UNIT 2 AREA UNDER A CURVE

Structure

2.1 INTRODUCTION

When we introduced you to integration, we mentioned that the origin of the method of integration lies in the attempt to estimate the areas of regions bounded by plane curves. In this unit we shall see how to calculate the area under a given curve, when the equation of the curve is given in the Cartesian or polar or parametric form. This process is also called **quadrature.** We shall also study two methods of numerical integration. These are helpful when the antiderivative of the integrand cannot be expressed in terms of known functions, and the given definite integral cannot be exactly evaluated.

Objectives

After reading this unit you should be able to :

- use your knowledge of integration to find the area under a given curve whose equation is given in the Cartesian or polar or parametric form,
- recognise the role of numerical integration in solving some practical problems when some values of the function are known, but the function, as a whole, is not known.
- use trapezoidal and Simpson's rules to find approximate values of some definite integrals,
- compare the two rules of numerical integration.

2.2 AREA UNDER A CURVE

In this section we shall see how the area under a curve can be calculated when the equation of the curve is given in the

i) Cartesian form
ii) polar form
iii) parametric form.

Some curves may have a simple equation in one form, but complicated ones in others. So, once we have considered all these forms, we can choose an appropriate form for a given curve, and then integrate it accordingly. Let us consider these forms of equations one by one.

2.2.1 Cartesian Equation

We shall quickly recall what we studied in earlier. Let $y = f(x)$ define a continuous function of x on the closed interval $[a, b]$. For simplicity, we make the assumption that $f(x)$ is positive for $x \in [a, b]$. Let R be the plane region in Fig. 1 (a) bounded by the graphs of the four equations:

$y = f(x), y = 0, x = a$ and $x = b$.

We divide the region R into n thin strips by lines perpendicular to the x-axis through the end points $x = a$ and $x = b$, and through many intermediate points which we indicate by $x_1, x_2, \ldots\ldots x_{n-1}$. Such a subdivision, as you have already seen in Sec. 2 of Block 3, is referred to as a **partition P_n** of the interval $[a, b]$ is indicated briefly by writing

$$P_n = [a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b]$$

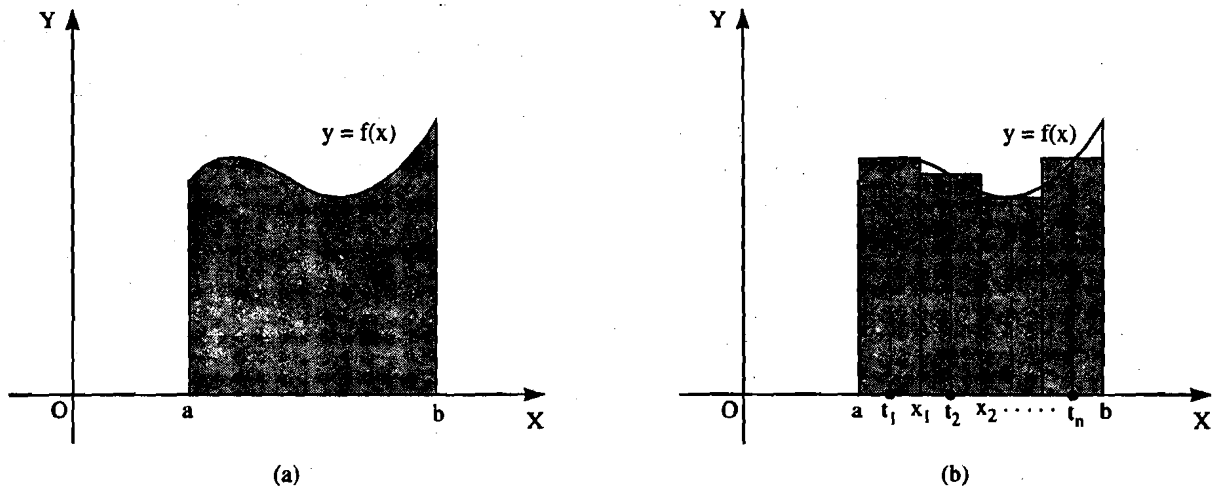


Fig. 1

We write

$$\Delta x_i = x_i - x_{i-1} \qquad \text{for } i = 1, 2, \ldots, n,$$

and take the set of n points on x-axis.

