UNIT 2 CRAMER'S RULE

Structure

2.1 INTRODUCTION

In the previous unit we introduced you to linear systems and two methods for solving these. In this unit we discuss a method for solving a particular type of linear system. We shall first briefly introduce you to an efficient notation for dealing with systems of linear equations, namely, a matrix. You can find a detailed study in our course on linear algebra.

After that we shall explain a concept that is intimately linked with a certain type of matrix, and hence with the solution of certain systems of linear equations. The concept is that of a determinant, which seems to have been first used by ancient Chinese mathematicians for solving simultaneous linear equations. In 1683 the Japanese mathematician Seki Kowa started developing the theory of determinants for the same purpose. About the same time the German mathematician Leibniz also defined determinants and developed their use for solving simultaneous linear equations. So, you can see that mathematicians through the ages and around the globe have felt that determinants are very important. Nowadays scientists and social scientists also increasingly feel the need to understand and use this concept.

We shall end this unit with a discussion on a method which uses determinants for solving certain systems of linear equations. This method is due to the eighteenth century mathematician Cramer. It only applies to some of those linear systems in which the number of variables equals the number of equations.

Let us now list the objectives of this unit.

Objectives

After studying this unit you should be able to

- define a matrix, and a square matrix, in particular;
- evaluate any determinant of order 1, 2 or 3;
- identify a non-singular matrix;
- identify the linear systems which can be solved by using Cramer's rule, and apply the rule to solve them.

And now let us look at a convenient way of representing a linear system.

2.2 WHAT IS A MATRIX?

Consider the set of linear equations

$$\left.\begin{aligned} 2x + y - z &= 5 \\ x + 5y - 3z &= -6 \\ -x + 2y + 2z &= 1 \end{aligned}\right] \qquad \text{....(1)}$$

While writing (1) we have had to write each variable three times. Wouldn't it be more satisfactory if there were a notation that would enable us to avoid this repetition? After all, it is the coefficients that really matter in obtaining their solutions. Let us do away with writing x, y, z each time, and only write their coefficients in a table in the following manner :

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 5 & -3 \\ -1 & 2 & 2 \end{bmatrix}$$

How did we prepare this table? The first row consists of the coefficients of x, y and z, respectively, in the first equation; the second row consists of the coefficients in the second equation; and the third row consists of the coefficients in the third equation.

In fact, we can symbolically rewrite (1) as

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 5 & -3 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 1 \end{bmatrix} \qquad ...(2)$$

Each group of numbers or variables enclosed in the square brackets is an example of a matrix.

Writing systems of equations in matrix notation leads to a saving of effort in dealing with them, especially as the number of equations grows. Nowadays computers are being used increasingly for solving large systems of linear equations. Their efficiency increases considerably if matrix methods are used. You may be interested to know that matrices (plural of "matrix") were first used in 250 B.C. for solving systems of linear equations in the Chinese mathematics text "Nine Chapters on the Mathematical Art". But the development of matrix theory is mainly due to the 19th century British mathematicians Arthur Cayley and J.J. Sylvester. We will discuss matrix theory in great detail in our course on linear algebra. In this section we shall only acquaint you with matrices. Let us start with the definition.

Definition : A matrix is a rectangular arrangement of numbers in the form of horizontal and vertical lines.

The numbers occurring in a matrix are called its **elements** or **entries.**

The set of entries in one horizontal line of a matrix is called a **row** of the matrix; and the set of entries in a vertical line is called a **column** of the matrix.

For example,

$\begin{bmatrix} 1 & 3 & -4 \\ 2 & \frac{1}{3} & 5 \end{bmatrix}$ is a matrix with two rows and three columns, and $[-1]$ is a matrix with one row and one column.

Note that each row of a matrix has the same number of elements. Similarly, each column of a matrix has the same number of elements. This is why we say that the arrangement of numbers in a matrix is 'rectangular'.

Now a few words about notation.

As you have seen in the examples of matrices given so far, **we use square brackets** to enclose the entries of matrix.

We usually denote matrices by capital letters. For instance, the general matrix with m rows and n columns is

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{22} \cdots\cdots\cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots\cdots\cdots & a_{mn} \end{bmatrix} = \left[a_{ij}\right], \text{ in brief.}$$

Here a_{11} denotes the element lying in the 1st row and the 1st column, a_{12} denotes the element lying in the 1st row and the 2nd column, and, in general, a_{ij} denotes the element lying in the ith row and the jth column. We also say that **a_{ij} is the (i, j)th entry of A.**

Thus, the (1, 3)th entry of $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 3 & 0.3 & -1 \end{bmatrix}$ is 0, and the (3, 1)th entry is 3.

