UNIT 3 FURTHER APPLICATIONS OF INTEGRAL CALCULUS

Structure

3.1 INTRODUCTION

In the last unit we have seen how definite integrals can be used to calculate areas. In fact, this application of definite integrals is not surprising. Because, as we have seen earlier, the problem of finding areas was the motivation behind the definition of integrals. In this unit we shall see that the length of an arc of a curve, the volume of a cone and other solids of revolution, the area of a sphere and other surfaces of revolution, can all be expressed as definite integrals. This unit also brings us to the end of this course on calculus.

Objectives

After reading this unit you should be able to :

- find the length of an arc of a given curve whose equation is expressed in either the Cartesian or parametric or polar forms,
- find the volumes of some solids of revolution,
- find the areas of some surfaces of revolution.

3.2 LENGTH OF A PLANE CURVE

In this section we shall see how definite integrals can be used to find the lengths of plane curves whose equations are given in the Cartesian, polar or parametric form. A curve whose length can be found is called a **rectifiable curve** and the process of finding the length of a curve is called **rectification.** You will see here that to find the length of an arc of a curve, we shall have to integrate an expression which involves not only the given function, but also its derivative. Therefore, to ensure the existence of the integral which determines the arc length, we make an assumption that the function defining the curve is derivable, and is deriative is also continuous on the interval of integration.

Let's first consider a curve whose equation is given in the Cartesian form.

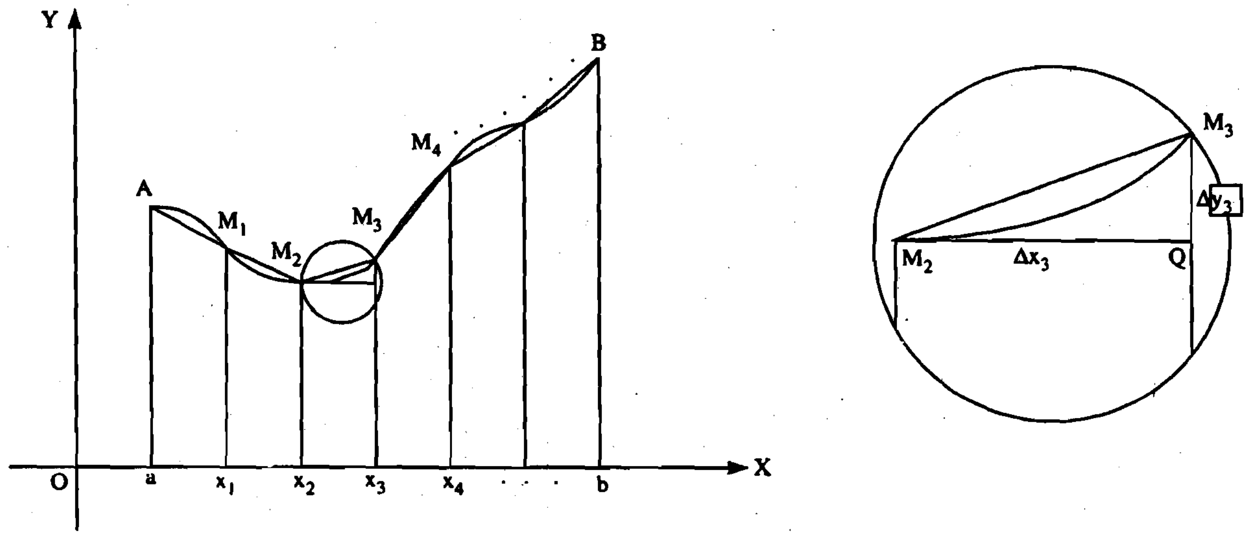
3.2.1 Cartesian Form

Let $y = f(x)$ be defined on the interval $[a, b]$. We assume that f is derivable and its derivative f' is continuous. Let us consider a partition P_n of $[a, b]$, given by

$P_n = [a = x_0 < x_1 < x_2 < ..... < x_n = b]$

The ordinates $x = a$ and $x = b$ determine the extent of the arc AB of the curve $y = f(x)$ [Fig. 1 (a)]. Let $M_i = 1, 2, ..., n-1$, be the points in which the lines $x = x_i$ meet the curve.

Join the successive points A, M_1, M_2, M_3, ..., M_{n-1}, B by straight line segments. Here we have approximated the given curve by a series of line segments.



(a) (b)

Fig. 1

If we can find the length of each line segment, the total length of this series will give us an approximation to the length of the curve. But how do we find the length of any of these line segments? Take M_2, M_3, for example. Fig. 1(b) shows an enlargement of the encircled portion in Fig. 1(a). Looking at it we find that

$$M_2M_3 = \sqrt{(\Delta x_3)^2 + (\Delta y_3)^2}$$

where $\Delta x_3 = M_2Q$ is the length $(x_3 - x_2)$, and
$\Delta y_3 = M_3Q = f(x_3) - f(x_2) = y_3 - y_2$.

In this way we can find the lengths of the chords AM_1, M_1M_2, ..., $M_{n-1}B$, and take their sum

$$S_n = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2},$$

S_n gives an approximation to the length of the arc AB. When the number of division points is increased indefinitely, and the length of each segment tends to zero, we obtain the length of the arc AB as

$$L_A^B = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}, \qquad ...(1)$$

provided this limit exists.

Our assumptions that f is derivable on [a,b], and that f′ is continuous, permit us to apply the mean value theorem [Theorem 3, Block 2].

Thus, there exists a point P_i^* (x_i^*, y_i^*) between the points M_{i-1} and M_i on the curve, where the tangent to the curve is parallel to the chord $M_{i-1} M_i$. That is,

$$f'(x_i^*) = \frac{\Delta y_i}{\Delta x_i}$$

or $\quad \Delta y_i = f'(x_i^*)\Delta x_i$

Hence we can write (1) as

$$L_B^A = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + [f'(x_i^*)\Delta x_i]^2}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)^2}\ \Delta x_i$$

This is nothing but the definite integral

$$\int_a^b \sqrt{(1 + [f'(x)^2]}\ dx.$$

Therefore,

$$L_A^B = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad ...(2)$$

Remark 1: It is sometimes convenient to express x as a single valued function of y. In this case we interchange the roles of x and y, and get the length

$$L_A^B = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy, \qquad ...(3)$$

where the limits of integration are with respect to y. Note that the length of an arc of a curve is invariant since it does not depend on the choice of coordinates, that is, on the frame of reference. Our assumption that f′ is continuous on [a, b] ensures that the integrals in (2) and (3) exist, and their value L_A^B is the length of the curve y = f(x) between the ordinates x = a and x = b.

The following example illustrates the use of the formulas given by (2) and (3).

Example 1 : Suppose we want to find the length of the arc of the curve y = lnx intercepted by the ordinates x = 1 and x = 2.

We have drawn the curve y = lnx in Fig. 2.

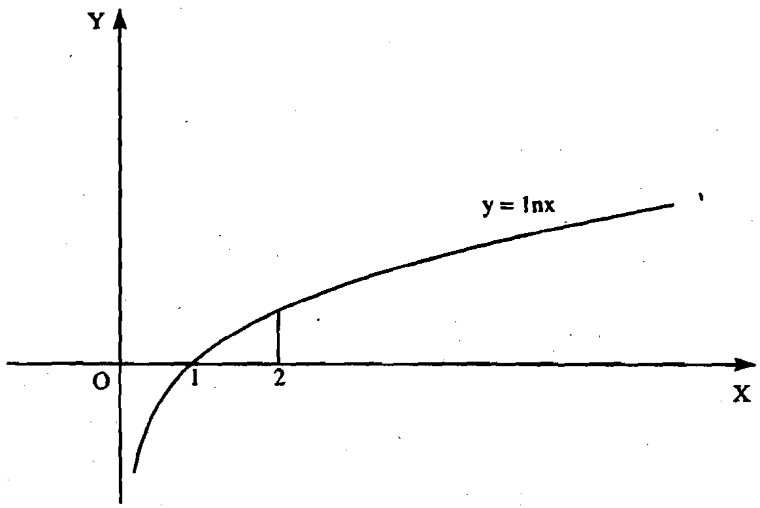


Fig. 2

Using (2), the required length L_1^2 is given by

$$L_1^2 = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{\left(1+\frac{1}{x^2}\right)}\, dx, \text{ since } \frac{dy}{dx} = \frac{1}{x}$$

$$= \int_1^2 \frac{\sqrt{1+x^2}}{x}\, dx$$

If we put $1 + x^2 = t^2$, we get $\frac{dx}{dt} = \frac{t}{x}$, and therefore,

$$L_1^2 = \int_{\sqrt{2}}^{\sqrt{5}} \left(1+\frac{1}{(t^2-1)}\right) dt$$

$$= \int_{\sqrt{2}}^{\sqrt{5}} dt + \int_{\sqrt{2}}^{\sqrt{5}} \frac{1}{t^2 - 1}\, dt$$

