UNIT 1 DEFINITE INTEGRAL

Structure

1.1 INTRODUCTION

We have seen in Unit 3 of Block 1 that one of the problems which motivated the concept of a derivative was a geometrical one—that of finding a tangent to a curve at a point. The concept of integration was also similarly motivated by a geometrical problem— that of finding the areas of plane regions enclosed by curves. Some recently discovered Egyptian manuscripts reveal that the formulas for finding the areas of triangles and rectangles were known even in 1800 B.C. Using these one could also find the area of any figure bounded by straight line segments. But no method for finding the area of figures bounded by curves had evolved till much later.

In the third century B.C. Archimedes was successful in rigorously proving the formula for the area of a circle. His solution contained the seeds of the present day integral calculus. But it was only later, in the seventeenth century, that Newton and Leibniz were able to generalise Archimedes' method and also to establish the link between differential and integral calculus. The definition of the definite integral of a function, which we shall give in this unit was first given by Riemann in 1854. In Unit 2 of this block, we will acquaint you with various methods of integration.

You have probably studies integration before. But in this unit we shall adopt a new approach towards integration. When you have finished the unit, you should be able to tie in our treatment with your previous knowledge.

Objectives

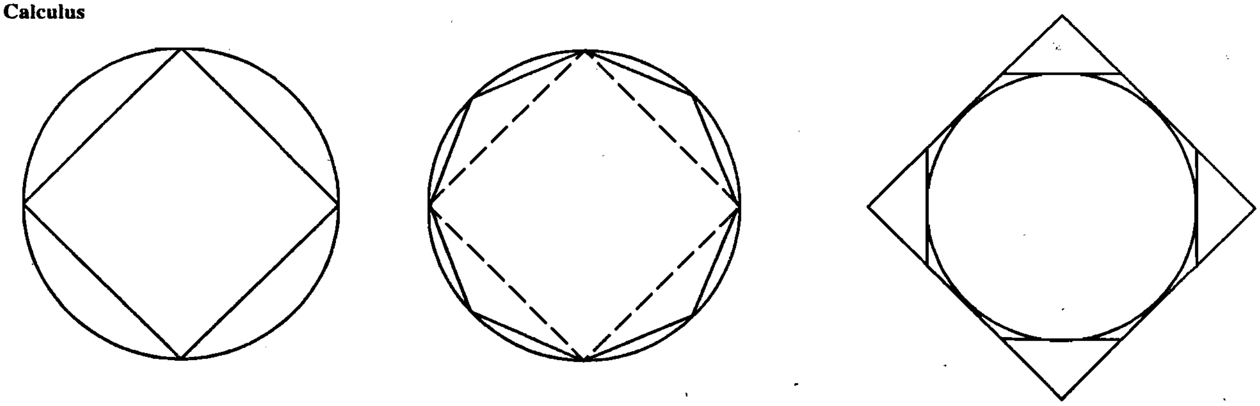
After reading this unit you should be able to :

- define and calculate the lower and upper sums of some simple functions defined on [a,b], corresponding to a partition of [a,b],
- define the upper and lower integrals of a function,
- define the definite integral of a given function and check whether a given function is integrable or not,
- state and prove the Fundamental Theorem of Calculus,
- use the Fundamental Theorem to calculate the definite integral of an integrable function.

1.2 PRELIMINARIES

We have mentioned in the introduction that Archimedes was able to find the formula for the area of a circle. For this he approximated a circle by an inscribed regular polygon (See Fig. 1 (a)).

Further, we can see from Fig. 1(b) that this approximation becomes better and better as we increase the number of sides of the polygon. Archimedes also tried to approximate the area of the circle by a number of circumscribed polygons as in Fig. 1(c). The area of the circle was thus compressed between the inscribed and the circumscribed polygons.



(a) (b) **Fig. 1** (c)

We shall follow a similar procedure for finding the area of the shaded region shown in Fig. 2. We begin with the concept of a partition.

1.2.1 Partition of a Closed Interval

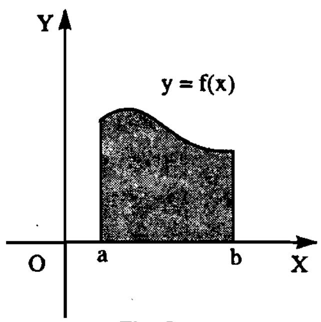


Fig. 2

Let us consider the closed interval $[a, b] \subset \mathbf{R}$. Then we have the following definition.

Defintion 1 Let $x_0, x_1, x_2 \ldots\ldots, x_{n-1}, x_n$ be numbers in [a,b] such that

$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$

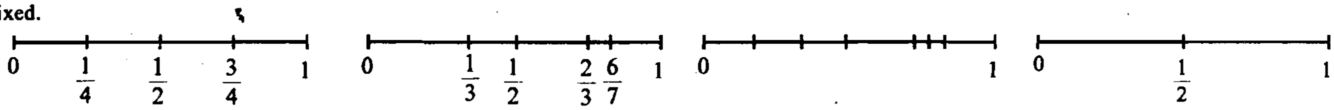
Then the ordered set $P = \{x_0, x_1\ x_2 \ldots\ldots\ldots\ldots\ x_n\}$ is called a **partition** of [a,b]

By an **ordered** set we mean a set, in which, the order in which its elements occur is fixed.

Example 1 $P_1 = \{0, 1/4, 1/2, 3/4, 1\}$ and $= P_2\{0, 1/3, 1/2, 2/3, 6/7, 1\}$, both are partitions of [0, 1].

Moreover,

$P_1 \cup P_2 = \{0, 1/4, 1/3, 1/2, 2/3, 3/4, 6/7, 1\}$ and $= P_1 \cap P_2 = \{0, 1/2, 1\}$ are also partitions of [0,1]. See Fig. 3 (a), (b), (c) and (d).



(a) (b) (c) (d)

Fig. 3

A partition of $P = \{x_0, x_1, x_2 \ldots\ldots\ldots\ldots x_n\}$ of [a,b] divides [a,b] into n closed sub-intervals, $[x_0, x_1]$, $[x_1, x_2], \ldots\ldots\ldots\ldots, [x_{n-1}, x_n]$,

A set J is called a sub-interval of an interval I, if
i) J is an interval, and
ii) $J \subseteq I$

with the n + 1 partitioning points as end-points. The interval $[x_{i-1}, x_i]$ is called the **ith sub-interval** of the partition. The length of the ith sub-interval, denoted by Δx_i, is defined by

$\Delta x_i = x_i - x_{i-1}.$

'Δ' is read as delta.

It follows that

$$\sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = b - a$$

We call partition P **regular** if every sub-interval has the same length, that is, if $x_1 - x_0$, $x_2 - x_1, \ldots\ldots\ldots, x_n - x_{n-1}$, are all equal. In this case, the length of [a, b], that is b − a, is equally divided into n parts and we get

$$x_1 - x_0 = x_2 - x_1, \ldots\ldots = x_n - x_{n-1} = \frac{b-a}{n}$$

Thus a regular partition of [a, b] may be written as

$\{a, a + h, a + 2h, \ldots\ldots\ldots\ldots\ldots, a + nh\}$, where a + nh = b: We shall denote this partition by $\{a + ih\}^{n}_{i=0}$.

For $P = \{1, 3/2, 2, 5/2, 3, 7/2, 4\}$, $\Delta x_1 = x_1 - x_0 = 3/2 - 1 = 1/2$, $\Delta x_2 = x_2 - x_1 = 2 - 3/2 = 1/2$. If you calculate $\Delta x_3, \Delta x_4, \Delta x_5$ and Δx_6, you will see that P is a regular partition of [1,4].

E E1) See Example 1. Which partitions among P_1, P_2, $P_1 \cup P_2$ and $P_1 \cap P_2$ are regular ? What are the lengths of the third sub-intervals in P_1 and in P_2 ?

E E2) Write down a regular partition for each of the following intervals

a) [0, 2] with 7 partitioning points.

b) [2, 9] with 11 partitioning points.

Definition 2 Given two partition P_1 and P_2 of [a,b], we say that P_2 is a refinement of P_1 (or P_2 is finer than P_1) if $P_2 \supset P_1$.

In other words, P_2 is a refinement of P_1 if each sub-interval of P_2 is contained in some sub-interval of P_1.

Example 2 Consider the partitions

$P_1 = \{1, 5/4, 3/2, 7/4, 2\}$,
$P_2 = \{1, 6/5, 5/4, 3/2, 19/10, 2\}$,
$P_3 = \{1, 5/4, 3/2, 2\}$

P_1 and P_2 are both finer than P_3, as $P_1 \supset P_3$ and $P_2 \supset P_3$. However, neither is P_1 a refinement of P_2 nor is P_2 a refinement of P_1.