$$T_n = \{t_1, t_2, \ldots\ldots, t_{n-1}, t_n\},$$

such that $x_{i-1} \le t_i \le x_i$ for $i = 1, 2, \ldots, n$. We now construct the n rectangles (Fig. 1 (b)) whose bases are the n sub-intervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots\ldots, n$ induced by the partition P_n, and whose altitudes are $f(t_1), f(t_2), \ldots, f(t_1), \ldots, f(t_{n-1}), f(t_n)$. The

$$\text{sum} \sum_{i=1}^{n} f(t_i)\,\Delta x_i$$

of the areas of these n rectangles will be an approximation to the "area of R". Notice (Fig. 2(a) and (b)) that if we increase the number of sub-intervals, and decrease the length of each sub-interval, we obtain a closer approximation to the "area of R".

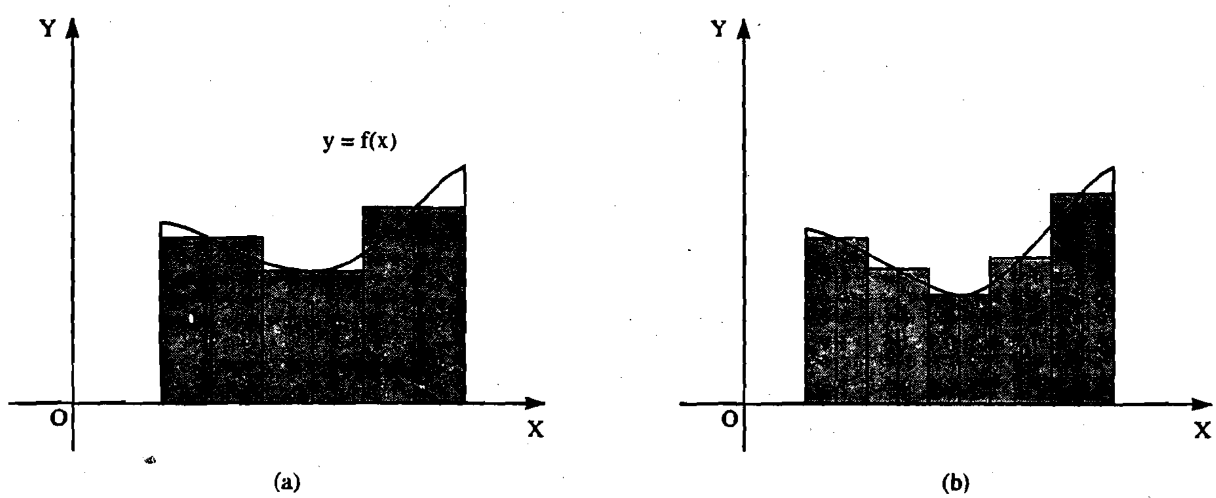


Fig. 2

Thus, we have

Definition 1: Let f be a real valued function continuous on [a, b], and let

$f(x) \geq 0 \ \forall \ x \in [a, b]$. If the limit of $\sum_{t=1}^{n} f(t_i)\, \Delta x_i$ exists as the lengths of the sub-intervals, $\Delta x_i \to 0$, then that limit is the area A of the region R.

That is, $A = \lim_{\substack{\Delta x_i \to 0 \\ i = 1, 2, \ldots, n}} \sum_{i=1}^{n} f(t_i)\, \Delta x_i$

Compare this definition with that of a definite integral given in Block 3. Over there we had seen that the definite integral,

$\int_a^b f(x)\, dx$ is the common limit of $\sum_{i=1}^{n} m_i\, \Delta x_i$ and $\sum_{i=1}^{n} M_i\, \Delta x_i$ as the Δx_i's $\to 0$.

