# UNIT 1 CLASSICAL PROBLEMS AND PUZZLES

## Structure



# 1.0 INTRODUCTION

Problem solving is by definition a creative activity: more correctly, for most of us, it's a re-creative activity, especially when we solve text-book problems. But it can also be a recreative activity, as you will discover in working through many of the problems and puzzles which follow below. Though occasionally (for some of us!) a pastime indulged in leisure hours, solving problems and puzzles is important in its own right, for much new mathematics has had its genesis in plain curiosity.

The problem of the Konigsberg bridges is a case in point: at the centre of this town, which lies on the river Pregel, is the island of Kneiphof, connected to each bank of the river by two bridges. A fifth bridge runs to a neighbouring (smaller) island, which has also one bridge linking it to each bank. The citizens of Konigsberg liked to stroll on the bridges, and the question was proposed: is it possible to cross all seven bridges without crossing any one of them twice?

In 1736 the great Swiss mathematician Leonhard Euler (pronounced OILER) settled the question, by establishing general principles that may be used to solve all network problems. But one famous network - and essentially recreative - problem that has defied solution is the four-colour problem, discussed by Arthur Cayley in 1879, but supposedly first posed by August De Morgan in a letter to Sir William Rowan Hamilton (1852).

The problem is simple to state: A map must be coloured so that every country is coloured differently from every country that it borders. Prove that four colours are sufficient to colour any map on a sphere or in a plane. Nobody has produced a proof, or a map that requires five colours. (However Kenneth Appel and Wolfgang Haken have given a "computer proof" in which some two hundred thousand maps of all possible types, involving millions of cases, were examined in an exercise that involved over a thousand hours of processing time on a supercomputer at the University of Illinois: so we can now say with certainty that four colours do suffice.

Though computer generated proofs are not "elegant" - they are essentially proofs by exhaustion, in both senses of the word! - there may be several propositions that can for the moment be established only by computer methods.)

Mathematicians study their subject not to solve puzzles merely: but chiefly because it affords satisfactions that are unequalled by any other branch of knowledge - the excitement of the chase, the thrill of discovery, the achievement of insight and the elation of success. Curiosity lies at the heart of all great discovery. But mathematical curiosity is a gift given only to geniuses of the highest ability, and in the pages that follow we will look at some of the problems that have excited the curiosity of the great makers of mathematics.

De Morgan Was famous for the puzzles he invented so to whet your appetite for what's in store, it is fitting to end this introduction with one which he presented in 1864: I was x years old in the year $x^2$ When was I born?

# 1.1 OBJECTIVES

After studying this Unit you should he able to understand the approaches to some famous problems:
the Konigsberg Bridges problem
the Cannibals and Missionaries Problem
Decanting Problems
Problems solved by the use of Decision Trees
Classical Conundrums

# 1.2 CROSSING THE KONIGSBERG BRIDGES

We begin this Unit by introducing the idea of a graph - not the kind which expresses a functional relationship between two varying quantities, but rather a discrete mathematical structure of points and connecting lines, which can be used to model a variety of situations. The subject of graph theory began in the year 1736 when the Swiss mathematician Leonhard Euler solved an entertaining problem, which was posed to him by the inhabitants of Konigsberg, formerly a Prussian city, but now a part of Russia, where it is known today as Kaliningrad. The river Pregel runs through the city, and includes two islands which are connected by seven bridges as shown in Fig. 1 below:

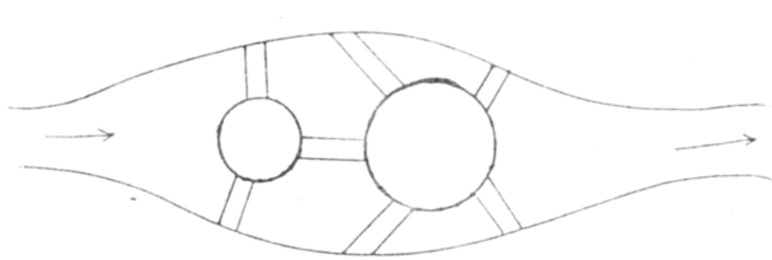

**Fig. 1 : The Bridges of Konigsberg**

Is it possible, Euler was asked, for person to return to the starting point after having crossed the seven bridges once, and only once?

Euler converted the problem to a graphic representation, in which each land mass becomes a vertex, and each bridge and edge between two vertices(Fig.2(a). In graph-theoretic terms, the problem reduces to the question: Can one trace the figure starting at one of the vertices in a continuous fashion, traverse each edge only once, and return to the starting vertex? Try this without lifting your pencil from the paper.

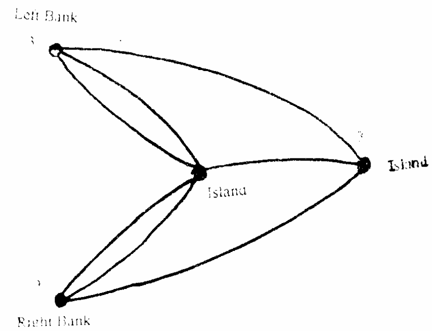


**Figure 2 (a)**

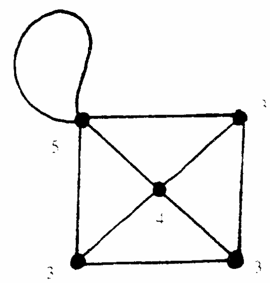

**Figure 2 (b)**

Euler proved that a closed trail of this kind - named an Eulerian circuit in his honour - does not exist for the graph of Fig. 2 (a). He proved in fact: a graph G (which we may define as a composition of two sets: vertices and edges) contains a circuit if every vertex of G is of even degree. The degree of a vertex is the number of edges which meet at it. Note that in Fig. 2 (a) the degree of each vertex is written alongside it. For each vertex in the circuit there must be an edge by which the vertex is reached, and another by which it is exited. A loop (Fig. 2 (b) contributes two, and not one, to the degree of a vertex.