$$= \left[t + \frac{1}{2} \ln \frac{t-1}{t+1} \right]_{\sqrt{2}}^{\sqrt{5}}$$

$$= \sqrt{5} - \sqrt{2} + \frac{1}{2} \ln \frac{\sqrt{5}-1}{\sqrt{5}+1} - \frac{1}{2} \ln \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

$$= \sqrt{5} - \sqrt{2} + \ln \frac{2}{\sqrt{5}+1} - \ln \frac{1}{\sqrt{2}+1}$$

$$= \sqrt{5} - \sqrt{2} + \ln \frac{2(\sqrt{2}+1)}{\sqrt{5}+1}$$

We can also use (3) to solve this example. For this we write the equation $y = \ln x$ as $x = e^y$. The limits $x = 1$ and $x = 2$, then correspond to the limits $y = 0$ and $y = \ln 2$, respectively. Hence, using (3), we obtain

$$L_0^{\ln 2} = \int_0^{\ln 2} \sqrt{1 + e^{2y}}\, dy$$

$$= \int_{\sqrt{2}}^{\sqrt{5}} \frac{u^2}{u^2 - 1}\, du, \text{ on putting } 1 + e^{2y} = u^2$$

$$= \int_{\sqrt{2}}^{\sqrt{5}} \left(1 + \frac{1}{u^2 - 1}\right) du, = \sqrt{5} - \sqrt{2} + \ln \frac{2(\sqrt{2}+1)}{\sqrt{5}+1},$$

as we have seen earlier. This verifies our observation in Remark 1 that both (2) and (3) give us the same value of arc length.

Now, here are some exercises for you to solve.

E E1) Find the length of the line $x = 3y$ between the points (3, 1) and (6, 2). Verify your answer by using the distance formula.

E E2) Find the length of the curve $y = \ln \sec x$ between $x = 0$ and $x = \pi/2$.

E E3) Find the length of the arc of the catenary $y = C \cosh (x/c)$ measured from the vertex $(0, c)$ to any point (x, y) on the catenary.

E E4) Find the length of the semi-cubical parabola $ay^2 = x^3$ from the vertex to the point (a, a).

E E5) Show that the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$ is $a(\ln 2 + 15/16)$.

In the next sub-section we shall consider curves whose equations are expressed in the parametric form.

3.2.2 Parametric Form

Sometimes the equation of a curve cannot be written either in the form $y = f(x)$ or in the form $x = g(y)$. A common example is a circle $x^2 + y^2 = a^2$. In such cases, we try to write the equation of the curve in the parametric form. For example, the above circle can be represented by the pair of equations $x = a \cos t$, $y = a \sin t$. Here, we shall derive a formula to find the length of a curve given by a pair of parametric equations.

Let $x = \phi(t)$, $y = \psi(t)$, $\alpha \leq t \leq \beta$ be the equation of a curve in parametric form. As in the previous sub-section, we assume that the functions ϕ and ψ are both derivable and have continuous derivatives ϕ' and ψ' on the interval $[\alpha, \beta]$. We have

$$\frac{dx}{dt} = \phi'(t), \text{ and } \frac{dy}{dt} = \psi'(t).$$

Hence, $\dfrac{dy}{dx} = \dfrac{\psi'(t)}{\phi'(t)}$, and

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{\psi'(t)}{\phi'(t)}\right)^2}$$

$$= \frac{[\phi'(t)]^2 + [\psi'(t)]^2}{\phi'(t)} \qquad \text{(we assume that } \phi'(t) \neq 0\text{).}$$

Now, using (3) we obtain the length

$$L = \int_{x=\phi(\alpha)}^{x=\phi(\beta)} \sqrt{\left(1+\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{t=\alpha}^{t=\beta} \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\, \frac{\phi'(t)}{\phi'(t)}\, dt$$

Thus, $L = \displaystyle\int_{\alpha}^{\beta} \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\, dt$ ...(4)

The following example shows that sometimes it is more convenient to express the equation of a given curve in the parametric form in order to find its length.

Example 2: Let us find the whole length of the curve

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1.$$

By substitution, you can easily check that $x = a\cos^3 t$, $y = b\sin^3 t$ is the parametric form of the given curve. The curve lies between the lines $x = \pm a$ and $y = \pm b$ since $-1 \le \cos t \le 1$, and $-1 \le \sin t \le 1$. The curve is symmetrical about both the axes since its equation remains unchanged if we change the signs of x and y. The value $t = 0$ corresponds to the point (a, 0) and $t = \pi/2$ corresponds to the point (0, b). By applying the curve tracing methods discussed in Unit 9 we can draw this curve (see Fig. 3).

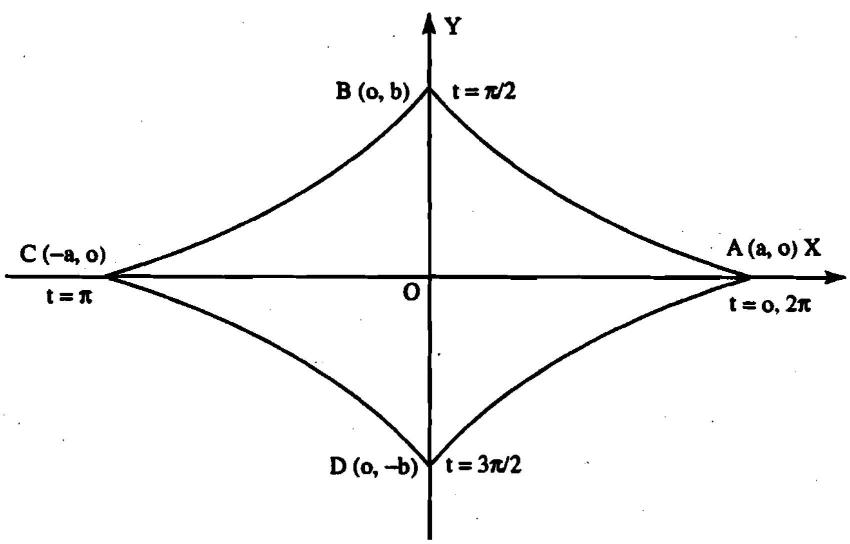


Fig. 3

Since the curve is symmetrical about both axes, the total length of the curve is four times its length in the first quadrant.

Now, $\frac{dx}{dt} = -3a\cos^2 t \sin t$; $\frac{dy}{dt} = 3b\sin^2 t\cos t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^2 t\cos^2 t\,(a^2\cos^2 t + b^2\sin^2 t)$$

Hence, the length of the curve is

$$L = 4\int_0^{\pi/2} 3\sin t\cos t\,\sqrt{a^2\cos^2 t + b^2\sin^2 t}\;dt$$

$$= 12\int_0^{\pi/2} \sin t\cos t\,\sqrt{a^2\cos^2 t + b^2\sin^2 t}\;dt$$

Putting $u^2 = a^2\cos^2 t + b^2\sin^2 t$, we obtain

$$2u = (2b^2 - 2a^2)\sin t\cos t\,\frac{dt}{du}$$

and the limits $t = 0$, $t = \pi/2$ correspond to $u = a$, $u = b$, respectively.

Thus, we have

$$= 12\int_a^b \frac{u^2\,du}{b^2-a^2} = \frac{12}{b^2-a^2}\left[\frac{u^3}{3}\right]_a^b$$

$$= \frac{12}{b^2-a^2}\,\frac{b^3-a^3}{3} = \frac{4(a^2+b^2+ab)}{a+b}$$

Now you can apply equation (4) to solve these exercises.

E E6) Find the length of the cycloid
$x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$

E E7) Show that the length of the arc of the curve
$x = e^t \sin t$, $y = e^t \cos t$ from $t = 0$ to $t = \pi/2$ is $\sqrt{2}\,(e^{\pi/2} - 1)$.

3.2.3 Polar Form

In this sub-section we shall consider the case of a curve whose equation is given in the polar form.

Let $r = f(\theta)$ determine a curve as θ varies from $\theta = \alpha$ to $\theta = \beta$, i.e., the function f is defined in the interval $[\alpha, \beta]$ (see Fig. 4). As before, we assume that the function f is derivable and its derivative f' is continuous on $[\alpha, \beta]$. This assumption ensures that the curve represented by $r = f(\theta)$ is rectifiable.

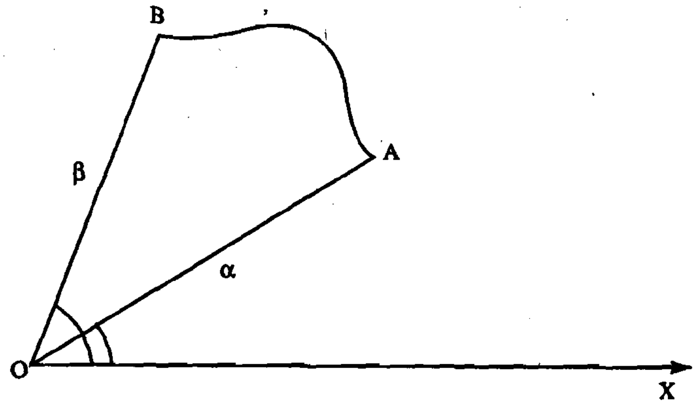


Fig. 4

Transforming the given equation into Cartesian coordinates by taking $x = r\cos\theta$, $y = r\sin\theta$, we obtain $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$.

