

UNIT 3 GENERAL THEORY OF CONICS

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3.1 INTRODUCTION

So far you have studied the standard equations of a parabola, an ellipse and a hyperbola. We defined these curves and other conics by the focus-directrix property of a conic. This defining property was discovered by Pappus (approx. 320 AD) long after the definition of conic sections by the ancient Greeks. In his book "Conics", the ancient Greek mathematician Apollonius defined these curves to be the intersection of a plane and a cone. You will study cones later, in unit 6, but let us show you how conics are planar sections of a cone, with the help of diagrams (see Fig.1)

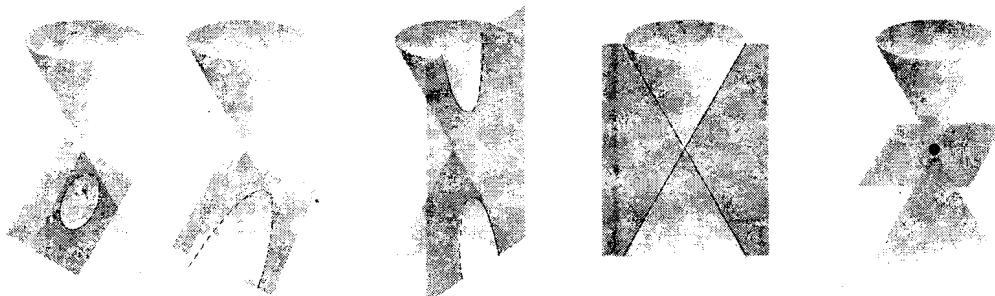


Fig. 1: A planar section of a cone can be (a) an ellipse, (b) a parabola, (c) a hyperbola
(d) a pair of lines, (e) a point.

In this unit we will prove a result that may surprise you. According to this result, the general second degree equation $ax^2 + hxy + by^2 + gx + fy + c = 0$ always represents a conic section. You will see how to identify it with the various conics, depending on the conditions satisfied by the coefficients.

In Unit 2 you saw one way of classifying conics. There is another way of doing so, which you will study in Sec. 3.3. We shall discuss the geometric properties of the different types of conics, and see how to trace them. After that, we shall discuss the tangents of a conic. And finally we shall see what curves can be obtained when two conics intersect.

With this unit we end our discussion on conics. But in the next two blocks you will be coming across them again. So, the rest of the course will be easier for you to grasp if you ensure that you have achieved the unit objectives given below.

Objectives

After studying this unit you should be able to

- identify the conic represented by a quadratic expression;
- find the centre (if it exists) and axes of a conic;

- trace any given conic;
- find the tangent and normal to a given conic at a given point;
- obtain the equations of conics which pass through the points of intersection of two given conics.

3.2 GENERAL SECOND DEGREE EQUATION

In Unit 2 you must have noticed that the standard equation of each conic is a second degree equation of the form

$$ax^2 + hxy + by^2 + gx + fy + c = 0$$

for some $a, b, c, f, g, h \in \mathbb{R}$ and where at least one of a, h, b is non-zero.

We write the coefficients of xy, x and y as $2h, 2g$, and $2f$ to have simpler expressions later on, as you will see.

In this section we will show you that the converse is also true. That is, we will prove that the general second degree equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots(1)$$

where at least one of a, h, b is non-zero, can be transformed into a standard equation of a conic. We achieve this by translating and rotating the coordinate axes. Let us see how.

We first get rid of the term containing xy by rotating the XY-system through a "suitable" angle θ about origin. You will see how we choose θ a little further on. Now, by (16) and (17) of Unit 1, we see that (1) becomes

$$\begin{aligned} & a(x' \cos \theta - y' \sin \theta)^2 + 2h(x' \cos \theta - y' \sin \theta)(x' \sin \theta + y' \cos \theta) + \\ & b(x' \sin \theta + y' \cos \theta)^2 + 2g(x' \cos \theta - y' \sin \theta) + 2f(x' \sin \theta + y' \cos \theta) + c = 0 \\ \Rightarrow & (a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta)x'^2 - 2(a - b) \sin \theta \cos \theta - \\ & h(\cos^2 \theta - \sin^2 \theta)x'y' + (a \sin^2 \theta - 2h \sin \theta \cos \theta + b \cos^2 \theta)y'^2 + \\ & (2g \cos \theta + 2f \sin \theta)x'^2 + (2f \cos \theta - 2g \sin \theta)y' + c = 0. \end{aligned}$$

The $x'y'$ term will disappear if $(a - b) \sin \theta \cos \theta = h(\cos^2 \theta - \sin^2 \theta)$, that is,

$$\frac{1}{2}(a - b) \sin 2\theta = h \cos 2\theta.$$

So, to get rid of the $x'y'$ term, if $a = b$ we can choose $\theta = \frac{\pi}{4}$; otherwise we,

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \end{aligned}$$

$$\text{can choose } \theta = \frac{1}{2} \tan^{-1} \left(\frac{2}{a - b} \right).$$

$\left(\text{We can always choose such a } \theta \text{ lying between } -\frac{\pi}{4} \text{ and } \frac{\pi}{4} \right).$

For this choice of θ the $x'y'$ term becomes zero.

So, if we rotate the axes through an angle $\theta = \frac{1}{2} \tan^{-1} \left(\frac{2h}{a - b} \right)$, then (1) transforms into the second degree equation

$$Ax'^2 + By'^2 + 2Gx' + 2Fy' + C = 0, \quad \dots(2)$$

where $A = a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta$ and

$$B = a \sin^2 \theta - 2h \sin \theta \cos \theta + b \cos^2 \theta.$$

Thus, $A + B = a + b$.

Also, with a bit of computation, you can check that $ab - h^2 = AB$. Now various situations can arise.

Case 1 ($ab - h^2 = 0$): In this case we see that either $A = 0$ or $B = 0$. So, let us assume

that $A = 0$. Then we claim that B must be non-zero. Do you agree? What would happen if $A = 0$ and $B = 0$? In this case we would get $a = 0$, $b = 0$ and $h = 0$, which contradicts our assumption that (1) is a quadratic equation.

So, let $A = 0$ and $B \neq 0$. Then (2) can be written as

$$B\left(y' + \frac{F}{B}\right)^2 = -2G' - C + \frac{F^2}{B} \quad \dots(3)$$

Now, if $G = 0$, then the above equation is

$$\left(y' + \frac{F}{B}\right)^2 = \frac{F^2 - BC}{B^2}, \text{ that is,}$$

$$y' + \frac{F}{B} = \pm \sqrt{\frac{F^2 - BC}{B^2}}$$

This represents a pair of parallel lines if $F^2 \geq BC$, and the empty set if $F^2 < BC$. On the other hand, if $G \neq 0$, then we write (3) as

$$\left(y' + \frac{F}{B}\right)^2 = \frac{-2G}{B} \left(x' + \frac{C}{2G} - \frac{F^2}{2BG}\right)$$

Now if we shift the origin to $\left(\frac{F^2}{2BG} - \frac{C}{2G}, \frac{-F}{B}\right)$, then the equation becomes

$$y^2 = -\frac{2G}{B} X,$$

where X, Y are the current coordinates. From Sec. 2.3. You know that this represents a parabola with $\left(-\frac{G}{2B}, 0\right)$ and directrix $X = \frac{G}{2B}$.

Now let us look at the other case.

Case 2 ($ab - h^2 \neq 0$): Now both A and B are non-zero. We can write (2) as

$$A\left(x' + \frac{G}{A}\right)^2 + B\left(y' + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C, \text{ which is a constant } K, \text{ say.}$$

Let us shift the origin to $\left(-\frac{G}{A}, -\frac{F}{B}\right)$. Then this equation becomes

$$AX^2 + BY^2 = K, \quad \dots(4)$$

where X and Y are the current coordinates.

Now, what happens if $K = 0$? Well, if both A and B have the same sign, that is, If $AB = ab - h^2 > 0$, then (4) represents the point $(0, 0)$.

And, if $ab - h^2 < 0$, then (4) represents the pair of lines,

$$X = \pm \sqrt{-\frac{B}{A}} Y.$$

And, what happens if $K \neq 0$? Then we can write (4)

$$\frac{X^2}{K/A} + \frac{Y^2}{K/B} = 1. \quad \dots(5)$$

Does this equation look familiar? From Sec. 2.4. You can see that this represents an ellipse if both $\frac{K}{A}$ and $\frac{K}{B}$ are positive, that is, if $K > 0$ and $AB = ab - h^2 > 0$.

But, what if $K/A < 0$ and $K/B < 0$? In this case $K < 0$ and $ab - h^2 > 0$. And then (5) represents the empty set.

And, if $\frac{K}{A}$ and $\frac{K}{B}$ are of opposite signs, that is, if $AB = ab - h^2 < 0$, then what will (5) represent? A hyperbola.

So we have covered all the possibilities for $ab - h^2$, and hence for (1). Thus, we have proved the following result.

Theorem 1: The general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a conic.

While proving this theorem you must have noticed the importance we gave expression $ab - h^2$. Let us tabulate the various types of non-degenerate and degenerate conics that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents, according to the way $ab - h^2$ behaves. (Recall from Unit 2 that a degenerate conic is a conic whose focus lies on the corresponding directrix.)

Table 1 : Classification of Conics.

Condition	Types of Conics	
	Non-degenerate	Degenerate
$ab - h^2 = 0$	parabola	pair of parallel lines, or empty set
$ab - h^2 > 0$	ellipse	point, or empty set
$ab - h^2 < 0$	hyperbola	pair of intersecting lines

Table 1 tells us about all the possible conics that exist. This is what the following exercise is about.

- E1) a) Write down all the possible types of conics there are. Which of them are degenerate?
 b) If (1) represents a circle, will $ab - h^2 = 0$?

Now let us use the procedure in the proof above in some examples.

Example 1: Find the conic represented by
 $9x^2 - 24xy + 16y^2 - 124x + 132y + 324 = 0$.

Solution: The given equation is of the form (1), where $a = 9$, $b = 16$, $h = -12$.

