UNIT 3 CUBIC AND BIQUADRATIC EQUATIONS

Structure

3.1 INTRODUCTION

In this unit we will look at an aspect of algebra that has exercised the minds of several mathematicians through the ages. We are talking about the solution of polynomial equations over **R**. The ancient Hindu, Arabic and Babylonian mathematicians had discovered methods of solving linear and quadratic equations. The ancient Babylonians and Greeks had also discovered methods of solving some cubic equations. But, as we have said in unit 2, they had not thought of complex numbers. So, for them, a lot of quadratic and cubic equations had no solutions.

In the 16th century various Italian mathematicians were looking into the geometrical problem of trisecting an angle by straight edge and compass. In the process they discovered a method for solving the general cubic equation. This method was divulged by Girolamo Cardano, and hence, is named after him. This is the same Cardano who was the first to introduce complex numbers into algebra. Cardano also publicised a method developed by his contemporary, Ferrari, for solving quartic equations. Later, in the 17th century, the French mathematician Descartes developed another method or solving 4th degree equations.

In this unit we will acquaint you with the solutions due to Cardano, Ferrari and Descartes. But first we will quickly cover methods for solving linear and quadratic equations. In the process we will also touch upon some general theory of equations.

There are several reasons, apart from a mathematician's natural curiosity, for looking at cubic and biquadratic equations. The material covered in this unit is also useful for mathematicians, physicists, chemists and social scientists.

After going through the unit, please check to see if you have achieved the following objectives.

Objectives

After studying this unit, you should be able to

- solve a linear equation;
- solve a quadratic equation;
- apply Cardanio's method for solving a cubic equation;

- apply Ferrari's or Descartes method for solving a quartic equation;
- use the relation between roots and the coefficients of a polynomial equation for obtaining solutions.

3.2 LET US RECALL

You may be familiar with expressions of the from $2x+5$, $-5x^2+\frac{7}{2}$, $\sqrt{2}\,x^3+x^2+1$, etc.

All these expressions are polynomials in one variable with coefficients in **R**. In general, we have the following definitions.

Definitions : An expression of the form $a_0x^0+a_1x^1+a_2x^2+\ldots\ldots\ldots+a_nx^n$, where

$n \in N$ and $a_i \in C\ \forall i=1,\ldots\ldots n$, is called a **polynomial** over C in the variable x.

$a_0\,a_1,\ldots\ldots a_n$ are called the **coefficients** of the polynomial.

If $a_n \neq 0$ we say that the **degree of the polynomial** is n and the **leading term** is a_nx^n. While discussing polynomials we will observe the **following conventions.**

Conventions : We will

i) Write x^0 as 1, so that we will write a_0 for a_0x^0,

ii) Write x^1 as x,

iii) Write x^m instead of $1.x^m$ (i.e. when $a_m=1$),

iv) omit terms of the type $0.x^m$.

Thus, the polynomial $2+3x^2-x^3$ is $2x^0+0.x^1+3x^2+(-1)x^3$.

We usually denote polynomials in x by f(x), g(x), etc. If the variable x is understood, then we often write f instead of f(x). We denote the degree of a polynomial f(x) by **deg f(x)** or **deg f.**

Note that the degree of f(x) is the highest power of x occurring in f(x). For example,

i) $3x+6x^2+\frac{5}{2}ix^3$ is a polynomial of degree 3,

ii) x^5 is a polynomial of degreee 5, and

iii) $2+i$ is a polynomial of degree 0, since $2+i=(2+i)x^0$.

Remark 1: If f(x) and g(x) are two polynomials, then

$\deg(f(x)+g(x)) \le \max(\deg f(x), \deg g(x))$

$\deg(f(x).\,g(x)) \le \deg f(x)+\deg g(x)$

We say that f(x) is a polynomial over **R** if its coefficients are real numbers, and f(x) is **over Q** if its coefficients are rational numbers. For example, $2x+3$ and x^2+3 are polynomials over **Q** as well as **R** (of degrees 1 and 2, respectively). On the other hand $\sqrt{3}$ is a polynomial (of degree 0) over **R** but not over **Q**. In this course **we shall almost always be dealing with polynomials over R.**

Note that any non- zero element of **R** is a polynomial of degree 0 over **R.**

We define the **degree of 0 to be** $-\infty$

Now, if we put a polynomial of degree n equal to zero, we get a **polynomial equation of degree n**, or an nth degree equation.

For exeample,

(i) $2x + 3 = 0$ is a polynomial equation of degree 1 and

(ii) $3x^2 + \sqrt{2}\,x - 1 = 0$ is a polynomial equation of degree 2.

If $f(x) = a_0 + a_1 x + \ldots\ldots + a_n x^n$ is a polynomial and $a \in C$, we can substitute a for x to get f(a), the value of the polynomial at $x = a$. Thus, $f(a) = a_0 + a_1 a + a_2 a^2 + \ldots\ldots\ldots + a_n a^n$.

For example, if $f(x) = 2x + 3$, then $f(1) = 2.1 + 3 = 5$, $f(i) = 2i + 3$, and

$$f\left(\frac{-3}{2}\right) = 2\left(\frac{-3}{2}\right) + 3 = 0.$$

Since $f\left(\frac{-3}{2}\right) = 0$ we say that $\frac{-3}{2}$ is a root of f (x).

Definition : Let f (x) be a non - zero polynomial, $\alpha \in C$ is called a root (or a **zero)** of f (x) if

$f(\alpha) = 0$.

In this case we also say that α is a **solutuion** (or a **root)** of the equation $f(x)=0$.

A polynomial equation can have several solutions. For example, the equation $x^2 - 1 = 0$

The set of solutions of an equation is called its **solution set.** Thus, the solution set of $x^2 + 1 = 0$ is $\{i, -i\}$.

Another definition that you will need quite often is the following.

Definition: Two polynomials $a_0 + a_1 x + \ldots\ldots + a_n x^n$ and $b_0 + b_1 x \ldots\ldots + b_m x^m$ are called **equal** if $n = m$ and $a_i = b_i$, $\forall i = 0, 1, \ldots\ldots, n$.

Thus, two polynomials are equal if they have the same degree and at their corresponding coefficients are equal. Thus, $2x^3 + 3 = ax^3 + bx^2 + cx + d$ iff $a = 2, b = 0, c = 0, d = 3$.

Let us now take a brief look at polynomials over **R** whose degrees are 1 or 2 and at their solutions sets. We start with degree 1 equations.

3.2.1 Linear Equations

Consider any polynomial $ax + b$ with $a, b \in R$ and $a \neq 0$. We call such a polynomial a linear **polynomial.** If we put it equal to zero, we get a **linear equation.**

Thus,

$ax + b = 0, a, b \in \mathbf{R}, a \neq 0,$

is the most general form of a linear equation.

You know that this equation has a solution in **R**, namely, $x = \frac{-b}{a}$; and that this is the only solution.

Sometimes you may come across equations that don't appear to be linear, but, after simplification they become linear.

Let us look at some examples.

Example 1 : Solve $\frac{3p-1}{3} - \frac{2p}{p-1} = p$. (Here we must assume $p \neq 1$.)

Solution: At first glance, this equation in p does not appear to be linear. But, through cross-multiplication, we get the following equivalent equation:

Two equations are equivalent if their solution sets are equal.

$(3p - 1)(p - 1) - 3(2p) = 3(p - 1)p.$

On simplifying this we get

$3p^2 - 4p + 1 - 6p = 3p^2 - 3p$,

that is, $7p - 1 = 0$.

The solution set of this equation is $\left\{\frac{1}{7}\right\}$. Thus, this is the solution set of the equation we started with.

Example 2 : Suppose I buy two plots of land for total Rs. 1,20,000, and then sell them. Also suppose that I have made a profit of 15% on the first plot and a loss of 10% on the second plot. If my total profit is Rs. 5500, how much did I pay for each piece of land ?

Solution : Suppose the first piece of land cost Rs. x . Then the second piece cost

Rs. (1,20,000 – x). Thus, my profit is Rs. $\frac{15}{100}$ x and my loss is Rs. $\frac{10}{100}$ (1, 20, 000 – x).

$$\therefore \frac{15}{100} x - \frac{10}{100} (1{,}20{,}000 - x) = 5500$$

$\Leftrightarrow 25x - 1{,}750{,}000 = 0$

$\Leftrightarrow x = 70{,}000$.

Thus, the first piece cost Rs. 70,000 and the second polt cost Rs. 50,000.

You may like to try these exercises now.

E1) Solve each of the following equations for the variable indicated. Assume that all denominators are non - zero.

a) $J\left(\frac{x}{k} + a\right) = x$ for x where J, k and a are constants.

b) $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ for R, keeping r_1 and r_2 constant.

c) $C = \frac{5}{9}(F - 32)$ for F, keeping C constant.

E2) An isosceles triangle has a perimeter of 30 cm. Its equal sides are twice as long as the third side. Find the lenghs of the three sides.

E3) A student cycles from her home to the study centre in 20 minutes. The return journey is uphill and takes her half an hour. If her rate is 8 km per hour slower on the return trip, how far does she live from the study centre?

E4) Simple interest is directly proportional to the principal amount as well as the time for which the amount is invested. If Rs. 1000, left at interest for 2 years, earns Rs 110, find the amount of interest earned by Rs. 5000 for 3 years.

(**Hint :** S = k Pt, where k is the constant of proportionality, S is the simple interest, P is the principal and t is the time.)

Now that we have looked at first degree equations, let us consider second degree equations, that is, equations of degree 2.

3.2.2 Quadratic Equations

Consider the general polynomial in x over **R** of degree 2:

$ax^2 + bx + c$, where a, b, c $\in$ **R**, a $\neq 0$.

We call this polynomial a **quadratic polynomial.** On equating polynomial to zero, we get a **quadratic equation** in standard from.

Can you think of an example of a quadratic equation ? One is $x^2 = 5$, Which is the same as $x^2 - 5 = 0$. Another is the equation Cardano tried to solve, namely, $x^2 - 10x + 40 = 0$ (see sec. 2.1). We are sure you can think of several others.

The word 'quadratic' comes from The latin-word 'quadratum' meaning 'square'

Various methods for solving such equations have been known since Babylonian times (2000 B.C.). Brahmagupta, in 628 A.D. approximtely, also gave a rule for solving quadraratic equations. The method that can be used for any quadratic equation is "completing the square". Using it we get the quadratic formula. Let us state this formula.

Quadratic Formula : The solutions of the quadratic equation $ax^2+bx+c=0$, where $a, b, c \in \mathbf{R}$ and $a \neq 0$, are

$$x = \frac{-b + \sqrt{b^2-4ac}}{2a}$$

The expression b^2-4ac is called the **discriminant** of $ax^2 + bx + c = 0$.

Note that this formula tells us that a quadratic equation has only two roots. These roots may be equal or they may be distinct; they may be real or complex.

Convention : We call a root that lies in $C \setminus \mathbf{R}$ a **complex root**, that is ,a root of the form $a + ib$, $a, b \in \mathbf{R}$, $b \neq 0$, is a complex root.

Let us consider some examples.

Example 3 : Solve

i) $x^2-4x+1=0$

ii) $4x^2+25=20x$

iii $x^2-10x+40=0$

Solution : i) This equation is in standard form. So we can apply the quadratic formula immediately. Here $a = 1$, $b = -4$, $c = 1$. Substituting these values in the quadratic formula, we get the two roots of the equation to be.

$$x = \frac{-(-4) + \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \frac{4+\sqrt{12}}{2} = 2 + \sqrt{3}, \text{ and}$$

$$x = \frac{-(-4) - \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = 2 - \sqrt{3}.$$

Thus, the solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$, two distinct elements of $\mathbf{R}$.

Note that in this case the discriminant was positive.

ii) In this case let us first rewrite the equation in standard form as

$4x^2-20x+25=0$.

Now, putting $a=4$, $b=-20$, $c=25$ in the quadratic formula, we find that

$$x = \frac{20 + \sqrt{400-4(4)(25)}}{2(4)} = \frac{20+\sqrt{0}}{8} = \frac{5}{2}, \text{ and}$$

$$x = \frac{20 - \sqrt{400-4(4)(25)}}{2(4)} = \frac{5}{2}.$$

Here we find that both the roots coincide and are real.

Note that in this case the discriminant is 0.

iii) Using the quadratic formula, we find that the solutions are

$$x = \frac{10 \pm \sqrt{100-160}}{2} = 5 \pm \frac{\sqrt{-60}}{2} = 5 \pm \frac{\sqrt{4(-15)}}{2}$$

$$= 5 \pm \sqrt{-15}$$

$$= 5 \pm i\sqrt{15}$$

By a complex root we mean a root in $C \setminus R$

Thus, in this case we get two distinct complex roots $5 + i\sqrt{15}$ and $5 - i\sqrt{15}$.

Note that in this case the discriminant is negative.

In the example above do you see a relationship between the types of roots of a quadratic equation and the value of its discriminant? There is such a relationship, which we now state.

