UNIT 3 INEQUALITIES

Structure

3.1 INTRODUCTION

So far we have discussed equations of various kinds. Now we shall consider some **inequalities**; not of the social kind, but between real numbers. A mathematical inequality is a mathematical expression of the condition that of two quantities one is **greater than, greater than or equal to**, less than or less than or equal to the other. An inequality that holds for every real number is called an **absolute inequality.** In this unit we shall restrict ourselves to such inequalities.

We will discuss six famous absolute inequalities. We have divided them into two sections—those that have been used for centuries and those that were discovered by some famous nineteenth century European mathematicians. These inequalities have several applications also. We will discuss a few of them. You may come across some applications in other courses too, at which time we hope that you will find that you didn't study this unit in vain!

Let us list our unit objectives now.

Objectives

After reading this unit you should be able to prove and apply

- the inequalities of the means;
- the triangle inequality;
- the Cauchy-Schwarz (Bunyakovskii) inequality:
- Weierstrass' inequalities:
- Tchebychev's inequalities.

Let us discuss the inequalities one by one.

3.2 INEQUALITIES KNOWN TO THE ANCIENTS

In this section we shall discuss two inequalities handed down to us by ancient mathematicians. But first we will give a list of some properties of inequalities you must be familiar with. They are the following:

for a, b, c, d $\in$ **R**

i) $a \geq b, c \geq 0 \Rightarrow ac \geq bc$

ii) $a \geq b \Leftrightarrow -a \leq -b$

iii) $a \geq b \Leftrightarrow \frac{1}{a} \leq \frac{1}{b}$ provided $a \neq 0, b \neq 0$.

iv) $a \geq b, c \geq d \Rightarrow a+c \geq b+d$

v) $a^n \geq b^n, a \geq 0 \Rightarrow a \geq b$, where $n \in \mathbf{N}$.

We will often use these properties implicitly while proving the inequalities mentioned in the unit objectives.

Now let us discuss the inequality that relates three averages.

3.2.1 Inequality of the Means

An important part of arithmetic that can be traced back to the Babylonians and Pythagoreans (approximately 6th century B.C.) is the theory of means or averages. The word "average" comes from the Latin word "havaria", which was the insurance paid to compensate for damage to goods in transit in the olden days. All of us are familiar with compensate for damage to goods in transit in the olden days. All of us are familiar with the term "average". In fact, all of us must have often calculated the average of a finite set of numbers by adding them up and dividing the sum by the total number of these numbers. But this is only one of many types of averages. We will discuss three of these types here. Let us start with the "usual" average.

Definition: the **arithmetic mean (AM)** of n real numbers $x_1, x_2, ..., x_n$ is

$$\frac{x_1 + x_2 + \ldots + x_n}{n}, \text{ that is } \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right).$$

For example, the AM of $\frac{1}{2}, \frac{-1}{3}$ and 0 is $\frac{\frac{1}{2} - \frac{1}{3} + 0}{3} = \frac{1}{18}$.

The AM is often used in statistics for studying data.

Another type of average is the geometric mean. This is the best mean to use if we want to find the mean of any finite set of positive numbers that follow geometric progression. Thus, this mean is very useful for studying population growth. Let us see how the geometric mean is defined.

Definition: The **geometric mean (GM)** of n positive real numbers

$x_1, x_2, \ldots\ldots, x_n$ is

$$(x_1 x_2 \ldots\ldots x_n)^{1/n}, \text{ that is, } \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$$

For example, the GM of 3 and 4 is $\sqrt{3 \times 4} = \sqrt{12}$, and the GM of 2, 4 and 8 is $(2 \times 4 \times 8)^{1/3} = 4$.

Yet another kind of average of numbers is their harmonic mean, which we now define.

Definition: The **harmonic mean (HM)** of n non-zero real numbers

$x_1, x_2, \ldots. x_n$ is

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

Thus, the HM of $x_1, x_2, \ldots\ldots, x_n$ is the inverse of the AM of $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$.

For example, the HM of $-2, \frac{1}{3}$ and 7 is $\frac{3}{-\frac{1}{2}+3+\frac{1}{7}} = \frac{42}{37}$.

The HM is the most appropriate type of average to use when we want to find the average rate of a set of varying rates. Thus, it is the best average to use for obtaining the average velocity of a vehicle covering various distances at different speeds.

At this point we would like to make a remark.

Note: We can obtain the AM of **any** n real numbers. But, we only define the GM of n **positive** real numbers; and the HM of n **non-zero** real numbers.

This proof is due to Cauchy] who you will meet again in Sec. 6.3.

Now let us look at the three different means together. To do so, we clearly need to restrict ourselves to positive real numbers. What is the AM of 2, 4 and 8? How is it related to their GM? And, how is their GM related to their HM? The following result answers these questions.

Theorem 1: Let $\{x_1, x_2, ..., x_n\}$ be any finite set of positive real numbers, and let A, G and H denote their arithmetic, geometric and harmonic means, respectively. Then

$A \geq G \geq H$,

and $A = G = H$ iff $x_1 = x_2 = ... = x_n$.

We will only give a broad outline of the proof here. The inequality $A \geq G$ is first proved by induction (see Unit 2) for all those integers n that are powers of two. That is,

$$\frac{x_1 + x_2 + ... + x_{2^m}}{2^m} \geq (x_1 x_2 .... x_{2^m})^{2^{-m}}, m \in N \qquad ...(1)$$

and equality holds iff $x_1 = x_2 = ... = x_2m$.

Now, given any $n \in N$, we can always choose $r \in N$ such that $2^r > n$.

We apply (1) to the 2^r numbers $x_1, x_2, ..., x_n, A, ..., A$, where the number of A's is $2^r - n$ We get

$$\frac{x_1 + x_2 + ... + x_n + A + A + ... + A}{2^r} \geq (x_1 x_2 ... x_n A ... A)^{2^{-r}}$$

(with equality iff $x_1 = x_2 = ... = x_n = A$.)

$$\Rightarrow \quad \frac{nA + (2^r - n)A}{2^r} \geq (G^n A^{(2^r - n)})^{2^{-r}}, \quad \text{since } \sum_{i=1}^{n} x_i = nA.$$

$$\Rightarrow \quad A^{2^r} \geq G^n \; A^{2^r - n}$$

$$\Rightarrow \quad A^n \geq G^n$$

$\Rightarrow$ $A \geq G$, since A and G are positive real numbers.

Note that $A = G$ iff $x_1 = x_2 = ... = x_n$.

Thus, the result is true $\forall\, n \in N$.

Now let us consider the n positive numbers $\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}$.

Since their AM is greater than or equal to their GM, we get

$$\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \geq \left(\frac{1}{x_1} . \frac{1}{x_2} ... \frac{1}{x_n}\right)^{\frac{1}{n}}$$

$$\Rightarrow \quad \frac{1}{H} \geq \frac{1}{G}$$

$$\Rightarrow \quad G \geq H.$$

Note that $H = G$ iff $\frac{1}{x_1} = \frac{1}{x_2} = \ldots = \frac{1}{x_n}$, that is, $x_1 = x_2 = \ldots = x_n$.

Thus, $A \geq G \geq H$, with equality iff $x_1 = x_2 = \ldots = x_n$.

In about 320 A.D. the geometer Pappus of Alexandria gave a geometric construction of the AM, GM and HM of two numbers. His construction is as follows:

Draw a semicircle with a+b as diameter (see Fig. 1). Let its diameter be AC, with mid-point O.

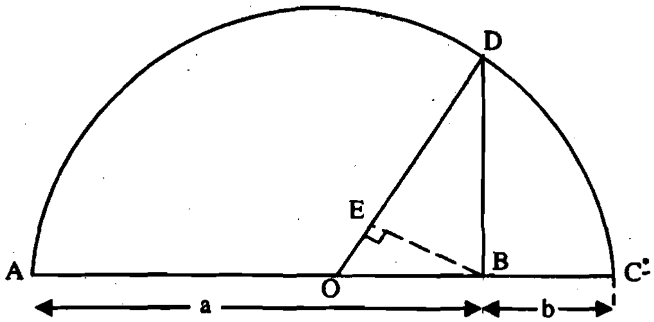

Fig. 1. The AM, GM and HM of a and b are DO, DB and DE, respectively.

