UNIT 4 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS USING RUNGE-KUTTA METHODS

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4.0 INTRODUCTION

In unit 3, we considered the IVPs

$$y' = f(t, y), \qquad y'(t_0) = y_0 \qquad (1)$$

and developed Taylor series method and Euler's method for its solution. As mentioned earlier, Euler's method being a first order method, requires a very small step size for reasonable accuracy and therefore, may require lot of computations. Higher order Taylor series require evaluation of higher order derivates either manually or computationally. For complicated functions, finding second, third and higher order total derivatives is very tedious. Hence, Taylor series methods of higher order are not of much practical use in finding the solutions of IVPs of the form given by Eqn. (1).

In order to avoid this difficulty, at the end of nineteenth century, the German mathematician, Runge observed that the expression for the increment function Ø (t, y, h) in the single step methods [see Eqn. (24) of Sec. 7.3, Unit 7]

$$y_{n+1} = y_n + h\,ø\,(t_n, y_n, h) \qquad (2)$$

can be modified to avoid evaluation of higher order derivatives. This idea was further developed by Runge and Kutta (another German mathematician) and the methods given by them are known as Runge-Kutta methods. Using their ideas, we can construct higher order methods using only the function f(t, y) at selected points on each subinterval. We shall, in the next section, derive some of these methods.

4.1 OBJECTIVES

After going through this unit, you should be able to:

- State the basic idea used in the development of Runge-Kutta methods;
- Obtain the solution of IVPs using Runge-Kutta methods of second, third and fourth order, and
- Compare the solutions obtained by using Runge-Kutta and Taylor series methods.

4.2 RUNGE-KUTTA METHODS

We shall first try to discuss the basic idea of how the Runge-Kutta methods are developed.

Consider the $O(h^2)$ singlestep method

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}\, y''_n \tag{3}$$

If we write Eqn. (3) in the form of Eqn. (2) i.e., in terms of $\text{Ø}\,[t_n, y_n, h]$ involving partial derivatives of f(t, y), we obtain

$$\text{Ø}(t, y, h) = f(t_n, y_n) + \frac{h}{2}\left[f_t(t_n, y_n) + f(t_n, y_n)\, f_y(t_n, y_n)\right] \tag{4}$$

Runge observed that the r.h.s. of Eqn. (4) can also be obtained using the Taylor series expansion of $f(t_n + ph, y_n + qhf_n)$ as

$$f(t_n + ph, y_n + qhf_n) \approx f_n + ph\, f_t(t_n, y_n) + qhf_n\, f_y(t_n, y_n), \tag{5}$$

Taylor's expansion in two variables t, y

where $f_n = f(t_n, y_n)$

Comparing Eqns. (4) and (5) we find that p = q = ½ and the Taylor series method of $O(h^2)$ given by Eqn. (3) can also be written as

$$y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f_n\right) \tag{6}$$

Since Eqn. (5) is of $O(h^2)$, the value of y_{n+1} in (6) has the TE of $O(h^3)$. Hence the method (6) is of $O(h^2)$ which is same as that of (3).

The advantage of using (6) over Taylor series method (3) is that we need to evaluate the function f(t, y) only at two points (t_n, y_n) and $\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f_n\right)$. We observe that $f(t_n, y_n)$ denotes the slope of the solution curve to the IVP (1) at (t_n, y_n). Further,

$f\left[t_n + \frac{h}{2}, y_n + \left(\frac{h}{2}\right) f_n\right]$ denotes an approximation to the slope of the solution curve at

the point $\left[t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right]$ Eqn. (6) denotes geometrically, that the slope of the

solution curve in the interval $[t_n, t_{n+1}]$ is being approximated by an approximation to

the slop at the middle points $t_n + \frac{h}{2}$. This idea can be generalized and the slope of the solution curve in $[t_n, t_{n+1}]$ can be replaced by a weighted sum of slopes at a number of

points in $[t_n, t_{n+1}]$ (called off-step points). This idea is the basis of the Runge-Kutta methods.

Let us consider for example, the weighted sum of the slopes at the two points $[t_n, y_n]$ and $[t_n + ph, y_n + qhf_n]$, $0 < p, q < 1$ as

$$\phi(t_n, y_n, h) = W_1 f(t_n, y_n) + W_2 f[t_n + ph, y_n + qhf_n] \tag{7}$$

We call W_1 and W_2 as weights and p and q as scale factors. We have to determine the four unknowns W_1, W_2, p and q such that $\text{Ø}\,(t_n, y_n, h)$ is of $O(h^2)$. Substituting Eqn. (5) in (7), we have

$$\phi\,(t_n, y_n, h) = W_1 f_n + W_2\,[f_n + phf_t\,(t_n, y_n) + phf_n\,fy\,(t_n, y_n)]. \qquad (8)$$

putting this in (2) we get,

$$\begin{aligned} y_{n+1} &= y_n + h\left[W_1 f_n + W_2\left\{f_n + phf_t\left(t_n, y_n\right) + qhf_n fy\left(t_n, y_n\right)\right\}\right] \\ &= y_n + h\left(W_1 + W_2\right)f_n + h^2 W_2\left(pf_t + qf_n f_y\right)_n \end{aligned} \qquad (9)$$

where $(\ )_n$ denotes that the quantities inside the brackets are evaluated at (t_n, y_n). Comparing the r.h.s. of Eqn. (9) with Eqn. (3), we find that

$$\left.\begin{aligned} W_1 + W_2 &= 1 \\ W_2\,p = W_2\,q &= \frac{1}{2} \end{aligned}\right\} \qquad (10)$$