The equation $ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c, \in \mathbf{R}$ has two roots. They are

i) real and distinct if $b^2 - 4ac > 0$;

ii) real and equal if $b^2 - 4ac = 0$;

iii) complex and distinct if $b^2 - 4ac < 0$.

Now, is there a difference in the character of the roots of $ax^2 + bx + c = 0$ and

$dax^2 + dbx + dc = 0$, where d is a non- zero real number ? For example, if $b^2 - 4ac > 0$,

what is the sign of $(db)^2 - 4\,(da)\,(dc)$? It will also be positive. In fact, **the character of the roots of equivalent quadratic equations is the same.** Two equations are equivalent if one is obtained from the other by multiplying each term throughout by a non-zero constant.

Now let us consider some important remarks which will be useful to you while solving quadratic equations.

Remark 2 : α and β are roots of a quadratic equation $ax^2 + bx + c = 0$ if any only if $ax^2 + bx + c = a\,(x - \alpha)\,(x - \beta)$.

Thus, $\alpha \in C$ is a root of $ax^2 + bx + c = 0$ if and only if $(x - \alpha) \mid (ax^2 + bx + c)$.

Remark 3 : From the quadratic formula you can see that if $b^2 - 4\,ac < 0$, then the quadratic equation $ax^2 + bx + c = 0$ has 2 complex roots which are each other's conjugates.

Remark 4 : Sometimes a quadratic equation can be solved without resorting to the quadratic formula. For example the equation $x^2 = 9$ clearly has 3 and -3 as its roots.

Similarly, the equation $(x - 1)^2 = 0$ clearly has two coincident roots, both equal to 1 (see Remark 2).

Using what we have said so far, try and solve the following exercises

E5) A quadratic equation over **R** can have complex roots while a linear equation over **R** can only have a real root. True of false ? Why ?

E 6) Solve the following equations:

a) $x^2 + 5 = 0$

b) $(x + 9)(x - 1) = 0$

c) $x^2 - \sqrt{5}\ x = 1$

d) $pm^2 - 8qm + \frac{1}{r} = 0$ for m, where $p, q, r, \in \mathbf{R}$ and $p, r \neq 0$.

E 7) For what values of k will the equation

$kx^2 + (2k + 6)\ x + 16 = 0$ have coincident roots ?

E 8) Show that the quadratic equation $ax^2 + bx + c = 0$ has equal roots if

$(2ax + b) \mid (ax + bx + c)$.

E 9) Find the values of b and c for which the polynomial $x^2 + bx + c$ has $1 + i$ and $1 - i$ as its roots.

E 10) If α and β are roots of $ax^2 + bx + c = 0$, then show that $\alpha + \beta = -\frac{b}{a}$ and $\alpha\ \beta = \frac{c}{a}$.

E 11) Let $\alpha, \beta \in \mathbf{C}$ such that $\alpha + \beta = p \in \mathbf{R}$ and $\alpha\,\beta = q \in \mathbf{R}$. Show that α and β are the roots of $x^2 - px + q = 0$.

E11) is the converse of E10. We will use it in the next section.

Let us now consider some equations which are not quadratic, but whose solutions can be obtained from related quadratic equations. Look at the following example.

Exanple 4: Solve

i) $2x^4 + x^2 + 1 = 0$, and

ii) $x = \sqrt{15 - 2x}$.

Solution : i) $2x^4 + x^2 + 1 = 0$ can be written as $2y^2 + y + 1 = 0$, where $y = x^2$. Then, solving this for y we get $y = \frac{-1 \pm i\sqrt{7}}{4}$ that is, $x^2 = \frac{-1 \pm i\sqrt{7}}{4}$, two polynomials over C.

Thus, the four solutions of the original equation are

$$\sqrt{\frac{-1 + i\sqrt{7}}{4}}, -\sqrt{\frac{-1 + i\sqrt{7}}{4}}, \sqrt{\frac{-1 - i\sqrt{7}}{4}}, -\sqrt{\frac{-1 - i\sqrt{7}}{4}}$$

ii) $x = \sqrt{15-2x}$ is not a polynomial equations. We square both sides to obtain the polynomial equation $x^2 = 15 - 2x$.

Now, any root of $x = \sqrt{15-2x}$ is also a root the equation $x^2 = 15 - 2x$.

But the converse need not be true, since $x^2 = 15 - 2x$ can also mean $x = -\sqrt{15-2x}$.

So we will obtain the roots of $x^2 = 15 - 2x$ and see which of these satisfy $x = \sqrt{15-2x}$.

Now, the roots of the quadratic equation $x^2 = 15 - 2x$ are $x = -5$ and $x = 3$. We must put these values in the original equation to see if they satisfy it.

Now, for $x = -5$,

$$x - \sqrt{15 - 2x} = (-5) - \sqrt{15 + 10} = (-5) - 5 = -10 \neq 0.$$

So $x = -5$ is not a solution of the given equation. But it is a solution of $x^2 = 15 - 2x$. We call it an **extraneous solution.**

What happens when we put $x = 3$ in the given equation ? We get $3 = \sqrt{15 - 6}$ i.e., $3 = 3$, which is true. Thus, $x = 3$ is the solution of the given equation.

Now you may like to solve the following exercises. Remember that **you must check** if the solutions you have obtained satisfy the given equations. This will help you

i) to get rid of extraneous solutions, if any, and

ii) to ensure that your calculations are alright.

E 12) Reduce the following to quadratic and hence, solve them.

a) $4p^4 - 16p^2 + 5 = 0$

b) $(5x^2 - 6)^{1/4} = x$

c) $\sqrt{2x + 3} - \sqrt{x + 1} = 1$

E 13) Ameena walks 1 km per hour faster than Alka. Both walked from their village to the nearest library, a distance of 24 km. Alka took 2 hours more than ameena. What was Alka's average speed ?

In this section our aim was to help you recall the methods of solving linear and quadratic equations. Let us now see how to solve equations of degree 3.

3.3 CUBIC EQUATIONS

In this section we are going to discuss some mathematics to which the great 11th century Persian poet Omar Khayyam gave a great deal of thought. He, and Greek mathematicians before him, obtained solutions for third degree equations by considering geometric methods that involved the intersections of conics. But we will only discuss algebraic methods of obtaining solutions of such equations, that is, solutions obtained by using the basic algebraic operations and by radicals. Let us first see what an equation of degree 3, or a cubic equation, is.

Definition : An equation of the form.

$ax^3 + bx^2 + cx + d = 0$ with $a, b, c, d \in \mathbf{R}, a \neq 0$.

is the most general form of a **cubic equation** (or a third degree equation) over **R**.

For example $2x^3 = 0$, $\sqrt{3}\,x^3 + 5x^2 = 0$, $-2x = 5x^3 - 1$ and $x^3 + 5x^2 + 2x = -7$ are all cubic equations, since each of them can be written in the form $ax^3 + bx^2 + cx + d = 0$, with $a \neq 0$. On the other hand $x^4 + 1 = 0$, $x^3 + 2x^3 = x^3 - x$ and $x^3 + \sqrt{x} = 0$ are not cubic equations.

There are several situations in which one needs to solve cubic equations. For example, many problems in the social, physical and biological sciences reduce to obtaining the eigenvalues of a 3×3 matrix (which you can study about in the Linear Algebra course). And for this you need to know how to obtain the solutions of a cubic equation.

For obtaining solutions of a cubic equation, or any polynomial equation, we need some results about the roots of polynomial equations. We will briefly discuss them one by one.
We give the first one without proof.

Therorem 1 : The polynomial equation of degree n,

$a_0 + a_1 x + \ldots\ldots.. + a_n x^n = 0$, where $a_0, a_1, \ldots\ldots.. a_n \in \mathbf{R}$ and $a_n \neq 0$, has n roots, which are real or non - real complex numbers.

If $x_1, \ldots\ldots x_n$ are the n roots of the equation in Theorem 1, then

$a_0 + a_1 x + \ldots\ldots.. + a_n x^n = a_n (x - x_1)(x - x_2) \ldots\ldots (x - x_n)$.

(Note that the roots need not be distinct. For example, $1 + 2x + x^2 = (x + 1)^2$)

We will not prove this fact here; but we will now state a very important result which is used in the proof.

Theorem 2 (Division algorithm) : Given polynomials f (x) and g (x) and g (x) ($\neq 0$) over **R**, $\exists$ Polynomials q (x) and r (x) over **R** such that

$f(x) = g(x)\, q(x) + r(x)$ and $\deg r(x) < \deg g(x)$.

We will also use this theorem to prove the following result which tells us something about complex roots, that is, roots that are non- real complex numbers.

Theorem 3 : If a polynomial equation over **R** has complex roots, they occur in pairs. In fact, if $a + ib \in \mathbf{C}$ is a root, then $a - ib$ C is also a root.

Proof : Let $f(x) = a_0 + a_1 x + \ldots\ldots. + a_n x^n$ be a polynomial over **R** of degree n. Suppose $a + ib \in \mathbf{C}$ is a root of $f(x) = 0$, that is, $(x - (a + ib)) \mid f(x)$. We want to show that $(x - (a - ib)) \mid f(x)$ too.

Now, $(x - (a + ib))(x - (a - ib)) = (x - a)^2 + b^2$.

Also, by the division algorithin, $\exists$ polynomials g (x) and r (x) over **R** such that

$f(x) = \{(x - a)^2 + b^2\}\, g(x) + r(x)$, where $\deg (r(x)) < 2$.

$f(x) \mid g(x)$
$\Rightarrow \deg f(x) \leq \deg g(x)$
(see Remark 1).

Since $x - (a + ib)$ divide f (x) as well as $(x - a)^2 + b^2$, it divided $f(x) - \{(x - a)^2 + b^2\}\, g(x)$, that is, r (x).

But as deg (r(x)) < 2, therefore, r(x) is linear over R or a constant in R. So $(x - (a + ib))$ can only divide r (x) if $r(x) = 0$ ($\because$ polynomial R has real coefficients)

We find that

$f(x) = \{(x-a)^2 + b^2\}\, g(x).$

Since $x - (a - ib)$ divides the right hand side of this equation, it must divide $f(x)$.

Thus, $a - ib$ is a root of $f(x) = 0$ also.

Note that Theorem 3 **does not say** that $f(x) = 0$ must have a complex root. It only says that if it has a complex root, then the conjugate of the root is also a root.

Why don't you try following exercises now ? In these we are just recalling some facts that you are already aware of.

E 14) How many complex roots can a linear equation over **R** have ?

E 15) Under what circumstances does the quadratic equation over **R**, $x^2 + px + q = 0$, have complex roots ? If it has complex roots, how many are they and how are they related ?

Now let us look at Theorems 1 and 3 in the context of cubic equations. Consider the general cubic equation over **R**.

$ax^3 + bx^2 + cx + d = 0,\ a \neq 0.$

Any solution of this is also a solution of

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0,$$

and vice versa.

Thus, we can always assume that the most general equation of degree 3 over **R** is

$x^3 + px^2 + qx + r = 0$ with $p, q, r, \in$ **R**.

Theorem 1 says that this equation has 3 roots. Theorem 3 says that either all 3 roots are real or one is real and two are complex. Let us find these roots algebraically.

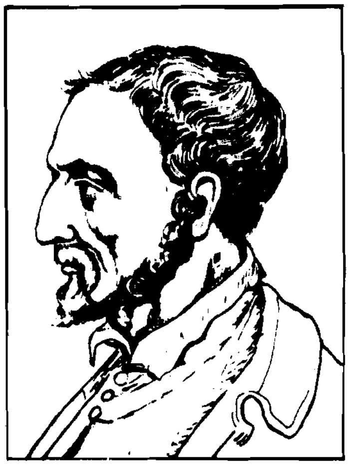

Fig. 1: Cardano

3.3.1 Cardano's Solution

The algebraic method of solving cubic equations is supposed to be due to the Italian, del Ferro (1465 - 1526). But it is called Cardano's method because it became known to people after the Italian, Girolamo Cardano, published it in 1545 in 'Ars Magna'.

Let us see what the method is. We will first look at a particular case.

Example 5 : Solve $2x^3 + 3x^2 + 4x + 1 = 0$

Solution : We first remove the second degree term by completing the cube in the following way.

