
UNIT 3 CONES AND CYLINDERS

Structure

3.1 Introduction

Objectives

3.2 Cones

3.3 Tangent Planes

3.4 Cylinders

3.5 Summary

3.6 Solutions/Answers

3.1 INTRODUCTION

In the previous unit we discussed a very commonly found three-dimensional object. In this unit we look at two more commonly found three-dimensional objects, namely, a cone and a cylinder. But, what you will see in this unit may surprise you — what people usually call a cone or a cylinder are only portions of very particular cases of what mathematicians refer to as a cone or a cylinder.

We shall start our discussion on cones by defining them, and deriving their equations. Then we shall concentrate on cones whose vertices are the origin. In particular, we will obtain the tangent planes of such cones.

The other surface that we will discuss in this unit is a cylinder. We shall define a general cylinder, and then focus on a right circular cylinder.

The contents in this unit are of mathematical interest, of course. But, they are also of interest to astronomers, physicists, engineers and architects, among others. This is because of the many applications that cones and cylinders have in various fields of science and engineering.

The surfaces that you will study in this unit are particular cases of conicoids, which you will study in the next block. So if you go through this unit carefully and ensure that you achieve the following objectives, you will find the next block easier to understand.

Objectives

After studying this unit you should be able to :

- obtain the equation of a cone if you know its vertex and base curve;
- prove and use the fact that a second degree equation in 3 variables represents a cone with vertex at the origin if it is homogenous;
- obtain the tangent planes to a cone;
- obtain the equation of a right circular cylinder if you know its axis and base curve.

Let us now see what is cone is.

3.2 CONES

When you see an ice-cream cone, do you ever think that it is a set of lines? That is exactly what it is, as you will see in this section.

Definition: A cone is a set of lines that intersect a given curve and pass through a fixed point which is not in the plane of the curve. The fixed point is called the **vertex** of the cone, and the curve is called the **base curve** (or **directrix**) of the cone.

Each line that makes up a cone is called a **generator** of the cone.

Thus, we can also define a cone in the following way.

Definition : A cone is a surface generated by a line that intersects a given curve and passes through a fixed point which doesn't lie in the plane of the curve.

For example, in Fig. 1 (a), we give the cone generated by a line passing through the point P, and intersecting the circle C. The base curves of the cones in Fig. 1 (b) and Fig. 1 (c) are an ellipse and a parabola, respectively.

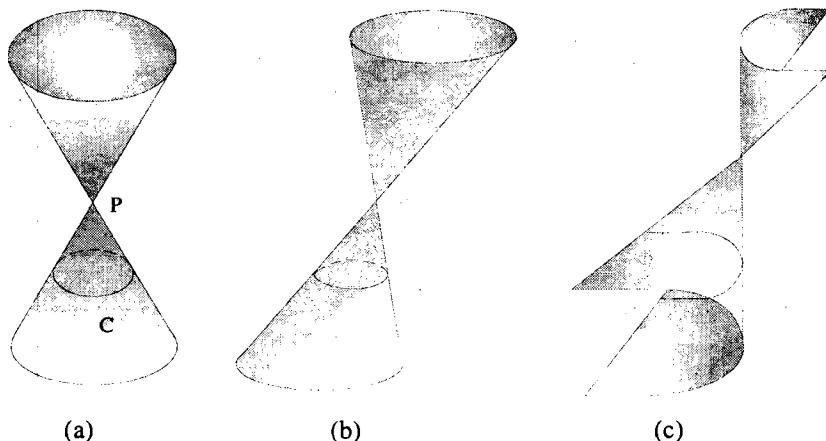


Fig. 1 : (a) A circular cone, (b) an elliptic cone, (c) a parabolic cone.

At this point we would like to make an important remark.

- **Remark 1 :** A cone is a set of lines passing through its vertex and base curve. Thus, it extends **beyond** the vertex and the base curve. So the cones in our diagrams are only a portion of the actual cones.

Now let us introduce some new terms.

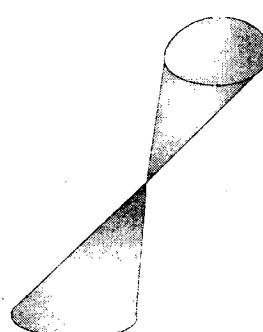


Fig. 2: A circular cone which is not right circular.

Definitions : A cone whose base curve is a circle is called a **circular cone**. The line joining the vertex of a circular cone to the centre of its base curve is called the **axis** of the cone. If the axis of a circular cone is perpendicular to the plane of the base curve, then the cone is called a **right circular cone**.

Thus, the cone in Fig. 1 (a) is a right circular cone, while the one in Fig. 2 is not.

Cones were given great importance by the ancient Greeks who were studying the problem of doubling the cube. A teacher of Alexander the Great, Manaechmus, is supposed to have geometrically proved the following result. This result is the reason for the continuing importance of cones.

Theorem 1 : Every planar section of a cone is a conic.

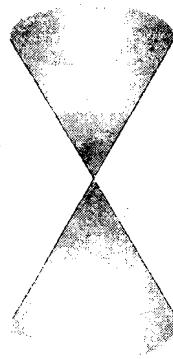


Fig. 3 : A pair of intersecting lines is a conic section.

This theorem is the reason for an ellipse, parabola or hyperbola to be called a conic section (see Fig. 1 of Unit 3). It was proved by the Greek astronomer Appolonius (approximately 200 B.C.). We will not give the proof here.

Now, according to Theorem 1 would you call a pair of intersecting lines a conic? If you cut a right circular cone by a plane that contains its axis, what will be the resulting curve be? See Fig. 3.

Let us now see how we can represent a cone algebraically. We shall first talk about a right circular cone, which we shall refer to as an r.c. cone.

So, let us take an r.c. cone. Let us assume that its vertex is at the origin, and its axis is the z-axis (see Fig. 4). Then the base curve, which is a circle of radius r (say), lies in a plane that is parallel to the XY-plane. Let this plane be $z = k$, where k is a constant. Then, any generator will intersect this curve in a point (a, b, k) , for some $a, b \in \mathbb{R}$. So the angle between the generator

and the axis of the cone will $\theta = \tan^{-1} \left(\frac{r}{k} \right)$, which is a constant. This is true for any generator of the cone.

Thus, every line that makes up the cone makes a fixed angle θ with the axis of the cone. This angle is called the semi-vertical angle (or generating angle) of the cone.

We can now define an r.c. cone in the following way.

Definition : A right circular cone is a surface generated by a line which passes through a fixed point (its vertex), and makes a constant angle with a fixed line through the fixed point.

Let us obtain the equation of an r.c. cone in terms of its semi-vertical angle. Let us assume that the vertex of the cone is $O(0, 0, 0)$ and axis is the z-axis. (We can always choose our coordinate system in this manner). Now take any point $P(x, y, z)$ on the cone (see Fig. 5). Then, the direction ratios of OP are x, y, z , and of the cone's axis are $0, 0, 1$. Thus, from Equation (9) of Unit 4, we get

$$\cos \theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\text{Thus, } x^2 + y^2 + z^2 = z^2 \sec^2 \theta$$

$$\Rightarrow x^2 + y^2 = z^2 \tan^2 \theta.$$

(1) is called the standard form of the equation of a right circular cone.