Now let us rotate the axes through an angle θ , where

$$\tan 2\theta = \frac{2h}{a-b} = \frac{24}{7}, \text{ that is, } \frac{2\tan\theta}{1-\tan^2\theta} = \frac{24}{7}, \text{ that is,}$$

$$12\tan^2\theta + 7\tan\theta - 12 = 0.$$

$$\text{So we can take } \tan\theta = \frac{3}{4}, \text{ and then } \sin\theta = \frac{3}{5} \text{ and } \cos\theta = \frac{4}{5}.$$

Then, in the new coordinate system the given equation becomes

$$25y'^2 - \frac{124}{5}(4x' - 3y') + \frac{132}{5}(3x' + 4y') + 324 = 0, \text{ that is,}$$

$$\left(y' + \frac{18}{5}\right)^2 = \frac{4}{5}x.$$

Now let us shift the origin to $(0, -\frac{18}{5})$. Then the equation becomes

$$Y^2 = \frac{4}{5}X,$$

where X and Y are the current coordinates.

Can you recognise the conic represented by this equation? From Unit 2 you know that this is a parabola. Since the transformations we have applied do not alter the curve, the original equation also represents a parabola.

Example 2 : Identify the conic $x^2 - 2xy + y^2 = 2$.

Solution: Over here, since $a = b = 1$, we choose $\theta = 45^\circ$. So, let us rotate the axes through 45° . The new coordinates x' and y' are given by

$$x = \frac{1}{\sqrt{2}}(x' - y') \text{ and } y = \frac{1}{\sqrt{2}}(x' + y')$$

Then, in the new coordinate system,

$x^2 - 2xy + y^2 = 2$ transforms to
 $y'^2 = 1$,

which represents the pair of straight lines $y' = 1$ and $y' = -1$.

You can do the following exercise on the same lines.

E 2) Identify the conic

- a) $x^2 - 2xy + y^2 + \sqrt{2}x = 2$,
 - b) $9x^2 - 6xy + y^2 - 40x - 20y + 75 = 0$.
-

So far you have seen that any second degree equation represents one of the following conics:

a parabola, an ellipse, a hyperbola, a pair of straight lines, a point, the empty set.

But, from Table 1 you can see that even if we know the value of $ab - h^2$, we can't immediately say what the conic is. So, each time we have to go through the whole procedure of Theorem 1 to identify the conic represented by a given equation. Is there a short cut? Yes, there is. We have a simple condition for (1) to represent a pair of lines. It can be obtained from the proof of Theorem 1 after some calculations, or independently. We shall only state it, and then see how to use it to cut short our method for identifying a given conic.

Theorem 2: The quadratic equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight lines if and only if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$,

that is, the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Further, if the condition is satisfied, then the angle between the lines is

$$\tan^{-1} \left(\frac{2\sqrt{h^2 - ab}}{a + b} \right)$$

The 3×3 determinant given above is called the **discriminant** of the given conic. You can see that the discriminant looks neater if we take $2h$, $2g$ and $2f$ as coefficients, instead of h , g and f .

Let us consider some examples of the use of Theorem 2.

Recall the definition of a determinant from Unit 5 of MTE-04.

Example 3: Show that $x^2 - 5xy + 6y^2 = 0$ represents a pair of straight lines. Find the angle between these lines.

Solution: With reference to Theorem 2, in this case $a = 1$, $h = -\frac{5}{2}$, $b = 6$, $g = 0 = f \neq c$. Thus, the related discriminant is

$$\begin{vmatrix} 1 & -\frac{5}{2} & 0 \\ -\frac{5}{2} & 6 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

, which is 0, as one row of the determinant is of all zeros.

Thus, the given equation represents a pair of lines.

$$\text{The angle between them is } \tan^{-1} \left(\frac{2}{7} \sqrt{\frac{25}{4} - 6} \right) = \tan^{-1} \frac{1}{7}.$$

Example 4: Find the conic represented by $2x^2 + 5xy + y^2 = 1$.

Solution : In this case $ab - h^2 = -23 < 0$. So, from Table 1 we know that the equation represents a hyperbola or a pair of lines. Further, in this case the discriminant becomes

$$\begin{vmatrix} 2 & \frac{5}{2} & 0 \\ \frac{5}{2} & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = - \begin{vmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 1 \end{vmatrix} = \frac{17}{4} \neq 0.$$

So, by Theorem 2 we know that the given equation doesn't represent a pair of lines. Hence, it represents a hyperbola.

Why don't you do these exercises now?

E3) Check whether $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$ represents a pair of lines.

E4) Show that the real quadratic equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines

E5) Under what conditions on a , b and h , will the equation in Theorem 2 represent
 a) a pair of parallel lines?
 b) a pair of perpendicular lines?

So far we have studied all the conics in a unified manner. Now we will categorise them according to the property of centrality.

3.3 CENTRAL AND NON-CENTRAL CONICS

In our discussion on the ellipse in unit 2, we said that the midpoint of the major axis was the centre of the ellipse. The reason that this point is called the centre is because of a property that we ask you to prove in the following exercise.

E6) Consider the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let (x_1, y_1) be a point on this ellipse and O be $(0, 0)$. Show that the line PO also meets the ellipse in $P' (-x_1, -y_1)$.

What you have just proved is that $O(0, 0)$ is bisects every chord of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ that passes through it. Similarly, any chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ through O(0, 0) is bisected by O. Hence, according to the following definition, O is the centre of the ellipse and hyperbola given above.

Definition: The **centre of a conic C** is a point which bisects any chord of C that passes through it.

Not all conics have centres, as you will see. A conic that has a centre is called a **central conic**. For example, an ellipse and a hyperbola are central conics.

Now, can a central conic have more than one centre? Suppose it has two centres C_1 and C_2 . Then the chord of the conic intercepted by the line $C_1 C_2$ must be bisected by both C_1 and C_2 , which is not possible. Thus,

a central conic has a unique centre.

Let us see how we can locate this point.

Consider the conic (1). Suppose it is central with centre at the origin. Then we have the following result, which we will give without proof.

Theorem 3: A central conic with centre at (0, 0) is of the form $ax^2 + 2hxy + by^2 = 1$, for some a, h, b in \mathbb{R} .

This result is used to prove the following theorem about any central conic. We shall not prove the theorem in this course but we will apply it very often.

Theorem 4: Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be a central conic. Then its centre is the intersection of the lines

$$ax + hy + g = 0 \text{ and } hx + by + f = 0.$$

What this theorem tells us is that if $ax + hy + g = 0$ and $hx + by + f = 0$ intersect, then the conic is central; and the point of intersection of these straight lines is the centre of the conic.

But what if the lines don't intersect? Then the conic under consideration can't be central; that is, non-central. Thus, the conic is **non-central** if the slopes of these lines are equal, that is, if $ab = h^2$.

So, we have the following result:

The conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is
 i) central if $ab \neq h^2$, and
 ii) non-central if $ab = h^2$.

Does this result and Table 1 tell you which conics are non-central? You can immediately tell that a parabola doesn't have a centre.

Let us see how we can apply the above results on centres of conics.

Example 5: Is the conic $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$ central? If it is, find its centre.

Solution : In this case $a = 17$, $b = 8$, $h = -6$, $g = 23$, $f = -14$.

So, $ab \neq h^2$. Hence, the conic is central. Its centre is the intersection of the lines
 $17x - 6y + 23 = 0$ and $3x - 4y + 7 = 0$, which is $(-1, 1)$.

Why don't you try some exercises now?

E7) Is the conic in E3 central? If yes, find its centre.

E8) Identify the conic $x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$. If it is central, find its centre.

E9) Which degenerate conics are central, and which are not?

One point that has been made in this sub-section is that a parabola is a non-central conic, while an ellipse and a hyperbola are central conics. Now let us see if this fact helps us to trace a conic corresponding to a given quadratic equation.

3.4 TRACING A CONIC

Suppose you are given a quadratic equation. Can you get enough geometric information from it to be able to draw its geometric representation? You are now in a position to check whether it is a pair of lines or not. You can also tell whether it is a central conic or not. But there is still one piece of information that you would need before you could draw the required conic. You need to know the equation of its axis, or axes, as the case may be. So let us see how to find the axes. We shall consider the central and non-central cases separately.

3.4.1 Central Conics

Suppose we are given the equation of a central conic. By translating the axes, if necessary, we can assume that its centre lies at $(0, 0)$. Then, by theorem 3, its equation is

$$ax^2 + 2hxy + by^2 = 1 \quad \text{where } a, h, b \in \mathbf{R}. \quad \dots(6)$$

In theorem 1 you saw that if we rotate the coordinate axes through an angle $\theta = \frac{1}{2} \tan^{-1} \frac{2h}{a-b}$, then the axes of the conic lie along the coordinate axes.

Therefore, the axes of the conic are inclined at the angle θ to the coordinate axes. (Here if $a = b$, we take $\theta = 45^\circ$) Now,

$$\begin{aligned} \tan 2\theta &= \frac{2h}{a-b} \\ \Rightarrow \frac{2 \tan \theta}{1 - \tan^2 \theta} &= \frac{2h}{a-b} \\ \Rightarrow \tan^2 \theta + \left(\frac{a-b}{h}\right) \tan \theta - 1 &= 0 \end{aligned}$$

This is a quadratic equation in $\tan \theta$, and hence is satisfied by two values of θ , say θ_1 and θ_2 . Then the slopes of the axes of the conic are $\tan \theta_1$ and $\tan \theta_2$. Note that the axes are mutually perpendicular, since $(\tan \theta_1)(\tan \theta_2) = -1$.

Now, to find the lengths of the axes of the conic, we write (6) in polar form (see Sec. 1.5). For this we substitute $x = r \cos \theta$, $y = r \sin \theta$ in (6). Then we get

$$\begin{aligned} r^2(a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta) &= 1 \\ \Rightarrow r^2 &= \frac{\cos^2 \theta + \sin^2 \theta}{a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta}, \text{ writing } 1 = \cos^2 \theta + \sin^2 \theta \\ &= \frac{1 + \tan^2 \theta}{a + 2h \tan \theta + b \tan^2 \theta} \quad \dots(7) \end{aligned}$$

A semi-axis is half the axis.

If we substitute $\tan \theta_1$ and $\tan \theta_2$ in (7), we will get the corresponding values of r , which will give the lengths of the corresponding semi-axes.

Let us use what we have just done to trace the conic in Example 5. Since $ab - h^2 > 0$, from Theorem 1 we know that the conic is an ellipse. You have already seen that its centre lies at $(-1, 1)$. Now, we need to shift the axes to the centre $(-1, 1)$, to get the equation in the form (6). The equation becomes

$$\frac{17}{20}x'^2 - \frac{3}{5}x'y' + \frac{2}{5}y'^2 = 1.$$

Now we can obtain the directions of the axes from

$$\tan^2 \theta - \frac{3}{2} \tan \theta - 1 = 0.$$

This gives us $\tan \theta = 2, -\frac{1}{2}$.