Let's see why an Eulerian circuit in a graph forces all its vertices to have even degree. Start at some vertex on the circuit, and follow the circuit from vertex to vertex, erasing each edge as you go along it. In passing through a vertex, you erase one edge entering and one edge exiting, or else you erase a loop. In each case you reduce the degree of the vertex by two. Eventually every edge gets erased, and all vertices are reduced to zero degree, So all vertices must have had even degree to begin with. The graph of the Konigsberg bridges has vertices of odd degree, and cannot include an Eulerian circuit.

# 1.3 CANNIBALS AND MISSIONARIES

Consider now the well-known problem involving three cannibals and three edible missionaries, who are all initially on the left bank of a river, and wish to cross it. The only boat available is one capable of carrying two persons; and the condition of safe crossing is that at no time can the cannibals outnumber the missionaries. How do they cross the river, with no one getting eaten?
It is convenient to label the "states" which represent the numbers of missionaries and cannibals on a given side of the river as the vertices of a graph. A state in which there are c cannibals and m missionaries is represented by the symbol (c, m). Clearly, some states are impossible:(3, 2) on the left bank, or (1, 2) on the right bank. Table 1 lists the set of sixteen possible states:

| States | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Left Bank (c, m) | 3,3 | 3,2 | 3,1 | 3,0 | 2,3 | 2,2 | 2,1 | 2,0 | 1,3 | 1,2 | 1,1 | 1,0 | 0,3 | 0,2 | 0,1 | 0,0 |
| Right Bank (c, m) | 0,0 | 0,1 | 0,2 | 0,3 | 1,0 | 1,1 | 1,2 | 1,3 | 2,0 | 2,1 | 2,2 | 2,3 | 3,0 | 3,1 | 3,2 | 3,3 |

**Table 1**

The letters l or r used to specify the position of the boat in any state; thus (l, 1) is the start , state in which three cannibals and three missionaries are standing on the left bank, where the boat is also located. From this state - or vertex in the graph of Fig. 3 (a) - it is possible to move to five states, only three of which are however acceptable: (r, 5), (r,6) and (r, 9). (Verify that (r, 2) and (r, 3) are not allowed states.)

| | | |
|---|---|---|
| | (r, 2) | illegal move |
| | (r, 3) | illegal move |
| ( *l, 1* ) | (r, 5) | |
| | (r, 6) | |
| | (r, 9) | |

**(r, 2) and (r, 3) are unacceptable states**

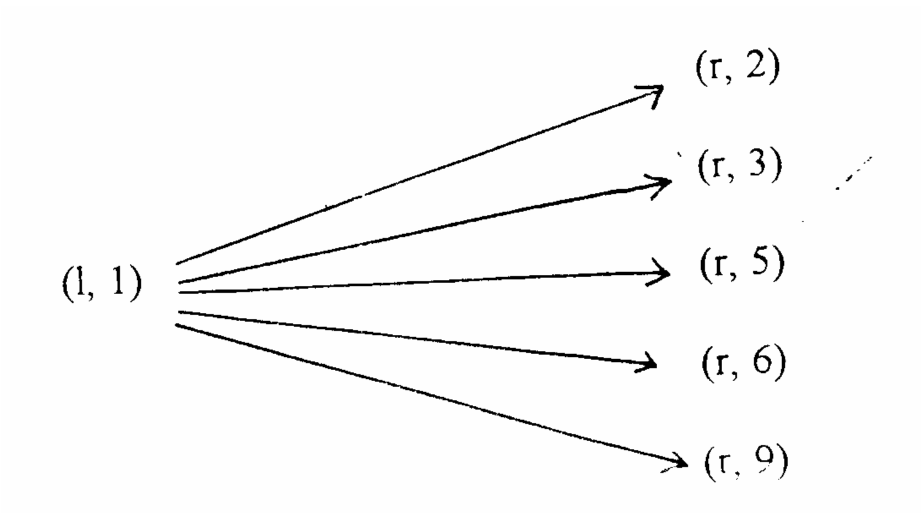


**Figure 3 ( a )**

From (r, 5) we can only return to (l, 1) so this is not a very useful move. From (r, 6) we can move to (l, 2), which is impossible, and to (l, 5); from (r,9) similarly to (l, 5). See Fig. 3 (b).

| | | | |
|---|---|---|---|
| | (r, 5) | (l, 2) | illegal move |
| (l, 1) | (r, 6) | (l, 5) | |
| | (r, 9) | | |

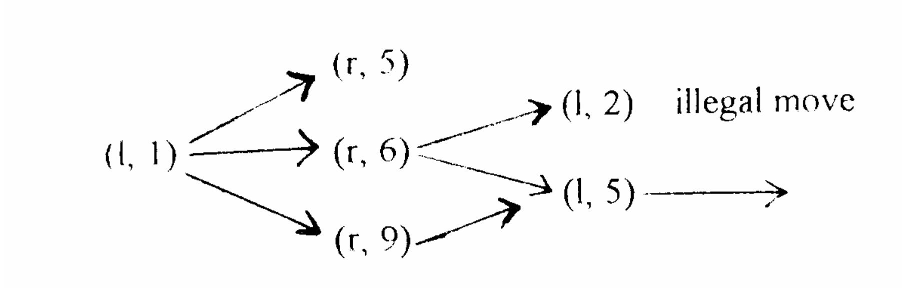


**Figure 3 ( b )**

From (l,5) the non-trivial legal move is (r,13).(Verify that (r,7) and (r,10) are illegal.)We thus have the situation of Fig.3(c).