If P_1 and P_2 are partitions of [a,b], then from Definition 2 it follows that

i) $P_1 \cup P_2$ is a refinement of both P_1 and P_2.

ii) P_1 and P_2 are both finer than $P_1 \cap P_2$.

Now, suppose for every $n \in N$ we define P_n as

$$P_n = \left\{a + i\,\frac{b-a}{2^n} : 0 \le i \le 2^n\right\}$$

This means P_n has $2^n + 1$ elements. We can see that P_n is a regular partition, with each sub-interval having length

$$\Delta x_i = x_i - x_{i-1} = a + i\,\frac{b-a}{2^n} - \left\{a + (i-1)\,\frac{b-a}{2^n}\right\} = \frac{b-a}{2^n}$$

Now $\dfrac{b-a}{2^{n+1}} = \dfrac{1}{2}\left(\dfrac{b-a}{2^n}\right)$

This means that the length of the sub-intervals corresponding to P_{n+1} is half the length of those corresponding to P_n. We can also see that $P_{n+1} \supset P_n$. In other words, P_{n+1} is finer than P_n (also see E3)). Thus we have defined a sequence of partitions $\{P_n\}$ of [a,b], such that P_{n+1} is a refinement of P_n for all n. Such a sequence $\{P_n\}$ is called a **sequence of refinements of partitions of [a,b].**

E E3) From the sequence of partitions $\{P_n\}$ defined above,

$$P_1 = \{a, \frac{a+b}{2}, b\}$$

a) Find P_2 and P_3.

b) Verify that $P_3 \supset P_2 \supset P_1$.

c). What are the lengths of the sub-intervals in each of these partitions?

E4) Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of partitions of [a,b], and let $P_1^* = P_1$, $P_2^* = P_1 \cup P_2$, $P_3^* = P_1 \cup P_2 \cup P_3$, and in general, $P_n^* = P_n \cup P_{n-1}^*$. Show that $\{P_n^*\}_{n=1}^{\infty}$ is a sequence of refinements of [a,b]

1.2.2 Upper and Lower Product Sums

By now, we suppose you are quite familiar with partitions. Here we shall introduce the concept of product sums. It is through this that we shall be in a position to probe the more subtle concept of a definite integral in the next section.

Let $f: [a,b] \to \mathbf{R}$ be a bounded function, and let

$P = \{x_0, x_1, x_2, \ldots\ldots. x_n\}$ be a partition of [a, b].

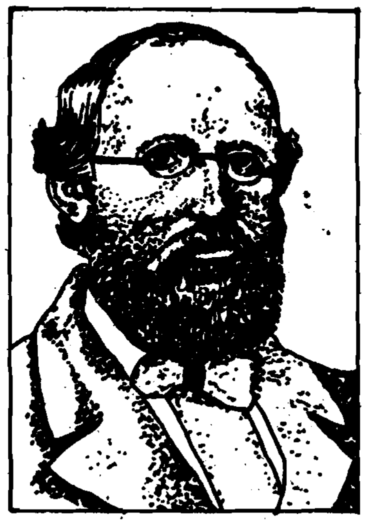
we have defined l.u.b. (supremum) and g.l.b. (infimum of a bounded set of real numbers in Unit 1.

Now for any sub-interval $[x_{i-1}, x_i]$, consider the set $S_i = \{f(x) : x \in [x_{i-1}, x_i]\}$.

Since f is a bounded function, S_i must be a bounded subset of **R**. This means, it has a supremum (or least upper bound) and infimum (or greatest lower bound). We write

$M_i = \sup S_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\}$, and

$m_i = \inf S_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\}$,



G.F.B. Riemann (1826-1866)

We now define the **upper product sum U (P,f) and the lower product sum L (P, f) by**

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i, \quad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i, \qquad \text{....(1)}$$

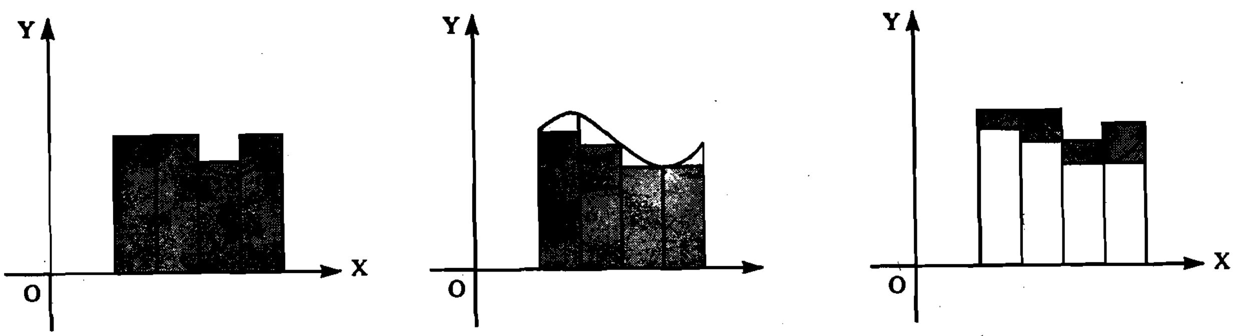
You must have come across this $\sum$ notation earlier. But let us state clearly what (1) means :

$U(P,f) = M_1 (x_1 - x_0) + M_2 (x_2 - x_1) + \ldots\ldots\ldots + M_n (x_n - x_{n-1})$, and

$L(P,f) = m_1 (x_1 - x_0) + m_2 (x_2 - x_1) + \ldots\ldots\ldots + m_n (x_n - x_{n-1})$.

Thus, to get U(p,f) we have multiplied the supremum in each sub-interval by the length of that sub-interval, and have taken the sum of all such products. Similarly, L(P,f) is obtained by summing the products obtained by multiplying the infimum in each sub-interval by the length of that sub-interval. U(P,f) and L(P,f) are also called **Riemann sums** after the mathematician George Friedrich Bernhard Riemann.

Riemann gave a definition of definite integral that, to this day, remains the most convenient



(a) (b) (c)

Fig.

and useful one.

We started this unit saying that we wanted to find the area of the shaded region in Fig. 2. Then what are we doing with partitions, U(P,f) and L(P,f)? Fig. 4 will give you a clue to the path which we are going to follow to achieve our aim.

Fig. 4 (a) and 4(b) give the geometric view of $M_i \Delta x_i$ and $m_i \Delta x_i$ as areas of rectangles with base Δx_i and heights M_i and m_i respectively.

The shaded rectangles in Fig 4(a) are termed as outer rectangles, while the shaded rectangles in Fig. 4(b) are called inner rectangles.

Thus, when f is a non-negative valued function ($f(x) \geq 0 \quad \forall x$),

U(P,f) = sum of the areas of outer rectangles as in Fig. 4(a).

L(P,f) = sum of the areas of inner rectangles as in Fig. 4(b), and

U(P,f) – L(P,f) = difference of the areas of the shaded rectangles along the graph of f as shown in Fig. 4(c).

As you see from Fig. 5, U(P,f) and L(P,f) depend upon the function

$f: \{a,b] \to \mathbf{R}$ (compare Fig. 5(a) and (b)), and the partition P of [a,b] (compare Fig. 5(c) and (d)).

