \to p} g(x) \neq 0$ Reciprocal rule

We can easily prove two more rules in addition to the three rules given in Theorem 3. These are :

iv) $\lim\limits_{x \to p} k = k$ Constant function rule

v) $\lim\limits_{x \to p} x = p$ Identity function rule

We shall only indicate the method of proving iv) and v):

iv) Here $| f(x) - L | = | k - k | = 0 < \varepsilon$, whatever be the value of δ.

v) $| f(x) - L | = | x - p | < \varepsilon$ whenever $| x - p | < \delta$, if we choose $\delta = \varepsilon$.

Using the properties that we have just stated, we will calculate the limit in the following example.

Example 5 Let us evaluate $\lim\limits_{x \to 2} \dfrac{3x^2 + 4x}{2x + 1}$

Now $\lim\limits_{x \to 2} 2x + 1 = \lim\limits_{x \to 2} 2x + \lim\limits_{x \to 2} 1$ by using i)

$= \lim\limits_{x \to 2} 2 \ \lim\limits_{x \to 2} x + \lim\limits_{x \to 2} 1$ by using ii)

$= 2 \times 2 + 1 = 5 \neq 0$ by using iv) and v)

$\therefore$ We can use (iii) of Theorem 3. Then the required limit is

$$\frac{\lim\limits_{x \to 2} (3x^2 + 4x)}{\lim\limits_{x \to 2} (2x + 1)} = \frac{\lim\limits_{x \to 2} 3x^2 + \lim\limits_{x \to 2} 4x}{\lim\limits_{x \to 2} 2x + \lim\limits_{x \to 2} 1} \quad \text{by using i)}$$

$$= \frac{\lim\limits_{x \to 2} 3 \lim\limits_{x \to 2} x \lim\limits_{x \to 2} x + \lim\limits_{x \to 2} 4 \lim\limits_{x \to 2} x}{\lim\limits_{x \to 2} 2 \lim\limits_{x \to 2} x + \lim\limits_{x \to 2} 1} \quad \text{by using ii)}$$

$$= \frac{3 \times 2 \times 2 + 4 \times 2}{2 \times 2 + 1} = \frac{20}{5} = 4$$

You can similarly calculate the limits in the following exercises.

E E 2) Show that $\lim\limits_{x \to 1} \dfrac{3}{x} = 3$

E E 3) Calculate $\lim\limits_{x \to 1} 2x + 5\left(\dfrac{x^2}{1 + x^2}\right)$

2.2.2 Limits as $x \to \infty$ (or $-\infty$)

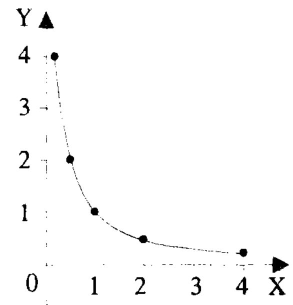


Fig. 3

Take a look at the graph of the function $f(x) = \frac{1}{x}$, $x > 0$ in Fig. 3. This is a decreasing function of x. In fact, we see from Fig. 3 that f(x) comes closer and closer to zero as x gets larger and larger. This situation is similar to the one where we have a function g(x) getting closer and closer to a value L as x comes nearer and nearer to some number p, that is when $\lim_{x\to p} g(x) = L$.

The only difference is that in the case of f(x), x is not approaching any finite value, and is just becoming larger and larger. We express this by saying that $f(x) \to 0$ as $x \to \infty$, or $\lim_{x\to\infty} f(x) = 0$.

Note that, ∞ is not a real number. We write $x \to \infty$ merely to indicate that x becomes larger and larger.

We now formalise this discussion in the following definition.

Definition 2 A function f is said to tend to a limit L as x tends to ∞ if, for each $\varepsilon > 0$ it is possible to choose $K > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > K$.

In this case, as x gets larger and larger, f(x) gets nearer and nearer to L. We now give another example of this situation.

Example 6 Let f be defined by setting $f(x) = 1/x^2$ for all $x \in \mathbf{R} \setminus \{0\}$. Here f is defined for all real values of x other than zero. Let us substitute larger and larger values of x in $f(x) = 1/x^2$ and see what happens (see Table 2).

Table 2

x	100	1000	100,000
$f(x) = 1/x^2$	.0001	.000001	.0000000001

We see that as x becomes larger and larger, f(x) comes closer and closer to zero. Now, let us choose any $\varepsilon > 0$. If $x > 1/\sqrt{\varepsilon}$, then $1/x^2 < \varepsilon$. Therefore, by choosing $K = 1/\sqrt{\varepsilon}$, we find that $x > K \Rightarrow |f(x)| < \varepsilon$. Thus, $\lim_{x\to\infty} f(x) = 0$.

Fig. 4 gives us a graphic idea of how this function behaves as $x \to \infty$.

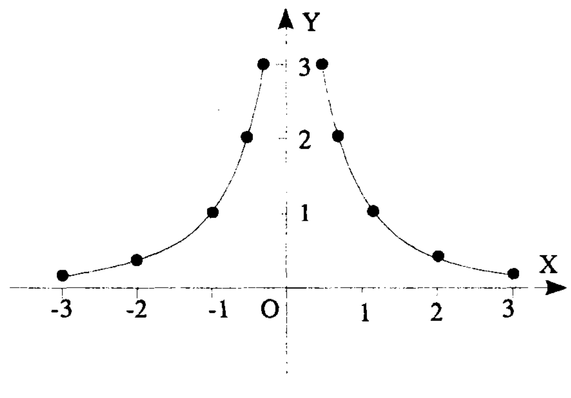


Fig. 4

Sometimes we also need to study the behaviour of a function f(x), as x takes smaller and smaller negative values. This can be examined by the following definition.