We can also denote a matrix **A** consisting of m rows and n columns by $A_{m \times n}$ or $A^{(m, n)}$; and we say that it **has order $m \times n$ or is an $m \times n$ matrix.**

We will often refer to the ith row of a matrix, meaning the ith row from the top. Similarly, the ith column of a matrix refers to the ith column, counting from left to right.

If the number of rows in a matrix equals the number of columns, the matrix is called a **square matrix**. Isn't the name appropriate?

Some examples of square matrices are [2], $\begin{bmatrix} 3 & 5 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Such matrices are very important in matrix theory.

You may like to **try some exercises** on matrices now.

E 1) Write the order of each of the following matrices.

$$\begin{bmatrix} 1 & 0 & 2 \\ -4 & i & 9 \\ 3 & 0 & 8i \end{bmatrix}, [5], \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 90 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, [2 \ \ -2 \ \ 1].$$

Also write their (2, 3)th and (1, 1) th entries and their 3rd columns, if they exist.

E 2) Write down a 3×4 matrix in which the (i, j)th entry is 0 whenever $i < j$, and non-zero otherwise.

E 3) a) Rewrite the linear systems that you obtained in E5 and E6 of Unit 4, in matrix notation.

b) What would happen to these matrices if the first and second equations in each of the linear systems were interchanged?

The **coefficient matrix** of a linear system is the matrix formed by taking the coefficients of the equations in the system.

Now, go back to the system of equations (1). We rewrote them in matrix notation in (2). We can also write (2) in shorthand notation as AX = B, where A is the 3×3 coefficient matrix, X is the 3×1 matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B is the 3×1 matrix $\begin{bmatrix} 5 \\ -6 \\ 1 \end{bmatrix}$.

If the number of equations in a linear system equals the number of variables, the coefficient matrix will be a square matrix. In this situation we can sometimes use the concept of a determinant to solve the system. Let us see what this concept is.

2.3 DETERMINANTS

You have just seen that we can represent a set of n linear equations in n variables by a matrix equation AX = B, where A is an $n \times n$ matrix. Associated with this square matrix A, we can define a unique number — its determinant. In this section we will discuss determinants of matrices whose elements are real numbers. We will also discuss some of their properties.

Let us start with a definition.

Definition: The **determinant of a $1\times 1'$ matrix** $A = [b]$, denoted by $|A|$ or **det (A)**, is b.

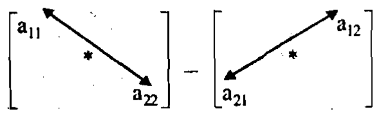
For example, if $A = [3]$, then $|A|$ is 3. Similarly, if $A = \left[-\frac{1}{2}\right]$, then $|A| = -\frac{1}{2}$.

Now let us consider the determinant of a 2×2 matrix.

Definition: The **determinant of the 2×2 matrix** $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is the number

$a_{11}\,a_{22} - a_{12}\,a_{21}$, and is denoted by $|A|$.

This is simply the cross multiplication



For example, if $A = \begin{bmatrix} 0 & 2 \\ -1 & 5 \end{bmatrix}$, then $|A| = 0\times 5 - 2\times(-1) = 2$

Another common way of writing $|A|$ is to write its elements within parallel vertical lines, instead of within square brackets. For example, we can write the determinant of

$\begin{bmatrix} 0 & 2 \\ -1 & 5 \end{bmatrix}$ as $\begin{vmatrix} 0 & 2 \\ -1 & 5 \end{vmatrix}$.

We will often see this way of writing a determinant.

You may like to **calculate some determinants now.**

E4) Evaluate

(a) $\begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix}$, (b) $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$, (c) $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}$,

(d) $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$, (e) $\begin{vmatrix} 2 & -2 \\ -1 & 0 \end{vmatrix}$, (f) $\begin{vmatrix} 1 & -1 \\ -4 & 2 \end{vmatrix}$,

Compare the determinants obtained in (a) and (e), and (a) and (f).

Now let us use determinants of 2×2 matrices to obtain the determinant of a 3×3 matrix.

Definition: The **determinant of the 3×3 matrix**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is}$$

The determinants of A is denoted by | A | or det (A)

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= (-1)^{1+1}a_{11}|A_{11}| + (-1)^{1+2}a_{12}|A_{12}| + (-1)^{1+3}a_{13}|A_{13}|,$$

where A_{1j} is the matrix obtained from A after deleting the first row and the jth column, for $j = 1, 2, 3$.

In obtaining $|A|$, we **expanded by (or along) the first row**. We could also have expanded by the second row, third row, or either of the columns. So, for example, expanding along the third column, we get

$|A| = (-1)^{1+3} a_{13} |A_{13}| + (-1)^{2+3} a_{23} |A_{23}| + (-1)^{3+3} a_{33} |A_{33}|$,

where A denotes the matrix obtained from A after deleting the ith row and the third column.