Therefore, we can take $\theta_1 = \tan^{-1} 2 = 63.43^\circ$ (approximately), and

$$\theta_2 = \frac{\pi}{2} + \tan^{-1} 2.$$

The lengths of the semi-axes, r_1 and r_2 , are given by substituting these values in (7).

So

$$r_1^2 = \frac{1+4}{\frac{17}{20} - \frac{6}{5} + \frac{8}{5}} = 4 \Rightarrow r_1 = 2, \text{ and}$$

$$r_2^2 = \frac{1+\frac{1}{4}}{\frac{17}{20} + \frac{3}{10} + \frac{1}{10}} = 1 \Rightarrow r_2 = 1.$$

Thus, the length of the major axis is 4, and that of the minor axis is 2.

So now we can trace the conic. We first draw a line $O'X'$ through $O'(-1, 1)$ at an angle of $\tan^{-1} 2$ to the x-axis (see Fig. 2). Then we draw $O'Y'$ perpendicular to $O'X'$. Now we mark off A' and A on $O'X'$ such that $A'O' = 2$ and $O'A = 2$.

Similarly, we mark off B and B' on $O'Y'$ such that $OB = 1$ and $O'B' = 1$.

The required ellipse has AA' and BB' as its axes. For further help in tracing the curve, we can check where it cuts the x and y axes. It cuts the x-axis in $(-4, 0)$, $(-2.2, 0)$, and the y-axis in $(0, 2.7)$ and $(0, .8)$. So the curve is what we have drawn in Fig. 2.

Now why don't you see if you've understood what has been done in this section?

E 10) Trace the conic in E8.

E 11) Under what conditions on the coefficients, will $x^2 + 2hxy + y^2 + 2fy = 0$ be central? And then, find its centre and axes.

So far you have seen how to trace a central conic. But what about a non-central conic? Let us look at this case now.

3.4.2 Parabola

In this sub-section we shall look at a method for finding the axis of a parabola, and hence tracing it. We will use the fact that if (1) is a parabola it can be written in the form

$$\left(\frac{Ax + By + C}{\sqrt{A^2 + B^2}} \right)^2 = k \left(\frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} \right) \quad (8)$$

$C = 0$ is the axis of the parabola and $A'x + B'y + C' = 0$ is the tangent at the vertex, and hence they are perpendicular to each other. The vertex (x_1, y_1)

If this parabola is the intersection of $Ax + By + C = 0$ and $A'x + B'y + C' = 0$

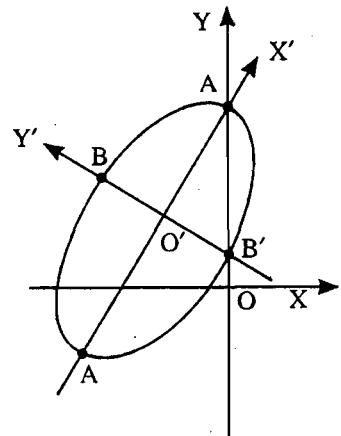


Fig. 2: The ellipse $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$.

$A'x + B'y + C' = 0$, k is the length of its latus rectum, and $F\left(x_1 + \frac{k}{4}\cos\theta, y_1 + \frac{k}{4}\sin\theta\right)$

is its focus, where $\tan\theta$ is the slope of the axis.

Let us see the method with the help of an illustration.

Example 6: Show that the conic $x^2 + 2xy + y^2 - 2x - 1 = 0$ is a parabola. Find its axis and trace it.

Solution: Here $a = 1, b = 1, h = 1$. $\therefore ab - h^2 = 0$.

$$\text{Further, the discriminant of the conic is } \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{vmatrix} = -1 \neq 0.$$

Hence, by Theorem 2, the equation does not represent a pair of straight lines. Thus, by Theorem 1, we know that the given conic is a parabola.

We can write the given equation as $(x + y)^2 = 2x + 1$.

Now we will introduce a constant c so that we can write the equation in the form (8). So, let us rewrite the equation as

$(x + y + c)^2 = 2x + 1 + 2cx + 2cy + c^2$, that is,

$$(x + y + c)^2 = 2(1 + c)x + 2cy + c^2 + 1. \quad \dots(9)$$

We all choose c in such a way that the lines $x + y + c = 0$ and $2(1 + c)x + 2cy + c^2 + 1 = 0$ are perpendicular. From Equation (13) of Unit 1, you know that the condition is

$$(-1) \left[\frac{-2(1+c)}{2c} \right] = -1 \Rightarrow c = -\frac{1}{2}.$$

Then (9) becomes

$$\left(x + y - \frac{1}{2}\right)^2 = x - y + \frac{5}{4}, \text{ that is,}$$

$$\left(\frac{x + y - \frac{1}{2}}{\sqrt{2}}\right)^2 = \frac{1}{\sqrt{2}} \left(\frac{x - y + \frac{5}{4}}{\sqrt{2}}\right).$$

This is in the form (8).

Thus, the axis of the parabola is $x + y - \frac{1}{2} = 0$, and the tangent at the vertex is

$$x - y + \frac{5}{4} = 0.$$

The vertex is the intersection of these two lines, that is, $\left(-\frac{3}{8}, \frac{7}{8}\right)$.

The length of the latus rectum of the parabola is $\frac{1}{\sqrt{2}}$,

Thus, the focus is at $\left(-\frac{3}{8} + \frac{1}{4\sqrt{2}}\cos\theta, \frac{7}{8} + \frac{1}{4\sqrt{2}}\sin\theta\right)$, where θ is the angle that the axis makes with the x-axis, that is, $\theta = \tan^{-1}(-1)$.

$$\therefore \sin \theta = -\frac{1}{\sqrt{2}}, \cos \theta = \frac{1}{\sqrt{2}}$$

Therefore, the focus $F\left(-\frac{1}{4}, \frac{3}{4}\right)$.

What are the points of intersection of the parabola and the coordinate axes? They are $(1 + \sqrt{2}, 0), (1 - \sqrt{2}, 0), (0, 1), (0, -1)$.

So, we can trace the parabola as in Fig. 3.

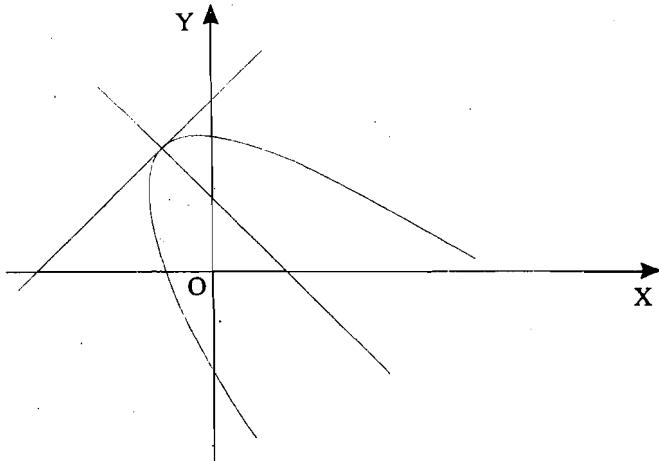


Fig. 3 : The parabola $x^2 + 2xy + y^2 - 2x - 1 = 0$

Has the example helped you to understand the method for tracing a parabola? The following exercise will help you to find out.

E 12) Trace the conic $4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0$.

Let us now see how to obtain the tangents of a general conic.

3.5 TANGENTS

In Unit I you studied the equations of tangents to the conics in standard form. Now we will discuss the equation of a tangent to the general conic (1)

So, consider two distinct points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

If $x_1 = x_2 = \alpha$, say, then the line PQ is $x = \alpha$.

Similarly, if $y_1 = y_2 = \alpha$, say, then the line PQ is $y = \alpha$.

Otherwise, the line PQ is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \dots(10)$$

Since P and Q lie on the conic,

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \dots(11)$$

$$\text{and } ax_2^2 + 2hx_2y_2 + by_2^2 + 2gx_2 + 2fy_2 + c = 0. \quad \dots(12)$$

Then (12) - (11)

$$a(x_2^2 - x_1^2) + 2h(x_2y_2 - x_1y_1) + b(y_2^2 - y_1^2) + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

$$\Rightarrow a(x_2^2 - x_1^2) + 2h(x_2y_2 - x_1y_2 + x_1y_2 - x_1y_1) + b(y_2^2 - y_1^2) +$$

$$2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

$$\Rightarrow (x_2 - x_1) \{a(x_1 + x_2) + 2hy_2 + 2g\} + (y_2 - y_1) \{b(y_1 + y_2) + 2hx_1 + 2f\} = 0$$

$$\Rightarrow \left(\frac{y_2 - y_1}{x_2 - x_1} \right) = \frac{[a(x_1 + x_2) + 2hy_2 + 2g]}{[b(y_1 + y_2) + 2hx_1 + 2f]}$$

Putting this in (10), we get

$$y - y_1 = - \left[\frac{a(x_1 + x_2) + 2hy_2 + 2g}{b(y_1 + y_2) + 2hx_1 + 2f} \right] (x - x_1) \quad \dots(13)$$

As (x_2, y_2) tends to (x_1, y_1) , (13) gives us the equation of the tangent to the given conic at (x_1, y_1) .

Thus, the equation of the tangent at $P(x_1, y_1)$ is

$$(y - y_1)(by_1 + hx_1 + f) + (x - x_1)(ax_1 + hy_1 + g) = 0$$

$$\Leftrightarrow x(ax_1 + hy_1 + g) + y(by_1 + hx_1 + f) + (gx_1 + fy_1 + c) = 0, \text{ using (11).}$$

$$\Leftrightarrow axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0. \quad \dots(14)$$

Thus, (14) is the equation of the tangent to the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) lying on the conic.

From (14) you can see that we can use the following rule of thumb to obtain the equation of a conic.

In the equation of the conic, replace x^2 by xx_1 , y^2 by yy_1 , $2x$ by $(x + x_1)$, $2y$ by $(y + y_1)$ and $2xy$ by $(xy_1 + yx_1)$, to get the equation of the tangent at (x_1, y_1)

For instance, the tangent to the parabola $y^2 - 4xy = 0$, at a point (x_1, y_1) is $yy_1 - 2a(x + x_1) = 0$.

We have already derived this in Sec. 2.3.2.

In fact, the equations of tangents to the ellipse and hyperbola given in standard form are also special cases of (14), as you can verify from Unit 2.

Now you may like to try your hand at finding tangents at some points.

