UNIT 1 APPLICATIONS OF DIFFERENTIAL CALCULUS

Structure

1.1 INTRODUCTION

In this unit we see some immediate applications of the concepts which we have studied in Blocks 1 and 2.

Differential Calculus have varied applications. You have already seen some applications to geometrical, physical and practical problems in Block 2. In this unit, we shall study some applications to the theory of real functions.

There are some questions in mathematics, that can be asked even before knowing differentiation, but can be solved with ease by using the theorems on differentiation. In other words, differentiation is a useful tool in solving problems that arise independent of this notion, for example.

a) Is the function $\sin x + \cos x$ increasing in $[0, \pi/4]$?

b) Do we have the inequality $e^x > 1 + x$ for all x?

c) What is the limit of $\frac{1-e^x}{\sin x}$ when x tends to 0?

d) What is the approximate value of $\cos 1°$?

Do you notice that these questions do not involve any concept that you have not studied earlier? You could have asked them earlier. You could have even answered some of them. Here we shall see how the theorems of Unit 7 can be systematically applied to yield solutions to such questions. Keep Block 2 ready with you since you will need to read the relevant portions from the units in that block.

Objectives

After studying this unit you should be able to :

- recognise the equivalence of some properties of functions (like monotonicity and positiveness or negativeness of its derivative),
- prove some inequalities using the mean value theorems,
- apply Taylor's series to obtain approximate values of certain functions at certain points.

1.2 MONOTONIC FUNCTIONS

In this section we employ differentiation to decide whether a given function is monotonic or not.

First we recall some terms from Unit 1. We say that a function from an interval I to R is monotonic if it is either an increasing function on I or a decreasing function on I.

It is said to be an increasing function on I, if $x \le y$ implies $f(x) \le f(y)$. Increasing functions may also be thought of as **order-preserving functions.**

Let us also recall that

- every constant function is an increasing function,
- the identity function is an increasing function,
- the function $f(x) = 2x + 3$ is an increasing function,
- the function $f(x) = \sin x$ is **not** an increasing function on **R**.

This is because even though $0 < 3\pi/2$, we have $\sin 0 = 0 > -1 = \sin 3\pi/2$. However, on the interval $[0, \pi/2]$, $\sin x$ is an increasing function.

Further, we know that a function f from I to **R** is said to be a decreasing function if $x \leq y$ implies $f(x) \geq f(y)$. Decreasing functions are the **order-reversing functions.**

Here are some examples:

- The function $f(x) = 4 - 2x$ is decreasing.
- The function $f(x) = \sin x$ is decreasing in the interval $[\pi/2, \pi]$.

Do you agree that each constant function is both increasing and decreasing?

Warning: It is incorrect to say that if a function is not increasing, then it is decreasing. It may happen that a function is neither increasing nor decreasing. For instance, if we consider the interval $[0, \pi]$, the function $\sin x$ is neither increasing nor decreasing. It is increasing on $[0, \pi/2]$ and decreasing on $[\pi/2, \pi]$. There are other functions that are even worse. They are not monotonic on any sub-interval also. But most of the functions that we consider are not so bad.

Usually, by looking at the graph of the function one can say whether the function is increasing or decreasing or neither. The graph of an increasing function does not fall as we go from left to right, while the graph of a decreasing function does not rise as we go from left to right. But if we are not given the graph, how do we decide whether a given function is monotonic or not? Theorem 1 gives us a criterion to do just that.

Theorem 1: Let I be an open interval. Let $f : I \to \mathbf{R}$ be differentiable. Then

a) f is increasing if and only if $f'(x) \geq 0$ for all x in I.

b) f is decreasing if and only if $f'(x) \leq 0$ for all x in I.

Proof: a) Let f be increasing. Let $x \in I$. Then,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$$

Since f is increasing, if $h > 0$, then $x + h > x$ and $f(x + h) \geq f(x)$. Hence $f(x + h) - f(x) \geq 0$. If $h < 0$, $x + h < x$, and $f(x + h) \leq f(x)$. Hence $f(x + h) - f(x) \leq 0$.

So either $f(x + h) - f(x)$ and h are both non-negative or they are both non-positive. Therefore,

$\frac{f(x+h) - f(x)}{h}$ is non-negative for all non-zero values of h.

Therefore, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ must also be non-negative.

Thus, $f'(x) \geq 0$.

Conversely, let $f'(x) \geq 0$, for all x in I. To show f is monotonically increasing. Let $a < b$ in I. We shall prove that $f(a) \leq f(b)$. By mean value theorem (Theorem 3, of Block 2),

$\frac{f(b) - f(a)}{b - a} = f'(c)$ for some $c \in]a, b[\subset I$.

Since $f'(c) \geq 0$, we have

$\frac{f(b) - f(a)}{b - a} \geq 0$. Also $b - a > 0$. It follows that $f(b) - f(a) \geq 0$, or $f(b) \geq f(a)$.

Thus, $a < b$ implies $f(a) \leq f(b)$. Therefore, f is increasing.