In the system of Eqns. (10), since the number of unknowns is more than the number of equations, the solutions is not unique and we have infinite number of solutions. The solution of Eqn. (10) can be written as

$$\begin{aligned} W_1 &= 1 - W_2 \\ p = q &= 1/(2W_2) \end{aligned} \qquad (11)$$

By choosing W_2 arbitrarily we may obtain infinite number of second order Runge-Kutta methods. If $W_2 = 1$, $p = q = \frac{1}{2}$ and $W_1 = 0$, then we get the method (6). Another choice is $W_2 = \frac{1}{2}$ which gives $p = q = 1$ and $W_1 = \frac{1}{2}$. With this choice we obtain from (7), the method

$$y_{n+1} = y_n + \frac{h}{2}\left[f\left(t_n, y_n\right) + f\left(t_n + h, y_n + hf_n\right)\right] \qquad (12)$$

which is known as **Heun's method.**

Note that when f is a function of t only, the method (12) is equivalent to the trapezoidal rule of integration, whereas the method (6) is equivalent to the midpoint rule of integration. Both the methods (6) and (12) are of $O(h^2)$. The methods (6) and (12) can easily be implemented to solve the IVP (1). **Method** (6) is usually known as **improved tangent method** or **modified Euler method**. Method (12) is also known as **Euler – Cauchy method**.

We shall now discuss the Runge-Kutta methods of $O(h^2)$, $O(h^3)$ and $O(h^4)$.

4.2.1 Runge-Kutta Methods of Second Order

The general idea of the Runge-Kutta (R-K) methods is to write the required methods as

$$y_{n+1} = y_n + h \text{ (weighted sum of the slopes).}$$

$$= y_n + \sum_{i=1}^{m} W_i K_i \qquad (13)$$

where m slopes are being used. These slopes are defined by

$K_1 = hf(t_n, y_n)$,
$K_2 = hf(t_n + C_2h, y_n + a_{21}K_1)$,
$K_3 = hf(t_n + C_3h, y_n + a_{31}K_1 + a_{32}K_3)$,
$K_4 = hf(t_n + C_4h, y_n + a_{41}K_1 + a_{42}K_2 + a_{43}K_3)$,
etc. In general, we can write

$$K_i = hf\left[t_n + C_i h, \sum_{j=1}^{i-1} a_{ij}K_j\right], \ i = 1, 2, \ldots, m \text{ with } C_1 = 0 \qquad (14)$$

The parameters C_i, a_{ij}, W_j are unknowns and are to be determined to obtain the Runge-Kutta methods.

We shall now derive the second order Runge-Kutta methods.

Consider the method as

$$Y_{n+1} = y_n + W_1K_1 + W_2K_2, \qquad (15)$$

where

$$K_1 = hf(t_n, y_n)$$
$$K_2 = hf(t_n + C_2h, y_n + a_{21}K_1), \qquad (16)$$

where the parameters C_2, a_{21}, $W_1$and W_2 are chosen to make y_{n+1} closer to $y(t_{n=1})$.

The exact solution satisfies the Taylor series

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \frac{h^3}{6}y'''(t_n) + \ldots. \qquad (17)$$

where

$$y' = f(t, y)$$
$$y'' = f_t + ff_y$$
$$y''' = f_{tt} + 2ff_{ty} + f_{yy}f^2 + f_y(f_t + ff_y)$$

We expand K_1 and K_2 about the point (t_n,y_n)

$$K_1 = hf(t_n, y_n) = hf_n$$
$$K_2 = hf(t_n + C_2h, y_n + a_{21}hf_n)$$

$$h\left\{f(t_n, y_n) + (C_2hf_t + a_{21}hf_nf_y) + \frac{1}{2!}(C_2^2h^2f_{tt} + 2C_2a_{21}h^2f_nf_{ty} + a_{21}^2h^2f_n^2f_{yy}) + \ldots\right\}$$

Substituting these values of K_1 and K_2 in Eqn. (15), we have

$$y_{n+1} = y_n + (W_1 + W_2)hf_n + h^2[W_2C_2f_t + W_2a_{21}f_nf_y]$$

$$+\frac{h^3}{2}W_2(C_2^2f_{tt} + 2C_2a_{21}f_nf_{ty} + a_{21}^2f_n^2f_{yy}) + \ldots. \qquad (18)$$