- E13) Obtain the equations of the tangent and the normal to the conic in E8 at the points where it cuts the y-axis.

In Unit 2 you have seen that not every line can be a tangent to a given standard conic. Let us now see which lines qualify for being tangents to the general conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. With your experience in Unit 2, can you tell the conditions under which the line $px + qy + r = 0$ will be a tangent to this conic?

Suppose it is a tangent at a point (x_1, y_1) to the conic. Now, either $p \neq 0$ or $q \neq 0$.

Let us suppose $p \neq 0$. Then we can substitute $x = -\frac{(qy + r)}{p}$ in the equation of the conic, to get

$$\frac{a}{p^2} (qy + r)^2 - \frac{2h}{p} (qy + r)y + by^2 - \frac{2g}{p} (qy + r) + 2fy + c = 0$$

$$\Rightarrow (aq^2 - 2hpq + bp^2)y^2 - 2y(prh + pqr - p^2f) + (ar^2 - 2gpr + cp^2) = 0$$

The roots of this quadratic equation in y give us the y -coordinates of the points of intersection of the given line and conic. The line will be a tangent if these points

coincide, that is, if the quadratic equation has coincident roots, that is, if $(ph + pq - aqr - p^2f)^2 = (aq^2 - 2hpq + bp^2)(ar^2 - 2gpr + cp^2)$ (15)

In terms of determinants (CS-60, Block 6), we can write this condition as

$$\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & 0 \end{vmatrix} = 0 \quad \dots(16)$$

Thus, (15) or the determinant condition (16) tell us if $px + qy + r = 0$ is a tangent to the general conic or not.

For example, the line $y = mx + c$ will touch the parabola $y^2 = 4ax$, if

$$\begin{vmatrix} 0 & 0 & -2a & m \\ 0 & 1 & 0 & -1 \\ -2a & 0 & 0 & c \\ m & -1 & c & 0 \end{vmatrix} = 0$$

$$\Rightarrow (-2a) \begin{vmatrix} 0 & 1 & -1 \\ -2a & 0 & c \\ m & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ -2a & 0 & 0 \\ m & -1 & c \end{vmatrix} = 0, \text{ expanding along the first row.}$$

$$\Rightarrow (-2a)(cm - 2a) - m(2ac) = 0$$

$$\Rightarrow c = \frac{a}{m}$$

This is the same condition that we derived in Sec. 2.3.2.

Why don't you try these exercises now?

E 14) Is $x + 4y = 0$ a tangent to the conic $x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0$?

Find all the tangents to this conic that are parallel to the given line.

E 15) a) Prove that the condition for $ax + by + 1 = 0$ to touch

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

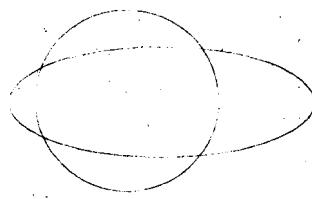
$$(ag + bf - 1)^2 = (a^2 + b^2)(g^2 + f^2 - c).$$

b) In particular, under what conditions on C will $y = Mx + C$ touch $x^2 + y^2 = A^2$?

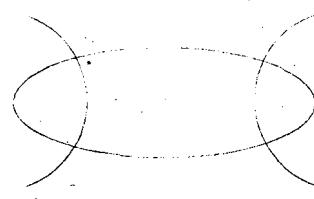
In this section you saw that a line and a conic intersect in at most two points. Now let us see what we get when two conics intersect.

3.6 INTERSECTION OF CONICS

Consider the intersection of an ellipse and a circle (Fig. 4 (a)) or of an ellipse and a hyperbola (Fig. 4 (b)).



(a)



(b)

Fig. 4: Intersecting conics.

You can see that these conics intersect in four points. But, do any two conics intersect in four points? The following result answers this question.

Theorem 5 : In general, two conics intersect in four points.

Proof : Let the equations of the two conics be
 $ax^2 + 2(hy + g)x + by^2 + 2fy + c = 0$, and

$$a_1x^2 + 2(h_1y + g_1)x + b_1y^2 + 2f_1y + c_1 = 0.$$

These equations can be considered as quadratic equations in x . If we eliminate x from them, we will get a fourth degree equation in y . This will have four roots. Corresponding to each of these roots, we will get a root of x . So there are, in general, four points of intersection for the two conics.

Since a fourth degree equation with real coefficients may have two or four complex roots (see CS-60, Block-5), two conics can intersect in

- i) Four real points,
- ii) two real and two imaginary points, or
- iii) Four imaginary points.

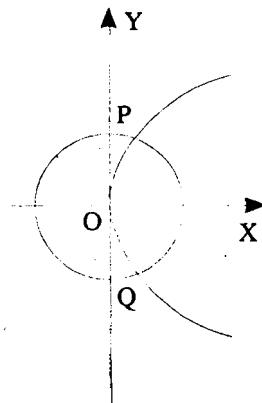


Fig. 5: $y^2 = 2x$ and $x^2 + y^2 = 1$ intersect in the points P and Q.

These points of intersection can be distinct, or some may coincide, or all of them may coincide.

Let us consider an example.

Example 7 : Find the points of intersection of the parabola $y^2 = 2x$ and the circle $x^2 + y^2 = 1$ (see Fig. 5).

Solution : If (x_1, y_1) is a point of intersection, then $x_1^2 + y_1^2 = 1$ and $y_1^2 = 2x_1$.

Eliminating y_1 from these equations, we get

$$x_1^2 + 2x_1 = 1, \text{ that is, } (x_1 + 1)^2 = 2.$$

$$\text{So } x_1 = -1 \pm \sqrt{2}.$$

Then $y_1^2 = 2x_1$ gives us

$$y_1 = \pm\sqrt{2}(\sqrt{2-1})^{\frac{1}{2}} \text{ if } x_1 = -1 + \sqrt{2}, \text{ and}$$

$$y_1 = \pm\sqrt{2}(-\sqrt{2+1})^{\frac{1}{2}} \text{ if } x_1 = -1 - \sqrt{2},$$

Thus, there are only two real points of intersection, namely,

$(\sqrt{2}-1, \sqrt{2}(\sqrt{2}-1)^{\frac{1}{2}})$ and $(\sqrt{2}-1, -\sqrt{2}(\sqrt{2}-1)^{\frac{1}{2}})$. This is why you see only two points of intersection in Fig. 4.

Here is an exercise for you now.

E 16) Find the points of intersection of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

You have seen that two conics intersect in four real or imaginary points. Now we will find the equation of any conic that passes through these points.

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, and

$$a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$$

be the equations of two conics.

Let us briefly denote them by $S = 0$ and $S_1 = 0$, respectively.

Then, for each $k \in \mathbb{R}$, $S + kS_1 = 0$ is a second degree equation in x and y . So it is a conic, for each value of k .

On the other hand, any point of intersection of the two conics satisfies both the equations $S = 0$ and $S_1 = 0$. Hence it satisfies $S + kS_1 = 0$. Thus, the conic $S + kS_1 = 0$ passes through all the points of intersection of $S = 0$ and $S_1 = 0$.

So we have proved.

Theorem : The equation of any conic passing through the intersection of two conics $S = 0$ and $S_1 = 0$ is of the form $S + kS_1 = 0$, where $k \in \mathbb{R}$.

For different values of k , we get different conics passing through the points of intersection of $S = 0$ and $S_1 = 0$. But, will all these conics be of the same type? If you do the following exercises, you may answer this question.

- E17) If $S = 0$ and $S_1 = 0$ are rectangular hyperbolas, then show that $S + kS_1 = 0$ is a rectangular hyperbola, for all real k .

(Hint: Recall that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is a rectangular hyperbola if $a + b = 0$.)

- E18) Let $S \equiv \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$ and $S_1 \equiv xy - 9 = 0$.

Under what conditions on k will $S + kS_1 = 0$ be

- an ellipse?
- a parabola?
- a hyperbola?

Now we have come to the end of our discussion on conics. Let us see what we have covered in this unit.

3.7 SUMMARY

In this unit we discussed the following points:

- The general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a conic. It is

i) a pair of straight lines iff $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Further, if the condition is satisfied, then the angle between the lines is

$$\tan^{-1} \left(\frac{2\sqrt{h^2 - ab}}{a+b} \right);$$

- a parabola if $ab - h^2 = 0$, and the determinant condition in (i) is not satisfied;
- an ellipse if $ab - h^2 > 0$;
- a hyperbola if $ab - h^2 < 0$.
- An ellipse and a hyperbola are central conics; a parabola is a non-central conic.
- A central conic with centre at the origin is of the form $ax^2 + 2hxy + by^2 = 1$, where $a, h, b \in \mathbb{R}$.
- $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a central conic if $ax + hy + g = 0$ and $hx + by + f = 0$ intersect. And then the centre of the conic is the point of intersection of these lines. The slopes of the axes of this conic are the roots of the equation.

$$\tan^2 \theta + \left(\frac{a-b}{h} \right) \tan \theta - 1 = 0.$$

- 5) Tracing a conic.
 6) The tangent to the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) is

$$axx_1 + h(xy_1 + x_1y) + bby_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Further, a line $px + qy + r = 0$ is tangent to the given conic if

$$\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & o \end{vmatrix} = 0$$

- 7) Two conics intersect in four points, which can be real or imaginary.
 8) The equation of a conic passing through the four points of intersection of the conics $S = 0$ and $S_1 = 0$ is $S + kS_1 = 0$, $k \in \mathbb{R}$.

3.8 SOLUTIONS/ANSWERS

- E1) a) There are 3 types of non-degenerate conics: parabola, ellipse, hyperbola.
 There are 5 types of degenerate conics: point, pair of intersecting lines, pair of distinct parallel lines, pair of coincident lines, empty set.
 b) A circle is a particular case of an ellipse. Thus, if (1) represents a circle then $ab - h^2 > 0$.