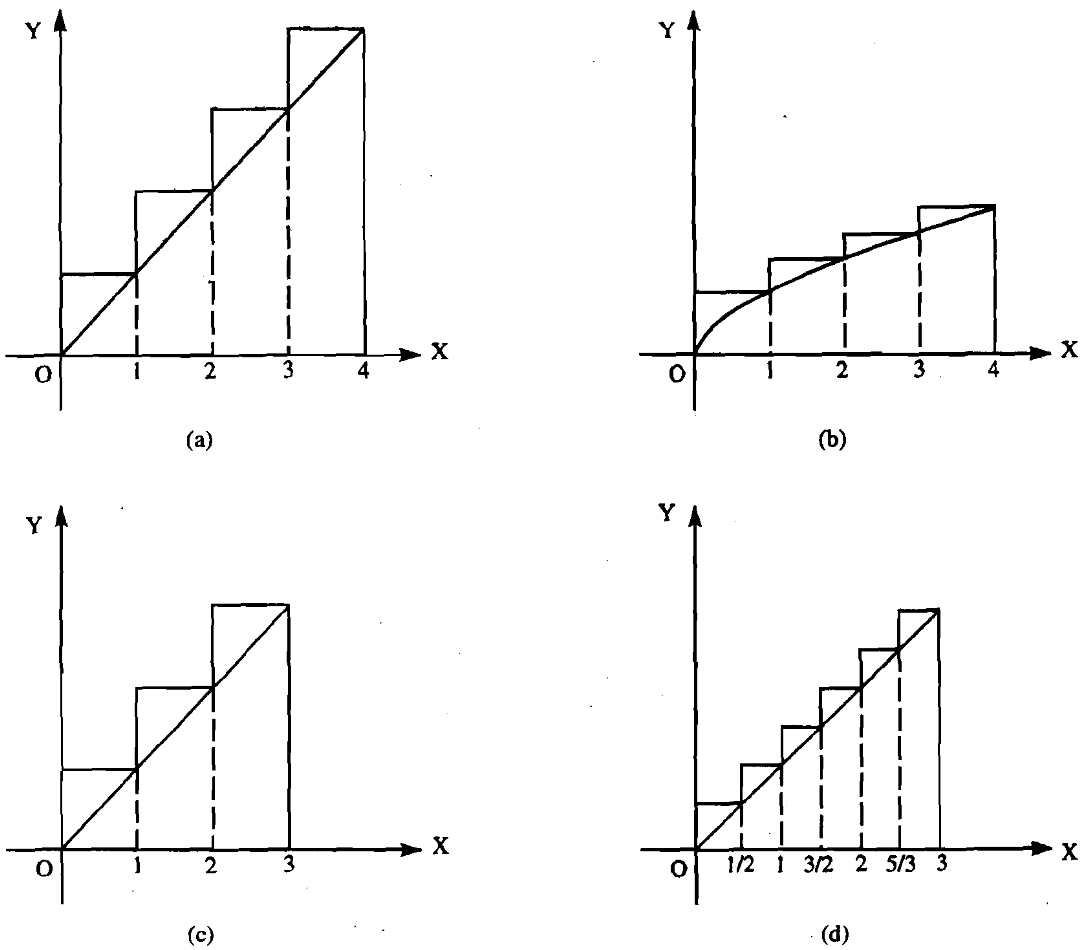


Fig. 5: (a) U(P,f) where y = x (b) U(P,f) where $y^2 = x$
(c) U(P,f) when P = {0, 1, 2, 3} (d) U(P,f) when P = {0, 1/2, 1, 3/2, 2, 5/2, 3}

If we denote the area between the curve given by y = f(x), the x-axis and the lines x = a and x = b, (the shaded area in Fig. 2) by A, then it is also quite clear from Fig. 4(a) and (b), that $L(P,f) \leq A \leq U(P,f)$.

The geometric view suggests the following theorem :

If $X \subset Y$, then $\sup X \leq \sup Y$ and $\inf X \geq \inf Y$.

Theorem 1 Let f: [a,b] R be a bounded function, and let P be a partition of [a,b]. If M and m are the supremum and the infimum of f, respectively in [a,b], then

$m(b-a) \le L(P,F) \le U(P,f) \le M(b-a)$.

Proof : Now $M = \text{Sup}\{f(x) : x \in [a,b]\}$, and

$M_i = \sup\{f(x): x \in [x_{i-1}, x_i]\}$. Hence $M_i \le M$.

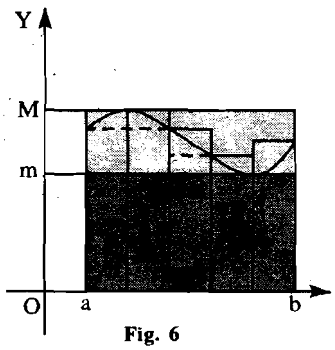


Fig. 6

Further, $m = \inf\{f(x) : x \in [a,b]\}$, and
$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. Thus, $m \le m_i$. This means
$m \le m_i \le M_i \le M$ ...... (2)

Once we have the inequalities (2), we can complete our proof in easy steps. (2) implies that

$m\,\Delta x_i \le m_i\,\Delta x_i \le M_i\,\Delta x_i \le M\,\Delta x_i$

This implies that if we take the sum over i = 1, 2..........., n, we get

$m\sum_{i=1}^{n} \Delta x_i \le L(P,f) \le U(P,f) \le M\sum_{i=1}^{n} \Delta x_i$

$\Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$,

since $m \sum_{i=1}^{n} \Delta x_i =$ the sum of the lengths of all sub-intervals

$=$ the length of [a,b]

$= b-a$.

Fig. 6 will help you understand this theorem better. Let us verify this theorem in the case of a given function.

Example 3 Let $f:[1,2] \to \mathbf{R}$ be a function defined by $f(x) = x^2$, and let

$P = \{1, 5/4, 3/2, 5/3, 2\}$ be a partition of [1,2]. The sub-intervals associated with partition P are [1, 5/4], [5/4, 3/2], [3/2, 5/3] and [5/3,2].

The function f is a bounded function on [1,2]. In fact, the image set of f is [1,4], which is obviously bounded.

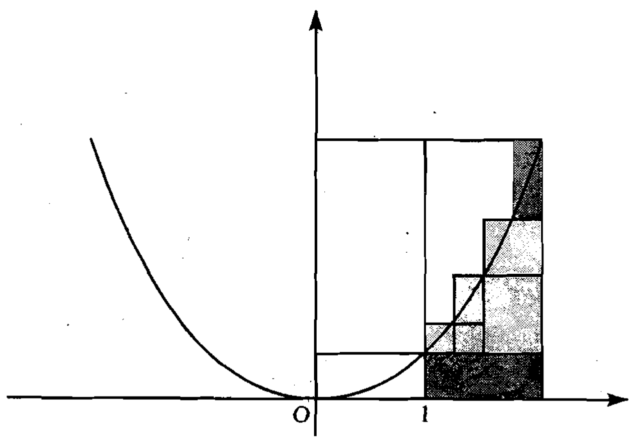


Fig. 7

Since f is an increasing function on each sub-interval (see Fig. 7) the supremum of f in $[x_{i-1}, x_i]$ will be attained at x_i and the infimum will be attained at x_{i-1} That is,

$$M_i = f(x_i) \text{ and}$$

$$m_i = f(x_{i-1}). \text{ Therefore, we can write}$$

$$U(P,f) = \Sigma M_i \Delta x_i = \Sigma f(x_i)\Delta x_i = \Sigma x_i^2 (x_i - x_{i-1})$$

$$= x_1^2(x_1 - x_0) + x_2^2(x_2 - x_1) + x_3^2(x_3 - x_2) + x_4^2(x_4 - x_3)$$

$$\therefore\ U(P,f) = \left(\frac{5}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{3}{2}\right)^2\left(\frac{1}{4}\right) + \left(\frac{5}{3}\right)^2\left(\frac{1}{6}\right) + (2)^2\left(\frac{1}{3}\right)$$

$$= \frac{25}{64} + \frac{9}{16} + \frac{25}{54} + \frac{4}{3}$$

$$= \frac{4751}{1728}$$

$$L(P.f) = \Sigma m_i \Delta x_i = \Sigma f(x_{i-1}) \Delta x_i.$$

$$L(P,f) = (1)^2 \left(\frac{1}{4}\right) + \left(\frac{5}{4}\right)^2 \left(\frac{1}{4}\right) + \left(\frac{3}{2}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{5}{3}\right)^2 \left(\frac{1}{3}\right)$$

$$= \frac{1}{4} + \frac{25}{54} + \frac{9}{24} + \frac{25}{27}$$

$$= \frac{3652}{1728}$$

Now, the supremum of f(x) in $[1,2] = M = f(2) = 2^2 = 4$, and
the infimum $= m = f(1) = 1$. Thus,
$M(b-a) = (4)(2-1) = 4$, and $m(b-a) = (1)(2-1) = 1$. Thus,
$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$.

We have noted that the upper and lower product sums depend on the partition of the given interval. Here we have a theorem which gives us a relation between the lower and upper sums corresponding to two partitions of an interval.

Theorem 2. Let $f:[a,b] \to \mathbf{R}$ be a bounded function and let P_1 and P_2 be partitions of $[a,b]$. If P_2 is finer than P_1, then

$L(P_1,f) \leq L(P_2,f) \leq U(P_2,f) \leq U(P_1,f)$.

Proof : For proving this theorem we look at Fig. 8(a) and (b).

Let $P_1 = \{x_0, x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$ and $P_2 = \{x_0, v_1, x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$ be two partitions of $[a,b]$.