Now we proceed as in the case of parametric equations, and get,

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}}{dx/d\theta}$$

Hence, the length of the arc of the curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$L = \int_{x=f(\alpha)\cos\alpha}^{x=f(\beta)\cos\beta} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

changing the variable x to θ.

$$= \int_{\alpha}^{\beta} \sqrt{[f'(\theta)\cos\theta - f(\theta)\sin\theta]^2 + [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2}\,d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{[f(\theta)^2 + [f'(\theta)]^2}\,d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \qquad ...(5)$$

We shall apply this formula to find the length of the curve in the following example.

Example 3: To find the perimeter of the cardioid $r = a(1+\cos\theta)$ we note that the curve is symmetrical about the initial line (Fig. 5). Therefore its perimeter is double the length of the arc of the curve lying above the x-axis.

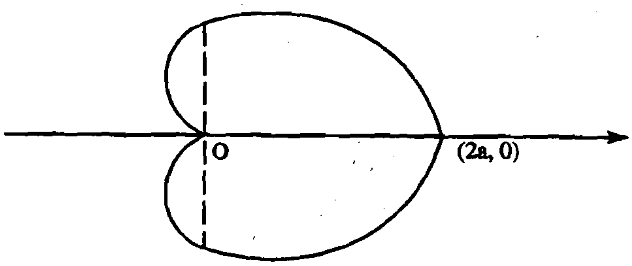


Fig. 5

Now, $\frac{dr}{d\theta} = -a\sin\theta$. Hence, we have

$$L = 2\int_0^{\pi} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = 2a\int_0^{\pi} \sqrt{(1+\cos\theta)^2+\sin^2\theta}\,d\theta$$

We know

$\frac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2}$

$$= 2a\int_0^{\pi}\sqrt{2(1+\cos\theta)}\,d\theta = 4a\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta$$

$$= 4a\left|2\sin\frac{\theta}{2}\right|_0^{\pi} = 8a.$$

In this section we have derived and applied the formulas for finding the length of a curve when its equation is given in either of the three forms: Cartesian, parametric or polar. Let us summarise our discussion in the following table.

Table 1: Length of an arc of a curve

Equation of the Curve	Length L
$y=f(x)$	$\int_a^b \sqrt{1+f'(x)^2}\,dx$
$x=g(y)$	$\int_c^d \sqrt{1+g'(y)^2}\,dy$
$x=\phi(t), y=\psi(t)$	$\int_\alpha^\beta \sqrt{\phi'(t)^2+\psi'(t)^2}\,dt$
$r=f(\theta)$	$\int_\alpha^\beta \sqrt{f(\theta)^2+f'(\theta)^2}\,d\theta$

Using this table you will be able to solve these exercises now.

E E8) Find the length of the curve $r = a\cos^3(\theta/3)$.

E E9) Find the length of the circle of radius 2 which is given by the equations $x=2\cos t+3, y=2\sin t+4, 0\le t\le 2\pi$.

E 10) Show that the arc of the upper half of the curve $r = a(1 - \cos\theta)$ is bisected by $\theta = 2\pi/3$.

E 11) Find the length of the curve $r = a(\theta^2 - 1)$ from $\theta = -1$ to $\theta = 1$.

3.3 VOLUME OF A SOLID OF REVOLUTION

Until now, in this course, we were concerned with only plane curves and regions. In this section we shall see how our knowledge of integration can be used to find the volume of certain solids. Look at the plane region in Fig. 6 (a). It is bounded by $x = a$, $x = b$, $y = f(x)$ and the x-axis. If we rotate this plane region about the x-axis, we get a solid. See Fig. 6 (b).

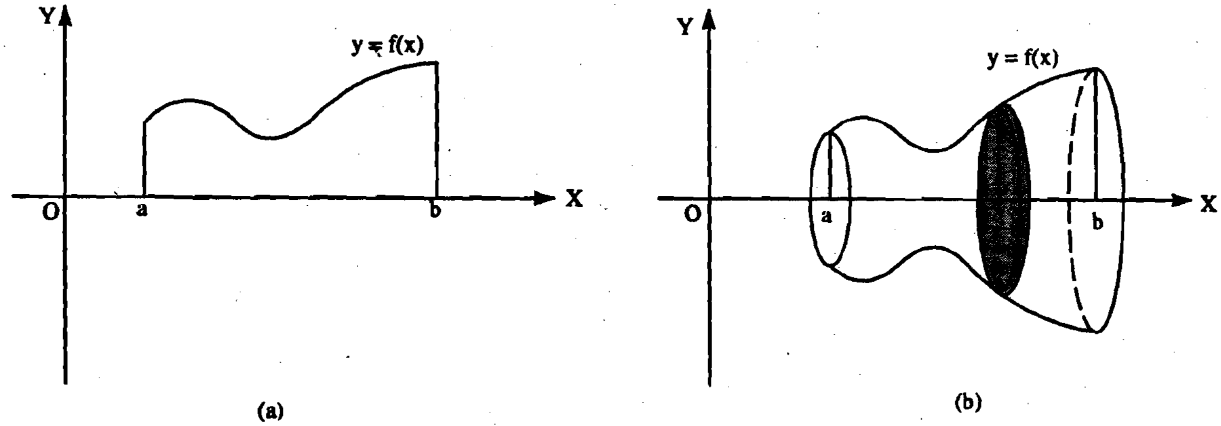


Fig. 6

Such solids are called solids of revolution. Fig. 7 (a) and Fig. 7(b) show two more examples of solids of revolution.

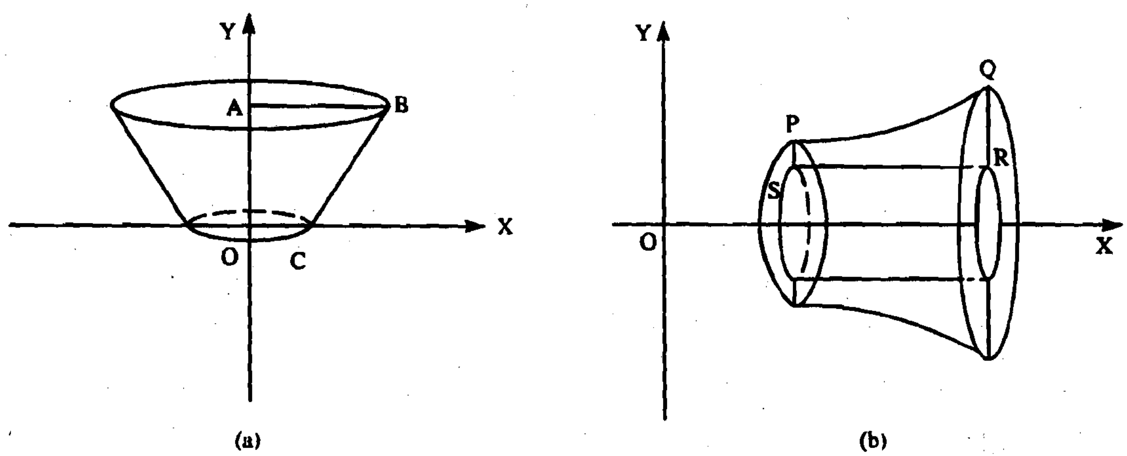
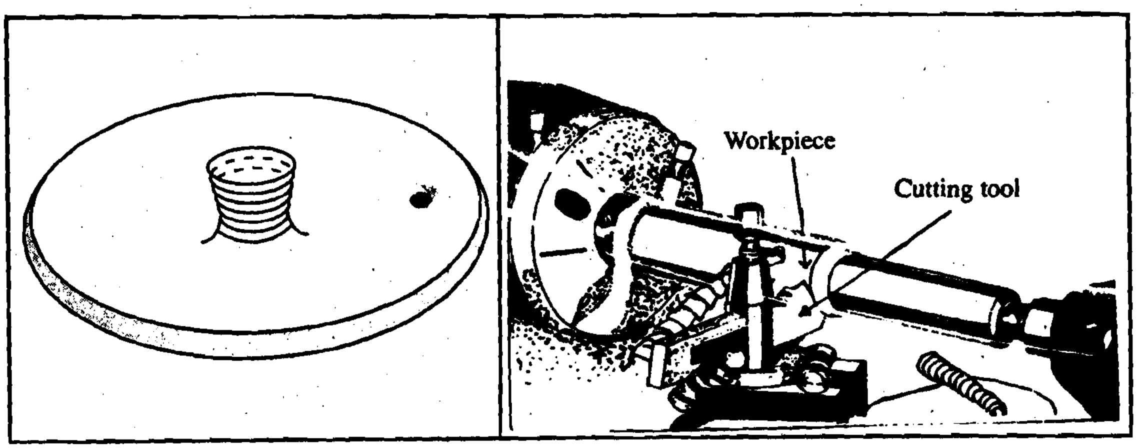


Fig. 7

The solid in Fig. 7(a) is obtained by revolving the region ABCO around the y-axis. The solid of revolution in Fig. 7(b) differs from the others in that its axis of rotation does not form a part of the boundary of the plane region PQRS which is rotated.