All 6 ways of obtaining | A | lead to the same value. We will not prove this here. However, let us consider an example.

Let $A = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 4 & 1 \\ 7 & 3 & 2 \end{bmatrix}$. Then, expanding by the first row, we get

$$\begin{vmatrix} 1 & 2 & 6 \\ 5 & 4 & 1 \\ 7 & 3 & 2 \end{vmatrix} = (-1)^{1+1}.1.\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} + (-1)^{1+2}.2.\begin{vmatrix} 5 & 1 \\ 7 & 2 \end{vmatrix} + (-1)^{1+3}.6.\begin{vmatrix} 5 & 4 \\ 7 & 3 \end{vmatrix}$$

$$= (4 \times 2 - 1 \times 3) - 2(5 \times 2 - 1 \times 7) + 6(5 \times 3 - 4 \times 7) = -79.$$

Now, why **don't you try this exercise?**

E5) Obtain | A |, for A in the example above, by expanding along the 3rd row and by expanding along the 2nd column.

As you have seen, we define the determinant of a 3×3 matrix in terms of the determinants of 2×2 matrices. In the same way we can obtain the determinant of any $n \times n$ square matrix ($n \geq 2$) in terms of the determinants of $(n-1) \times (n-1)$ square matrices.

Definition: The determinant of an n × n matrix $A = [a_{ij}]$, where $n > 1$, is given by

$|A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}| + \ldots + (-1)^{1+n} a_{1n} |A_{1n}|$,

where A_{1j} = matrix obtained from A on deleting the 1st row and the jth column, $\forall\ j = i, \ldots, n$.

What we have stated for the 3×3 case is true for the $n \times n$ case ($n \geq 2$), namely, that we can expand along any row or column to obtain the determinant of A. Thus,

$$|A| = (-1)^{i+1} . a_{i1} |A_{i1}| + (-1)^{i+2} . a_{i2} |A_{i2}| + \ldots + (-1)^{i+n} . a_{in} |A_{in}|\ \forall\ i = 1.., n,$$

and

$$|A| = (-1)^{1+j} . a_{1j} |A_{1j}| + (-1)^{2+j} . a_{2j} |A_{2j}| + \ldots + (-1)^{n+j} . a_{nj} |A_{nj}|\ \forall\ j = 1.., n,$$

We call the determinant of an $n \times n$ matrix **a determinant of order n** (or **size n**).

So far we have spent some time on evaluating determinants of orders 1, 2 and 3. In this course we shall not go to higher orders. (They are discussed in great detail in our course on linear algebra.) We will only introduce you to some elementary properties of determinants now. While solving E4 you may have realised some of them. These properties help us in evaluating a determinant in a shorter time. Let us see what they are.

Theorem 1 : Let A be a square matrix. Then | A | satisfies the following properties.

P1: If all the elements of any row or column of A are zero, then | A | = 0

P2: If B is the matrix obtained from A by interchanging any two rows (or any two columns), then | B | = – | A |.

P3: If B is the matrix obtained from A by multiplying all the elements of a row (or all the elements of a column) of A by a number c, then | B | = c | A |.

P4: If B is the matrix obtained from A by adding the multiple of a row to another row (or the multiple of a column to another column), then | B | = | A |.

A multiple of a row (or of a column) by a non-zero number K is the row (or column) obtained by multiplying each of its entries by K.

P5: If two rows (or two columns) of A are equal, then | A | = 0.

We shall not prove these properties here. If you are interested in the proofs, you can study Block 3 of our next level course on linear algebra. In this course we shall only see how to apply these properties. Let us first verify them in some cases.

If $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, then $|A| = 1 \times 0 - 2 \times 0 = 0$ (an example of P1).

If $A = \begin{bmatrix} 3 & 0 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 5 & -2 \end{bmatrix}$, that is, B is obtained by interchanging the columns of A, then $|A| = 15$ and $|B| = -15 = -|A|$ (an example of P2).

If $A = \begin{bmatrix} 1 & -1 \\ 2 & 6 \end{bmatrix}$ and B is obtained by multiplying the second row of A by 5, that is,

$B = \begin{bmatrix} 1 & -1 \\ 10 & 30 \end{bmatrix}$, then $|A| = 8$ and $|B| = 40 = 5|A|$ (an example of P3).

Now, let us take an example which satisfies the hypothesis of P4. Let $A = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}$

and $B = \begin{bmatrix} 4 & 3 \\ -3 & -1 \end{bmatrix}$. Can you make out if B is related to A in any manner suggested in P4?

How have we got B? We multiplied the elements in the second row of A by (– 1) and added them to the corresponding elements in the first row, that is, we subtracted the second row of A from the first row.