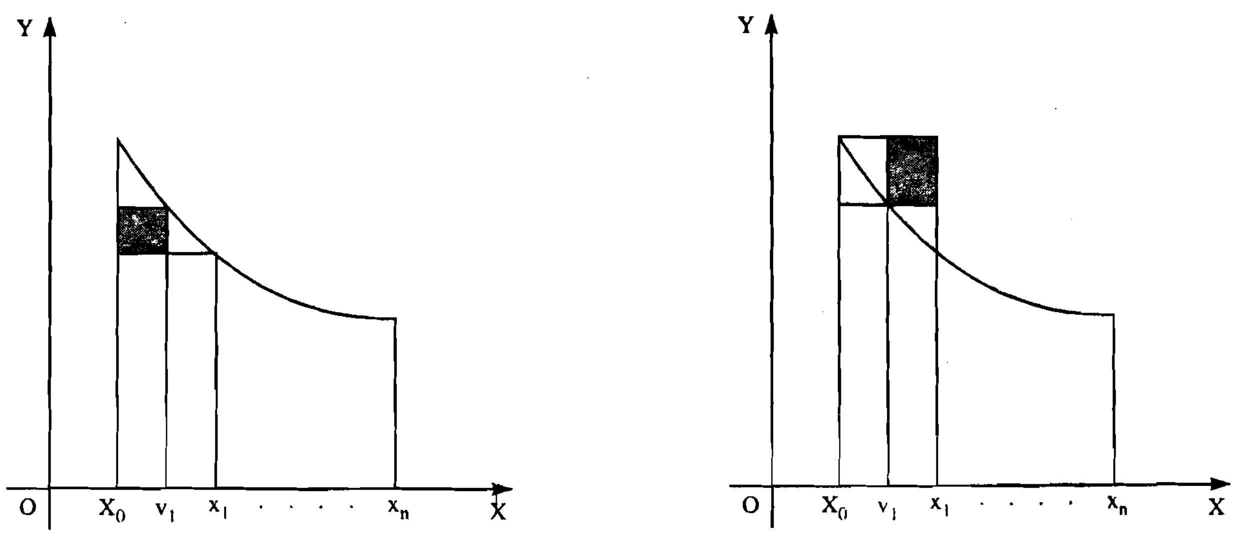
P_2 contains one element more than P_1, Namely v_1.

Therefore, P_2 is finer than P_1.

In fact, P_2 can be rightly called a simple refinement of P_1. We shall prove the theorem for this simple refinement here.

P_1 divides $[a,b]$ into n sub-intervals :

$[x_0, x_1], [x_1, x_2], \ldots\ldots\ldots\ldots [x_{n-1}, x_n]$.



(a) (b)

Fig. 8

Fig. 8(a) clearly shows that $L(P_1,f) \leq L(P_2,f)$ (by an amount represented by the area of the shaded rectangle).

Similarly, Fig 8(b) shows that $U(P_2,f) \leq U(P_1,f)$.

Since $L(P_2,f) \leq U(P_2,f)$, the conclusion of the theorem follows in this case.

If P_2 is not a simple refinement of P_1, and P_2 has m elements more than P_1. Then we can find

$(m-1)$ partitions $P_2^1, P_2^2, P_2^3 \ldots\ldots\ldots\ldots P_2^{m-1}$

such that

$P_1 \subset P_2^1 \subset P_2^2 \subset P_2^3 \subset \ldots\ldots\ldots\ldots \subset P_2^{m-1} \subset P_2$ and each partition in this sequence is a simple refinement of the previous one.

Theorem 2 then holds for each pair of successive refinements and we get

$L(P_1,f) \le L(P_2^1,f) \le L(P_2^2,f) \le \ldots\ldots\ldots\ldots\ldots\ldots \le L(P_2^{m-1},f) \le L(P_2,f)$ and

$U(P_2,f) \le U(P_2^{m-1},f) \le \ldots\ldots\ldots\ldots\ldots \le U(P_2^2,f) \le U(P_2^1,f) \le U(P_1,f)$

Thus, $L(P_1,f) \le L(P_2,f) \le U(P_2,f) \le U(P_1,f)$.

From Theorem 2 we conclude the following :

Let $f : [a,b] \to \mathbf{R}$ be a continuous and non-negative valued function, and let

$\{P_n\}_{n=1}^{\infty}$ be a sequence of refinements of $[a,b]$.

Then we have

$L(P_1,f) \le L(P_2,f) \le \ldots\ldots\ldots \le L(P_n,f) \le \ldots\ldots\ldots \le A \le \ldots\ldots\ldots \le U(P_n,f) \le \ldots\ldots\ldots \le U(P_2,f) \le U(P_1,f)$
where A is the area bounded by the curve, the x-axis and the lines $x = a$ and $x = b$.

E E5) Find the upper product sum and the lower product sum of the function f relative to the partition P, when

a) $f(x) = 1 + x^2$, $P = \{0, 1/2, 1, 3/2, 2\}$

b) $f(x) = 1/x$, $P = \{1, 2, 3, 4\}$

E E6) Verify Theorem 2 for the function $f(x) = 1/x$, $2 \le x \le 3$, and the partitions

$P_1 = \{2, 5/2, 3\}$ and $P_2 = \{2, 9/4, 5/2, 11/4, 3\}$ of $[2,3]$

In this sub-section we have seen that the area A in Fig. 2 can be approximated by means of the lower and upper sums corresponding to some partition of [a,b], further. Theorem 2 tells us that as we go on refining our partition, the lower and upper sums approach A from both sides. The lower sums underestimate A ($L(P,f) \le A$), while the upper sums overestimate A, i.e., $U(P,f) \ge A$. Let us go a step further in the next sub-section, and define lower and upper integrals.

1.2.3 Upper and Lower Integrals

Let $f:[a,b] \to \mathbf{R}$ be a non negative bounded function. Then to each partition P of [a,b], there corresponds the upper product sum U(P,f) and the lower product sum L(P,f).

Let $\boldsymbol{P}$ be the set of all partitions of [a,b], Then the set $\boldsymbol{u} = \{U(P,f) : P \in \boldsymbol{P}\}$ is a subset of $\mathbf{R}$ and is bounded below since $A \le U(P,f)\ \forall\ P \in \boldsymbol{P}$. Thus, it is possible to find the infimum of $\boldsymbol{u}$.

Recall (Unit 1) that every set which is bounded below has an infimum, and every set which is bounded above has a supremum.

Similarly the set $\boldsymbol{u}' = \{L(P,f) : P \in \boldsymbol{P}\}$ is bounded above, since $L(P,f) \le A\ \forall\ P \in \boldsymbol{P}$.

Hence we can find the supremum of $\boldsymbol{u}'$. The infimum of $\boldsymbol{u}$ and the supremum of $\boldsymbol{u}'$ are given special names as you will see from this definition.

Definition 3 If a function f is defined on [a,b] and if P denotes the set of all partitions of [a,b] then infimum of $\{U(p,f) : P \in \boldsymbol{P}\}$ is called the upper integral of f

on [a,b], and is denoted by $\overline{\int_a^b} f(x)dx$.

The symbol '$\int$' is read as integral.

The supremum of $\{L(P,f) : P \in \boldsymbol{P}\}$ is called the lower integral of f on [a,b], and is

denoted by $\underline{\int_a^b} f(x)dx$.

From Theorem 2 it follows that $\overline{\int_a^b} f(x)dx \ge A$ and $\underline{\int_a^b} f(x)dx \le A$.

Thus we have $\underline{\int_a^b} f(x)dx \le A \le \overline{\int_a^b} f(x)dx$.

Example 4. Let us find $\overline{\int_a^1} f(x)dx$ and $\underline{\int_0^1} f(x)dx$.

for the function f, defined by $f(x) \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational} \end{cases}$

Suppose $P = \{x_0, x_1, x_2, \ldots\ldots\ldots\ldots, x_n\}$ is a partition of [0,1].

Each sub-interval $[x_{i-1}, x_i]$ contains both rational and irrational numbers, this means, $M_i = 1$ and $m_i = 0$ for each i.

Thus,

$$U(P,f) = \sum_{i=1}^{n} M_i\ \Delta x_i = \sum_{i=1}^{n} (1)\ (x_i - x_{i-1}) = 1 - 0 = 1$$

and

$$L(P,f) = \sum_{i=1}^{n} m_i\ \Delta x_i = \sum_{i=1}^{n} (0)\ (x_i - x_{i-1}) = 0$$

Since P was any arbitrary partition of [0,1], this means that

$U(P,f) = 1$ and $L(P,f) = 0\ \forall\ P \in \boldsymbol{P}$.

Thus, $\boldsymbol{u} = \{U(P,f) : P \in \boldsymbol{P}\} = \{1\}$

and $\boldsymbol{u}' = \{L(P,f) : P \in \boldsymbol{P}\} = \{0\}$

Hence inf $\boldsymbol{u} = 1$ and sup $\boldsymbol{u}' = 0$. That is,

$$\overline{\int_0^1} f(x)\,dx = 1 \quad \text{and} \quad \underline{\int_0^1} f(x)\,dx = 0.$$

See if you can do these exercise now.

E E7) Find $\overline{\int_0^1} f(x)\,dx$ and $\underline{\int_0^1} f(x)\,dx$, for the function f defined as

$f(x) = 2$.