Definition 3 A function f is said to tend to a limit L as $x \to -\infty$ if, for each $\varepsilon > 0$, it is possible to choose $K > 0$, such that $|f(x) - L| < \varepsilon$ whenever $x < -K$.

The following example will help you in understanding this idea.

Example 7 Consider the function $f: \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \frac{1}{1 + x^2}$$

The graph of f is as shown in Fig. 5.

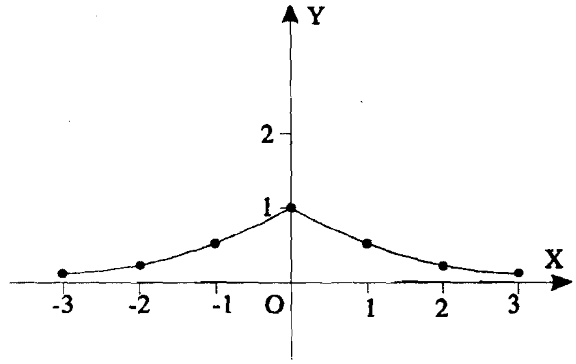


Fig. 5

What happens to f(x) as x takes smaller and smaller negative values? Let us make a table (Table 3) to get some idea.

Table 3

x	–10	–100	–1000
$f(x) = \frac{1}{1+x^2}$	1/101	1/10001	1/1000001

We see that x takes smaller and smaller negative values, f(x) comes closer and closer to zero. In fact $1/(1+x^2) < \varepsilon$ whenever $1 + x^2 > 1/\varepsilon$, that is, whenever $x^2 > (1/\varepsilon) - 1$, that is, whenever either

$x < -\left|\frac{1}{\varepsilon} - 1\right|^{1/2}$ or $x > \left|\frac{1}{\varepsilon} - 1\right|^{1/2}$. Thus, we find that if we take $K = \left|\frac{1}{\varepsilon} - 1\right|^{1/2}$, then

$x < -k \Rightarrow |f(x)| < \varepsilon$. Consequently, $\lim\limits_{x \to -\infty} f(x) = 0$.

In the above example we also find that $\lim\limits_{x \to \infty} f(x) = 0$.

Let us see how $\lim\limits_{x \to \infty} f(x)$ can be interpreted geometrically.

In the above example, we have the function $f(x) = 1/(1+x^2)$, and as $x \to \infty$, or $x \to -\infty$, $f(x) \to 0$. From Fig. 5 you can see that, as $x \to \infty$ or $x \to -\infty$, the curve $y = f(x)$ comes nearer and nearer the straight line $y = 0$, which is the x-axis.

Similarly, if we say that $\lim\limits_{x \to \infty} g(x) = L$, then it means that, as $x \to \infty$ the curve $y = g(x)$ comes closer and closer to the straight line $y = L$.

Example 8 Let us show that $\lim\limits_{x \to \infty} \frac{x^2}{(1+x^2)} = 1$.

Now, $|x^2/(1+x^2) - 1| = 1/(1+x^2)$. In the previous example we have shown that $|1/(1+x^2)\} < \varepsilon$ for $x > K$, where $K = |1/\varepsilon - 1|^{1/2}$. Thus, given $\varepsilon > 0$, we choose $K = |1/\varepsilon - 1|^{1/2}$, so that

$x > K \Rightarrow \left|\frac{x^2}{1+x^2} - 1\right| < \varepsilon$. This means that $\lim\limits_{x \to \infty} \frac{x^2}{1+x^2} = 1$.

We show this geometrically in Fig. 6.

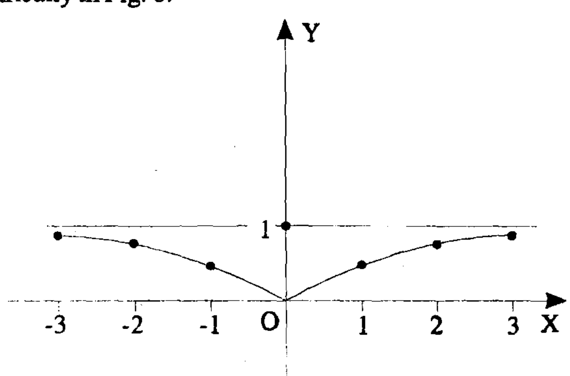


Fig. 6

You must be wondering if all the properties given in Theorem 3 also hold when we take limits as $x \to \infty$. **Yes, they do.**

You can use them to solve this exercise.

E4) Shos that a) $\lim_{x\to\infty} 1/x = 0$

b) $\lim_{x\to\infty} (1/x + 3/x^2 + 5) = 5$

Sometimes we cannot use Theorem 2 directly, as is clear in the following example, Let us see how to overcome this problem.

Example 9 Suppose, we want to find $\lim_{x\to\infty} \frac{3x+1}{2x+5}$.

We cannot apply Theorem 3 directly since the limits of the numerator and the denominator, as $x \to \infty$, cannot be found.

Instead, we rewrite the quotient by multiplying the numerator and denominator by $1/x$. for $x \neq 0$.

Then, $\frac{3x+1}{2x+5} = \frac{3+(1/x)}{2+(5/x)}$, for $x \neq 0$. Now we use

Theorem 3 and the fact that $\lim_{x\to\infty} 1/x = 0$ (see # 4 a), to get

$$\lim_{x\to\infty} \frac{3x+1}{2x+5} = \lim_{x\to\infty} \frac{3+(1/x)}{2+(5/x)}$$

$$= \frac{\lim_{x\to\infty} (3+1/x)}{\lim_{x\to\infty} (2+5/x)} = \frac{3+0}{2+0} = \frac{3}{2}$$

By now you must be used to the various definitions of limits, so can try this exercise.