Now since $m_i \leq f(t_i) \leq M_i \ \forall$ i, we have

$$\sum_{i=1}^{n} m_i\, \Delta x_i \leq \sum_{i=1}^{n} f(t_i) \Delta x_i \leq \sum_{i=1}^{n} M_i\, \Delta x_i$$

Hence if the limit of each of these as Δx_i's $\to 0$ exists, then by the Sandwich Theorem in Unit 2,

$$\lim_{\substack{\Delta x_i \to 0 \\ i = 1, 2, \ldots, n}} \sum_{i=1}^{n} m_i\, \Delta x_i \leq \lim_{\substack{\Delta x_i \to 0 \\ i = 1, 2, \ldots, n}} \sum_{i=1}^{n} f(t_i)\, \Delta x_i \leq \lim_{\substack{\Delta x_i \to 0 \\ i = 1, 2, \ldots, n}} \sum_{i=1}^{n} M_i\, \Delta x_i$$

Now, if $\int_a^b f(x)\, dx$ exists, then the first and the third limits here are equal, and therefore

we get $A = \int_a^b f(x)\, dx$. ...(1)

The equaltity in (1) is a consequence of the definitions of the area of R and the definite integral $\int_a^b f(x)\, dx$. Since f(x) is assumed to be continuous on the interval [a, b], the integral in (1) exists, and hence yields the area of the region R under consideration.

From the Interval Union Property (Sec. 3, Unit 10) of definite integrals, we have

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx, \ a \leq c \leq b. \qquad ...(2)$$

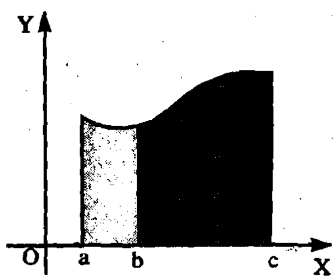


Fig. 3

This means if A_a^c, A_c^b, A_a^b denote the areas under the graph of y = f(x) above the x-axis from a to c, from c to b and from a to b, respectively, (Fig. 3) then, if c is in between a and b, then we have

$$A_a^c + A_c^b = A_a^b \qquad ...(3)$$

If we define $A_a^a = 0, A_b^b = 0$, then the above equation is true for c = a and c = b too.

Till now, we have assumed the function f(x) to be positive in the interval [a, b]. In general, a function f(x) may assume both positive and negative values in the interval [a, b]. To cover such a case, we introduce the convention about *signed* areas.

The area is taken to be positive above the x-axis as we go from left to right, and negative if we go from right to left. The function f(x) may be defined beyond the interval [a, b] also. In that case (3) is true even if c is beyond b, since according to our convention of signed areas, A_c^b will turn out to be a negative quantity (Fig. 4).

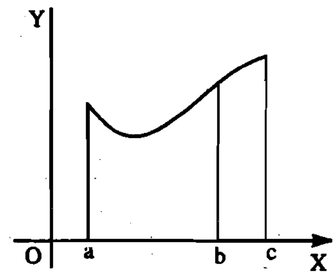


Fig. 4

Thus, $A_a^b = A_a^c + A_c^b = A_a^c - A_b^c$.

Or, $A_a^b + A_b^c = A_a^c$.

Now, if $f(x) \leq 0$ for all x in some interval [a, b], then by applying the definition of "area of R" to the function $-f(x)$, we get the area $A = -\int_a^b f(x)\,dx$.

If we do not take the negative sign, the value of the area will come out to be negative, since f(x) is negative for all $x \in [a, b]$. To avoid a "negative" area, we follow this convention. Thus, if $f(x) \leq 0$ for $x \in [a, b]$ (Fig. 5), then the area between the ordinates x = a and x = b will be

$$A = -\int_a^b f(x)\,dx$$

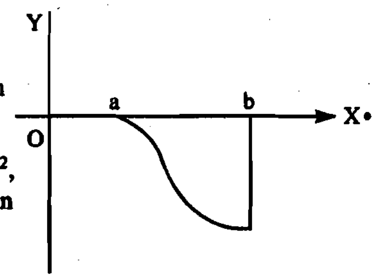


Fig. 5

The following examples will illustrate how our knowledge of evaluating definite integrals can be used to calculate certain areas.

Example 1: Suppose we want to find the area of the region bounded by the curve $y = 16 - x^2$, the x-axis and the ordinates x = 3, x = –3. The region R, whose area is to be found, is shown in Fig. 6.