$2x^3 + 3x^2 + 4x + 1 = 0$

$$\Rightarrow x^3 + \frac{3}{2}x^2 + 2x + \frac{1}{2} = 0$$

$$\Leftrightarrow \left[\left(x + \frac{1}{2}\right)^3 - \frac{3}{4}x - \frac{1}{8}\right] + 2x + \frac{1}{2} = 0$$

$$\Leftrightarrow \left(x + \frac{1}{2}\right)^3 + \frac{5}{4}x + \frac{3}{8} = 0$$

Put $x + \frac{1}{2} = y$. Then the equation becomes

$$y^3 + \frac{5}{4}y - \frac{1}{4} = 0.$$

Assume that the solution is $y = m + n$, where $m, n \in$ **C**. Then.

$$(m+n)^3 + \frac{5}{4}(m+n) - \frac{1}{4} = 0$$

$$\Leftrightarrow m^3 + 3mn(m+n) + n^3 + \frac{5}{4}(m+n) - \frac{1}{4} = 0$$

$$\Leftrightarrow m^3 + n^3 + \left(3mn + \frac{5}{4}\right)(m+n) - \frac{1}{4} = 0. \qquad \text{..........(1)}$$

Let us add a further condition on m and n namely,

$$3mn + \frac{5}{4} = 0; \text{ that is, } mn = -\frac{5}{12} \qquad \text{.........(2)}$$

Then (1) gives us $m^3 + n^3 = \frac{1}{4}$,

and (2) gives us $m^3 n^3 = -\frac{125}{1728}$,

Thus, using E11, we see that m^3 and n^3 are roots of

$$t^2 + \frac{1}{4}t - \frac{125}{1728} = 0.$$

Hence, by the quadratic formula we find that

$$m^3 = \frac{1}{8}\left(1 + \sqrt{\frac{152}{27}}\right) = \alpha, \text{say},$$

$$\text{and } n^3 = \frac{1}{8}\left(1 - \sqrt{\frac{152}{27}}\right) = \beta, \text{say}.$$

From Unit 2 (E 36) you know that α and β have real cube roots, say u and v, respectively, Thus, m can take the values u, ωu, $\omega^2 u$ and n can take the values v, ωv, $\omega^2 v$.

Now, ω and ω^2 are non - real complex numbers such that $\omega(\omega^2) = 1$.

Also, from (2) we know that $mn = -\frac{5}{12}$, a real number.

Thus, if m = u , n must be v ; if m = ωu, n must be ω^2 v ; if m = ω^2u, n must be v. Hence, the possible values of y are

$u + v$, $\omega u + \omega^2 v$ and $\omega^2 u + \omega v$.

To get the three roots of the original equation, we simply put these values of y in the relation

$$x = y - \frac{1}{2}.$$

This example has probably given you some idea about Cardano's method for solving a general cubic equation. Let us outline this method for solving the general equation

$$x^3 + px^2 + qx + r = 0, \ p, q, r \in \mathbf{R}. \qquad \text{.........(3)}$$

Step 1 : We first write $x^3 + px^2 = \left(x + \frac{p}{3}\right)^3 - \frac{p^2}{3}x - \frac{p^3}{27}$.

Then (3) becomes

$$\left(x + \frac{p}{3}\right)^3 + qx + r - \left(\frac{p^2}{3}x + \frac{p^3}{27}\right) = 0$$

$$\Leftrightarrow \left(x + \frac{p}{3}\right)^3 + \left(q - \frac{p^2}{3}\right)x + \left(r - \frac{p^3}{27}\right) = 0.$$

Now put $y = x + \frac{p}{3}$. Then $x = y - \frac{p}{3}$; and the equation becomes

$$y^3 + \left(q - \frac{p^2}{3}\right)\left(y - \frac{p}{3}\right) + \left(r - \frac{p^3}{27}\right) = 0, \text{ that is,}$$

$$y^3 + Ay + B = 0. \qquad \ldots\ldots\ldots\ldots (4)$$

where $A = q - \frac{p^2}{3}$ and $B = \frac{2p^3}{27} - \frac{pq}{3} + r.$

Step 2 : Now let us solve (4).

Let $y = \alpha + \beta$ be a solution. Putting this value of y in (4) we get

$(\alpha + \beta)^3 + A(\alpha + \beta) + B = 0$

$\Leftrightarrow \alpha^3 + 3\alpha\beta(\alpha + \beta) + \beta^3 + A(\alpha + \beta) + B = 0$

$$\Leftrightarrow \alpha^3 + \beta^3 + (3\alpha\beta + A)(\alpha + \beta) + B = 0 \qquad \ldots\ldots\ldots\ldots (5)$$

Now, we choose α and β so that, $3\alpha\beta + A = 0$. Then we have the two equations

$$(\alpha\beta)^3 = \left(-\frac{A}{3}\right)^3, \text{ that is, } \alpha^3\beta^3 = -\frac{A^3}{27}, \qquad \ldots\ldots\ldots\ldots (6)$$

and from (5)

$$\alpha^3 + \beta^3 = -B. \qquad \ldots\ldots\ldots\ldots (7)$$

Thus , using E 11, we find that α^3 and β^3 are roots of the quadratic equation

$$t^2 + Bt - \frac{A^3}{27} = 0. \qquad \ldots\ldots\ldots\ldots (8)$$

Hence, using the quadratic formula, we find that

$$\left.\begin{aligned} a^3 &= -\frac{B}{2} + \sqrt{\frac{B^2}{4} + \frac{A^3}{27}} = u, \text{ say, and} \\ b^3 &= -\frac{B}{2} \sqrt{\frac{B^2}{4} + \frac{A^3}{27}} = v, \text{ say,} \end{aligned}\right] \qquad \ldots\ldots\ldots\ldots (9)$$

Now, from Unit 2 (E36) we know that any complex number has three cube roots. We also know that if γ is a cube root, then the three roots are γ, $\omega\gamma$ and $\omega^2\gamma$.

Therefore, if a and b denote a cube root each of u and v, respectively, then α can be a, a ω or aω^2, and β can be b, bω or bω^2. Does this mean that $y = \alpha + \beta$ can take on 9 values?

Note that α and β also satisfy the relations $\alpha\beta = -\frac{A}{3} \in \mathbf{R}.$

Thus, since $\omega \in C$, $\omega^2 \in C$, $\omega^3 = 1 \in \mathbf{R}$, the only possibilities for y are

$a + b$, $a\omega + b\omega^2$, $a\omega^2 + b\omega$.

Step 3 : The 3 solutions of (3) are given by substituting each of these values of y in the equation $x = y - \frac{p}{3}$.

So, what we have just shown is that

> the roots of $x^3 + px^2 + qx + r = 0$ are $\alpha + \beta - \frac{p}{3}$, $\alpha\omega' + \beta\omega^2 - \frac{p}{3}$, $\alpha\omega^2 + \beta\omega - \frac{p}{3}$,
>
> where $\omega = \frac{-1 + i\sqrt{3}}{2}$, α is a cube root of $\left\{-\frac{B}{2} + \sqrt{\frac{B^2}{4} + \frac{A^3}{27}}\right\}$, β is a cube root of $\left\{-\frac{B}{2} - \sqrt{\frac{B^2}{4} + \frac{A^3}{27}}\right\}$, $A = q - \frac{p^2}{3}$, $B = \frac{2p^3}{27} - \frac{pq}{3} + r.$

The formula we have obtained is rather a complicated business. A calculator, would certainly ease matters, as you may find while trying the following exercise.

E16) Solve the following cubic equations :

a) $2x^3 + 3x^2 + 3x + 1 = 0$

b) $x^3 + 21x + 342 = 0$

c) $x^3 + 6x^2 + 6x + 8 = 0$

d) $x^3 + 29x - 97 = 0$

e) $x^3 = 30x - 133$

In each of the equations in E16, you must have found that $\frac{B^2}{4} + \frac{A^3}{27} \geq 0$.

But what happens if $\frac{B^2}{4} + \frac{A^3}{27} < 0$

This case is known as the irreducible case. In this case (9) tells us that α^3 and β^3 are complex numbers of the form a + ib and a - ib where b ≠ 0. From Unit 2 you know that if the polar from of a + ib is r (cosθ + i sin θ), then its cube roots are

$$r^{1/3}\left(\cos\frac{\theta + 2k\pi}{3} + i\sin\frac{\theta + 2k\pi}{3}\right), k = 0, 1, 2.$$

Similarly, the cube roots of a – i b are

$$r^{1/3}\left(\cos\frac{\theta + 2k\pi}{3} - i\sin\frac{\theta + 2k\pi}{3}\right), k = 0, 1, 2.$$

Hence , the 3 values of y in (4) are

$$2r^{1/3}\cos\frac{\theta + 2k\pi}{3}, \text{ where } k = 0, 1, 2.$$

All these are real numbers. Thus in this case all the roots of (3) are real, and are given by

$$2r^{1/3}\cos\frac{\theta}{3} - \frac{p}{3}, 2r^{1/3}\cos\frac{\theta + 2\pi}{3} - \frac{p}{3}, 2r^{1/3}\cos\frac{\theta + 4\pi}{3} - \frac{p}{3}.$$

This trigonometric from of the solutipn is due to Francois Viete (1550 - 1603).

Now try an exercise.

E17) Solve the equation $x^3 - 3x + 1 = 0$

So far, we have seen that a cubic equation has three roots. We also know that either all the roots are real, or one is real and two are complex conjugates. Can we tell the roots or the character of the roots by just inspecting the coefficients ? We shall answer this question now.

3.3.2 Roots And Their Relation With Coefficients.

In this sub - section we shall first look at the cubic analogue of E10 and E11. Over there we saw how closely the roots of a quadratic equation are linked with its cofficients. The same thing is true for a cubic equation. Why don't you try and prove the relationship that we give in the following exercise ?

E 18) Show that α, β and γ are the roots of the cubic equation

$ax^3 + bx^2 + cx + d = 0, a \neq 0$, if and only if

$$\alpha + \beta + \gamma = -\frac{b}{a},$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a},$$

$$\alpha\beta\gamma = -\frac{d}{a}.$$

(**Hint :** Note that the given cubic equation is equivalent to
$a(x-\alpha)(x-\beta)(x-\gamma) = 0.$)

The relationship in E18 allows us to solve problems like the following.

Example 6 : If α, β, γ are the roots of the equation

$x^3-7x^2+x-5=0$

find the equation whose roots are $\alpha+\beta, \beta+\gamma, \alpha+\gamma$.

Solution : By E 18 we know that

$$\left.\begin{aligned} &\alpha+\beta+\gamma=7 \\ &\alpha\beta+\beta\gamma+\alpha\gamma=1 \\ &\alpha\beta\gamma=5 \end{aligned}\right] \qquad \text{................ (10)}$$

Therefore, $(\alpha+\beta)+(\beta+\gamma)+(\alpha+\gamma)=2(\alpha+\beta+\gamma)=14.$ .... (11)

Also, $\alpha+\beta=7-\gamma$, $\beta+\gamma=7-\alpha$, $\gamma+\alpha=7-\beta$, so that

$(\alpha+\beta)(\beta+\gamma)+(\beta+\gamma)(\alpha+\gamma)+(\alpha+\gamma)(\alpha+\beta)$

$=\{49-7(\gamma+\alpha)+\gamma\alpha\}+\{49-7(\alpha+\beta)+\alpha\beta\}+\{49-7(\beta+\gamma)+\beta\gamma\}$

$=147-98+1$, using (10) and (11).

$=50$, and ....(12)

$(\alpha+\beta)(\beta+\gamma)(\alpha+\gamma)=(7-\gamma)(7-\beta)(7-\alpha)$

To evaluate the expression on the right hand side, we can use (10) or we can use the fact that

$x^3-7x^2+x-5=(x-\alpha)(x-\beta)(x-\gamma)$

$7^3-7.7^2+7-5=(7-\alpha)(7-\beta)(7-\gamma)$

Therefore, $(\alpha+\beta)(\beta+\gamma)(\alpha+\gamma)=2.$ ....(13)

Now, E 18, (11), (12) and (13) give us the required equation, which is,

$x^3-14x^2+50x-2=0.$

Why don't you try the following exercise now ?

E 19) Find the sum of the cubes of the roots of the equation $x^3-6x^2+11x-6=0$
Hence find the sum of the fourth powers of the roots.

Let us now study the character of the roots of a cubic equation. For this purpose we need to introduce the notion of the discriminant. In the case of a quadratic equation $x^2+bx+c=0$, you know that the discriminant is b^2-4c. Also, if α and β are the two roots of the equation, then $\alpha+\beta=-b$, $\alpha\beta=c$. Therefore.

$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=b^2-4c.$

Thus, the discriminant $=(\alpha-\beta)^2$, where α and β are the roots of the quadratic equation.

Now consider the general quadratic equation, $ax^2+bx+c=0$.

Let its roots be α and β. Then its discriminant is $b^2-4ac=a^2(\alpha-\beta)^2$.

We use this relationship to define the discriminant of any polynomial equation.

Definition : The discriminant of the nth drgree equation

$a_0+a_1x+a_2x^2+\ldots\ldots a_n+x^n$ is

$$a_n^{2(n-1)} \prod_{1\le i<j\le n} (\alpha_i-\alpha_j)^2,$$

where $\alpha_1 \ldots\ldots, \alpha_n$ are the roots of the polynomial equation.

In particular, if we consider the case n = 3 and $a_n = 1$, we find that

> The discriminant of the cubic $x^3 + Px^2 + qx + r = 0$ is
>
> $D = -(27B^2 + 4A^3)$, where $A = q - \frac{p^2}{3}$, $B = \frac{2p^3}{27} - \frac{pq}{3} + r$,

Now consider Cardano's solution of the cubic equation (3), namely,
$x^3 + px^2 + qx + r = 0$.