Comparing Eqn. (18) with (17), we have

$$W_1 + W_2 = 1$$

$$C_2W_2 = \frac{1}{2}$$

$$a_{21}W_2 = \frac{1}{2}$$

From these equations we find that if C_2 is chosen arbitrarily we have

$$a_{21} = C_2, W_2 = 1/(2C_2), \qquad W_1 = 1 - 1/(2C_2) \tag{19}$$

The R-K method is given by

$$y_{n+1} = y_n + h[W_1 f(t_n, y_n) + W_2 f(t_n + C_2 h, y_n + C_2 h f_n)]$$

and Eqn. (18) becomes

$$y_{n+1} = y_n + hf_n \frac{h^2}{2}(f_1 + f_n f_y) + \frac{C_2 h^3}{4}(f_{tt} + 2f_n t_{yy} + f_n{}^2 f_{yy}) + ... \tag{20}$$

Subtracting Eqn. (20) from the Taylor series (17), we get the truncation error as

$$TE = y(t_{n+1}) - y_{n+1}$$

$$h^3\left[\left(\frac{1}{6} - \frac{C_2}{4}\right)(f_{tt} + 2f_n f_{ty} + f_n^2 f_{yy}) + \frac{1}{6} fy(f_t + f_n f_y)\right] + ...$$

$$= \frac{h^3}{12}[(2 - 3C_2)y''' + 3C_2 f_y y''] + ... \tag{21}$$

Since the TE is of $O(h^3)$, all the above R-K methods are of second order. Observe that no choice of C_2 will make the leading term of TE zero for all f(t, y). The local TE depends not only on derivatives of the solution y(t) but also on the function f(t, y). This is typical of all the Runge-Kutta methods. Generally, C_2 is chosen between 0 and 1 so that we are evaluating f(t, y) at an off-step point in $[t_n, t_{n+1}]$. From the definition, every Runge-Kutta formula must reduce to a quadrature formula of the same order or greater if f(t, y) is independent of y, where W_i and C_i will be weights and abscissas of the corresponding numerical integration formula.

Best way of obtaining the value of the arbitrary parameter C_2 in our formula is to

i) choose some of W_i's zero so as to minimize the computations.
ii) choose the parameters to obtain least TE,
iii) choose the parameter to have longer stability interval.

Methods satisfying either of the condition (ii) or (iii) are called optimal **Runge-Kutta methods**.

We made the following choices:

i) $C_2 = \frac{1}{2}$, $\quad \therefore a_{21} = \frac{1}{2}, W_1 = 0, W_2 = 1$, then

$$y_{n+1} = y_n + K_2,$$

$$K_1 = hf(t_n, y_n),$$

$$K_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right) \qquad (22)$$

which is the same as **improved tangent** or **modified Euler's method.**

ii) $C_2 = 1, \qquad \therefore a_{21} = 1, W_1 = W_2 = \frac{1}{2}$, then

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2),$$
$$K_1 = hf(t_n, y_n),$$
$$K_2 = hf(t_n + h, y_n + K_1) \qquad (23)$$

which is same as the **Euler-Cauchy method.**

iii) $C_2 = \frac{2}{3}, \qquad \therefore a_{21} = \frac{2}{3}, W_1 = \frac{1}{4}, W_2 = \frac{3}{4}$, then

$$y_{n+1} = y_n + \frac{1}{4}(K_1 + 3K_2),$$

$$K_1 = hf(t_n, y_n),$$

$$K_2 = hf\left(t_n + \frac{2h}{3}, y_n + \frac{2K_1}{3}\right) \qquad (24)$$

which is the **optimal R-K method.**

Method (24) is optimal in the sense that it has minimum TE. In other words, with the above choice of unknowns, the leading term in the TE given by (21) is minimum. Though several other choices are possible, we shall limit our discussion with the above three methods only.

In order to remember the weights W and scale factors C_i and a_{ij} we draw the following table:

$$\begin{array}{c|cc} C_2 & a_{21} & \\ \hline & W_1 & W_2 \end{array}$$

General form

$$\begin{array}{c|cc} 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

Improved tangent method

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

Heun's method

$$\begin{array}{c|cc} 2/3 & 2/3 & \\ \hline & 1/4 & 1/4 \end{array}$$

Optimal method

We now illustrate these methods through an example.

Example 1: Solve the IVP $y' = -ty^2, y(2) = 1$ and find y(2.1) and y(2.2) with h = 0.1 using the following R-K methods of O(h^2)

a) Improved tangent method [modified Euler method (22)]
b) Heun's method [Euler-Cauchy method (23)]
c) Optimal R-K method [method (24)]
d) Taylor series method of O(h^2).

