UNIT 2 METHODS OF INTEGRATION

Structure

2.1 INTRODUCTION

In the last unit we have seen that the definite integral $\int_a^b f(x)\, dx$ represents the signed area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. The Fundamental Theorem of Calculus gives us an easy way of evaluating such an integral, by first finding the antiderivative of the given function, whenever it exists. Starting from this unit, we shall study various methods and techniques of integration. In this unit, we shall consider two main methods: the method of substitution and the method of integration by parts. The next two units will cover some special integrals, which can be evaluated using these two methods.

Objectives

After reading this unit you should be able to :

- define the indefinite integral of a funciton
- evaluate certain standard integrals by finding the antiderivatives of the integrands
- use the rules of the algebra of integrals to evaluate some integrals
- use the method of substitution to simplify and evaluate certain integrals
- integrate a product of two functions, by parts.

2.2 BASIC DEFINITIONS

We have seen in Unit 1, that the antiderivative of a function is not unique. More precisely, we have seen that if a function F is an antiderivative of a function f, then F+c is also an antiderivative of f, where c is any arbitrary constant. Now we shall introduce a notation here :

We shall use the symbol $\int f(x)\, dx$ to denote the class of all antiderivatives of f. We call it the **indefinite integral** or just the **integral of** f. You must have noticed that we use the same sign $\int$, here that we have used for definite integrals in Unit 1. Thus, if F(x) is an antiderivative of f(x), then we can write $\int f(x)\,dx = F(x) + c$.

This c is called the constant of integration. As in the case of definite integrals, f(x) is called the integrand and dx indicates that f(x) is integrated with respect to the variable x. For example, in the equation

$$\int (av + b)^4 \, dv = \frac{(av+b)^5}{5a} + c,$$

$(av + b)^4$ is the integrand, v is the variable of integration, and $\frac{(av+b)^5}{5a} + c$ is the integral of the integrand $(av + b)^4$.

You will also agree that the indefinite integral of cosx is sinx + c, since we know that sin x is an antiderivative of cosx. Similarly, the indefinite integral of e^{2x} is $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$, and the indefinite integral of $x^3 + 1$ is $\int (x^3 + 1)dx = \frac{x^4}{4} + x + c$. You have seen in Unit 1 that the definite integral $\int_a^b f(x)\,dx$ is a uniquely defined **real number** whose value depends on a, b and the function f.

On the other hand, the indefinite integral $\int f(x)\,dx$ is a **class of functions** which differ from one another by constants. It is not a definite number; it is not even a definite function. We say that the indefinite integral is unique upto an arbitrary constant.

Unlike the definite integral which depends on a, b and f, the indefinite integral depends only on f.

All the symbols in the notation $\int_a^b f(x)$ f(x) dx for the definite integral have an interpretation.

The symbol $\int$ reminds us of summation, a and b give the limits for x for the summation. And f(x) dx shows that we are not considering the sum of just the function values, rather we are considering the sum of function values multiplied by small increments in the values of x.

In the case of an indefinite integral, however, the notation $\int f(x)$ dx has no similar interpretation. The inspiration for this notation comes from the fundamental Theorem of Calculus.

Thus, having defined an indefinite integral, let us get acquainted with the various techniques for evaluating integrals.

2.2.1 Standard Integrals

Integration would be a fairly simple matter if we had a list of integral formulas, or a table of **integrals,** in which we could locate any integral that we ever needed to evaluate. But the diversity of integrals that we encounter in practice, makes it impossible to have such a table. One way to overcome this problem is to have a short table of integrals of elementary functions, and learn the techniques by which the range of applicability of this short table can be extended. Accordingly, we build up a table (Table 1) of standard types of integral formulas by inverting formulas for derivatives, which you have already studied in Block 1. Check the validity of each entry in Table 1, by verifying that the derivative of any integral is the given corresponding function.

Table 1

S. No.	Function	Integral
1.	x^n	$\frac{x^{n+1}}{n+1} + c,\ n \neq -1$
2.	sinx	$-\cos x + c$
3.	cosx	$\sin x + c$
4.	$\sec^2 x$	$\tan x + c$
5.	$\mathrm{cosec}^2 x$	$-\cot x + c$
6.	secx tanx	$\sec x + c$
7.	cosec x cotx	$-\mathrm{cosec}x + c$
8.	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x + c$, or $-\cos^{-1} x + c$
		$-\cot^1 x + c$
9.	$\frac{1}{1+x^2}$	$\tan^{-1}x + c$ or $-\cot^{-1} x + c$
10.	$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1}x + c$ or $-\cot^{-1} x + c$
11.	$\frac{1}{x}$	$\mathrm{In}\lvert x \rvert + c$
12.	e^x	$e^x + c$
13.	a^x	$(a^x/\ln a).+c$
14.	sinhx	$\cosh x + c$
15.	coshx	$\sinh x + c$
16.	$\mathrm{sech}^2 x$	$\tanh x + c$
17.	$\mathrm{cosech}^2 x$	$-\cot x + c$
18.	sechx tanhx	$-\mathrm{sech}x + c$
19.	cosechx cothx	$-\mathrm{cosech}x + c$

Now let us see how to evaluate some functions which are linear combinations of the functions listed in Table 1.

2.2.2 Algebra of Integrals

You are familiar with the rule for differentiation which says

$$\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)]$$

There is a similar rule for integration:

Rule 1 $\int[af(x) + bg(x)dx = a\int f(x)dx + b\int g(x)dx$

This rule follows from the following two theorems.

Theorem 1 If f is an integrable function, then so is kf(x) and

$$\int kf(x)dx = k\int f(x)dx$$

Proof Let $\int f(x)dx = F(x) + c.$

Then by definition, $\frac{d}{dx}[F(x) + c] = f(x)$

$$\therefore \frac{d}{dx}[k\{F(x) + c\}] = kf(x)$$

Again, by definition, we have

$$\int kf(x)dx = k[F(x) + c] = k\int f(x)dx$$

Theorem 2 If f and g are two integrable functions, then f+g is integrable, and we have

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Proof Let $\int f(x)dx = F(x) + c, \int g(x)dx = G(x) + c$

Then, $\frac{d}{dx}[\{F(x) + c\} + \{G(x) + c\}] = f(x) + g(x)$

Thus, $\int [f(x) + g(x)]dx = [F(x) + c] + [G(x) + c]$

$$= \int f(x)dx + \int g(x)dx$$

Rule (1) may be extended to include a finite number of functions, that is, we can write

Rule 2 $\int [k_1 f_1(x) + k_2 f_2(x) + \dots\dots\dots\dots + k_n f_n(x)]dx$

$= k_1 \int f_1(x)dx + k_2 \int f_2(x)dx + \dots\dots\dots\dots\dots\dots + k_n \int f_n(x)dx$

We can make use of rule (2) to evaluate certain integrals which are not listed in Table 1.

Example 1 Let us evaluate $\int (x + \frac{1}{x})^3 dx$

We know that $(x + \frac{1}{x})^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$. Therefore,

$\int (x + \frac{1}{x})^3 dx = \int (x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}) dx$

$= x^3 dx + 3\int xdx + 3\int \frac{dx}{x} + \int \frac{dx}{x^3}$ ....**Rule 2**

Using integral formulas 1 and 11 from Table 1, we have

$\int (x + \frac{1}{x})^3 dx = \left(\frac{x^4}{4} + c_1\right) + 3\left(\frac{x^2}{2} + c_2\right) + 3(\ln|x| + c_3) + (\frac{x^{-2}}{-2} + c_4)$

$= \frac{x^4}{4} + \frac{3}{2}x^2 + 3 \ln|x| - \frac{1}{2x^2} + (c_1 + 3c_2 + 3c_3 + c_4)$

$= \frac{1}{4}x^4 + \frac{3}{2}x^2 + 3 \ln|x| - \frac{1}{2x^2} + c$

Note that $c_1 + 3c_2 + 3c_3 + c_4$ has been replaced by a single arbitrary constant c.

Example 2 Suppse we want to evaluate $\int (2 + 3 \sin x + 4e^x)dx$

This integral can be written as

$2\int dx + 3\int \sin x\, dx + 4\int e^x dx$

$= 2x - 3\cos x + 4e^x + c$

Note that $\int dx = \int 1dx = \int x^0 dx = x + c$

Example 3 To evaluate the definite integral $\int_0^1 (x + 2x^2)^2 dx$, we first find the indefinite integral $\int (x + 2x^2)^2 dx$

Thus, $\int (x + 2x^2) dx = \int (x^2 + 4x^3 + 4x^4) dx$

$= \int x^2 dx + 4\int x^3 dx + 4\int x^4 dx$

$= \frac{1}{3}x^3 + x^4 + \frac{4}{5}x^5 + c$

According to our definition of indefinite integral, this gives an antiderivative of $(x+2x^2)^2$ for a given value of c. By using the fundamental Theorem of Calculus we can now evaluate the definite integral.

FTC says that if G(x) is an antiderivative of f(x), then

$$\int_a^b f(x)\,dx = G(x)\Big|_a^b = G(b) - G(a).$$

$$\int_0^1 (x+2x^2)^2\,dx = \left(\frac{1}{3}x^3 + x^4 + \frac{4}{5}x^5 + c\right)\Bigg]_0^1$$

$$= \left(\frac{1}{3} + 1 + \frac{4}{5} + c\right) - c\Big) = \frac{32}{15}$$

Note that for the purpose of evaluating a definite integral, we could take the antiderivative corresponding to $c = 0$, that is,

$\frac{1}{3}x^3 + x^4 + \frac{4}{5}x^5$, as the constants cancel out.

See if you can do these exercises now.