The part b) of the theorem can be proved similarly. We leave it as an exercise. It can also be deduced by applying part a) to the function, $-f$.

E E1) Prove Part b) of Theorem 1.

From the class of increasing functions we can separate out functions which are strictly increasing. The following definition gives the precise meaning of the term "strictly increasing function".

Definition 1: $f : I \to \mathbf{R}$ is said to be **strictly increasing if $a < b$ implies that $f(a) < f(b)$.**

We can similarly say that a function defined on I is strictly decreasing if $a < b$ implies $f(a) > f(b)$. For example, a constant function is not strictly increasing, nor is it strictly decreasing. The function $f(x) = [x]$ too, is increasing, but not strictly increasing, whereas the function $f(x) = x$ is strictly increasing.

Fig. 1 shows the graphs of these three functions. In Fig. 1 (a) the graph is horizontal. In Fig. 1(b) there are parts of the graph which are horizontal. But the graph in Fig. 1(c) has no horizontal portions, and rises as we go from left to right.

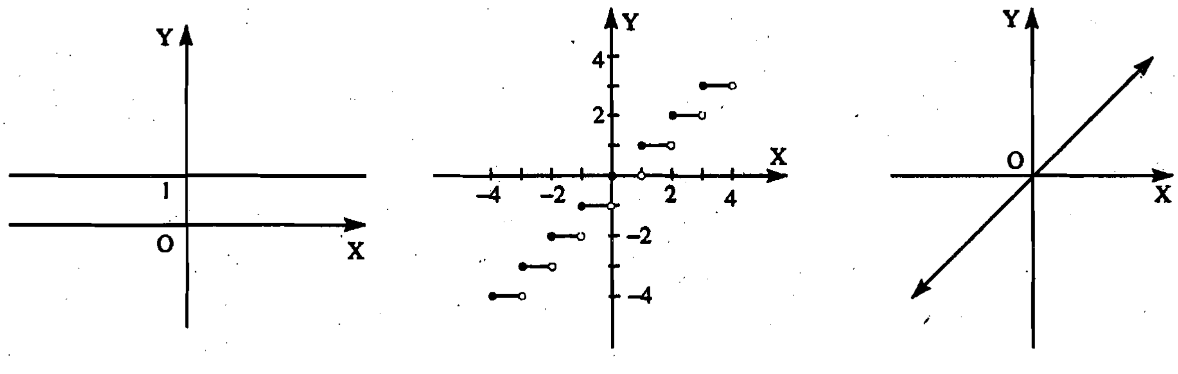


Fig. 1: (a) graph of $f(x) = 1$ (b) graph of $f(x) = [x]$ (c) graph of $f(x) = x$.

Now let us see whether strict monotonicity of a function is reflected by its derivative. We have the following theorem.

Theorem 2 : a) Let f' be positive on I. Then f is strictly increasing on I.

b) Let f' be negative on I. Then f is strictly decreasing on I.

Proof: a) By Theorem 1, we know that since $f' > 0$ on I, f must be increasing on I. That is, $x < y \Rightarrow f(x) \leq f(y)$. We have only to prove that if $x < y$, then $f(x)$ cannot be equal to $f(y)$. Let, if possible, $f(x) = f(y)$, where $x < y$ in I.

Then, by Rolle's Theorem (Theorem 2, Unit 7) applied to the function f on the interval [x, y], we have $f'(c) = 0$ for some $x < c < y$.

But this contradicts the assumption that f' is strictly positive on I. Hence f(x) cannot be equal to f(y) for any $x < y$ in 1. Thus, $x < y \Rightarrow f(x) < f(y)$. That is, f is strictly increasing.

b) We indicate two different proofs for this part. One way is to imitate the proof of part a) by changing the symbol '<' to the symbol '>', wherever it occurs.

Another way is to consider –f, apply Theorem 2 a) to it and use the facts: $(-f)' = -f'$. Therefore f' is negative if and only if $(-f)'$ is positive. Also f is strictly decreasing if and only if – f is strictly increasing. Combining the results for increasing and decreasing functions we get the following corollary.

Corollary 1: f is strictly monotonic on the interval I if f' is of the same sign throughout I, assuming zero is neither positive nor negative.

You may have noticed that there is a difference between the statements of Theorem 1 and Theorem 2. We said:

"f is increasing if and only if f' is non-negative".

"If $f' > 0$, then f' is strictly increasing".

It is natural to ask:

Can we have "If and only if" in Theorem 2 also? That is, if f is strictly increasing, does it follow that $f' > 0$? Unfortunately, we have a negative answer as shown in Example 2 below.

But before that, our first example shows how among many methods available to prove the monotonicity, the one using differentiation is the simplest (provided the function is differentiable, of course).

Example 1: Let $f(x) = x^3$ for all x in **R**. We shall prove that the function f is increasing.

First Method: Let $x < y$. We want to prove that $x^3 \leq y^3$. Consider two cases.