The expression under the root sign is $\frac{B^2}{4} + \frac{A^3}{27} = \frac{-D}{108}$, where D is the discriminant.

Now, (9) tells us that the sign of the discriminant is closely related to the characters of the roots of the equation. Let us look at the different possibilities for the roots α, β and γ of (3).

1) The roots of (3) are all real and distinct. Then $(\alpha - \beta)^2 (\beta - \gamma)^2 (\beta - \gamma)^2$, that is D, must be positive.

2) Only one root of (3) are real. Let this root be α. Then β and γ are complex conjugates. $\therefore \beta - \gamma$ is purely imaginary $\therefore (\beta - \gamma)^2 < 0$.

   Also, $\alpha - \beta$ and $\alpha - \gamma$ are conjugates.

   Therefore, their product is positive.

   Hance, in this case $D < 0$.

3) Suppose $\alpha = \beta$ and $\gamma \neq \alpha$. Since $\alpha - \beta = 0$, $D = 0$.

   Also, $B \neq 0$. Why ? Because if $B = 0$, then $A = 0$ (since $D = 0$).

   But $A = 0 \Rightarrow q = \frac{p^2}{3}$, that is $\alpha(\alpha + 2\gamma) = \frac{(2\alpha + \gamma)^2}{3}$,

   [Over here we have used the relationship between the roots,

   since $p = -(\alpha + \beta + \gamma) = -(2\alpha + \gamma)$ and $q = \alpha\beta + \beta\gamma + \alpha\gamma = \alpha(\alpha + 2\gamma)$].

   On simplifying we get $\alpha = \gamma$, a contradiction.

   Thus, $B \neq 0$.

   So, if exactly two roots of (3) are equal, then $D = 0$ and $B \neq 0$, and hence, $A \neq 0$.

4) If all the roots of (3) are equal, then $D = 0$, $B = 0$, and hence $A = 0$.

   Let us summarise the different possibilities for the character of the roots now.

> Consider the cubic equation $x^3 + px^2 + qx + r = 0$, p, q, r $\in$ **R**,
>
> and let $B = \frac{2p^3}{27} - \frac{pq}{3} + r$ and $A = q - \frac{p^2}{3}$. Then
>
> 1) all its roots are real and distinct iff $\frac{B^2}{4} + \frac{A^3}{27} < 0$,
>
> 2) exactly one root is real iff $\frac{B^2}{4} + \frac{A^3}{27} > 0$,
>
> 3) exactly two roots are equal iff $\frac{B^2}{4} + \frac{A^3}{27} = 0$ and $B \neq 0$.
>
>   In this case all the roots are real.
>
> 4) all three roots are equal iff $\frac{B^2}{4} + \frac{A^3}{27} = 0$ and $B = 0$.

You may now like to try the following problems to see if you've understood what we have just discussed.

E20) Under what conditions on the coefficients of
$ax^3 + 3bx^2 + 3cx + d = 0, a \neq 0,$
will the equation have complex roots ?

E 21) Will all the roots of $x^3 = 15x + 126$ be real ? Why?

So far we have introduced you to a method of solving cubic equations and we have studied the solutions in some depth. We shall study them some more in Unit 6, as an application of the cauchy - schwarz inequality. Now let us go on to a discussion of polynomial equations of degree 4.

3.4 BIQUADRATIC EQUATIONS

As in the case of cubic equations, biquadratic equations have been studied for a long time. The ancient Arabs were known to have studied them from a geometrical point of view. In this section we will discuss two algebraic methods of solving such equations. Let us first see what a biquadratic equatiop is.

Definition: An equation of the form
$ax^4 + bx^3 + cx^2 + dx + e = 0$, where $a, b, c, d, e \in \mathbf{R}$ and $a \neq 0$.
is the most general form of a **biquadratic equation** (or a **quartic equation,** or **a fourth degree equation)** over **R.**

Can you think of examples of quartic equations over **R** ? What about $x^4 + 5 = \sqrt{2}\, x - x^2$?

This certainly is a quartic equation, as it is equivalent to $x^4 + x^2 - \sqrt{2}\, x + 5 = 0$.

What about $\sqrt{x} = x^4 + 1$? This isn't even a polynomial equation. So it can't be a quartic.

Let us now consider various ways in which we can solve an equation of degree 4. In some cases, as you have seen in Example 4, such an equation can be solved by solving related quadratic equations. But most biquadratic equations can't be solved in this manner. Two algebraic methods for obtaining the roots of such equations were developed in the 16th and 17th centuries. Both these methods depend on the solving of a cubic equation. Let us see what they are.

3.4.1 Ferrari's Solution

The first method for solving a biquadratic equation that we will discuss is due to the 16th century Italian mathematician Ferrari, who worked with Cardano. Let us see what the method is with the help of an example.

Example 7 : Solve the equation

$x^4 - 2x^3 - 5x^2 + 10x - 3 = 0.$

Solution : We will solve this in several steps.

Step 1 : Add the quadratic polynomial $(ax + b)^2 = a^2x^2 + 2abx + b^2$ to both sides. We get

$$x^4 - 2x^3 + (a^2 - 5)x^2 + 2(ab + 5)x + b^2 - 3 = (ax + b)^2. \quad \text{..........(14)}$$

Step 2 : Choose a and b in **R** so that the left hand side of (14) becomes a perfect square, say $(x^2 - x + k)^2$, where k is an unknown. Thus, we need to choose a and b so that

$$x^4 - 2x^3 + (a^2 - 5)x^2 + 2(ab + 5)x + b^2 - 3 = x^4 + x^2 + k^2 - 2x^3 - 2kx + 2kx^2 = (x^2 - x + k)^2$$

Equating the coefficients of x^2, x and the constant term on both sides, we get

$$a^2 - 5 = 2k + 1 \quad \text{.............(15)}$$

$$2(ab + 5) = -2k \quad \text{.............(16)}$$

$$b^2 - 3 = k^2 \quad \text{.............(17)}$$

$(15) \Rightarrow a^2 = 2k+6$

Also, $(16) \Rightarrow a = -\frac{1}{b}(k+5)$

Thus, $2k+6 = \frac{1}{b^2}(k+5)^2$

Then $(17) \Rightarrow k^2 + 3 = \frac{(k+5)^2}{2k+6}$ .... (18)

$\Rightarrow 2k^3 + 5k^2 - 4k - 7 = 0$

This cubic equation is called the **resolvent cubic** of the given biquadratic equation. We have obtained it by **eliminating a and b** from the equations (15), (16) and (17).

We choose any one root of the cubic. One real solution of (18) is $k = -1$. (It is easy to see this by inspection. Otherwise you can apply Cardano's method.)

Then, from (15), (16) and (17) we get

$a^2 = 4, b^2 = 4, ab = -4.$

$a = 2$ and $b = -2$ (or $a = -2$ and $b = 2$) satisfy these equations. We need only one set of values of a and b. Either will do. Let us take $a = 2$ and $b = -2$.

Step 3: Put these values of k, a and b in $(x^2 - x + k)^2 = (ax + b)^2$. On taking square roots, we get two quadratic equation, namely,

$x^2 - x - 1 = \pm(2x - 2)$, that is,

$x^2 - 3x + 1 = 0$ and $x^2 + x - 3 = 0$.

Applying the quadratic formula to these equations we get

$$x + \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \frac{-1+\sqrt{13}}{2}, \frac{-1-\sqrt{13}}{2}.$$

Does Example 7 give you some idea of the general method developed by Ferrari ? Let us see what it is.

We want to solve the general 4th degree equation over **R**, namely,

$x^4 + px^3 + qx^2 + rx + s = 0, p, q, r, s, \in \mathbf{R}.$ .... (19)

The idea is to express this equation as a difference of squares of two polynomials. Then this difference can be split into a product of two quadratic factors, and we can solve the two quadratic equations that we obtain this way. Let us write down the steps involved.

Sept 1: Add $(ax + b)^2$ to each side of (19), where a and b will be chosen so as to make the left hand side a perfect square. So (19) becomes

$x^4 + px^3 + (q + a^2)x^2 + (r + 2ab)x + s + b^2 = (ax + b)^2$ .... (20)

Step 2 : We want to choose a and b so that the left hand is a perfect square, say

$(x^2 + \frac{p}{2}x + k)^2$, where k is an unknown.

Note that the coefficient of x is necessarily $\frac{p}{2}$, since the coefficient of x^3 in (20) is p. So we see that

$$x^4 + px^3 + (q + a^2)x^2 + (r + 2ab)x + s + b^2 = x^4 + px^3 + \left(\frac{p^2}{4}\right)x^2 + 2kx^2 + pkx + k^2.$$

Comparing coefficients of x^2, x and the constant term, we have

$\frac{p^2}{4} + 2k = q + a^2, pk = r + 2ab, k^2 = s + b^2.$

Eliminating a and b from these equations, we get the **resolvent cubic**

$$(pk - r)^2 = 4\left(\frac{p^2}{4} + 2k - q\right)(k^2 - s), \text{ that is,}$$

$8k^3 - 4qk^2 + 2(pr - 4s)k + (4qs - p^2 s - r^2) = 0.$

From Sec. 3.3 you know that this cubic equation has at least one real root, say α.

Then, we can find a and b in terms of α.

Step 3 : Our assumption was that

$$(x^2 + \frac{p}{2}x + k)^2 = (ax+b)^2.$$

Now, putting $k = \alpha$ and substituting the values of a and b, we get the quadratic equations

$$x^2 + \frac{p}{2}x + \alpha = \pm(ax + b), \text{ that is,}$$

$$x^2 + (\frac{p}{2} - a)x + (\alpha - b) = 0, \text{ and}$$

$$x^2 + (\frac{p}{2} + a)x + (\alpha + b) = 0.$$

Then, using the quadratic formula we can obtain the 4 roots of these equations, which will be the roots of (20), and hence of (19).

The following exercise gives you a chance to try out this method for yourself.

E 22) Solve the following equations :

a) $x^4 - 3x^2 - 42x - 40 = 0$

b) $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$

c) $x^4 + 12x = 5$

Let us now consider the other classical method for solving quartic equations.

3.4.2 Descartes' Solution

The second method for obtaining an algebraic solution for a quartic was given by the mathematician and philosopher Rene Descartes in 1637. In this method we write the biquadratic polynomial as a product of two quadratic polynomials. Then we solve the resultant quadratic equations to get the 4 roots of the original quartic.

Let us consider an example. In fact, let us solve the problem in example 7 by this method.

Thus, we want to solve

$$x^4 - 2x^3 - 5x^2 + 10x - 3 = 0 \qquad \text{............(21)}$$

Step 1 : Remove the cube term. For this we rewrite $x^4 - 2x^3$ as

$$\left(x - \frac{1}{2}\right)^4 - \frac{3}{2}x^2 + \frac{1}{2}x - \frac{1}{16}.$$

Thus, the given equation becomes

$$\left(x - \frac{1}{2}\right)^4 - \frac{13}{2}x^2 + \frac{21}{2}x - \frac{49}{16} = 0$$

Now, put $x - \frac{1}{2} = y$. We get

$$y^4 - \frac{13}{2}(y + \frac{1}{2})^2 + \frac{21}{2}(y + \frac{1}{2}) - \frac{49}{16} = 0$$

$$\Rightarrow y^4 - \frac{13}{4}y^2 + 4y + \frac{9}{16} = 0. \qquad \text{............(22)}$$

Step 2 : Write the left hand side of (22) as product of quadratic polynomials. For this, let us assume that

$$y^4 - \frac{13}{2}y^2 + 4y + \frac{9}{16} = (y^2 + ky + m)(y^2 - ky + n).$$

(Note that the coefficients of y in each of these factors are k and $-k$, respectively, since the product does not contain any term with y^3.)

Equating coefficients, we get

$$m+n-k^2=-\frac{13}{2},\ k(n-m)=4,\ mn=\frac{9}{16} \qquad \text{...........(23)}$$

Eliminating m and n from these equations we get

$$\left(k^2-\frac{13}{2}-\frac{4}{k}\right)\left(k^2-\frac{13}{2}+\frac{4}{k}\right)=\frac{9}{4}, \text{ that is,}$$

$k^6-13k^4+40k^2-16=0.$

If we put $k^2=t$, then this becomes the resolvent cubic

$t^3-13t^2+40t-16=0.$

This has one real root; in fact, it has a **positive** real root, because of the following result, that we give without proof.

> Every polynomial equation, whose leading coefficient is 1 and degree is an odd number, has at least one real root whose sign is opposite to that of its last term.

So, using this result, we see that we can expect to get one positive value of t. By trial, we see that $t=4$ is a root, that is, $k^2=4$, that is $k=\pm 2$. Any one of these values is sufficient for us, so let us take $k=2$.