Compare the results with the exact solution

$$y(t) = \frac{2}{t^2 - 2}$$

Solution: We have the exact values
y(2.1) = 0.82988 and y(2.2) = 0.70422

a) Improved tangent method is

$$y_{n+1} = y_n + K_2$$

$$K_1 = hf(t_n, y_n)$$

$$K_2 = hf\left(t_n + \frac{h}{2} y_n + \frac{K_1}{2}\right).$$

For this problem f(t, y) = − t y^2 and

$$K_1 = (0.1)[(-2)(1)] = -0.2$$

$$K_2 = (0.1)[(-2.05)\ (1 - 0.1)^2] = -0.16605$$

y(2.1) = 1-0.16605 = 0.83395

Taking t_1 = 2.1 and y_1 = 0.83395, we have

$$K_1 = hf(t_1, y_1) = (0.1)[(-2.1)(0.83395)^2] = -0.146049$$

$$K_2 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right).$$

$$= (0.1)[-(2.15)(0.83395 - 0.0730245)^2] = -0.124487$$

y(2.2) = y_1 +k_2 = 0.83395 − 0.124487 = 0.70946

b) Heun's method is :

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$$

$$K_1 = hf(t_n, y_n) = -0.2$$

$$K_2 = hf(t_n + h, y_n + K_1) = -0.1344$$

y(2.1) = 0.8328

Taking $t_1 = 2.1$ and $y_1 = 0.8328$, we have
$K_1 = -0.14564$, $K_2 = -0.10388$
$y(2.2) = 0.70804$

c) Optimal method is:

$$y_{n+1} = y_n + \frac{1}{4}(K_1 + 3K_2)$$

$$K_1 = hf(t_n, y_n) = -0.2$$
$$K_2 = hf\left(t_n + \frac{2h}{3}, y_n + \frac{2k_1}{3}\right) = 0.15523$$

$y(2.1) = 0.83358$

Taking $t_1 = 2.1$ and $y_1 = 0.83358$, we have
$K_1 = -0.1459197$, $K_2 = -0.117463$

$y(2.2) = 07090$

d) Taylor series method of $O(h^2)$:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n$$

$$y' = -ty^2, y'' = -y^2 - 2tyy'$$

$y(2) = 1$, $y'(2) = -2$, $y''(2) = 7$
$y(2.1) = 0.8350$

with $t_1 = 2.1$, $y_1 = 0.835$, we get
$y'(2.1) = -1.4641725$, $y''(2.1) = 4.437627958$
$y(2.2) = 0.71077$

We now summarise the results obtained and give them in Table 1.

Table 1

Solution and errors in solution of $y' = -t y^2$, $y(2) = 1$, $h = 0.1$. Number inside brackets denote the errors.

T	Method (22)	Method (23)	Method (24)	Method Taylor $O(h^2)$	Exact Solution
2.1	0.83395 (0.00405)	0.8328 (0.0029)	0.83358 (0.00368)	0.8350 (0.0051)	0.8299
2.2	0.70746 (0.0033)	0.708.4 (0.00384)	0.7090 (0.0048)	0.71077 (0.00657)	0.7042

You may observe her that all the above numerical solutions have almost the same error. You may now try the following exercises:

Solve the following IVPs using Heun's method of $O(h^2)$ and the optimal R-K method of $O(h^2)$.

E1) 10y' = t2+y2, y(0) = 1. find y (0.2) taking h = 0.1.

E2) $y' = 1+y2$, $y(0) = 0$. Find $y(0.4)$ taking $h = 0.2$. iven that the exact solution is $y(t) = \tan t$, find the errors.

Also compare the errors at $t = 0.4$, obtained here with the one obtained by Taylor series method of O(h2)

E3) $y' = 3t + \frac{1}{2}y, y(0) = 1$. Find $y(0.2)$ taking $h = 0.1$. Given $y(t) = 13e^{t/2} - 6t - 12$, find the error.

Let us now discuss the R-K methods of third order.

4.2.2 Runge-Kutta Methods of Third Order

Here we consider the method as

$$y_{n+1} = y_n + W_1K_1 + W_2K_2 + W_3K_3 \qquad (25)$$

where

$$K_1 = h\, f(t_n, y_n)$$

$$K_2 = hf\,(t_n + C_2h, y_n + a_{21}K_1)$$

$$K_3 = hf\,(t_n + C_3h, Y_n + a_{31}K_1 + a_{32}K_2)$$

Expanding K_2, K_3 and Y_{n+1} by Taylor series, substituting their values in Eqn. (25) and comparing the coefficients of powers of h, h^2 and h^3, we obtain

$$\begin{aligned}
a_{21} &= C_2 & C_2W_2 + C_3W_3 &= \frac{1}{2} \\
a_{31} + a_{32} &= C_3 & C_2^2W_2 + C_3^2W_3 &= \frac{1}{3} \qquad (26)\\
W_1 + W_2 + W_3 &= 1 & C_2a_{32}W_3 &= \frac{1}{6}
\end{aligned}$$

We have 6 equations to determine the 8 unknowns (3 W′s + 2C′s + 3a′s). Hence the system has two arbitrary parameters. Eqns. (26) are typical of all the R-K methods. Looking at Eqn. (26), you may note that the sum of a_{ij}'s in any row equals the corresponding C_i's and the sum of the Wi's is equal to 1. Further, the equations are linear in w_2 and w_3 and have a solution for W_2 and W_3 if and only if

$$\begin{vmatrix} C_2 & C_3 & -1/2 \\ C^2 & C^2 & -1/3 \\ 0 & C_{2^{a32}} & -1/6 \end{vmatrix} = 0$$

(Ref. Sec. 8.4.2, Unit 8, Block-2, MTE-02, IGNOU material).