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | (r,6) | | | | | | | | (l,11) | |
| (l,1) | | (l,5) | (r,13) | (l,9) | (r,11) | (l,6) | (r,8) | (l,4) (r,12) | | (r,16) |
| | (r,9) | | | | | | | | (l,8) | |

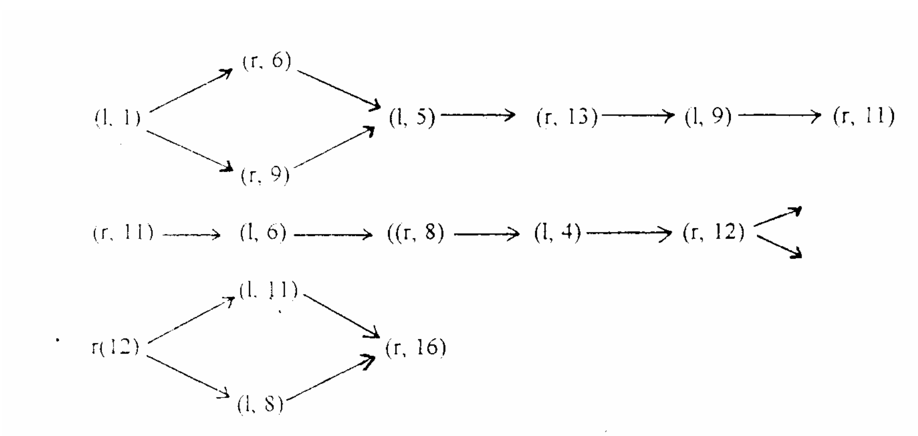


**Figure 3 (c)**

# Check Your Progress 1

1. Verify that the next legal steps are indeed (r,13),(l,9),(r,11),(l,6),(r,8),(l,4),(r,12),(l,8) and (r,16).

2. A boatsman must ferry across a river a wolf, a sheep, and a bundle of fodder. His boat is too small to carry more than himself and one of the "passengers" at a time. He cannot leave the wolf alone with the sheep, or the sheep alone with the fodder. Show he can cross without any detriment to either sheep or fodder.

3. Three married couples are to cross a river in a boat which can hold only two persons ,one of whom must row the boat. Each of the three husbands is a jealous man, who will not allow his wife to be left in the company of other men, unless he is also present. Show how the jealous husbands and their wives can accomplish the trip.

# 1.4 DECANTING PROBLEMS

Suppose that you are given three jugs A,B AND C, with capacities 8,5 and 3 litres respectively, but none are calibrated. Jug A is filled with 8 litres of wine. By a series of pouring back and forth among the three jugs, divide the 8 litres into two equal parts:4 litres in jug A, and 4 litres in jug B.

Again we represent the vertices of our graph as "states" or ordered triples such as (a, b, c),where a, b and c represent the amounts of wine in the jugs A,B,C respectively. The initial state is(8,0,0), and the desired state is (4,4,0).Possible moves from the start state are (5,0,3),(0,5,3) and (3,5,0).From (5,0,3) one can pour to reach (5,3,0) and then(2,3,3).From (3,5,0) a possible move is (3,2,3),and subsequently (6, 2, 0). The graph representing these moves is sketched in Fig. 4.

| | | | |
|---|---|---|---|
| | (5, 0, 3) | (5, 3, 0) | (2, 3, 3) |
| (8, 0, 0) | (0, 5, 3) | | |
| | (3, 5, 0) | (3, 2, 3) | (6, 2, 0) |

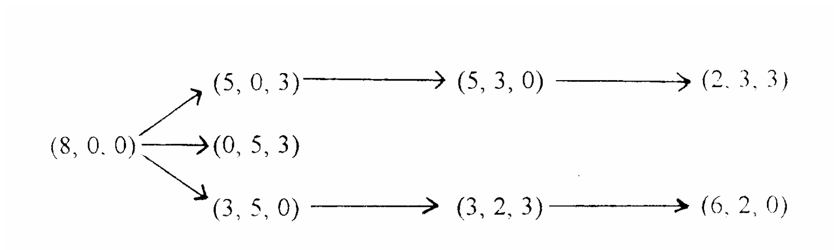


**Figure 4**

Decanting problems are most directly solved using trilinear co-ordinates, in which three sets of parallel lines divide the plane into a tesselation of equilateral triangles. In this system the co-ordinates (x, y, z) of' a point P are defined as the distances of the point from the sides of an (appropriately large) equilateral Triangle ABC, with side a and altitude h. In Fig. 5, x, y and z are the distances of P from BC, CA and AB respectively. (Observe that the vertices A, B and C have the co-ordinates (h, 0, 0), (0, h, 0) and (0, 0, h)). The co-ordinates are regarded as positive when the point is within the triangle. The co-ordinates (x, y, z) of P are its distances from BC, CA and AB respectively.