Case 1: x and y are of the same sign. (Either both are positive or both are negative). In this case $xy > 0$. Now,

$y^3 - x^3 = (y - x)(y^2 + xy + x^2) \geq 0$ since both $y - x$ and $y^2 + yx + x^2$ are non-negative.

Case 2: Let x and y be not of the same sign. Since $x < y$, this means that $x < 0 < y$. (Note that if either x or y is zero, it comes under Case 1.) Therefore, $x^3 < 0 < y^3$. Because, the cube of a negative number is negative, and the cube of a positive number is positive.

Thus, in both case $x^3 \leq y^3$. In fact, we have the strict inequality $(x^3 < y^3)$, indicating that $x \to x^3$ is a strictly increasing function.

Second Method: Let $x < y$. We want to prove that $x^3 \leq y^3$.

Now $(y^3 - x^3) = (y - x)(y^2 + yx + x^2)$

Here $y - x > 0$ (since $x < y$). Also, $y^2 + yx + x^2 \geq 0$ because

$$y^2 + yx + x^2 = \frac{1}{2}[(y^2 + x^2) + (y^2 + 2yx + x^2)].$$

$$= \frac{1}{2}[y^2 + x^2 + (y + x)^2] \geq 0$$

since the square of any number is non-negative. Thus $y^3 - x^3$ is a product of two non-negative numbers and, hence is non-negative.

Third Method: (Using Differentiation): Let $f(x) = x^3$. Then $f(x) = 3x^2$. This is always non-negative. Therefore, using Theorem 1 we can say that f is an increasing function.

Example 2: Here we give an example of a strictly increasing function whose derivative is not strictly positive.

Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = x^3$ (see Fig. 21). It is strictly increasing because,

$$x < y \Rightarrow y - x > 0 \text{ and } x^2 + y^2 > 0$$

$$\Rightarrow y^3 - x^3 = (y - x)(y^2 + yx + x^2)$$

$$= \frac{1}{2}(y - x)[(x^2 + y^2) + (x + y)^2] > 0$$

$$\Rightarrow x^3 < y^3.$$

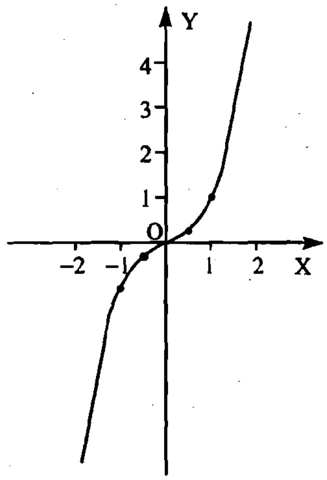


Fig. 2

Its derivative is not strictly positive because $f'(0) = 0$. Our next example describes two different ways in which non-monotonicity of a function can be proved.

Example 3: To prove that the function $f : x \to \sin x + \cos 2x$ is not monotonic on the interval $[0, \pi/4]$, we can proceed as follows:

First Method: We shall consider three points, $0, \pi/10$ and $\pi/6$ belonging to $[0, \pi/4]$.

Then, $f(0) = \sin 0 + \cos 0 = 1$

$$f(\pi/10) = \sin\frac{\pi}{10} + \cos\frac{\pi}{5} = 0.3090 + 0.8090 > 1$$

$$f(\pi/6) = \sin\frac{\pi}{6} + \cos\frac{\pi}{3} = \frac{1}{2} + \frac{1}{2} = 1$$

We have $0 < \pi/10 < \pi/6$ and $f(0) < f(\pi/10) > f(\pi/6)$.

Therefore, f is neither increasing, nor decreasing on $[0, \pi/4]$. Or, We can say that f is not monotonic on $[0, \pi/4]$.

Second Method: (Using Differentiation)

Let $f(x) = \sin x + \cos 2x$

Then $f'(x) = \cos x - 2\sin 2x$

Now, $f'(0) = 1 - 0 = 1$ and $f'(\pi/4) = \frac{1}{\sqrt{2}} - 2 \times 1 < 0$.

Thus, f' is of different signs at 0 and $\pi/4$.
Therefore f is not monotonic on $[0, \pi/4]$.
Our next example warns us in dealing with functions not defined at some points.

Example 4: If the function $f : x - \tan x$ is defined on an interval, we can prove that it is an increasing function there. See Fig. 3.