Expanding the determinant and simplifying we obtain

$$C_2(2 - 3C_2)a_{32} - C_3(C_3 - C_2) = 0, C_2 \neq 0 \qquad (27)$$

Thus we choose C_2, C_3 and a_{32} satisfying Eqns. (27).

Since two parameters of this system are arbitrary, we can choose C_2, C_3 and determine a_{32} from Eqn. (27) as

$$a_{32} = \frac{C_3(C_3 - C_2)}{C_2(2 - 3C_2)}$$

If $C_3 = 0$, or $C_2 = C_3$ then $C_2 = \frac{2}{3}$ and we can choose $a_{32} \neq 0$, arbitrarily. All C_i's should be chosen such that $0 < C_i < 1$. Once C_2 and C_3 are prescribed, W_i's and a_{ij}'s can be determined from Eqns. (26).

We shall list a few methods in the following notation

C_2	a_{21}		
C_3	A_{31}	A_{32}	
	W_1	W_2	W_3

i) Classical third order R-K method

1/2	1/2		
1	-1	2	
	1/6	4/6	1/6

$$Y_{n+1} = Y_n + \tfrac{1}{6}(K_1 + 4k_2 + K_3) \tag{28}$$

$$K_1 = hf(t_n, y_n)$$

$$K_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$$

$$K_3 = hf(t_n - \frac{h}{2}; y_n - K_1 + 2K_2)$$

ii) Heun's Method

1/3	1/3		
2/3	0	2/3	
	1/4	0	3/4

$$y_{n+1} = y_n + \frac{1}{4}(K_1 + 3K_3) \tag{29}$$

$$K_1 = h\,f(t_n, y_n)$$

$$K_2 = h\,f\left(t_n + \frac{h}{3}, y_n + \frac{K_1}{3}\right)$$

$$K_3 = h\,f\left(t_n + \frac{2h}{3}, y_n + \frac{2K_2}{3}\right)$$

iii) Optimal method

½	1/2		
¾	0	¾	
	2/9	3/9	4/9

$$y_{n+1} = y_n + \frac{1}{9}(2K_1 + 3K_2 + 4K_3) \tag{30}$$

$$K_1 = h\,f(t_n, y_n),$$

$$K_2 = f(t_n + \frac{h}{2}, y_n + \frac{K_1}{2}),$$
$$K_3 = h f\left(t_n + \frac{3h}{4}, y_n + \frac{3K_2}{4}\right).$$

We now illustrate the third order R-K methods by solving the problem considered in Example 1, using (a) Heun's method (29) (b) optimal method (30).

a) Heun's method

$$y_{n+1} = y_n + \frac{1}{4}(K_1 + 3K_3)$$
$$K_1 = h f(t_n, y_n)$$
$$= -0.2$$
$$K_2 = h f\left(t_n + \frac{h}{3}, y_n + \frac{K_1}{3}\right)$$
$$= -0.17697$$
$$K_3 = h f\left(t_n + \frac{2h}{3}, y_n + \frac{2k_2}{3}\right)$$
$$= -0.16080$$
$$y(2.1) = 0.8294$$
Taking $t_1 = 2.1$ and $y_1 = 0.8294$, we have
$$K_1 = -0.14446$$
$$K_2 = -0.13017$$
$$K_3 = -0.11950$$
$$y(2.2) = 0.70366$$

b) Optimal method

$$y_{n+1} = y_n + \frac{1}{9}(2K_1 + 3K_2 + 4K_3)$$
$$K_1 = -0.2$$
$$K_2 = -0.16605$$
$$K_3 = -0.15905$$
$$y(2.1) = 0.8297$$
Taking $t_1 = 2.1$ and $y_1 = 0.8297$, we have
$$K_1 = -0.14456$$
$$K_2 = -012335$$
$$K_3 = -0.11820$$
$$y(2.2) = 0.70405$$

You can now easily find the errors in these solutions and compare the results with those obtained in Example 1.
And now here is an exercise for you.

E4) Solve the IVP
$Y' = y - t \qquad y(0) = 2$
Using third order Heun's and optimal R-K methods. Find (02) taking h = 0.1. Given the exact solution to be $y(t) = 1 + t + e^t$, find the errors at t = 0.2.

We now discuss the fourth order R-K methods.