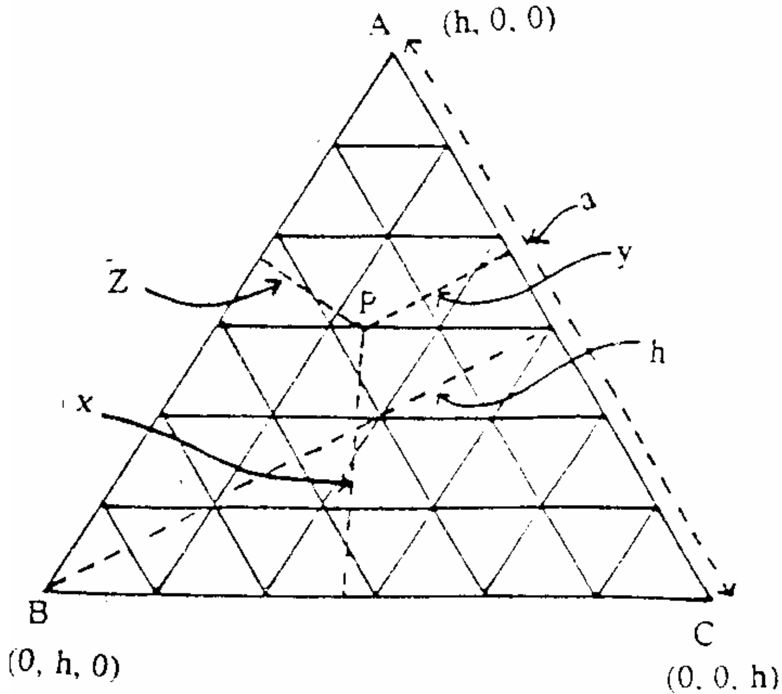


**Figure 5**

Since the area of any triangle is half the base times the altitude,

we see that:

Area(ABC) = Area(PBC) + Area(PCA) + Area(PAB)

OR

$$1/2\ (ah) = 1/2\ (ax) + 1/2\ (ay) + 1/2\ (az)$$

OR

$$x + y + z = h$$

Thus triliner co-ordinates can be useful when three variable quantities have a constant sum, and the application to pouring problems in immediate.

When one of the quantities stays fixed while the other two vary (with the sum of the three remaining constant), the point P(x, y, z) moves along a line parallel to one side of the triangle.

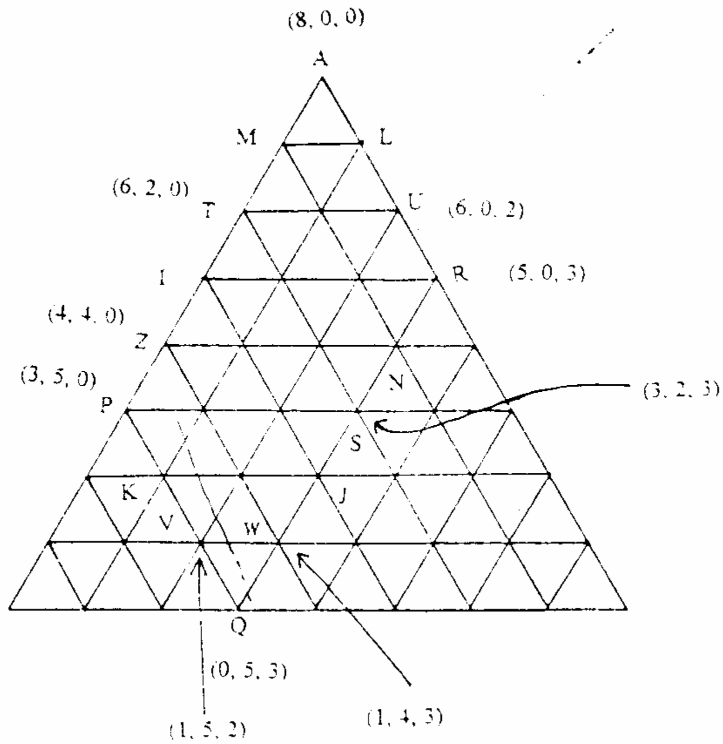


**Figure 6**

Consider again the previous example, in which we are given a vessel filled with eight litres of wine, and two empty vessels of capacities five and three litres respectively, the aim being to divide the wine into two equal parts. The initial state is represented by the vertex A(8,0,0) of the triangle in Fig. 6, and the domain of the solution is bounded by the vertices P(3, 5, 0), Q(0, 5, 3), and R(5, 0, 3) which lie on a parallelogram bounded by the lines y = 0, y = 5, z = 0, z = 3. The figure shows how the desired point (4, 4, 0) can be reached. Suppose the first move is to fill the five-litre jug, so the new state becomes (3, 5, 0). The solution path runs along a direction parallel to one side of the triangle of reference, and bends whenever it reaches a vertex or side of the parallelogram.

We may proceed now from P to Q, but this achieves nothing spectacular: the three litres which remain in the eight-litre jug are transferred to the three-litre jug. A more useful approach may be to move from P to S, parallel to the base of the triangle: the contents of the five-litre jug are poured into the three-litre jug, so that the state (3, 2, 3) is obtained. Proceed now from S to T, transferring the contents of the three-litre jug back to the eight-litre vessel, achieving (6, 2, 0); move from T to U, by transferring the contents of the five-litre jug to the three-litre jug, obtaining (6, 0, 2), thence to V by pouring from the eight-litre vessel into the five-litre vessel, achieving (1, 5, 2). Proceed from V to W obtaining (1, 4, 3), then to Z obtaining (4, 4, 0). The solution path is zig - zag, following the motion of a billiard ball as it reflects from the sides of a parallelogram-shaped billiard table: A P S T U V W Z.