Then, from the equations in (23) we get

$$m=-\frac{9}{4},\ n=-\frac{1}{4}.$$

Thus, (22) is equivalent to

$$\left(y^2+2y-\frac{9}{4}\right)\left(y^2-2y-\frac{1}{4}\right)=0$$

Step 3 : Solve the quadratic equations

$$y^2+2y-\frac{9}{4}=0 \text{ and } y^2-2y-\frac{1}{4}=0.$$

By the quadratic formula we get

$$y=\frac{-2\pm\sqrt{13}}{2} \text{ and } y=\frac{2\pm\sqrt{5}}{2}.$$

Step 4 : Put these values in $x=y+\frac{1}{2}$ to get the four roots of (21).

Thus, the roots of (21) are :

$$\frac{-1+\sqrt{13}}{2},\ \frac{-1-\sqrt{13}}{2},\ \frac{3+\sqrt{5}}{2},\ \frac{3-\sqrt{5}}{2}.$$

Let us write down the steps in this method of solution for the general quartic equation

$$x^4+ax^3+bx^2+cx+d=0,\ a,b,c,d \in \mathbb{R}. \qquad \text{............(24)}$$

Step 1: Reduce the equation to the form

$$x^4+qx^2+rx+s=0. \qquad \text{............(25)}$$

Step 2: Assume that

$x^4+qx^2+rx+s=(x^2+kx+m)(x^2-kx+n).$

Then, on equating coefficients, we get

$m+n-k^2=q,\ k(n-m)=r,\ mn=s.$

From these equations we get

$$m+n=k^2+q,\ n-m=\frac{r}{k}.$$

Therefore, $2m=k^2+q-\frac{r}{k},\ 2n=k^2+q+\frac{r}{k}.$

Substituting in $mn = s$, we get

$(k^3 + qk - r)(k^3 + qk + r) = 4sk^2$, that is,

$k^6 + 2qk^4 + (q^2 - 4s)k^2 - r^2 = 0$, that is,

$t^3 + 2qt^2 + (q^2 - 4s)t - r^2 = 0$, putting $k^2 = t$.

This is a cubic with at least one positive real root. Then, with a known value of t, we can determine the values of k, m and n. So, (25) is equivalent to

$(x^2 + kx + m)(x^2 - kx + n) = 0$

Step 3 : Solve the quadratic equations

$x^2 + kx + m = 0$ and $x^2 - kx + n = 0$.

This will give us the 4 roots of (25), and hence, the 4 roots of (24).

Now, why don't you try the following exercises to see if you have grasped Descartes' method ?

E 23) Solve the following equations by Descartes' method:

a) $x^4 - 2x^2 + 8x - 3 = 0$

b) $x^4 + 8x^3 + 9x^2 - 8x = 10$

c) $x^4 - 3x^2 - 6x - 2 = 0$

d) $x^4 + 4x^3 - 7x^2 - 22x + 24 = 0$.

E 24) Reduce the equation $2x^8 + 5x^6 - 5x^2 = 2$ to a biquadratic. Hence solve it.

While solving quartic equations you may have realised that the methods that we have discussed appear to be very easy to use; but, in practice, they can become quite cumbersome. This is because Cardano's method for solving a cubic often requires the use of a calculator.

Well, so far we have discussed methods of obtaining algebraic solutions for polynomial equations of degrees 1, 2, 3 and 4. You may think that we are going to do something similar for quintic equations, that is, equations of degree 5. But, in 1824 the Norwegian algebraist Abel (1802 – 1829) published a proof of the following result :

> There can be no general formula, expressed in explicit algebraic operations on the coefficients of polynomial equation, for the roots of the equation, if the degree of the equation is greater than 4.

This result says that polynomial equations of degree > 4 do not have a general algebraic solution. But there are methods that can give us the value of any real root to any required degree of accuracy. We will discuss these methods in our course on Numerical Analysis. There are, of course, special polynomial equations of degree ≥ 5 that can be solved (as in E 24.)

Let us now look a little closely at the roots of a biquadratic equation. We shall see how they are related to the coefficients of the equation, just as we did in the case of the cubic.

3.4.3 Roots and Their Relation with Coefficients

In the two previous sub-sections we have shown you how to explicitly obtain the 4 roots of a biquadratic equation. Let us go back to Theorems 1 and 3 for a moment. Theorem 1 tells us that a quartic has 4 roots, which may be real or complex. By Theorem 3, the possibilities are

i) all the roots are real, or

ii) two are real and two are complex conjugates of each other, or

iii) the roots are two pairs of complex conjugates, that is, $a + ib$, $a - ib$, $c + id$, $c - id$ for some $a, b, c, d, \in \mathbf{R}$.

Now if r_1, r_2, r_3, r_4 are the roots of the quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$ax^4+bx^3+cx^2+dx+e=a(x-r_1)(x-r_2)(x-r_3)(x-r_4)$$
$$=x^4-(r_1+r_2+r_3+r_4)x^3+(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)x^2$$
$$-(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)x+r_1r_2r_3r_4.$$

Comparing the coefficients, we see that

$$r_1+r_2+r_3+r_4=-\frac{b}{a}$$
$$r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4=\frac{c}{a}$$
$$r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4=-\frac{d}{a}$$
$$r_1r_2r_3r_4=\frac{e}{a}$$

This means that

$$\text{sum of the roots} = -\frac{\text{coeff. of } x^3}{\text{coeff. of } x^4}$$

coeff. is short for coefficient.

$$\text{sum of the product of roots taken two at a time} = \frac{\text{coeff. of } x^2}{\text{coeff. of } x^4}$$

$$\text{sum of the product of roots taken three at a time} = -\frac{\text{coeff. of } x}{\text{coeff. of } x^4}$$

$$\text{product of the roots} = \frac{\text{coeff. of } x^0}{\text{coeff. of } x^4}, \text{ that is } \frac{\text{constant term}}{\text{coeff. of } x^4}$$

These four equations constitute a particular case of the following result that relates the roots of a polynomial equation with its coefficients.

Theorem 4 : Let $\alpha_1, \ldots\ldots, \alpha_n$ be the n roots of the equation
$a_0 x^n + a_1 x^{n-1} + \ldots.. + a_n = 0$, $a_i \in \mathbf{R} \,\forall i = 0, 1, \ldots.., n$, $a_0 \neq 0$. Then

$\sum_{i=1}^{n} A_1 = A_i + \ldots. A_n$

$$\sum_{i=1}^{n} \alpha_1 = -\frac{a_1}{a_0}$$

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} \alpha_i \alpha_j = \frac{a_2}{a_0}$$

$\vdots$

$$\sum_{i_1<i_2<\ldots<i_t}^{n} \alpha_{i2} \ldots \alpha_{i_t} = (-1)^t \frac{a_t}{a_n}$$

$\vdots$

$\sum_{i=1}^{n} A_1 = A_1, A_2 \ \ldots. A_n$

$$\prod_{i=1}^{n} \alpha_1 = (-1)^n \frac{a_n}{a_0}$$

In E10 and E18 you have already seen that this result is true for n = 2 and 3.

Theorem 4 is very useful in several ways. Let us consider an application in the case n = 4.

Example 8 : If the sum of two roots of the equation

$4x^4-24x^3+31x^2+6x-8=0$

is zero, find all the roots of the equation.

Solution : Let the roots be a, b, c, d, where a + b = 0.

Then $a+b+c+d=\frac{24}{4}=6$ ....(26)

$\therefore c+d=6$

Also $ab+ac+ad+bc+bd+cd=(a+b)(c+d)+ab+cd=\frac{31}{4}$

$\therefore\ ab + cd = \frac{31}{4}$ ....(27)

Further, $(a + b)cd + ab(c + d) = acd + bcd + abc + abd = -\frac{3}{2}$

$\therefore\ (26) \Rightarrow ab = -\frac{1}{4}$ ....(28)

Finally, $abcd = -2$

$\therefore\ (28) \Rightarrow cd = 8$ ....(29)

Now using E11, (26) and (29) tell us that c and d are roots of $x^2 - 6x + 8 = 0$.

thus, by the quadratic formula, $c = 2, d = 4$.

Similarly, a and b are roots of $x^2 - \frac{1}{4} = 0 \therefore a = \frac{1}{2}, b = -\frac{1}{2}$.

Thus, the roots of the given quartic are

$\frac{1}{2}, -\frac{1}{2}, 2, 4.$

Try the following problems now.

E 25) Solve the equation

$x^4 + 15x^3 + 70x^2 + 120x + 64 = 0,$

given that the roots are in G.P., i.e., geometrical progression.

(**Hint**: If four numbers a, b, c, d, are in G. P., then ad = bc.)

E26) Show that if the sum of two roots of $x^4 - px^3 + qx^2 - rx + s = 0$ (where $p, q, r, s \in \mathbf{R}$) equals the sum of the other two, then $p^3 - 4pq + 8r = 0$.

We have touched upon relations between roots and coefficients for n = 2, 3, 4. But you can apply Theorem 4 for any $n \in \mathbf{N}$. So, in future whenever you need to, you can refer to this theorem and use its result for equations of degree ≥ 5.
Let us now wind up this unit with a summary of what we have done in it.

3.5 SUMMARY

In this unit we have introduced you to the theory of lower degree equations. Specifically, we have covered the following points:

1) The linear equation $ax + b = 0$ has one root, namely, $x = \frac{-b}{a}$.

2) The quadratic equation $ax^2 + bx + c = 0$ has 2 roots given by the quadratic fcrmula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

3) Every polynomial equation of degree n over **R** has n roots in **C**.

4) If $a + ib \in \mathbf{C}$ is a root of a real polynomial, then so is $a - ib$.

5) Cardano's method for solving a cubic equation.

6) A cubic equation can have :

   i) three distinct real roots, or

   ii) one real root and two complex roots, which are conjugates, or

   iii) three real roots, of which exactly two are equal, or

   iv) three real roots, all of which are equal.

7) Methods due to Ferrari and Descartes for solving a quartic equation. Both these methods require the solving of one cubic and two quadratic equations.

8) A quartic equation can have four real roots, or two real and two complex roots, or 4 complex roots.

9) If the n roots of the nth degree equation $a_0 x^n + a_1 x^{n-1} + ...+ a_{n-1} \ x + a_n = 0$, are $p_1, p_2 ......p_n$, then

$$\sum_{i=1}^{n} p_i = \frac{a_1}{a_0}$$

$$\sum_{\substack{i<j \\ i,j=1,2,\ldots,n}} p_i \, p_j = \frac{a_2}{a_0}$$

$$\vdots \qquad \vdots$$

$$\prod_{i=1}^{n} p_i = (-1)^n \frac{a_n}{a_0}$$

That is, the sum of the product of the roots taken k at a time is

$$(-1)^k \frac{a_k}{a_0} \ \forall k=1, ...,n.$$

As in our other units, we give our solutions and / or answers to the exercises in the unit in the following section. You can go through them if you like. After that please **go back to Section 3.1** and see if you have achieved the objectives.

3.6 SOLUTIONS/ANSWERS

E 1) a) This has a solution provided $J \neq k$.

$$J\left(\frac{x}{k}+a\right) = x \Leftrightarrow x\left(\frac{J}{k}-1\right) + Ja = 0 \Leftrightarrow x = \frac{-Ja}{\left(\frac{j}{k}-1\right)} = \frac{-Jak}{J-k}$$

b) $\frac{1}{R} = \frac{r_1 + r_2}{r_1 \, r_2} \Leftrightarrow R = \frac{r_1 \, r_2}{r_1 + r_2}$

c) $F = \frac{9}{5}C + 32$

E2) Let the third side be x cm.

Then the other two sides are each 2x cm long.

Therefore, $x + 2x + 2x = 30 \Rightarrow x = 6$.

Thus, the lengths of the sides are 6 cm, 12 cm and 12 cm.

E 3) Let her rate of travel to the study centre be x km per hour. Thus, the distance from her home to the study centre is $\frac{x}{3}$ km. While returning, her rate is (x - 8) km / hr.

$$\therefore \frac{1}{2}(x-8) = \frac{x}{3} \Leftrightarrow x = 24.$$

Thus, the distance is $\frac{24}{3}$ km = 8 km.

E4) $S = k\,p\,t$.

We know that $110 = k \times 1000 \times 2 \Rightarrow k = \frac{11}{200}$.

$$\therefore s = \frac{11}{200} \text{ Pt.}$$

So, the required interest is

$\frac{11}{200} \times 500 \times 3$, that is, Rs. 825.

E5) True . For example, $x^2 + 1 = 0$ has complex roots. Any linear equation $ax + b = 0$ over $\mathbf{R}$ has only one root, namely, $\frac{-b}{a} \in \mathbf{R}$.