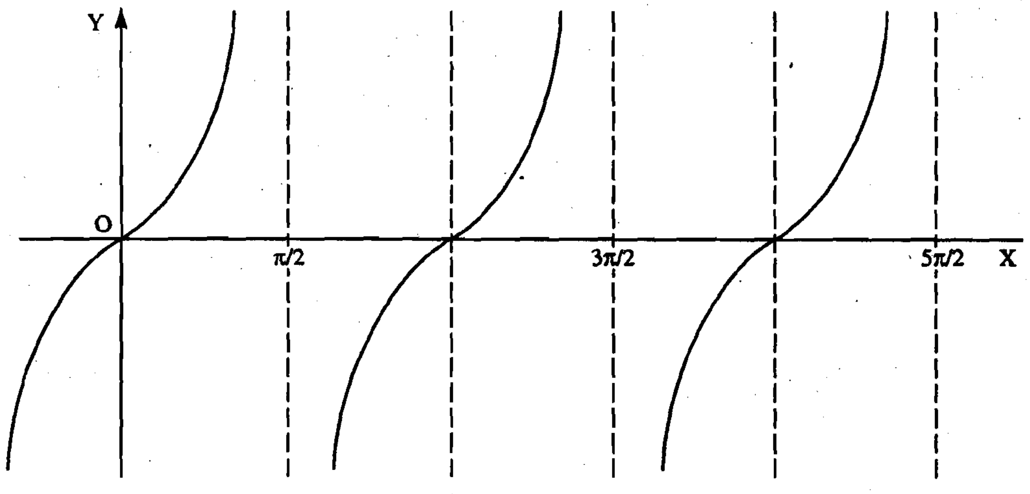


Fig 3

Now consider the interval $[0, \pi]$. Can we prove that tan x is increasing on this interval? Suppose we argue as follows:

$f'(x) = \sec^2 x$, and $\sec^2 x \geq 0 \ \forall \ x$ since the square of any quantity is non-negative. Hence by Theorem 1, f is an increasing function on $[0, \pi]$.

But if we take two points $\frac{\pi}{4}$ and $\frac{2\pi}{3}$ in $[0, \pi]$.

then $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} = 1$, and $f\left(\frac{2\pi}{3}\right) = \tan\frac{2\pi}{3} = -\sqrt{3}$.

Thus $\frac{\pi}{4} < \frac{2\pi}{3}$, but $\tan\frac{\pi}{4} \not\leq \tan\frac{2\pi}{3}$.

This indicates that tan x is not an increasing function on $[0, \pi]$.
So where did we go wrong? We can explain it as follows:

In the interval $[0, \pi]$, there is a point, namely $\frac{\pi}{2}$, where tan is not defined. Hence, its derivative does not exist at that point, and therefore we can not apply Theorem 1.

What we proved is that this function is increasing in an interval, provided it is defined throughout this interval. It is only when it is defined, that we can differentiate it and apply our theorems.

In the next example we use an additional property of continuous functions in the first method, and repeated differentiation in the second method.

Example 5: Let us prove that the function $x \to \sin x + \cos x$ is increasing on $[0, \pi/4]$ and decreasing on $[\pi/4, \pi/2]$.

First Method: Recall that for all x in the first quadrant (i.e., if $0 \leq x \leq \pi/2$), sin x, cos x and tan x are non-negative.

Let $f(x) = \sin x + \cos x$, then $f'(x) = \cos x - \sin x$
We note that $f'(0) = 1$,
$f'(\pi/4) = 0$,
$f'(\pi/2) = -1$.
$x = \pi/4$ is the only point in $[0, \pi/2]$ at which $f'(x) = 0$.
Because $f'(x) = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1$. But tan x is strictly increasing on $[0, \pi/2]$, by Example 4. So, it cannot take the value 1 at any point other than $\pi/4$.

Also, f' is a continuous function because the sine and cosine functions are continuous.

Therefore, f cannot take negative values in $[0, \pi/4]$.

Explanation: If a continuous function takes a positive value at 0 and a negative value at some x, then it must be zero somewhere in between 0 and x. In this problem, the continuous function f' cannot take the value zero between 0 and x if $x < \pi/4$.

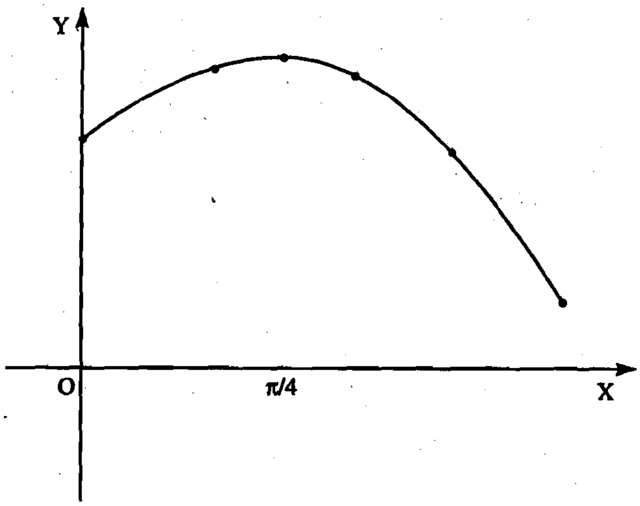
Therefore, since f' is non-negative on $[0, \pi/4]$, f is increasing on, $[0, \pi/4]$.

Similarly, f' cannot take positive values in $[\pi/4, \pi/2]$, because its value in $[\pi/4, \pi/2]$ is negative. It follows that f is decreasing on $[\pi/4, \pi/2]$.