# Check Your Progress 2

1. By examining Fig. 6 prove that another solution to the (8, 5, 3) problem is along the path A R I J K L M N Z.

2. We are given a 12-litre vessel filled with milk, and two empty vessels with capacities of 9 litres and 5 litres. How can we divide the milk into two equal portions?

3. Three highwaymen robbed a gentleman of Ujjain of a bottle containing 24 cc of a precious, hitherto unpatented, aviation lubricant. While fleeing, they met a glass blower, from whom they purchased three vessels. On reaching a secluded place, they wished to divide their booty, but found that their vessels could hold 13, 11 and 5 cc respectively. How could they divide their loot into equal portions?

4. You are given two uncalibrated jugs, one of capacity five litres and the other of capacity three litres, and an unlimited supply of beer (wait: on second thought, perhaps water may be a better choice). By a sequence of filling and emptying the jugs, and pouring back and forth between the two jugs, show how you can obtain exactly four litres in the first jug.

# 1.5 DECISION TREES

When alternative choices can be made based on discrete answers - usually true or false to questions, and then further choices are made based on those answers, the graphical representation of the situation results in a special kind of a graph called a decision tree. Such graphs are called trees because they have the appearance of an inverted tree held up by its root, the root representing the first choice or decision, the "branches" the consequences of further choices, and the "leaves" the final solution. For an example of the use of decision trees in the solution of puzzles, consider the following well-known problem:
You are given twelve apparently identical coins, one of which is counterfeit, weighing less than the eleven others, which are all of the same weight. Given only an equal-arm balance, determine the counterfeit coin in no more than three weighings.

To solve the problem, we regard each weighing as a vertex - more properly, node - of a tree (Fig, 7); the possible outcomes are the branches pointing to subsequent weighings. The first weighing is the root of the tree: thus suppose that we weigh six coins in the left pan of the balance against six coins in the right pan: then either the left pan will be lighter, or it will be heavier: the node divides into two branches. Similarly, if we choose to weigh four coins in the left pan against four others in the right, there will be three possibilities to consider at the subsequent step: the first two, and the additional possibility that all eight coins chosen were genuine.

Let's begin by weighing six coins in the left pan against six in the right. For convenience we label the twelve coins by the letters A through L. In this trial, the left pan holds the set (A - F), while the right hand pan holds (G - L). The weighing has two outcomes. It narrows us down to six coins in which the counterfeit coin may lie. Suppose that these are the set(A-F).For the next weighing choose three from these (say A,B,C) in the left pan, and three in the right(D,E,F).If the left hand pan contains the lighter coin, weigh A again B. There are now three possibilities: A is lighter, B is lighter, or both weight the same, in which case C is the counterfeit coin. And similarly if the right hand pan contains the lighter coin. The decision tree shown in Fig. 7.

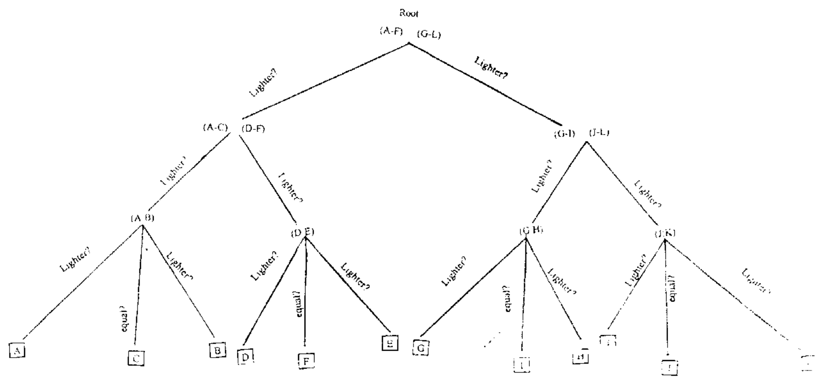


**Figure 7**

# Check Your Progress 3

1. Suppose that you have eight coins and are told that one is counterfeit and has a different weight (heavier or lighter) than the other seven. Determine which coin is different in no more than three weighings. (Hint: To start with, weigh three coins in the left pan against three in the right.)

2. If you are given the information that the counterfeit coin is heavier than the others, how will you pick it in only two weighings?

3. If you have twelve coins, one of which is *different* from the rest (heavier or lighter), how will you pick it in three weighings? (Hint: Start by weighing in groups of four.)

4. Given 13 coins, one of them heavier than the rest, how can you determine it in no more than three weighings?

# 1.6 CLASSICAL CONUNDRUMS

To challenge our faculties, let's now solve some trivial problems: trivial only in the sense that pondering over these questions will not lead to earth shattering discoveries; nor will the solutions themselves be seen to fit into any general theory. But the act of creation - discovering the solution by the application of logical arguments, drawn from our stock of accumulated knowledge - that is never trivial, and the ensuing sense of achievement is invariably worth the effort. We solve problems because they are there; because they excite us and challenge us; because they draw divinity from us; because of the feeling of well-being that derives from accomplishment.

So here are some problems: think of them as "warm ups"- most are simple, a few are difficult, while some have no (hitherto known) solutions at all! Grapple with them, wrestle with them, worry over them. think about them, until they're under your belt. [Note : Many of the problems below have been adapted from the MENUS collections and from Raymond Smullyan's "What is the name of this Book?"; Further lots more challenging problems are to be found in the Mathematical Games Section of *Scientific American,* and in the ENIGMA column of *New Scientist.*

1 .Five horses ran at the Delhi Race Club. There were no ties. Sikandar did not come first. Star of India was neither first nor last. Mughal Glory came in one place after Sikandar. Zulfikar was not second. Rangila was two places below Zulfikar. In what order did the horses finish?