Then OA is the radius of the circle. Mark off the point B on AC such that AB = a. Then BC = b. Draw BD $\perp$ AC to meet the semicircle in D. Then draw BE $\perp$ DO, as in Fig. 1.
Then Pappus proved that

DO is the AM of a and b.
DB is the GM of a and b.
DE is the HM of a and b.

Since DO $\geq$ DB $\geq$ DE, this gives us a geometric proof for Theorem 1, when n = 2.

Now let us apply Theorem 1 to prove some more inequalities.

Example 1: Show that $\left(\sum_{i=1}^{n} i^r\right)^n > n^n (n!)r$, where n! denotes factorial n and $r > 0$.

Solution: Let r be a fixed positive real number. Consider the n positive number $1^r, 2^r, \ldots, n^r$. By Theorem 1

$$\frac{1^r + 2^r + \ldots + n^r}{n} \geq (1^r . 2^r \ldots n^r)^{1/n} = ((n!)^r)^{1/n}.$$

Since the numbers $1^r, 2^r, \ldots, n^r$ are not equal, their AM is strictly greater than their GM. Thus

$$\left(\frac{1^r + 2^r + \ldots + n^r}{n}\right)^n \geq (n!)^r$$

$$\Rightarrow \left(\sum_{i=1}^{n} i^r\right)^n > n^n (n!)^r$$

We can prove several inequalities, which are particularly useful in mathematics, by using Theorem 1. We ask you to **prove some of them in the following exercises.**

E 1) Show that $(ab + xy)(ax + by) \geq 4abxy$, where a, b, x, y are positive real numbers. Under what conditions on a, b, x and y would the equality hold?

E2) For any $n \in \mathbf{N}$ and positive real numbers x and y, show that

a) $(xy^n)^{1/n+1} \le \frac{x+ny}{n+1}$,

b) $\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$,

c) $\left(1+\frac{1}{m}\right)^m < \left(1+\frac{1}{n+1}\right)^n$, where $m \in \mathbf{N}$ such that $m < n$.

E3) Is Theorem 1 true if we remove the condition that the numbers are positive? Why?

Now, you know that the inequalities in Theorem 1 become equalities when $x_1 = x_2 = ... = x_n$. When this happens, then $x_i = A = G = H \ \forall \ i = 1, ..., n$. Thus, we see that

> if $x_1, x_2, ..., x_n$ are n positive real numbers such that $x_1 + x_2 + ... + x_n$ is a constant, then their arithmetic mean attains its lowest value and their geometric mean attains its maximum value when $x_1 = x_2 = ... = x_n = A = G$

Let us see how to use this fact for obtaining some maximum and minimum values. For convenience, we shall denote the set of positive real numbers by $\mathbf{R}^+$.

Example 2: Find the greatest value of xyz, where $x, y, z \in \mathbf{R}^+$ are subject to the condition $yz + zx + xy = 12$.

Solution: xyz has greatest value when $(xyz)^2 = (yz)(zx)(xy)$ has greatest value. Since $yz+zx+xy$ is a constant, we know that the maximum value of $(yz)(zx)(xy)$ is attained when $yz = zx = xy$, that is, when $x = y = z$.

Then, $yz + zx + xy = 12 \Rightarrow x = y = z = 2$.

Hence, the maximum value of xyz is $2^3 = 8$.

Example 3: If the sum of the sides of a triangle is the constant 2s, prove that the area is greatest when the triangle is equilateral.

Solution: Let a, b, c be the sides of the triangle, where

$a+b+c= 2s$,

and let Δ denote the area of the triangle.

Then $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

So, Δ will be greatest when $(s-a)(s-b)(s-c)$ is maximum.

Now $(s-a) + (s-b) + (s-c) = s$, a constant.

Thus, $(s-a)(s-b)(s-c)$ is maximum when

$s-a = s-b = s-c$, that is, $a = b = c$.

Thus, the area is maximum when the triangle is an equilateral triangle.

Why don't you try these exercises now?

E4) a) Prove that if the sum of two positive numbers is given, their product is greatest when they are equal.

b) Is (a) true if the words 'sum' and 'product' are interchanged? Why?

E5) Find the greatest value of $(5+x)^3(5-x)^4$, for $-5 < x < 5$.

(**Hint:** The greatest value of $(5+x)^3(5-x)^4$ occurs when the greatest value of $\left(\frac{5+x}{3}\right)^3\left(\frac{5-x}{3}\right)^4$ occurs.)

E 6) When does a cuboid, with dimensions x, y and z such that x+y+z is fixed, have maximum volume?

E 7) Under what conditions on the dimensions, will a cuboid with fixed volume have minimal surface area?
(**Hint:** Use the inequality $G \geq H$.)

You can study other techniques for obtaining maximum values in our course on calculus.

Let us now consider another inequality, which follows from Theorem 1

Theorem 2: If $x_1, ..., x_n \in \mathbf{R}^+$ such that not all of them are equal, and $m \in \mathbf{Q}, m \neq 0, m \neq 1$, then

$$\frac{x_1^m + x_2^m + ... + x_n^m}{n} \leq \left(\frac{x_1 + x_2 + ... + x_n}{n}\right)^m, \text{ if } 0 < m < 1, \text{ and}$$

$$\frac{x_1^m + x_2^m + ... + x_n^m}{n} \geq \left(\frac{x_1 + x_2 + ... + x_n}{n}\right)^m, \text{ if } m < 0 \text{ or } m \geq 1, \text{ and}.$$

The proof of this result uses Theorem 1. We shall not give it here.

A result that follows from Theorem 2 (and Theorem 1) is that

If $x_1 + x_2 + ... + x_n = c$, a constant, then

for $0 < m < 1$ the maximum value of $\sum_{i=1}^{n} x_i^m$ is $n^{1-m}c^m$, and

for $m < 0$ or $m \geq 1$ the minimum value of $\sum_{i=1}^{n} x_i^m$ is $n^{1-m}c^m$.

These values are attained when $x_1 = x_2 = ... = x_n$

Again, we shall not prove this result in this course. But let us consider an example of its use for finding some maximum and minimum values.

Example 4: Find the least value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, where $x, y, z \in \mathbf{R}^+$ and $x + y + z = 27$.

Solution: $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is of the form $x^m + y^m + z^m$, where $m = -1 < 0$. Since $x + y + z = 27$, the least value is obtained when $x = y = z$.

And then $x + y + z = 27$ gives us $x = y = z = 9$.

Thus, the minimum value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is $\frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$.

Note that we could have also obtained the answer by applying the inequality $G \geq H$ (of Theorem 1), exactly on the lines of the solution of E7.

Now for some exercises.

E 8) Show that the sum of the mth powers of the first n even numbers is greater than $n(n+1)^m$, if $m > 1$.

E 9) Show that $\sqrt{1} + \sqrt{2} + ... + \sqrt{n} < n\sqrt{\frac{n+1}{2}}$, where $n \in \mathbf{N}$.

E 10) Let $a_1, a_2, ..., a_n \in \mathbf{R}^+$ and $p, q \in \mathbf{N}$ such that $p > q$. Show that

$$a_1^q + a_2^q + ... + a_n^q < n^{p-q}\left(a_1^p + a_2^p + ... + a_n^p\right)$$

(**Hint:** Put $m = \frac{-q}{p}$, $x_i = a_i^p$ in Theorem 2.)

So far we have discussed various inequalities related to the arithmetic, geometric and harmonic means. Now let us consider an inequality that had its origin in ancient Greek geometry.

3.2.2 Triangle Inequality

If you look up any translation of the ancient Greek mathematician Euclid's "Elements", you will find that Proposition 20 of Book 1 says:

"In any triangle, two sides taken together in any manner are greater than the remaining one.".

This result is the basis of the triangle inequality, which is a statement about the absolute value of numbers.