4.2.3 Runge-Kutta Methods of Fourth Order

Consider the method as

$$
\begin{aligned}
y_{n+1} &= y_n + W_1 K. + W_2 K_2 + W_3 K_3 + W_4 K_4 \\
K_1 &= h\,f(t_n, y_n), \\
K_2 &= h\,f(t_n + C_2 h, y_n + a_{21} K_1), \qquad (31) \\
K_3 &= h\,f(t_n + C_3 h, y_n + a_{31} K_1 + a_{32} K_2), \\
K_4 &= h\,f(t_n + C_4 h, y_n + a_{41} K_2 + a_{43} K_3).
\end{aligned}
$$

Since the expansions of K_2, K_3, K_4 and y_{n+1} in Taylor series are completed, we shall not write down the resulting system of equations for the determination of the unknowns. It may be noted that the system of equations has 3 arbitrary parameters. We shall state directly a few R-K methods of $O(h^4)$. The R-K methods (31) can be denoted by

C_2	a_{21}			
C_3	a_{31}	a_{32}		
C_4	a_{41}	a_{42}	a_{43}	
	W_1	W_2	W_3	W_4

For different choices of these unknowns we have the following methods :

i) **Classical R-K method**

½	1/2			
½	0	½		
1	0	0	1	
	1/6	2/6	2/6	1/6

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4), \\
K_1 &= h\,f(t_n, y_n), \\
K_2 &= h\,f\left(t_n \frac{h}{2}, y_n + \frac{K_1}{2}\right), \qquad (32) \\
K_3 &= h\,f\left(t_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right), \\
K_4 &= h\,f(t_n + h, y_n + K_3).
\end{aligned}
$$

This is the widely used method due to its simplicity and moderate order. We shall also be working out problems mostly by the classical R-K method unless specified otherwise.

1/2	1/2			
1/2	$\frac{(\sqrt{2}-1}{2}$	$\frac{(2-\sqrt{2}}{2}$		
1	0	$\frac{\sqrt{2}}{2}$	$1+\frac{\sqrt{2}}{2}$	
	1/6	$\frac{(2-\sqrt{2}}{6}$	$\frac{(2+\sqrt{2}}{6}$	1/6

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + (2-\sqrt{2})\,K_2 + (2+\sqrt{2})\,K_3 + K_4) \qquad (33)$$

$$K_1 = h\,f(t_n, y_n),$$

$$K_2 = h\,f\left(t_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right),$$

$$K_3 = h\,f\left(t_n + \frac{h}{2}, y_n + \left(\frac{\sqrt{2}-1}{2}\right)K_1 + \left(\frac{2-\sqrt{2}}{2}\right)K_2\right),$$

$$K_4 = h\,f\left(t_n + h, y_n - \frac{\sqrt{2}}{2}K_2 + \left(1 + \frac{\sqrt{2}}{2}\right)K_3\right),$$

The Runge-Kutta-Gill method is also used widely. But in this unit, we shall mostly work out problems with the classical R-K method of $O(h^4)$. Hence, whenever we refer to R-K method of $O(h^4)$ we mean only the classical R-K method of $O(h^4)$ given by (32). We shall now illustrate this method through examples.

Example 2 : Solve the IVP $y' = t+y$, $y(0) = 1$ by Runge-Kutta method of $O(h^4)$ for $t \in (0,0.5)$ with $h = 0.1$. Also find the error at $t = 0.5$, if the exact solution is $y(t) = 2e^t - t - 1$.

Solution : We use the R-K method of $O(h^4)$ given by (32).

Initially, $t_0 =$, $t_0 = 1$

We have

$$K_1 = hf(t_0, y_0) = (0.1)[0+1] = 0.1$$

$$K_2 = hf\left(t_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0.1)[0.05 + 1 + 0.05] = 0.11$$

$$K_3 = hf\left(t_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0.1)[0.05 + 1 + 0.055] = 0.1105$$

$$K_4 = hf(t_0 + h, y_0 + K_3) = (0.1)[0.1 + 1 + 0.1105] = 0.12105$$

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6}\ [1 + 0.22 + 0.2210 + 0.12105] = 1.11034167$$

Taking $t_1 = 0.1$ and $y_1 = 1.11034167$, we repeat the process.

$$K_1 = hf(t_1, y_1) = (0.1)\ [0.1 + 1.11034167] = 0.121034167$$

$$K_2 = hf\ \left(t_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = (0.1)$$

$$\left[0.1 + 0.05 + 1.1103416 + \frac{(0.121034167)}{2}\right] = 0.132085875$$

$$K_3 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = (0.1)$$

$$\left[0.1+0.05+1.1103417+\frac{(0.132085875)}{2}\right] = 0.132638461$$

K_4 = hf (t1 + h, y1 + K3) = (0.1) [0.1 + 0.05 + 1.11034167 + $\frac{(0.132085875)}{2}$

= 0.144303013

$y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

= 1.11034167 + $\frac{1}{6}$ [(0.121034167 + 2(0.132085875) + 2(0.132638461)

+ 0.1443030131 = 1.24280514

Rest of the values y_3, y_4, y_5 we give in Table 2.