E E8) If the function f and g are bounded non-negative valued functions in [a,b] and if $f(x) \le g(x)$ in [a,b], prove that $\overline{\int_a^b} f(x)dx \le \overline{\int_a^b} g(x)dx$ and

$\underline{\int_a^b} f(x)\,dx \le \underline{\int_a^b} g(x)\,dx$.

1.3 DEFINITE INTEGRAL

In the last section we had restricted our discussion to non-negative valued functions. But we can easily extend our definition of L(P,f), U(P,f) and the lower and upper integrals to all bounded functions. However we shall have to modify our interpretation of these sums as areas. For this purpose, we introduce the concept of **signed area.** If R is any region, its signed area is defined to be the area of its portion lying above the x-axis, minus the area of its portion lying below the x-axis (see Fig 9).

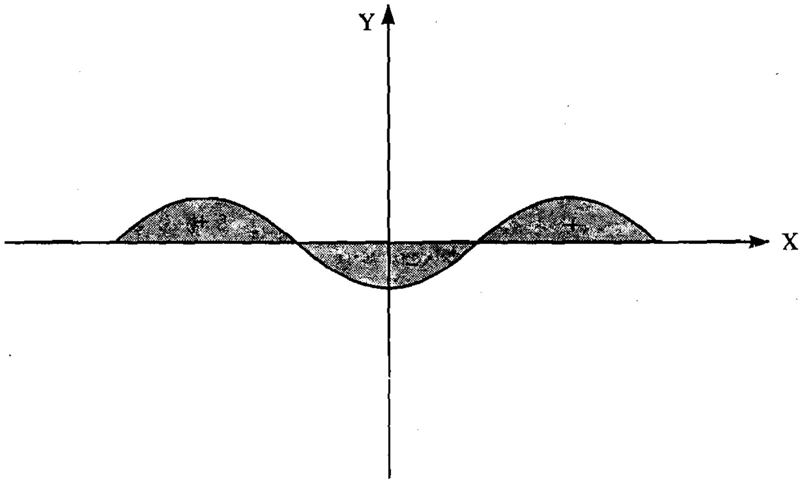


Fig. 9

With this definition then, we can interpret L(P,f) as the signed area of a polygon inscribed inside the given region, and U(P,f) as the signed area of a polygon circumscribed about the region. Thus, for any bounded function on a closed interval [a,b], we can define

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P \in \boldsymbol{P}\} \text{ and}$$

Now we are in a position to discuss the definite integral for a bounded function on a closed interval. (The adjective 'definite' anticipates the study of indefinite integral later).

Definition 4. Let $f:[a,b] \to R$ be a bounded function. f is said to be **integrable** over [a,b] if, and only if,

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

This common value is called the definite integral of f over the interval of integration [a,b], and is denoted by $\int_a^b f(x)dx$.

In this notation for the definite integral, f(x) is called the **integrand,** a is called the **lower limit** and b is called the **upper limit** of integration.

The symbol dx following f(x) indicates the independent variable. Here x is merely a dummy variable, and we may replace it by t or v, or any other letter. This means,

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(v)\,dv.$$

The symbol $\int$ reminds us of S which is appropriate, because a definite integral is in some sense, the limit of a sum. In fact it is the common value (when it exists) of the lower and upper integrals which are themselves infimum and supremum **sums.**

The use of f(x) dx reminds us that we do not take the sum of function values, rather we take the sum of terms, each of which is the product of the supremum or infimum of the function in an interval multiplied by the length of the sub-interval.

The definition of definite integral above, applies only if $a < b$, but it would be appropriate to include the cases $a = b$ and $a > b$ as well. In such cases we define

$$\int_a^b f(x)\,dx = 0$$

and $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Provided the right hand integral exists.

In Example 4, we have seen that if

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational, then} \end{cases}$$

$$\underline{\int_0^1} f(x)dx = 0, \text{ and } \overline{\int_0^1} f(x)\,dx = 1$$

Since the lower and upper integrals for this function are not equal, we conclude that it is not integrable.

E E9) Check whether the function given in E7) is integrable or not.

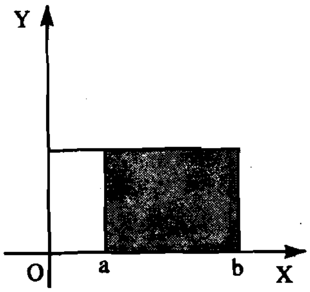


Fig. 10

Now we shall list some basic properties of definite integrals.

I) Integral of a constant function f(x) = c

$$\int_a^b c\,dx = c(b-a)$$

This is intuitively obvious since the area represented by the integral is simply a rectangle with base b – a and height c, (see Fig. 10).

Now let us consider a function f which is integrable over [a,b].

II) Constant Multiple Property

$$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx.$$

III) Interval Union Property

If $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b t(x)\,dx$$

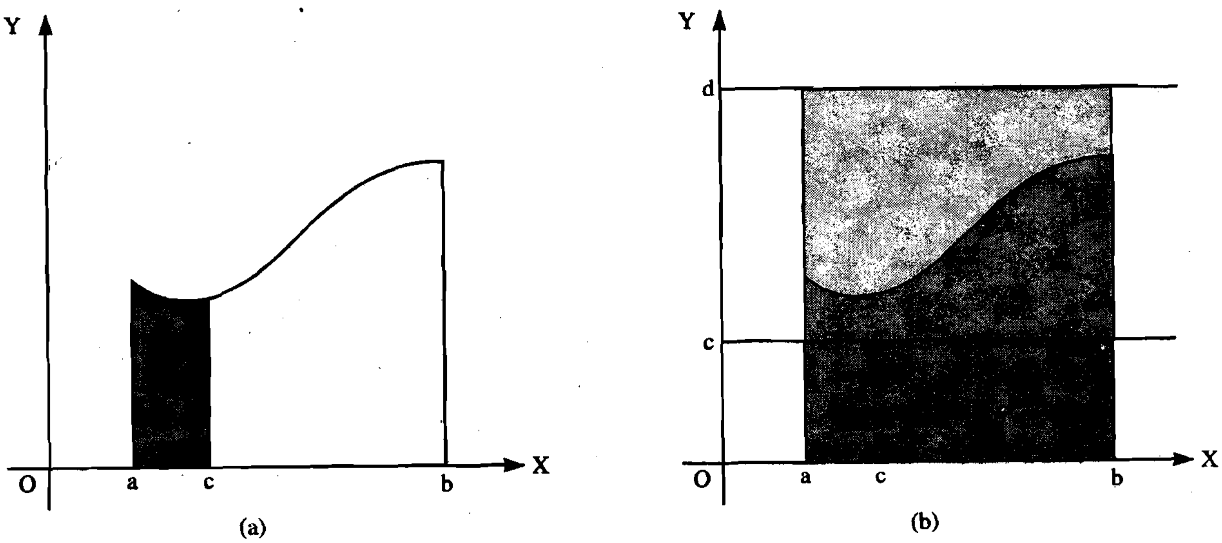


Fig. 11

Its geometrical interpretation is shown in Fig. 11(a).

IV) Comparison Property

If c and d are constants such that $c \leq f(x) \leq d$ for all x in [a,b], then

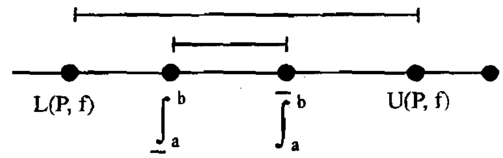
$$c(b-a) \leq \int_a^b f(x)\,dx \leq d(b-a)$$

Fig. 11(b) makes this statement clearer. Note that c and d are not necessarily the minimum and maximum values of f(x) on [a,b]. The value c may be less than the minimum, and d may be greater than the maximum.

The following theorem gives a criterion for a function to be integrable.

Theorem 3 A bounded function f is integrable over [a,b] if and only if, for every $\varepsilon > 0$, there exists a partition P of [a,b] such that $0 \le U(P,f) - L(P,f) < \varepsilon$.

Proof : We know that for any partition P of [a,b],



$$L(P,f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P,f)$$

$$\Rightarrow 0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(P,f) - L(P,f).$$

If the function f has the property that for every $\varepsilon > 0$ there exists a partition P of [a,b] such that $U(P,f) - L(P,f) < \varepsilon$, we conclude that

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon \text{ for every } \varepsilon > 0.$$

From this it follows that $\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0$, and hence f is integrable over [a,b].

On the other hand, if f is integrable over [a,b],

$\int_a^b f(x)\,dx = \sup\{L(P,f) : P \in \boldsymbol{P}\} = \inf\{U(P,f) : P \in \boldsymbol{P}\}$. Thus, for every $\in > 0$ we can find partitions P' and P'' of [a, b], such that

$0 \le \int_a^b f(x)\,dx - L(P',f) < \varepsilon/2$, and $0 \le U(P'',f) - \int_a^b f(x)\,dx < \varepsilon/2$ (see Sec, 2 of Unit 1).