- E2) a) $x^2 - 2xy + y^2 + \sqrt{2}x - 2 = 0$. Here $a = 1 = b$, $h = -1$.
 If we rotate the axes through $\pi/4$, then the new coordinates x' and y' are given by

$$x = \frac{1}{\sqrt{2}}(x' - y') \text{ and } y = \frac{1}{\sqrt{2}}(x' + y')$$

Thus, the given equation becomes

$$2y'^2 + x' - y' = 2.$$

$$\Leftrightarrow y'^2 - \frac{1}{2}y' + \frac{1}{2}x' = 1$$

$$\Leftrightarrow \left(y' - \frac{1}{4}\right)^2 = -\frac{1}{2}x' + \frac{17}{16} = -\frac{1}{2}\left(x' - \frac{17}{8}\right)$$

Now, if we shift the origin to $\left(\frac{17}{8}, \frac{1}{4}\right)$ the equation becomes the parabola

$$Y^2 = -\frac{1}{2}X, \text{ where } X \text{ and } Y \text{ are the new coordinates.}$$

- b) $9x^2 - 6xy + y^2 - 40x - 20y + 75 = 0$.
 Here $a = 9$, $b = 1$, $h = -3$.
 So, let us rotate the axes through θ , where

$$\theta = \frac{1}{2} \tan^{-1} \left(-\frac{6}{8} \right). \therefore \tan 2\theta = -\frac{3}{4}$$

$$\text{So we can take } \tan \theta = 3 \text{ so that } \sin \theta = -\frac{3}{\sqrt{10}}, \cos \theta = \frac{1}{\sqrt{10}}.$$

Then the equation in the $X'Y'$ -system becomes

$$\frac{59}{5}y'^2 - 10\sqrt{10}x' - 10\sqrt{10}y' + 75 = 0.$$

which can be transformed and seen to be the equation of a parabola.

E 3) In this case $a = 3$, $b = 2$, $c = 2$, $f = \frac{5}{2}$ = g , $h = \frac{7}{2}$.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 3 & \frac{7}{2} & \frac{5}{2} \\ \frac{7}{2} & 2 & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} & 2 \end{vmatrix} = 0$$

Hence the given equation represents a pair of lines.

E 4) Here the discriminant concerned is

$$\begin{vmatrix} a & h & 0 \\ h & b & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

Thus, the given equation represents a pair of lines.

E 5) a) The lines will be parallel if $\sqrt{h^2 - ab} = 0$, that is, $ab - h^2 = 0$.

b) The lines will be perpendicular if $a + b = 0$.

E 6) Since P lies on the ellipse, os does P'.

The equation of PO is $\frac{y - y_1}{-y_1} = \frac{x - x_1}{-x_1}$, that is, $x_1(y - y_1) = y_1(x - x_1)$.

P' also lies on this since $(-x_1, -y_1)$ satisfies this equation.

Hence, we have shown the result.

E 7) In this case $ab \neq h^2$. So the conic is central. Its centre is the intersection of

$$3x + \frac{7}{2}y + \frac{5}{2} = 0 \text{ and } \frac{7}{2}x + 2y + \frac{5}{2} = 0,$$

That is, $\left(-\frac{3}{5}, -\frac{1}{5} \right)$,

E 8) In this case $a = 1 = b$, $h = -\frac{3}{2}$

$$\therefore ab - h^2 = -\frac{5}{4} < 0.$$

So the given equation is central, and can be a hyperbola or a pair of intersecting lines.

$$\text{Since } \begin{vmatrix} 1 & -\frac{3}{2} & 5 \\ -\frac{3}{2} & 1 & -5 \\ 5 & -5 & 21 \end{vmatrix} \neq 0,$$

using Theorem 2 we can say that the equation represents a hyperbola. Its centre is the intersection of

$$x - \frac{3}{2}y + 5 = 0 \text{ and } -\frac{3}{2}x + y - 5 = 0, \text{ that is, } (-2, 2).$$

- E 9) The central degenerate conics: point, pair of intersecting lines. The non-central degenerate conics : pair of distinct parallel lines, pair of coincident lines. The empty set is both central and non-central.
- E 10) The equation represents a hyperbola with centre $(-2, 2)$. If we shift the origin to $(-2, 2)$, the equation becomes
- $$-x'^2 + 3x'y' - y'^2 = 1.$$

Here $a = -1$, $b = -1$, $h = \frac{3}{2}$.

Thus, the axes of the conic are at an angle of $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ to the x -axis,

So, putting these values of θ in (7), we get the lengths r_1 and r_2 , of the semi-axes, on solving $r_1^2 = 2$ and $r_2^2 = -\frac{2}{5}$.

Thus, $r_1 = \sqrt{2}$ and $r_2 = \sqrt{\frac{2}{5}}$

Note that over here, though r_2^2 is negative, we only want its magnitude to compute the length of the axis.

Now, you know that if e is the eccentricity of the hyperbola then

$$r_2 = r_1 \sqrt{e^2 - 1}, \text{ that is, } \sqrt{\frac{2}{5}} = \sqrt{2} \sqrt{e^2 - 1},$$

$$\Rightarrow e = \sqrt{\frac{6}{5}}.$$

Now let us also see where the hyperbola cuts the x and y axes. Putting $y = 0$ in the given equation, we get

$$x^2 + 10x + 21 = 0 \Rightarrow x = -3, -7.$$

So, the hyperbola intersects the x -axis in $(-3, 0)$ and $(-7, 0)$. Similarly, putting $x = 0$ in the given equation and solving for y , we see that the hyperbola intersects the y -axis in $(0, 3)$ and $(0, 7)$.

With all this information the curve is as given in Fig. 6.

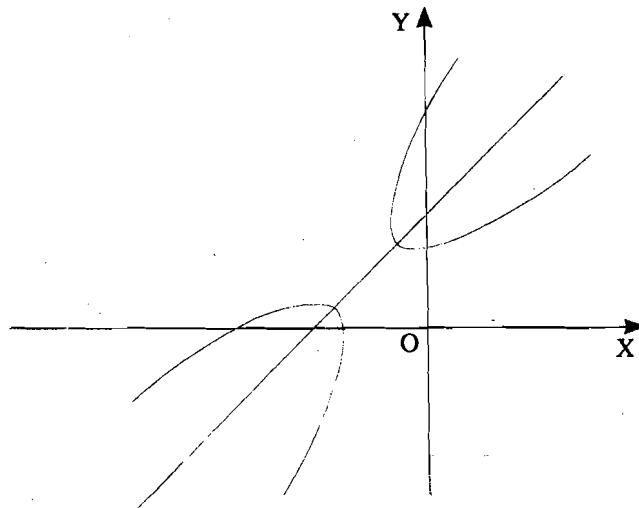


Fig. 6

- E 11) It will be central if $h^2 \neq 1$. And then its centre will be the intersection of $x + hy = 0$ and $hx + y + f = 0$, which is

$$\left(\frac{hf}{1-h^2}, \frac{-f}{1-h^2} \right).$$

If we shift the origin to this point, the given equation is transformed to

$$\begin{aligned} X^2 - 2hXY + Y^2 &= \frac{f^2}{1-h^2} \\ \Leftrightarrow \frac{1-h^2}{f^2} X^2 - \frac{2h(1-h^2)}{f^2} XY + \frac{1-h^2}{f^2} Y^2 &= 1. \end{aligned}$$

This is in the standard form $AX^2 + 2HXY + BY^2 = 1$ of a central conic.

Here $A = B = \frac{1-h^2}{f^2}$. Therefore, the axes of the conic are at an angle of 45° and 135° to the x-axis. Since they pass through the centre, their equations are

$$y + \frac{f}{1-h^2} = X - \frac{hf}{1-h^2} \text{ and}$$

$$y + \frac{f}{1-h^2} = -\left(X - \frac{hf}{1-h^2} \right).$$

- E 12) The conic is a parabola since $ab = h^2$, and the determinant condition for it to represent a pair of lines is not satisfied.

We can rewrite the equation as

$$(2x - y)^2 = 8x + 6y - 5.$$

We introduce a constant c to the equation, to get

$$\begin{aligned} (2x - y + c)^2 &= 8x + 6y - 5 + 4cx - 2cy + c^2. \\ \Leftrightarrow (2x - y + c)^2 &= 4(2+c)x + 2(3-c)y + c^2 - 5 \end{aligned}$$

We choose c in such a way that

$$2\left(\frac{4(2+c)}{2(3-c)}\right) = -1 \Rightarrow c = -1.$$

Then the equation of the curve becomes

$$(2x - y - 1)^2 = 4(x + 2y - 1)$$

$$\Leftrightarrow \left(\frac{2x - y - 1}{\sqrt{5}} \right)^2 = \frac{4}{\sqrt{5}} \left(\frac{x + 2y - 1}{\sqrt{5}} \right).$$

The vertex of this parabola is the intersection of $2x - y - 1 = 0$ and

$$x + 2y - 1 = 0, \text{ that is, } \left(\frac{3}{5}, \frac{1}{5} \right). \text{ The focus lies at}$$

$$\left(\frac{3}{5} + \frac{1}{\sqrt{5}} \cos \theta, \frac{1}{5} + \frac{1}{\sqrt{5}} \sin \theta \right), \text{ where } \tan \theta = 2.$$

$$\therefore \sin \theta = \frac{2}{\sqrt{5}}, \cos \theta = \frac{1}{\sqrt{5}}.$$

$$\therefore \text{The focus lies at } \left(\frac{4}{5}, \frac{3}{5} \right)$$

The curve intersects the y-axis in $(0, 1)$ and $(0, 5)$. It doesn't intersect the x-axis. Thus, the shape of the parabola is as given in Fig. 7.

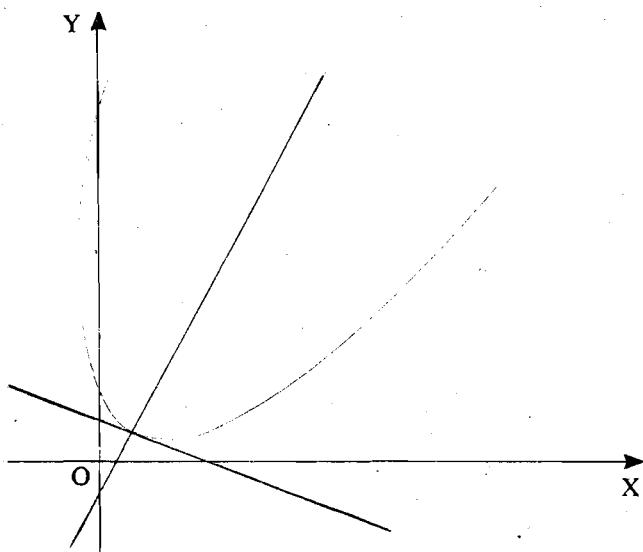


Fig. 7

E 13) The conic's equation is

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0.$$

From E10 you know it intersects the axes in $(-3, 0)$, $(-7, 0)$, $(0, 3)$, $(0, 7)$.