Table 2

t_n	y_n
0.0	1
0.1	1.11034167
0.2	1.24280514
0.3	1.39971699
0.4	1.58364848
0.5	1.79744128

Now the exact solution is

$$y(t) = 2e^t - t - 1$$

Error at t = 0.5 is

$$\begin{aligned} y(0.5) - y_5 &= (2e^{0.5} - 0.5 - 1) - 1.79/44128 \\ &= 1.79744254 - 1.79744128 \\ &= 0.000001261 \\ &= 0.13 \times 10^{-5}. \end{aligned}$$

Let us consider another example

Example 3 : Solve the IVP

$$y' = 2y + 3e^t, y(0) = 0 \text{ using}$$
a) classical R – K method of $O(h^4)$
b) R – K Gill method of $O(h^4)$,

Find y (0.1), y(0.2), y(0.3) taking h = 0.1. Also find the errors at t = 0.3, if the exact solution is $y(t) = 3(e^{2t} - e^t)$.

Solution: a) Classical R-K method is

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Here $t_0 = 0$, $y_0 = 0$, $h = 0.1$

$K_1 = h\, f(t_0, y_0) = 0.3$

$K_2 = h\, f\left(t_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.3453813289$

$K_3 = h\, f\left(t_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.3499194618$

$K_4 = h\, f\ (t_0 + h, y_0 + K_3) = 0.4015351678$

$y_1 = 0.3486894582$

Taking $t_1 = 0.1, y_1 = 0.3486894582,$ we repeat the process and obtain

K_1= 0.4012891671	K_2 = 0.4584170812
K_3 = 0.4641298726	K_4 = 0.6887058455
Y(0.2) = 0.8112570941	

Taking $t_2 = 0.2$, $y_2 = 0.837870944$ and repeating the process we get

K_1= 0.53399502	K_2 = 0.579481565
K_3 = 0.61072997	K_4 = 0.694677825
$\therefore$ y (0.3) = 1.416807999	

b) R-K gill method is

$$y_{n+1} = y_n + \frac{1}{6}\left(K_1 + (2 + (2 - \sqrt{2}\right)K_2 + (2 + \sqrt{2})\,K_3 + K_4)$$

Taking $t_0 = 0$, $y_0 = 1$ and $h = 0.1$, we obtain

K_1= 0.3	K_2 = 0.3453813289
K_3 = 0.3480397056	K_4 = 0.4015351678
y (0.1) = 0.3486894582	

Taking $t_1 = 0.1$, $y_1 = 0.3486894582$, we obtain

K_1= 0.4012891671	K_2 = 0.4584170812
K_3 = 0.4617635569	K_4 = 0.5289846936
y (0.2) = 0.8112507529	

Taking $t_2 = 0.2$, $y_2 = 0.8112507529$, we obtain

K_1= 0.528670978	K_2 = 0.6003248734
K_3 = 0.6045222614	K_4 = 0.6887058455
y (0.3) = 1.416751936	

From the exact solution we get

y(0.3) = 1.416779978

Error in classical R-K method (at t = 0.3) = 02802×10^{-04}

Error in R-K Gill method (at t = 0.3) = 0.2804×10^{-04}

You may now try the following exercises

Solve the following IVPs using R-K method of $O(h^4)$

E5) $y' = \frac{y-t}{y+t}, y(0) = 1$. Find y(0.5) taking h = 0.5.

E6) $y' \ 1 - 2ty$, y(0.2) = 0.1948. Find y(0.4) taking h = 0.2.

E7) $10ty' + y^2 = 0, y(4) = 1$. Find y(4.2) taking h = 0.2. Find the error give n the exact solution is $y(t) = \frac{1}{c + 0.1 \text{in } t'}$ where c = 0.86137

E8) $y' = \frac{1}{t^2} - \frac{y}{t} - y^2, y(1) = -1$. Find y (1.3) taking h = 0.1. Given the exact solution to be y(t) = $\frac{1}{t'}$ find the error at t = 1.3.

We now end this unit by giving a summary of what we have covered in it.

4.3 SUMMARY

In this unit we have learnt the following :

1) Runge-Kutta methods being singlestep methods are self-starting methods.

2) Unlike Taylor series methods, R-K methods do not need calculation of higher order derivatives of f(t, y) but need only the evaluation of f(t, y) at the off-step points.

3) For given IVP of the form

$$y' = f(t, y), y'(t_0) = y_0, t\varepsilon[t_0, b]$$

where the mesh points are $t_j = t_0 + jh, j = 0, 1, ..., n.$

$t_n = b = t_0 + nh, R - K$ methods are obtained by writing

$y_{n+1} = y_n + h$ (weighted sum of the slopes)

$$= y_n + \sum_{i=1}^{m} W_i K_i$$

where in slopes are used. These slopes are defined by

$$K_i = f\left[t_n + C_i h, \sum_{i=1}^{i-1} a_{ij} k_j\right], i = 1, 2, ..., m, C_1 = 0.$$

Here is the order of the method. The unknowns C_i, a_{ij} and W_j are then obtained by expanding K_i's and y_{n+1} in Taylor series about the point (t_n, y_n) and comparing the coefficients of different powers of h.