E E 1) Write down the integrals of the following using Table 1 and Rule 2.

a) (i) x^4 (ii) $x^{-3/2}$ (iii) $4x^{-2}$ (iv) 3

b) (i) $1-2x+x^2$ (ii) $(x-\frac{1}{2})^2$ (iii) $(1+x)^3$

c) (i) $e^x + e^{-x} + 4$ (ii) $4\cos x - 3\sin x + e^x + x$ (iv) $4\,\text{sech}^2 x + e^x - 8x$

d) (i) $\frac{2}{\sqrt{1-x^2}} + \frac{5}{x}$ (ii) $\frac{2x^2+5}{x^2+1}$

e) (i) $ax^3 + bx^2 + cx + d$ (ii) $(\sqrt{x} - \frac{1}{\sqrt{x}})^2$

f) (i) $\frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x}$ (ii) $(2+x)(3-\sqrt{x})$

E2) Evaluate the following definite integrals.

a) c i) $\int_5^6 x^4 \, dx$ (ii) $\int_1^2 \frac{1+x}{x^2} \, dx$

b) i) $\int_2^4 (x + \frac{1}{2})^2 \, dx$ (ii) $\int_0^1 (x+1)^3 \, dx$

You have seen that with the help of Rule 2 we could evaluate a number of integrals. But still there are certain integrals like $\int \sin 2x \, dx$ which cannot be evaluated by using Rule 2. The method of substitution which we are going to describe in the next section will come in handy in these cases.

2.3 INTEGRATION BY SUBSTITUTION

In this section we shall study the first of the main methods of integration dealt with in this unit: the method of substitution. This is one of the most commonly used techniques of integration. We shall illustrate its application through a number of examples.

2.3.1 Method of Substitution

The following theorem will lead us to this method.

Theorem 3. If $\int f(v)dv = F(v) + c$, then on substituting g(x) for v, we get

$$\int f[g(x)]g'(x)dx = \int f(v)dv.$$

Proof We shall make use of the chain rule for derivatives **(Unit 3)** to prove this theorem.

Since $\int f(v)dv = F(v) + c$, we can write $\frac{dF(v)}{dv} = f(v)$. Now if we write v as a function of x, say v = g(x), then

$$\frac{d}{dx} F[g(x) = \frac{dF[g(x)]}{dg(x)} \cdot \frac{dg(x)}{dx} \text{ by chain rule}$$

$$= f[g(x)]. \frac{dg(x)}{dx} \text{ since } v = g(x)$$

$$= f[g(x)]. g'(x)$$

This shows that F[g(x)] is an antiderivative of f[g(x)g′(x). This means that

$$\int f[g(x)]g'(x)\,dx = F[g(x)] + c = F(v) + c = \int f(v)dv.$$

The statement of this theorem by itself may not seem very useful to you. But it does simplify our task of evaluating integrals. For example, to evaluate $\int \sin 2x\,dx$, we could take v = g(x) = 2x and get

$$\int \sin 2x\,dx = \frac{1}{2}\int \sin 2x\ (2)\ dx$$

$$= \frac{1}{2}\int \sin v\,dv \text{ by Theorem 3, since } g(x) = 2x \text{ and } g'(x) = 2.$$

$$= \frac{-\cos v}{2} + c$$

$$= -\frac{\cos 2x}{2} + c$$

We make a special mention of the following three cases which follow from theorem 3.

Case i) If $f(v) = v^n, n \neq -1$ and v = g(x), then by Formula 1 of Table 1.

$$\int [g(x)]^n\ g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + c$$

Case ii) If f(v) = 1/v and v = g(x).
then, by formula 11 of Table 1

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + c$$

Case iii) If $\int f(x)\ dx = F(x) + c$, then

$$\int_a^b f[g(x)]g'(x)\,dx = \int_{g(a)}^{g(b)} f(v)\,dv, \text{ where } v = g(x)$$ [The limits of integration are g(a) and g(b)

$$= F(v)\Big]_{g(a)}^{g(b)}$$

Since x = a => v = g(x) = g(a), and
x = b => v = g(x) = g(b).]

We shall be using these three cases very often. Their usefulness is evident from the following examples.

Example 4 Let us integrate $(2x + 1)(x^2 + x + 1)^5$

For this we observe that $\frac{d}{dx}(x^2 + x + 1) = 2x + 1$

Thus, $\int(2x+1)(x^2 + x + 1)^5\, dx$ is of the form $\int [g(x)]^n g'(x)\, dx$ and hence can be evaluated as in i) above.

Therefore, $\int (2x + 1)(x^2 + x + 1)^5\, dx = \frac{1}{6}(x^2 + x + 1)^6 + c.$

Alternatively, to find $\int (2x+1)(x^2 + x + 1)^5\, dx$, we can substitute $x^2 + x + 1$ by u. This means

$$\frac{du}{dx} = 2x + 1$$

Therefore, $\int (2x+1)(x^2 + x + 1)^5\, dx = \int u^5\, du$ by Theorem 3

$= \frac{1}{6}u^6 + c$ by Formula 1 from Table 1.

$$= \frac{1}{6}(x^2 + x + 1)^6 + c$$

Example 5 Let us evaluate $\int (ax + b)^n\, dx$

$$= \int (ax + b)^n\, dx = \frac{1}{a}(ax + b)^n\, dx$$

Therefore, when $n \neq -1$,

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,$$

and when $n = -1$

$$\int (ax+b)^n\, dx = \int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b| + c$$

Example 6 Suppose we want to evaluate the definite integral

$$\int_0^2 \frac{x+1}{x^2 + 2x + 3}\, dx$$

We put $x^2 + 2x + 3 = u$. This implies $\frac{du}{dx} = 2(x+1)$. Further, when $x = 0, u = 3$, and when $x = 2, u = 11$. Thus,

$$\int_0^2 \frac{x+1}{x^2+2x+1}\, dx = \frac{1}{2}\int_0^2 \frac{1}{u}\frac{du}{dx}\, dx = \frac{1}{2}\int_3^{11} \frac{du}{u} = \frac{1}{2} \ln |u|]_3^{11}$$

$$= \frac{1}{2}(\ln 11 -- \ln 3) = \frac{1}{2} \ln \frac{11}{3}$$

Example 7 To evaluate $\int xe^{2x^2}\, dx$, we substitute $2x^2 = u$. Since $\frac{du}{dx} = 4x$, we can write,

$$\int xe^{2x^2}\, dx = \frac{1}{4}\int e^{2x^2}\, 4x\, dx = \frac{1}{4}\int e^u \frac{du}{dx}\, dx$$

$$= \frac{1}{4}\int e^u\, du = \frac{1}{4}e^u + c.$$

$$= \frac{1}{4}e^{2x^2} + c$$

On the basis of the rules discussed in this section, you will be able to solve this exercise.

E E3) Evaluate

a) $\int \sqrt{5x-3}\,dx$ b) $\int (2x+1)^6\,dx$ c) $\int_1^3 \frac{dx}{4+5x}$

d) $\int \frac{5\,dx}{10x+7}$ e) $\int \frac{x+1}{x^2+2x+7}\,dx$ f) $\int_2^3 \frac{3x^2+2x+1}{x^3+x^2+x-8}$

g) $\int x^{1/3}\sqrt{x^{4/3}-1}\,dx$ h) $\int \frac{x\,dx}{1-3x^2}$

Now we shall use the method of substitution to integrate some trigonometric function. Let's start with sin ax.

Example 8 To evaluate $\int \sin ax\, dx$, we proceeded in the same manner as we did for

$\int \sin 2x\, dx$. We make the substitution $ax = u$

This gives $\frac{du}{dx} = a$. Thus,

$$\int \sin ax\, dx = \frac{1}{a}\int \sin u.\frac{du}{dx}.dx$$

$$= \frac{1}{a}\int \sin u\ du = \frac{-\cos u}{a} + c$$

$$= -\frac{1}{a}\cos ax + c$$

E E4) Proceeding exactly as in Example 8, fill up the blanks in the table below.

S.No.	f(x)	$\int$ f(x) dx
1.	sin ax	$-\frac{1}{a}\cos ax + c$
2.	cos ax	$\frac{1}{a}\sin ax + c$
3.	$\sec^2 ax$	....
4.	$\text{cosec}^2 ax$	....
5.	cosec ax cot ax	....
6.	$\sec^{ax}$ tanax	....
7.	e^{nx}	....
8.	a^{nx}	....

Example 9 Suppse we want to evaluate

i) $\int \cot x\, dx$, ii) $\int \tan x\, dx$ and iii) $\int \text{cosec} 2x\, dx$

i) We can write

$\int \cot x\, dx, = \frac{\cos x}{\sin x} dx$. Now since $\frac{d}{dx}\sin x = \cos x$, this integral falls in the category of case ii) mentioned earlier, and thus, $\int \cot x\, dx = \text{In}\,|\sin x| + c$

ii) To evaluate $\int \tan x\, dx$, we write

$$\int \tan x\ dx = \frac{\sec x \tan x}{\sec x}\, dx$$

$$= \text{In}|\sec x| + c, \text{ as } \frac{d}{dx}\sec x = \sec x\ \tan x$$

iii) To integrate cosec 2x we write

$$\int \text{cosec}\, 2x\ dx = \frac{1}{2}\int \frac{2\,\text{cosec}2x\,(\text{cosec}2x - \cot 2x)}{\text{cosec}2x - \cot 2x}\, dx$$

Here again, $\frac{d}{dx}(\text{cosec}2x - \cot 2x) = 2\,\text{cosec}2x(\text{cosec}2x - \cot 2x)$

This means $\int \text{cosec}2\ dx = \frac{1}{2}\ \ln\ |\text{cosec}2x - \cot 2x| + c$

In this example we have used some 'tricks' to put the integrand in some standard form. After you study various examples and try out a number of exercises, you will be able to decide on the particular substitution or the particular trick which will reduce the given integrand to one of the known forms. Let's look at the next example now.