E 6) a) $x^2 = -5 \Rightarrow x = i\sqrt{5}$ and $-i\sqrt{5}$.

b) This is $(x-(-9))(x-1) = 0$. Thus, by Remark 2, -9 and 1 are the roots.

c) We rewrite the given equation is standard from as

$$x^2 - \sqrt{5}\,x - 1 = 0$$

$$\Rightarrow x = \frac{\sqrt{5} \pm \sqrt{5+4}}{2} = \frac{\sqrt{5} \pm 3}{2}$$

$$\therefore x = \frac{\sqrt{5}+3}{2} \text{ and } \frac{\sqrt{5}-3}{2}.$$

d) $$m = \frac{8q \pm \sqrt{64q^2 - \frac{4p}{r}}}{2p} = \frac{4q}{p} \pm \frac{1}{p}\sqrt{16q^2 - \frac{p}{r}}$$

E7) The roots are

$$x = \frac{-(2k+6) \pm \sqrt{(2k+6)^2 - 64k}}{2\,k}$$

The roots will coincide if the discriminant is zero, that is, $(2k+6)^2 - 64\,k = 0$.

This will happen when $k^2 - 10\,k + 9 = 0$, that is,

$k = 1$ or $k = 9$.

E8) $(2ax+b) \mid (ax^2+bx+c)$

$\Rightarrow$ the root of $2ax + b = 0$ is a root of $ax^2 + bx + c = 0$.

($\because$ then $ax^2 + bx + c = (2ax+b)(\ldots.)$.)

$\Rightarrow x = \frac{-b}{2a}$ is a root of $ax^2 + bx + c = 0$.

$$\Rightarrow a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c = 0.$$

$\Rightarrow b^2 - 4ac = 0$

$\Rightarrow ax^2 + bx + c = 0$ has coincidental roots.

E 9) By Remark 2, we must have

$x^2 + bx + c = (x-(1+i))(x-(1-i))$

$= x^2 - 2x + 2$

Thus, comparing the coefficients of x^1 and x^0, we get

$b = -2, c = 2$

E10) α and β are roots of $ax^2 + bx + c = 0$

$\Leftrightarrow ax^2 + bx + c = a(x-\alpha)(x-\beta)$

$\Leftrightarrow ax^2 + bx + c = a\{x^2 - (\alpha+\beta)x + \alpha\beta\}$

$\Leftrightarrow b = -a(\alpha+\beta)$ and $c = a\,\alpha\,\beta$

$\Leftrightarrow \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

E11) Substituting $x = \alpha$ in $x^2 - px + q$, we get

$\alpha^2 - p\alpha + q = \alpha^2 - (\alpha+\beta)\alpha + \alpha\beta = 0$, since $\alpha + \beta = p$ and $\alpha\beta = q$.

$= 0$

$\therefore \alpha$ is a root of $x^2 - px + q = 0$

Similarly, β is a root of $x^2 - px + q = 0$

E 12) a) $4p^4 - 16p^2 + 5 = 0$.

Put $p^2 = x$. Then the equation becomes

$4x^2 - 16x + 5 = 0$.

Its roots are $2 + \frac{\sqrt{11}}{2}$ and $2 - \frac{\sqrt{11}}{2}$.

Now, $p^2 = 2 + \frac{\sqrt{11}}{2} \Rightarrow p = \pm\sqrt{2 + \frac{\sqrt{11}}{2}}$

and $p^2 = 2 - \frac{\sqrt{11}}{2} \Rightarrow p = \pm\sqrt{2 - \frac{\sqrt{11}}{2}}$.

These 4 values of p are the required roots.

b) $(5x^2 - 6)^{\frac{1}{4}} = x$.

Every root of this is a root of

$5x^2 - 6 = x^4$

$\Leftrightarrow x^4 - 5x^2 + 6 = 0$

Put $x^2 = y$. Then

$y^2 - 5y + 6 = 0$.

Its roots are $y = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2} = 3, 2$.

Now $x^2 = 3 \Rightarrow x = \sqrt{3}$ or $-\sqrt{3}$,

and $x^2 = 2 \Rightarrow x = \sqrt{2}$ or $-\sqrt{2}$.

Putting these 4 values of x in the given equation, we find that $\sqrt{3}$ and $\sqrt{2}$ are its solutions.

c) Separating the radicals, we get

$\sqrt{2x+3} = 1 + \sqrt{x+1}$.

Squaring both sides, we get

$2x + 3 = 1 + (x+1) + 2\sqrt{x+1}$

$\Leftrightarrow x + 1 = 2\sqrt{x+1}$.

Again squaring both sides, we get

$x^2 - 2x - 3 = 0$.

Its roots are $x = 3$ and $x = -1$

Substituting these values of x in the given equation, we get

$\sqrt{2(3)+3} - \sqrt{3+1} = 1$, and $\sqrt{2(-1)+3} - \sqrt{-1+1} = 1$.

Thus, both $x = 3$ and $x - 1$ are roots of the given equation.

E 13) Let Alka's rate be x km per hour. Then Ameena's is $(x+1)$ km per hour.

The time taken by Ameena of walk to the library $= \frac{24}{x+1}$ hours. Thus, the time taken by Alka $= \left(\frac{24}{x+1} + 2\right)$ hours.

$$\therefore \frac{24}{x} = \frac{24}{x+1} + 2$$

$\Rightarrow x(x+1) = 12$

$\Rightarrow x = -4$ or $x = 3$.

Since (-4) can't be the rate, it is an extraneous solution. Thus, the required speed must be 3km per hour.

E 14) None, since $ax + b = 0$, $a, b \in \mathbf{R} \Rightarrow x = -\frac{b}{a} \in \mathbf{R}$

E15) If $p^2 - 4q < 0$.

There will be two such roots, and they will be conjugates.

E16) a) $2x^3 + 3x^2 + 3x + 1 = 0$

$$\Leftrightarrow x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + \frac{1}{2} = 0$$

Referring to Cardano's formula, we see that in this case

$p = \frac{3}{2}, q = \frac{3}{2}, r = \frac{1}{2}$.

$\therefore A = \frac{3}{4}, B = 0$

$\therefore \alpha = \frac{1}{2}, \beta = -\frac{1}{2}$.

$\therefore$ the roots are $-\frac{1}{2}, \frac{\omega - \omega^2}{2} - \frac{1}{2}, \frac{\omega^2 - \omega}{2} - \frac{1}{2}$, that is,

$-\frac{1}{2}, \omega, \omega^2$ (since $1 + \omega + \omega^2 = 0$)

b) $x^3 + 21x + 342 = 0$.

Here we don't need to apply Step 1 of Cardano's Method, since there is no term containing x^2. Now, with reference to Cardano's formula,

$A = 21 - 0 = 21, B = 0 - 0 + 342 = 342$.

$$\therefore \alpha = \left\{\frac{-342}{2} + \sqrt{\frac{(342)^2}{2} + \frac{(21)^3}{27}}\right\}^{\frac{1}{3}} = (-171 + 172)^{\frac{1}{3}} = 1$$

and $\beta = (-171 - 172)^{\frac{1}{3}} = -7$.

Thus, the roots of the equation are

$1 - 7, \omega - 7\omega^2, \omega^2 - 7\omega$, that is $-6, \omega - 7\omega^2, \omega^2 - 7\omega$.

c) $x^3 + 6x^2 + 6x + 8 = 0$.

Here $p = 6, q = 6, r = 8$.

$$\therefore A = q - \frac{p^2}{3} = -6, \ B = \frac{2p^3}{27} - \frac{pq}{3} + r = 12.$$

$$\therefore \alpha = (-6 + \sqrt{36 - 8})^{\frac{1}{3}} = (-6 + 2\sqrt{7})^{\frac{1}{3}} = -0.891$$

and $\beta = (-6 - 2\sqrt{7})^{\frac{1}{3}} = -2.243$

(We have used a calculator to evaluate α and β to 3 decimal places.)

Then the required roots are

$\alpha + \beta - 2, \alpha\omega + \beta\omega^2 - 2, \alpha\omega^2 + \beta\omega - 2$

d) $x^3 + 29x - 97 = 0$.

Here $p = 0, q = 29, r = -97$.

$$\therefore \alpha = \left\{\frac{97}{2} + \sqrt{\frac{(97)^2}{4} + \frac{(29)^3}{27}}\right\}^{\frac{1}{3}} = (64.557)^{\frac{1}{3}} = 4.01, \text{and}$$

$\beta = (-8.557)^{\frac{1}{3}} = -2.045$.

Then the roots are

$\alpha + \beta, \alpha\omega + \beta\omega^2, \alpha\omega^2 + \beta\omega$.

e) $x^3 - 30x + 133 = 0$.

Here $p = 0, q = -30, r = 133$.

$\therefore A = -30, B = 133$.

$$\therefore \alpha = \left\{-\frac{133}{2} + \sqrt{\frac{(133)^2}{4} - \frac{(30)^3}{27}}\right\}^{\frac{1}{3}} = (-8)^{\frac{1}{3}} = -2, \text{ and}$$

$$\beta = (-66.5 - 58.5)^{\frac{1}{3}} = -5$$

$\therefore$ the roots are $-7, -2\omega - 5\omega^2, -2\omega^2 - 5\omega$.

E 17) $x^3 - 3x + 1 = 0$.

Here $p = 0, q = -3, r = 1$.

$\therefore A = -3, B = 1$.

$$\therefore \frac{B^2}{4} + \frac{A^3}{27} = -\frac{3}{4} < 0.$$

So we are in the irreducible case.

$$\text{Now, } \frac{-B}{2} + \sqrt{\frac{B^2}{4} + \frac{A^3}{27}} = \frac{-1}{2} + \frac{i\sqrt{3}}{2} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}.$$

Thus, the solutions of the given equation are $2\cos\left(\frac{\frac{2\pi}{3} + 2k\pi}{3}\right)$, where $k = 0, 1, 2$, that is

$$2\cos\frac{2\pi}{9},\ 2\cos\frac{8\pi}{9},\ \cos\frac{14\pi}{9}.$$

E 18) α, β, γ are the roots iff

$$ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$$
$$= a\{x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x + \alpha\beta\gamma\}$$

On comparing coefficients, we get

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}.$$

E 19) Let the roots be α, β, γ

Then $\alpha + \beta + \gamma = 6,\ \alpha\beta + \beta\gamma + \alpha\gamma = 11,\ \alpha\beta\gamma = 6$.

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$$
$$= 36 - 22 = 14.$$

$$\therefore \alpha^3 + \beta^3 + \gamma^3 = (\alpha+\beta+\gamma)^3 - 3\alpha^2(\beta+\gamma) - 3\beta^2(\alpha+\gamma)$$
$$-3\gamma^2(\alpha+\beta) - 6\alpha\beta\gamma$$
$$= 6^3 - 3\alpha^2(6-\alpha) - 3\beta^2(6-\beta) - 3\gamma^2(6-\gamma) - 6\times 6$$
$$= 180 - 18(\alpha^2+\beta^2+\gamma^2) + 3(\alpha^3+\beta^3+\gamma^3)$$

$$\therefore 4(\alpha^3+\beta^3+\gamma^3) = 180 - 18\times 14 = -72$$

$$\therefore \alpha^3 + \beta^3 + \gamma^3 = -18$$

Now, each of α, β, γ satisfy

$x^3 - 6x^2 + 11x - 6 = 0$.

Thus, they will satisfy $x^4 - 6x^3 + 11x^2 - 6x = 0$ also.

$\therefore \alpha^4 - 6\alpha^3 + 11\alpha^2 - 6\alpha = 0$

$\beta^4 - 6\beta^3 + 11\beta^2 - 6\beta = 0$

$\gamma^4 - 6\gamma^3 + 11\gamma^2 - 6\gamma = 0.$

Adding these equations, we get

$(\alpha^4 + \beta^4 + \gamma^4) - 6\,(\alpha^3 + \beta^3 + \gamma^3) + 11\,(\alpha^2 + \beta^2 + \gamma^2) - 6\,(\alpha + \beta + \gamma) = 0$

$\Rightarrow \alpha^4 + \beta^4 + \gamma^4 = 6\,(-18) - 11\,(14) + 6\,(6) = -226$

20) Here $p = \frac{3b}{a}, q = \frac{3c}{a}, r = \frac{d}{a}.$

$$\therefore A = \frac{3c}{a} - \frac{1}{3}\left(\frac{3b}{a}\right)^2 = \frac{3(ac - b^2)}{a^2}, \text{ and}$$

$$B = \frac{2}{27}\left(\frac{3b}{a}\right)^3 - \frac{1}{3}\left(\frac{3b}{a}\right)\left(\frac{3c}{a}\right) + \frac{d}{a} = \frac{2b^3 - 3abc + a^2 d}{a^3}$$

Therefore, the equation has complex roots if

$$\frac{B^2}{4} + \frac{A^3}{27} > 0, \text{ that is,}$$

$$\frac{(2b^3 - 3abc + a^2 d)^2}{4a^6} + \frac{(ac - b^2)^3}{a^6} > 0, \text{ that is,}$$

$a^2 d^2 - 3b^2 c^2 - 6\,abcd + 4b^3 d + 4\,ac^2 > 0.$

E21) Here $B = -126, A = -15.$

$$\therefore \frac{B^2}{4} + \frac{A^2}{27} = 3844 > 0.$$

Thus, the equation has 1 real and 2 complex roots.