E5) a) If for some $\varepsilon > 0$, and for ever K, $\exists\, x > K$ s.t. $|f(x) - L| > \varepsilon$, what will you infer?

b) If $\lim_{x\to p} f(x) \neq L$, how can you express it in the $\varepsilon - \delta$ form?

We end this section with the following important remark.

Remark 2 In case we have to show that a function f does not tend to a limit L as x approaches p, we shall have to negate the definition of limit (also see E5(b)). Let us see what this means.

Suppose we want to prove that $\lim_{x \to p} f(x) \neq L$. Then, we should find some $\varepsilon > 0$ such that for every $\delta > 0$, there is some $x \in]p + \delta$ for which $|(x) - L| > \delta$. Through our next example we shall illustrate the negation of the definition of the limit of f(x) as $x \to \infty$.

Example 10 To show that $\lim_{x \to \infty} 1/x \neq 1$, we have to find some $\varepsilon > 0$ such that for any K (howsoever large) we can always find an $x > K$ such that $|1/x - 1| > \varepsilon$.

Take $\varepsilon = 1/4$. Now, for any $K > 0$, if we take $x = \max\{2, K + 1\}$, we find that $x > K$ and $|1/x - 1| > 1/4$. This clearly show that $\lim_{x \to \infty} 1/x \neq 1$.

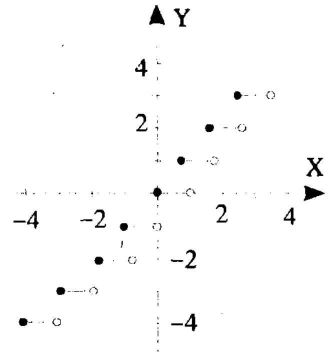


Fig. 7

2.2.3 One-sided Limits

If we consider the graph of the function $f(x) = [x]$, shown in Fig. 7, we see that f(x) does not see to approach any fixed value as x approaches 2. But from the graph we cansay that if x approaches 2 from the left then f(x) seems to tend to 1. At the same time, if x approaches 2 from the right, then f(x) seems to tend to 2. This means that the limit of f exists if x approaches 2 from only one side (left or right) at a time. This example suggests that we introduce the idea of a one-sided limit.

Definition 4 Let f be a function defined for all x in the interval]p, q[. f is said to approach a **limit L as x approaches p from right** if, given any $\varepsilon > 0$, there exist a number $\delta > 0$ such that $p < x < p + \delta \Rightarrow |f(x) - L| < \varepsilon$.

In symbols we denote this limit by $\lim_{x \to p^+} f(x) = L$.

Similarly, the function $f :]a, p[\to \mathbf{R}$ is said to approach a limit L as **x approaches p from the left** if, given any $\varepsilon > 0$, $\exists\, \delta > 0$ such that $p - \delta < x < p \Rightarrow |f(x) - L| < \varepsilon$.

This limit is denoted by $\lim_{x \to p^-} f(x)$.

Note that in computing these limits the values of f(x) for x lying on only **one side** of p are taken into account.

Let uf apply this definition to the function $f(x) = [x]$, we know that for $x \in [1, 2[$, $[x] = 1$. That is, [x] is a constant function on [1, 2[. Hence $\lim_{x \to 2^-} [x] = 1$. Arguing similarly, we find that since $[x] = 2$ for all $x \in [2, 3[$, [x] is, again, a constant function on [2, 3[, and $\lim_{x \to 2^+} [x] = 2$.

Let us improve our understanding of the definition of one-sided limits by looking at some more examples.

Example 11 Let f be defined on **R** by setting

$$f(x) = \frac{|x|}{x}, \text{ when } x \neq 0$$

$$f(0) = 0$$

We will show that $\lim_{x \to 0^-} f(x)$ equals -1.

When $x < 0$, $|x| = -x$, and therefore, $f(x) = (-x)/x = -1$. In order to show that $\lim_{x \to 0^-} f(x)$ exists and equals -1, we have to start with any $\varepsilon > 0$ and then find a $\delta > 0$ such that, if $-\delta < x < 0$, then $|f(x) - (-1)| < \varepsilon$.

Since $f(x) = -1$ for all $x < 0$, $|f(x) - (-1)| = 0$ and, hence any number $\delta > 0$ will work. Therefore, whatever $\delta > 0$ we may choose, if $-\delta < x < 0$, then $|f(x) - (-1)| = 0 < \varepsilon$. Hence $\lim_{x \to 0^-} f(x) = -1$.

Example 12 f is a function defined on **R** by setting

$f(x) = x - [x]$, for all $x \in \mathbf{R}$.

Let us examine whether $\lim_{x \to 1^-} f(x)$ exists.

Recall (Unit 1) that this function is given by $f(x) = x$, if $0 \le x < 1$.

$f(x) = x - 1$ if $1 \le x < 2$, and, in general

$f(x) = x - n$ if $n \le x < n + 1$ (see Fig. 8).