Example 10 Let us evaluate $\int e^{\sin^2 x} \sin 2x \, dx$

If we put $\sin^2 x = u$ the $\frac{du}{dx} = 2 \sin x \cos x = \sin 2x$

Therefore, $\int e^{\sin^2 x} \sin 2x \, dx = \int e^u \, du$

$= e^u + c = e^{\sin^2 x} + c$

See if you can solve this exercise now.

E E5) Evaluate the following integrals

a) $\int \sec x \, dx$ b) $\int_0^{\pi/2} \sin^2 x \cos x \, dx$ c) $\int e^{\tan x} \sec^2 x \, dx$

2.3.2 Integrals using Trigonometric Formulas

In this section, we shall evaluate integrals with the help of the following trigonometric formulas:

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

$$\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

In each of these formulas you will find that on the left hand side we have either a power of a trigonometric function or a product of two trigonometric functions. And on the right hand side we have a sum (or difference) of two trigonometric functions. You will realise that the functions on the right hand side can be easily integrated by making suitable substitutions.

The following examples will illustrate how we make use of the above formulas in evaluating certain integrals.

Example 11 To evaluate $\int \cos^3 ax\ dx$. We write

$$\int \cos^3 ax\ dx = \int (\frac{3}{4}\cos ax + \frac{1}{4}\cos 3ax)dx$$

$$= \frac{3}{4}\int \cos ax\,dx + \frac{1}{4}\int \cos 3ax\ dx$$

$$= \frac{3}{4a}\sin ax + \frac{1}{12a}\sin 3ax + c \qquad \text{(see E4)}$$

Example 12 Let us evaluate i) $\int \sin 3x \cos 4x$ and ii) $\int \sin x \sin 2x \sin 3x\, dx$

Here the integrand is the form of a product of trigonometric functions. We shall write it as a sum of trigonometric functions so that it can be integrated easily.

i) $\int \sin 3x \cos 4x\, dx = \int \frac{1}{2}(\sin 7x - \sin x)\ dx$

$$\frac{1}{2} = (\sin 7x\ dx - \frac{1}{2}\int \sin x$$

$$= -\frac{1}{14}\cos 7x + \frac{1}{2}\cos x + c$$

ii) To evaluate $\int \sin 2x \cos 3x\, dx$, again we express the product $\sin x \sin 2x \sin 3x$ as a sum of trigonometric functions.

$$\sin x \sin 2x \sin 3x = \frac{1}{2}\sin x\ (\cos x - \cos 5x)$$

$$= \frac{1}{2}\sin x \cos x - \frac{1}{2}\sin x \cos 5x$$

$$= \frac{1}{4}\sin 2x - \frac{1}{4}(\sin 6x - \sin 4x)$$

Therefore, $\int \sin x \sin 2x \sin 3x \, dx$

$$= \frac{1}{4}\int \sin 2x dx + \frac{1}{4}\int \sin 4x \, dx - \frac{1}{4}\int \sin 6x \, dx$$

$$= -\frac{1}{8}\cos 2x - \frac{1}{16}\cos 4x + \frac{1}{24}\cos 6x + c$$

Try to do some exercises now. You will be able to solve them either by applying the trigonometic formulas mentioned in the beginning of this section or by using the method of substitution. Don't be scared by the number of integrals to be evaluated. The more integrals you evaluate, the more skilled you will become. You have to practise a lot to be able to decide on the best method to be applied for evaluating any given integral.

E6) Evaluate each of the following integrals.

a) i) $\int \sin^5 x \cos x \, dx$, ii) $\int \frac{\cos x}{\sin^3 x} dx$ iii) $\int_{\pi/6}^{\pi/3} \cot 2x \operatorname{cosec}^2 2x \, dx$

iv) $\int \sin 2\theta \, e^{\cos 2\theta} \, d\theta$ v) $\int_0^{\pi/2} \sin\theta(1 + \cos^4\theta) d\theta$

b) i) $\int (1+\cos\theta)^4 \sin\theta \, d\theta$ ii) $\int_0^{\pi/3} \frac{\sin^2\theta d\theta}{(1-5\tan\theta)^3}$

iii) $\int_0^{\pi/4} \sec\theta \tan\theta(1 + \sec\theta)^3 \, d\theta$

c) i) $\int \sin^4\theta \, d\theta$ ii) $\int \sin 3\theta \cos\theta \, d\theta$

iii) $\int_0^{\pi/2} \cos 5\theta \cos\theta d\theta$ iv) $\int_0^{\pi/2} \cos\theta \cos 2\theta \cos 4\theta \, d\theta$

E E 7) The cost of a transistor radio is Rs. 700/-. Its value is depreciating with time according to the formula $\frac{dv}{dt} = \frac{-500}{(1+t^2)}$ where Rs. v is its value t years after its purchase. What will be its value 3 years after its purchase? (Don't forget the constant of integration! Think how you can find it which the help of the given information.)

2.3.3 Trigonometric and Hyperbolic Substitution

Various trigonometric and hyperbolic identities like $\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$, $\tan h\,\theta \frac{\sinh\theta}{\cosh\theta}$ and so on, prove very useful while evaluating certain integrals. In this section we shall see how.

A trigonometric or hyperbolic substitution is generally used to integrate expressions involving $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, or $a^2 + x^2$. We suggest the following substitutions.

Expression involved	Substitution
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$ or $a \sinh\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$ or $a \cosh\theta$
$a^2 + x^2$	$x = a \tan\theta$

Thus to evaluate $\int \frac{dx}{\sqrt{a^2 - x^2}}$, put $x = a \sin\theta$. Then we know that $\frac{dx}{d\theta} = a \cos\theta$. This means we can write

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta \, d\theta}{\sqrt{a^2 - a^2 \sin^2\theta}}$$

$$= \int \frac{a\cos\theta \, d\theta}{a\cos\theta} = \int d\theta = \theta + c$$

$$= \sin^{-1}(x/a) + c$$

Similarly to evaluate $\int \frac{dx}{a^2 + x^2}$, we shall put $x = a \tan\theta$

Since $\frac{dx}{d\theta} = a \sec^2\theta \, d\theta$, we get

$$\int \frac{dx}{a^2 + x^2} = \int \frac{a \sec^2\theta d\theta}{a^2 + a^2 \tan^2\theta}$$

$$= \int \frac{a \sec^2 d\theta d\theta}{a^2 \sec^2\theta} = \frac{1}{a} \int d\theta = \frac{\theta}{a} + c$$

$$= \frac{1}{a} \tan^{-1}(x/a) + c$$

We can also evaluate $\int \frac{dx}{\sqrt{a^2 + x^2}}$, by substituting $x = a \tan\theta$

This gives $\frac{dx}{d\theta} = a \sec^2\theta$

$$\text{Thus, } \int \frac{dx}{\sqrt{a^2 + x^2}} = \int \frac{a \sec^2\theta d\theta}{\sqrt{a^2 + a^2 \tan^2\theta}} = \int \sec\theta d\theta$$

$$= \text{In}|\sec\theta + \tan\theta| + c$$

$$= \text{In}\left|\frac{x + \sqrt{a^2 + x^2}}{a}\right| + c$$

We can also evaluate this integral by putting $x = a\sinh\theta$. With this substitution we get,

$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}(x/a) + c$, and we know that

$\sinh^{-1}(x/a) = \text{In}. \frac{x + \sqrt{x^2 + a^2}}{a}$ (see Unit 5)

Similarly,

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}(x/a) + c,$$

$$= \text{In}. \frac{x + \sqrt{x^2 - a^2}}{a} + c$$

and $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}(x/a) + c$

Let us put these results in the form of a table

Table 3

S.No.	f(x)	$\int f(x)dx$
1.	$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}(x/a) + c$
2.	$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\tan^{-1}(x/a) + c$
3.	$\frac{a}{x\sqrt{x^2 - a^2}}$	$\frac{1}{a}\sec^{-1}(x/a) + c$
4.	$\frac{1}{\sqrt{a^2 + x^2}}$	$\ln\left\|\frac{x + \sqrt{x^2 + a^2}}{a}\right\| + c$ or $\sinh^{-1}(x/a) + c$
5.	$\frac{1}{\sqrt{x^2 - a^2}}$	$\ln\left\|\frac{x + \sqrt{x^2 - a^2}}{a}\right\| + c$ or $\cosh^{-1}(x/a) + c$

Sometimes the integrand does not seem to fall in any of the types mentioned in Table 3, but it is possible to modify or rearrange it so that it conforms to one of these types. We shall illustrate this through some examples.

Example 13 Suppose we want to evaluate $\int_1^2 \frac{dx}{\sqrt{2x - x^2}}$

Let us try to rearrange the terms in the integrand $\frac{1}{\sqrt{2x - x^2}}$ to suit us. You will see that

$$\int_1^2 \frac{dx}{\sqrt{2x - x^2}} = \int_1^2 \frac{dx}{\sqrt{1-(x-1)^2}}$$

If we put $x - 1 = v$, $\frac{dv}{dx} = 1$ and

$\int_1^2 \frac{dx}{\sqrt{2x - x^2}} = \int_0^1 \frac{dv}{\sqrt{1 - v^2}}$. Note the new limits of integration.

This integrals is finally in the form that we want and using the first formula in Table 3 we get

$$\int_1^2 \frac{dx}{\sqrt{2x - x^2}} = \sin^{-1} v \Big]_0^1$$

$$= \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Example 14 The integration in $\int \frac{x^2}{1 + x^6} dx$ does not again

fall into the types mentioned in Table 3. But let's see what we can do.