Thus, $B = \begin{bmatrix} 1 + (-3)(-1) & 2 + (-1)(-1) \\ -3 & -1 \end{bmatrix}$.

Now, $|A| = (1)(-1) - (2)(-3) = 5$ and $|B| = 5 = |A|$. So P4 seems to work in this case.

Now, suppose we add 3 times the first column of A to the second column. We get the matrix

$$C = \begin{bmatrix} 1 & 2+3 \\ -3 & (-1)+(-9) \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -3 & -10 \end{bmatrix}.$$

Then, $|C| = |A|$, verifying P4 again.

Now let us see if P5 holds for a general 2×2 matrix satisfying the hypothesis. One such matrix is

$$A = \begin{bmatrix} a & a \\ b & b \end{bmatrix}, \; a, b \in \mathbf{R}.$$

Then $|A| = ab - ab = 0$, verifying P5.

Now, let us try and obtain the determinant of the 3×3 matrix

$$A = \begin{bmatrix} 3 & -5 & 5 \\ 2 & 1 & -1 \\ 3 & 9 & -9 \end{bmatrix}$$

The third column is (–1) times the second column. So let us add the second column to the third column. By P4 we get

$$|A| = \begin{bmatrix} 3 & -5 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 0 \end{bmatrix} = 0, \text{ by P1}.$$

So, you have seen some ways in which P1 to P5 can be used for calculating determinants. **Why don't you try this exercise now?**

E6) Using P1 to P5, evaluate the determinants of

a) $A = \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ d & e & f \end{bmatrix}$, where $a, b, c, d, e, f \in \mathbf{R}$

b) $B = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 6 & 3 \\ 1 & 5 & 1 \end{bmatrix}$,

c) $C = \begin{bmatrix} 1 & 5 & 1 \\ -3 & 6 & 3 \\ -1 & 2 & 1 \end{bmatrix}$

d) $D = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 6 & 3 \\ 2 & 5 & 1 \end{bmatrix}$

e) $E = \begin{bmatrix} a & 2a & d \\ b & 2b & e \\ c & 2c & f \end{bmatrix}$, where a, b, c, d, e, f $\in$ **R**

As we said earlier, the importance of the properties P1 to P5 lies in their use for decreasing the effort involved in computing a determinant. You must have realised this while doing E6. Let us look at another example of their use. For our convenience we **will use R_i to denote the ith row and C_i to denote the ith column.**

Let $A = \begin{bmatrix} 2 & 3 & 6 \\ 6 & 5 & -5 \\ 1 & 1 & 3 \end{bmatrix}$. To evaluate $|A|$ by expanding along any row or column will require us to evaluate 3 determinants of order 2. But, if we use P4, we can multiply R_3 by (-2) and add it to R_1, to get

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 5 & -5 \\ 1 & 1 & 3 \end{bmatrix}.$$

Now, R_1 has two zeros in it. So we can obtain $|B|$, which has the same value as $|A|$ by expanding along R_1. This means that we only need to evaluate $|B_{12}|$.
Thus, $|A| = |B| = (-1).1.|B_{12}| = -23$.

The following remark will be very useful to you for evaluating a determinant.

Remark 1: Whenever you have to compute the determinant of a matrix, it is best to expand along the row or column with the maximum number of zeros. Therefore, one should use the property P4 so as to get as many zeros as possible in some row or column.

You may like to use the remark above to obtain the following interesting results.

E 7) Let $A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ and $B = \begin{bmatrix} p & 0 & 0 \\ q & s & 0 \\ r & t & u \end{bmatrix}$,

where a, b, ..., f, p, q, ..., u $\in$ **R**.

A and B are the general forms of 3×3 **triangular matrices** over **R**.

Show that

a) $|A| = adf$, that is, the product of the elements lying on the **principal diagonal**

b) $|B| = psu$.

E 8) Let $C = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b, c $\in$ **R**.

C is the general 3×3 **diagonal matrix** over **R**.

Show that $|C| = abc$.

E 9) Obtain $\begin{vmatrix} 3 & -2 & 4 \\ 6 & 8 & 1 \\ -9 & 6 & 12 \end{vmatrix}$

E 10) Show that the analogues of E7 and E8 are true for general 2 × 2 triangular and diagonal matrices.

Now consider $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. What is $|I|$? We use E8, and find $|I| = 1$.

And $1 \neq 0$. So $|I| \neq 0$. Because of this property I belongs to the class of matrices that we will now define.

Definition: A square matrix A is called **non-singular** if $|A| \neq 0$. Otherwise A is called **singular.**

You can check that some more examples of non-singular matrices are [5],

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$; and $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ is an example of a singular matrix.

Why don't you try some exercise now?