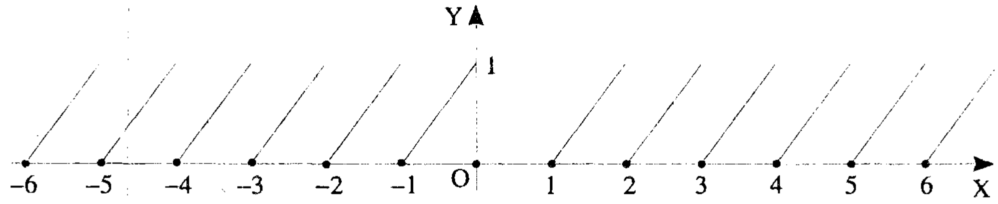


Fig. 8

Since $f(x) = x$ for values of x less than 1 but close to 1, it is reasonable to expect that $\lim_{x \to 1^-} f(x) = 1$. Let us prove this by taking any $\varepsilon > 0$ and choosing $\delta = \min | 1, \varepsilon |$. We find $1 - \delta < x < 1 \Rightarrow f(x) = x$ and $| f(x) - 1 | = | x - 1 | < \delta \leq \varepsilon$.

Therefore, $\lim_{x \to 1^-} f(x) = 1$.

Proceeding exactly as above, the noting that $f(x) = x - 1$ if $1 \leq x < 2$, we can similarly prove that $f(x) = 0$.

E E6) Prove that a) $\lim_{x \to 3^-} x\,[x] = 1$

b) $\lim_{x \to 0^+} \frac{| x |}{x} = 1$

c) $\lim_{x \to 0^-} \frac{(x^2 + 2)|x|}{x} = -2$

Before going further, let us see how the concepts of one-sided limit and limit are connected.

Theorem 4 The following statements are equivalent.

i) $\lim_{x \to p} f(x)$ exists

ii) $\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-}$ exist and are equal.

Proof : To show that i) and ii) are equivalent, we have to show that i) $\Rightarrow$ ii) and ii) $\Rightarrow$ i).

We first prove that i) $\Rightarrow$ ii). For this we assume that $\lim_{x \to p} f(x) = L$. Then given $\Rightarrow \varepsilon > 0$.
$\exists\, \delta > 0$ such that $| f(x) - L | < \varepsilon$ for $0 < | x - p | < \delta$.

Now, $0 < | x - p | < \delta \Rightarrow p < x < p + \delta$ and $p - \delta < x < p$. Thus, we have $| f(x) - L | < \varepsilon$ for $p < x < p + \delta$ and $p - \delta < x < p$. This means that $\lim_{x \to p^-} f(x) = L = \lim_{x \to p^+} f(x)$.

We now prove the converse, that is, ii) $\Rightarrow$ i). For this, we assume that

$\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x) = L$. Then, given $\varepsilon > 0, \exists\, \delta_1, \delta_2 > 0$.
such that
$| f(x) - L | < \varepsilon$ for $p - \delta_1 < x < p$
$| f(x) - L | < \Rightarrow \varepsilon$ for $p < x < p + \delta_2$

Let $\delta = \min \{\delta_1, \delta_2\}$. Then for both $p - \delta < x < p$ and $p < x < p + \delta$, we have $| f(x) - L | < \varepsilon$.

This means that $| f(x) - L \} < \varepsilon$ whenever $0 < | x - p | < \delta$.

Hence, $\lim_{x \to p} f(x) = L$.

Thus, we have shown that i) $\Rightarrow$ ii) and ii) $\Rightarrow$ i), proving that they are equivalent.

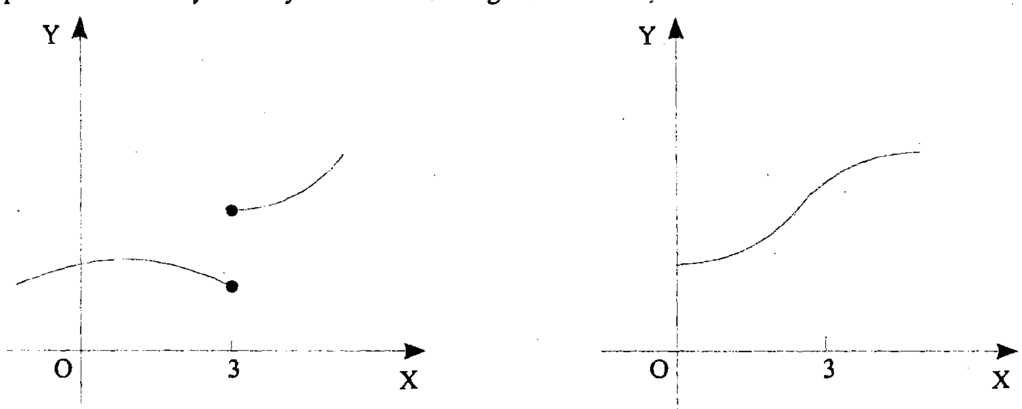
From Theorem 4, we can conclude that if $\lim_{x \to p} f(x)$ exists, then $\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-} f(x)$ also exist and further.

$$\lim_{x \to p} f(x) = \lim_{x \to p^+} f(x) = \lim_{x \to p^-} f(x)$$

Remark 3 If you apply Theorem 4 to the function $f(x) = x - [x]$ (see Example 12), you will see that $\lim_{x \to 1} \{x - [x]\}$ does not exist as $\lim_{x \to 1^+} \{x - [x]\} \neq \lim_{x \to 1^-} \{x - [x]\}$,
We shall use this concept of one-sided limits to define continuous functions in the nest section.

2.3 CONTINUITY

A continuous process is one that goes on smoothly without any abrupt change. Continuity of a function can also be interpreted in a similar way. Look at Fig. 9. The graph of the function f in Fig. 9 (a) has an abrupt cut at the point x = 3, whereas the graph of the function g in Fig. 9 (b) proceeds smoothly. We say that the function g is continuous, while f is not.