Now, why don't you try the following exercises?

E1) Show that the equation of the r.c. cone with vertex at (a, b, c) , axis $\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$

$$= \frac{z-c}{\gamma} \text{ and semi-vertical angle } \theta \text{ is}$$

$$[\alpha(x-a) + \beta(y-b) + \gamma(z-c)]^2 (\alpha^2 + \beta^2 + \gamma^2) \{ (x-a)^2 + (y-b)^2 + (z-c)^2 \} \cos^2 \theta \dots (2)$$

E2) Can you deduce (1) from (2)?

E3) Find the equation of the r.c. cone whose axis is the x-axis, vertex is the origin and

$$\text{semi-vertical angle is } \frac{\pi}{3}.$$

Let us now look at a cone whose vertex is the origin. In this situation we have the following result.

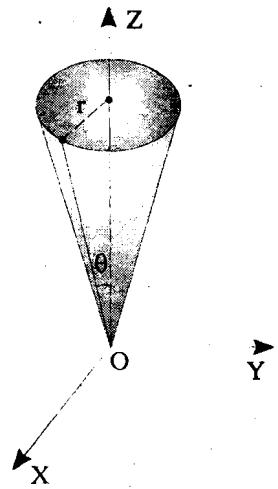


Fig. 4 : A right circular cone with vertex at the origin and base curve $x^2 + y^2 = r^2, z = k$.

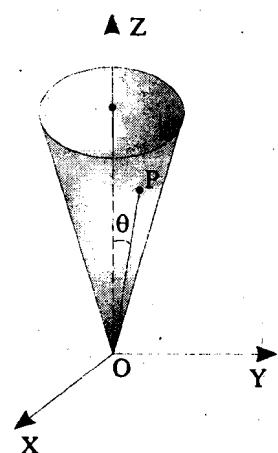


Fig. 5: $x^2 + y^2 = z^2 \tan^2 \theta$

An equation is homogeneous of degree 2 if each of its terms is of degree 2.

Theorem 2: The equation of a cone whose base curve is a conic and whose vertex is $(0, 0, 0)$ is a homogeneous equation of degree 2 in 3 variables.

Proof: Let us assume that the base curve is the conic.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = k.$$

Any generator of the cone passes through $(0, 0, 0)$. Thus, it is of the form

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}. \quad \dots(3)$$

This line intersects the plane $z = k$ at the point $\left(\frac{\alpha k}{\gamma}, \frac{\beta k}{\gamma}, k\right)$.

This point should lie on the conic. Thus,

$$\frac{k^2}{\gamma^2} (a\alpha^2 + 2h\alpha\beta + b\beta^2) + \frac{k}{\gamma} (2g\alpha + 2f\beta) + c = 0.$$

Eliminating α, β, γ from this equation and (3), we get

$$k^2 \left(a \frac{x^2}{z^2} + 2h \frac{xy}{z^2} + \frac{y^2}{z^2} \right) + k \left(2g \frac{x}{z} + 2f \frac{y}{z} \right) + c = 0.$$

$$\Rightarrow ax^2 + 2hxy + by^2 + 2gx \frac{z}{k} + 2fy \frac{z}{k} + \frac{cz^2}{k^2} = 0.$$

This is the equation of the cone. As you can see, it is homogeneous of degree 2 in the 3 variables x, y , and z .

For example, the equation of the cone whose base curve is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in the plane $z = 5$, and whose vertex is the origin, is

$$\left| \frac{x^2}{4} \right| + \left| \frac{y^2}{9} \right| = \left| \frac{z^2}{25} \right|$$

Do you see a pattern in the way we obtain the equation of the cone from the equation of the base curve? The following remark is about this.

Remark 2: To find the equation of the cone with vertex at $(0, 0, 0)$ and base curve in the plane $Ax = By + Cz = D, D \neq 0$, we simply homogenise the equation of the curve, as

follows. We multiply the linear terms by $\frac{Ax + By + Cz}{D}$, and the constant term by

$$\left(\frac{Ax + By + Cz}{D} \right)^2; \text{ and we leave the quadratic terms as they are. The equation that we get by}$$

this process is a homogeneous equation of degree 2, and is the equation of the cone. Let us look at a few examples of cones with their vertices at the origin.

Example 1: Show that the equation of the cone with the axes as generators is $fyz + gzx + hxy = 0$, where $f, g, h \in \mathbb{R}$.

Solution. By Theorem 2, the equation of the cone is

$$ax^2 + by^2 + cz^2 + 2fxy + 2gzy + 2hxy = 0, \text{ for some } a, b, c, f, g, h \in \mathbb{R}.$$

Since the x -axis is a generator, $(1, 0, 0)$ lies on it. Therefore, $a = 0$. Similarly, as it passes through $(0, 1, 0)$ and $(0, 0, 1)$, $b = c = 0$. So the equation becomes $fyz + gzx + hxy = 0$.

Example 2: Find the equation of the cone with vertex at the origin, and whose base curve is the circle $x^2 + y^2 + z^2 = 16, x + 2y + 2z = 9$.

Solution : On homogenising the equation of the sphere, we get

Cones and Cylinders

$$x^2 + y^2 + z^2 = 16 \left(\frac{x+2y+2z}{9} \right)^2$$

This is a second degree homogeneous equation in x, y, z and passes through the circle. Hence, it is the required equation of the cone.

The following exercise will give you some practice in homogenising equations.

E4) Find the equation of the cone with vertex at the origin and base curve

a) the parabola $y^2 = 4ax, z = 3$,

b) the ellipse $\left| \frac{y^2}{3} \right| + \left| \frac{z^2}{5} \right| = 1, x = -2$.

E5) Find the equation of the cone passing through $2x^2 + 3y^2 + 4z^2 = 1$ and $x + y + z = 1$.

Let us go back to Theorem 2 now. Do you think its converse is true? Consider the following result.

Theorem 3: A homogenous equation of the second degree in 3 variables represents a cone whose vertex is at the origin.

Proof: Let the given equation be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \dots(4)$$

Let $P(\alpha, \beta, \gamma)$ be a point on this surface and O the origin. Then OP is given by the equations

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = r \text{ (say).}$$

So any point on OP is $(r\alpha, r\beta, r\gamma)$. Since P lies on (4),

$$a(r\alpha)^2 + b(r\beta)^2 + c(r\gamma)^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0. \quad \dots(5)$$

Multiplying (5) throughout by r^2 , we get

$$a(r\alpha)^2 + b(r\beta)^2 + c(r\gamma)^2 + 2f(r\beta)(r\gamma) + 2g(r\gamma)(r\alpha) + 2h(r\alpha)(r\beta) = 0.$$

Thus, $(r\alpha, r\beta, r\gamma)$ also lies on (4), for any $r \in \mathbb{R}$. In particular, O lies on (4). So, the line OP lies on the surface given by (4). In other words, OP is a generator of (4). Thus, the surface (4) is generated by lines through the origin. Each of these lines will also pass through any curve obtained by intersecting (4) by a plane, and any of these curves can be treated as a base curve. Thus, (4) represents a cone with the origin as vertex.