If we put $x^3 = u$, $\frac{du}{dx} = 3x^2$, Thus,

$x = \Rightarrow u = x^3 = 0$
and $x = 1 \Rightarrow u = 1$.

$$\int_0^1 \frac{x^2}{1 + x^6} dx = \frac{1}{3}\int_0^1 \frac{3x^2}{1 + x^6} dx$$

$$= \frac{1}{3}\int_0^1 \frac{1}{1 + u^2}\frac{du}{dx} dx$$

$$= \frac{1}{3}\int_0^1 \frac{1}{1 + u^2} du, \text{ by Theorem 3 } (\because u = 1 \text{ when } x = 1 \text{ and } u = 0 \text{ when } x = 0)$$

Here the integrand $\frac{1}{1+u^2}$ can be evaluated using formula 2 in Table 3.

Thus, we get $\frac{1}{3}\int_0^1 \frac{3x^2}{1+x^6}\,dx = \frac{1}{3}\tan^{-1}u\Big]_0^1$

$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{12}$

If you have followed this disscussion, you will certainly be able to solve this exercise.

E E8) Integrate each of the following with respect to the corresponding variables.

i) $\frac{1}{\sqrt{9-x^2}}$ ii) $\frac{1}{\sqrt{u^2-4}}$ iii) $\frac{1}{\sqrt{1+4x^2}}$ iv) $\frac{1}{\sqrt{2x^2+5}}$

v) $\frac{x}{\sqrt{x^4-1}}$ vi) $\frac{t^2}{\sqrt{t^6+16}}$ vii) $\frac{u^2}{\sqrt{4-u^6}}$

viii) $\frac{1}{\sqrt{2x-x^3}}$ ix) $\frac{1}{\sqrt{1+x+x^2}}$ x) $\frac{1}{y^2+6y+5}$

xi) $\frac{x^2}{1+x^2}$ **(Hint:** $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$)

2.3.4 Two Properties of Definite Integrals

We have already derived some properties of the definite integrals in Unit 11. These are the

i) Constant Function Property : $\int_a^b c\,dx = c(b-a)$

ii) Constant Multiple Property : $\int_a^b kf(x)dx = k\int_a^b f(x)\,dx$

iii) Interval Union Property : $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a \le c \le b$.

iv) Comparison Property : If if $c \le f(x) \le d \;\forall\; x \in [a,b]$,

then $c(b-a) \le \int_a^b f(x)\,dx \le d(b-a)$.

Now we shall use the method of substitution to derive two more properties to add to this list. Let's consider them one by one

v) $\int_0^a f(x)dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$ for any integrable function f.

We already know that

$$\int_0^a f(x)dx = \int_0^{a/2} f(x)\,dx + \int_{a/2}^a f(x)\,dx.$$

Now if we put $x = a-y$ in the second integral on the right hand side, then since

$\frac{dy}{dx} = -1$, we get

$$\int_{a/2}^a f(x)dx = \int_{a/2}^0 -f(a-y)\,dy = \int_0^{a/2} f(a-y)\,dy = \int_0^{a/2} f(a-x)dx$$

since x is a dummy variable.

Thus $\int_0^a f(x)dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$.

The usefulness of this property will be clear to you from the following example.

Example 15 Let us evaluate i) $\int_0^{\pi} \sin^4 x \cos^5 x \, dx$ and ii) $\int_0^{2\pi} \cos^3 x \, dx$

i) Using property v), we can write

$$\int_0^{\pi} \sin^2 x \cos^5 x \, dx = \int_0^{\pi/2} \sin^4 x \cos^5 x \, dx + \int_0^{\pi/2} \sin^4(\pi - x)\cos^5(\pi - x)dx$$

$$= \int_0^{\pi/2} \sin^4 x \cos^5 x \, dx + \int_0^{\pi/2} \sin^4 x(-\cos x)^5 dx$$

$\sin(\pi - x) = \sin x$, and $\cos(\pi - x) = -\cos$

$$= \int_0^{\pi/2} \sin^4 x \cos^5 x \, dx - \int_0^{\pi/2} \sin^4 x(\cos^5 x) \, dx$$

$$= 0$$

$$\text{ii)} \int_0^{2\pi} \cos^3 x \, dx = \int_0^{\pi} \cos^3 x \, dx + \int_0^{\pi} \cos^3 (2\pi - x \, dx)$$

$\cos(2\pi - x) = \cos x$

$$= \int_n^{\pi} \cos^3 x \, dx + \int_0^{\pi} \cos^3 - x \, dx$$

$$= 2\int_0^{\pi} \cos^3 x \, dx$$

$$= 2[\int_0^{\pi/2} \cos^3 x \, dx + \int_0^{\pi/2} \cos^3(\pi - x) \, dx]$$

$$= 2[\int_0^{\pi/2} \cos^3 x \, dx - \int_0^{\pi/2} \cos^3 x \, dx] = 0$$

Our next property greatly simplifies some integrals when the integrands are even or odd functions.

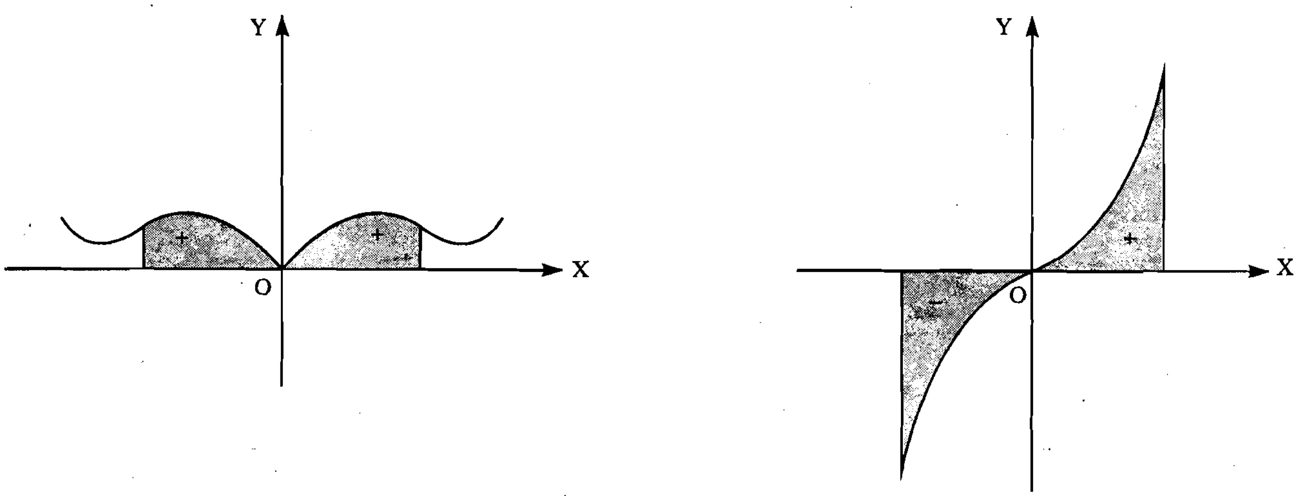
vi) If f is an even function of x, i.e, $f(-x) = f(x)$, then

$$\int_a^a f(x) \, dx = 2\int_0^a f(x) \, dx$$

and if f is an odd function, i.e., $f(-x) = -f(x)$, then

$$\int_a^a f(x) \, dx = 0$$

This is also obvious from Fig. 1(a) and (b)



(a) (b)

Fig. 1

We shall prove the result for even functions. The result for odd functions follows easily and is let to you as an exercise. (see E9)c)).

So let f be an even function of x in [-a,a], that is $f(-x) = f(x) \ \forall \ x \in [-a,a]$.

Then, $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

If we put $x = -y$ in the first integral on the right hand side, we get

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-y) \ -dy = \int_{0}^{a} f(y)\,dy = \int_{0}^{a} f(x)\,dx.$$

Thus $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

Using this property we can directly say that

$$\int_{-\pi/2}^{\pi/2} \sin x\,dx = 0 \quad \int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_{0}^{-\pi/2} \cos x\,dx = 2\sin x\Big]_{0}^{\pi/2} = 2$$

E E9) a) Evaluate

$$\int_{0}^{\pi} \sin^5 x \cos^3 x\,dx$$

b) Show that $\int_{0}^{\pi/2} \sin 2x \ \text{In}(\tan x)\,dx = 0$

c) Prove that $\int_{-a}^{a} f(x)\,dx = 0$ if f is an odd function of x.

In this section we have seen how the method of substitution enables us to substantially increase our list of integrable functions. (Here by "integrable function" we mean a function which we can integrate!) We shall discuss another technique in the next section.

2.4 INTEGRATION BY PARTS

In this section we shall evolve a method for evaluating integrals of the type

$\int u(x)v(x)dx$, in which the integrand $u(x)v(x)$ is the product of two functions. In other words, we shall first evolve the integral analogue of

$$\frac{d}{dx}[u(x)v(x)] = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x)$$

and then use that result to evaluate some standard integrals.

2.4.1 Integrals of a Product of Two Functions

We can calculate the derivative of the product of two functions by the formula

$$\frac{d}{dx}[u(x)v(x)] = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x)$$

Let us rewrite this as

$$u(x)\frac{d}{dx}v(x) = \frac{d}{dx}[u(x)v(x)] - v(x)\frac{d}{dx}u(x)$$

Integrating both the sides with respect to x, we have

$$\int u(x)\frac{d}{dx}(v(x))\,dx = \int \frac{d}{dx}(u(x)v(x))\,dx - \int v(x)\frac{d}{dx}(u(x)dx. \text{ or}$$

$$\int u(x)\frac{d}{dx}(v(x))\,dx = u(x)v(x) - \int v(x)\frac{d}{dx}(u(x))\,dx \qquad \text{...............(1)}$$

To express this in a more symmetrical form, we replace $u(x)$ by $f(x)$, and put

$\frac{d}{dx}v(x) = g(x)$. This means $v(x) = \int g(x)\,dx$.