(a) (b)

Fig. 9 : (a) Graph of f (b) Graph of g

Continuous functions play a very important role in calculus. As you proceed, you will be able to see that many theorems which we have stated in this course are true only for continuous functions. You will also see that continuity is a necessary condition for the derivability of a function, and that it is a sufficient condition for the integrability of a function. But let us give a precise meaning to "a continuous function" now.

2.3.1 Definitions and Examples

In this section we shall give you the definition and some examples of a continuous function. We shall also give you a short list of conditions which a function must satisfy in order to be continuous at a point.

Definition 5 Let f be a function defined on a domain D, and let r be a positive real number such that the interval $]p-r, p+r[\subset D$. **f is said to be continuous at x = p** if $\lim\limits_{x \to p} f(x) = f(p)$.

By the definition of limit this means that f is continuous at p is given $\varepsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ whenever $|x - p| < \delta$. To clarify this concept let us look at an example.

Example 13 Let us check the continuity of the function

$f : L\, \mathbf{R} \to \mathbf{R}$ such that $f(x) = x$ at the point $x = 0$.

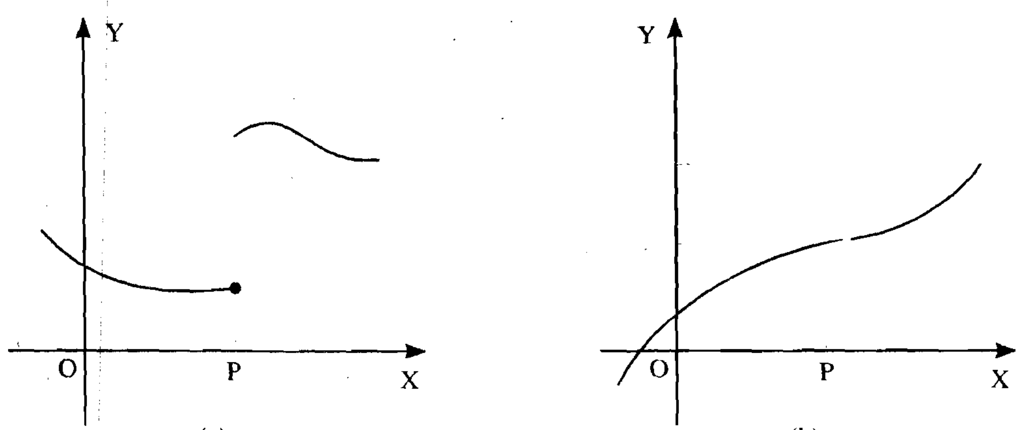
Now, $f(0) = 0$. Thus we want to know if $\lim\limits_{x \to 0} f(x) = 0$.

This is true because given $\varepsilon > 0$, , we can choose $\delta = \varepsilon$ and verify that $|x| < \delta \Rightarrow |f(x)| < \varepsilon$. Thus f is continuous at $x = 0$.

Remark 4 f is continuous at x = p provided the following two criteria are met :

i) $\lim\limits_{x \to p} f(x)$ exists.

ii) $\lim\limits_{x \to p} f(x) = f(p)$.



(a) (b)

Fig. 10 : (a) Graph of f (b) Graph of g

Fig. 10 shoiws two discontinuous functions f and g. Criterion i) is not met by f, whereas g fails to meet criterion ii). If you read Remark 3 again, you will find that $f(x) = x - [x]$ is not continuous as $x = 1$. But we have see that we can calculate one-sided limits of $f(x) = x - [x]$ at $x = 1$. This leads us to the following definition.

Definition 6 A function $f :]p, q[\to \mathbf{R}$ is said to be continuous from the right at $x = p$ if $\lim\limits_{x \to p^+} f(x) = f(p)$. We say that f is continuous from the left at q if $\lim\limits_{x \to q^-} f(x) = f(q)$.

Thus, $f(x) = x - [x]$ is continuous from the right but not from the left at $x = 1$ since $\lim\limits_{x \to 1^-} f(x) \neq f(1)$ and $\lim\limits_{x \to 1^+} f(x) = f(1) = 0$.

E E7) Give $\varepsilon - \delta$ definition of continuity at a point from the right as well as from the left.

E E8) Show that function f: $]p-r,, p+r[\to \mathbf{R}$ is continuous at $x = p$ if and only if f is continuous from the right as well as from the left at $x = p$ (Use Theorem 4).

Now that you know how to test the continuity of a function at a point, let us go a step further, and define continuity of a function on a set.

Definition 7 A function f defined on a domain D is said to be **continuous on D**, if it is continuous at every point of D.

Let us see some more examples.

Example 14 Let $f(x) = x^n$ for all $x \in \mathbf{R}$ and any $n \in \mathbf{Z}^+$. Show that f(x) is continuous at $x = p$ for all $p \in \mathbf{R}$.

$\mathbf{Z}^+$ is the set of positive integers.

We know that $\lim\limits_{x \to p} x = p$ for any $p \in \mathbf{R}$. Then, by the product rule in Theorem 3, we get

$$\lim_{x \to p} x^n = (\lim_{x \to p} x)(\lim_{x \to p} x) \ldots\ldots\ldots (\lim_{x \to p} x) \qquad \text{(n times)}$$

$= pp \ldots\ldots\ldots\ldots p$ (n times) $= p^n$. Therefore, $\lim\limits_{x \to p.} f(x)$ exists and equals f(p). Hence f is continuous at $x = p$. Since p was any arbitrary number in **R**, we can say that f is continuous on **R**.