Taking some partition P which is finer than both P' and P'', and adding the two inequalities, we have

$0 \le U(P,f) - L(P,f) < \varepsilon$

This completes the proof.

Now arises a natural question : Which are the functions which satisfy the above criterion? The following theorems provide an answer.

Theorem 4 A function that is monotonic (increasing or decreasing) on [a,b], is integrable over [a,b].

Proof Let the function $f:[a,b] \to \mathbf{R}$ be increasing. Then

$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$.

For each positive integer n, let $P_n = \{a, a+h, \ldots\ldots\ldots\ldots, a+nh = b\}$, where $h = \dfrac{b-a}{n}$, be a regular partition of [a,b]. Then

$$U(P,f) = \sum_{i=1}^{n} M_i\,\Delta x_i = \sum_{i=1}^{n} M_i\,h = h\sum_{i=1}^{n} M_i = \frac{b-a}{n}\sum_{i=1}^{n} f(a+ih)$$

since the supremum of f(x) in $[a+(i-1)h, a+ih]$ is $f(a+ih)$.

$$\text{and } L(Pn,f) = h\sum_{i=1}^{n} m_i = \frac{b-a}{n}\,h\sum_{i=1}^{n} f(a+(i-1)h)$$

Therefore

$$U(P_n, f) - L(P_n, f) = \frac{b-a}{n}[f(a+h)+f(a+2h)+\dots\dots\dots+f(a+nh)$$
$$-f(a)-f(a+h)\dots.-f(a+(n-1)h)]$$
$$= \frac{b-a}{n}[f(a+nh) - f(a)]$$
$$= \frac{b-a}{n}[f(b) - f(a)] \quad \dots(1)$$

Let $\varepsilon > 0$. Can we choose an n which will make $U(P_n,f) - L(P_n,f) < \varepsilon$?

$$n > \frac{(b-a)[f(b)-f(a)]}{\varepsilon}$$
$$\Rightarrow h = \frac{b-a}{n} < \frac{\varepsilon}{f(b)-(a)}$$

Yes, we can, Try some $n > \dfrac{(b-a)[f(b)-f(a)]}{\varepsilon}$. If we substitute this value of n in (1), we ge

$$U(P_n,f) - L(P_n,f) < \frac{\varepsilon(b-a)\,[f(b)-f(a)]}{(b-a)\,[f(b)-f(a)]} < \varepsilon$$

Thus, applying Theorem 3, we can conclude that f is integrable.

Theorem 4 leads us to the following useful result.

Corollary 1 If f is increasing or decreasing on [a,b], then

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + \dots\dots.f(a+(n-1)h)]$$

$$= \lim_{h\to 0} h[f(a+h) + f(a+2h) + \dots\dots\dots+ f(a+nh)], \text{ where } h = \frac{b-a}{n}$$

We shall illustrate the usefulness of Corollary 1 through some examples. But before that we state another theorem, which identifies one more class of integrable functions.

Theorem 5 If a function $f:[a,b) \to \mathbf{R}$ is continuous, then f is integrable.

The proof of this theorem is beyond the scope of this course. We shall prove it in a later course on real analysis.

In Sec. 5 in Unit 3, we have seen that differentiability implies continuity, Now we can write differentiability $\Rightarrow$ continuity $\Rightarrow$ integrability

Now, let us evaluate some definite integrals with the help of Corollary 1.

Example 5 To evaluate $\int_a^b \cos x\,dx,\ 0 \le a \le b \le \pi/2$, we observe that

$f: x \to \cos x$ is a decreasing function of [a,b]. Therefore, by Corollary 1

$$\int_a^b \cos x\,dx = \lim_{h\to 0} h[\cos(a+h) + \cos(a+2h) + \dots\dots. + \cos(a+nh)],\ a + nh = b.$$

Now

$2\sin(h/2)[\cos(a+h) + \cos(a+2h) + \dots\dots\dots + \cos(a+nh)]$

$= 2\sin(h/2)\cos(a+h) + 2\sin(h/2)\cos(a+2h) + \dots\dots + 2\sin(h/2)\cos(a+nh).$

$$= [\sin(a+\frac{3h}{2} - \sin(a+\frac{h}{2})] + [\sin(a+\frac{5h}{2}) - \sin(a + \frac{3h}{2})] + \dots\dots\dots +$$
$$[\sin(a+(\frac{2n+1}{2})h) - \sin(a+(\frac{2n-1}{2})h)]$$
$$= \sin(a+(\frac{2n+1}{2})h) - \sin(a+\frac{h}{2})$$
$$= [\sin(b+\frac{h}{2}) - \sin(a+\frac{h}{2}), \text{ since } a+nh=b$$
$$\Rightarrow \cos(a+h) + \cos(a+2h) + \dots\dots\dots\dots+ \cos(a+nh) = \frac{\sin(b+h/2) - \sin(a+h/2)}{2\sin h/2}$$

Thus

$$\int_a^b \cos x\,dx = \lim_{h\to 0} [\sin(b+\frac{h}{2}) - \sin(a+\frac{h}{2})]\frac{h/2}{\sin(h/2}$$

$= \sin b - \sin a.$

Example 6 Suppose we want to evaluate $\int_1^2 (x+x^2)\,dx$

Here, f: x $\rightarrow$ x + x^2 is an increasing function on [1,2].
Therefore,

$$\int_1^2 (x+x^2)\,dx = \lim_{h\to 0} h\sum_{i=1}^{n} f(1+ih),\ h = 1/n$$

$$\lim_{h\to 0} h\sum_{i=1}^{n} [(1+ih)+(1+ih)^2]$$

$$\lim_{h\to 0} h\sum_{i=1}^{n} (2+3hi)+h^2i^2)$$

$$\lim_{h\to 0} [2h\sum_{=1}^{n} 1+3h^2\sum_{n=1}^{n} i + h^3 \sum_{i=1} i^2]$$

$$\lim_{h\to 0} [2nh+\frac{3}{2}h^2n(n+1)+\frac{1}{6}h^3n(n+1)(2n+1)]$$

$$\lim_{h\to 0} [2+\frac{3}{2}(1+h)+\frac{1}{6}(1+h)(2+h),\ \text{since } nh = 1.$$

$$= 2+\frac{3}{2}+\frac{1}{3}=\frac{23}{6}.$$

Recall that

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} 1 = \frac{n(n+1)(2n+1)}{6}$$

In this section we have noted that a continuous function is integrable. We have also proved that a monotone function is integrable. corollary 1 gives us a method of finding the integral of a monotone function. One condition which is very essential for the integrability of a function in an interval, is its boundness in that interval. If a function is unbounded, it cannot be integrable. In fact, if a function is not bounded, we cannot talk of M_i or m_i, and thus cannot form the upper or lower product sums. Now on the basis of the criteria discussed in this section you should be able to solve this exercise.

E E10) State whether or not each of the following functions is integrable in the given interval. Give reasons for each answer.

a) $f(x) = x^2 - 2x + 2$ in $[-1,5]$

b) $f(x) = \sqrt{x}$ in $[1, a]$

c) $f(x) = 1/x$ in $[-1, 1]$

d) $f(x) = [x]$ in $[0, 4]$

e) $f(x) = |x-1|$ in $[0, 3]$

f) $f(x) = \dfrac{x^2+1}{2x+1}$ in $[-4, 0]$

$$f(x) = \begin{cases} x+1 \text{ when } x < 0 \\ 1-x \text{ when } x \le 0, \end{cases} \quad \text{in } [-1,1]$$

$$f(x) = \begin{cases} x+1 \text{ when } x < 1 \\ 2x+1 \text{ when } x \ge 1, \end{cases} \quad \text{in } [0,1]$$

E E11) Use corollary 1 to evaluate the following definite integral.

$$\int_0^2 (1+x)\,dx$$

1.4 FUNDAMENTAL THEOREM OF CALCULUS

As you have already read in the introduction, the basic concepts of definite integral were used by the ancient Greeks, mainly Archimedes (287-212 B.C.), more than 2000 years ago. This was long before calculus was invented. But in the seventeenth century Newton and Leibniz developed a procedure for evaluating a definite integral by antidifferentiation. This procedure is emboded in the Fundamental Theorem of Calculus (FTC).

Before we state this theorem, we introduce the notions of the **average value** of a function and the antiderivative of a fuction.

Definition 5 Let f be integrable over [a,b]. The average value $\bar{y}$ of $y = f(x)$ over [a,b] is

$$\bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The following theorem tells us that every continuous function on a closed interval attains its average value at some point of the interval. We shall not give its proof here.

Theorem 6 (Average Value Theorem) If f: $[a,b] \to \mathbf{R}$ is continuous, then

$$f(\bar{x}) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

for some $\bar{x} \in [a,b]$

We shall now define the antiderivative of a function.