So, from what you have seen so far in this section; whenever you come across a homogeneous equation in 3 variables of degree 2, you know that it represents a cone.

Remark 3: If $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$, then (4) can be written as a product of two linear

expressions. Thus, in this case (4) represents a pair of planes containing the origin. We shall consider this case as a **degenerate cone**, and any point on the line of intersection of the two planes can be considered as its vertex.

Using Theorem 3, we can show that if $\alpha, \beta, \gamma \in \mathbb{R}$, then a homogeneous equation in $x - \alpha, y - \beta, z - \gamma$ represents a cone with vertex at (α, β, γ) . (We shall discuss this kind of shifting in detail in Unit 7.)

Why don't you try some exercises now?

E6) If $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ is a generator of the cone given by the homogeneous equation (4), then show that (α, β, γ) lies on (4).

E7) Which of the following equations represents a cone?

$$3x + 4y + 5z = 0; x^2 + y^2 + z^2 = 9; 3(x^2 + y^2 + z^2) = xy, xyz = yz + zx + xy.$$

E8) If $\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d$, show that $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represents a cone.

Let us now go back to Example 1. This is an example of a cone with three mutually perpendicular generators. Its equation has no term containing x^2 , y^2 or z^2 . Does this mean that whenever a cone has three mutually perpendicular generators, its equation must have no term with x^2 , y^2 or z^2 ? The following theorem helps us to answer this question.

Theorem 4: If the cone $ax^2 + by^2 + cz^2 + 2fxy + 2gzx + 2hxy = 0$ has 3 mutually perpendicular generators, then $a + b + c = 0$.

Proof: Let the direction cosines of the three mutually perpendicular generators be l_i, m_i, n_i , where $i = 1, 2, 3$. Since they are mutually perpendicular, we can rotate our coordinate system so that these lines become the new coordinate axes.

Then the direction cosines of the previous coordinate axes with respect to the new axes are $l_1, l_2, l_3; m_1, m_2, m_3$ and n_1, n_2, n_3 , respectively.

So Unit 4 (Equations (3) and (10)) tell us that

$$\left. \begin{array}{l} l_1^2 + l_2^2 + l_3^2 = 1 \\ m_1^2 + m_2^2 + m_3^2 = 1 \\ n_1^2 + n_2^2 + n_3^2 = 1 \\ l_1m_1 + l_2m_2 + l_3m_3 = 0 \\ m_1n_1 + m_2n_2 + m_3n_3 = 0 \\ n_1l_1 + n_2l_2 + n_3l_3 = 0 \end{array} \right\} \quad \dots(6)$$

Further, since the perpendicular lines are generators of the cone, using E6 we get

$$al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gn_1l_1 + 2hl_1m_1 = 0$$

$$al_2^2 + bm_2^2 + cn_2^2 + 2fm_2n_2 + 2gn_2l_2 + 2hl_2m_2 = 0$$

$$al_3^2 + bm_3^2 + cn_3^2 + 2fm_3n_3 + 2gn_3l_3 + 2hl_3m_3 = 0$$

Adding these equations, and using (6), we get $a + b + c = 0$.

Actually, the converse of this result is also true. The proof uses a fact that you have already seen in Fig. 3 in the case of an r.c. cone, namely,

any plane through the vertex of a cone intersects the cone in two lines, which may or may not be distinct.

The following result, which we shall not prove, tells us about the angle between the lines of intersection.

Theorem 5: The angle between the two lines in which the plane $ux + vy + wz = 0$ intersects the cone $C(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is

If the lines of intersection of a cone and a plane through the cone's vertex are imaginary, the intersection reduces to a single point, namely, the cone's vertex (as in Fig. 1(e) of Unit 3).

$$\tan^{-1} \left| \frac{2P\sqrt{u^2 + v^2 + w^2}}{(a+b+c)(u^2 + v^2 + w^2) - C(u, v, w)} \right|, \quad \dots(7)$$

$$\text{where } P^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

Looking at (7), can you give the condition under which the angle will be $\pi/2$?

The lines of intersection of the plane and the cone will be perpendicular iff

$$C(u, v, w) = (a+b+c)(u^2 + v^2 + w^2) \quad \dots(8)$$

Let us use (8) to solve the following example, which includes the converse of Theorem 4.

Example 3: Show that if $a+b+c=0$, then the cone

$$C(x, y, z) \int ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

has infinitely many sets of three mutually perpendicular generators.

Solution : Let $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ be any generator of the cone. Then, by E6, we know that

$C(\alpha, \beta, \gamma) = 0$. Therefore, using the fact that $a+b+c=0$ and (8), we see that the plane $\alpha x + \beta y + \gamma z = 0$ intersects the cone in two mutually perpendicular generators, say L and L' .

Now $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ is normal to the plane $\alpha x + \beta y + \gamma z = 0$. Thus, it is perpendicular to both L

and L' . Thus, these three lines form a set of three mutually perpendicular generators of the cone.

Note that we choose $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ arbitrarily. Thus, for each generator chosen we get a set of

three mutually perpendicular generators. Hence, the cone has infinitely many such sets of generators.

Why don't you try some exercises now?

E9) Find the angle between the lines of intersection of
 $3x+y+5z=0$ and $6yz-2zx+5xy=0$.

E10) Prove that $ax+by+cz=0$, where $abc \neq 0$, cuts the cone $yz+zx+xy=0$ in
 perpendicular lines iff $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

E11) Prove that if a right circular cone has three mutually perpendicular generators, its semi-vertical angle is $\tan^{-1} \sqrt{2}$.

Let us now discuss the intersection of a line and a cone.

3.3 TANGENT PLANES

In the previous unit you saw that a line can intersect a sphere in at most two points. What do you expect in the case of the intersection of a line and a cone? Let's see.

Let the equation of the cone be (4), that is, $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

Note that, by shifting the origin if necessary, we can always assume this equation as the cone's equation.

For convenience, we will write $C(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$.

Now consider the line $\frac{x - x_1}{\alpha} = \frac{y - y_1}{\beta} = \frac{z - z_1}{\gamma}$. Any point on this line is given by

$(x_1 + r\alpha, y_1 + r\beta, z_1 + r\gamma)$, for some $r \in \mathbb{R}$. Thus, the line will intersect the cone, if this point lies on the cone for some $r \in \mathbb{R}$.

This happens if

$$a(x_1 + r\alpha)^2 + b(y_1 + r\beta)^2 + c(z_1 + r\gamma)^2 + 2f(y_1 + r\beta)(z_1 + r\gamma) + 2g(z_1 + r\gamma)(x_1 + r\alpha) + 2h(x_1 + r\alpha)(y_1 + r\beta) = 0.$$

$$\Leftrightarrow r^2 C(\alpha, \beta, \gamma) = 2r \{ \alpha(ax_1 + hy_1 + gz_1) + \beta(hx_1 + by_1 + fz_1) + \gamma(gx_1 + fy_1 + cz_1) \} + C(x_1, y_1, z_1) = 0. \quad \dots(9)$$

Now, if (x_1, y_1, z_1) doesn't lie on the cone, then (9) is a quadratic in r , and hence has two roots. Each root corresponds to a point of intersection of the line and the cone. Thus, we have just proved the following result.