Remark 5 Using Example 14 and Theorem 3, we can also prove that polynomial $a_0 + a_1x + \ldots\ldots\ldots\ldots + a_nx^n$, where $a_0, a_1 \ldots\ldots\ldots\ldots\ldots a_n \in \mathbf{R}$, is continuous on **R**, that is,

$\lim\limits_{x \to p} (a_0 + a_1x + \ldots\ldots\ldots\ldots\ldots\ldots + a_nx^n) = a_0 + a_1p + \ldots\ldots\ldots\ldots\ldots + a_np^n$ for all $p \in \mathbf{R}$.

Example 15 The greatest integer function $f : \mathbf{R} \to \mathbf{R} : f(x) = [\,x\,]$ is discontinuous at $x = 2$.

To prove this we recall our discussion in Sec. 2 in which we have proved that $\lim\limits_{x \to 2^-} f(x) = 1$ and $\lim\limits_{x \to 2^+} f(x) = 2$. Thus, since these two limits are not equal $\lim\limits_{x \to 2} f(x)$ does not exist.

Therefore, f is not continuous at $x = 2$ because the first criterion laid down in Remark 4 is not met.

Example 16 Let $f(x) = |\,x\,|$ for all $x \in \mathbf{R}$. This f is continuous at $x = 0$.

Here $f(x) = x$, if $x \geq 0$, and $f(x) = -x$ if $x < 0$. You can show that

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0 = f(0)$ and

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-x) = 0 = f(0)$. Thus $\lim\limits_{x \to 0} f(x)$ exists and equals f(0). Hence f is continuous at $x = 0$.

Note : f is also continuous at every other point of **R**. (Check this statement).

Example 17 Suppose we want to find whether $f(x) = \dfrac{x^2 - 1}{x - 1}$ is continuous at $x = 0$.

We can write $f(x) = x + 1$ for all $x \neq 1$.

In Fig. 11 you see the graph of f. It is the line $y = x + 1$ except for the point **(1, 2)**

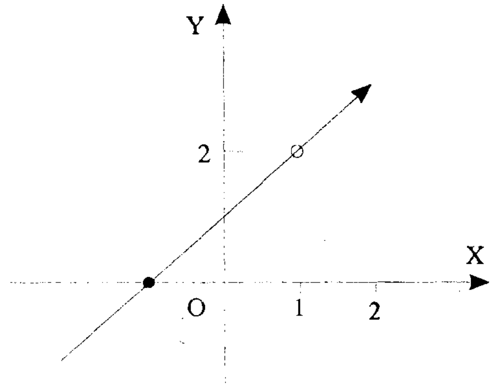


Fig. 11

$f(0) = (0^2 - 1)/(0 - 1) = 1$ and

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 - 1)/(x - 1) = \lim_{x \to 0} (x + 1) = 1$$

$$\lim_{x \to 0} f(x) = f(0), \text{ so that f is continuous at } x = 0.$$

The exponential function $f(x) = e^x$ and the logarithmic function $f(x) = \ln x$ are continuous functions. You can check this by looking at their graphs in Unit 1. Similarly, $x \to \sin x$ and $x \to \cos x$ are continuous, $x \to \tan x$ is continuous in $]-\pi/2, \pi/2[$. This fact is quite obvious from the graphs of these functions (We have given their graphs in Unit 1). We shall not attempt a rigorous proof of their continuity here.

Caution : Checking the continuity of a function from the smoothness of its graph is not a fool-proof method. If you look at the graph (Fig. 12) of the function $x \to x \sin(1/x)$, you will find that it has no breaks in the neighbourhood of $x = 0$. But this function is not continuous. Observe that the graph oscillates wildly near zero.

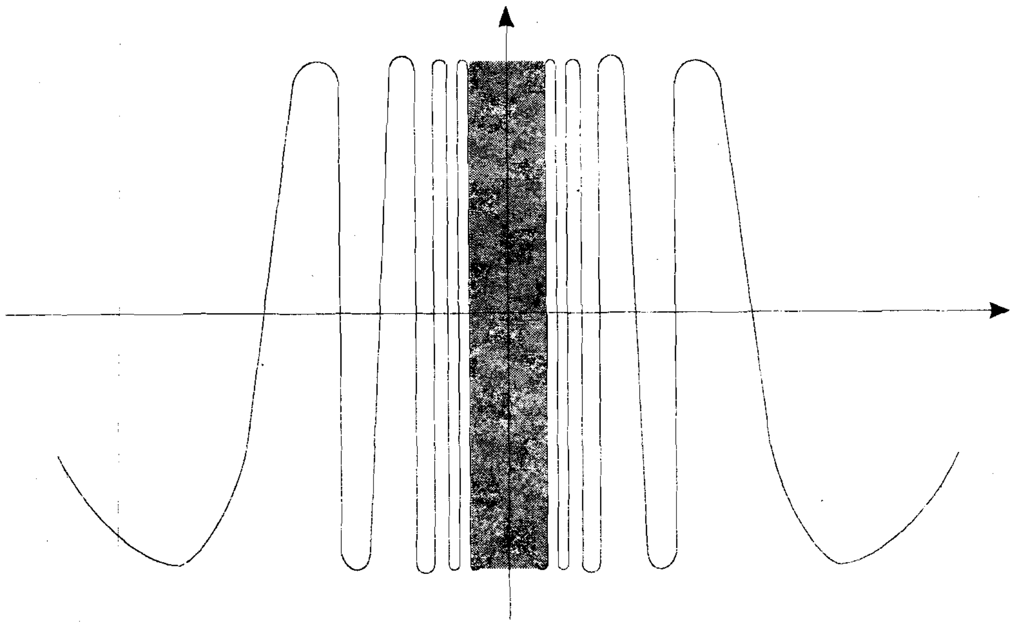


Fig. 12

E9) Show that the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 1/(x^2 - 9)$ is continuous at all points of $\mathbf{R}$ except at $x = 3$ and $x = -3$.