As a result of this substitution, (1) takes the form

$$\int \mathbf{f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int \{f'(x)\int g(x)\,dx\}\,dx}$$

This formula may be read as:

The integral of the product of two functions = First factor × integral of second factor −integral of (derivative of first factor × integral of second factor)

It is called the **formula for integration by parts.** This formula may appear a little complicated to you. But the success of this method depends upon choosing the first factor in such a way that the second term on the right-hand side may be easy to evaluate. It is also essential to choose the second factor such that it can be easily integrated.

The following examples will show you the wide variety of integrals which can be evaluated by this technique. You should carefully study our choice of first and second functions in each example. You may also try to evaluate the integrals by reversing the order of functions. This will make you realise why we have chosen these functions the way we have.

Example 16 Let us use the method of integration by parts to evaluate $\int xe^x\,dx$.

In the integrand xe^x we choose x as the first factor and e^x as the second factor. Thus, we get

$$\int xe^x\,dx = x\int e^x\,dx - \{\frac{d}{dx}(x)\int e^x\,dx\}\,dx$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + c$$

Example 17 To evaluate $\int_0^{\pi/2} x^2 \cos x \, dx$, We shall take x^2 as the first factor and cos x as the second. Let us first evaluate the corresponding indefinite integral.

$$\int x^2 \cos x \, dx = x^2 \int \cos x \, dx - \int \{\frac{d}{dx}(x^2) \int \cos x dx\} \, dx$$
$$= x^2 \sin x - \int 2x \sin x \, dx$$
$$= x^2 \sin x - 2 \int x \sin x \, dx$$

We shall again use the formula of integration by parts to evaluate $\int x \sin x \, dx$. Thus

$$\int x \sin x \, dx = x(-\cos x) - \int (1)(-\cos x) \, dx \quad \text{as } (f(x) = x, g(x) = \sin(x)$$
$$= -x \cos x + \int \cos x \, dx$$
$$= -x \cos x + \sin x + c$$

Hence,

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin + c$$

Note that we have written the arbitrary constant as c instead of 2c

$$\text{Now } \int_0^{\pi/2} x^2 \cos x \, dx = (x^2 \sin x + 2x \cos x - 2 \sin x + c) \Big]_0^{\pi/2}$$
$$= \frac{\pi^2}{4} - 2$$

Example 18 Let us now evaluate $\int x \ln |x| dx$

Here we take ln |x| as the first factor since it can be differentiated easily, but cannot be integrated that easily. We shall take x to be the second factor.

$$\int x \ln |x| dx = \int \ln|x| \, x \, dx$$
$$= (\ln |x| \frac{x^2}{2} - \int \left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right) dx$$
$$= \frac{1}{2}x^2 \ln |x| - \frac{1}{2} \int x dx$$
$$= \frac{1}{2} x^2 \ln |x| - \frac{1}{4} x^2 + c$$

While choosing ln | x | as the first factor, we mentioned that it cannot be integrated easily. The method of integration by parts, in fact, helps us in integrating ln x too.

Example 19 We can find $\int \ln x \, dx$ by taking lnx as the first factor and 1 as the second factor.

Thus,

$$\int \ln x \, dx = \int (\ln x)(1) dx$$
$$= \ln x \int 1 dx - \int (\frac{1}{x} \int 1 \, dx) \, dx$$
$$= (\ln x)(x) - \int \frac{1}{x}(x) \, dx$$
$$= x \ln x - \int dx = x \ln x - x + c$$
$$= x \ln x - x \ln e + c \text{ since } \ln e = 1$$
$$= x \ln(x/e) + c$$

The trick used in Example 19, that is, considering 1 (unity) as the second factor, helps us to evaluate many integrals which could not be evaluated earlier.

You will be able to solve the following exercises by using the method of integration by parts.

E E 10) Evaluate

a) $\int x^2 \ln x \, dx$ Take $f(x) = \ln x$ and $g(x) = x^2$

b) $\int (1+x)e^x \, dx$ Take $f(x) = 1 + x$ and $g(x) = e^x$

c) $\int (1 + x^2)e^x \, dx$

d) $\int x^2 \sin x \cos x \, dx$ Take $f(x) = x^2$ and $g(x) = \sin x \cos x$

$$= \frac{1}{2} \sin 2x$$

E E11) Evaluate the following integrals by choosing 1 as the second factor.

a) $\int \sin^{-1} x \, dx$ b) $\int_0^1 \tan^{-1} x \, dx$ c) $\int \cot^{-1} x \, dx$

E E 12) Integrate

a) $x \sin^{-1} x$ b) $\ln(1+x^2)$ w.r.t. x.

2.4.2 Evaluations of $\int e^{ax} \sin bx$ and $\int e^{ax} \cos bx\, dx$

To evaluate $\int e^{ax} \sin bx\, dx$ and $\int e^{ax} \cos bx\, dx$, we use the formula for integration by parts.

$$\int e^{ax} \sin bx\, dx = (e^{ax})\left(-\frac{1}{b}\cos bx\right) - \int (ae^{ax})\left(-\frac{1}{b}\cos bx\right) dx$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx\, dx$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b}[e^{ax}) \left(\frac{a}{b}\sin bx\right) - \int e^{ax} \frac{a}{b} \sin bx\, dx$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx\, dx$$

Therefore, you will notice that the last integral on the right hand side is the same as the integral on the left hand side. Now we transfer the third term on the right to the left hand side, and obtain,

$$\left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \sin bx\, dx = e^{ax} \left(\frac{a}{b^2} \sin bx - \frac{1}{b} \cos bx\right)$$

This means,

$$\int e^{ax} \sin bx\, dx = \frac{1}{a^2 + b^2} e^{ax} (a \sin bx - b \cos bx) + c$$

We can similarly show that

$$\int e^{ax} \cos bx\, dx = \frac{1}{a^2 + b^2} e^{ax} (a \cos bx + b \sin bx) + c$$

If we put $a = r\cos\theta$, $b = r\sin\theta$, these formulas become

$$\int e^{ax} \sin bx\, dx = \frac{1}{\sqrt{a^2 + b^2}} e^{ax} \sin (bx - \theta) + c$$

$$\int e^{ax} \cos bx\, dx = \frac{1}{\sqrt{a^2 + b^2}} e^{ax} \cos(bx - \theta) + c, \text{ where } \theta = \tan^{-1} \frac{b}{a}.$$

Example 20 Using the formulas discussed in this sub-section, we can easily check that

i) $\int e^{x} \sin x\, dx = \frac{1}{\sqrt{2}} e^{x} \sin \left(x - \frac{\pi}{4}\right) + c$

and

ii) $\int e^{x} \cos \sqrt{3x}\, dx = \frac{1}{2} e^{x} \cos \left(\sqrt{3x} - \frac{\pi}{3}\right) + c$

Example 21 To evaluate $\int e^{2x} \sin x \cos 2x\, dx$, we shall first write

$\sin x \cos 2x = \frac{1}{2}(\sin 3x - \sin x)$ as in sec. 3.

Therefore,

$$\int e^{2x} \sin x \cos^{2x} dx = \frac{1}{2} \int e^{2x} \sin 3x\, dx - \frac{1}{2} \int e^{2x} \sin x\, dx$$

Now the two integrals on the right hand side can be evaluated. We see that

$$\int e^{2x} \sin 3x\, dx = \frac{1}{\sqrt{13}} e^{2x} \sin \left(3x - \tan^{-1} \frac{3}{2}\right) + c$$

and

$$\int e^{2x} \sin x\, dx = \frac{1}{\sqrt{5}} e^{2x} \sin \left(x - \tan^{-1} \frac{1}{2}\right) + c'$$

Hence

$$\int e^{2x} \sin x \cos 2x\, dx = e^{2x} \left[\frac{1}{\sqrt{13}} \sin \left(3x - \tan^{-1} \frac{3}{2}\right) - \frac{1}{\sqrt{5}} \sin \left(x - \tan^{-1} \frac{1}{2}\right)\right] + c$$

Example 22 Suppose we want to evaluate $\int x^3 \sin (a \ln x)\, dx$

Let $\ln x = u$. This implies $x = e^u$ and $du/dx = 1/x$

Then, $\int x^3 \sin (a\ln x)\, dx = \int x^4 \sin (a\ln x)\, (1/x)\, dx$

$$= \int e^{4u} \sin au\, du$$

$$= \frac{1}{\sqrt{16+a^2}} e^{4u} \sin (au - \tan^{-1} (a/4)) + c$$

$$= \frac{1}{\sqrt{16+a^2}} x^4 \sin (a \ln x - \tan^{-1} \frac{a}{4} + c$$

Why don't you try some exercises now.