2. When Alexander the Great attacked the forces of Porus, an Indian soldier was captured by the Greeks. He had displayed such bravery in battle, however, that the enemy offered to let him choose how he wanted to be killed. They told him, " If you tell a lie, you will be put to the sword, and if you tell the truth you will be hanged." The soldier could make only one statement. He made that statement, and went free. What did he say?

3. I went shopping with Rs. 6000. I spent 1/4 on books, Rs. 3000 for software for my home computer, and 10% of the original amount in a restaurant. How much did I have left?

4. You are working in a store that stocks bangles. Three boxes of bangles have been incorrectly labelled. The labels say Red Bangles, Green Bangles, and Red and Green Bangles. How can you relabel the boxes correctly, by taking out only one bangle from one box?

5. My friend collects antique stamps. She purchased two, but found that she needed to raise money urgently. So she sold them for Rs. 8000 each. On one she made 20%, and on the other she lost 20%. How much did she gain or lose in the entire transaction?

6. There are four seats in a row at a cinema. Sudhir will sit next to Shirin, but not next to Ganesh. If Ganesh will not sit next to Geeta, who is sitting next to Geeta?

7. A man asks his friend to meet him at the airport at 3:00 p.m., to drive him to an appointment He catches an earlier plane, and arrives at the airport at 2:00 p.m. He decides to start walking, and is picked up en route by his friend. He arrives twenty minutes early for his appointment. How long did he walk?

8. Put the appropriate plus or minus signs between the numbers below, in the correct places, so that the value of the expression on the left will equal the value on the right:

   0 1 2 3 4 5 6 7 8 9 = 1

9. A man went into a fast food restaurant, and ate a meal costing Rs. 105, giving the accountant a Rs. 500 note. He kept the change, came back a few minutes later, and had some food packed for his girl friend. He gave the accountant a Rs. 100 note, and received Rs. 20 in change. Later the bank told the accountant that both the Rs. 500 and the Rs. 100 notes were counterfeit. How much money did the restaurant lose?

10. Cubical dice have the following numbers written upon them:

    5 17 19 37 41 46 50 66

    The dice are tossed, and the winner is the person who first reaches 100 with the fewest dice possible, and with no repeats. What numbers does the winner need?

11. My wife and I went shopping for Diwali gifts together. We had a total of Rs. 1640 between us. My wife had Rs. 240 more to begin with, but she spent twice as much as I did, and ended up with two thirds as much money as I had left. How much did I spend?

12. Next week I am going to collect rent from my tenant, visit the new museum, go to the UTI office, and have my teeth checked at the dentist's. My tenant is not at home on Wednesday or Friday the UTI office is closed on weekends, and the museum is closed Mondays. My dentist can give me an appointment on Tuesday, Thursday or Saturday. What day can I do everything I've planned?

13. How many minutes is it before six o'clock, if fifty minutes ago it was four times as many minutes past three o'clock?

14. At what time after 4:00 p.m. is the minutes hand of a clock exactly aligned with the hour hand?

15. A man just finished having his house built, and needs something more. At the hardware store the shopkeeper shows him what he wants and says, "Each is Rs. 10." "OK", says the man, "I'll take 1243, so here's Rs. 40." What had he bought?

16. How many odd-numbered pages are there in a book 749 pages long?

17. Multiply by 7 the number of 8's followed by 3, but not by 8, in the number below.

    381654783298514285838385737983256941837408326

18. Visiting a new town, you discover that there are only two barber shops, with only one hair-dresser in either. In need of a haircut, you explore further and find that one barber is messy and disorganized and has a terrible haircut, while the other is neat and tidy and has a beautiful haircut. Which one do you choose to cut your hair?

19. In the kindergarten class which I teach, a pupil came in excitedly this morning and said that it was the birthday of both her father and her grandfather, and what is more, she said, they're both the same age! Could she be right?

20. There is a family party consisting of two fathers, two mothers, two sons, one father-in-law, one mother-in-law, one daughter-in-law, one grandfather, one grandmother and one grandson. What is the minimum number of persons required so that this is possible?

21. My doctor gave me eight pills, and told me to take one every half-hour. How long does it take me to use up all of the pills?

22. Which word becomes smaller when you append more letters to it?

23. A carpenter cuts a board of wood twelve feet long into one foot pieces without stacking. How many cuts must he make?

24. A fly is flying between two trains, each travelling towards each other on the same track at 60 kmph. The fly reaches one engine, reverses itself immediately, and flies back to the other engine, repeating the process each time. The fly is flying at 90 kmph. If the fly flies 180 km before the trains meet, how far apart were the trains initially?

25. Ashok Trivedi lives on the twenty-fourth floor of an apartment building in Bombay. Every morning when he goes to work, he takes the lift down to the ground floor. Every evening, when he's the only person in the lift, he gets off on the eighth floor, and walks up several flights of stairs to reach his apartment. He would prefer to use the lift all the way, if he could. Why can't he?

26. A frog is climbing out of a well, which is twenty feet deep. Every day the frog jumps up three feet, but slips back two feet in the night. In how many days will the frog be able to jump out of the well?