We see many examples of solids of revolution in every day life. The various kinds of pots made by a potter using his wheel are solids of revolution. See Fig. 8(a). Some objects manufactured with the help of lathe machine are also solids of revolution. See Fig. 8 (b).



(a) (b)

Fig. 8

Now, let us try to find the volume of a solid of revolution. The method which we are going to use is called the method of slicing. The reason for this will be clear in a few moments.

Let $T_n = \{a = x_0 < x_1 < x_2 < ...... < x_{n-1}, x_n = b\}$ be a partition of the interval [a, b] into n sub-intervals.

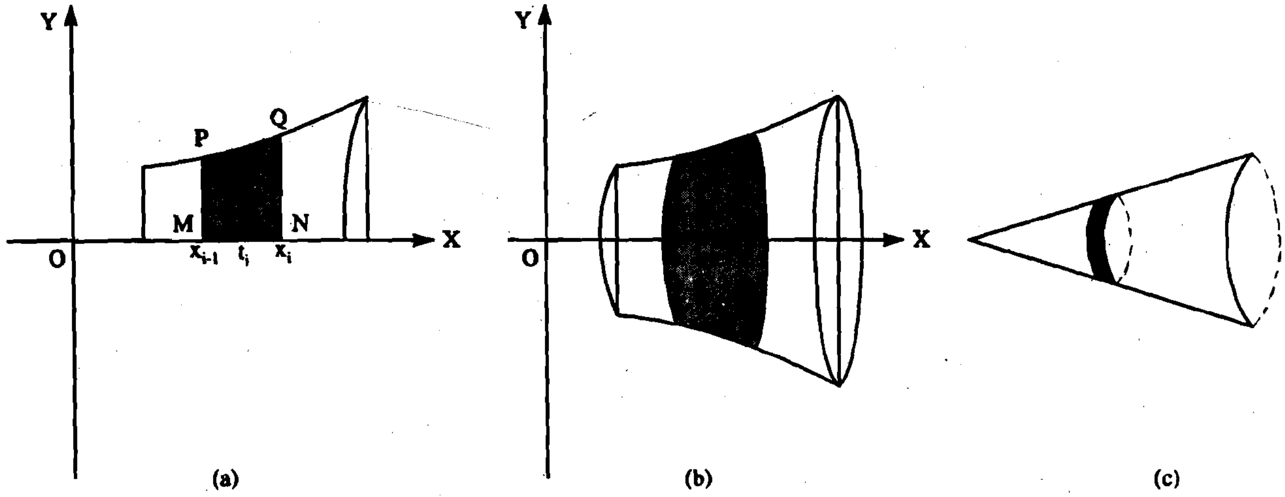


Fig. 9

Let Δx_i denote the length of the ith sub-interval $[x_{i-1}, x_i]$. Further, let P and Q be the points on the curve, $y = f(x)$ corresponding to the ordinates $x = x_{i-1}$ and $x = x_i$, respectively. Then, as the curve revolves about the x-axis, the shaded strip PQNM (Fig. 9(a)) generates a disc of thickness Δx_i. In general, the ordinates PM and QN may not be of equal length. Hence, the disc is actually the frustum of a cone with its volume Δv_i, lying between $\pi\, PM^2 MN$ and $\pi\, QN^2 MN$, that is, between

$\pi[f(x_{i-1})]^2 \Delta x_i$ and $\pi [f(x_i)]^2 \Delta x_i$ (Fig. 9(b) and (c))

πPM^2. MN is the volume of the disc with radius PM and thickness MN.
$\pi QN^2 MN$ is the volume of the disc with radius QN and thickness MN.

If we assume that f is a continuous function on [a, b], we can apply the intermediate value theorem (Theorem 7, Block 1, also see margin remark), and express this volume as

If f is continuous on [a, b], $f(a) = c$ and $f(b) = d$, and z lies between c and d, then $\exists x_0 \in]a, b[$ s.t. $f(x_0) = z$.

$\Delta v_i = \pi\{f(t_i)\}^2 \Delta x_i$, where t_i is a suitable point in the interval $[x_{i-1}, x_i]$. Now summing up over all the discs, we obtain

$$V_n = \sum_{i=1}^{n} \Delta v_i = \sum_{1}^{n} \pi\, [f(t_i)]^2 \ \Delta x_i, \qquad x_{i-1} \le t_i \le x_i \text{ as an approximation}$$

to the volume of the solid of revolution. As we have observed earlier while defining a definite integral, the approximation gets better as the partition P_n gets finer and finer and Δx_i tends to zero. Thus, we get the volume of the solid of revolution as

$$V = \lim_{n\to\infty} V_n = \lim_{n\to\infty} \sum_{i=1}^{n} \pi\, [f(t_i)]^2 \ \Delta x_i$$

$$= \pi \int_a^b [f(x)]^2\, dx = \pi \int_a^b y^2\, dx \qquad ...(6)$$

We shall use this formula to find the volume of the solid described in the following example.

Example 4: Let us find the volume of the solid of revolution formed when the arc of the parabola $y^2 = 4ax$ between the ordinates $x = 0$, and $x = a$ is revolved about its axis. The solid of revolution is the parabolic cap in Fig. 10.

The volume V of the cap is given by

$$V = \int_0^a \pi y^2\, dx = \pi \int_0^a 4ax\, dx = 4\pi a \left[\frac{x^2}{2}\right]_0^a = 2\pi a^3$$

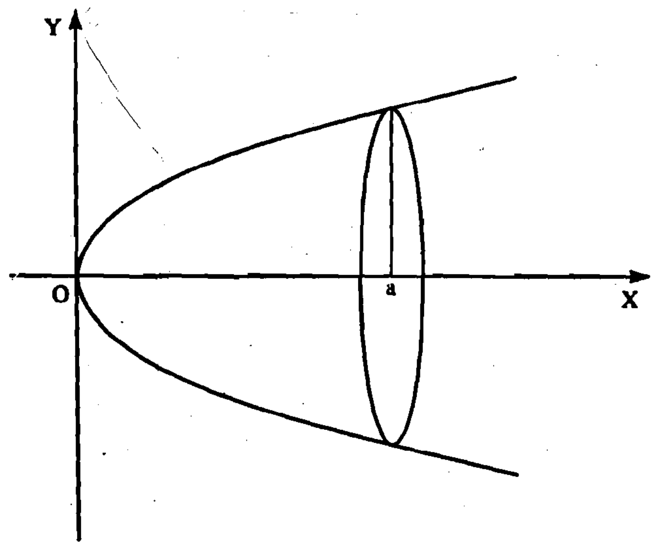


Fig. 10

Our next example illustrates a slight modification of Formula (6) to find the volume of a solid obtained by revolving a plane region about the y-axis.

Example 5: Suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ is revolved about the minor axis, AB (see Fig. 11). Let us find the volume of the solid generated.

In this case the axis of rotation is the y-axis. The area revolved about the y-axis is shown by the shaded region in Fig. 11. You will agree that we need to consider only the area to the right of the y-axis.

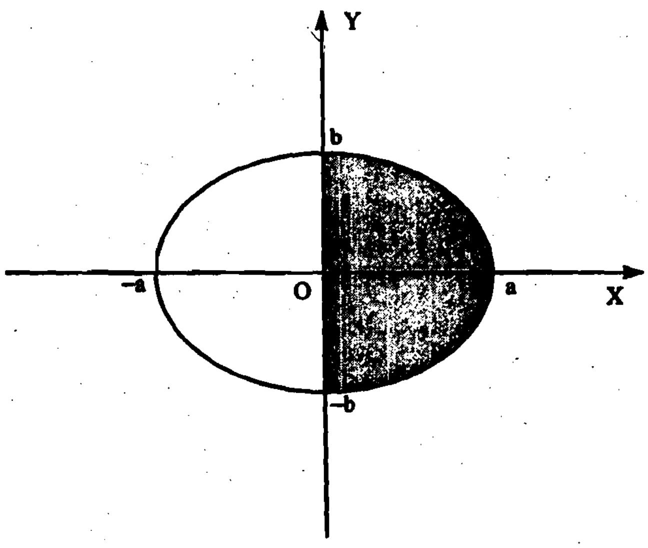


Fig. 11

To find the volume of this solid we interchange x and y in (6) and get

$$V = \int_{-b}^{b} \pi x^2\, dy = \int_{-b}^{b} \pi a^2 \left(1 - \frac{y^2}{b^2}\right) dy, \text{ since } x^2 = a^2\left(1 - \frac{y^2}{b^2}\right).$$

$$= 2\pi a^2 \int_0^b \left(1 - \frac{y^2}{b^2}\right) dy, \text{ since } 1 - \frac{y^2}{b^2} \text{ is an even function of y.}$$

$$= 2\pi a^2 \left[y - \frac{y^3}{3b^2}\right]_0^b$$

$$= \frac{4}{3}\pi a^2\, b.$$

We can also modify Formula (6) to apply to curves whose equations are given in the parametric or polar forms. Let us tackle these one by one.