Second Method: Let $f(x) = \sin x + \cos x$, and $f'(x) = \cos x - \sin x$. To prove that f is increasing on $[0, \pi/4]$, we have to prove that f' is non-negative on $[0, \pi/4]$. We first note that $f'(0) = 1$ and $f'(\pi/4) = 0$. It is enough to prove that f' is decreasing on $[0, \pi/4]$ (for then, all values of f' in this interval will be between 0 and 1). For this purpose we consider $f''(x) = -\sin x - \cos x$. Also see Fig. 4(a) and (b).

We note that $f''(x) \leq 0$ for all x in the first quadrant, and in particular for all x in $[0, \pi/4]$. Therefore, f' is decreasing on $[0, \pi/4]$. Therefore, (since $f'(0) = 1$ and $f'(\pi/4) = 0$), f' is non-negative on $[0, \pi/4]$. Therefore f is increasing on $[0, \pi/4]$.

Next, we shall prove that f' is non-positive on $[\pi/4, \pi/2]$. First, we note that $f'(\pi/4) = 0$ and $f'(\pi/2) = -1$. Also, f' is decreasing on $[\pi/4, \pi/2]$, since $f''(x) \leq 0$ on $[\pi/4, \pi/2]$. Therefore all the values of f' on this interval are between 0 and -1 and hence cannot be positive. Therefore f is decreasing on $[\pi/4, \pi/2]$.



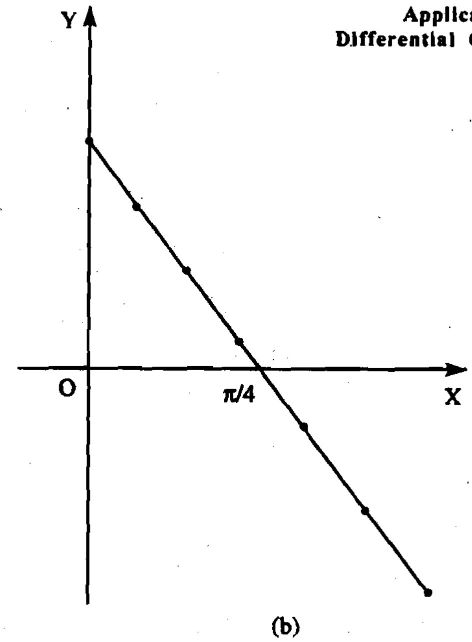


Fig. 4: (a) Graph of $f(x) = \sin x + \cos x$, (b) Graph of $f(x) = \cos x - \sin x$

If you have followed the arguments in these examples, you should not have any difficulty in solving these exercises.

E E2) Assuming that e^x never takes negative values, prove that it is an increasing function on **R**.

E E3) Prove that $\ln x$ is an increasing function on $]0, \infty[$.

E E4) Using the fact that sin x and cos x are never negative in the first quadrant, prove by differentiation that sin x is an increasing function and cos x is a decreasing function on $[0, \pi/2]$.

E E5) Which of the following functions are increasing on the interval given? Which of them are decreasing?

a) $x^2 - 1$ on $[0, 2]$

b) $2x^2 + 3x$ on $[-1/2, 1/2]$

c) e^{-x} on $[0, 1]$

d) $x(x-1)(x+1)$ on $[-2, -1]$

e) $x \sin x$ on $[0, \pi/2]$

f) $\tan x + \cot x$ on $[0, \pi/4]$

E E6) Prove that the following functions are not monotonic in the intervals given

a) $2x^2 + 3x - 5$ on $[-1, 0]$

b) $x(x-1)(x+1)$ on $[0, 2]$

c) $x \sin x$ on $[0, \pi]$

d) $\tan x + \cot x$ on $[0, \pi/2]$

E E7) Give an example of a cubic polynomial that decreases on $]-\infty, 2]$, increases on $[2, 3]$ and again decreases on $[3, \infty]$.

(*Hint*: The derivative should change sign while passing through 2, and again while passing through 3).

1.3 INEQUALITIES

Another application on differential calculus is to prove certain inequalities. In the three examples below we illustrate how some inequalities can be deduced from Taylor's series and the mean value theorem.

Example 6: Suppose we want to prove $e^x \geq 1 + x$ for all x in R.

Let $f(x) = e^x - x$, Then $f'(x) = e^x - 1$, $f''(x) = e^x$

We know that $e^x > 0$ for all x.

Therefore, $f'(x)$ is a strictly increasing function.

Also $f'(0) = 0$.

Therefore there is no other point where f' vanishes.

Also, $f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$.

Therefore f(x) is increasing on $]0, \infty[$.

So $x > 0$ implies $f(x) > f(0)$. This means $e^x - x > e^0 - 0 = 1$.

This proves $e^x > 1 + x$, if $x > 0$.

It remains to prove this for negative values of x also. For this purpose we let $g(x) = e^x - xe^x$.

Then $g'(x) = e^x - (xe^x + e^x) = -xe^x < 0$ wherever $x > 0$.

$\Rightarrow$ g is strictly decreasing on $]0, \infty[$

$\Rightarrow g(x) < g(0)$ whenever $x > 0$.

$\Rightarrow e^x - xe^x < e^0 - 0.e^0 = 1$ for all $x > 0$.