27. Jack had some beans that grew into trees very quickly. The trees double in height every hour. At 10 a.m. the trees were four thousand feet high. When were they two thousand feet high? What was their height at 5:00 am.?

28. I was very tired and went to bed at 8:00 p.m. However, I had an important engagement at 10:00 a.m. the following day, and did not wish to oversleep, so I set the alarm for 9:30 o'clock. For how long did I sleep?

29. On a train, Kapoor, Tandon and Mehra are the fireman, brakeman and engineer, but not respectively. Also aboard are three businessmen who have the same names: a Mr. Kapoor, a Mr. Tandon and a Mr. Mehra.

1. Mr. Tandon lives in Delhi.
2. The brakeman lives exactly halfway between Jammu and Delhi.
3. Mr. Mehra earns exactly Rs. 20,000 per year.
4. The brakeman's nearest neighbour, one of the passengers, earns exactly three times as much as the brakeman.
5. Kapoor beats the fireman at table tennis.
6. The passenger with the same name as the brakeman lives in Jammu. Who is the engineer?

30. You have two identical-looking iron bars, but no other equipment (such as a thread, or compass, or a steady current flowing through a galvanometer). One of the bars is a magnet; how can you tell which one is the magnet?

31. A glass contains 250 ml of water, another identical glass contains 250 ml of wine. A teaspoon of water from the first glass is poured into the wine, the mixture is thoroughly stirred, and a teaspoon from it is poured back into the water. Is there now more water in the wine, or more wine in the water?

32. A man was looking at a portrait. Someone asked him, " Whose picture are you looking at?" He replied, pointing at the portrait: " Brothers and sisters have I none, but this man's father is my father's son." Whose picture was the man looking at? Suppose the man had answered, "Brothers and sisters have I none, but this man's son is my father's son." Now whose picture is the man looking at?

33. In the town called Hirsute, the following facts are true

(1) No two inhabitants have exactly the same number of hairs.
(2) No inhabitant has exactly 2025 hairs.
(3) There are more inhabitants than there are hairs on the head of any one inhabitant.

What is the largest possible number of the inhabitants of Hirsute?

34. Two men were being tried for a murder. The jury found one of them guilty and the other not guilty. The judge turned to the guilty one and said, "This is the strangest case I have ever come across! Though your guilt has been established beyond all doubt, and you should ordinarily have been sentenced to death, I am compelled to set you free." Why did the judge say this?

35. A man owned no watch, but he had an accurate clock, which, however, he sometimes forgot to wind. Once when this happened he went to the house of a friend, passed the evening with him, went back home, and set his clock. How could he do this without knowing beforehand the length of the trip.

36. A certain street contains 1000 buildings. A signmaker is contracted to number the houses from 1 to 1000. How many zeros will he need?

37. The Protagoras Paradox: Protagoras was a teacher of law in ancient Greece, who once had a poor but talented student. Protagoras agreed to teach the student without a fee,

provided that on completion of his studies the student would pay to Protagoras a certain amount from the earnings of the first case that he won. The student accepted this condition, completed his studies, but did not take any cases. Time passed, Protagoras grew impatient, and sued the student for the amount due.

Here are the arguments that each gave before the judge:

Student: If I win the case, then naturally I do not have to pay. If I lose, then I have not won my first case, and nothing can be due from me, in accordance with the agreement with Protagoras. So no matter that I win or lose, I don't have to pay a drachma.

Protagoras: If he loses the case, then of course he must pay. (What else can it mean to lose the case?) If he wins the case, then he will have won his first case, and so must pay me my fee. In either case, I should receive the amount due.

Who is right?

Here are some more problems connected with paradoxes:

38. Suppose I declare: "I am now lying." Am I lying or telling the truth?

39. A version of the liar's paradox was given by the English logician P.E.B. Jourdain as follows. On one side of a card is written:

THE SENTENCE ON THE OTHERSIDE OF THIS CARD IS TRUE.

On turning the card over you find:

THE SENTENCE ON THE OTHERSIDE OF THIS CARD IS FALSE.

Is the maker of the card telling the truth?

40. A card contains the following three sentences:

1. THIS SENTENCE CONTAINS FIVE WORDS.
2. THIS SENTENCE CONTAINS TWO VERBS.
3. EXACTLY ONE SENTENCE ON THIS CARD IS TRUE.

Is sentence 3 true or false?

41. A barber in a certain small town shaves all the men who do not shave themselves, but never any who do shave themselves. Does the barber shave himself If he shaves himself, he violates the rule of shaving one who shaves himself. If he does not shave himself, he violates the rule of shaving one who does not shave himself!

42. A shopper came one afternoon into a store owned equally by two partners. (One attended to the shop during the day, the other during the night.) The shopper gave to the shopkeeper a hundred rupee note, thinking it was a fifty. The daytime shopkeeper too thought it was a fifty, and returned the change accordingly. Several hours later he discovered that it was a hundred. Should he tell this to his partner?

43. A friend (not a very close one, as you may surmise from what follows) and I were having tea in a posh Delhi restaurant. The waiter brought a plate of two pastries, one larger than the other. My friend said, "Please help yourself." I said, "Okay", and helped myself to the larger pastry. After a tense silence, my companion said, " If you had offered me first

choice, I would have chosen the smaller one." I replied, "What's your problem? You've got it now!" Was my friend justified in complaining?

44. The population of an island in the South Pacific consists of two, and only two, types of people: the knights, who invariably tell the truth, and the knaves, who always lie.