Parametric Form

If a curve is given by $x = \phi(t)$, $y = \psi(t)$, $\alpha \le t \le \beta$, then the volume of the solid of revolution about the x-axis can be found by substituting x and y in Formula (6) by $\phi(t)$ and $\psi(t)$, respectively. Thus,

$$V = \pi \int_{\alpha}^{\beta} [\psi(t)]^2 \frac{dx}{dt} dt$$

$$\text{or } V = \pi \int_{\alpha}^{\beta} [\psi(t)]^2 \ \phi'(t)\, dt$$

We'll now derive the formula for curves given by polar equations.

Polar Form

Suppose a curve is given by $r = f(\theta)$, $\theta_1 \le \theta \le \theta_2$. The volume of the solid generated by rotating the area bounded by $x = a$, $x = b$, the x-axis and $r = f(\theta)$ about the x-axis is

$$V = \pi \int_{\theta_1}^{\theta_2} (r \sin\theta)^2 \frac{d}{d\theta}(r\cos\theta)\, d\theta$$

Thus, $$V = \pi \int_{\theta_1}^{\theta_2} [f(\theta)\sin\theta]^2 \, [f'(\theta)\cos\theta - f(\theta)\sin\theta]\, d\theta$$

Let's use this formula to find the volume of the solid generated by a cardioid about its initial line.

Example 6: The cardioid shown in Fig. 12 is given by $r = a(1 + \cos\theta)$.

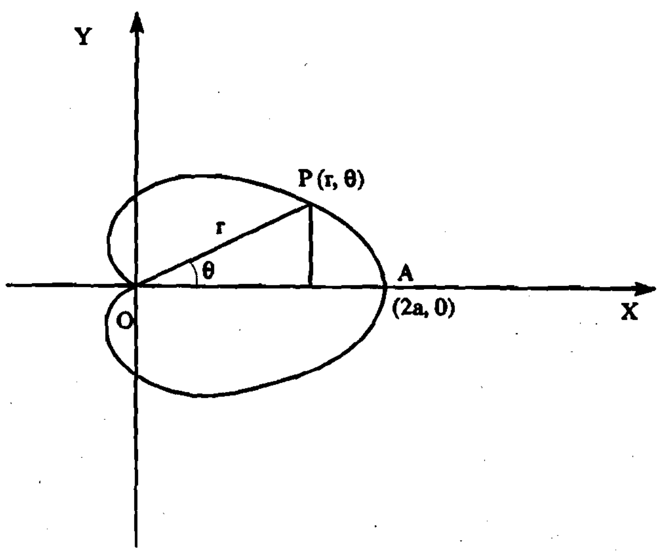


Fig. 12

The points A and O correspond to $\theta = 0$ and $\theta = \pi$, respectively. Here, again, we need to consider only the part of the cardioid above the initial line. Thus,

$$V = \int_{\pi}^{0} \pi (r\sin\theta)^2 \frac{d}{d\theta}(r\cos\theta)\, d\theta$$

$$= \pi a^3 \int_{0}^{\pi} (1 + \cos\theta)^2 \sin^3\theta \,(1 + 2\cos\theta)\, d\theta, \text{ since } r = a(1 + \cos\theta)$$

$$= \pi a^3 \int_0^a 8 \sin^3 \frac{\theta}{2} \cos^3 \frac{\theta}{2} 4 \cos^4 \frac{\theta}{2} \left(4 \cos^2 \frac{\theta}{2} - 1\right) d\theta$$

$$= 128\,\pi a^3 \int_0^\pi \sin^3 \frac{\theta}{2} \cos^9 \frac{\theta}{2}\, d\theta - 32\pi a^3 \int_0^\pi \sin^3 \frac{\theta}{2} \cos^7 \frac{\theta}{2}\, d\theta$$

$$= 256\,\pi a^3 \int_0^{\pi/2} \sin^3 \phi \cos^9 \phi\, d\phi - 64\pi a^3 \int_0^{\pi/2} \sin^3 \phi \cos^7 \phi\, d\phi, \text{ where } \phi = \theta/2$$

$$= \frac{64\,\pi a^3}{15} - \frac{8\pi a^3}{5} \text{ on applying a reduction formula from Unit 12.}$$

In all the examples that we have seen till now, the axis of rotation formed a boundary of the region which was rotated. Now we take an example in which the axis touches the region at only one point.

Example 7: Let us find the volume of the solid generated by revolving the region bounded by the parabolas $y = x^2$ and $y^2 = 8x$ about the x-axis. We have shown the area rotated and the solid in Fig. 13 (a) and (b), respectively.

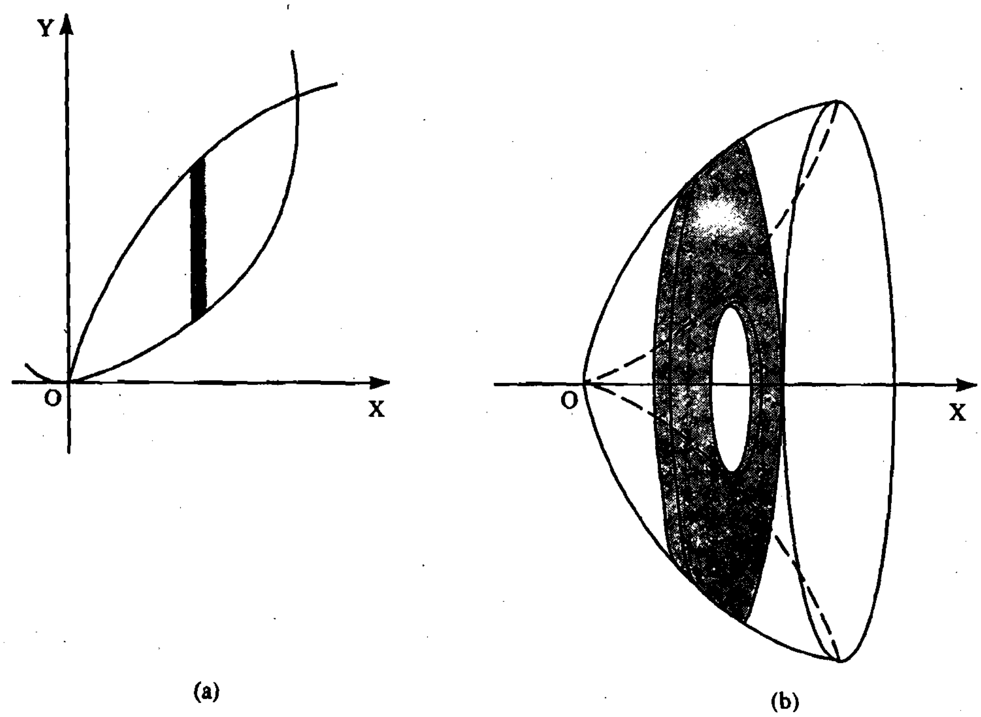


Fig. 13

Here, the required volume will be the difference between the volume of the solid generated by the parabola $y^2 = 8x$ and that of the solid generated by the parabola $y = x^2$.

$$\text{Thus, } V = \pi \left[\int_0^2 8x\, dx - \int_0^2 x^4\, dx \right] = \pi \left[4x^2 - \frac{x^5}{5} \right]_0^2 = \frac{48\pi}{5}$$

Note the limits of integration.

Here, we list some exercises which you can solve by applying the formulas derived in this section.

E E 12) Find the volume of the right circular cone of height h and radius of the circular base r.

(*Hint:* The cone will be generated by rotating the triangle bounded by the x-axis and the line $y = (r/h)\, x$).

E 13) Show that the volume of the solid generated by revolving the curve $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis is $32\pi a^3/105$.

E 14) The arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ in $[0, 2\pi]$ is rotated about the y-axis. Find the volume generated.
(*Caution:* The rotation is about the y-axis.)

E E 15) Find the volume of the solid obtained by revolving the limacon $r = a + b \cos \theta$ about the initial line.

E E 16) The semicircular region bounded by $y - 2 = \sqrt{9 - x^2}$ and the line $y = 2$ is rotated about the x-axis. Find the volume of the solid generated.

3.4 AREA OF SURFACE OF REVOLUTION

Instead of rotating a plane region, if we rotate a curve about an x-axis, we shall get a surface of revolution. In this section we shall find a formula for the area of such a surface. Let us start with the case when the equation of the curve is given in the Cartesian form.