4.4 SOLUTIONS/ANSWERS

E1) Heun's method : $y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$

Starting with $t_0 = 0, y_0 = 1, h = 0.1$

$\therefore\ K_1 = 0.01$

$K_2 = 0.010301$

$y(0.1) = 1.0101505$

Taking $t_1 = 0.1$, $y_1 = 1.0101505$

$K_1 = 0.0103040403$
$K_2 = 0.0181327468$
$y(0.2) = 1.020709158$

Optimal R-K, method : $y_{n+1} = y_n + \frac{1}{4}(K_1 + 3K_2)$

$t_0 = 0, y_0 = 1, h = 0.1$
$K_1 = 0.01, K_2 = 0.01017823$
$y(0.1) = 1.010133673$
$t_1 = 0.1, y_1 = 1.010133673$
$K_i = 0.0103037, K_2 = 0.010620.$
$y(0.2) = 1.020675142$

E2) Heun's method:

$K_1 = 0.2$, $K_2 = 0.208$
$y(0.2) = 0.204$
$K_1 = 0.2083232$, $K_2 = 0.2340020843$
$y(0.4) = 0.4251626422$

Optimal R-K, method:
$K_1 = 0.2$, $K_2 = 0.2035556$
$y(0.2) = 0.2026667$
$K_1 = 0.2082148$, $K_2 = 0.223321245$
$y(0.4) = 0.42221134$

Taylor Series method
$y' = 1 + y^2$, $y'' = 2yy'$
$y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$
$y(0.2) = 0.2$
$y'(0.2) = 1.04$, $y''(0.2) = 0.416$
$y(0.4) = 0.41632$

Now the exact solution is $y(t) = \tan t$
Exact $y(0.4) = 0.422793219$

Error in Heun's method = 0.422793219

Error in Optimal R-K method = 0.236×10^{-2}

Error in Optimal R-K method = 0.582×10^{-3}

Error in Taylor series method = 0.647×10^{-2}

E3) Heun's method:

$K_1 = 0.05$, $K_2 = 0.0825$
$y(0.1) = 1.06625$
$K_1 = 0.0833125$
$y(0.2) = 1.166645313$

Optimal R-K, method

$K_1 = 005$, $K_2 = 0.071666667$
$y(0.2) = 1.166645313$
Exact $y(0.2) = 1.167221935$

Error in both the methods is same and = 0.577×10^{-3}

E4) Heun's method : $y_{n+1} = y_n + \frac{1}{4}(K_1 + 3K_3)$

Starting with $t_0 = 0$, $y_0 = 2$, $h = 0.1$, we have

$K_i = 0.2$, $K_2 = 0.203334, K_3 = 0.206889$
$y(0.1) = 2.205167$
$t_1 = 0.1$, $y_1 = 2.205167$ we have
$K_1 = 0.210517$, $K_2 = 0.214201$, $K_3 = 0.218130$
$y(0.2) = 2.421393717$

Optimal R − K method: $y_{n+1} = y_n + \frac{1}{9}(2K_1 + 3K_2 + 4K_3)$

$K_1 = 0.2$, $K_2 = 0.205$, $K_3 = 0.207875$
$y(0.1) = 2.205167$
$t_1 = 0.1, y_1 = 2.205167$
$K_1 = 0.2105167$, $K_2 = 0.2160425$, $K_3 = 0.219220$
$y(0.2) = 2.421393717$
exact $y(0.2) = 2.421402758$
Since $y(0.2)$ is same by both the mehods

Error $= 0.9041 \times 10^{-5}$ in both the methods at t = 0.2.

E5) $K_1 = 0.5$, $K_2 = 0.333333$
$K_3 = 0.3235294118$, $K_4 = 0.2258064516$
$y(0.5) = 1.33992199$.

E6) $K_1 = 0.184416$, $K_2 = 0.16555904$
$K_3 = 0.1666904576$, $K_4 = 0.1421615268$
y (0.4) = 0.3599794203.

E7) $K_1 = -0.005$, K2 = -0.004853689024
$K_3 = -0.0048544$, K4 = -0.004715784587
y (4.2) = 0.9951446726.
Exact y (4.2) = 0.995145231, Error = 0.559×10^{-6}

E8) K_1 0.1, $K_2 = 0.09092913832$
$K_3 = 0.9049729525$, $K_4 = 0.8260717517$
$y(1.1) = -0.909089993$
$K_1 = 0.08264471138$, $K_2 = 0.07577035491$
$K_3 = 0.07547152415$, $K_4 = 0.06942067502$
$y(1.2) = -0.8333318022$
$K_1 = 0.06944457204$, $K_2 = 0.06411104536$
$K_3 = 0.06389773475$, $K_4 = 0.0591559551$
$y(1.3) = -0.7692307692$
Exact $y(1.3) = -0.7692287876$
Error = 0.19816×10^{-5}