Theorem 6 : A straight line, passing through a point not on cone, meets the cone in at most two points.

Now suppose that the line $\frac{x - x_1}{\alpha} = \frac{y - y_1}{\beta} = \frac{z - z_1}{\gamma}$ is a tangent to the cone (4) at (x_1, y_1, z_1) .

Then, since (x_1, y_1, z_1) lies on the cone, $C(x_1, y_1, z_1) = 0$. So (9) becomes
 $r^2 C(\alpha, \beta, \gamma) + 2r \{ \alpha(ax_1 + hy_1 + gz_1) + \beta(hx_1 + by_1 + fz_1) + \gamma(gx_1 + fy_1 + cz_1) \} = 0$

This equation must have coincident roots, since the line is a tangent to the cone at (x_1, y_1, z_1) . The condition for this is

$$\alpha(ax_1 + hy_1 + gz_1) + \beta(hx_1 + by_1 + fz_1) + \gamma(gx_1 + fy_1 + cz_1) = 0. \quad \dots(10)$$

So, (10) is the condition for $\frac{x - x_1}{\alpha} = \frac{y - y_1}{\beta} = \frac{z - z_1}{\gamma}$ to be tangent to the cone
 $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

Note that (10) is satisfied by infinitely many values of α, β, γ . Thus,

at each point of a cone we can draw infinitely many tangents to the cone.

Now, from Sec. 4.3.3, you know that (10) tells us that each of these lines is perpendicular to the line with direction ratios.

$$ax_1 + hy_1 + gz_1, hx_1 + by_1 + fz_1, gx_1 + fy_1 + cz_1.$$

Thus, the set of all the tangent lines at (x_1, y_1, z_1) is the plane

$$(x - x_1)(ax_1 + hy_1 + gz_1) + (y - y_1)(hx_1 + by_1 + fz_1) + (z - z_1)(gx_1 + fy_1 + cz_1) = 0 \\ \Rightarrow x(ax_1 + hy_1 + gz_1) + y(hx_1 + by_1 + fz_1) + z(gx_1 + fy_1 + cz_1) = 0. \quad \dots(11)$$

since $C(x_1, y_1, z_1) = 0$.

This plane is defined to be the tangent plane to the cone at (x_1, y_1, z_1) .

Thus, (11) is the equation of the tangent plane at (x_1, y_1, z_1) to the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

There is a very simple working rule for writing (11).

Rule of thumb: To write the equation of the tangent plane at any point (α, β, γ) on the cone (4),

replace x^2 by αx , y^2 by βy , z^2 by γz , $2yz$ by $\gamma y + \beta z$, $2zx$ by $\alpha z + \gamma x$ and $2xy$ by $\beta x + \alpha y$.

For example, the tangent plane to the cone $2x^2 + y^2 - 2xz = 0$ at $(1, 0, 1)$ is $2x(1) + y(0) - (x + z) = 0$, that is, $x = z$.

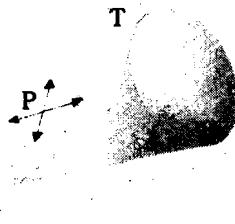
So far we have only found the tangent plane to a cone whose vertex is at the origin. What about a general cone? The following remark is about this.

Remark 4: We can find the tangent plane to a cone with vertex at (a, b, c) in the same manner as above. All we need to do is to shift the origin to (a, b, c) and find the tangent plane in the new coordinate system. And then we can shift back to the old coordinate system, making the required transformations in the equation of the plane. This will give us the required equation.

Now, if you look closely at (11), you will notice that $(0, 0, 0)$ satisfies it. Thus, **the tangent plane to a cone passes through the vertex of the cone**.

Therefore, the tangent plane at $P(x_1, y_1, z_1)$ contains P as well as the vertex O of the cone. Hence, it contains the generator OP of the cone. Thus,

the tangent plane to a cone touches the cone along the generator passing through the point of contact.



This generator is called the **generator of contact** of the plane. In Fig. 6 OP is the generator of contact of the tangent plane T .

You can try some exercises now.

Fig. 6 : T is the tangent plane to the cone at P .

E 12) Find the equation of the tangent plane at the point $\left(\frac{1}{7}, \frac{1}{4}, 1\right)$ to the cone
 $5yz - 8zx - 3xy = 0$.

E 13) Use Theorem 5 to obtain the condition under which a given plane is tangent to a cone.

If you've solved E13, you would have seen that the condition for $ux + vy + wz = 0$ to be tangent to the cone (4), that is, $ax^2 + by^2 + cz^2 + 2hxy + 2fyx + 2gzx = 0$ is

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0, \text{ that is,}$$

$$Au^2 + Bv^2 + Cw^2 + 2Huv + 2Fvw + 2Gwu = 0,$$

where $A = bc - f^2$, $B = ca - g^2$, $C = ab - h^2$, $F = gh - af$, $G = hf - bg$, $H = fg - ch$.

Thus, the line $\frac{x}{u} = \frac{y}{v} = \frac{z}{w}$, which is the normal at $(0, 0, 0)$ to the tangent plane, is a generator of the cone $Ax^2 + By^2 + Cz^2 + 2Hxy + 2Fyz + 2Gzx = 0$ (12)

Thus, (12) is the cone generated by the normals to the tangent planes at the vertex $(0, 0, 0)$ of the cone (4). Since it is homogeneous, its vertex is also $(0, 0, 0)$.

Note that (12) is nothing but the determinant equation.

$$\begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & 0 \end{vmatrix} = 0$$

Now, if we consider the surface generated by the normals at $(0, 0, 0)$ to the tangent planes to (12), what do we get? On calculating, you will get a surprise! The cone is (4), because $BC - F^2 = a\Delta$, $CA - G^2 = b\Delta$, $AB - H^2 = c\Delta$, $GH - AF = f\Delta$, $HF - BG = g\Delta$, $FG - CH = h\Delta$, where,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Because of this relationship between (4) and (12) we call them reciprocal cones.

Actually, the following example shows why the name is appropriate.

Example 4: Show that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are reciprocal.
(Here $abc \neq 0$).

Solution: The reciprocal cone of $ax^2 + by^2 + cz^2 = 0$ is given by the determinant equation

$$\begin{vmatrix} a & 0 & 0 & x \\ 0 & b & 0 & y \\ 0 & 0 & c & z \\ x & y & z & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow a \begin{vmatrix} b & 0 & y \\ 0 & c & z \\ y & z & 0 \end{vmatrix} - x \begin{vmatrix} 0 & b & 0 \\ 0 & 0 & c \\ x & y & z \end{vmatrix} = 0$$

$$\Leftrightarrow x^2bc + y^2ac + z^2ab = 0$$

$$\Leftrightarrow \left| \frac{x^2}{a} \right| + \left| \frac{y^2}{b} \right| + \left| \frac{z^2}{c} \right| = 0, \text{ dividing throughout by } abc.$$

This is the required equation.

Now you can do the following exercises. This will help you to understand reciprocal cones better.

E 14) Find the cone on which the perpendiculars drawn from the origin to tangent planes to the cone $19x^2 + 11y^2 + 3z^2 + 6yz - 10zx - 26xy = 0$ lie.