E E 13) Evaluate the following integrals

a) $\int e^{2x} \cos 4x\, dx$

b) $\int e^{3x} \sin 3x\, dx$

c) $\int e^{4x} \cos x \cos 2x\, dx$

d) $\int e^{2x} \cos^2 x\, dx$

e) $\int \cosh ax \sin bx\, dx$ (write cosh ax in terms of the exponential function)

f) $\int x e^{ax} \sin bx\, dx$

2.4.3 Evaluation of $\int \sqrt{a^2 - x^2}\, dx$, $\int \sqrt{a^2 + x^2}\, dx$, and $\int \sqrt{x^2 - a^2}\, dx$

In this sub-section, we shall see that integrals like $\int \sqrt{a^2 - x^2}\, dx$, $\int \sqrt{a^2 + x^2}\, dx$,

and $\int \sqrt{x^2 - a^2}\, dx$ can also be evaluated with the help of the formula for integration by parts

and Table 3.

$$\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 + x^2}\,(1)\, dx$$

$$= \sqrt{a^2 - x^2} \times x - \int \left(\frac{-x}{\sqrt{a^2 - x^2}} \times x\right) dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{(a^2 - x^2)\, a^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - \int \sqrt{a^2 - x^2}\, dx$$

Shifting the last term on the right hand side to the left we get

$$2\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$

Using the formula,

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$, we obtain

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\, x\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$$

Similarly, we shall have,

$$\int \sqrt{a^2 + x^2}\, dx = \frac{1}{2}\, x\sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + c$$

$$= \frac{1}{2}\, x\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln \frac{x + \sqrt{a^2 + x^2}}{a} + c$$

and

$$\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2}\, x\sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{x}{a}\right) + c$$

$$= \frac{1}{2}\, x\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \frac{x + \sqrt{x^2 - a^2}}{a} + c$$

Example 23 Let us evaluate $\int_0^1 \sqrt{x + x^2}\, dx$

Now $\int_0^1 \sqrt{x + x^2}\, dx = \int_0^1 \sqrt{(x + 1/2)^2 - 1/4}\, dx$

Let $x + \frac{1}{2} = u$,

$$\int_0^1 \sqrt{x + x^2}\, dx = \int_{1/2}^{3/2} \sqrt{u^2 - 1/4}\, du$$

$$= \left\{\frac{1}{2} u\sqrt{u^2 - 1/4} - \frac{1}{8} \ln \frac{u + \sqrt{u^2 - 1/4}}{1/2}\right\}\Bigg]_{1/2}^{3/2}$$

$$= \frac{3\sqrt{2}}{4} - \frac{1}{8} \ln (3 + 2\sqrt{2})$$

Surely, you will be able to do these exercises now.

E 14) Verify that

a) $\int \sqrt{a^2 + x^2}\, dx = \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{a^2}{2} \ln \frac{x + \sqrt{a^2 + x^2}}{a} + c$

b) $\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \frac{x + \sqrt{x^2 - a^2}}{a} + c$

In the next sub-section we shall consider another type of integrand which occurs quite frequently in mathematics.

2.4.4 Integrals of the Type $\int e^x [f(x) + f'(x)]\, dx$

We first prove the formula $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$ and see how it can be used in integrating some functions.

By the formula for integration by parts

$$\int e^x f(x)\, dx = \int f(x) e^x\, dx$$

$$= f(x) e^x - \int f'(x)\, e^x\, dx + c$$

This implies

$$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$$

Example 24 Let us evaluate the following integrals.

i) $\displaystyle\int \frac{1+x}{(2+x)^2}\, e^x\, dx$ (ii) $\displaystyle\int \frac{\sqrt{1-\sin x}}{1+\cos x}\, e^{-x/2}\, dx$

We take up (i) first,

$$\int \frac{1+x}{(2+x)^2} e^x\, dx = \int \frac{(2+x)-1}{(2+x)^2} e^x\, dx$$

$$= \int \left[\frac{1}{2+x} + \frac{-1}{(2+x)^2} \right] e^x\, dx$$

$$= \frac{1}{2+x}\, e^x + c, \text{ since } \frac{-1}{(2+x)^2} = \frac{d}{dx}\left(\frac{1}{2+x}\right)$$

Now we shall evaluate ii)

ii) $\displaystyle\int \frac{\sqrt{1-\sin x}}{1+\cos x}\, e^{-x/2}\, dx$

$$= \int \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{2\cos^2\frac{x}{2}}\, e^{-x/2}\, dx$$

$$= \frac{1}{2}\int \sec\frac{x}{2}\, e^{-x/2}\, dx - \frac{1}{2}\int \tan\frac{x}{2}\sec\frac{x}{2}\, e^{-x/2}\, dx$$

Now

$$\int \sec\frac{x}{2}\, e^{-x/2}\, dx = \left(\sec\frac{x}{2}\right)\left(-2e^{-x/2}\right) - \int \left(\frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2}\right)\left(-2e^{-x/2}\right) dx$$

$$= -2\sec\frac{x}{2}\, e^{-x/2} + \int \sec\frac{x}{2}\tan\frac{x}{2}\, e^{-x/2}\, dx$$

Thus,

$$\int \frac{\sqrt{1-\sin x}}{1+\cos x}\, e^{-x/2}\, dx$$

$$= -\sec\frac{x}{2} e^{-x/2} + \frac{1}{2}\int \sec\frac{x}{2}\tan\frac{x}{2}\, dx - \frac{1}{2}\int \sec\frac{x}{2}\tan\frac{x}{2}\, e^{-x/2}\, dx$$

$$= -\sec\frac{x}{2} e^{-x/2} + c$$

In this unit we have exposed you to various methods of integration. You have also had a fair amount of practice in using these methods. We are now giving you some additional exercises. You may like to try your hand at these too. To solve these you will have to first identify the method which will suit the particular integrand the best. This is the crucial step. The next step where you apply the chosen method to get the answer is relatively easy. If you have studied this unit thoroughly, neither of these steps should pose any problem. So good luck!

E E 15) Evaluate the following integrals :

a) $\displaystyle\int (2x^3+2x+3)\,dx$ b) $\displaystyle\int \frac{x^2+2}{x}\, dx$

c) $\displaystyle\int \sinh(x/2)\cosh(x/2)\, dx$

d) $\displaystyle\int (e^x - e^{-x})^2\, dx$ e) $\displaystyle\int_2^4 \frac{x^2}{\sqrt{x^3+1}}\, dx$

f) $\displaystyle\int \frac{x}{(x^2+2)^8}\, dx$ g) $\displaystyle\int \sin x e^{\cos x}\, dx$

h) $\displaystyle\int \frac{1}{1+9x^2}\, dx$ i) $\displaystyle\int_0^{\pi/2} \frac{\sin x\cos x}{(1+\sin x)^3}\, dx$

j) $\int (x^2 + 2)^6 x^3 \, dx$

k) $\int x \sqrt{x^4 + 2x^2 + 2} \, dx$

l) $\int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} \, dx$

m) $\int \cos^{-1} (\frac{1-x^2}{1+x^2}) \, dx$

n) $\int e^x (\ln \sin x + \cot x) \, dx$

E E16) Prove that

$$\int u \frac{d^2v}{dx^2}\,dx = u\frac{dv}{dx} - v\frac{du}{dx} + \int v \frac{d^2u}{dx^2}\,dx,$$

and use it to evaluate $\int x^3 \sin x\,dx$

Before we end this unit, here are some general remarks about the existence of integrals.

The result

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

where F(x) is the antiderivative of f(x), will make sense only if f(x) exists at every point of the interval. Hence we have to be careful in using this result.

Thus,

$$\int \frac{1}{2}\,dx = [\ln|x|]_a^b = \ln\frac{|b|}{|a|}$$

But 1/x is not defined at x = 0, and ln | x | is also not differentiable at x = 0. As such, at this stage, we should use the result only if the inrerval [a,b] does not include x = 0

Thus, $\int_{-1}^{2} \frac{1}{x}\,dx = \ln\frac{|2|}{|-1|} = \ln 2$ is **not** valid.

$\int_{-2}^{-1} \frac{1}{x}\,dx = \ln\frac{|-1|}{|-2|} = \ln\frac{1}{2}$ is valid.

Again, consider

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = [\sin^{-1}x]_0^1 = \frac{\pi}{2}.$$

We have used this in Example 3. However $\frac{1}{\sqrt{1-x^2}}$ does not exist at x = 1, and $\sin^{-1}x$ is not differentiable at x = 1. $L(\sin^{-1}x)'$ exists at x = 1, but $R(\sin^{-1}x)'$ does not exist, since $\sin^{-1}x$ itself does not exist when x > 1.

However, the above result is true in some sense. This sense will be clear to you in your course on analysis.

The antiderivative of every function need not exist, i.e., it need not be any of the functions we are familiar with. For example, there is no function known to us whose derivative is e^{-x^2}

However, the value of the definite integral $\int_a^b f(x)\,dx$ of every function, where f(x) is continuous on the interval [a,b], can be found out by numerical methods to any degree of approximation. You can study these methods in detail if you take the course on numerical analysis. You will study two simple numerical methods in Block 4 too. Thus, we cannot find the antiderivative of e^{-x^2}, but still, we can find the approximate value of

$\int_a^b e^{-x^2}\,dx$, for all real values of a and b. In fact, this integral is very important in probability theory and you will use it very often if you take the course on probability and statistics.

That brings us to the end of this unit. Let us summarise what we have studied so far.

3.5 SUMMARY

In this unit we have covered the following points.

1) If F(x) is an antiderivative of f(x), then the indefinite integral (or simply, integral) of f(x) is

   $\int f(x)\,dx = F(x) + c$, where c is an arbitrary constant.