In other words, $e^x < \dfrac{1}{1-x}$ for all $x > 0$.

Putting $y = -x$, we get $e^{-y} < \dfrac{1}{1+y}$, or

$e^y > 1 + y$ for $y < 0$. In other words, $e^x > 1 + x$ for all $x < 0$.

When $x = 0$, $e^x = 1 = 1 + x$.

Thus, the inequality $e^x \geq 1 + x$ is true for all values of x.

In the next example we give an inequality that is still better.

Example 7: We prove that $e^x > 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$ for all $x > 0$.

We have seen in Unit 6 that the Taylor's series expansion.

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots\ldots\ldots$ is valid for all x.

This proves the inequality

$e^x > 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$ whenever $x > 0$.

Fig. 5 represents the results of Examples 6 and 7.

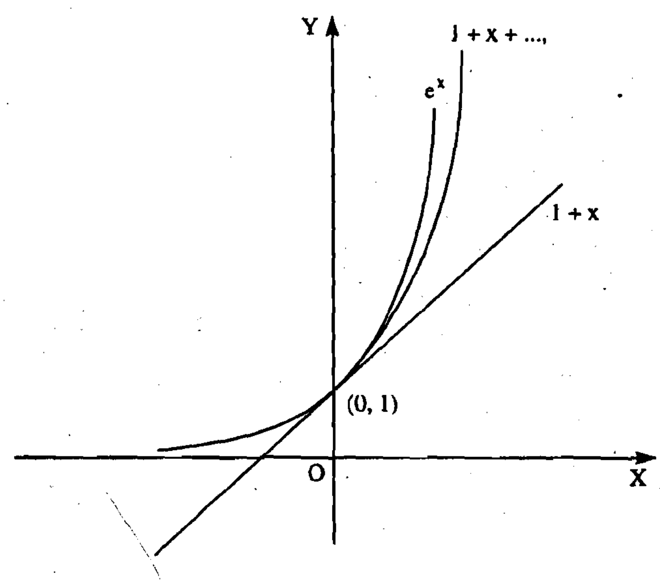


Fig. 5

Example 8: To prove $b^n - a^n < n\,b^{n-1}(b-a)$, wherever $0 < a < b$ and $n > 1$, we consider the function $f : x \to x^n$ on the interval $[a, b]$. It is continuous there. It is also differentiable in $]a, b[$. Therefore, by the mean value theorem, there is some c, $a < c < b$ such that

$$nc^{n-1} = f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cross-multiplying we get, $b^n - a^n = nc^{n-1}(b-a)$.

Therefore, it suffices to prove that $e^{n-1} < b^{n-1}$.

This is true because $0 < c < b$, and $n > 1$.

You can try these exercises now.

E8) Prove the following inequalities using the methods indicated alongside in brackets.

a) $\ln(1+x) < x$ for all positive x
(first prove that $x - \ln(1+x)$ is increasing).

b) $\tan^{-1} x < x$ for all positive x
(by mean value theorem on $[0, x]$ for $\tan^{-1} x$).

c) $e^x + e^{-x} > 2 \ \forall \ x$. (writing $e^x + e^{-x} - 2$ as a perfect square).

d) $e^x + e^{-x} > 2 \ \forall \ x$. (using the already proved result $e^x \geq 1 + x$).

e) $e^x - e^{-x} \geq 2x \ \forall \ x > 0$. (using the inequality in d) and differentiation).

1.4 APPROXIMATE VALUES

In the previous two sections we have seen how the concept of derivatives can be used in proving the monotonicity of a given differentiable function; and how this knowledge can be applied to prove some inequalities.

In this section we shall see how Taylor's series can be used to find approximate values of some functions at some points. We use the symbol $\approx$ to mean approximately equal to.

Example 9: Taking the first two non-zero terms in Maclaurin's series for sin x, we shall prove that sin 20° is approximately equal to 0.342 (in symbols, sin 20° $\approx$ 0.342). Remember that in the formula

$$\sin x = x + \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots\dots\dots$$

the angle x is always measured in radians. The same holds for cos x, tan x, and so on.

Now, $20° = \frac{\pi}{9}$ radians. Therefore,

$$\sin\frac{\pi}{9} = \frac{\pi}{9} - \frac{(\pi/9)^3}{3!} + \dots\dots\dots$$

$$\sin\frac{\pi}{9} \cong \frac{\pi}{9} - \frac{(\pi/9)^3}{6}$$

Taking $\pi = 3.142$, we get this quantity to be

$$\frac{3.142}{9} - \frac{.0425}{6} \approx 0.349 - 0.007 = 0.342.$$

If you look into a table of sines, you will find sin 20° = 0.342. This shows that our approximation is really a good approximation. In fact, the tables are written by using precisely these methods.

Example 10: Let us find the approximate value of $(0.99)^{5/2}$ by taking three terms of Maclaurin's series for $(1-x)^{5/2}$

Maclaurin's series for $(1-x)^{5/2}$ is

$$1 - \frac{5}{2}x + \frac{(5/2)(3/2)}{2}x^2 + \dots\dots\dots$$

We can write $(0.99)^{5/2}$ as $(1-0.01)^{5/2}$.