E 15) Prove that the cone (4) has three mutually perpendicular tangent planes iff $bc + ca + ab = f^2 = g^2 = h^2$.

And now let us shift our attention to another surface that is generated by lines.

3.4 CYLINDERS

You must have come across several examples of the surface that we are going to discuss in this section. For instance, a drain pipe is cylindrical in shape, and so is a pencil. But for us, the pencil will not be a cylinder, only its surface will, according to the following definition.

Definition: A cylinder is the set of all lines which intersect a given curve and which are parallel to a fixed line which does not lie in the plane of the curve. The fixed line is called the axis of the cylinder and the curve is called the base curve (or directrix) of the cylinder.

All the figures in Fig. 7 represent portions of cylinders.

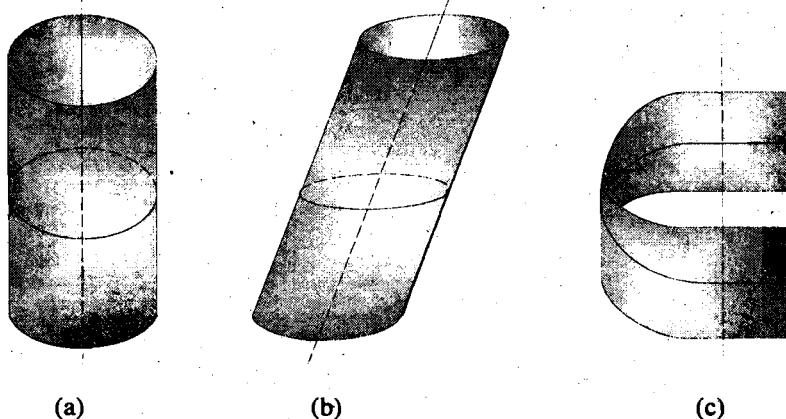


Fig. 7: (a) A circular cylinder, b) an elliptic cylinder, c) a parabolic cylinder.

In Fig. 7 (b) the cylinder's base curve is an ellipse, while it is a parabola in Fig. 7 (c). Fig. 7 (a) is an example of a right circular cylinder according to the following definition.

Definition: A cylinder whose base curve is a circle, and whose axis passes through the centre of the base curve and the perpendicular to the plane of the base curve, is called a right circular cylinder.

As you can see, in common parlance when people talk of a cylinder, they mean a portion of a right circular cylinder.

Henceforth, in this section, by a cylinder we shall mean a right circular cylinder.

Let us now find the equation of a cylinder. We shall first assume that its axis is the z-axis, and its base curve is the circle $x^2 + y^2 = r^2$, $z = 0$ (see Fig. 8).

Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Let the generator through P intersect the plane of the base curve (that is, the XY-plane) in M . Then the perpendicular distance of P from the axis is $OM = \sqrt{x_1^2 + y_1^2}$.

But this is also r . Thus,

$$r^2 = x_1^2 + y_1^2.$$

This equation is true for every point $P(x_1, y_1, z_1)$ on the cylinder. Thus, the equation of the cylinder is

$$x^2 + y^2 = r^2. \quad (13)$$

You may wonder why z is not figuring in the equation. This is because whatever value of z you take, the equation of the cylinder remains $x^2 + y^2 = r^2$.

What does this mean geometrically? It says that whatever plane parallel to the XY-plane you take, say $z = t$, and take its intersection with the cylinder, you will always get the circle $x^2 + y^2 = r^2$.

Thus, in a sense, the cylinder is made up of infinitely many circles, each piled up on the other!

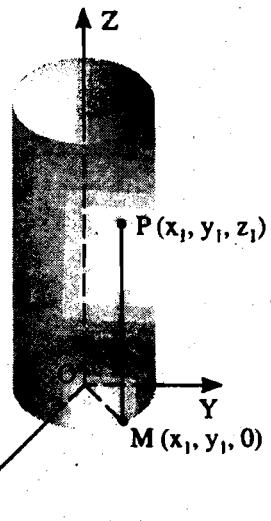


Fig. 8: The cylinder $x^2 + y^2 = r^2$.

The radius of a cylinder is the radius of its base curve.

Note that the plane $z = t$ is perpendicular to the axis of the cylinder $x^2 + y^2 = r^2$.

Also note that the length of the perpendicular from any point on a cylinder to its axis is equal to its radius.

Using this fact, let us find the equation of a cylinder of radius r and whose axis is

$$\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma} \text{ (see Fig. 9).}$$

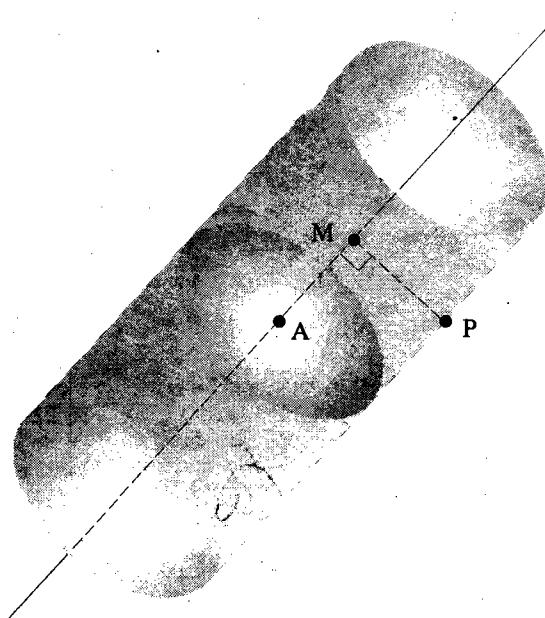


Fig. 9 : A right circular with axis AM.

Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Let A be the point (a, b, c) , which lies on the axis, and M be the foot of the perpendicular from P onto the axis. Then $PM = r$.

Also, $AM = AP \cos \theta$, where θ is the angle between the lines AM and AP .

$$\therefore AM = \frac{(x_1 - a)\alpha + (y_1 - b)\beta + (z_1 - c)\gamma}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}, \text{ using Equation (9) of Unit 4.}$$

Since AMP is right-angled triangle, we get $AP^2 = AM^2 + MP^2$. Thus,

$$(x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2 = \frac{(x_1 - a)\alpha + (y_1 - b)\beta + (z_1 - c)\gamma)^2}{\alpha^2 + \beta^2 + \gamma^2} + r^2.$$

This equation holds for any point (x_1, y_1, z_1) on the cylinder.

Thus, the equation of the right circular cylinder with radius r and axis $\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$

$$is \{(x-a)^2 + (y-b)^2 + (z-c)^2 - r^2\} (\alpha^2 + \beta^2 + \gamma^2) = \{(x-a)\alpha + (y-b)\beta + (z-c)\gamma\}^2. \quad \dots(14)$$

Let us look at an example.

Example 5: Find the equation of the cylinder having for its base the circle $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$.

Solution : The centre of the sphere is $(0, 0, 0)$, and radius 3. The distance between $(0, 0, 0)$ and the plane $x - y + z = 3$ is $\sqrt{3}$. So the radius of the base circle is $\sqrt{9 - 3} = \sqrt{6}$ (see Fig. 10).

The axis of the cylinder is perpendicular to the plane $x - y + z = 3$ and passes through $(0, 0, 0)$.