2) $\int [k_1 f_1(x) + k_2 f_2(x) + \ldots\ldots + k_n f_n(x)]dx =$

$$k_1 \int f_1(x)dx + k_2 \int f_2(x)\,dx + \dots + k_n \int f_n(x)\,dx$$

3) The method of substitution gives :

$$\int_a^b f[g(x)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du, \text{ if } u = g(x).$$

In particular,

$$\int [f(x)^n f'(x)\,dx = \frac{[f(x)^{n+1}}{n+1} + c, \; n \neq -1, \text{ and}$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f \text{ is even} \\ 0, & \text{if } f \text{ is odd.} \end{cases}$$

4) Standard formulas :

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\frac{x + \sqrt{a^2 + x^2}}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\frac{x + \sqrt{x^2 - a^2}}{a} + c$$

5) Integration of a product of two functions (integration by parts):

$$\int u(x)v(x)dx = u(x)\int v(x)dx - \int \{u'(x)\} \int v(x)dx\}\,dx$$

This leads us to :

$$\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$\int \sqrt{a^2 + x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\ln\frac{x + \sqrt{a^2 + x^2}}{a} + c$$

$$\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\frac{x + \sqrt{x^2 - a^2}}{a} + c$$

$$\int e^{ax}\sin bx\,dx = \frac{1}{\sqrt{a^2 + b^2}}e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

$$\int e^{ax}\cos bx\,dx = \frac{1}{\sqrt{a^2 + b^2}}e^{ax}\cos\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

$$\int e^x[f(x) + f'(x)]dx = e^x f(x) + c$$

3.6 SOLUTIONS AND ANSWERS

E1) a) i) $\frac{x^5}{5} + c$ ii) $-2x^{-1/2} + c$ iii) $-4x^{-1} + c$ iv) $3x + c$

b) i) $x - x^2 + \frac{x^3}{3} + c$ ii) $\frac{x^3}{3} - 2x - \frac{1}{x} + c$

iii) $x + \frac{3x^2}{2} + x^3 + \frac{x^4}{4} + c$

c) i) $e^x - e^{-x} + 4x + c$ ii) $4\sin x + 3\cos x + e^x + \frac{x^2}{2} + c$

iii) $4\tanh x + e^x - 4x^2 + c$

d) i) $2\sin^{-1}x + 5\ln|x| + c$

ii) $\int \frac{2(x^2+1)+3}{x^2+1}dx = 2\int dx + 3\int \frac{1}{x^2+1}dx$

$= 2x + 3\tan^{-1}x + c$

e) i) $\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx + c$ ii) $\frac{x^2}{2} - 2x + \ln|x| + c$

f) i) $\int \frac{\sin^4 x + \cos^4 x}{\sin^2 x\cos^2 x}dx$

$= \int \frac{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}$

$= \int \frac{1}{\sin^2 x}dx + \int \frac{1}{\cos^2 x}dx - 2\int dx$

$= -\cot x + \tan x - 2x + c$

iii) $6x + \frac{3x^2}{2} - \frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2} + c$

E2) a) i) $\frac{6^5}{5} - 5^4$ ii) $\frac{1}{2} + \ln 2$

b) i) $\frac{275}{12}$ ii) $\frac{15}{4}$

E3) a) $\int (5x-3)^{1/2}dx = \frac{1}{5}\int 5(5x-3)^{1/2}dx$ if $5x-3 = u, \frac{du}{dx} = 5$

$= \frac{1}{5}\int u^{1/2}du = \frac{1}{5}\frac{u^{3/2}}{3/2} + c = \frac{2}{15}(5x-3)^{3/2} + c$

b) $\frac{1}{14}(2x+1)^7 + c$ c) $\frac{1}{5}\ln\frac{19}{9}$ d) $\frac{1}{2}\ln|10x+7| + c$

e) $\frac{1}{2}\ln|x^2+2x+7| + c$ f) $\ln|x^3+x^2+x-8|\Big]_2^3 = \ln\frac{31}{6}$

g) $\frac{(3/4)(x^{4/3}-1)^{3/2}}{3/2} + c = \frac{1}{2}(x^{4/3}-1)^{3/2} + c$ h) $-\frac{1}{3}\sqrt{1-3x^2} + c$

E4)

S. No.	f(x)	$\int f(x)\,dx$
1.	sin ax	$-\frac{1}{a}\cos ax + c$
2.	cos ax	$\frac{1}{a}\sin ax + c$
3.	$\sec^2 ax$	$\frac{1}{a}\tan ax + c$
4.	$\mathrm{cosec}^2\, ax$	$-\frac{1}{a}\cot ax + c$
5.	sec ax tan ax	$\frac{1}{a}\sec ax + c$
6.	cosec ax cot ax	$-\frac{1}{a}\mathrm{cosec}\, ax + c$
7.	e^{ax}	$\frac{1}{a}e^{ax} + c$
8.	a^{nx}	$\frac{1}{m}\frac{a^{mx}}{\ln a} + c$

E5) a) $\int \sec x \, dx + \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \ln|\sec x + \tan x| + c$

b) $\int_0^{\pi/2} \sin^2 x \cos x dx = \left.\frac{\sin^3 x}{3}\right]_0^{\pi/2} = \frac{1}{3}$

c) if $u = \tan x$, $\frac{du}{dx} = \sec^2 x$

$\Rightarrow \int e^{\tan x} \sec^2 x \, dx = \int e^u \, du = e^u + c = e^{\tan x} + c$

E6) a) i) $\frac{\sin^6 x}{6} + c$ ii) $\frac{-2}{\sin^2 x} + c$

iii) $\int_{\pi/6}^{\pi/3} \cot 2x \operatorname{cosec}^2 2x \, dx = \frac{1}{2} \int_{\pi/6}^{\pi/3} \cot 2x \, (2 \operatorname{cosec}^2 2x) \, dx$

$= \frac{1}{2} \times \left.\frac{\cot^2 2x}{2}\right]_{\pi/6}^{\pi/3} = 0$

iv) Put $\cos 2\theta = u$. Then $\frac{du}{d\theta} = -2\sin 2\theta$

$\int \sin 2\theta \, e^{\cos 2\theta} \, d\theta = -\frac{1}{2} e^u \, du = -\frac{1}{2} e^u + c$

$= -\frac{1}{2} e^{\cos 2\theta} + c$

v) $\int_0^{\pi/2} \sin\theta \, (1 + \cos^4\theta) \, d\theta = \int_0^{\pi/2} \sin\theta \, d\theta + \int_0^{\pi/2} \sin\theta \cos^4\theta \, d\theta$

$= -\cos\theta\Big]_0^{\pi/2} - \left.\frac{\cos^5\theta}{5}\right]_0^{\pi/2}$

$= 1 + \frac{1}{5} = \frac{6}{5}$

b) i) $-\frac{(1+\cos\theta)^5}{5} + c.$

ii) $\left.\frac{1}{10} \frac{1}{(1-5\tan\theta)^2}\right]_0^{\pi/3}$

$= \frac{1}{2} \frac{2\sqrt{3}-15}{(1-5\sqrt{3})^2}$

iii) $\left.\frac{(1+\sec\theta)^4}{4}\right]_0^{\pi/4} = \frac{(1+\sqrt{2})^4 - 2^4}{4} = \frac{1+12\sqrt{2}}{4}$

c) i) $\int \sin^4\theta \, d\theta = \int \sin^3\theta \sin\theta \, d\theta$

$= \int (\frac{3}{4}\sin^2\theta - \frac{1}{4}\sin\theta \sin 3\theta) \, d\theta$

$= \frac{3}{8} \int \{1 - (1 - 2\sin^2\theta)\} \, d\theta - \frac{1}{8} \int (\cos 2\theta - \cos 4\theta) \, d\theta$

$= \frac{3}{8} \int d\theta - \frac{3}{8} \int \cos 2\theta \, d\theta - \frac{1}{8} \int \cos 2\theta \, d\theta + \frac{1}{8} \int \cos 4\theta \, d\theta$

$= \frac{3}{8}\theta - \frac{1}{2}\frac{\sin 2\theta}{2} + \frac{1}{8}\frac{\sin 4\theta}{4} + c$

$= \frac{1}{4}(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta) + c$

ii) $\int \sin 3\theta \cos\theta \, d\theta = \frac{1}{2}[\int \sin 4\theta \, d\theta + \int \sin 2\theta \, d\theta]$

$$= \frac{1}{4}\left[\frac{-\cos 4\theta}{2} - \cos 2\theta + c\right]$$

iii) $\displaystyle\int_0^{\pi/2} \cos 5\theta \cos\theta \, d\theta = \left.\frac{\sin 4\theta}{8} - \frac{\sin 6\theta}{12}\right]_0^{\pi/2} = 0$

iv) $\displaystyle\int_0^{\pi/2} \cos\theta \cos 2\theta \cos 4\theta \, d\theta = \frac{19}{105}$

ii) $\displaystyle\int \frac{dx}{\sqrt{1 + x + x^2}} = \int \frac{dx}{\sqrt{(3/4 + (x+1/2)^2}} = \sinh^{-1}\left(\frac{x+(1/2)}{\sqrt{3}/2}\right) + c$

$$\text{or} \quad \ln\left|\frac{(x+1/2) + \sqrt{3/4 + (x+1/2)^2}}{\sqrt{3}/2}\right| + c$$

$$= \ln\left|\frac{(x+1/2) + \sqrt{x^2 + x + 1}}{\sqrt{3}/2}\right| + c$$

$$= \ln\left|\frac{2x + 1 + 2\sqrt{x^2 + x + 1}}{\sqrt{3}}\right| + c$$

iii) $\displaystyle\int \frac{dy}{\sqrt{y^2 + 6y + 5}} = \int \frac{dy}{\sqrt{(y+3)^2 - 4}} = \cosh^{-1}\left(\frac{y+3}{2}\right) + c$

iv) $\displaystyle\int \frac{x^2}{1+x^2} dx = \int dx - \int \frac{1}{1+x^2} dx$

$$= x - \tan^{-1}x + c$$

E7) $\displaystyle v = \int \frac{-500}{1+t^2} dt + c$

$$= -500 \tan^{-1} t + c$$

$v(0) = 700 = -500 \tan^{-1}0 + c = c$

$\Rightarrow c = 700$

$v(3) = 700 - 500 \tan^{-1} 3.$

E8) For solution, see P. 104

E9) a) $\displaystyle\int_0^{\pi} \sin^5 x \cos^3 x \, dx = \int_0^{\pi/2} \sin^5 x \cos^3 x \, dx + \int_0^{\pi/2} \sin^5(\pi - x)\cos^3(\pi - x) \, dx$

$$= \int_0^{\pi/2} \sin^5 x \cos^3 x \, dx - \int_0^{\pi/2} \sin^5 x \cos^3 x \, dx = 0$$

b) $\displaystyle\int_0^{\pi/2} \sin 2x \ln \tan x \, dx = \int_0^{\pi/4} \sin 2x \ln \tan x \, dx$

$$= \int_0^{\pi/4} \sin 2\left(\frac{\pi}{2} - x\right) \ln \tan\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/4} \sin 2x \ln \tan x \, dx + \int_0^{\pi/4} \sin 2x \ln \cot x \, dx$$

$$= \int_0^{\pi/4} \sin 2x \ln (\tan x \cot x) \, dx$$

$$= \int_0^{\pi/4} \sin 2x \ln 1 dx = 0$$

c) $\displaystyle\int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_0^{a} f(x) \, dx$