The tangent at $(-3, 0)$ is

$$-3x - \frac{3}{2}(x \cdot 0 - 3y) + y \cdot 0 + 5(x - 3) - 5(y - 0) + 21 = 0.$$

$$\Leftrightarrow 2x - \frac{1}{2}y + 6 = 0.$$

Its slope is 4.

Thus, the slope of the normal at $(-3, 0)$ is $-\frac{1}{4}$. Hence, its equation is

$$y = -\frac{1}{4}(x + 3).$$

You can similarly check that the tangents at $(-7, 0)$, $(0, 3)$ and $(0, 7)$ are respectively,

$$4x - 11y + 28 = 0,$$

$$x - 4y + 12 = 0,$$

$$11x - 4y + 28 = 0.$$

The normals at these points are respectively,

$$y = \frac{4}{11}(x + 7)$$

$$y - 3 = \frac{1}{4}x,$$

$$y - 7 = \frac{11}{4}x.$$

E 14) $x + 4y = 0$ will be a tangent to the given conic if

$$\begin{vmatrix} 1 & 2 & -\frac{5}{2} & 1 \\ 2 & 3 & -3 & 4 \\ -\frac{5}{2} & -3 & 3 & 0 \\ 1 & 4 & 0 & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow - \begin{vmatrix} 2 & 3 & -3 \\ -\frac{5}{2} & -3 & 3 \\ 1 & 4 & 0 \end{vmatrix} + 4 = \begin{vmatrix} 1 & 2 & -\frac{5}{2} \\ -\frac{5}{2} & -3 & 3 \\ 1 & 4 & 0 \end{vmatrix} = 0$$

$\Leftrightarrow 4 = 0$, which is false.

Thus, the given line is not a tangent to the given conic.

Any line parallel to the given line is of the form $x + 4y + c = 0$. This will be a tangent to the given conic if (15) is satisfied, that is,

$$(5c + 28)^2 = 3(3c^2 + 24c + 48)$$

$$\Leftrightarrow c = -5 \text{ or } -8.$$

Thus, the required tangents are

$$x + 4y - 5 = 0 \text{ and } x + 4y - 8 = 0.$$

E 15) Using (15), we see that the condition is

$$\begin{vmatrix} 1 & 0 & g & a \\ 0 & 1 & f & b \\ g & f & c & 1 \\ a & b & 1 & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow b(f - bc) - (1 - bf) + g \{af + b(bg - af)\} + a \{(ac - g) + f(bg - af)\} = 0$$

$$\Leftrightarrow b^2g^2 + a^2f^2 + 2bf - 2abfg - b^2c - 1 + 2ag - a^2c = 0.$$

Adding a^2g^2 on both sides and simplifying, we get the given condition.

b) In (a) we put $g = 0 = f$, $c = -A^2$, $-\frac{a}{b} = M$, $-\frac{1}{b} = C$.

So the condition for $y = Mx + C$ to touch $x^2 + y^2 = A^2$ is
 $C^2 = A^2(M^2 + 1)$

Thus, $C = A \sqrt{M^2 + 1}$.

E 16) Substituting $x^2 = a^2 \left(1 - \frac{y^2}{b^2}\right)$ in $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we get

$$\frac{a^2}{b^2} \left(1 - \frac{y^2}{b^2}\right) + \frac{y^2}{a^2} = 1 \Leftrightarrow y^2 = \frac{a^2 b^2}{a^2 + b^2}$$

$$\therefore y = \pm \frac{ab}{\sqrt{a^2 + b^2}}.$$

$$\text{Then } x^2 = a^2 \left(1 - \frac{a^2}{a^2 + b^2}\right) = \frac{a^2 b^2}{a^2 + b^2} \Rightarrow x = \pm \frac{ab}{\sqrt{a^2 + b^2}}.$$

Thus the 4 points of intersection are

$$\left(\frac{ab}{\sqrt{a^2 + b^2}}, \frac{ab}{\sqrt{a^2 + b^2}}\right), \left(\frac{-ab}{\sqrt{a^2 + b^2}}, \frac{ab}{\sqrt{a^2 + b^2}}\right), \left(\frac{ab}{\sqrt{a^2 + b^2}}, \frac{-ab}{\sqrt{a^2 + b^2}}\right)$$

$$\text{and } \left(\frac{-ab}{\sqrt{a^2 + b^2}}, \frac{-ab}{\sqrt{a^2 + b^2}}\right).$$

We have drawn the situation in Fig. 8.

E 17) Let $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$\text{and } S_1 \equiv a_1 x^2 + 2h_1 xy + b_1 y^2 + 2g_1 x + 2f_1 y + c_1 = 0$$

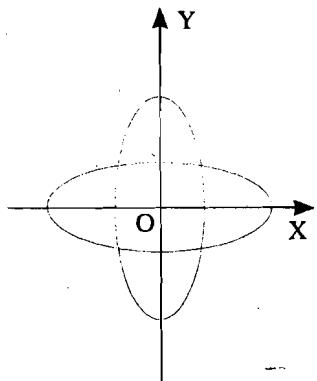


Fig. 8

be rectangular hyperbolas. Then

$$a + b = 0 \text{ and } a_1 + b_1 = 0.$$

$$\therefore (a+b) + k(a_1 + b_1) = 0 \forall k \in \mathbb{R}$$

$$\Leftrightarrow (a + ka_1) + (b + kb_1) = 0 \forall k \in \mathbb{R}$$

$\Leftrightarrow S + kS_1 = 0$ is a rectangular hyperbola $\forall k \in \mathbb{R}$

$$E18) S + kS_1 = 0$$

$$\Leftrightarrow \frac{x^2}{9} - kxy + \frac{y^2}{4} - (1+9k) = 0.$$

a) This conic will be an ellipse if

$$\left(\frac{1}{9}\right)\left(\frac{1}{4}\right) - \frac{k^2}{4} > 0, \text{ that is, } k^2 < \frac{1}{9}.$$

b) The conic will be a parabola if

$$k^2 = \frac{1}{9} \text{ and } \begin{vmatrix} \frac{1}{9} & -\frac{k}{2} & 0 \\ -\frac{k}{2} & \frac{1}{4} & 0 \\ 0 & 0 & -(1+9k) \end{vmatrix} \neq 0, \text{ that is,}$$

$$\text{If } k = \pm \frac{1}{3} \text{ and } (1+9k) \left(\frac{1}{36} - \frac{k^2}{4} \right) \neq 0.$$

But this can't be true.

So the conic can't be a parabola

But it will be a pair of lines if $k = \pm \frac{1}{3}$.

c) The conic will be a hyperbola if $k^2 > \frac{1}{9}$.

MISCELLANEOUS EXERCISES

(This section is optional)

In this section we have gathered some problems related to the contents of this block. You may like to do them to get a better understanding of conics. Our solutions to the questions follow the list of problems, in case you'd like to counter check your answers.

- 1) Find the equation of the path traced by a point P, the sum of the squares of the distances from (1, 0) and (-1, 0) of which is 8.

A path traced by a moving point is called its locus.

- 2) Find the equation of the circle which passes through (1, 0), (0, -6) and (1, 4).
(Hint: The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$).

- 3) Prove the reflecting property for a parabola.

(Hint: Show that $\alpha = \beta$ in Fig. 9, Unit 2.)

- 4) Prove Theorem 2 of Unit 3.

- 5) A circle cuts the parabola $y^2 = 4ax$ in the points $(at_i^2, 2at_i)$ for $i = 1, 2, 3, 4$.

Prove that $t_1 + t_2 + t_3 + t_4 = 0$.

(Hint: $t_1 + t_2 + t_3 + t_4$ are the solutions of the quadric equation obtained by putting $x = 0$, $y = 2at$ in the equation of a circle.)

- 6) Trace the curves $xy = 0$ and $xy - 4x - 5y + 20 = 0$.

- 7) What relations must hold between the coefficients of $ax^2 + by^2 + cx + cy = 0$ for it to represent a pair of straight lines?

- 8) Find the angle through which the axes should be rotated so that the equation $Ax + By + C = 0$ is reduced to the form $x = \text{constant}$, and find the value of the constant.

- 9) Prove that $y^2 + 2Ax + 2By + C = 0$ represents a parabola whose axis is parallel to the x-axis. Find its vertex and the equation of its latus rectum.

- 10) Prove that the set of midpoints of all chords of $y^2 = 4ax$ which are drawn through its vertex is the parabola $y^2 = 2ax$.

- 11) a) Prove that $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$ is negative, zero or positive, according as the point

(x_1, y_1) lies inside, on or outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

- b) Is the point (4, -3) inside, or outside the ellipse $5x^2 + 7y^2 = 11$?

- 12) A line segment of fixed length $a + b$ moves so that its ends are always on two fixed perpendicular lines (see Fig. 1). Prove that the path traced by a point which divides this segment in the ratio $a : b$ is an ellipse.

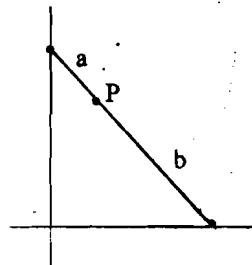


Fig. 1

- 13) Find the equation of the common tangent to the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

- 14) A normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the x and y axes in M and N, respectively. The lines through M and N drawn perpendicular to the x and y-axes, respectively, meet in the point P. Prove that type locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.

- 15) Consider the hyperbola in Fig. 20 of Unit 2. Through A and A' draw parallels to the conjugate axis, and through B and B' draw parallels to the transverse axis. Show that the diagonals of the rectangle so formed lie along the asymptotes of the hyperbola.

A method for drawing asymptotes of a hyperbola.

Conics

- 16) Which conics are represented by the following equations?
- $(x - y)^2 + (x - a)^2 = 0$,
 - $r \sin^2 \theta = 2a \cos \theta$,
 - $\frac{1}{r} = 1 + \cos \theta + \sqrt{3} \sin \theta$.
- 17) Trace the conics
- $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$
 - $xy - y^2 = a^2$
 - $(3x - 4y + 1)(4x + 3y + 1) = 1$.
- 18) Find the equation to the conic which passes through $(1, 1)$ and the intersection of $x^2 + 2xy + 5y^2 - 7x - 8y + 6 = 0$ with the pair of straight lines $2x - y - 5 = 0$ and $3x + y - 11 = 0$.

Solutions

- 1) Let P be (x, y) . Then

$$\{(x-1)^2 + y^2\} + \{(x+1)^2 + y^2\} = 8$$

$$\Leftrightarrow 2x^2 + 2y^2 = 6$$

$\Leftrightarrow x^2 + y^2 = 3$, which is a circle with centre $(0, 0)$ and radius $\sqrt{3}$.