So its equations are $\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$.

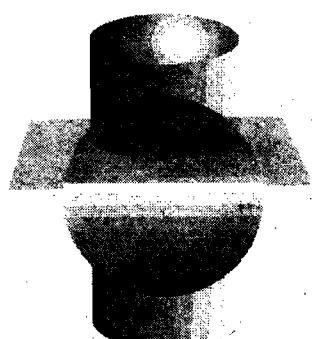


Fig. 10

Thus, using (14), we get the required equation as

$$\begin{aligned} 3(x^2 + y^2 + z^2 - 6) &= (x - y + z)^2 \\ \Rightarrow x^2 + y^2 + z^2 + xy + yz - zx - 9 &= 0. \end{aligned}$$

Why don't you try an exercise now?

E16) Find the equation of the cylinder

a) whose axis is $x = 2y = -z$ and radius is 4.

b) whose axis is $\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$ and radius is 2.

We shall end our discussion on cylinders here. Let us now briefly review what we have covered in this unit.

3.5 SUMMARY

In this unit we have discussed the following points :

- 1) A cone is a surface generated by a line passing through a fixed point (its vertex) and intersecting a given curve (its base curve), such that the vertex does not lie in the plane of the base curve.
- 2) A cone whose base curve is a circle, and for which the line joining its vertex to the centre of the base curve is perpendicular to the plane of the base curve, is called a right circular cone.
- 3) A planar section of a cone is a conic.
- 4) The equation of a right circular cone with semi-vertical angle θ is $x^2 + y^2 = z^2 \tan^2 \theta$.
- 5) A second degree equation in x, y, z represents a cone with vertex at the origin if it is homogeneous.
- 6) The cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ has 3 mutually perpendicular generators if $a + b + c = 0$.
- 7) Any plane through the vertex of a cone intersects the cone in two distinct or coincident lines. The angle between the lines obtained by intersecting $ux + vy + wz = 0$ with $C(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is

$$\tan^{-1} \left| \frac{2P\sqrt{u^2 + v^2 + w^2}}{(a+b+c)(u^2 + v^2 + w^2) - C(u, v, w)} \right|,$$

$$\text{where } P^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

Thus, the plane is tangent to the cone iff $P^2 = 0$.

- 8) The equation of the tangent plane to the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ at the point $P(x_1, y_1, z_1)$ is

$$(ax_1 + hy_1 + gz_1)x + (hx_1 + by_1 + fz_1)y + (gx_1 + fy_1 + cz_1)z = 0.$$

This contains the line OP , where O is the vertex of the cone.

- 9) The cone formed by the normals to the tangent planes to a given cone at its vertex is the reciprocal of the given cone. The reciprocal cone of the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is given by

$$\begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & 0 \end{vmatrix} = 0$$

- 10) A cylinder is a surface generated by a line which is parallel to a fixed line (its axis) and which cuts a given curve (its base curve), such that the line and curve are not in the same plane.
- 11) A right circular cylinder is a cylinder whose base curve is a circle and axis is the line through the centre of the circle and perpendicular to its plane.
- 12) The equation of a right circular cylinder with base curve a circle of radius r and centre $(0, 0, 0)$ in the plane $z = 0$ is $x^2 + y^2 = r^2$.
- 13) The equation of a right circular cylinder of radius r and axis $\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$ is $[(x-a)^2 + (y-b)^2 + (z-c)^2 - r^2](\alpha^2 + \beta^2 + \gamma^2) = [(x-a)\alpha + (y-b)\beta + (z-c)\gamma]^2$.

And now, you may like to go back to Sec. 6.1 to see if you have achieved the objectives listed there. You must have solved the exercises as you came to them in the unit. In the next section we have given our answers to the exercises. You may like to have a look at them.

3.6 SOLUTIONS/ANSWERS

- E1) Let $P(x, y, z)$ be any point on the cone. Since $V(a, b, c)$ is its vertex, the direction ratios of PV are $x-a, y-b, z-c$. Also, the direction ratios of the axis of the cone are α, β, γ .

$$\therefore \cos \theta = \frac{\alpha(x-a) + \beta(y-b) + \gamma(z-c)}{\sqrt{\alpha^2 + \beta^2 + \gamma^2} \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

Hence, we get (2).

- E2) Yes. Just take $\alpha = 0 = \beta, \gamma = 1, a = b = c = 0$ in (2), and you will get (1).

- E3) The direction ratios of the axis are $1, 0, 0$ and the vertex is $(0, 0, 0)$. If (x, y, z) is any point on the cone, then

$$\cos \frac{\pi}{3} = \frac{x \cdot 1 + y \cdot 0 + z \cdot 0}{\sqrt{x^2 + y^2 + z^2}}$$

$$\Rightarrow x^2 + y^2 + z^2 = 4x.$$

which is the required equation.

E4) a) $y^2 = 4ax \left(\frac{z}{3}\right) \Leftrightarrow 3y^2 - 4azx = 0.$

b) $\frac{x^2}{4} - \frac{y^2}{3} - \frac{z^2}{5} = 0.$

E5) $2x^2 + 3y^2 + 4z^2 = (x+y+z)^2$
 $\Leftrightarrow x^2 + 2y^2 + 3z^2 - 2xy - 2yz - 2zx = 0.$

- E6) Let $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = r$, say. Then putting $x = r\alpha, y = r\beta, z = r\gamma$ in (4), and dividing throughout by r^2 , we get

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0.$$

$\therefore (\alpha, \beta, \gamma)$ lies on (4).

E7) Only $3(x^2 + y^2 + z^2) = xy$.

E8) Substituting the value of d in the equation, we can write it as

$$a\left(x + \frac{u}{a}\right)^2 + b\left(y + \frac{v}{b}\right)^2 + \left(z + \frac{w}{c}\right)^2 = 0.$$

which is a homogeneous equation of degree 2 in $x + \frac{u}{a}, y + \frac{v}{b}, z + \frac{w}{c}$.

Thus, it is a cone with vertex at $\left(-\frac{u}{a}, -\frac{v}{b}, -\frac{w}{c}\right)$.

E9) The required angle is

$$\alpha = \tan^{-1} \left| \frac{2P\sqrt{3^2 + 1^2 + 5^2}}{0 - 6(1)(5) + 2(5)(3) - 5(3)(1)} \right|,$$

$$\text{where } P^2 = \begin{vmatrix} 0 & \frac{5}{2} & -1 & 3 \\ \frac{5}{2} & 0 & 3 & 1 \\ -1 & 3 & 0 & 5 \\ 3 & 1 & 5 & 0 \end{vmatrix} = \frac{225}{4}$$

$$\therefore P = \frac{15}{2}.$$

$$\therefore \alpha = \tan^{-1} \sqrt{35}.$$

E10) In this situation (8) tells us that the lines will be perpendicular iff $bc + ca + ab = 0$.

$$\Leftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

E11) Let its semi-vertical angle be θ .

Then the equation of the cone is (1), that is, $x^2 + y^2 = z^2 \tan^2 \theta$.

Since this has three mutually perpendicular generators, Theorem 4 tells us that $1 + 1 - \tan^2 \theta = 0 \Rightarrow \theta = \tan^{-1} \sqrt{2}$.