Now that we know how to check whether a function is continuous or not, let us go further, and talk about the continuity of some combinations of functions.

2.3.2 Algebra of Continuous Functions

Let f and g be functions defined and continuous on a common domain $D \subseteq \mathbf{R}$, and let k be any real number. In Unit 1, we defined the functions $f + g$, fg/ f/g (provided $g(x) \neq 0$ any where in D), kif and | f |. The following theorem tells us about the continuity of these functions.

Theorem 5 Let f and g be functions defined and continuous on a common domain D, and let k be any real number. The functions $f + g$, kf, | f | and fg are all continuous on D. If $g(x) \neq 0$ anywhere in D, then the function f/g is also continuous on D.

We shall not prove this theorem here.

In Unit 1, you have studied the important concept of composite functions. In Theorem 6, we will talk about the continuity of the composite of two continuous functions. Here again, we shall state the theorem without giving proof as it is beyond the level of this course.

Theorem 6 Let $f : D_1 \to D_2$ and $g : D_2 \to D_3$ be continuous on their domains. Then gof is continuous on D_1, $(D_1, D_2, D_3 \subseteq \mathbf{R})$.

Example 18 To prove that $f : \mathbf{R} \to \mathbf{R} : f(x) = (x^2 + 1)^3$ is continuous at $x = 0$, we consider the functions $g : \mathbf{R} \to \mathbf{R} : g(x) = x^3$ and $h : \mathbf{R} \to \mathbf{R} : h(x) = x^2 + 1$. You can check that $f(x) = goh(x)$. Further, by Remark 5, h is continuous on **R**, and g is also continuous on **R**. Thus, goh = f is continuous on **R**.

Let us see if the converse of the above theorems are true. For example, if f and g are defined on an interval [a, b] and if f + g is continuous on [a, b], does that mean that f and g are continuous on [a, b] ?

No. Consider the functions f and g over the interval [0, 1] given by

$$f(x) = \begin{cases} 0, & 0 \leq x \leq 1/2 \\ 1, & 1/2 \leq 1 \text{ and} \end{cases}$$

$$g(x) = \begin{cases} 1, & 0 \leq x \leq 1/2 \\ 0, & 1/2 \leq x \leq 1 \end{cases}$$

Then neither f nor g is continuous at $x = 1/2$. (Why?) But $(f + g)(x) = 1 \ \forall \ x \in [0, 1]$. Therefore, $f + g$ is continuous on [0, 1].

Now, if | f | is continuous at a point p, must f also be continuous at p? Again, the answer is **No**. Take, for example, the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} -1, & \text{for } x \leq 0 \\ 1, & \text{for } > 0 \end{cases}$$

Then $| f(x) | = 1$ in **R** and hence | f | is continuous.
But f is not continuous at $x = 0$ (Why?)

E E10) If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = f(x) = \begin{cases} 1, & \text{if } x \in Z \\ -1, & \text{if } x \notin Z \end{cases}$
is f continuous at a) $x = 1$ b) $x = -3/2$?

One again, we won't prove this theorem here. But try to understand its statement because we shall be using it in subsequent units.

Theorem 7 (Intermediate Value Theorem) Let f be continuous on the closed interval [a, b]. Suppose c is a real number lying between f(a) and f(b). (That is, $f(a) < c < f(b)$ or $f(a) > c > (b)$). Then there exists some $x_0 \in]a, b[$, such that $f(x_0) = c$.

How can we interpret this geometrically? We have already seen that the graph of a continuous function is smooth. It does not have any breaks or jumps. This theorem says that, if the points (a, f(a)) and (b, f(b)) lie on two opposite sides of a line y = c (see Fig. 13), then the graph of f must cross the line y = c.

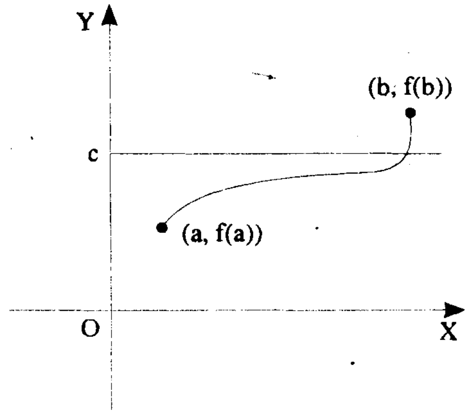


Fig. 13

Note that this theorem guarantees only the existence of the number x_0. It does not tell us how to find it. Another thing to note is that this x_0 need not be unique.
That brings us to the end of this unit.

2.4 SUMMARY

We end this unit by summarising what we have covered in it.

1. The limit of a function f at a point p of its domain is L is given $\varepsilon > 0$, $\exists\ \delta > 0$, such that $| f(x) - L | < \varepsilon$ whenever $| x - p | < \delta$.
2. One-sided limits
3. $\lim\limits_{x \to p} f(x)$ exists if and only if

   $\lim\limits_{x \to p^+} f(x)$ and $\lim\limits_{x \to p^-} f(x)$ both exist and are equal.
4. A function f is continuous at a point x = p if

   $\lim\limits_{x \to p} f(x) = f(p)$
5. If the function f and g are continuous on D, then so are the functions f + g, fg, | f |, kf (where $k \in \mathbf{R}$) f/g (where $g(x) \neq 0$ in D)
6. The Intermediate Value Theorem : If f is continuous on [a, b] and if $f(a) < c < f(b)$ (or $f(a) > c > f(b)$), then
   $\exists x_0 \in]a, b[$ such that $f(x_0) = c$.