So when x = 0.01, taking the first three terms of Maclaurin's series, we get

$$(1-0.01)^{5/2} \approx 1 - \frac{5}{2}(0.01) + \frac{15}{8}(0.001)$$

That is, $(0.99)^{5/2} = 0.975$.

The error is the difference between the actual value and the approximate value.

Example 11: We know that $\cos\frac{\pi}{6} = \sqrt{3}/2$. If the first two non-zero terms of Maclaurin's series for cos x are taken to approximate it, let us calculate the error, rounded off to two decimal places.

Maclaurin's series for cos x is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\dots\dots\dots$$

If we take the first two terms alone and put x = π/6, then

$$\cos\frac{\pi}{6} = 1 - \frac{(\pi/6)^2}{2} = 1 - \frac{1}{2}\left(\frac{3.142}{6}\right)^2$$

$$= 1 - 0.274/2 = 0.863$$

The actual value is $\cos\frac{\pi}{6} = \sqrt{3}/2$. We know that $\sqrt{3}/2 = 0.866$ when rounded off to three decimal places.

We have found that $1 - \frac{(\pi/6)^2}{2} = 0.863$ when rounded off to three decimal places.

The error is $0.866 - 0.863 = 0.003$.

When rounded off to two decimal places, the error is 0.00. (This means that the error is so negligible that there is no error at all when rounded off to two decimal places.)

See if you can find the approximate values in the following exercises.

E E9) Find the approximate value of $\sin 31°$ by taking the first two non-zero terms of its Maclaurin's series.

E E10) If the first three non-zero terms of Maclaurin's series for $\cos x$ are used to approximate $\cos \frac{\pi}{2}$, show that the error is less than 1/50.

E E11) Find the value of $\cos 59°$, rounded off to one decimal place.

E 12) Find $(1.01)^{1/2}$ upto two decimal places.

That brings us to the end of this unit. Let us summarise what we have studied in it.

1.5 SUMMARY

In this unit we have studied the following results.

1)

If f is	then f ′ is
increasing	non-negative
decreasing	non-positive
constant	identically zero
monotonic	of same sign throughout

2)

If f ′ is	then f is
non-negative	increasing
non-positive	decreasing
(strictly) positive	strictly increasing
(strictly) negative	strictly decreasing
identically zero	constant
of same sign throughout	monotonic

3) Differentiation can be used

- to test whether a function is monotonic or not,
- to prove some inequalities,
- and to find some approximate values.

1.6 SOLUTIONS AND ANSWERS

E 1) Let f be a decreasing function. If $x \in 1$, then $f'(x)$ exists, and

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$h > 0 \Rightarrow f(x+h) \leq f(x) \Rightarrow f(x+h) - f(x) \leq 0$$

$$h < 0 \Rightarrow f(x+h) \geq f(x) \Rightarrow f(x+h) - f(x) \geq 0.$$

So, $\dfrac{f(x+h) - f(x)}{h} \leq 0 \ \forall \ h \neq 0$

Hence, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} \leq 0.$

Now, let $f'(x) \leq 0$ in 1, and $a, b \in I$ s.t. $a < b$. Then $\exists \ c \in I$ s.t.

$$\frac{f(b) - f(a)}{b - a} = f'(c) \leq 0.$$

$b-a>0 \Rightarrow f(b)-f(a) \le 0$ or $f(b) \le f(a)$.

$\Rightarrow f$ is a decreasing function.

E2) Let $f(x)=e^x, f'(x)=e^x \ge 0 \ \forall\ x \in \mathbf{R}$.

$\Rightarrow f(x)=e^x$ is an increasing function on $\mathbf{R}$ (by Theorem 1).

E3) If $f(x)=\ln x, f'(x)=\frac{1}{x}$. Now $\frac{1}{x}>0 \ \forall\ x \in]0,\infty[$.

Hence $\ln x$ is an increasing function on $]0,\infty[$.

E4) If $f(x)=\sin x$, then $f'(x)=\cos x$.

$\cos x \ge 0 \ \forall\ x \in [0,\pi/2]$.

Hence $\sin x$ is an increasing function on $[0,\pi/2]$. Similarly for $\cos x$.

E5) a) $f'(x)=2x \ge 0$ on $[0,2]$: f is increasing on $[0,2]$.

b) $f'(x)=4x+3 \ge 0$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$: f is increasing on $\left[-\frac{1}{2},\frac{1}{2}\right]$

c) $f'(x)=-e^{-x} \le 0$ on $[0,1] \Rightarrow f$ is decreasing on $[0,1]$

d) increasing

e) increasing

f) $f'(x)=\sec^2 x-\operatorname{cosec}^2 x=\frac{-\cos 2x}{\sin^2 x \cos^2 x} \le 0$ since $\cos 2x \ge 0. \ \forall\ x \in [0,\pi/4]$.

$\therefore$ f is decreasing.