E12) The required equation is

$$x \left\{ -\frac{3}{2} \left(\frac{1}{4}\right) - 4(1) \right\} + y \left\{ -\frac{3}{2} \left(\frac{1}{7}\right) + \frac{5}{2}(1) \right\} + z \left\{ (-4) \left(\frac{1}{7}\right) + \frac{5}{2} \left(\frac{1}{4}\right) \right\} = 0.$$

$$= -245x + 128y + 3z = 0.$$

E13) A tangent plane must touch the cone along a generator. Thus, the two lines of intersection of the plane and the cone must coincide. Thus, the angle between these lines must be 0.

Thus, from Theorem 5, we see that $ux + vy + wz = 0$ is a tangent to the cone $C(x, y, z) = 0 \Leftrightarrow P = 0$ (since $u^2 + v^2 + w^2 \neq 0$).

$$\Leftrightarrow \begin{vmatrix} a & h & g & u \\ i & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0$$

E14) The required cone is the reciprocal of the given cone. Thus its equation is

$$\begin{vmatrix} 19 & -13 & -5 & x \\ -13 & 11 & 3 & y \\ -5 & 3 & 3 & z \\ x & y & z & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow 3x^2 + 4x^2 + 5z^2 + 2yz + 4zx + 6xy = 0.$$

E15) The cone will have three mutually perpendicular tangent planes iff the reciprocal cone has three mutually perpendicular generators. Using Theorem 4 and its converse, we see that this happens iff in (12), $A + B + C = 0$, that is, iff $(bc - t^2) + (ca - g^2) + (ab - h^2) = 0$, that is iff $bc + ca + ab = t^2 + g^2 + h^2$.

E16) a) The equation is $(x^2 + y^2 + z^2 - 16) \left(1 + \frac{1}{4} + 1\right) = \left(x + \frac{y}{2} - z\right)^2$

$$\Leftrightarrow 5x^2 + 8y^2 + 5z^2 - 4xy + 4yz + 8xy - 144 = 0.$$

b) The required equation is

$$14 \{(x-1)^2 + y^2 + (z-3)^2 - 4\} = \{2(x-1) + 3y + (z-3)\}^2$$

$$\Leftrightarrow 10x^2 + 5y^2 + 13z^2 - 12xy - 6yz - 4xz - 8x + 30y - 74z + 59 = 0.$$

MISCELLANEOUS EXERCISES

(This section is optional)

In this section we have gathered some problems related to the contents of this block. You may like to do them to get a better understanding of these contents. Our solutions to the questions follow the list of problems, in case you'd like to counter-check your answers.

- 1) Find the equations to the planes through the line $\frac{x-2}{2} = \frac{y-3}{4} = \frac{z-4}{5}$, which are parallel to the coordinate axes.

- 2) Find the equation of the plane passing through $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and the point $(0, 7, -7)$. Also check if $x = \frac{2-y}{3} = \frac{z+2}{2}$ lies in the plane.

- 3) The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axes in the points A, B and C. Find the equations determining the circumcircle of the triangle ABC (see Fig. 1).
- 4) Prove that if every planar section of a surface given by a quadratic equation is a circle, the surface must be a sphere.

- 5) Find an equation of the set of points which are twice as far from the origin as from $(-1, 1, 1)$.

- 6) If the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ cuts $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ in a great circle then show that $2(uu' + vv' + ww') - (d + d') = 2r'^2$, where r' is the radius of the second sphere.

- 7) Find the equation of the sphere inscribed in the tetrahedron whose faces are $x=0, y=0, z=0, x+y+z=1$ (see Fig. 2).

- 8) Show that the sum of the squares of the intercepts made by a given sphere on any three mutually perpendicular lines through a fixed point is constant.
(Hint: Take the fixed point to be the origin.)

- 9) Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$.

- 10) If $x = \frac{y}{z} = z$ represents one out of a set of three mutually perpendicular generators of the cone $11yz + 6zx - 14xy = 0$, find the equations of the other two.

(Hint: Take a plane through the given line, and apply the condition for the lines of intersection of this plane and the cone to be perpendicular.)

- 11) Find the equation of the right circular cone with vertex $(1, 1, 3)$, axis parallel to the line

$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$ and with one of its generators having direction ratios $2, 1, -1$.

- 12) Find the equation of the cone which passes through the common generators of the cones $x^2 + 2y^2 + 3z^2 = 0$ and $5xy - yz + 5xz = 0$ and the line with direction ratios $1, 0, 1$.

(Hint: The cone passing through the common generators of the cone $C_1 = 0$ and $C_2 = 0$ is $C_1 + kC_2 = 0$, where $k \in \mathbb{R}$.)

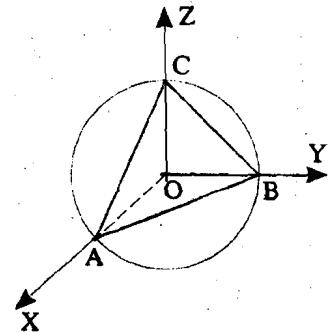


Fig. 1

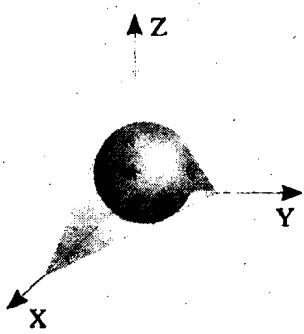


Fig. 2



Fig. 3: The cylinder
envelops the sphere.



Fig. 4 : The plane is tangent
to the cylinder.

- 13) Find the equation of the right circular cylinder that is generated by lines which are parallel to $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, and which are tangent to the sphere $x^2 + y^2 + z^2 = r^2$ (see Fig. 3).
(Note: Such a cylinder is called the **enveloping cylinder** of the sphere.)
- 14) The axis of a cylinder of radius 3 has equations $\frac{x}{2} = \frac{y+1}{2} = \frac{z-1}{1}$. Find the equation of the cylinder.
- 15) Prove that for a cylinder the tangent plane at any point to its axis (see Fig. 4).

SOLUTIONS

- 1) The equation to any plane through the given line is $a(x-2) + b(y-3) + c(z-4) = 0$, where $2a + 4b + 5c = 0$
If this is parallel to the x-axis, we must have $a(1) + b(0) + c(0) = 0 \Rightarrow a = 0$.
Thus, the equation of such a plane is $5y - 4z + 1 = 0$.
Similarly, you can check that the planes parallel to the y and z axes are $5x - 2z - 2 = 0$ and $2x - y - 1 = 0$, respectively.

- 2) The plane will be $a(x-1) + b(y-3) + c(z+2) = 0$, ... (1)
where $-3a + 2b + c = 0$ (2)
Since $(0, 7, -7)$ lies on it, we have $a + 4b - 5c = 0$ (3)
Eliminating a, b and c from (2) and (3), we get

$$\frac{a}{-10-4} = \frac{b}{1-15} = \frac{c}{-12-2} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1}$$

$$\therefore (1) \text{ becomes } 1(x+1) + 1(y-3) + 1(z+2) = 0$$

$$\Rightarrow x + y + z = 0.$$

The line $\frac{x}{1} = \frac{y-2}{-3} = \frac{z+2}{2}$ will lie on this plane, if it is parallel to the plane and any point on it lies on the plane. Since $1(1) + (-3)(1) + 2(1) = 0$, the line is parallel to the plane. Also, $(0, 2, -2)$ is a common point. Thus, the line lies in the plane.