Cartesian Form

Suppose that the curve $y = f(x)$ [Fig. 14] is rotated about the x-axis. To find the area of the area of the generated surface, we consider a partition P_n of the interval [a, b], namely,

$P_n = \{a = x_0 < x_1 < x_2 < \dots\dots < x_{n-1} < x_n = b\}$

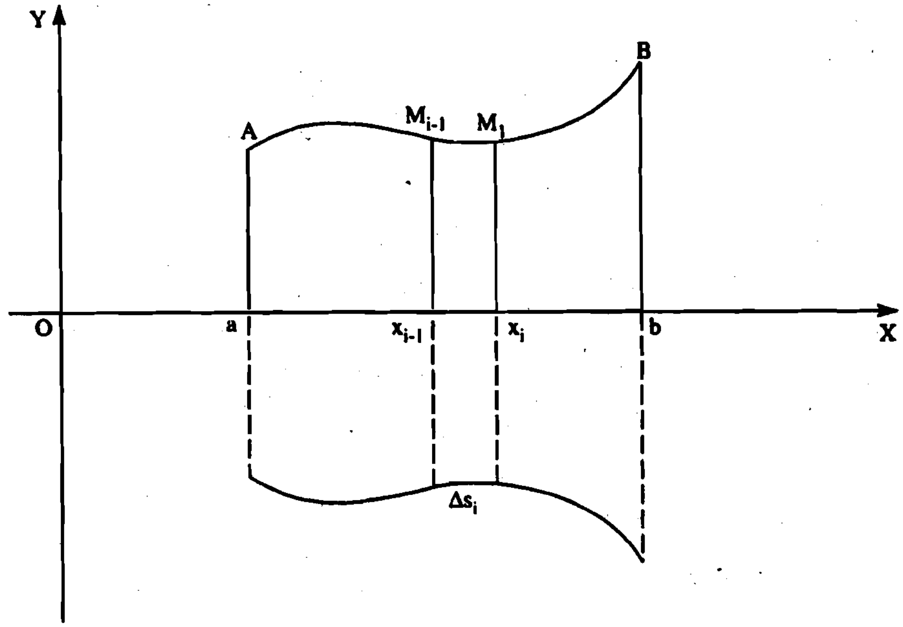


Fig. 14

Let the lines $x = x_i$ intersect the curve in points M_i, $i = 1, 2, \ldots\ldots n$. If we revolve the chord M_{i-1} Mi about the x-axis, we shall get the surface of the frustum of a cone of thickness $\Delta x_i = x_i\, x_{i-1}$. Let Δs_i be the area of the area of the surface of this frustum. Then the total surface area of all the frusta is

$$S_n = \sum_{i=1}^{n} \Delta s_i$$

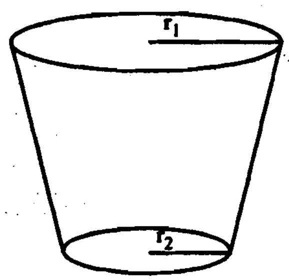


Fig. 15

This S_n is an approximation to the area of the surface of revolution. The area of the surface of revolution generated by the curve $y = f(x)$, is the limit of S_n (if it exist), as $n \to \infty$ and each $\Delta x_i \to 0$.

To find the area A of the curved surface of a typical frustum, we use the formula $A = \pi (r_1 + r_2)\, l$, where l is the slant height of the frustum and r_1 and r_2 are the radii of its bases (Fig. 15).

In the frustum under consideration the radii of the bases are the ordinates $f(x_{i-1})$ and $f(x_i)$, while the slant height $M_{i-1} M_i$ is given by $\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$, where $\Delta y_i = f(x_i) - f(x_{i-1})$. We assume that f is derivable on [a, b] and f′ is continuous. Then by the mean value theorem (Theorem 3, Block 2), we obtain $\Delta y_i = f'(t_i)\, x_i$, for some $t_i \in [x_{i-1}, x_i]$.

Therefore,

$$\Delta s_i = 2\pi \frac{[f(x_{i-1}) + f(x_i)]}{2} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

where $(f(x_{i-1}) + f(x_i))/2$ is the mean radius of revolution

$$= 2\pi \frac{y_{i-1} + y_i}{2} \sqrt{1 + [f'(t_i)]^2}\, \Delta x_i$$

and $S_n = 2\pi \sum_{i=1}^{n} \frac{y_{i-1} + y_i}{2} \sqrt{1 + [f'(t_i)]^2}\, \Delta x_i$.

Proceeding to the limit as $n \to \infty$, and each $\Delta x_i \to 0$, we have

$$S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

$\therefore$ in the lim $\Delta x_i \to 0$, $y_i \to f(x)$, $y_{i-1} \to f(x)$, and $f'(t_i) \to f(x)$

$$= 2\pi \int_a^b y \sqrt{1 + (dy/dx)^2}\, dx \qquad \ldots(7)$$

We shall now illustrate the use of this formula.

Example 8: Let us find the area of the surface of revolution obtained by revolving the parabola $y^2 = 4ax$ from $x = a$ to $x = 3a$, about the x-axis.

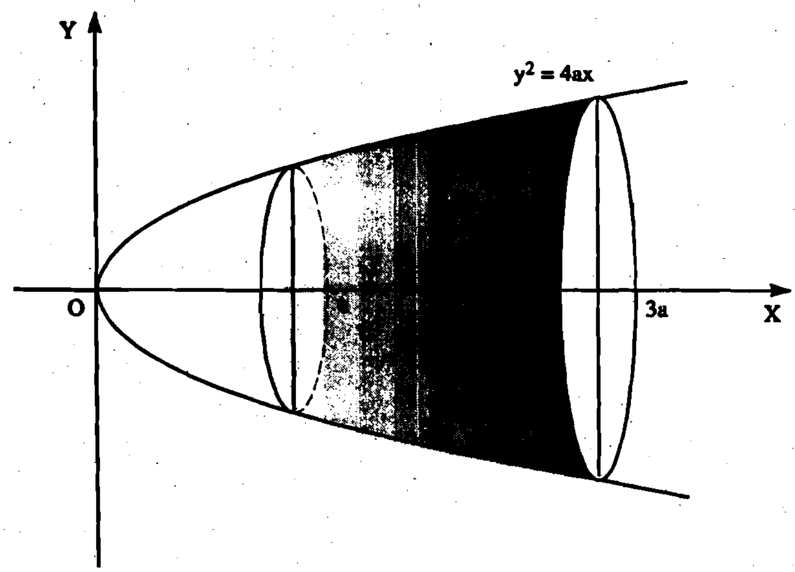


Fig. 16

The area of the surface of revolution.

$$S = 2\pi \int_a^{3a} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

where $y^2 = 4ax$, $\frac{dy}{dx} = \frac{2a}{y}$. Hence,

$$S = 2\pi \int_a^{3a} y \sqrt{1 + 4a^2/y^2}\, dx$$

$$= 2\pi \int_a^{3a} \sqrt{y^2 + 4a^2}\, dx = 2\pi \int_a^{3a} \sqrt{4ax + 4a^2}\, dx$$

$$= 4\pi \sqrt{a} \int_a^{3a} \sqrt{x + a}\, dx = 4\pi \sqrt{a}\, \frac{2}{3} \left[(x + a)^{3/2}\right]_a^{3a}$$

$$= \frac{8\pi a^2}{3} \left[4^{3/2} - 2^{3/2}\right]$$

Instead of revolving the given curve abour the x-axis, if we revolve it about the y-axis, we get another surface of revolution. The area of the surface of revolution generated by the curve $x = g(y)$, $c \le y \le d$, as it revolves about the y-axis is given by,

$$S = 2\pi \int_c^d x \sqrt{1 + (dx/dy)^2}\, dy$$

Now let us look at curves represented by parametric equations.

Parametric Form

Suppose a curve is given by the parametric equations $x = \phi(t)$, $y = \psi(t)$, $t \in [\alpha, \beta]$. Then we know that

$$\frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)}$$

Substituting this in formula (10), we get the area of the surface of revolution generated by the curve as it revolves about the x-axis, to be

$$S = 2\pi \int_{\alpha}^{\beta} \psi(t) \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\, dt.$$

Now we shall state the formula for the surface generated by a curve represented by a polar equation.

Polar Form

If $r = h(\theta)$ is the polar equation of the curve, then the area of the surface of revolution generated by the arc of the curve for $\theta_1 \le \theta \le \theta_2$, as it revolves about the initial line, is

$$S = 2\pi \int_{\theta_1}^{\theta_2} r \sin\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$$

Study the following examples carefully before trying the exercises given at the end of this section.

Example 9: Suppose the astroid $x = a \sin^3 t$, $y = a \cos^3 t$, is revolved about the x-axis. Let us find the area of the surface of revolution. You will agree that we need to consider only the portion of the curve above the x-axis.