2.5 SOLUTIONS AND ANSWERS

E 1) a) Given any $\varepsilon > 0$, if we choose $\delta = \min \{\varepsilon/2, 1/2\}$ then
$| x - 1 | < \delta \leq 1/2 \Rightarrow x > 1/2$.

and $| 1/x - 1 | = \left| \dfrac{x-1}{x} \right| < \left| \dfrac{x-1}{1/2} \right|$

$= 2 | x - 1 | < 2\delta < \varepsilon$.

That is, $| x - 1 | < \delta \Rightarrow | 1/x - 1 | < \varepsilon$

Hence $\lim_{x\to 1} 1/x = 1$

b) $\frac{x^3-1}{x-1} - 3 = \frac{x^3-3x+2}{x-1} = (x-1)(x+2), \text{ if } x \neq 1.$

Given $\varepsilon > 0$, if we choose $\delta = \min|(2/7)\varepsilon, 1/2]$, then

$|x-1| < 1/2 \Rightarrow x < 3/2 \Rightarrow x+2 < 7/2$ and

$$\left|\frac{x^3-1}{x-1} - 3\right| = |(x-1)(x+2)| < (7/2)|x-1| < (7/2).2/7.\varepsilon = \varepsilon.$$

That is, $|x-1| < \delta \Rightarrow \left|\frac{x^3-1}{x-1} - 3\right| < \varepsilon$

Hence, $\lim_{x\to 1} \frac{x^3-1}{x-1} = 3$

E2) $\lim_{x\to 1} 3/x = \frac{\lim_{x\to 1} 3}{\lim_{x\to 1} x} = 3/1 = 3$

E3) $\lim_{x\to 1} 2x + 5\left(\frac{x^2}{1+x^2}\right) = 2\lim_{x\to 1} x + \frac{5\lim_{x\to 1} x^2}{1+\lim_{x\to 1} x^2}$

$= 2 + \frac{5\times 1}{1+1} = 2 + 5/2 = 9/2$

E4) a) Given $\varepsilon > 0$ if we choose $K = 1/\varepsilon$, then

$x > K \Rightarrow |1/x - 0| = |1/x| < 1/K = \varepsilon$

Thus, $\lim_{x\to\infty} 1/x = 0$

b) Given $\varepsilon > 0$, if we choose $K = 1/\sqrt{\varepsilon}$, then

$x > K \Rightarrow |1/x^2 - 0| = |1/x^2| < 1/K^2 = \varepsilon.$

Hence, $\lim_{x\to\infty} 1/x^2 = 0$

Now, $\lim_{x\to\infty} (1/x + 3/x^2 + 5)$

$= \lim_{x\to\infty} 1/x + 3\lim_{x\to\infty} 1/x^2 + \lim_{x\to\infty} 5 = 0 + 3\times 0 + 5 = 5$

E5) a) $\lim_{x\to\infty} f(x) \neq L$

b) $\exists \varepsilon > 0$, s.t. $\delta > 0\ \exists x$ s.t. $|x-p| < \delta$ and $|f(x) - L| > \varepsilon$.

E6) a) Since $x - [x] = x - 2, 2 \leq x < 3,$

$\lim_{x\to 3^-} x - [x] = \lim_{x\to 3^-} x - 2 = 1$

b) $\lim_{x\to 0^+} |x|/x = \lim_{x\to 0^+} x/x = 1, |x| = x$ for $x > 0$.

c) $\lim_{x\to 0^-} \frac{(x^2+2)|x|}{x} = \lim_{x\to 0^-} \frac{(x^2+2)(-x)}{x}$ since $|x| = -x$ for $x < 0$

$= \lim_{x\to 0^-} -(x^2+2) = -2$

E7) f is continuous from the right at $x = p$ if $\forall\ \varepsilon > 0$ there exists a $\delta > 0$ s.t.
$p < x < p + \delta \Rightarrow |f(x) - f(p)| < \varepsilon$.
f is continuous from the left at $x = p$ if $\forall\ \varepsilon > 0$ there exists a $\delta > 0$ s.t.
$p - \delta < x < p \Rightarrow |f(x) - f(p)| < \varepsilon$.

E8) f is continuous at $x = p \Rightarrow \lim_{x\to p} f(x) = f(p)$

$\Rightarrow \lim_{x\to p^+} f(x) = f(p)$ and $\lim_{x\to p^-} f(x) = f(p)$ by Theorem 4.

$\Rightarrow$ f is continuous from right and from left at $x = p$.
If f is continuous from right and left,

$\Rightarrow \lim_{x \to p^+} f(x) = f(p) = \lim_{x \to p^-} f(x)$

$\Rightarrow \lim_{x \to p} f(x)$ exists and $= f(p)$

$\Rightarrow$ f is continuous at p,

E9) $\lim_{x \to a} f(x) = \dfrac{1}{\lim_{x \to 1} x^2 - 9} = \dfrac{1}{a^2 - 9} = f(a)$ for all a except $a = 3 \pm 3$

Hence f is continuous at all points except at ± 3, f is not defined at ± 3.

E10) a) f is not continuous at $x = 1$. For $\varepsilon = 1$ and any $\delta > 0$, if x is any non-integer $\in \,]1 - \delta, 1 + \delta[$, then
$|f(x) - f(1)| = |-1 - 1| = 2 > \varepsilon$.

b) f is continuous at $-3/2$. Since given $\varepsilon > 0$, if we choose $\delta < 1/2$, then
$|x - (-3/2)| < 1/2 \Rightarrow -2 < x < -1 \Rightarrow x \notin Z$ and hence $|f(x) - f(-3/2)| = 0 < \varepsilon$.