- 3) The circumcircle will be the intersection of the given plane with any sphere passing through A, B and C. So, let us take the sphere through O, A, B and C. The coordinates of these points are $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. You can check that the equation is $x^2 + y^2 + z^2 - ax - by - cz = 0$.
Thus, the equations that give the circumcircle are

$$x^2 + y^2 + z^2 - ax - by - cz = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

- 4) Let the equation of the surface be $ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyx + 2ux + 2vy + 2wz + d = 0$.
It intersects $z = 0$ in the conic $ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0$.
This will be a circle iff $a = b$ and $h = 0$.
Similarly, on intersecting with $x = 0$ and $y = 0$ we will get $a = b = c$ and $f = 0 = g = h$.
Thus, the equation of the surface reduces to $a(x^2 + y^2 + z^2) + 2ux + 2vy + 2wz + d = 0$, which represents a sphere.

- 5) Let (x, y, z) be any point in the set. Then

$$\sqrt{x^2 + y^2 + z^2} = 2\sqrt{(x+1)^2 + (y-1)^2 + (z-1)^2}$$

$$\Rightarrow 3(x^2 + y^2 + z^2) + 8x - 8y - 8z + 12 = 0, \text{ which represents a sphere.}$$

- 6) If the two spheres are $S = 0$ and $S_1 = 0$, then $(-u', -v', -w')$ lies on $S - S_1 = 0$, that is, on $2(u-u')x + 2(v-v')y + 2(w-w')z + d - d' = 0$.
 $\therefore 2(u-u')u' + 2(v-v')v' + 2(w-w')w' = d - d'$.
 $\Rightarrow 2(uu' + vv' + ww') - d + d' = 2(u'^2 + v'^2 + w'^2) = 2r^2 + 2d'$
 $\Rightarrow 2(uu' + vv' + ww') - (d + d') = 2r^2$.

- 7) Let the equation be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

Since the given planes are tangent to it, the distance of $(-u, -v, -w)$ from these planes is $r = \sqrt{u^2 + v^2 + w^2 - d}$. Thus, we see that

$$u = v = w = -r \text{ and } |-u - v - w - 1| = \sqrt{3}r$$

Solving these equations, we get $r = \frac{3 + \sqrt{3}}{6}$.

Thus, the equation of the sphere is,

$$x^2 + y^2 + z^2 - 2r(x + y + z) + 2r^2 = 0, \text{ where } r = \frac{3 + \sqrt{3}}{6}.$$

- 8) Let us assume that the fixed point is $(0, 0, 0)$ and the three lines are the axes. Then let the sphere be given by $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

Its intercept on the x-axis, that is, $y = 0 = z$, is $2\sqrt{u^2 - d}$.

Similarly, the other intercepts are $2\sqrt{v^2 - d}$ and $2\sqrt{w^2 - d}$.

$$\text{Now, } (2\sqrt{u^2 - d})^2 + (2\sqrt{v^2 - d})^2 + (2\sqrt{w^2 - d})^2 = 4(u^2 + v^2 + w^2 - 12d).$$

which is a constant, since the sphere is a given one.

- 9) Let a line of intersection be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. Then $2l + m - n = 0$ and $ul^2 - m^2 + 3n^2 = 0$.

Solving these equations, we get

$$m = -2l, n = 0 \text{ or } m = -4l, n = -2l.$$

Thus, the two lines are

$$\frac{x}{1} = \frac{y}{-2}, z = 0 \text{ and } \frac{x}{1} = \left| \begin{array}{c} \frac{y}{-4} \\ \end{array} \right| = \left| \begin{array}{c} z \\ -2 \end{array} \right|.$$

- 10) $2x - y + k(y - 2z) = 0, k \in \mathbb{R}$, gives any plane through the given line. This will cut the given cone in perpendicular lines if

$$11(k-1)(-2k) + 6(-2k)(2) - 14(2)(k-1) = 0 \Rightarrow k = -2, 7/11.$$

Thus, the planes are $2x - 3y + 4z = 0$ and $11x - 2y - 7z = 0$.

Now, $2x - 3y + 4z = 0$ intersects the cone in two perpendicular lines of which one is the given one which lies in the plane. Therefore, the other one has to be the normal to the

plane at $(0, 0, 0)$. This is $\frac{x}{2} = \left| \begin{array}{c} \frac{y}{-3} \\ \end{array} \right| = \frac{3}{4}$. So this will be another of the required set of mutually perpendicular generators.

Similarly, the third generators will be the normal to $11x - 2y - 7z = 0$ at $(0, 0, 0)$, that is,

$$\frac{x}{11} = \frac{y}{-2} = \frac{z}{-7}.$$

- 11) If θ is its semi-vertical angle, then

$$\cos \theta = \frac{2+2-2}{3\sqrt{6}} = \frac{2}{3\sqrt{6}}.$$

Also, the axis is given by $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-3}{2}$.

Thus, the equation of the r.c. cone is

$$\{(x-1) + 2(y-1) + 2(z-3)\}^2 - (x-1)^2 + (y-1)^2 + (z-3)^2 = \frac{4}{54}$$

$$\Leftrightarrow x^2 + 10y^2 + 10z^2 + 12xy + 24yz + 12xz - 50x - 104y - 96z + 221 = 0.$$

- 12) Let the cone be $(x^2 + 2y^2 + 3z^2) + k(5xy - yz + 5xz) = 0$, where $k \in \mathbb{R}$. Since the line with direction ratios 1, 0, 1 lies on it, $(1, 0, 1)$ must satisfy it. This gives us $k = -\frac{4}{5}$.

Thus, the required cone is

$$5(x^2 + 2y^2 + 3z^2) - 4(5xy - yz + 5xz) = 0.$$

- 13) Let (α, β, γ) be any point on the cylinder. A generator through this will be given by

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}.$$

This line intersects the sphere in $(ak + \alpha, bk + \beta, ck + \gamma)$, where k is given by $(ak + \alpha)^2 + (bk + \beta)^2 + (ck + \gamma)^2 = r^2$.

This quadratic equation in k gives two values of k , which correspond to two points of intersection. Thus, the generator will be a tangent to the sphere if these points coincide, that is, iff

$$(a\alpha + b\beta + c\gamma)^2 = (a^2 + b^2 + c^2)(\alpha^2 + \beta^2 + \gamma^2 - r^2).$$

Thus, the locus of (α, β, γ) , which is the equation of the enveloping cylinder, is

$$(ax + by + cz)^2 = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2 - r^2).$$

- 14) $3x^2 + 3y^2 + 8xy + 4yz + 4xz + 4x + 2y + 4z + 8 = 0$.

- 15) We can always assume the equation of the cylinder to be $x^2 + y^2 = r^2$.

Its axis is the z-axis, that is, $x = 0, y = 0$.

As in the case of a cone, you can show that its tangent plane at a point (x_1, y_1, z_1) is $xx_1 + yy_1 = r^2$.

This is parallel to the z-axis. Hence, the result.