E6) a) $f'(x)=4x+3$

$f'(x) \le 0$ if $x \in [-1,-3/4]$ and $f'(x) \ge 0$ if $x \in [-3/4,0]$.

b) $f'(x)=3x^2-1$. $f'(x) \le 0$ if $x \in \left[0,\frac{1}{\sqrt{3}}\right]$

and $f'(x) \ge 0$ if $x \in \left[\frac{1}{\sqrt{3}},2\right]$

c) $f'(x)=\sin x+x\cos x \begin{cases} \ge 0 \text{ for } x \in [0,\pi/2] \\ \le 0 \text{ for } x \in [3\pi/4,\pi] \end{cases}$

d) similar argument

E7) Let $f(x)=ax^3+bx^2+cx+d$
$f'(x)=3ax^2+2bx+c$, where a, b and c are to be determined so that the desired cubic polynomial is obtained.

f has extreme at $x=2$ and $x=3$.

$f'(2)=0 \Rightarrow 12a+4b+c=0$

and $f'(3)=0 \Rightarrow 27a+6b+c=0$

$\therefore 15a+2b=0$

or $b=\frac{-15a}{2}$. $\quad \therefore c=18a$

Suppose $a=-2$ and $d=0$.

Then $f(x)=-2x^3+15x^2-36x$ satisfies the given conditions.

E8) a) $f(x)=x-\ln(1+x)$. $f(x)=1-\frac{1}{1+x} \ge 0$ for all $x>0$.
$\therefore\ x-\ln(1+x)$ is increasing on $]0,\infty[$
$\Rightarrow \ln(1+x)<x \ \forall\ x \in]0,\infty[$.

b) $f(x) = \tan^{-1} x$.

Let $x > 0$. By the mean value theorem $\exists\, y \in\,]0, x[$ s.t.

$$\frac{\tan^{-1} x - \tan^{-1} 0}{x - 0} = f'(y)$$

or $\frac{\tan^{-1} x}{x} = \frac{1}{1+y^2} < 1$ for $y \in\,]0, x[$.

$\tan^{-1} x < x \;\forall\; x > 0$.

c) $(e^x + e^{-x} - 2) = (e^{x/2} + e^{-x/2})^2 > 0 \;\forall\; x$.

d) $e^x \geq 1 + x$ and $e^{-x} = e^y \geq 1 + y = 1 - x$ (if $y = -x$)
$\therefore e^x + e^{-x} \geq 1 + x + 1 - x = 2$.

e) Let $f(x) = e^x - e^{-x} - 2x$. Then $f'(x) = e^x + e^{-x} - 2 \geq 0$.
$\Rightarrow f$ is an increasing function
$\Rightarrow f(x) \geq f(0)$ for all $x > 0$
$\Rightarrow e^x - e^{-x} - 2x \geq 0$ for all $x > 0$.
$\Rightarrow e^x - e^{-x} \geq 2x$ for all $x > 0$.

E9) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$$31^\circ = \frac{31\pi}{180} \text{ radians.}$$

$$\therefore \sin \frac{31\pi}{180} \cong \frac{31\pi}{180} - \left(\frac{31\pi}{189}\right)^3 \frac{1}{6}$$

$$= \frac{31 \times 3.142}{180} - \left(\frac{31 \times 3.142}{180}\right)^3 \cdot \frac{1}{6}$$

$$= .5411 - (.5411)^3 \frac{1}{6}$$

$= .5411 - .0264$
$= .5147.$

E 10) If $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$
$f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$

$$f(x) = \cos x \cong f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0)$$

$$\cos \frac{\pi}{2} \cong \cos 0 + \frac{\pi}{2} \times 0 - \left(\frac{\pi}{2}\right)^2 \frac{1}{2!} \times 1 + \left(\frac{\pi}{2}\right)^3 \times \frac{1}{3!} \times 0 + \left(\frac{\pi}{2}\right)^4 \frac{1}{4!} \times 1$$

$$\cong 1 - \frac{\pi^2}{8} + \frac{\pi^4}{16 \times 24}$$

$$\cong 1 - \frac{(3.142)^2}{8} + \frac{(3.142)^4}{16 \times 24}$$

$\cong 1 - 1.234 + 0.235$
$= 0.001$

We know that $\cos \frac{\pi}{2} = 0$

$\therefore$ Error $= 0.001 < \frac{1}{50}$

E11) $\cos \frac{59\pi}{180} \cong 1 - \left(\frac{59\pi}{180}\right)^2 \frac{1}{2!} \times 1$

$$= 1-\left(\frac{3.142 \times 59}{180}\right)^2 \frac{1}{2!}$$

$$\cong 1-0.530$$

$$= 0.470$$

$$\cong 0.5$$

E12) $(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \dots$

$$\therefore\ (1.01)^{1/2} = (1+0.01)^{1/2} \cong 1 + \frac{1}{2}(0.01) + \frac{\frac{1}{2}\cdot\frac{-1}{2}}{2!}(0.01)^2$$

$$= 1 + 0.005 - 0.00001 \cong 1.00$$