E22) a) $x^4 - 3x^2 - 42x - 40 = 0.$

Adding $(ax + b)^2$ to both sides, we get

$x^4 + (a^2 - 3)\,x^2 + (2ab - 42)\,x + b^2 - 40 = (ax + b)^2$

Assume that the left hand side is $(x^2 + k)^2$.

(Note that the coefficient of x in the given equation is 0.)

Then $x^4 + (a^2 - 3)\,x^2 + (2ab - 42)\,x + b^2 - 40 = x^4 + k^2 + 2k\,x^2.$

Comparing coefficients, we get

$$\left.\begin{array}{l} a^2 - 3 = 2k \\ 2ab - 42 = 0 \\ b^2 - 4 - 0 = k^2 \end{array}\right] \Rightarrow \left[\begin{array}{l} a^2 = 2k + 3 \\ ab = 21 \\ b^2 = k^2 + 40. \end{array}\right.$$

Eliminating a and b we get

$(21)^2 = (2k + 3)(k^2 + 40) = 2k^3 + 3k^2 + 80k + 120$

$\therefore 2k^3 + 3k^2 + 80k - 321 = 0.$

3 is a root of this equation. With this value of k we get

$a^2 = 9, b^2 = 49, ab = 21.$

These equations are satisfied by $a = 3, b = 7$.

Thus, solving the given quartic reduces to solving the following quadratic equations:

$x^2 + 3 = 3x + 7$ and $x^2 + 3 = -(3x + 7)$, that is,

$x^2 - 3x - 4 = 0$ and $x^2 + 3x + 10 = 0.$

Thus, the required roots are $4, -1$ and $\frac{-3 \pm i\sqrt{31}}{2}$.

b) The given equation is equivalent to

$$x^4 - 5x^3 + \frac{33}{4}x^4 - 5x + 1 = 0$$

The resolvent cubic is $8k^3 - 33k^2 + 42 - 17 = 0$.

One real root is 1.

With this value of k, we find that

$a = 0, b = 0$.

Thus, the given equation becomes

$$(x^2 - \frac{5}{2}x + 1)^2 = 0.$$

Therefore, the given equation has the roots

$2, \frac{1}{2}, 2, \frac{1}{2}$, that is, two pairs of equal roots.

c) $x^4 + 12x - 5 = 0$. The resolvent cubic is $k^3 + 5k - 18 = 0$.

A real root is $k = 2$.

Then, solving the given equation reduces to solving

$(x^2 + 2) = \pm(2x - 3)$, that is

$x^2 - 2x + 5 = 0$ and $x^2 + 2x - 1 = 0$.

Thus, the required roots are

$$\frac{2 \pm \sqrt{4-20}}{2} \text{ and } \frac{-2 \pm \sqrt{4+4}}{2}, \text{ that is,}$$

$1 + 2i, 1 - 2i, -1 + \sqrt{2}, -1 - \sqrt{2}$.

E23) a) $x^4 - 2x^2 + 8x - 3 = 0$.

Since there is no x term, we don't need to apply Step 1. Now assume

$x^4 - 2x^2 + 8x - 3 = (x^2 + kx + m)(x^2 - kx + n)$.

Then $m + n - k^2 = -2, k(n - m) = 8, mn = -3$.

Thus, eliminating m and n we get

$k^6 - 4k^4 + 16k^2 - 64 = 0$.

$k^2 = 4$ is a root of this cubic in k^2.

Thus, $k = 2$ is a solution. For this value of k, we get $n = 3, m = -1$

Thus, the roots of the given equation are the roots of $x^2 + 2x - 1 = 0$ and

$x^2 - 2x + 3 = 0$, that is, the roots are $-1 \pm \sqrt{2}$ and $-1 \pm i\sqrt{2}$.

b) This equation can be rewritten as

$(x + 2)^4 - 15x^2 - 40x - 26 = 0$.

Putting $x + 2 = y$, we get

$y^4 - 15y^2 + 20y - 6 = 0$.

Then the cubic in k^2 is

$k^6 - 30k^4 + 249k^2 - 400 = 0$, that is.

$t^3 - 30t^2 + 249k^2 - 400 = 0$, putting $k^2 = t$.

One real positive root is $t = 16$. So we can take $k = 4$.

Then we need to solve the quadratic equations

$y^2 - 4y + 3 = 0$ and $y^2 + 4y - 2 = 0$

Thus, $y = 3, 1, -2 \pm \sqrt{6}$.

Thus, the roots of the given equation are $(y - 2)$, that is, $1, -1, -4 \pm \sqrt{6}$.

c) $x^4-3x^2-6x-2=0$.

The cubic in k^2 is $k^6-6k^4+17k^2-36=0$.

$k^2=4$ is a root. So we can take $k=2$.

Then we need to solve the equations.

$x^2+2x+2=0, x^2-2x-1=0$.

$x=-1\pm i, 1\pm\sqrt{2}$.

d) $1, 2, -3, -4$.

E 24) Putting $x^2=y$ in the equation, we get

$2y^4+5y^3-5y-2=0$.

Then, by either Ferrari's or Descartes' method, we can find the four values of y, which are

$$1,\ -1,\ -2,\ \frac{-1}{2}.$$

Putting these values in $x^2=y$, and solving, we get the 8 roots of the given equation. Thus , the required roots are

$$\pm\sqrt{1},\ \pm\sqrt{-1},\ \pm\sqrt{-2},\ \pm\sqrt{\frac{-1}{2}},\ \text{that is},$$

$$1, -1, i, -i, i\sqrt{2}, -i\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}.$$

E25) Let the roots be a, b, c, d. Then ad = bc.

Now, we know that

i) $a+b+c+d=-15 \Rightarrow (a+d)+(b+c)=-15$ ....(30)

ii) $ab+ac+ad+bc+bd+cd=70$

$\Rightarrow (a+d)(b+c)+ad+bc=70$ ....(31)

iii) $abc+abd+acd+bcd=-120$ ....(32)

iv) $abcd=64$

Now, (32) $\Rightarrow ad(b+c)+bc(a+d)=-120 \Rightarrow ad(a+b+c+d)=-120$

$\Rightarrow -15\,ad=-120 \Rightarrow ad=8$. Thus, $ad=8=bc$.

Then (31) $(a+d)(b+c)=70-16=54$.

This, with (30) tells us that $a+d$ and $b+c$ are roots of $x^2+15x+54=0$.

Thus , by the quadratic formula,

$$a+d=\frac{-15+3}{2}=-6 \text{ and } b+c=\frac{-15-3}{2}=-9.$$

Then, $ad=8$ and $bc=8$ tell us that a and d are zeros of $x^2+6x+8=0$,

and b and c are zeros of $x^2+9x+8=0$.

$$\therefore\ a=\frac{-6+2}{2}=-2\ d=-4,\ b=\frac{-9+7}{2}=-1\ c=-8.$$

E26) Let the roots be a, b, c, d, where

$a+b=c+d$. ....(33)

We know that

$a+b+c+d=p$ ....(34)

$(a+b)(c+d)+ab+cd=q$ ....(35)

$ab(c+d)+(a+b)cd=r$ ....(36)

$abcd=s$

(33) and (34) $\Rightarrow a+b=\frac{p}{2}=c+d.$

Then (36) $\Rightarrow \frac{p}{2}\,(ab+cd)=r \;\Rightarrow ab+cd=\frac{2r}{p}$

Then (35) $\Rightarrow \frac{p}{2}.\frac{p}{2}+\frac{2r}{p}=q.$

$\Rightarrow p^3+8r=4pq$

$\Rightarrow p^3-4pq+8r=0.$

MISCELLANEOUS EXERCISES

This section is optional

We have listed some problems related to the material covered in this block. You may like to do them to get more practice in solving problems. We have also given our solutions to these questions, because you may like to counter - check your answers.

1) Let 1, ω and ω^2 be the cube roots of unity. Evaluate

a) $(1-\omega+\omega^2)(1+\omega-\omega^2)$

b) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)$

c) $\prod_{i=1}^{5}(1-\omega^i)$

2) Give the equations, in standard form, whose roots are

a) $2, -\frac{3}{2}, 9$

b) $\sqrt{2}, -\sqrt{3}, \pm i.$

3) For what value of m ($\neq -1$) will the equation $\frac{x^2-bx}{ax-c} = \frac{m-1}{m+1}$ have roots equal in magnitude but opposite in sign ? Here $a, b \in \mathbf{R}$ and $a+b \neq 0$.

4) Solve

a) $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$, where $a, b \in \mathbf{R}$.

b) $(\sqrt{2})^x + \frac{1}{(\sqrt{2})^x} = 2.$

5) $x^4+9x^3+12x^2-80x-192=0$ has a pair of equal roots. Obtain all its roots.

6) If a, b, c are the roots of $x^3-px^2+r=0$, find the equation whose roots are

$\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}.$

7) Solve $x^4-4x^2+8x+35=0$, given that one root is $2+\sqrt{-3}$

8) From the cubic whose roots are a, b, c, where

$a+b+c=3,$

$a^2+b^2+c^2=5$, and

$a^3+b^3+c^3=11.$

Hence evaluate $a^4+b^4+c^4$.

9) Find the equation whose roots are 4 less in value then the roots of

$x^4-5x^3+7x^2-17x+11=0.$

(**Hint:** Write the equation as an equation in ($x-4$).)

10) From the polynomial equation over **R** of lowest degree which is satisfied by $1-i$ and $3+2i$. Is it unique ?

11) Solve $x^4+9x^3+16x^2+9x+1=0.$

(**Hint :** Note that in this equation the coefficients of xr and x4 – r are the same

^{n}r = 0, 1, 2, 3, 4,. So we can divide throughout by x2 and then write the equation as a

quadratic in $x+\frac{1}{x}=y$ say. Now solve for y, and then for x.)

If you are interested in doing more exercises on the material covered in this block, please refer to the book 'Higher Algebra' by Hall and Knight. A copy is available in your study centre.

Solutions

1) a) We know that $1+\omega+\omega^2=0$

$\therefore\ 1-\omega+\omega^2=-2\omega$ and $1+\omega-\omega^2=-2\omega^2$.

$\therefore\ (1-\omega+\omega^2)\ (1+\omega+\omega^2)=(-2\omega)\ (-2\omega^2)=4\omega^3=4,$

since $\omega^3=1$.

b) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)$

$=(1-\omega)^2(1-\omega^2)^2$, since $\omega^3=1$.

$=(1-2\omega+\omega^2)(1-2\omega^2+\omega^4)$

$=(-3\omega)(-3\omega^2)$

$=9$

C) 0, since $1-\omega^3=0$.

2) a) The equation is

$$(x-2)\left(x+\frac{3}{2}\right)(x-9)=0$$

$\Leftrightarrow (x-2)\ (2x+3)\ (x-9)=0$

$\Leftrightarrow 2x^3-19x^2+3x+54=0$, in standard form.

b) $x^4\ (\sqrt{3}-\sqrt{2})\ x^3+(1-\sqrt{6})\ x^2+(\sqrt{3}-\sqrt{2})\ x-\sqrt{6}=0$

3) The equation is equivalent to

$(1+m)x^2+\{(a-b)-m(a+b)\}\ x+(m-1)c=0.$

The roots are equal in magnitude and opposite in sign iff their sum is zero.

$\therefore\ (a-b)-m(a+b)=0$, that is,

$$m=\frac{a-b}{a+b}$$

4) Let $y=\sqrt{\frac{x}{a}}$. Then the given equation is equivalent to

$$2y+\frac{3}{y}=\frac{b^2+6a^2}{ab}, \text{ that is,}$$

$$2y^2-\frac{b^2+6a^2}{ab}y+3=0.$$

Its roots are $\frac{b}{2a}, \frac{3a}{b}$.

Thus, the roots of the given equation are

$$\left(\sqrt{a}\,\frac{b}{2a}\right)^2 \text{ and } \left(\sqrt{a}\,\frac{3a}{b}\right)^2, \text{ that is, } \frac{b^2}{4a} \text{ and } \frac{9a^3}{b^2}$$

B) Put $y=\sqrt{2^x}$. Then our equation in y is

$$y+\frac{1}{y}=2 \Leftrightarrow y^2-2y+1=0 \Leftrightarrow (y-1)^2=0.$$

$\therefore\ y=1$

$\therefore\ \sqrt{2^x}=1$

$x=0$ is the required root

5) Let its roots be a, a, b, c.

Then, the relations between the roots and the coefficients are

$2a+b+c=-9$ ...(1)

$a^2 + 2ab + 2ac + bc = 12$ ...(2)

$a^2 b + a^2 c + 2abc = 80$ ...(3)

$a^2 bc = -192$ ...(4)

Using (1), (2) and (3), we get

$4a^3 + 27a^2 + 24a = 80.$

$a = -4$ is a solution.

Then (1) $\Rightarrow b + c = -1$, and (4) $\Rightarrow bc = -12$.