- 2) Let the equation be $x^2 + y^2 + 2gx + 2fy + c = 0$. Since $(1, 0), (0, -6)$ and $(3, 4)$ lie on it,

$$1 + 2g + c = 0,$$

$$36 - 12f + c = 0,$$

$$9 + 16 + 6g + 8f + c = 0.$$

Solving these three linear equations in g, f and c , we get

$$g = -\frac{71}{4}, f = \frac{47}{8}, c = \frac{69}{2}.$$

$$\text{Thus the equation is } x^2 + y^2 - \frac{71}{2}x + \frac{47}{4}y + \frac{69}{2} = 0.$$

- 3) The parabola is $y^2 = 4ax$. The tangent T at a point $P(x_1, y_1)$ is

$$yy_1 = 2a(x + x_1).$$

$$\text{So } \tan \alpha = \frac{2a}{y_1}$$

$$\text{The line } PF, \text{ where } F(a, 0) \text{ is the focus, is } \frac{y - y_1}{-y_1} = \frac{x - x_1}{a - x_1}.$$

$$\text{Its slope is } \frac{y_1}{x_1 - a}.$$

$$y_1 = \frac{2a}{x_1 - a}$$

$$\text{Thus, } \tan \beta = \frac{x_1 - a - \frac{2a}{x_1 - a}}{1 + \frac{2a}{x_1 - a}}, \text{ using (11) of Unit 1.}$$

$$= \frac{2a}{y_1}, \text{ using the fact that } y_1^2 = 4ax_1.$$

Thus $\tan \alpha = \tan \beta$ and α and β are both less than or equal to 90° .

$$\therefore \alpha = \beta$$

- 4) We want to show that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

can be written as a product of two linear factors iff its discriminant is 0.

If $a \neq 0$, we multiply (1) throughout by a and arrange it in decreasing powers of x . We get

$$a^2x^2 + 2ax(hy + g) = -aby^2 - 2afy - ac.$$

On completing the square on the left hand, we get

...(1)

$$\Leftrightarrow ax + hy + g = \pm \sqrt{y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac}$$

From this we can obtain x in terms of y , only involving the first degree iff the quantity under the square root sign is a perfect square, that is, iff $(gh - af)^2 = (h^2 - ab)(g^2 - ac)$,

$$\Leftrightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

$$\Leftrightarrow \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

- 5) Let the circle's equation be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Substituting $x = at^2$, $y = 2at$ in this, we get

$$a^2t^4 + 4a^2t^2 + 2agt^2 + 4aft + c = 0.$$

We know that it has 4 roots t_1, t_2, t_3, t_4 . So, from (CS-60/Block 5 & 6) you know that the

sum of the roots will be $\frac{1}{a^2}$ (coefficient of t^3) = 0.

$$\therefore t_1 + t_2 + t_3 + t_4 = 0.$$

- 6) $xy = 0$ is the pair of lines $x = 0$ and $y = 0$. We have traced it in Fig. 2.

$xy - 4x - 5y + 20 = 0$ is a pair of lines since its discriminant is 0. In fact, we can easily factorise it as $(x - 5)(y - 4) = 0$.

Thus, it represents the pair of lines $x = 5$ and $y = 4$, which we have traced in Fig. 3.

- 7) $ax^2 + by^2 + cx + cy = 0$ represents a pair of lines iff

$$\begin{vmatrix} a & 0 & \frac{c}{2} \\ 0 & b & \frac{c}{2} \\ \frac{c}{2} & \frac{c}{2} & 0 \end{vmatrix} = 0 \Leftrightarrow (a+b)\frac{c^2}{4} = 0 \Leftrightarrow a = -b \text{ or } c = 0.$$

- 8) Let us rotate the axes through θ . Then the equation becomes

$$A(x' \cos \theta - y' \sin \theta) + B(x' \sin \theta + y' \cos \theta) + C = 0.$$

$$\Leftrightarrow x'(A \cos \theta - B \sin \theta) + y'(B \cos \theta - A \sin \theta) + C = 0.$$

This will reduce to the form $x' = \text{constant}$ iff $B \cos \theta = A \sin \theta$,

$$\text{that is, } \theta = \tan^{-1} \frac{B}{A}.$$

And then the equation becomes

$$x' \left(A \frac{A}{\sqrt{A^2 + B^2}} + B \frac{A}{\sqrt{A^2 + B^2}} \right) + C = 0$$

$$\Leftrightarrow x' = \frac{-C}{\sqrt{A^2 + B^2}}$$

Thus, the constant is $\frac{-C}{\sqrt{A^2 + B^2}}$

- 9) We rewrite the given equation as

$$y^2 = -(2Ax + 2By + c)$$

$$\Leftrightarrow (y + k)^2 = -2Ax - 2By - C + 2ky + k^2, \text{ where } k \text{ is a constant.}$$

$$\Leftrightarrow (y + k)^2 = -2Ax + 2(k - B)y + k^2 - C$$

We choose k so that

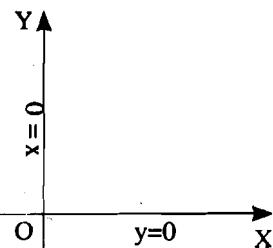


Fig. 2

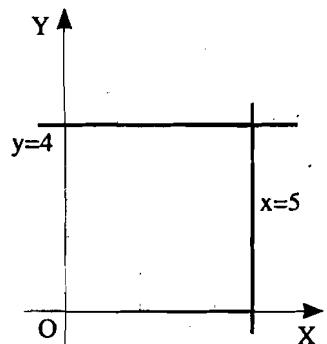


Fig. 3

$Ax + (B - k)y + \frac{C - k^2}{2} = 0$ is parallel to the y-axis, that is,

$k = B$. Then the equation becomes

$$(y + B)^2 = -2A\left(x + \frac{C - B^2}{2A}\right).$$

Its axis is $y + B = 0$, vertex is $\left(\frac{B^2 - C}{2A}, -B\right)$ and the equation of

$$\text{its latus rectum is } x = \frac{B^2 - A^2 - C}{2A}$$

- 10) The midpoint of any chord through $P(x_1, y_1)$ and $O(0, 0)$ is

$$Q\left(\frac{x_1}{2}, \frac{y_1}{2}\right). \text{ Since } y_1^2 = 4ax_1, \left(\frac{y_1}{2}\right)^2 = 2a\left(\frac{x_1}{2}\right).$$

Thus, the set of all such Q is $y^2 = 2ax$.

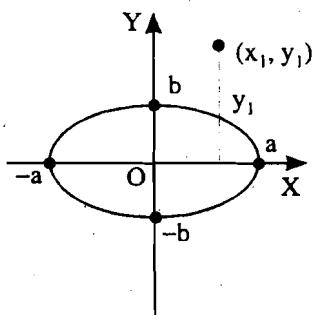


Fig. 4

- 11) a) Firstly, if (x_1, y_1) lies on the ellipse, then clearly

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

Now, if (x_1, y_1) lies outside the ellipse (see Fig. 4) then either $|x_1| > a$ or $|y_1| > b$

$$\therefore x_1^2 > a^2 \text{ or } y_1^2 > b^2$$

$$\therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} > 1.$$

Similarly, you can show that if (x_1, y_1) lies inside the ellipse,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < 1.$$

- b) Since $5(16) + 7(9) = 143 > 11$, the point lies outside the ellipse.

- 12) Let the perpendicular lines be the coordinate axes. Let the segment intersect the axes in $(x, 0)$ and $(0, y)$. Then the coordinates of the point P are

$$(X, Y) = \left(\frac{ax}{a+b}, \frac{by}{a+b}\right).$$

Now, since $(x^2 + y^2) = (a+b)^2$

$$\Rightarrow \frac{x^2}{(a+b)^2} + \frac{y^2}{(a+b)^2} = 1$$

$$\Rightarrow \left(\frac{ax}{a+b}\right)^2 \frac{1}{a^2} + \left(\frac{by}{a+b}\right)^2 \frac{1}{b^2} = 1$$

$$\Rightarrow \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1,$$

Thus, the path traced by P is the ellipse $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

- 13) Any tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = mx + \sqrt{a^2 m^2 - b^2}$, and any

tangent to $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is $x = m_1 y + \sqrt{a^2 m_1^2 - b^2}$.

For these two lines to be the same, we must have $\frac{1}{m_1} = m$ and

$$\sqrt{a^2 m^2 - b^2} = -\frac{1}{m} \sqrt{a^2 m_1^2 - b^2}$$

$$\Leftrightarrow a^2 m^2 - b^2 = a^2 - m^2 b^2$$

$$\Leftrightarrow m^2 = 1, \quad a^2 \neq b^2.$$

Thus the common tangents are $y = x + \sqrt{a^2 - b^2}$ and
 $y = -x + \sqrt{a^2 - b^2}$.

- 14) See Fig. 5 for a diagram of the situation. The normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

at (x_1, y_1) is

$$\frac{a^2}{x_1}(x - x_1) + \frac{b^2}{y_1}(y - y_1) = 0.$$

Thus, M is $\left(\frac{(a^2 + b^2)x_1}{a^2}, 0\right)$ and N is $\left(0, \frac{(a^2 + b^2)y_1}{b^2}\right)$.

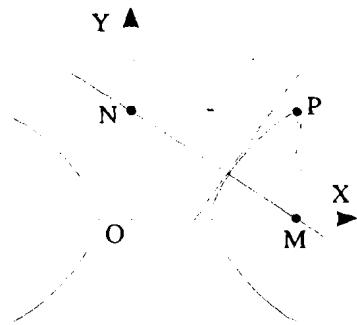


Fig. 5

Thus, the coordinates of P are

$$\left(\left(\frac{a^2 + b^2}{a^2}\right)x_1, \left(\frac{a^2 + b^2}{b^2}\right)y_1\right).$$

Now, since (x_1, y_1) lies on the hyperbola,

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

$$\Leftrightarrow \frac{a^2}{(a^2 + b^2)^2} \left(\frac{a^2 + b^2}{a^2}\right)^2 x_1^2 - \frac{b^2}{(a^2 + b^2)^2} \left(\frac{a^2 + b^2}{b^2}\right)^2 y_1^2 = 1$$

$$\Leftrightarrow a^2 x^2 - b^2 y^2 = (a^2 + b^2)^2, \text{ Where } X = \frac{a^2 + b^2}{a^2} x_1 \text{ and } Y = \frac{a^2 + b^2}{b^2} y_1$$

Now, as P varies, X and Y vary; but always satisfy the relationship

$$a^2 x^2 - b^2 y^2 = (a^2 + b^2)^2. \text{ Thus, this is the locus of the point P.}$$

- 15) The lines meet in (a, b) , $(-a, b)$ and $(-a, -b)$. Thus the diagonals of the rectangles lie along $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$, which are the asymptotes of the hyperbola.