Recall that absolute value of $x \in \mathbf{R}$ is defined by

$$|x| = x, \text{ if } x \geq 0$$
$$= -x, \text{ if } x < 0.$$

Thus it satisfies the following properties

i) $|x| = |-x| \ \forall\, x \in \mathbf{R}$ and ii) $x \leq |x| \ \forall\, x \in \mathbf{R}$.

You can study the absolute value of real numbers in more detail in our course on calculus.

Now let us state the triangle inequality.

Theorem 3: Let $x_1, x_2, ..., x_n \in \mathbf{R}$. Then

$|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|$.

Moreover, equality holds only when all the non-zero x_i's have the same sign.

Proof: Let us prove the result for $n = 2$ first.

Now

$$(|x_1 + x_2|)^2 = (x_1 + x_2)^2, \text{ since } |x|^2 = x^2 \ \forall\ x \in \mathbf{R}$$
$$= x_1^2 + 2x_1 x_2 + x_2^2$$
$$\leq |x_1|^2 + 2|x_1||x_2|| + |x_2|^2, \text{ since } x \leq |x| \ \forall\ x \in \mathbf{R}.$$
$$= (|x_1| + |x^2|)^2.$$

Now we take the square root on both sides, keeping in mind that $|x| \geq 0 \ \forall\, x \in \mathbf{R}$. We get

$|x_1 + x_2| \leq |x_1| + |x_2|$, which is what we wanted to prove.

Note that if $x_1 < 0$, say $x_1 = -a$, and $x_2 > 0$ say $x_2 = b$, where $a, b > 0$, then

$|x_1 + x_2| = |b - a|$, while $|x_1| + |x_2| = a + b$. Thus, when x_1 and x_2 have opposite signs

$|x_1 + x_2| < |x_1| + |x_2|$.

So, Theorem 3 is true for $n = 2$.

Now, let us prove the result for the general case, by induction. So let $n > 2$ and assume that Theorem 3 is true for any $n-1$ numbers. Now consider

$$|x_1 + x_2 + ... + x_n| = |x_1 + x_2 + ... + x_{n-1}) + x_n|$$
$$\leq |x_1 + x_2 + ... + x_{n-1}| + |x_n|$$
$$\leq |x_1| + |x_2| + ... + |x_{n-1}| + |x_n|, \text{ since the result is true for } n-1 \text{ numbers.}$$

Thus, $|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n| \ \forall\, n \in \mathbf{N}$.

Further, just as we have shown for the case n = 2, strict inequality holds if all the non-zero x_i's don't have the same sign.

Theorem 3 is not only true for real numbers. In our course on linear algebra we have proved that if $z_1, z_2, ..., z_n \in \mathbf{C}$, then $|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|$,

where $|z|$ is the modulus of z.

Let us verify Theorem 3 for the numbers $-2, 1, 5, 0$.

Since $|-2 + 1 + 5 + 0| = |4| = 4$, and

$|-2| + |1| + |5| + |0| = 2 + 1 + 5 + 0 = 8$,

we find that strict inequality in Theorem 3 is true in this case. Note that -2 and 1 have opposite signs.

Why don't you try some exercises now?

E 11) The absolute value of the AM of n numbers is less than or equal to the AM of their absolute values. True or false. Why?

E 12) Prove or disprove that

$$|x - y| \le |x| - |y| \ \forall\ x, y \in \mathbf{R}.$$

(To disprove a statement means to show that it is false. See the appendix of Block 1.)

E 13) Prove or disprove that

$$|x - y| \ge \big||x| - |y|\big| \ \forall\ x, y \in \mathbf{R}.$$

(**Hint:** Write $|x| = |(x - y) + y|$, and also use the fact that $|x| = |-x| \ \forall\ x \in \mathbf{R}$.)

Now let us discuss some "newer" inequalities.

3.3 LESS ANCIENT INEQUALITIES

In this section we shall discuss four important inequalities which are due to some mathematical giants of the nineteenth century. We start an inequality due to three mathematicians.

3.3.1 Cauchy - Schwarz Inequality

Augustin - Louis Cauchy, the famous French mathematician, was responsible for developments in infinite series, function theory, differential equations, determinants, probability and several other areas of mathematics. One of his contributions was result, which was later generalised by the German mathematician H.A. Schwarz (1848-1921). We now state this result, which was also proved independently by the Russian mathematician Bunyakovskii.

Fig. 2 : Cauchy (1789-1857)

Theorem 4 (Cauchy-Schwarz Inequality): Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n \in \mathbf{R}$.

Then

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \le (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2),$$

with equality iff $a_i = cb_i \ \forall\ i = 1, ..., n$, where c is a fixed real number.

When $a_i = cb_i \ \forall\ i = 1, .., n$, where c is a constant, we say that the n-tuples $(a_1 ..., a_n)$ and $(b_1 ... b_n)$ are **proportional.**

Proof: To help you understand the proof we shall prove it for n = 3 first. Then you can try and generalise it (see E14).

Now

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$= (a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2\, b_1\, b_2) + (a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3\, b_2\, b_3) + (a_3^2b_1^2 + a_1^2b_3^2 - 2a_3a_1\, b_3\, b_1)$

$= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 \geq 0.$

When will the equality sign hold? Equality holds iff $a_1b_2 - a_2b_1 = 0$, $a_2b_3 - a_3b_2 = 0$ and $a_3b_1 - a_1b_3 = 0$, that is, $a_1 = cb_1$, $a_2 = cb_2$, $a_3 = cb_3$, for a fixed real number c.

Thus, we have proved the result for n = 3.

Now, to complete the proof of Theorem 4, why don't you try this exercise?

E 14) Prove Theorem 4 for any $n \in \mathbf{N}$.

Let us consider an application of Theorem 4 for locating the roots of a polynomial. Before going further, you may like to keep Unit 3 nearby for easy reference.

Theorem 5: If all the roots of the real polynomial equation

$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0$ are real, then they lie between $\frac{-a_1}{n} - \frac{n-1}{n}\sqrt{a_1^2 - \left(\frac{2n}{n-1}\right)a_2}$

and $\frac{-a_1}{n} + \frac{n-1}{n}\sqrt{a_1^2 - \left(\frac{2n}{n-1}\right)a_2}$

Proof: From Theorem 1 of Unit 3, you know that the given equations has n roots. Let x be a root. If $x_1, \dots, x_{n-1}$ are the other roots, then by Theorem 4 of Unit 3,

$x + x_1 + \dots + x_{n-1} = -a_1$

$$\Rightarrow \quad (a_1 + x)^2 = (x_1 + \dots + x_{n-1})^2 \leq \underbrace{(1^2 + 1^2 + \dots + 1^2)}_{(n-1)\text{ times}}\left(x_1^2 + \dots + x_{n-1}^2\right), \text{ by Theorem 4.}$$

Also, by Theorem 4 of Unit 3,

$$x^2 + x_1^2 + \dots + x_{n-1}^2 = a_1^2 - 2a_2$$

$$\therefore \quad (a_1 + x)^2 \leq (n-1)(a_1^2 - 2a_2 - x^2)$$

$$\Rightarrow \quad nx^2 + 2a_1x - (n-2)a_1^2 + 2a_2(n-1) \leq 0$$

$$\Rightarrow \quad \left[x - \left(\frac{-a_1}{n} + \frac{n-1}{n}\sqrt{a_1^2 - \left(\frac{2n}{n-1}\right)a_2}\right)\right]\left[x - \left(\frac{-a_1}{n} - \frac{n-1}{n}\sqrt{a_1^2 - \left(\frac{2n}{n-1}\right)a_2}\right)\right] \leq 0,$$

by the quadratic formula.

This holds for all x such that

$$\frac{-a_1}{n} - \frac{n-1}{n}\sqrt{a_1^2 - \left(\frac{2n}{n-1}\right)a_2} \leq X \leq \frac{-a_1}{n} + \frac{n-1}{n}\sqrt{a_1^2 - \left(\frac{2n}{n-1}\right)a_2}$$

Thus, any root of the given polynomial equation must lie between the bounds of the given statement of the theorem.

Before giving an example of the use of Theorem 5, we shall make some related observations.

Remark 1 : Consider the polynomial equation $a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0$, where $a_i \in \mathbf{Z}$.