Thus, b and c are roots of

$x^2 + x - 12 = 0.$

$\therefore b = 3, c = -4.$

Thus, the roots are $-4, -4, -4, 3$.

6) We know that

$a + b + c = p$ ...(5)

$ab + ac + bc = 0$ ...(6)

$abc = r$ ...(7)

Now, $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} = \frac{bc(p-a)+ac\,(p-b)+ab(p-c)}{abc}$, using (5)

$= -3$

$\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right) + \left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right) + \left(\frac{a+b}{c}\right)\left(\frac{b+c}{a}\right) = \frac{a^3+b^3+c^3}{abc}$, using (6).

Now, a, b, and c satisfy $x^3 - px^2 + r = 0$.

Thus, on substituting each of a, b and c in this equation, and summing up, we get

$a^3 + b^3 + c^3 - p(a^2 + b^2 + c^2) + 3r = 0.$

$\therefore a^3 + b^3 + c^3 = p\{(a + b + c)^2 - 2(ab + ac + bc)\} - 3r$

$= p^3 - 3r$

$$\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right) + \left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right) + \left(\frac{a+b}{c}\right)\left(\frac{b+c}{a}\right) = \frac{p^3-3r}{r} \quad ....(9)$$

Also $\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right) = -\frac{r}{r}$, using (6) and (7)

$= -1$ ...(10)

Using (8), (9) and (10), we see that the required equation is

$$x^3 + 3x^2 + \left(\frac{p^3 - 3r}{r}\right) x + 1 = 0.$$

7) $2 + i\sqrt{3}$ is one root. So another must be $2 - i\sqrt{3}$.

Thus, $(x^2 - 4x + 7) \mid (x^4 - 4x^2 + 8x + 35)$.

By long division, or by inspection, we see that

$x^4 - 4x^2 + 8x + 35 = (x^2 - 4x + 7)(x^2 + 4x + 5).$

Thus the other two roots of the given equation are those of $x^2 + 4x + 5 = 0$, that is $-2 \pm i$.

8) Now, $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc)$

$\therefore ab + ac + bc = \frac{9-5}{2} = 2$

Also, $a^3 + b^3 + c^3 = (a + b + c)^3 - 3[(a^2 + b^2)c + (b^2 + c^2)a + (c^2 + a^2)b] - 6abc$

$\therefore abc = \frac{2}{3}$.

Thus a, b, c are the roots of

$x^3 - 3x^2 + 2x - \frac{2}{3} = 0$, that is,

$3x^3 - 9x^2 + 6x - 2 = 0$.

Thus a, b and c satisfy this equation, as well as,

$3x^4 - 9x^3 + 6x^2 - 2x = 0$.

$\therefore 3(a^4 + b^4 + c^4) - 9(a^3 + b^3 + c^3) + 6(a^2 + b^2 + c^2) - 2(a + b + c) = 0$

$\Rightarrow a^4 + b^4 + c^4 = 25$.

9) Put $y = x - 4$, that is, $x = y + 4$ in the given equation. Then the equation that we get in y will be the required equation. Thus, the required equation is

$(y+4)^4 - 5(y+4)^3 + 7(y+4)^2 - 17(y+4) + 11 = 0$

$\Leftrightarrow y^4 + 11y^3 + 43y^2 + 55y - 9 = 0$

10) If $1 - i$ is a root, so must $1 + i$ be. Similarly, $3 + 2i$ and $3 - 2i$ are roots of the equation. Thus, the equation of lowest degree is the quartic

$[x-(1-i)]\,[x-(1+i)]\,[x-(3+2i)]\,[x-(3-2i)] = 0$, that is,

$x^4 - 8x^3 + 27x^2 - 38x + 26 = 0$.

This is unique, up to equivalence. That is, any other polynomial equation that satisfies our requirements must be equivalent to this equation.

11) $x^4 + 9x^3 + 16x^2 + 9x + 1 = 0$

$\Leftrightarrow \left(x^2 + \frac{1}{x^2}\right) + 9\left(x + \frac{1}{x}\right) + 16 = 0$, by dividing throughout by x^2.

$\Leftrightarrow \left(x + \frac{1}{x}\right)^2 + 9\left(x + \frac{1}{x}\right) + 14 = 0$.

Putting $x + \frac{1}{x} = t$, we get

$t^2 + 9t + 14 = 0$

Thus, $t = \frac{-9 \pm \sqrt{81 - 56}}{2} = \frac{-9 \pm 5}{2} = -2, -7$.

Thus, we get two equations in x, namely

$x + \frac{1}{x} = -2$ and $x + \frac{1}{x} = -7$, that is,

$x^2 + 2x + 1 = 0$ and $x^2 + 7x + 1 = 0$

On solving these quadratic equations, we get the four solutions of the original equation, which are $1, 1, \frac{-7 \pm \sqrt{45}}{2}$.

APPENDIX: SOME MATHEMATICAL SYMBOLS AND TECHNIQUES OF PROOF

To be able to do any mathematical study, you need to know the language of mathematics. In this appendix we shall introduce you to some symbols and their meaning. We shall also briefly discuss some paths that you will often take to reach conclusions.

Symbols

1) **Implication** (denoted by $\Rightarrow$) : We say that a statement A implies a statement B if B follows from A.

   We write this as the compound statement, '$A \Rightarrow B$' or of A, then B'.

   For example, consider A and B, where

   A : Triangles ABC and DEF are congruent.

   B: Triangles ABC and DEF have the same area.

   Then $A \Rightarrow B$ ....(1)

   In this case '$A \Rightarrow B$' is a true statement.

   Another way of saying $A \Rightarrow B$ is that 'A **only if B**', or that 'A is **sufficient** for B'.

   The **converse** of the satement 'if A, then B' is the statement 'if B, then A', that is, $B \Rightarrow A$ (which is the same as $A \Leftarrow B$).

   For example, the converse of (1) is

   'if two triangles have the same area, then they are congruent.'

   While studying geometry you must have proved that this statement is false. (For example, the right-angled triangles with sides 2, 3, $\sqrt{13}$ cm., and 1,6, $\sqrt{37}$ cm. have the same area; but they are incongruent.) Thus, (1) is true, but its converse is not.

   Another way of saying $A \Leftarrow B$ is that 'A is **necessary** for B'.

2) **Two - way implication** (denoted by $\Leftrightarrow$): Sometimes we find two statements A and B for which $A \Rightarrow B$ and $B \Rightarrow A$. In this situation we save space and write $A \Leftrightarrow B$.

   This statement is the same as:

   'A is **equivalent** to B' ; or

   'A **if and only if** B', which we abbreviate to 'A iff B', or

   'A is **necessary and sufficient** for B'.

   For example, let

   $A : x + 2 = 3$ and

   $B : x = 1$.

   Then $A \Rightarrow B$ and $B \Rightarrow A$. Therefore, $A \Leftrightarrow B$.

   Note that for the composite statement 'A iff B' to be true, both $A \Rightarrow B$ and $B \Rightarrow A$ should be true. Hence the statement.

   'Two triangles are congruent iff they have the same area' is a false statement.

3) **For all / for every** (denoted by $\forall$) : Sometimes a statement involving a variable x, say P (x), is true for every value that x takes. We write this statement as :

   '$\forall x P(x)$ or '$P(x) \ \forall x$',

   meaning that P (x) is a true statement for every value of x.

   For example. let P (x) be the statement '$x > 0$'.

   Then $P(x) \ \forall x \in \mathbf{N}$.

4) There exist /there exists (denoted by $\exists$) : If a statement depending on a variable x, say P (x), is true for some value of x, then we write

   $\exists$ x such that P (x).

   This says that there is some x for which P (x) is true.

For example, $\exists\, x \in \mathbf{R}$ such that $x - 3 = 2$.

Now, consider the two statements

$\forall\, x \in \mathbf{R}\ \exists\, y \in \mathbf{R}$ such that $x = 2y$, and ....(3)

$\exists y \in \mathbf{R}$, such that $\forall\, x \in \mathbf{R}, x = 2y$. ....(4)

Is there a difference in them ? What does (3) mean ? It means that for any real number x, we can find a real number y for which $x = 2y$.

In fact, $y = \frac{x}{2}$ serves the purpose.

Now look at (4). It says that there is some real number y such that whatever real number x we take, $x = 2y$ is true. This is clearly a false statement.

This shows that we have to be very careful when dealing with mathematical symbols.

So far we have looked at the meaning and use of some common logical symbols. Let us now consider some common te hniques of proof.

Methods of proof

In any mathematical theory, we assume certain facts called axioms. Using these axioms, we arrive at certain results (theorems) by a sequence of logical deductions. Each such sequence forms a proof of a theorem. We can give proofs in several ways.

1) **Direct proof :** A direct proof, or step in a proof, takes the following form:

A is true and the statement '$A \Rightarrow B$' is true, therefore B is true.

For example,

ABC is equilateral and (If a triangle is equilateral, then it is an isosceles triangle.),

therefore, ABC is an isosceles triangle.

One kind of result that you will often meet in this course and other methematics courses is a theorem that asserts the equivalence of a number of statements, say A, B, C. We can prove this by proving $A \Rightarrow B, B \Rightarrow A, A \Rightarrow C, C \Rightarrow A$ and $B \Rightarrow C, C \Rightarrow B$. But, if $A \Rightarrow B$ and $B \Rightarrow C$ are both true, then so is $A \Rightarrow C$. So, a shorter proof could consist of the steps $A \Rightarrow B, B \Rightarrow C, C \Rightarrow A$. We write this in short as $A \Rightarrow B \Rightarrow C \Rightarrow A$.

Or, the proof could be $A \Rightarrow C \Rightarrow B \Rightarrow A$. Thus the order doesn't matter, as long as the path brings us back to the starting point, and all the satements are covered.

Whenever you meet such a result in our course, we shall indicate the path we shall follow.

2) **Contrapositive Proof :** This is an indirect method of proof. It uses the fact that '$A \Rightarrow B$' is equivalent to its contrapositive, namely, 'not $B \Rightarrow$ not A', that is, if B does not hold, then A does not hold.

(For example, $x = -2 \Rightarrow x^2 = 4$ is equivalent to its contrapositive,

$x^2 \neq 4 \Rightarrow x \neq -2$.)

Sometimes, it is easier to prove the contrapositive of a given result. In such situations we use this method of proof. So, how does this method work ? To prove '$A \Rightarrow B$', we prove 'not $B \Rightarrow$ not A', that is , we assume that B does not hold, and then, through a sequence of logical steps, we conclude that A does not hold.

Let us look at an example. Suppose we want to prove that ' If two triangles are not similar, then they are not congruent'. We prove its contrapositive, namely, ' If two triangles are congruent, then they are similar', which is easy to prove.

3) **Proof by contradiction:** This method is also called *reductio and absurdum*, a Latin phrase. In this method, to prove that a statement A is true, we start by assuming that A is false. Then by logical steps we arrive at a known false statement. So we reach a contradiction. Thus, we are forced to conclude that A cannot be false. Hence, A is true.

For example, to prove that $\sqrt{2} \notin \mathbf{Q}$, we start by assuming that $\sqrt{2} \in \mathbf{Q}$.

Then $\sqrt{2} = \frac{p}{q}$ for some $p,q \in \mathbf{Z}$, $q \neq 0$, and $(p,q) = 1$.

$$\Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2 \Rightarrow 2 \mid p^2 \Rightarrow 2 \mid p.$$

Let $p \Rightarrow 2m$

Then $2q^2 = p^2 = 4m^2$.

$\Rightarrow q^2 = 2m^2 \Rightarrow 2 \mid q^2 \Rightarrow 2 \mid q$. Thus 2/p and 2/q and hence 2/(p, q), which is not possible because we assumed that $(p, q) = 1$.

Thus, we arrive at a contradiction. Hence, we conclude that $\sqrt{2} \notin \mathbf{Q}$.

4) **Proof by counter-example:** Consider a statement P (x) depending on a variable x. Suppose we want to disprove it, that is, prove that is false. One way is to produce an x for which p (x) is false. Such an x is called a **counter-example** to P (x).

For example, let P (x) be the statment

'Every natural number is a product of distinct primes'.

Then $x = 4$ is a counter- example ,since $4 \in N$ and $4 = 2 \times 2$ is not a product of distinct primes. In fact, we have several counter- examples in this case.

This is not always the best method for disproving a statement . For example, suppose you want to check the truth of the statement

'Given $a, b, c \in \mathbf{Z}$, $\exists\, n \in \mathbf{Z}$ such that $an^2 + bn + c$ is not a prime number'.

If you try to look for counter- examples, then you're in trouble because you will have to find infinitely many - one for each triple $(a, b, c) \in \mathbf{R}^3$. So, why not try a direct proof, as the one below.

Proof : For fixed $a, b, c \in \mathbf{Z}$, take any $n \in \mathbf{Z}$ and let $an^2 + bn + c = t$.

Then $a(n+t)^2 + b(n+t) + c = t(2an + b + at + 1)$, which is a proper multiple of t. Thus our statement is true.