- 16) a) $(x - y)^2 + (x - a)^2 = 0$

$$\Leftrightarrow 2x^2 - 2xy + y^2 - 2ax + a^2 = 0.$$

Here $a = 2$, $b = 1$, $h = -1$, $g = -a$, $f = 0$, $c = a^2$.

$\therefore ab - h^2 > 0$. Thus, the conic is an ellipse.

- b) $r \sin^2 \theta = 2a \cos \theta$.

Changing to Cartesian coordinates, this equation is $y^2 = 2ax$, which is a parabola.

- c) $\frac{1}{r} = 1 + \cos \theta + \sqrt{3} \sin \theta$

$$\Leftrightarrow 1 = \sqrt{x^2 + y^2} + x + \sqrt{3}y, \text{ since } x = r \cos \theta, y = r \sin \theta.$$

$$\Leftrightarrow 2y^2 + 2\sqrt{3}xy + 2x + \sqrt{3}y + 1 = 0$$

Here $ab - h^2 < 0$ and its discriminant is

$$\begin{vmatrix} 0 & \sqrt{3} & 1 \\ \sqrt{3} & 2 & \sqrt{3} \\ 1 & \sqrt{3} & 1 \end{vmatrix} = 1 \neq 0.$$

Thus, the curve represents a hyperbola.

- 17) a) You can check that $ab - h^2 = 0$ and the discriminant is non-zero. Thus, the equation represents a parabola. We can write it as $(3x - 4y)^2 = 18x + 101y - 19$.

$$\Leftrightarrow (3x - 4y + c)^2 = (6c + 18)x + y(101 - 8c) + c^2 - 19, \text{ where we}$$

Choose the constant c so that

$$3(6c + 18) - 4(101 - 8c) = 0$$

$$\Leftrightarrow c = 7.$$

Then the given equation becomes

$$(3x - 4y + 7)^2 = 15(4x + 3y + 2)$$

$$\Leftrightarrow \left(\frac{3x - 4y + 7}{5}\right)^2 = 3\left(\frac{4x + 3y + 2}{5}\right).$$

Thus, the axis of the parabola is $4x + 3y + 2 = 0$. The vertex is the intersection of $3x - 4y + 7 = 0$ and $4x + 3y + 2 = 0$;

$$\text{that is, } \left(-\frac{29}{25}, \frac{22}{25}\right).$$

The length of its latus rectum is 3. Its focus F lies

$$\text{at } \left(-\frac{29}{25} + \frac{3}{4} \cos \theta, \frac{22}{25} + \frac{3}{4} \sin \theta\right), \text{ where } \tan \theta = -\frac{4}{3}.$$

$$\therefore F \text{ is } (-0.71, 0.28).$$

The curve intersects the y-axis in

$$\frac{101 \pm \sqrt{(101)^2 - 64 \times 19}}{32}$$

that is, approximately, $\frac{49}{8}$ and $\frac{3}{16}$.

It doesn't intersect the x-axis.

We have traced it in Fig. 6.

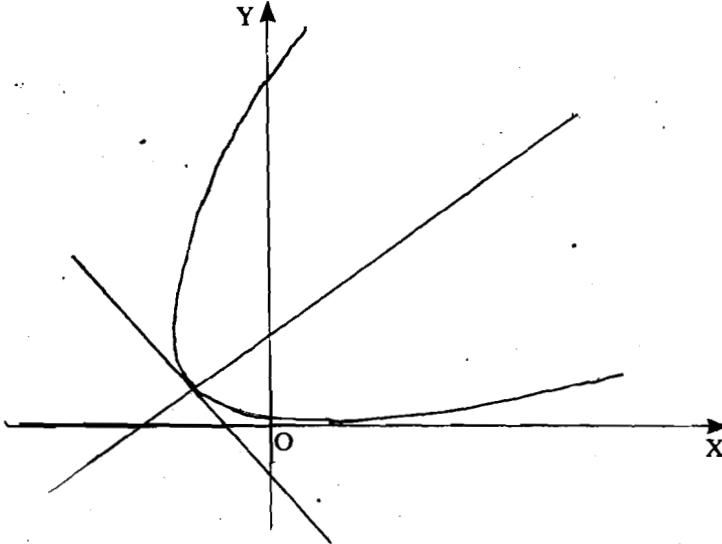


Fig. 6

- b) $xy - y^2 = 1$

This is a hyperbola. Its centre is the intersection of

$$-\frac{1}{2}y = 0 \text{ and } -\frac{1}{2}x + y = 0, \text{ that is, } (0, 0).$$

Its axes are inclined to the coordinate axes at an angle of θ , where $\tan 2\theta = 1$. Thus, the slope of the

transverse axis is $\theta_1 = \frac{\pi}{8}$, and of the conjugate axis is $\theta_2 = \frac{5\pi}{8}$. Since

$\tan \theta_1 = .41$, the length of the transverse axis, r_1 , is given by

Miscellaneous Exercises

$$r_1^2 = \frac{1 + (.41)^2}{-(.41) + (.41)^2} = \frac{1.168}{.758} = 1.54.$$

$\therefore r_1 = 1.24$, approximately.

We similarly find $r_2 = .91$.

Thus its eccentricity is 1.24.

It does't intersect the axes. With all this information we have traced the curve in Fig. 7.

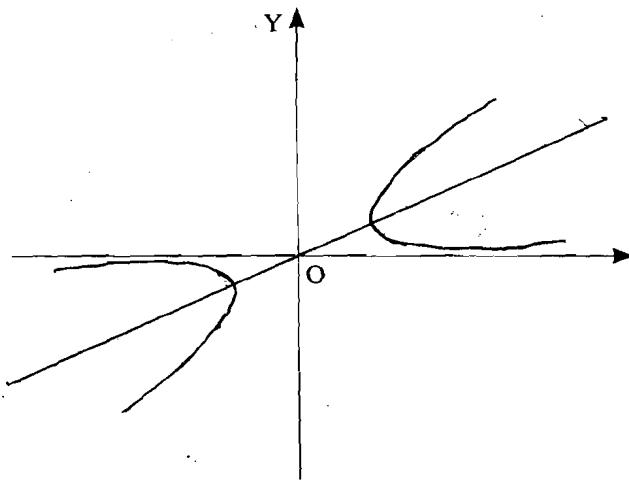


Fig. 7

- c) The equation is a hyperbola whose centre is the intersection of $3x - 4y + 1 = 0$

and $4x + 3y + 1 = 0$, that is, $\left(-\frac{7}{25}, \frac{1}{25}\right)$.

Here $a = 12$, $b = -12$, $h = -\frac{7}{2}$.

Thus, its axes are inclined at angles θ_1 and θ_2 to the coordinate axes, where

$\tan \theta_1$ and $\tan \theta_2$ are roots of the equation

$$\tan^2 \theta + \frac{a-b}{h} \tan \theta - 1 = 0$$

$$\Leftrightarrow \tan^2 \theta - \frac{48}{7} \tan \theta - 1 = 0$$

$$\Leftrightarrow \tan \theta_1 = 7 \text{ and } \tan \theta_2 = -\frac{1}{7} \Rightarrow \theta_1 = 81.9^\circ \text{ (approx.) and}$$

$$\theta_2 = 171.9^\circ \text{ (approx.)}.$$

The length of its axes are r_1 and r_2 , where

$$r_1^2 = \frac{1+49}{12-7 \times 7-12 \times 49} = -\frac{2}{25} \Rightarrow r_1 = .28$$

$$r_2^2 = \frac{1+\frac{49}{49}}{12-7\left(-\frac{1}{7}\right)-\frac{12}{49}} = -\frac{2}{25} \Rightarrow r_2 = .28,$$

The curve intersects the coordinate axes in $(0, 0)$, $\left(-\frac{7}{12}, 0\right)$, $\left(0, -\frac{1}{12}\right)$,

So, its curve is as given in Fig. 8.

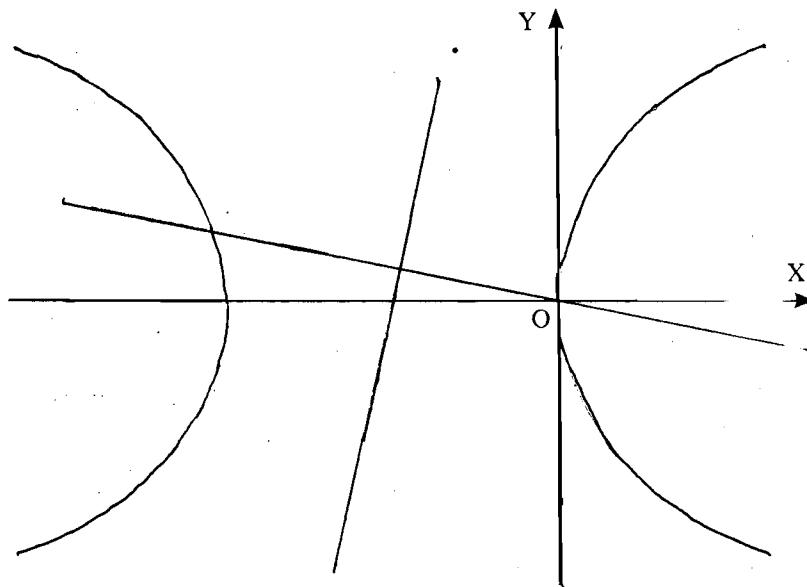


Fig. 8

18) Let $S_1 \equiv x^2 + 2xy + 5y^2 - 7x - 8y + 6 = 0$ and

$$S_2 \equiv (2x - y - 5) - (3x + y - 11) = 0.$$

Then the required conic is $S_1 + kS_2 = 0$, where we choose k so that

$(1, 1)$ lies on the curve. Thus, the curve is

$$(1 + 6k)x^2 + (2 - k)xy + (5 - k)y^2 - (7 + 37k)x - (8 - 6k)y + (6 + 55k) = 0.$$

Since $(1, 1)$ lies on it, $k = \frac{1}{28}$.

Thus, the conic is

$$34x^2 + 55xy + 139y^2 - 233x - 218x - 218y + 223 = 0.$$