$\forall\ i = 0, ..., n$, and $a_n \neq 0$.

Then any rational root of this equation is of the form $\frac{d}{a_n}$, where d is a factor of a_0.

Remark 2: For cubic cases, we know from the discriminant (Sec. 3.3.2) when the roots are all real. And then Theorem 5 can be very useful, especially if we know that the roots are rational.

Let us now consider an example of the use of Theorem 5.

Example 5: Solve $x^3 - 23x^2 + 167x - 385 = 0$.

Solution: The discriminant of this equation (see Sec. 3.3.2) is positive. Hence the given equation has three distinct real roots. If the roots are rational they must be integral factors of -385. Thus, they must belong to the set

$\{\pm 1, \pm 5, \pm 7, \pm 11, \pm 35, \pm 55, \pm 77, \pm 385\}$.

But, by Theorem 5 the roots must lie between $\frac{23}{3} - \frac{4}{3}\sqrt{7}$ and $\frac{23}{3} + \frac{4}{3}\sqrt{7}$. Thus, if they are rational, they can only be 5, 7, 11. On substituting these values in the equation, we find that they are indeed roots of the given equation. Also, you know that the equation can only have 3 roots. Hence, these values are the only roots.

You may now like to try to apply Theorem 5 yourself.

E15) Solve $x^3 - 2x^2 - x + 2 = 0$.

Now let us consider another example of the use of Theorem 4. In this example, we shall apply the Cauchy-Schwarz inequality twice to get an inequality that we want.

Example 6: Let $x, y, z \in \mathbf{R}^+$ such that $x^2 + y^2 + z^2 = 27$. Show that

$x^3 + y^3 + z^3 \geq 81$.

Solution: Let us first apply Theorem 4 to the two triples of real numbers, $(x^{3/2}, y^{3/2}, z^{3/2})$ and $(x^{1/2}, y^{1/2}, z^{1/2})$. We get

$(x^{3/2}x^{1/2} + y^{3/2}y^{1/2} + z^{3/2}z^{1/2})^2 \leq (x^3 + y^3 + z^3)(x+y+z)$, that is

$$(x^2 + y^2 + z^2)^2 \leq (x^3 + y^3 + z^3)(x + y + z) \qquad ...(2)$$

Now let us apply the Cauchy-Schwarz inequality to the triples

(x, y, z) and $(1, 1, 1)$. We get

$(x.1 + y.1 + z.1)^2 \leq (x^2 + y^2 + z^2)(1^2 + 1^2 + 1^2)$, that is

$(x + y + z)^2 \leq 3(x^2 + y^2 + z^2) = 81$

$\Rightarrow x + y + z \leq 9$

Thus, by (2)

$(x^2 + y^2 + z^2) \leq 9(x^3 + y^3 + z^3)$

But $x^2 + y^2 + z^2 = 27$. Thus,

$$(x^3 + y^3 + z^3) \geq \frac{(27)^2}{9}$$

$(x^3 + y^3 + z^3 \geq 81$.

Why don't you try some exercises now?

E16) If a, b, x, y $\in$ **R** such that $a^2 + b^2 = 1$ and $x^2 + y^2 = 1$, then prove that $ax + by \leq 1$.

E17) Prove that if $a_1, ..., a_n \in \mathbf{R}^+$, then

a) $(a_1 + a_2 + ... + a_n)\left[\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right] \geq n^2.$

b) $(a_1 + a_2 + ... + a_n)^2 \leq n(a_1^2 + a_2^2 + ... + a_2^2)$

c) $(\sqrt{a_1} + \sqrt{a_2} + ... + \sqrt{a_n})^2 \leq n(a_1 + a_2 + ... + a_n)$

E18) (Another form of the triangle inequality) If a, b, x, y $\in$ **R**, then show that

$\sqrt{(a-b)^2 + (x-y)^2} \leq \sqrt{a^2 + x^2} + \sqrt{b^2 + y^2}$

(**Hint:** Write $(a-b)^2 + (x-y)^2 = (a^2 + x^2) + (b^2 + y^2) - 2(ab + xy)$, and then apply Theorem 4 to (a, x) and (b, y).

E19) If x, y, z $\in \mathbf{R}^+$ such that $x^3 + y^3 + z^3 = 81$, then prove that $x + y + z \leq 9$.

E20) Prove or disprove the following generalisation of Theorem 4 :

Let $p \in$ **N**, $p \neq 1$, and $a_1, ..., a_n, b_1, b_2, , ..., b_n \in$ **R**.

Then $(a_1b_1 + a_2b_2 + ... + a_nb_n)^p \leq (a_1^p + a_2^p + ... + a_n^p)(b_1^p + b_2^p + ... + b_n^p)$

Fig. 3 : Karl Theodor Weierstrass

The Cauchy-Schwarz inequality has several applications in physics and mathematics, especially in the context of inner product spaces.

Now let us consider another useful set of inequalities.

3.3.2 Weierstrass' Inequalities

It is generally through that a good mathematician must have started serious mathematical studies at an early age. But the German mathematician Weierstrass (1815 – 1897) is an exception to this rule. This outstanding mathematician started serious mathematics at the age of forty. He was responsible for making analysis more rigorous, and is considered to be the "father of modern analysis". He is responsible for the following result.

Theorem 6 (Weierstrass' Inequalities): Let $a_1, a_2, ..., a_n$ be positive real numbers less than 1 and $S_n = a_1 + a_2 + ... + a_n$. Then

(i) $1 - S_n \leq (1-a_1)(1-a_2)...(1-a_n) < \frac{1}{1+S_n}$,

(ii) $1 + S_n \leq (1+a_1)(1+a_2)...(1+a_n) < \frac{1}{1-S_n}$ where it is assumed that $S_n < 1$.

Proof: We prove (i) by induction on n, a principle that we introduced you to in Unit 2.

If n = 1, then $S_1 = a_1$, and hence, $(1 - S_1) = (1 - a_1)$.

Also, since $0 < a_1^2 < 1$, $(1-a_1)(1+a_1) < 1$, that is, $(1-a_1) < \frac{1}{1+S_1}$.

So, (i) is true when n = 1.

Let us assume that (i) is true for n = m, where m $\in$ **N**.

We will see if it is also true for n = m+1.

Now $s_{m+1} = a_1 + ... + a_{m+1} = (a_1 + ... + a_m) + a_{m+1} = S_m + a_{m+1}$.

also $(1-a_1)(1-a_2)\dots(1-a_m) \ge 1 - S_m$, by our assumption.

Thus, $(1-a_1)(1-a_2)\dots(1-a_m)(1-a_{m+1}) \ge (1-S_m)(1-a_{m+1})$

R.H.S. $= 1-(S_m + a_{m+1}) + S_m a_{m+1}$

$= 1 - S_{m+1} + S_m a_{m+1}$

$> 1 - S_{m+1}$, since $S_m a_{m+1} > 0$.

So, $(1-a_1)\dots(1-a_{m+1}) > 1 - S_{m+1}$ ...(3)

Further, Since $(1-a_1)(1-a_2)\dots(1-a_m) < \frac{1}{1+S_m}$, by assumption, and $(1-a_{m+1}) < \frac{1}{1+a_{m+1}}$, we find that

$$(1-a_1)(1-a_2)\dots(1-a_{m+1}) < \frac{1}{(1+S_m)(1+a_{m+1})}$$

$$= \frac{1}{1+S_{m+1}+S_m a_{m+1}}$$

$$< \frac{1}{1+S_{m+1}} \qquad \dots(4)$$

(3) and (4), taken together, tell us that (i) is true for $n = m+1$. Hence, by induction, (i) is true $\forall\, n \in \mathbf{N}$.

Now, to complete the proof you can try E21.

E21) Prove (ii) of Theorem 6.

E22) For $0 < a_1, a_2, \dots, a_n < 1$, prove that

$$1 - \prod_{i=1}^{n} a_i < n - \sum_{i=1}^{n} a_i$$

(**Hint**: $0 < 1 - a_i < 1$.)

E23) Does Theorem 6 hold if $a_i < 0$ or $a_i > 1$ for any $i = 1, \dots, n$? Give reasons for your answer.