Three of the inhabitants, called A, B and C were standing together. A newcomer to the island asked A, "Are you a knight or a knave?" A mumbled his answer rather indistinctly, so the stranger could not quite make out what he had said. The stranger then asked B, "What did A say?" B replied, "A said that he was a knave." Whereupon C said, "Don't believe B, he's lying."

What are B and C?

45. Suppose that the stranger asked A, instead, "How many knights are among you?" Again A replies indistinctly. So the stranger asks B, "What did A say" B replies, "A said that there is one knight among us." Then C says, "Don't believe B, he's lying!" Now what are B and C?

46. There are only two inhabitants, A and B. A says, "At least one of us is a knave." What are A and B?

47. Suppose A says, "Either I am a knave, or B is a knight." What are A and B?

48. Consider once more A, B and C, each of whom is either a knight or a knave. A says, "All of us are knaves." B says, "Exactly one of us is a knight." What are A, B and C?

49. Professors Ahmad and Chaturvedi are extremely strange persons. Prof. Ahmad lies on Mondays, Tuesdays and Wednesdays, but tells the truth on other days of the week. Prof. Chaturvedi lies on Thursdays, Fridays and Saturdays, but tells the truth on other days of the week. They made the following statements:

Prof. Ahmad: "Yesterday was one of my lying days."
Prof. Chaturvedi: "Yesterday was one of my lying days too."
What day of the week was it?

50. Both Professors looked very alike, and one day they said to a visitor to their department:
First Prof: "I'm Ahmad."
Second Prof: "I'm Chaturvedi."

Which was which?

5 1. On another occasion, both Professors made the following statements:

First Prof: (1) "I lie on Saturdays."
(2) "I lie on Sundays."
Second Prof: "I will lie tomorrow."
What day of the week was it?

52. Ali Baba had four sons, to whom he bequeathed his 39 camels, with the proviso that the legacy be divided in the following way: the oldest son was to receive one half the property, the next a quarter, the third an eighth and the youngest one tenth. The four brothers were at a loss as to how to divide the inheritance between themselves, without cutting up a camel, until a stranger appeared upon the scene. Dismounting from his camel, he asked if he might help, for he knew just what to do. The brothers gratefully accepted his offer.

Adding his own camel to Ali Baba's 39, he divided the 40 as per the will. The oldest son received 20, the next 10, the third five, and the youngest, four. One camel remained: this was his, which he mounted and rode away.

Scratching their heads in amazement, they started calculating. The oldest thought: is not 20 greater than the half of 39? Someone must have received less than his proper share! But each brother discovered that he had received more than his due. They decided to consult a hermit to explain how this came to be. How did the hermit resolve the puzzle?

53. Five rooms are interconnected with themselves and an outside verandah by doors on every wall, as shown in Fig. 8. Is it possible to start in one room, or the outside, walk through every doorway exactly once, and return to the starting point?

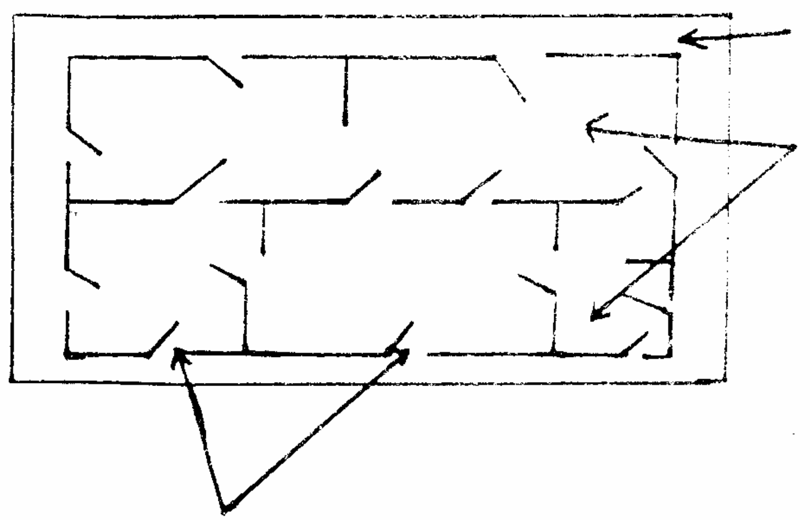

**Figure 8**

54. J.E. Littlewood, the eminent English mathematician, once published a paper in a French journal. he had almost no knowledge of French, so he requested his friend Prof. Riesz to translate this paper form the English.. When it appeared, Littlewood's paper contained, in order, (in French) the following three footnotes:

I am greatly indebted to Prof. Riesz for translating the present paper.

I am indebted to Prof. Riesz for translating the preceding footnote.

I am indebted to Prof Riesz for translating the preceding footnote.

Why did Prof. Littlewood not require an infinite series of footnotes similar to the last?

# 1.7 SUMMARY

In this Unit you have seen through several examples that problem-solving involves analysing the given data, arranging them in logical order, invoking other information that has a bearing on the problem, imagining and conceptualising, thus reasoning towards the solution. In this way Euler transformed the Konigsberg bridges problem into one involving graph traversals; and his technique helped generate ideas to attack a number of apparently unrelated problems, e.g. the cannibals and missionaries problems. Similarly, the realisation (or the remembrance from high-school geometry) that the sum of the lengths of the perpendiculars from a point inside a triangle to its sides remains a constant becomes useful in solving pouring problems in which the amount of liquid remains unchanged through the decantation. The amount of liquid is represented by the sum of the perpendicular distances. Finally, the twelve coins problem shows how organising the data in a logical structure such as a decision tree helps in arriving at its solution.