E11) Which of the following matrices are non-singular? Give reasons for your choice.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}, [-3], \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ -4 & 9 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 6 \\ \sqrt{2} & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}.$$

E12) When are $[a]$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} a & b & c \\ d & e & d \\ f & g & f \end{bmatrix}$ singular? here $a, b, c, d, e, f, g \in \mathbf{R}$.

Non-singular matrices form an important part of matrix theory. In the next section we shall introduce you to a rule for solving any system of linear equations whose coefficient matrix is non-singular.

2.4 CRAMER'S RULE

Fig. 1 : Cramer (1704-1752)

In 1750 the German mathematician Gabriel Cramer published a rule for solving a set of n linear equations in n unknowns simultaneously. Though this rule is named after Cramer, it seems to have been discovered by the British mathematician Colin Maclaurin twenty years earlier.

Let us see what this rule is.

Consider the general system of 2 equations in 2 unknowns:

$$ax + by + c = 0$$
$$dx + ey + f = 0$$

where $ae - db \neq 0$.

Then, if you use the substitution method, what solution do you get? We get

$$x = \frac{bf - ce}{ae - db}, \; y = \frac{cd - af}{ae - db}$$

Notice that this is the same as

$$x = \frac{\begin{vmatrix} -c & b \\ -f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}, \; y = \frac{\begin{vmatrix} a & -c \\ d & -f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

But how did we get x and y in the determinant form? What we did was to first write the system of equations as

$$ax + by = -c$$
$$dx + ey = -f,$$

that is, AX = B, where A is the coefficient matrix $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$, X is $\begin{bmatrix} x \\ y \end{bmatrix}$ and B is the matrix of constant terms. $\begin{bmatrix} -c \\ -f \end{bmatrix}$.

Then we calculated D = | A |, which is given to be non-zero. After that we calculated D_1, the determinant of the matrix obtained from A by replacing the first column by B; thus, $D_1 = \begin{bmatrix} -c & b \\ -f & e \end{bmatrix}$. Similarly, we calculated D_2, the determinant of the matrix obtained from A by replacing the second column by B; Thus, $D_2 = \begin{bmatrix} a & -c \\ d & -f \end{bmatrix}$. Then

$$x = \frac{D_1}{D} \text{ and } y = \frac{D_2}{D}.$$

Cramer extended this result to a system of n linear equations in n unknowns.

Let us consider his general rule.

Cramer's Rule: Consider the following linear system of n equations in n unknowns:

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n &= b_n \end{aligned}$$

that is, AX = B, where

$$A = \left[a_{ij}\right], x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

For i = 1, ..., n, define A_i to be the matrix obtained from A after substituting B for the ith column of A.

Define $D_i = |A_i| \;\forall\; i = 1, \dots, n$, and

D = | A |.

Then, **if D ≠ 0,** the system has the unique solution

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D}.$$

Does all this seem too much to take in? Don't worry. In this course we shall only be applying the rule for n = 2 or 3.

Just remember that

Cramer's rule can only be applied if i) the number of equations in the linear system equals the number of variables; and ii) the determinant of the coefficient matrix is non-zero.

Let us apply Cramer's rule in an example. Consider the system

$$2x - 3y + z = 1$$
$$x + y + z = 2$$
$$3x - 4z - 17 = 0$$

We first rewrite the system as

$$\begin{aligned} 2x-3y+z &= 1 \\ x+y+z &= 2 \\ 3x-0.y-4z &= 17 \end{aligned}$$

This is of the form $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 17 \end{bmatrix}$$

Now let us first see whether $|A| = 0$ or not. Let us expand along the third row. We get

$$D = |A| = 3\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -32 \neq 0.$$

So we can go ahead and apply Cramer's rule. For this we evaluate

$$D_1 = \begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & 1 \\ 17 & 0 & -4 \end{vmatrix} = 96$$

$$D_2 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 17 & -4 \end{vmatrix} = -32$$

$$D_3 = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 17 \end{vmatrix} = 64.$$

Then $x = \frac{D_1}{D} = 3, y = \frac{D_2}{D} = 1, z = \frac{D_3}{D} = -2.$

On **verifying**, we find that (3, 1, –2) is indeed the solution of the given system.

You may like to try your hand at applying Cramer's rule now.

E 13) Solve the following system by Cramer's rule, if applicable. Otherwise use the Gaussian elimination method (see Sec. 4.4).

a) $x+y+1=0$
$2x-y=7.$

b) $x+y-z+2=0$
$2x-y+z+5=0$
$x-2y+3z-4=0.$

c) $3x+5y+2z=1$
$4x+y-7=0$
$9x+15y+6z=3.$

E 14) Consider the $n \times n$ linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= 0 \end{aligned}$$

If A is the coefficient matrix and $|A| \neq 0$, can we apply cramer's rule to solve the system? If so, obtain the solution set.