Fig. 4 : Tchebychev (1821-1894)

When you study mathematical analysis, you will find that Weierstrass' inequalities and their generalisations are very useful.

And finally, we shall discuss some inequalities due to a leading Russian mathematician.

3.3.3 Tchebychev's Inequalities

The mathematician Pafnutii L. Tchebychev (pronounced Che-bee-cheff) is most known for his tremendous work in analytic number theory and the theory of orthogonal polynomials. Over here we shall prove and apply some inequalities that are named after him.

Theorem 7 (Tchebychev's Inequalities): If $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbf{R}$ such that

i) $a_1 \le a_2 \le \dots \le a_n$, $b_1 \le b_2 \le \dots \le b_n$, then

$n(a_1b_1 + a_2b_2 + \dots + a_nb_n) \ge (a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n)$.

ii) $a_1 \ge a_2 \ge \dots \ge a_n$, $b_1 \le b_2 \le \dots \le b_n$, then

$n(a_1b_1 + a_2b_2 + \dots + a_nb_n) \le (a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n)$.

Proof: Let us prove (i) for the case $n = 3$, so that you can understand the proof more easily.

$(a_1 - a_2)(b_1 - b_2) \geq 0$. Therefore,

$$a_1b_1 + a_2b_2 \geq a_1b_2 + a_2b_1 \qquad ...(5)$$

Similarly, we get

$$a_2b_2 + a_3b_3 \geq a_2b_3 + a_3b_2 \qquad ...(6)$$

$$a_3b_3 + a_1b_1 \geq a_3b_1 + a_1b_3 \qquad ...(7)$$

Adding (5), (6) and (7), we get

$$2(a_1b_1 + a_2b_2 + a_3b_3) \geq a_1(b_2 + b_3) + a_2(b_3 + b_1) + a_3(b_1 + b_2)$$

Now we add $a_1b_1 + a_2b_2 + a_3b_3$ to both sides and simplify to get

$$3(a_1b_1 + a_2b_2 + a_3b_3) \geq (a_1 + a_2 + a_3)(b_1 + b_2 + b_3).$$

The proof of (i) for any $n \in N$ is on exactly the same lines. Let us prove it by induction on n.

The result is true for $n = 3$ (and, in fact, for $n = 1$ and 2).

Assume that it is true for $n - 1$. Then

$$(n-1)(a_1b_1 + ... + a_{n-1}b_{n-1}) \geq (a_1 + a_2 + ... + a_{n-1})(b_1 + b_2 + ... + b_{n-1})$$

Also $a_1b_1 + a_nb_n \geq a_1b_n + a_nb_1$, since $(a_1 - a_n)(b_1 - b_n) \geq 0$.

Similarly $a_2b_2 + a_nb_n \geq a_2b_n + a_nb_2$

$$\vdots$$

$$a_{n-1}b_{n-1} + a_nb_n \geq a_{n-1}b_n + a_nb_{n-1}$$

Adding up the left hand sides and the right hand sides of these n inequalities, we get

$$n(a_1b_1 + ... + a_nb_n) - a_nb_n \geq (a_1 + ... a_{n-1} + a_n)(b_1 + ... + b_{n-1} + b_n) - a_nb_n.$$

$$\Rightarrow n(a_1b_1 + ... + a_nb_n) \geq (a_1 + ... + a_n)(b_1 + ... + b_n).$$

And now, to complete the proof of the theorem, try E24.

E24) Prove (ii) of Theorem 7.

(**Hint:** Put $x_i = -a_i \; \forall \; i = 1, ... n$, and use (i))

Now let us consider an application of Tchebychev's inequalities.

Example 7: Show that

$$\sqrt{1} + \sqrt{2} + ... + \sqrt{n} \leq n\sqrt{\frac{n+1}{2}}.$$

Solution: Put $a_i = b_i = \sqrt{i} \; \forall \; i = 1, ..., n$ in Theorem 7. Then we get

$$n(\sqrt{1}\sqrt{1} + \sqrt{2}\sqrt{2} + ... + \sqrt{n}\sqrt{n}) \geq (\sqrt{1} + \sqrt{2} + ... + \sqrt{n})^2, \text{ that is,}$$

$$n\left[\frac{n(n+1)}{2}\right] \geq (\sqrt{1} + \sqrt{2} + ... + \sqrt{n})^2, \text{ since } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Taking the square root on both sides, we get

$$\sqrt{1}+\sqrt{2}+\ldots+\sqrt{n} \le n\sqrt{\frac{n+1}{2}}.$$

Now you can try some exercises.

E25) Show that

$$\frac{1}{\sqrt{n}}\left(\sqrt{1}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right) \quad \le (2n-1)^{1/4}.$$

(Hint: First apply Tchebychev's inequality to the n-tuples

$\left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right)$ and $\left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right)$; and then apply it again to the n-tuples

$\left(\sqrt{\frac{1}{1}}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right)$ and $\left(\sqrt{\frac{1}{1}}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right)$.)

E 26) If a, b, c, $\in \mathbf{R}^+$, then show that.

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}.$$

(Hint: See if it is possible to apply Theorem 7 to b+c, c+a, a+b and their inverses.)

With this inequality we come to the end of this unit. This doesn't mean that we've exhausted all the inequalities, or even all the important ones. We have just exposed you to a few elementary ones and some of their applications. As you study more mathematics you will come across these and several others.

Now let us quickly go through what we have covered in this unit.

3.4 SUMMARY

We have discussed several inequalities and their applications in this unit. Let us list them one by one,

1) The inequality of the means: The AM of any finite set of elements of $\mathbf{R}^+$ is greater than or equal to their GM, which is greater than or equal to their HM.

2) If $x_1, \ldots, x_n \in \mathbf{R}$ such that not all of them are equal, and $m \in \mathbf{Q}$, $m \ne 0, 1$, then

$$\frac{1}{n}\left(\sum_{i=1}^{n} x_i^m\right) \le \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^m \text{ if } 0<m<1, \text{ and}$$

$$\frac{1}{n}\left(\sum_{i=1}^{n} x_i^m\right) \ge \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^m \text{ if } m<0 \text{ or } m>1.$$

3) The triangle inequality : For $x_1, x_2, \ldots, x_n \in \mathbf{R}$

$|x_1+x_2+\ldots+x_n| \le |x_1|+|x_2|+\ldots+|x_n|$.

The inequality is strict in case all the non-zero x_is don't have the same sign.

4) Cauchy-Schwarz (or Bunyakovaskii) inequality: If $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{R}$, then

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

with equality iff $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ are proportional.

5) Weierstrass' inequalities: For $0 < a_1, ..., a_n < 1$ and $S_n = \sum_{i=1}^{n} a_i$

$$1 - S_n \le \prod_{i=1}^{n}(1-a_i) < \frac{1}{1+S_n},$$

$$1 + S_n \le \prod_{i=1}^{n}(1+a_i) < \frac{1}{1-S_n} \text{ (here } S_n < 1 \text{ is assumed).}$$

6) Tchebychev's inequalities: If $(a_1, ..., a_n), (b_1, ..., b_n) \in \mathbf{R}^n$ such that

i) $a_1 \le a_2 \le ... \le a_n$, $b_1 \le b_2 \le ... \le b_n$, then

$$n(a_1b_1 + a_2b_2 + ... + a_nb_n) \ge (a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n)$$

ii) $a_1 \le a_2 \le ... \le a_n$, $b_1 \ge b_2 \ge ... \ge b_n$, then

$$n(a_1b_1 + a_2b_2 + ... + a_nb_n) \le (a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n)$$

As usual, we suggest that you go back to the beginning of the unit and see if you have achieved the objectives. We have given our solutions to the exercises in the unit in the last section. Please go through them too.

With this we have come to the end of this course. We hope you have enjoyed it, and will find it of use in the future.

3.5 SOLUTIONS/ANSWERS

E 1) Applying the inequality $A \ge G$ to the positive real numbers ab and xy, we get

$$ab + xy \ge 2\sqrt{abxy} \qquad (8)$$

Now we apply $A \ge G$ to ax and by, and we get

$$ax + by \ge 2\sqrt{abxy} \qquad (9)$$

(8) and (9) $\Rightarrow (ab + xy)(ax + by) \ge 4\sqrt{abxy}\sqrt{abxy} = 4abxy$.