Definition 6 Let f: $[a,b] \to \mathbf{R}$ and F: $[a,b] \to \mathbf{R}$ be two functions such that

$\frac{d}{dx}F(x) = F'(x) = f(x)$ for each $x \in]a,b[$, We call F(x) an **antiderivative** (for inverse derivative) of f(x).

For example, $\frac{x^3}{3}$ is an antiderivative of x^2, since $\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2$;

$-\cos x$ is an antiderivative of sin, since $\frac{d}{dx}(-\cos x) = \sin x$.

Is $\frac{x^2}{4} - x^2$ an antiderivative of $x^3 - 2x$?

Consider the two functions $f(x) = x^2$ and $g(x) = x^2 + 5$. Both these are antiderivatives of the function $h(x) = 2x$. This means that antiderivative of a function is not unique.

In fact, if F(x) is an antiderivative of f(x), then F(x) + c is also an antiderivative of f(x). This follows from the fact that

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}(F(x)+c) = f(x).$$

We can also say that any two antiderivatives of a function differ only by a constant. Because, if F(x) and G(x) are two antiderivatives of f(x), then

$F'(x) = F'(x) = f(x)$. That is, $[F(x) - G(x)]' = 0$

We have noted in Unit 7 that if the derivative of a function is zero on an interval, then that function must be a constant. Thus $F(x) - G(x) = c$.

The interval [a, b] on which f and its antiderivative are defined, so that $F'(x) = f(x)$ $\forall\, x \in\,]a, b[$, is implicit in our discussion here.

Now having defined the average value and the antiderivative, we are in a position to state the Fundamental Theorem of Calculus. We shall give this theorem in two parts.

Theorem 7 (FTC): Let $f: [a,b] \to \mathbf{R}$ be a continuous function.

Part 1 If the function $F : [a,b] \to \mathbf{R}$ is defined by

$$F(x) = \int_a^x f(t)dt,$$

Then F is an antiderivative of f, that is, $F'(x) = f(x)$ for all x in]a,b[.

Part 2 If G is an antiderivative of f in]a,b[, then

$$\int_a^b f(x)\,dx = G(x)\Big]_a^b = G(b) - G(a).$$

Proof of Part 1.

By the definition of derivative, $F'(x) = \lim\limits_{h\to 0} \dfrac{F(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right]$$

$$= \lim \frac{1}{h}\int_x^{x+h} f(t)\,dt,$$

by the interval union property of definite integrals.

But, by the Average Value Theorem (Theorem 6)

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(\bar{t}) \text{ for some } \bar{t} \in [x, x+h].$$

Therefore, $F'(x) = \lim\limits_{h\to 0} f(\bar{t})$. We know that $\bar{t} \in [x, x+h]$. This means that as

$h \to 0$, $\bar{t} \to x$ Therefore,

$F'(x) = \lim\limits_{t\to x} f(\bar{t}) = f(x)$, since f is a continuous function.

Hence, F as defined by (3), is an antiderivative of f.

Proof of Part 2

G is given as an antiderivative of f in]a,b[. Also, as shows in Part 1, F defined by (3) is an antiderivative of f in]a,b[. Therefore,

$G(x) = f(x) + c$ on]a,b[for some constant c.

To evaluate c, we substitute x = a, and obtain

$c = G(a) - F(a) = G(a) - 0 = G(a)$.

$F(a) = \int_b^a f(t)\,dt = 0$

Hence $G(x) = F(x) + G(a)$, or

$F(x) = G(x) - G(a)$

If we put x = b, we get

$$F(b) = \int_a^b f(x)\,dx = G(b) - G(a)$$

The Fundamental Theorem of Calculus tells us that differentiation and integration are inverse processes, because Part 1 may be rewritten as

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x), \text{ if f is continuous.}$$

That is, if we first integrate the continuous function f with the variable x as the upper limit of integration and then differentiate with respect to x, the result is the function f again. So differentiation offsets the effect of integration.

On the other hand, if we assume that G' is continuous, then Part 2 of FTC may be written as

$$\int_a^x G'(t)\,dt = G(x) - G(a)$$

Here we can say that if we first differentiate the function G and then integrate the result from a to x, the result can differ from the original function G(x) only by the constant G(a). If G is so chosen that G(a) = 0, then integration offsets the effect of differentiation.

Till now we had evaluated the integrals of some functions by first finding the lower and upper sums, and then taking their supremum and infimum, respectively. This is a tedious procedure and we cannot apply it easily to all functions. But now. FTC gives us an easy method of evaluating definite integrals. We shall illustrate this through some examples.

Example 7 Suppose we want to evaluate $\int_2^3 (ax^2 + bx + c)\,dx$.

Since $f : x \to ax^2 + bx + c$ is continuous on [2,3], it is integrable over [2,3].

$G(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$ is an antiderivative of $f(x)$.

Hence, by FTC (Part 2)

$$\int_2^3 (ax^2 + bx + c)\,dx = G(x)]_2^3$$

$$= G(3) - G(2)$$

$$= (9a + 9b/2 + 3c) - (8a/3 + 2b + 2c)$$

$$= [19\frac{a}{3} + 5\frac{b}{2} + c]$$

Example 8 Let us evaluate $\int_0^{\pi/4} \cos 2x\,dx$

$$\int_0^{\pi/4} \cos 2x\,dx = \frac{\sin 2x}{2}]_0^{\pi/4}$$

$$= \frac{\sin(\pi/2)}{2} - \frac{\sin 0}{2} = \frac{1}{2}$$

Example 9 To evaluate $\frac{d}{dx}\int_0^{x^2} \sin t\,dt$, we put $x^2 = u$

Then $\frac{d}{dx}\int_0^x \sin t\,dt = \frac{du}{dx}\int_0^u \sin t\,dt = \frac{d}{du}\left(\int_0^u \sin t\,dt\right)\frac{du}{dx}$

Now $\frac{du}{dx} = 2x$. and using FTC (Part 1), we get

$\frac{d}{du}\int_0^u \sin t\,dt = \sin u = \sin x^2$. Thus. $\frac{d}{dx}\int_0^{x^2} \sin t\,dt = 2x.\sin x^2$

Example 9 suggests the following formula.

$$\frac{d}{dx}\left(\int_0^{g(x)} f(t)dt\right) = f(g(x))g'(x).$$

If you have followed these examples, you should be able to solve the exercises below. Remember that the main thing in evaluating a definite integral is to find an antiderivative of the given function.

E E12) The second column in the table below consists of some functions which are antiderivatives of the functions given in column 1. Match a function with its antiderivative by pairing appropriate numbers.

For example, we can match x^n with $\frac{x^{n+1}}{n+1}$ since $\frac{x^{n+1}}{n+1}$ is an antiderivative of x^n. We shall indicate this by iii) → viii)

Function		Antiderivative	
i)	$\sin x$	i)	$-\ln \cos x$
ii)	$\cos x$	ii)	$\ln \cosh x$
iii)	x^n	iii)	$\operatorname{sech} x$
iv)	e^{ax}	iv)	$-\cos x$
v)	$\tan x$	v)	$\sin x$
vi)	a	vi)	$\frac{1}{a}e^{ax}$
vii)	$\tanh x$	vii)	ax
viii)	$\operatorname{sech} x \tanh x$	viii)	$\frac{x^{n+1}}{n+1}$

E E 13) Evaluate the following integrals by using FTC

a) $\int_1^3 2x^3\,dx$

b) $\int_1^3 (2x^2+2x+1)dx$

c) $\int_1^2 x(x+1)^2 dx$

d) $\int_0^{\pi/4} \sec^2 x\,dx$

e) $\int_b^2 x(x^2+1)^2\,dx$

f) $\int_{-/2}^{\pi/2} (x+\sin x)\,dx$

g) $\int_0^{\pi} (x-\cos x)dx$

h) $\int_0^4 e^{2x}\,dx$

i) $\int_0^1 \sinh x \cosh x\,dx$

j) $\int_{-1}^1 (\sinh x - \cosh x)dx$

E 14) Find $\frac{d}{dx}$ [F(x)] when F(x) is defined by the following definite integrals.

a) $\int_a^x \sqrt{1+t^2}\, dt$

b) $\int_0^{x^2} \sqrt{\sin t + \cos t}\, dt$

c) $\int_0^{\sqrt{1+x^2}} (t - 2t + 1)\, dt$

d) $\int_x^{x^2} \cos t^2\, dt$

e) $\int_{\sqrt{x}}^{x^2} t\sqrt{1-t^2}\, dt$

1.5 SUMMARY

In this unit we have conered the following points:

1) A partition P of a closed interval [a,b] is a set $\{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$] such that $x_0 < x_1 < x_2 < \ldots < x_n$.