E15) I have Rs. 2480/- in five, ten and twenty rupee notes. The total number of notes is 290 and all the ten rupee notes add up to Rs. 60/- more than the sum of the twenty rupee notes. How many of each type of note do I have?

In this unit and previous one we have discussed three methods for solving linear systems. The examples and exercises we attempted involved a maximum of 3 equations and a maximum of four unknowns. But, in practical applications in the sciences and social sciences, one needs to solve very large systems. These may consist of 15, 20 or more equations in as many or more variables. As you may have guessed, these systems require computers for solving them. Then the best method to apply is the Gaussian elimination method. We have discussed this method as well as Cramer's rule in greater detail in our course on linear algebra.

Now why don't you **try solving the following exercise** by any of the three methods we have covered in this unit and the previous one?

E 16) Which of the following linear systems are consistent? Obtain the solution sets, wherever possible.

a) $2x-5y+7z=6$
$x-3y+4z=3$
$3x-8y+11z=11.$

b) $2x-y+3z-5w=-7$
$-7y+3z-7w=-13$
$3x+4y+2z=0.$

c) $x-y+z=0$
$-3x+y-4z=0$
$7x-3y-9z=0$
$4x-2y-5z=0.$

d) $x-2y+z=6$
$3x+y-4z=-7$
$5x-3y+2z=5.$

Let us now summarise the contents of this unit.

2.5 SUMMARY

In this unit we

1) defined an $m \times n$ matrix, and a square matrix, in particular.
2) introduced you to the concept of a determinant of a square matrix.
3) discussed some properties of determinants.
4) used the definition and properties of determinants to evaluate determinants of orders 1, 2 and 3.
5) defined a non-singular matrix.
6) applied Cramer's rule for solving a linear system of equations whose coefficient matrix is non-singular.

With this unit we finish our discussion on simultaneous linear equations. In the next unit we shall look at some commonly used inequalities. But before going to Unit 6, **please go back to the objectives given in Sec. 5.1** and check if you have achieved them. You may also like to go through the next section, in which we have given our solutions to the exercises in the unit. This may be useful to you for counterchecking your solutions.

2.6 SOLUTIONS/ANSWERS

E1) Their orders are 3×3, 1×1, 3×3, 3×1 and 1×3, respectively.
The (2, 3)th and (1, 1)th entries of the first matrix are 9 and 1.
The (1, 1)th entry of the second one is 5. It has no (2, 3)th entry.
The (2, 3)th and (1, 1)th entries of the third one are go and 1.
The (1, 1) th entry of the fourth one is 4; it has no (2, 3)th entry.
The (1, 1)th entry of the fifth one is 2; it has no (2, 3)th entry.

Only the first and third and fifth matrices have third columns, which are $\begin{bmatrix} 2 \\ 9 \\ 8i \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

and [1] respectively.

E2) The required matrix will be of the form

$$\begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \end{bmatrix}, \text{ where } a, ..., f \in \mathbf{R} \setminus \{0\}.$$

E3) a) $\begin{bmatrix} 3 & 5 & 2 \\ 1 & 7 & 3 \\ 0 & 2 & 0 \\ 1 & 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 55 \\ 45 \\ 10 \\ 45 \end{bmatrix}$ gives us the system in E5.

$\begin{bmatrix} 7 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 150 \\ 30 \end{bmatrix}$ gives us the system in E6.

b) The first two rows of both the coefficient matrices, as well as of the matrices on the right hand sides, would be interchanged.

E4) a) $1 \times 0 - (-1)(-1) = -1$

b) $1 \times 2 - 1 \times 2 = 0$

c) 0

d) $-1^2 = 1$

E5) By expanding along the 3rd row, we get

$|A| = (-1)^{3+1}.7.|A_{31}| + (-1)^{3+2}.3.|A_{32}| + (-1)^{3+3}.2.|A_{33}|.$

$$= 7\begin{vmatrix} 2 & 6 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 6 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix}$$

$= -79.$

By expanding along the second column, we get

$|A| = (-1)^{2+1}.2.|A_{21}| + (-1)^{2+2}.4.|A_{22}| + (-1)^{2+3}.3.|A_{23}| = -79.$

E6) a) 0 by P1.

b) $|B| = 3\begin{vmatrix} -1 & 2 & 1 \\ -1 & 2 & 1 \\ 1 & 5 & 1 \end{vmatrix}$, by P3

$= 3.0$, by P5
$= 0.$

c) C is obtained from B above, by interchanging the first and third rows.
$\therefore |C| = -|B| = 0.$

d) $D = 0$, as in (b).

e) $|E| = 2\begin{vmatrix} a & a & d \\ b & b & e \\ c & c & f \end{vmatrix} = 0$, by P3.