Put $x = -y$ in $\displaystyle\int_{-a}^{0} f(x) \, dx$

$$\int_{-a}^{a} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0$$

E10 a) $\int x^2 \ln x\,dz = \ln x \int x^2\,dx - \int (\frac{1}{x}\int x^2\,dx)\,dx$

$$= \ln x\,\frac{x^3}{3} - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$$

b) $xe^x + c$

c) $\int (1+x^2)\,e^x\,dx = (1+x^2)\,e^x - 2\int xe^x\,dx = (1+x^2)e^x - 2[xe^x - \int e^x\,dx]$

$$= (1+x^2)\,e^x - 2xe^x + 2e^x + c$$

$$= e^x\,(x^2 - 2x + 3) + c$$

d) $\frac{1}{4}[-x^2\cos 2x + x\sin 2x + \frac{1}{2}\cos 2x] + c$

E11) a) $\int 1.\sin^{-1}x\,dx = \sin^{-1}x.x - \int \frac{1}{\sqrt{1-x^2}}\,x\,dx$

$$= x\sin^{-1}x + \sqrt{1-x^2} + c$$

b) $\frac{\pi}{4} - \frac{1}{2}\ln 2$

c) $x\cot^{-1}x + \frac{1}{2}\ln(1+x^2)$

E12) a) $\int x\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

Put $x = \sin u$ in $\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2 u}{\cos u}\cos u\,du$

$$= \int \sin^2 u\,du = \int \frac{1-\cos 2u}{2}\,du$$

$$= \frac{1}{2}u - \frac{1}{4}\sin 2u + c = \frac{1}{2}u - \frac{1}{2}\sin u\cos u + c$$

$$= \frac{1}{2}[\sin^{-1}x - x\cos(\sin^{-1}x)] + c$$

$$\therefore \int x\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}[\sin^{-1}x - x\sqrt{1-x^2}] + c$$

b) $\int \ln(1+x^2)\,dx = \int 1.\ln(1+x^2)\,dx$

$$= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$

$$= x\ln(1+x^2) - \int 2[1 - \frac{1}{1+x^2}]\,dx$$

$$= x\ln(1+x^2) - 2[x - \tan^{-1}x] + c$$

E13 a) $\frac{1}{20}e^{2x}(2\cos 4x + 4\sin 4x) + c$

b) $\frac{1}{18}e^{3x}(3\sin 3x - 3\cos 3x) + c$

c) $\int e^{4x}\cos x\cos 2x\,dx = \frac{1}{2}\int e^{4x}(\cos 3x + \cos x)\,dx$

$$= \frac{1}{2}[\int e^{4x}\cos 3x\,dx + \int e^{4x}\cos x\,dx] -$$

$$= \frac{1}{2}\left[\frac{1}{25} e^{4x} (4\cos 3x + 3\sin 3x) + \frac{1}{17} e^{4x} (4\cos x + \sin x)\right] + c$$

d) $\int e^{2x} \cos^2 x \, dx = \int e^{2x} \left(\frac{\cos 2x + 1}{2}\right) dx$

$$= \frac{1}{2}\left[\int e^{2x} \cos 2x \, dx + \int e^{2x} \, dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{8} e^{2x} (2\cos 2x + 2\sin 2x) + \frac{1}{2} e^{2x}\right] + c$$

e) $\int \cosh ax \sin bx \, dx = \int \left(\frac{e^{ax} + e^{-ax}}{2}\right) \sin bx \, dx$

$$= \frac{1}{2}\left[\int e^{ax} \sin bx \, dx + \int e^{-ax} \sin bx \, dx\right]$$

$$= \frac{1}{2} \frac{1}{(a^2 + b^2)} \left[e^{ax} (a\sin bx - b\cos bx) + e^{-ax} (-a\sin bx - b\cos bx)\right] + c$$

f) $\int xe^{ax} \sin bx \, dx = x \int e^{ax} \sin bx \, dx$

$$-\int \frac{1}{a^2 + b^2} e^{ax} (a\sin bx - b\cos bx) \, dx$$

$$= \frac{x}{a^2 + b^2} e^{ax} (a\sin bx - b\cos bx)$$

$$- \frac{1}{(a^2 + b^2)^2} [ae^{ax} (a\sin bx - b\cos bx) -$$

$$be^{ax} (a\cos bx + b\sin bx)] + c$$

E14 a) $\int \sqrt{a^2 + x^2} \, dx = x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}} dx$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2}{\sqrt{a^2 + x^2}} dx + \int \frac{a^2}{\sqrt{a^2 + x^2}} dx$$

$$= x\sqrt{a^2 + x^2} + a^2 \ln \frac{x + \sqrt{a^2 + x^2}}{a} - \int \sqrt{a^2 + x^2} \, dx + c$$

$$\therefore \int \sqrt{a^2 + x^2} \, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln \frac{x + \sqrt{a^2 + x^2}}{a} + c$$

b) $x\int \sqrt{x^2 - a^2} \, dx = x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} dx$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} \, dx - \int \frac{a^2}{\sqrt{x^2 - a^2}} dx$$

$$\therefore \int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \frac{x + \sqrt{x^2 - a^2}}{a} + c$$

E15 a) $\frac{x^4}{2} + x^2 + 3x + c$

b) $\frac{x^2}{2} + 2\ln|x| + c$

c) $\frac{1}{2} \cosh x + c$

d) $\frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2} + c$

e) $\frac{2}{3} \sqrt{x^3 + 1} \Big]_2^4 = \frac{2}{3} (\sqrt{65} - \sqrt{9})$

f) $\frac{-1}{2} \frac{(x^2 + 2)^{-7}}{7} + c$

g) $-e^{\cos x} + c$

h) $\frac{1}{3}\tan^{-1}(3x) + c$

i) $$\int_0^{\pi/2} \frac{\sin x \cos x}{(1+\sin x)^3}\,dx = \int_0^1 \frac{t}{(1+t)^3}\,dt = \int_0^1 \frac{1+t-1}{(1+t)^3}$$

$$= \int_0^1 \frac{dt}{(1+t)^2} - \int_0^1 \frac{dt}{(1+t)^3}$$

$$= \frac{-1}{1+t} + \frac{1}{2(1+t)^2}\Bigg]_0^1 = \frac{1}{8}$$

j) $$\int (x^2+2)^6 x^3\,dx = \frac{1}{2}\int t^6 (t-2)\,dt$$

$$= \frac{1}{2}\left[\int t^7\,dt - 2\int t^6\,dt\right]$$

$$= \frac{t^8}{16} - \frac{t^7}{7} + c$$

$$= \frac{(x^2+2)^8}{16} - \frac{(x^2+2)^7}{7} + c$$

k) $$\int x\sqrt{x^4+2x^2+2}\,dx = \int x\sqrt{(x^2+1)^2+1}\,dx = \frac{1}{2}\int\sqrt{t^2+1}\,dt$$

$$= \frac{1}{4}t\sqrt{1+t^2} + \frac{1}{4}\sinh^{-1} t + c$$

$$= \frac{1}{4}(x^2+1)\sqrt{x^4+2x^2+2} + \frac{1}{4}\sinh^{-1}(x^2+1) + c$$

l) $$\int \frac{x\tan^{-1} x}{(1+x^2)^{3/2}}\,dx = \int \theta\sin\theta\,d\theta, \text{ if } x = \tan\theta$$

$$= -\theta\cos\theta + \int\cos\theta\,d\theta \quad \text{(inegration by parts)}$$

$$= -\theta\cos\theta + \sin\theta + c$$

m) Put $x = \tan\theta$ in $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$

Answer $= 2[\theta\tan\theta + \ln|\cos\theta|] + c$ where $\theta = \tan^{-1} x$

n) $$\int e^x(\ln\sin x + \cot x)\,dx = \int e^x \ln\sin x\,dx + \int e^x\cot x\,dx$$

$$= \ln\sin x\, e^x - \int \cot x\, e^x\,dx + \int e^x\cot x\,dx$$

$$= e^x\ln\sin x.$$

E16) $$\int u\frac{d^2v}{dx^2}dx = u\frac{dv}{dx} - \int \frac{du}{dx}, \frac{dv}{dx}\,dx$$

$$= u\frac{dv}{dx} - v\frac{du}{dx} + \int v\frac{d^2v}{dx^2}\,dx$$

$$\int x^3\sin x\,dx = \int x^3\frac{d^2}{dx^2}(-\sin x)\,dx$$

$$= -x^3\cos x + 3x^2\sin x - 6\int x\sin x\,dx$$

$$= -x^3\cos x + 3x^2\sin x - 6\left[-x\cos x + \int \cos x\,dx\right]$$

$$= -x^3\cos x + 3x^2\sin x + 6(x\cos x - \sin x) + c$$