2) A partition P_1 of $[a,b]$ is finer than a partition P_2, if $P_1 \supseteq P_2$,
3) If M and m are the supremum and the infimum of a bounded function f in $[a,b]$, then, given any partition P of $[a,b]$, $m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$.
4) The lower integral of a bounded function is less than or equal to its upper integral.
5) A bounded function f is integrable over $[a,b]$ if and only if its lower and upper integrals are equal. In such a situation the lower (or upper) integral is called the definite integral of f over $[a,b]$, denoted by $\int_a^b f(x)dx$.
6) If f is monotonic or continuous on $[a,b]$, then f is integrable over $[a,b]$.
7) If f is continuous on $[a,b]$, then $\int_a^b f(x)\,dx$ represents the signed area of the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.
8) If f is monotonic on $[a,b]$, then

$$\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{i=1}^{n} f(a+ih), = \lim_{h\to 0} h\sum_{i=1}^{n} f(a+(i-1)h),$$

where $h = \frac{b-a}{n}$

9) The Fundamental Theorem of Calculus:
   i) If f is continuous on $[a,b]$, then for $x \in]a,b[$

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

   ii) If f is continuous on $[a,b]$ and $F'(x) = f(x)$ for $x \in]a,b[$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

1.6 SOLUTIONS AND ANSWERS

E1) $P_1, P_1 \cap P_2$ are regular.
$\Delta x_3 = 1/4$ in P_1, $\Delta x_3 = 1/6$ in P_2

E2) a) $\{0, \frac{1}{3}, \frac{2}{3}, 1, 1\frac{1}{3}, 1\frac{2}{3}, 2\}$

b) $\{2, 2\frac{7}{10}, 3\frac{2}{5}, 4\frac{1}{10}, 4\frac{4}{5}, 5\frac{1}{2}, 6\frac{1}{5}, 6\frac{9}{10}, 7\frac{3}{5}, 8\frac{3}{10}, 9\}$

E3) a) $P_2 = \{a, \frac{3a+b}{4}, \frac{a+b}{2}, \frac{a+3b}{4}, b\}$

$P_3 = \{a, \frac{7a+b}{8}, \frac{3a+b}{4}, \frac{5a+3b}{8}, \ldots\ldots\ldots b\}$

c) Δx in P_2 is $\frac{b-a}{4}$

Δx in P_3 is $\frac{b-a}{8}$

E4) $P^*_{n+1} = P^*_{n+1} \cup P^*_n \Rightarrow P^*_n \subseteq P^*_{n+1} \Rightarrow P^*_{n+1}$ is a refinement of $P^*_n \forall n$.

E5) a) f(x) is an increasing function on $[0,2]$. Hence

$$L(P,f) = 1.\frac{1}{2} + \frac{5}{4}.\frac{1}{2} + 2.\frac{1}{2} + \frac{13}{4}.\frac{1}{2}$$

and $$U(P,f) = \frac{5}{4}.\frac{1}{2} + 2.\frac{1}{2} + \frac{13}{4}.\frac{1}{2} + 5.\frac{1}{2}$$

b) f(x) is a decreasing function on $[1,4]$. Hence

$$L(P,f) = \frac{1}{2}.1 + \frac{1}{3}.1 + \frac{1}{4}.1$$

$$U(P,f) = 1.1 + \frac{1}{2}.1 + \frac{1}{3}.1$$

E6) $L(P_1, f) = \frac{2}{5} \cdot + \frac{1}{2} \quad \frac{1}{3} \cdot \frac{1}{2} = \frac{11}{30}$

$U(P_1,f) = \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{1}{2} = \frac{9}{20}$

$L(P_2 f) = \frac{4}{9} \cdot \frac{1}{4} + \frac{2}{5} \cdot \frac{1}{4} + \frac{4}{11} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4}$

$= \frac{2289}{5940}$

$U(P_2 f) = \frac{1}{2} \cdot \frac{1}{4} + \frac{4}{9} \cdot \frac{1}{4} + \frac{2}{5} \cdot \frac{1}{4} + \frac{4}{11} \cdot \frac{1}{4}$

$= \frac{1691}{3960}$

$L(P_1, f) \leq L(P_2,f) \leq U(P_2,f) \leq U(P_1,f)$.

E7) If $P = \{x_0, \ldots\ldots\ldots\ldots x_n\}$ is a partition of $\{0, 1\}$,

$L(P,f) = U(P,f) = \Sigma m_i . \Delta x_i = \Sigma 2 \Delta x_i = 2 \Sigma \Delta x_i = 2.1 = 2$

Hence $\int_0^1 f(x)\, dx = \int_0^1 f(x)\, dx = 2$.

E8) If $P = \{a = x_0, x_1, \ldots\ldots\ldots\ldots x_n = b\}$ is any partition of $[a,b]$,

then $L(P,f) = \Sigma m_{i,f} \Delta x_i \leq \Sigma m_{i,g} \Delta x_i = L(P,g)$,

where $m_{i,f} = \inf \{f(x): x \,\varepsilon\, [x_{i-1}, x_i]\}$ and

$m_{i,g} = \inf \{f(x): x \,\varepsilon\, [x_{i-1}, x_i]\}$

and $m_{i,f} \leq = m_{i,g}$ since $f(x) \leq g(x)$ for all x.

Similarly, $U(P,f) \leq U(P,g)$ for all P.

The result follows.

E9) $f(x) = 2$ is integrable

E 10 a), b), e) and g) are integrable as these are continuous.

c), f) are not integrable as they are not bounded.

d) h) are integrable as these are increasing functions.

E11) $\int_1^2 (1+x)\, dx = \lim_{h \to 0} h[(1+1) + (1+1+h) + (1+1+2h) + \ldots\ldots\ldots\ldots + (1+1+\overline{n-1}]$

$= \lim_{h \to 0} h\, [2n + h\, (0+1+\ldots\ldots\ldots + \overline{n-1}$ as $n\,h = 1$

$= \lim_{h \to 0} [2.1 + h^2 . \frac{n(n-1)}{2}]$

$= \lim_{h \to 0} [2 + \frac{nh.\,(nh-h)}{2}]$

$= \lim_{h \to 0} [2 + \frac{1.\,(1-h)}{2}]$

$= 2 + \frac{1}{2} = 5/2$

E12)

i)	→	iv)
ii)	→	v)
iii)	→	viii)
iv)	→	vi)
v)	→	i)
vi)	→	vii)
vii)	→	ii)
viii)	→	iii)

E13 a) $\frac{x^4}{2}$ is an antiderivative of $2x^3$. Hence

$$\int_2^3 2x^3\, dx \left.\frac{x^4}{2}\right]_1^3 = \frac{81}{2} - \frac{1}{2} = 40$$

b) $\int_1^3 (2x^2 + 2x + 1)\, dx = \left.\frac{2x^3}{3} + x^2 + x\right]_1^3$

$= 18 + 9 + 3 - (\frac{2}{3} + 1 + 1) = 27\frac{1}{3}$

c) $\frac{119}{12}$, d) 1, e) $\frac{62}{3}$, f) 0, g) $\frac{\pi^2}{2}$

d) $\frac{1}{2}(e^8 - 1)$, i) $\left.\frac{\sinh^2 x}{2}\right]_0^1 = \frac{1}{8}(e^2 + e^{-2} - 2)$

i) $e^{-1} - e$

E14 a) $\sqrt{1 + x^2}$

b) $\frac{d}{dx}\int_0^{x^2} \sqrt{\sin t + \cos t}\,dt = \frac{d}{dx^2}\int_0^{x^2} \sqrt{\sin t + \cos}\,dt.\ \frac{dx^2}{dx}$

$= \sqrt{\sin x^2 + \cos x^2}.\ 2x$

c) $\frac{d}{dx}\int_0^{\sqrt{1-x^2}} (t^3 - 2t + 1)\, dt = [(\sqrt{1-x^2})^3 - 2\sqrt{1-x^2 + 1}].\ \frac{d}{dx}\sqrt{1-x^2}$

$= x(1 + x^2 - \frac{1}{\sqrt{1-x^2}})$

d) $\frac{d}{dx}\int_x^{x^2} \cos t^2\, dt = \frac{d}{dx}[\int_0^{x^2} \cos t^2 dt - \int_0^x \cos t^2\, dt]$

$= 2x\cos x^4 - \cos x^2$

e) $\frac{d}{dx}(\int_{\sqrt{x}}^{x^2} t\sqrt{1-t^2}\, dt) = \frac{d}{dx}[\int_0^{x^2} t\sqrt{1-t^2}\, dt] - \int_0^{\sqrt{x}} t\sqrt{1-t^2}\, dt]$

$= 2x^3\sqrt{1-x^4} - \frac{1}{2}\sqrt{1-x}$