E7) a) We expand along a row or column which has the maximum number of zeros. So, expanding along C_1, we get.

$|A| = (-1)^{1+1}.a.|A_{11}| = a = \begin{vmatrix} d & e \\ 0 & f \end{vmatrix} = adf.$

b) We can expand along R1 to get

$|B| = psu$.

Note that we would get the same answer if we'd expanded along any other row or column.

E 8) Expand along R_1 to get $|C| = abc$.

E 9) Let $A = \begin{bmatrix} 3 & -2 & 4 \\ 6 & 8 & 1 \\ -9 & 6 & 12 \end{bmatrix}$

We want to make some entries zero. Looking at C_1, you may have noticed that replacing R_2 by $R_2 + (-2)R_1$ and R_3 by $R_3 + 3R_1$ will make the (2, 1)th and (3, 1)th entries zero. Then, by P_4 we get

$$|A| = \begin{vmatrix} 3 & -2 & 4 \\ 6+(-2)3 & 8+(-2)(-2) & 1+(-2)(4) \\ -9+3(3) & 6+3(-2) & 12+3(4) \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -2 & 4 \\ 0 & 12 & -7 \\ 0 & 0 & 24 \end{vmatrix}$$

Now, by E7, $|A| = 3 \times 12 \times 24 = 864$.

E 10) The general 2×2 real triangular matrix $A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ or $B = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, where $a, b, c \in \mathbf{R}$. In both cases their determinant is ac, the product of the diagonal elements. The general 2×2 real diagonal matrix is $C = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a, b \in \mathbf{R}$.

$|C| = ab$, the product of the diagonal elements.

The diagonal elements of an non matrix are its (i, i)th entries $\forall\ i = 1 \ldots n$

E 11) $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$; hence, the first matrix is non-singular.

$\begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$ is singular, since its determinant is zero.

$|[-3]| = -3 \neq 0$; thus, $[-3]$ is non-singular

$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ -4 & 9 \end{bmatrix}$ is not a square matrix, and hence can't be non-singular. Note that it is not singular either, since a singular matrix has to be square too.

The last matrix has zero determinant, and hence is singular.

E 12) $[a]$ is singular iff $a = 0$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is singular iff $ad - bc = 0$.

$\begin{vmatrix} a & b & c \\ d & e & d \\ f & g & f \end{vmatrix} = (a-c)(ef-dg)$. This is zero iff $a = c$ or $ef = dg$. That is, the given matrix is singular iff $a = c$ or $\begin{bmatrix} e & d \\ g & f \end{bmatrix}$ is singular.

E 13) a) $x + y = -1$

$2x - y = 7$

is the same as $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$.

Since $D = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3 \neq 0$, Cramer's rule can be applied.

Now, $D_1 = \begin{vmatrix} -1 & 1 \\ 7 & -1 \end{vmatrix} = -6$ and

$D_2 = \begin{vmatrix} 1 & -1 \\ 2 & 7 \end{vmatrix} = 9$.

Then $x = \dfrac{D_1}{D} = 2$ and $y = \dfrac{D_2}{D} = -3$.

Thus, the solution is $(2, -3)$.

b) The given system is equivalent to

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 4 \end{bmatrix}$$

Now $D = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -2 & 3 \end{bmatrix} = -3 \neq 0$. So we can apply Cramer's rule. We calculate

$$D_1 = \begin{bmatrix} -2 & 1 & -1 \\ -5 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix} = 7,$$

$$D_2 = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -5 & 1 \\ 1 & 4 & 3 \end{bmatrix} = -22,$$

$$D3 = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & -5 \\ 1 & -2 & 4 \end{bmatrix} = -21.$$

Therefore, $x = -\frac{7}{3}, y = \frac{22}{3}, z = 7$.

c) The system is equivalent to

$$\begin{bmatrix} 3 & 5 & 2 \\ 4 & 1 & 0 \\ 9 & 15 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$$

Note that the 3rd row of the coefficient matrix is 3 times the 1st row. Thus, its determinant is zero. So we can't apply Cramer's rule. Let us solve the system by successive elimination.

The system is

$3x + 5y + 2z = 1$ ...(3)

$4x + y \quad = 7$ ...(4)

$9x + 15y + 6z = 3$ ...(5)

Note that (5) is equivalent to (3). So we can drop (5). Now let us eliminate y from (3) and (4). For this we calculate (3) – 5 × (4), which is

$-17x + 2z = -34$ ...(6)

Now, we can't eliminate any further, between (3), (4) and (6). So let us use (4) and (6) to write y and z in terms of x.

(4) $\Rightarrow y = 7 - 4x$, and

(6) $\Rightarrow z = \frac{17}{2}x - 17$.