For this portion $y > 0$, and thus t varies from $-\pi/2$ to $\pi/2$.

$$\frac{dx}{dt} = 3a \sin^2 t \cos t, \quad \frac{dy}{dt} = -3a \cos^2 t \sin t$$

Therefore, $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2 \sin^2 t \cos^2 t$

We therefore get,

$$\begin{aligned} S &= 2\pi \int_{-\pi/2}^{\pi/2} a\cos^3 t \sqrt{9a^2 \sin^2 t \cos^2 t}\, dt \\ &= 2\pi \int_{-\pi/2}^{\pi/2} a\cos^3 t\, |3a \sin t \cos t|\, dt \\ &= 6\pi a^2 \int_{-\pi/2}^{\pi/2} \cos^4 t\, |\sin t|\, dt \\ &= 12\pi a^2 \int_{0}^{\pi/2} \cos^4 t \sin t\, dt = -12\pi a^2 \left| \frac{\cos^5 t}{5} \right|_0^{\pi/2} \\ &= \frac{12}{5}\pi a^5 \end{aligned}$$

Example 10: Suppose we want to find the area of the surface generated by revolving the cardioid $r = a(1 + \cos\theta)$ about its initial line.

Notice that the cardioid is symmetrical about the initial line, and extends above this line from $\theta = 0$ to $\theta = \pi$. The surface generated by revolving the whole curve about the initial line is the same as that generated by the upper half of the curve. Hence,

$$S = 2\pi \int_{0}^{\pi} r \sin\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$$

$$= 2\pi \int_0^{\pi} a(1+\cos\theta)\sin\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$$

Since $r = a(1+\cos\theta)$, and $\frac{dr}{d\theta} = -a\sin\theta$, we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta = 4a^2\cos^2\frac{\theta}{2}$$

Therefore,

$$S = 2\pi \int_0^{\pi} a(1+\cos\theta)\sin\theta\, 2a\cos\frac{\theta}{2}\, d\theta$$

$$= 4\pi a^2 \int_0^{\pi} 4\sin\frac{\theta}{2}\cos^4\frac{\theta}{2} d\theta$$

$$= 32\pi a^2 \int_0^{\pi/2} \sin\phi\cos^4\phi\, d\phi, \text{ where } \phi = \theta/2$$

$$= 32\pi a^2 \left| \frac{-\cos^5\phi}{5} \right|_0^{\pi/2} = \frac{32\,\pi a^2}{5}$$

E E 17) Find the area of the surface generated by revolving the circle $r = a$ about the x-axis, and thus verify that the surface area of a sphere of radius a is $4\pi a^2$.

E E 18) The arc of the curve $y = \sin x$, from $x = 0$ to $x = \pi$ is revolved about the x-axis. Find the area of the surface of the solid of revolution generated.

E E 19) The ellipse $x^2/a^2 + y^2/b^2 = 1$ revolves about the x-axis. Find the area of the surface of the solid of revolution thus obtained.

E E 20) Prove that the surface of the solid generated by the revolution about the x-axis of the loop of the curve $x = t^2$, $y = \left(t - \frac{t^3}{3}\right)$ is 3π.

E E 21) Find the surface area of the solid generated by revolving the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, about the line $y = 0$.

Now let us quickly recall what we have covered in this unit.

3.5 SUMMARY

In this unit we have seen how to find

1) the lengths of curves
2) volumes of solids of revolution and
3) the areas of surfaces of revolution.

In each case we have derived formulas when the equation of the curve is given in either the Cartesian or parametric or polar form. We give the results here in the form of the following tables.

Length of an arc of a curve

Equation	Length
$y = f(x)$	$\int_a^b \sqrt{1+[f'(x)]^2}\, dx$
$x = g(y)$	$\int_c^d \sqrt{1+[g'(y)]^2}\, dy$
$x = \phi(t)$ $y = \psi(t)$	$\int_\alpha^\beta \sqrt{[\phi'(t)]^2+[\psi'(t)]^2}\, dy$
$r = f(\theta)$	$\int_\alpha^\beta \sqrt{[f(\theta)]^2+[f'(\theta)]^2}\, d\theta$

Volume of the solid of revolution

Equation	Volume
$y = f(x)$ about x − axis	$\pi \int_a^b y^2\, dx$
$x = g(y)$ about y − axis	$\pi \int_c^d x^2\, dx$
$x = \phi(t)$, $y = \psi(t)$ about x − axis	$\pi \int_\alpha^\beta [\psi(t)]^2\, \phi'(t)\, dt$
$r = h(\theta)$ about the initial line	$\pi \int_{\theta_1}^{\theta_2} [h(\theta)\sin\theta]^2\, [h'(\theta)\cos\theta - h(\theta)\sin\theta]\, d\theta$

Area of the surface of revolution

Equation	Area
$y = f(x)$ about x – axis	$2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$
$x = g(y)$ about y – axis	$2\pi \int_c^d g(y)\sqrt{1+[g'(y)]^2}\,dy$
$x = \phi(t),\ y = \psi(t)$ about x – axis	$2\pi \int_\alpha^\beta \psi(t)\sqrt{[\phi'(t)]^2+[\psi'(t)]^2}\,dt$
$r = h(\theta)$ about the initial line	$2\pi \int_{\theta_1}^{\theta_2} r\sin\theta\sqrt{r^2+(dr/d\theta)^2}\,d\theta$

3.6 SOLUTIONS AND ANSWERS

E1) $L = \int_c^d \sqrt{1+[dx/dy]^2}\,dy$

$= \int_1^2 \sqrt{1+(3)^2}\,dy$

$= \sqrt{10}\int_1^2 dy = \sqrt{10}.$

By distance formula,

$L = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{(3-6)^2+(1-2)^2}$

$= \sqrt{(-3)^2+(-1)^2}$

$= \sqrt{10}.$

E2) $L = \int_a^b \sqrt{1+(dy/dx)^2}\,dx \quad \left(\frac{dy}{dx} = \frac{1}{\sec x}.\sec x\tan x = \tan x\right)$

$= \int_0^{\pi/3} \sqrt{1+\tan^2 x}\,dx$

$= \int_0^{\pi/3} \sec x\,dx \quad = \ln|\sec x+\tan x|\Big|_0^{\pi/3}$

$= \ln\left|\frac{\sec\pi/3+\tan\pi/3}{\sec 0+\tan 0}\right|$

$= \ln(2+\sqrt{3})$

E3) $$L = \int_0^x \sqrt{1+\sinh^2 (x/c)}\, dx$$

$$= \int_0^x \cosh (x/c)\, dx$$

$$= c \sinh (x/c)\Big]_0^x = c \sinh (x/c)$$

E4) $$y = \sqrt{\frac{x^3}{a}} \quad \therefore dy/dx = (3/2)\sqrt{\frac{x}{a}}$$

$$L = \int_0^a \sqrt{1+\frac{9x}{4a}}\, dx$$

$$= \frac{1}{2\sqrt{a}} \int_0^a \sqrt{4a+9x}\, dx$$

$$= \frac{1}{27\sqrt{a}} (4a+9x)^{3/2}\Big]_0^a$$

$$= \frac{1}{27\sqrt{a}} [(13a)^{3/2} - (4a)^{3/2}] = \frac{a}{27}(13^{3/2} - 8)$$

E5) $3y = 8x \Rightarrow y = \frac{8x}{3}$. Substituting this in $y^2 = 4ax$ we get

$$\frac{64x^2}{9} = 4ax$$

i.e., $64x^2 - 36ax = 0$

$$\Rightarrow x = 0 \text{ or } x = \frac{9a}{16}$$

$$\Rightarrow y = 0 \text{ or } y = \frac{3a}{2}$$

Hence (0, 0) and $\left(\frac{9a}{16}, \frac{3a}{2}\right)$ are the points of intersection

Now $4ax = y^2 \Rightarrow \frac{dx}{dy} = \frac{y}{2a}$

$$L = \int_0^{3a/2} \sqrt{1+\frac{y^2}{4a^2}}\, dy$$

$$= \frac{1}{2a} \int_0^{3a/2} \sqrt{4a^2+y^2}\, dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{4a^2+y^2} + 2a^2 \ln\left| y + \sqrt{4a^2+y^2}\right|\right]_0^{3a/2}$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2 \ln 2\right]$$

$$= \left(\frac{15}{16} + \ln 2\right) a$$

E6) $$\frac{dx}{d\theta} = a(1-\cos\theta),\ \frac{dy}{d\theta} = a \sin\theta$$

$$\therefore \quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2[1 + \cos^2\theta - 2\cos\theta + \sin^2\theta]$$

$$= 2a^2(1 - \cos\theta)$$

$$= 4a^2 \sin^2(\theta/2)$$

$$\therefore \quad L = 2a \int_0^{2\pi} \sin(\theta/2)\, d\theta$$

$$= 4a \int_0^{\pi} \sin\phi\, d\phi$$

$$= 8a \int_0^{\pi/2} \sin\phi\, d\phi = 8a$$

E 7) $\frac{dx}{dt} = e^t(\cos t + \sin t)$, $\frac{dy}{dt} = e^t(\cos t - \sin t)$

$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2e^{2t}$$

$$\therefore L = \sqrt{2} \int_0^{\pi/2} e^t\, dt = \sqrt{2}\, e^t \Big]_0^{\pi/2}$$

$$= \sqrt{2}\,(e^{\pi/2} - 1)$$

E 8) $r = a\cos^3\frac{\theta}{3} \Rightarrow \frac{dr}{d\theta} = -a\cos^2\frac{\theta}{3}\sin\frac{\theta}{3}$

$$\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\cos^6\frac{\theta}{3} + a^2\cos^4\frac{\theta}{3}\sin^2\frac{\theta}{3}$$

$$= a^2\cos^4\frac{\theta}{3}$$

$$\therefore L = 2a \int_0^{3\pi/2} \cos^2\frac{\theta}{3}\, d\theta = 6a \int_0^{\pi/2} \cos^2\phi\, d\phi$$

$$= \frac{3a\pi}{2}$$

E 9) $\frac{dx}{dt} = -2\sin t \; \frac{dy}{dt} = 2\cos t$

$$\therefore \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2\sqrt{\sin^2 t + \cos^2 t} = 2$$

$$\therefore L = 2\int_0^{2\pi} dt = 4\pi$$

Note that $L = 2\pi r$ since, here, $r = 2$.