Note that equality holds iff $ab = xy$ and $ax = by \Leftrightarrow a = y$ and $b = x$.

E 2) a) Applying the inequality $A \ge G$ to the n+ 1 numbers

x, y, y, ..., y (n times), we get

$$\frac{x + ny}{n+1} \ge \left(x.y^n\right)^{\frac{1}{n+1}}$$

Note that equality holds iff $x = y$.

b) Put $x = 1$ and $y = 1 + \frac{1}{n}$ in (a). Then we get

$$\left[1\cdot\left(1+\frac{1}{n}\right)^n\right]^{\frac{1}{n+1}} < \frac{1+n\left(1+\frac{1}{n}\right)}{n+1}$$

$$\Leftrightarrow \left(1.\frac{1}{n}\right)^n < \left(1+\frac{1}{n}\right)^{n+1}$$

c) Let m, n ∈ N and m < n. Then, by (b),

$$\left(1+\frac{1}{m}\right)^m < \left(1+\frac{1}{m+1}\right)^{m+1} < \left(1+\frac{1}{m+2}\right)^{m+1} < \ldots < \left(1+\frac{1}{n}\right)^n$$

Hence the result.

E3) No, since the GM of negative numbers is not defined. Even A ≥ H does not remain true any longer.
For example, take the three numbers – 2, 1 and 3.

Their AM is $\frac{2}{3}$ and their HM is $\frac{3}{\frac{-1}{2}+1+\frac{1}{3}} = \frac{18}{5}$, and $\frac{2}{3} \ngeq \frac{18}{5}$.

E4) a) Let $x, y \in \mathbf{R}^+$ such that $x+y = c$, a constant. Then, xy is maximum when $\sqrt{xy}$ is maximum, that is, when $x = y$.

b) No. For example, let $x, y \in \mathbf{R}^+$ such that $xy = 1$.

If $x = y$, then $x = 1 = y$. And then $x + y = 2$. But this is not the maximum value of $x + y$, since $x = 5$ and $y = \frac{1}{5}$, for example, give a larger value of $x + y$.

E5) Since $-5 < x < 5$, $5 + x > 0$ and $5 - x > 0$.

Since $3\left(\frac{5+x}{3}\right) + 4\left(\frac{5-x}{4}\right) = 10$, a constant, the maximum value of

$\left(\frac{5+x}{3}\right)^3 + 4\left(\frac{5-x}{4}\right)^4$ occurs when $\frac{5+x}{3} = \frac{5-x}{4}$, that is, when $x = \frac{-5}{7}$.

Thus $(5 + x)^3 (5 - 3)^4$ is maximum when $x = \frac{-5}{7}$.

E6) Let $x + y + z = c$.

The volume of the cuboid is xyz.

This is maximum when $x = y = z$, since $x + y + z = c$.

Thus, the cube is a cuboid with maximum volume, under the given conditions.

E7) Let x, y and z be dimensions of the cuboid, where $xyz = c$, a constant.

Now, the surface area of the cuboid is $2(xy + yz + zx)$. This is minimum when $xy + yz + zx$ is minimum, that is, when

$xyz\left(\frac{1}{z}+\frac{1}{x}+\frac{1}{y}\right)$ is minimum.

Now, xyz=c, and the HM of x, y and z is $\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$.

By Theorem 1 we know that the HM is maximum when $x = y = z$. Thus $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is minimum when $x = y = z$. Thus the surface area of a cuboid with fixed volume is minimum when the cuboid is a cube.

E8) We apply Theorem 2 to the n numbers 2, 4, 6, ..., 2n.

We get

$$\frac{2^m + 4^m + 6^m + \ldots + (2n)^m}{n} > \left(\frac{2+4+6+\ldots+2n}{n}\right)^m = 2^m\left(\frac{1+2+\ldots+n}{n}\right)^m$$

$$\Leftrightarrow 2^m + 4^m + \ldots + (2n)^m > n \,.\, 2^m \,.\, \left(\frac{n(n+1)}{2n}\right)^m, \text{ since } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$= n(n+1)^m.$$

E9) Applying Theorem 2 to 1, 2, ..., n, we get

$$\frac{\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n}}{n} < \sqrt{\frac{n(n+1)}{2n}}$$

Hence the result.

E 10) Put $m = \frac{q}{p}$. Then $0 < m < 1$. Now we apply Theorem 2 to the n positive numbers

$a_1^p, a_2^p, \ldots, a_n^p$. We get

$$\frac{\left(a_1^p\right)^m + \left(a_2^p\right)^m + \ldots + \left(a_n^p\right)^m}{n} \le \left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{n}\right)^m$$

$$\Leftrightarrow \frac{a_1^q + a_2^q + \ldots + a_n^q}{n} \le \left(\frac{a_1^p + a_2^p + \ldots + a_3^p}{n}\right)^{q/p}$$

$$\Leftrightarrow a_1^q + a_2^q + \ldots + a_n^q \le (n^{p-q})^{1/q}\left(a_1^p + a_2^p + \ldots + a_n^p\right)^{q/p}$$

$$< n^{p-q}\left(a_1^p + a_2^p + \ldots + a_n^p\right), \text{ since } \frac{1}{p} < 1 \text{ and } \frac{q}{p} < 1.$$

E 11) Let $x_1, \ldots, x_n$ be n numbers.

Then their AM, $A = \frac{x_1 + x_2 + \ldots + x_n}{n}$.

$$\therefore |A| = \left|\frac{x_1 + x_2 + \ldots + x_n}{n}\right| \le \frac{|x_1| + |x_2| + \ldots + |x_n|}{n}, \text{ by theorem 3.}$$

$= \text{AM of } |x_1|, |x_2|, \ldots, |x_n|$.

Hence the statement is true.

E 12) False. For example, take $x = 1, y = -3$.

Then $|x - y| = |4| = 4$,

and $|x| - |y| = 1 - 3 = -2$.

E 13) $|x| = |(x - y) + y| \le |x - y| + |y|$. ...(10)

$\therefore |x| - |y| \le |x - y|$

Similarly, $|y| = |x + (y - x)| \le |x| + |y - x| = |x| + |x - y|$,

since $|x| = |-x|$.

Therefore, $|\,y\,| - |\,x\,| \le |\,x - y\,|$ ...(11)

(10) and (11) $\Rightarrow \big|\,|\,x\,| - |\,x\,|\,\big| \le |\,x - y\,|$

E 14) $\left(a_1^2+\ldots+a_n^2\right)\left(b_1^2+\ldots+b_n^2\right)-\left(a_1b_1+a_2b_2+\ldots+a_nb_n\right)^2$

$$= \sum_{\substack{i,\,j=i \\ i\ne j}}^{n}\left(a_ib_j - a_jb_i\right)^2 \ge 0$$

with equality iff $a_ib_j = a_jb_i \;\forall\; i, j = 1, \ldots, n,\; i \ne j$,

that is, iff $\dfrac{a_i}{b_i} = \dfrac{a_j}{b_j} \;\forall\; i \ne j$.

that is, iff $a_i = cb_i \;\forall\; i = 1, \ldots, n$, where c is a constant.

E 15) The discriminant of the equation is positive. Thus, the given equation has distinct real roots. They must lie between $\frac{2}{3}(1-\sqrt{7})$ and $\frac{2}{3}(1+\sqrt{7})$. If they are rational, they have to be factors of 2. Hence, they can be 1, –1, 2 or –2. Of these, ± 1 and 2, lie within the given bounds. On substitution we find that they actually are the roots. Since the equation has only 3 roots, – 1, 1 and 2 are its roots.