Thus, we get a 1-parameter set of infinitely many solutions.

$$\left\{(x, 7-4x, \frac{17}{2}x - 17) \mid x \in \mathbf{R}\right\}.$$

E 14) Since $|A| \neq 0$, we can apply Cramer's rule. Now, to apply this rule, we calculate D_i

for $i = 1, \ldots, n$ by substituting the ith column with the constant column $\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$.

Thus, each $D_i = 0$, by P1 of Theorem 1.

Thus, $x_i = 0 \ \forall \ i = 1, \ldots, n$.

The solution $(0, 0, \ldots, 0)$ is called the **trivial solution.**

E 15) Let x, y, z denote the number of five, ten and twenty notes, respectively. Then we know that

$5x + 10y + 20z = 2480$

$x + y + z = 290$

$10y - 20z = 60$

We can solve this by Cramer's rule to get

$x = 164, y = 86, z = 40$.

E16) a) This is a 3×3 system. First let us see if we can apply Cramer's rule. Since the determinant of the coefficient matrix is zero, we can't apply Cramer's rule.

So let us try to solve it by elimination.

Adding the first two equations, we get

$3x - 8y + 11z = 9$.

Subtracting this from the third equation in the system, we get $0 = 2$, a false statement.

Thus the system is inconsistent.

b) The given system is

$$2x - y + 3z - 5w = 7 \quad \ldots(7)$$

$$-7y + 3z - 7w = -13 \quad \ldots(8)$$

$$3x + 4y + 2z = 0 \quad \ldots(9)$$

We shall try and solve by elimination.

To eliminate w from (7) and (8), we calculate $7 \times (7) - 5 \times (8)$

We get

$14x + 28y + 6z = 16$, that is,

$$7x + 14y + 3z = 8 \quad \ldots(10)$$

Eliminating z from (9) and (10), we get

$$5x + 16y = 16 \quad \ldots(11)$$

Now, we can't eliminate any further. So we shall try and obtain all the variables in terms of the minimum number of variables possible.

$$(11) \Rightarrow x = \frac{16}{5}(1 - y).$$

$$\text{Then } (9) \Rightarrow \frac{48}{5}(1-y) + 4y + 2z = 0 \Rightarrow z = \frac{14y - 24}{5}.$$

$$\text{Then } (8) \Rightarrow -7y + \frac{3}{5}(14y - 24) - 7w = -13 \Rightarrow w = \frac{1}{5}(y - 1).$$

Thus, we get a 1-parameter set of solutions, namely,

$$\left\{\left(\frac{16(1-y)}{5}, y, \frac{14y - 24}{5}, \frac{y-1}{5}\right) \mid y \in \mathbf{R}\right\}.$$

To check that these are the solutions, we substitute the 4-tuple

$\left(\frac{16}{5}(1-y), y, \frac{1}{5}(14y - 24), \frac{1}{5}(y-1)\right)$ in each of the equations of the system and find that it satisfies them.

(c) Since the system is a 4×3 system, we shall apply the Gaussian elimination method.

$$x - y + z = 0 \quad \ldots(12)$$

$$-3x + y - 4z = 0 \quad \ldots(13)$$

$7x-3y-9z=0$ ...(14)

$4x-2y-5z=0$ ...(15)

$3\times(12)+(13)\Rightarrow -2y-z=0\Rightarrow 2y+z=0$ ...(16)

$7(12)-(14)\Rightarrow -4y+16z=0\Rightarrow y-4z=0$ ...(17)

$4(16)+(17)\Rightarrow y=0$

Then $(17)\Rightarrow z=0$.

Then $(12)\Rightarrow x=0$.

We check that (0, 0, 0) satisfies all the equations. Thus the system only has the trivial solution.

d) Since the determinant of the coefficient matrix is non-zero, we can apply Cramer's rule as well as the elimination method. Let us apply Cramer's rule.
For this we calculate

$$D=\begin{bmatrix}1 & -2 & 1\\ 3 & 1 & -4\\ 5 & -3 & 2\end{bmatrix}=28.$$

$$D_1=\begin{bmatrix}6 & -2 & 1\\ -7 & 1 & -4\\ 5 & -3 & 2\end{bmatrix}=-32,$$

$$D_2=\begin{bmatrix}1 & 6 & 1\\ 3 & -7 & -4\\ 5 & 5 & 2\end{bmatrix}=-100,$$

$$D_3=\begin{bmatrix}1 & -2 & 6\\ 3 & 1 & -7\\ 5 & -3 & 5\end{bmatrix}=0$$

$\therefore$ $x=\frac{-32}{28}$, $y=\frac{-100}{28}$, $z=0$, that is, the unique solution is $\left(\frac{-8}{7},\frac{-25}{7},0\right)$.