E 10) $r = a(1 - \cos\theta)$, $\frac{dr}{d\theta} = a\sin\theta$

$$\therefore \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2a\sin\frac{\theta}{2}$$

The length of the curve in the upper half $= \int_0^{\pi} 2a\sin(\theta/2)\, d\theta$

$= 4a.$

The length from $\theta = 0$ to $\theta = 2\pi/3$

$$= \int_0^{2\pi/3} 2a \sin\frac{\theta}{2}\, d\theta = 2a$$

The arc of the curve in the upper half is bisected by $\theta = 2\pi/3$.

E 11) $r = a(\theta^2 - 1)$, $\frac{dr}{d\theta} = 2a\theta$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 [\theta^4 - 2\theta^2 + 1 + 4\theta^2]$$

$$= a^2 (\theta^2 + 1)^2.$$

$$\therefore \quad L = a \int_{-1}^{1} (\theta^2 + 1)^2\, d\theta$$

$$= a\left(\frac{\theta^3}{3} + \theta\right)\Bigg]_{-1}^{1}$$

$$= a\left(\frac{1}{3} + 1 + \frac{1}{3} + 1\right) = \frac{8a}{3}$$

E 12) $$V = \pi \int_0^h \frac{r^2}{h^2} x^2\, dx$$

$$= \frac{\pi r^2}{h^2} \frac{x^3}{3}\Bigg]_0^h = \frac{1}{3}\pi r^2 h$$

E 13) $$V = 2\pi \int_0^a (a^{2/3} - x^{2/3})^3\, dx$$

$$= 2\pi \int_0^a (a^2 - 3a^{4/3} x^{2/3} + 3a^{2/3} x^{4/3} - x^2)\, dx$$

$$= 2\pi \left[a^2 x - \frac{9}{5} a^{4/3} x^{5/3} + \frac{9}{7} a^{2/3} x^{7/3} - x^3/3\right]_0^a$$

$$= \frac{32\pi a^3}{105}$$

Note that the total volume generated is equal to twice the volume generated by the arc of the curve between $x = 0$ and $x = a$.

E 14) $x^2 = a^2 (t - \sin t)^2$, $\frac{dy}{dt} = a \sin t$

$$\therefore \; V = \pi \int_0^{2\pi} a^2 (t - \sin t)^2\, a \sin t\, dt$$

$$= \pi a^3 \int_0^{2\pi} (t^2 \sin t - 2t \sin^2 t \sin^3 t)\, dt$$

$$= \pi a^3 (-4\pi^2 - 2\pi^2) = -6\pi^3 a^3.$$

E 15) $r = a + b\cos\theta \Rightarrow \dfrac{dx}{d\theta} = \dfrac{d(r\cos\theta)}{d\theta}$

$$= (a + b\cos\theta)(-\sin\theta) - b\sin\theta\cos\theta$$
$$= -a\sin\theta - 2b\sin\theta\cos\theta.$$

$$\therefore V = \pi\int_0^{\pi} y^2 \frac{dx}{d\theta}\,d\theta$$

$$= -\pi\int_0^{\pi} (a + b\cos\theta)^2 \sin^2\theta\ (a\sin\theta + 2b\sin\theta\cos\theta)\,d\theta$$

$$= -\pi\int_0^{\pi} (a^3\sin^3\theta + 4a^2b\sin^3\theta\cos\theta + 5ab^2\sin^3\theta\cos^2\theta + 2b^3\sin^3\theta\cos^3\theta)\,d\theta$$

$$= -\pi\left[2a^3\int_0^{\pi/2}\sin^3\theta\,d\theta + 10ab^2\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta\right]$$

(The other two integrals are equal to zero since $\cos(\pi - \theta) = -\cos\theta$.)

$$= -\pi\left[\frac{4a^3}{3} + 10ab^2 \cdot \frac{2}{5} \cdot \int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right]$$

(using a reduction formula)

$$= -\pi\left[\frac{4a^3}{3} - \frac{4ab^2}{3}\right] = \frac{4\pi a}{3}(b^2 - a^2)$$

E 16) $V = 2\pi\left[\displaystyle\int_0^3 (2 + \sqrt{9 - x^2})^2\,dx - \int_0^3 4\,dx\right]$

$$= 2\pi\left[\int_0^3 (4 + 9 - x^2 + 4\sqrt{9 - x^2})\,dx - 12\right]$$

$$= 2\pi\left[13x - \frac{x^3}{3} + 2x\sqrt{9 - x^2} + 18\sin^{-1}\frac{x}{3}\right]_0^3 - 24\pi$$

$$= 36\pi + 18\pi^2 = 18\pi(2 + \pi)$$

E 17) $r = a \Rightarrow \dfrac{dr}{d\theta} = 0$

$$S = 2\pi\int_0^{\pi} a\sin\theta\sqrt{a^2 + 0}\,d\theta$$

$$= 2\pi a^2\int_0^{\pi}\sin\theta\,d\theta$$

$$= 4\pi a^2\int_0^{\pi/2}\sin\theta\,d\theta \quad \text{since } \sin(\pi - \theta) = \sin\theta$$

$$= 4\pi a^2 \cos\theta\,\Big]_0^{\pi/2}$$

$$= 4\pi a^2.$$

E 18) $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = 1+\cos^2 x$$

$$\therefore \quad S = 2\pi \int_0^{\pi} \sin x \sqrt{1+\cos^2 x}\, dx$$

$$= 4\pi \int_0^{\pi/2} \sin x \sqrt{1+\cos^2 x}\, dx$$

$$= 4\pi \int_0^{1} \sqrt{1+t^2}\, dt, \text{ if we put } t = \cos x$$

$$= 4\pi \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left|t+\sqrt{1+t^2}\right|\right]_0^1$$

$$= 2\sqrt{2}\pi + 2\pi \ln(1+\sqrt{2}).$$

E 19) $y = \frac{b}{a}\sqrt{a^2-x^2}.$

$$\therefore \frac{dy}{dx} = \frac{-bx}{a\sqrt{a^2-x^2}}$$

$$S = 4\pi \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}\, dx$$

$$= \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4-(a^2-b^2)x^2}\, dx$$

$$= \frac{4\pi b\sqrt{a^2-b^2}}{a^2}\int_0^a \sqrt{\frac{a^4}{a^2-b^2}-x^2}\, dx$$

$$= \frac{4\pi b\sqrt{a^2-b^2}}{a^2}\left[\frac{x}{2}\sqrt{\frac{a^4}{a^2-b^2}-x^2} + \frac{a^4}{2(a^2-b^2)}\sin^{-1}\frac{x\sqrt{a^2-b^2}}{a^2}\right]_0^a$$

$$= 2\pi b^2 + \frac{2\pi a^2 b}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}$$

E 20) The loop is between $t = -\sqrt{3}$ and $t = \sqrt{3}$. Because of symmetry, it is enough to consider the curve between $t = 0$ and $t = \sqrt{3}$.

$$\therefore \quad S = 2\pi \int_0^{\sqrt{3}} \left(t-\frac{t^3}{3}\right)\sqrt{4t^2+(1-t^2)^2}\, dt$$

$$= 2\pi \int_0^{\sqrt{3}} \left(t+\frac{2t^3}{3}-\frac{t^5}{3}\right) dt$$

$$= 2\pi\left[\frac{t^2}{2}+\frac{t^4}{6}-\frac{t^6}{18}\right]_0^{\sqrt{3}}$$

$$= 3\pi.$$

E21) $$S = 2\pi \int_0^{2\pi} a(1-\cos\theta)\sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= 4\pi a^2 \int_0^{2\pi} (1-\cos\theta)\sin\frac{\theta}{2}\, d\theta$$

$$= 8\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^3\phi\, d\phi$$

$$= 32\pi a^2 \int_0^{\pi/2} \sin^3\phi\, d\phi$$

$$= 32\pi a^2 \cdot \frac{2}{3}$$

$$= \frac{64\pi a^2}{3}$$