E 16) Just applying Theorem 4 ot the numbers a, b, x, y, we get

$(ax + by)^2 \le (a^2 + b^2)(x^2 + y^2) = 1.$

$\therefore\; ax + by \le 1.$

E 17) (a) We apply the Cauchy-Schwarz inequality to the n-tuples

$\left(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}\right)$ and $\left(\dfrac{1}{\sqrt{a_1}}, \dfrac{1}{\sqrt{a_2}}, \ldots, \dfrac{1}{\sqrt{a_n}}\right)$ we get

$$\left(a_1.\frac{1}{\sqrt{a_1}} + \ldots + a_n\frac{1}{\sqrt{a_n}}\right)^2 \le (a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)$$

$$\Leftrightarrow n^2 \le \left(a_1 + a_2 + \ldots + a_n\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right).$$

b) Applying Theorem 4 to the n-tuples (1, 1, ..., 1) and $(a_1, a_2, \ldots, a_n)$, we get the result.

c) Applying Theorem 4 to the n-tuples (1, 1, ..., 1) and $(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n})$ we get the result.

E 18) $(a-b)^2 + (x-y)^2 = (a^2 + x^2) + (b^2 + y^2) - 2(ab + xy),$

$\le (a^2 + x^2) + (b^2 + y^2) + 2\,|\,(ab + xy)\,|$ ...(12)

Also, by Theorem 4

$(ab + xy)^2 \le (a^2 + x^2)(b^2 + y^2)$

$\Rightarrow |\,ab + xy\,|\; \sqrt{a^2 + x^2}.\sqrt{b^2 + y^2}$ ...(13)

(12) and (13) $\Rightarrow (a-b)^2 + (x-y)^2 \le \left[\sqrt{(a^2 + x^2)} + \sqrt{(a^2 + y^2)}\right]^2,$

which gives us the desired result.

E 19) By the Cauchy-Schwarz inequality applied to (x, y, z) and (1, 1, 1), we have

$(x + y + z)^2 \le 3\,(x^2 + y^2 + z^2)$ ...(14)

Again, applying Theorem 4 to ($\sqrt{x}, \sqrt{y}, \sqrt{z}$) and $\left(x^{3/2}, y^{3/2}, z^{3/2}\right)$

we get $(x^2+y^2+z^2)^2 \le (x+y+z)(x^3+y^3+z^3)$

$$= 81\,(x+y+z) \qquad \ldots(15)$$

(14) and (15) $\Rightarrow (x+y+z)^4 \le 9.81\,(x+y+z)$.

$\Rightarrow (x+y+z)^3 \le (9)^3$

$\Rightarrow (x+y+z) \le 9.$

E20) False. For example, take $p = 3$ and the pairs $(1, 0), (1, -1)$.

Then $\{(1)(1)+(0)(-1)\}^3 \not\le (1^3+0^3)(1^3+(-1)^3)$

E21) We use the principle of induction on n.

For $n = 1$, $a_1 = s_1$, and hence, $1 + s_1 \le 1 + a_1$.

Also, $1 - a_1^2 \le 1$. Therefore, $1 + a_1 \le \dfrac{1}{1-a_1}$

Assume that the result holds for n=m.

Then $(1+a_1)(1+a_2)\ldots(1+a_m) \ge 1 + S_m$

$\therefore (1+a_1)(1+a_2)\ldots(1+a_m)(1+a_{m+1}) \ge 1 + S_m + a_{m+1} + S_m a_{m+1}$

$$> 1 + S_{m+1}$$

Also, $(1+a_1)(1+a_2)\ldots(1+a_m) \le \dfrac{1}{1-S_m}$

$$\therefore (1+a_1)\ldots(1+a_m)(1+a_{m+1}) < \frac{1}{(1-S_m)(1-a_{m+1})}$$

$$< \frac{1}{1-S_{m+1}}$$

Thus, (ii) is true for $n = m+1$, and hence $\forall\ n \in N$.

E22). We apply Theorem 6 (i) to the n numbers $1-a_1, 1-a_2, \ldots, 1-a_n$.

Then

$1-(1-a_1+1-a_2+\ldots+1-a_n) \le a_1 a_2 \ldots a_n$.

$$\Rightarrow 1 - n + \sum_{i=1}^{n} a_i \le \prod_{i=1}^{n} a_i$$

$$\Rightarrow 1 - \prod_{i=1}^{n} a_i \le n \sum_{i=1}^{n} a_i$$

E23) No. For example, let us take $a_1 = -1$ and $a_2 = 2$.

Then $1-(a_1+a_2) \le (1-a_1)(1-a_2)$

$\Rightarrow 0 \le 2(-1) = -2$, which is false.

E24) If we put $x_i = -a_i$, then $x_1 \le x_2 \le \ldots \le x_n$. Also $b_1 \le b_2 \le \ldots \le b_n$

So, by Theorem 7 (i),

$n(x_1 b_1 + x_2 b_2 + \ldots + x_n b_n) \ge (x_1 + x_2 + \ldots + x_n)(b_1 + b_2 + \ldots + b_n)$

$$\Leftrightarrow -n(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n) \geq -(a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$$

$$\Leftrightarrow n(a_1 b_1 + \ldots + a_n b_n) \leq (a_1 + a_2 + \ldots + a_n)(b_1 + \ldots + b_n)$$

E25) Applying Tchebychev's inequality to $\left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right)$ and $\left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right)$,

we get

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)^2 \leq n\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)$$

$$\leq n\left(1+\frac{1}{1.2}+\ldots+\frac{1}{(n-1)n}\right) \text{ since } \frac{1}{i^2} \leq \frac{1}{(i-1)i} \forall i.$$

$$= n\left\{1+\left(1-\frac{1}{2}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)\right\}$$

$$= n\left(1+1-\frac{1}{n}\right).$$

$$\Rightarrow 1 + \frac{1}{2} + \ldots + \frac{1}{n} \leq \sqrt{2n-1} \qquad \ldots(16)$$

Again, applying Theorem 7 to $\left(\sqrt{\frac{1}{1}}, \sqrt{\frac{1}{2}}, \ldots, \sqrt{\frac{1}{n}}\right)$ and $\left(\sqrt{\frac{1}{1}}, \sqrt{\frac{1}{2}}, \ldots, \sqrt{\frac{1}{n}}\right)$

we get

$$\left(\sqrt{\frac{1}{1}}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right)^2 \leq n\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) \qquad \ldots(17)$$

$$(16) \text{ and } (17) \Rightarrow \frac{1}{\sqrt{n}}\left(\sqrt{\frac{1}{1}}+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right) \leq (2n-1)^{\frac{1}{4}}.$$

E26) Given a, b and c, they can always be ordered. Let us assume that $a \leq b \leq c$.

Then $a + b \leq a + c \leq b + c$.

Therefore, $\frac{1}{a+b} \geq \frac{1}{a+c} \geq \frac{1}{b+c}$.

We apply Theorem 7 (i) to $\frac{1}{a+b}, \frac{1}{a+c}, \frac{1}{b+c}$ and to c, b, a.

We get

$$3\left(\frac{c}{a+b}+\frac{b}{a+c}+\frac{a}{b+c}\right) \geq (a+b+c)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right) \qquad \ldots(18)$$

Also, applying Theorem 7 (ii) to b + c, a+c, a+b and their inverses, we get

$$3(1+1+1)(b+c+a+c+a+b)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right).$$

$$\Rightarrow 9 \leq 2(a+b+c)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right). \qquad \ldots(19)$$

$$(18) \text{ and } (19) \Rightarrow \left(\frac{c}{a+b}+\frac{b}{a+c}+\frac{a}{b+c}\right) \geq \frac{3}{2}.$$

MISCELLANEOUS EXERCISES

This section is optional

As in the previous block, this section contains some extra problems related to the material covered in this block. Doing them may give you a better understanding of simultaneous linear equations and inequalities. As before, our solutions to the questions follow the list of problems.

1) Solve the following linear system, if possible.

$$3x + 6y + 15z + 6t = 42$$

$$3x + 8y + 21z - 2t = -8$$

$$2x + 9y + 25z + 7t = 41$$

2) a) If $a + b + c = 0$, solve the equation

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

b) Solve the equation

$$\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$$

3) Use Cramer's rule to solve the system

$$x - y + z = 0$$

$$2x + 3y - 5z = 7$$

$$3x - 4y - 2z = -1$$

4) A company produces three products, each of which must be processed through three different departments. In Table 1 we give the number of hours that each unit of each product must stay for in each department. We also give the weekly capacity of each department.

Table 1

Department	Product			No. of hours available per week
	P_1	P_2	P_3	
1	6	2	2	80
2	7	4	1	60
3	5	5	3	100

What combination of the three products will use up the weekly labour availability in all departments?

5) Prove the following identities:

a) $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$$

b) $$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

6) A fish of species S_1 consumes 10 gm of food F_1 and 5 gm. of food F_2 per day. A fish of species S_2 consumes 6 gm. of F_1 and 4 gm. of F_2 per day. If a given environment has 2.2 kg of F1 and 1.3 kg of F2 available daily, what population sizes of the species S_1 and S_2 will consume exactly all of the available food?

7) Does the system

$$x+y+z+w=0$$

$$x+3y-2z+w=0$$

$$2x-3z+2w=0$$

have a non-trivial solution?

8) Obtain a solution of the following system, if it exists.

$$x+2y+4z+t=4$$

$$2x-z-3t=4$$

$$x-2y-z=0$$

$$3x-y-z-5t=5$$

9) Show that the following linear system has a two-parameter solution set.

$$2x_1+x_3-x_4+x_5=2$$

$$x_1+x_3-x_4+x_5=1$$

$$12x_1+2x_2+8x_3+2x_5=12.$$

10) Use Cramer's rule, if possible, for solving the following linear systems:

a) $3x+y=3$

$5x+2y=1$

b) $2x-3y+z=1$

$x+y-z=0$

$x-2y+z=-1$

11) If the coordinate axes in a plane are rotated through an angle , then we can express the old coordinates (x, y) in terms of the new coordinates (x′, y′) as

$$x=x'\cos\theta-y'\sin\theta$$

$$y=x'\sin\theta+y'\cos\theta$$

Use Cramer's rule to write (x′, y′) in terms of (x, y).

Solutions

1) We apply the Gaussian elimination process. We get the solution set

$$\{(4+z,-1-3z,z,6)\mid z\in \mathbf{R}\}$$

2 a) Adding the 2nd and 3rd columns to the first column of the determinant doesn't change its value. On doing this and using the fact that a + b + c = 0, we get

$$\begin{vmatrix} -x & c & b \\ -x & b-x & a \\ -x & a & c-x \end{vmatrix}=0$$

$$\Rightarrow (-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ \end{vmatrix}=0, \text{ by P3}$$

$$\Rightarrow (-x)\begin{vmatrix} 1 & c & b \\ 0 & b-x-c & a-b \\ 0 & a-c & c-x-b \end{vmatrix} = 0, \text{ applying P4 twice.}$$

$\Rightarrow (-x)\,[(b-c-x)(c-b-x)-(a-c)] = 0$, expanding along the first column.

On solving this equation, we find that $x = 0$ or

$(b-c)^2 - x^2 + a^2 - a(b+c) + bc = 0$, that is,

$2x^2 = 3(a^2 + b^2 + c^2)$, using the condition that $a + b + c = 0$.
Thus, the solution set is

$$\left(0, \pm\sqrt{\frac{3}{2}(a^2+b^2+c)}\right).$$

b) Using the properties P1 to P5 of determinants, we see that

$$\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 15-2x & 1 & 10 \\ 11-3x & 1 & 16 \\ 7-x & 1 & 13 \end{vmatrix} = 0$$

$\Rightarrow\ x = 4.$

3) In matrix notation the system is

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -5 \\ 3 & -4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ -1 \end{bmatrix}$$

$$\text{Here } D = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -5 \\ 3 & -4 & -2 \end{bmatrix} = -32 \neq 0.$$

Thus, we can apply Cramer's rule.

Now

$$D_1 = \begin{bmatrix} 0 & -1 & 1 \\ 7 & 3 & -5 \\ -1 & -4 & -2 \end{bmatrix} = -44,$$

$$D_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 7 & -5 \\ 3 & -1 & -2 \end{bmatrix} = -42,$$

$$D3 = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 7 \\ 3 & -4 & -1 \end{bmatrix} = 2$$

Thus, $x = \frac{D_1}{D} = \frac{11}{8}, y = \frac{D_2}{D} = \frac{21}{16}, z = \frac{D_3}{D} = -\frac{1}{16}.$

4) Let x, y and z be the required quantities of P_1, P_2 and P_3. Then we need to solve the system.

$6x + 2y + 2z = 80$

$7x + 4y + z = 60$

$5x + 5y + 3z = 100.$

By Gaussian elimination (or Cramer's rule) we get

$x = 5, y = 0, z = 25.$

So, the ideal combination is 5 units of P_1, 25 units of P_3, and none of P_2, per week.

5) a) $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix},$$ subtracting the first row from the second and third rows

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(c+a-b-a)$$

$$= (b-c)(c-a)(a-b)$$

b) $$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} -2a & c+a & a+b \\ -2p & r+p & p+q \\ -2x & z+x & x+y \end{vmatrix},$$ subtracting the second and third columns from the first column.

$$= (-2)\begin{vmatrix} a & c+a & a+b \\ p & r+p & p+q \\ x & z+x & x+y \end{vmatrix}$$

$$= (-2)\begin{vmatrix} a & c & b \\ p & r & q \\ x & z & y \end{vmatrix},$$ subtracting the first column from the second and third columns.

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix},$$ interchanging the second and third columns.

6) Let x and y denote the number of fish of species S_1 and S_2, respectively. We have the information given in Table 2 below.

Table 2

Fish	Food consumed per week (in gms)	
Species	F_1	F_2
S_1	10	5
S^2	6	4
Total food available per week	2200	1300

Thus, we need to solve the system

$10x + 6y = 2200$

$5x + 4y = 1300$.

Solving by any of the methods, we get

$x = 100, y = 200$.

Thus, the required sizes are 100 fish of species S_1 and 200 of species S_2.

7) By elimination we find that the system has infinitely many solutions $(x, 0, 0, -x)$, where $x \in \mathbf{R}$. Thus, for any $x \neq 0$, we would get a non-trivial solution.

8) By Gaussian elimination, we reach a stage where we get $0 = 8/7$. Hence, the given system is inconsistent.

9) We apply Gaussian elimination. After a few steps we get the following system

$x_1 = 1$

$x_2 + 4x_4 - 3x_5 = 0$

$x_3 - x_4 + x_5 = 0.$

Thus, $x_1 = 1, x_2 = -4x_4 + 3x_5, x_3 = x_4 - x_5$.

So, if $x_4 = s$ and $x_5 = t$, then our solution set is

$\{(1, -4s + 3t, s - t, s, t) \mid s, t \in \mathbf{R}\}$.

Thus, we have expressed the solutions in terms of the two parameters, s and t.

10) a) In matrix notation, the system is

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Since $D = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 1 \neq 0$, we can apply Cramer's rule.

Here $D_1 = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$, and

$$D_2 = \begin{vmatrix} 3 & 3 \\ 5 & 1 \end{vmatrix} = -12.$$

Thus, the solution is $x = 5, y = -12$.

b) Here the coefficient matrix is

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix} \text{ and } D = |A| = 1 \neq 0.$$

Thus, we can apply Cramer's rule.

$$D_1 = \begin{vmatrix} 1 & -3 & 1 \\ 0 & 1 & -1 \\ -1 & -2 & 1 \end{vmatrix} = -3,$$

$$D_2 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 5,$$

$$D_3 = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 1 & -0 \\ 1 & -2 & -1 \end{vmatrix} = -8.$$

$\therefore x = -3, y = -5, z = -8.$

11) Here we can write the equations as

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Since $D = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0,$

we can apply Cramer's rule.

Now, $D_1 = \begin{vmatrix} x & -\sin\theta \\ y & \cos\theta \end{vmatrix} = x\cos\theta + y\sin\theta$, and

$D_2 = \begin{vmatrix} \cos\theta & x \\ \sin\theta & y \end{vmatrix} = y\cos\theta - x\sin\theta$.

Thus $x' = x\cos\theta + y\sin\theta$, $y' = y\cos\theta - x\sin\theta$.