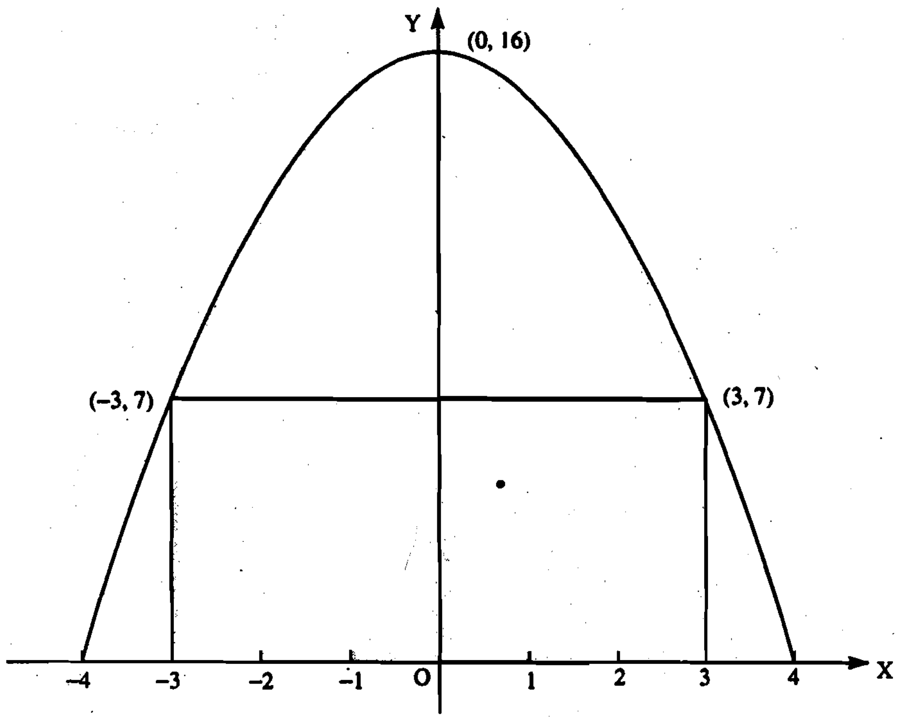


Fig. 6

The area A of the region R is given by

$$A = \int_{-3}^{3} (16 - x^2)\,dx$$

$$= \left[16x - \frac{x^3}{3}\right]_{-3}^{3}$$

$$= 78$$

Example 2: Consider the shaded region R in Fig. 7.

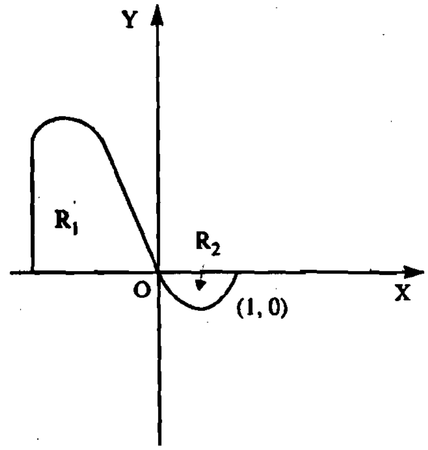


Fig. 7

R is composed of two parts, the region R_1 and the region R_2. We have
Area R = Area R_1 + Area R_2

The region R_1 is bounded above the x-axis by the graph of
$y = x^3 + x^2 - 2x$, $x = -2$ and $x = 0$.

Hence,

$$\text{Area } R_1 = \int_{-2}^{0} (x^3 + x^2 - 2x)\,dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{-2}^{1}$$

$$= \frac{8}{3}$$

The region R_2 is bounded below the x-axis by the graph of
$y = x^3 + x^2 - 2x$, $x = 0$ and $x = 1$.

Hence,

$$\text{Area } R_2 = -\int_{0}^{1} (x^3 + x^2 - 2x)\,dx$$

$$= -\left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{0}^{1} = \frac{5}{12}$$

Therefore, Area R $= \frac{8}{3} + \frac{5}{12} = \frac{37}{12}$.

In this example we had to calculate area R_1 and area R_2 separately, since the region R_2 was below the x-axis. Therefore, according to our convention

$$\text{Area } R_2 = -\int_{0}^{1} f(x)\,dx.$$

If we calculate $\int_{-2}^{1} f(x)\,dx$, it will amount to calculating

$\int_{-2}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$ = area R_1 − area R_2, which would be a wrong estimate of area R.

Example 3: Let us find the area of the smaller region lying above the x-axis and included between the circle $x^2 + y^2 = 2x$ and the parabola $y^2 = x$.

On solving the equation $x^2 + y^2 = 2x$ and $y^2 = x$ simultaneously, we get (0, 0), (1, 1), (1, –1) as the points of intersection of the given curves. We have to find the area of the region R bounded by OAPBO (Fig. 8).

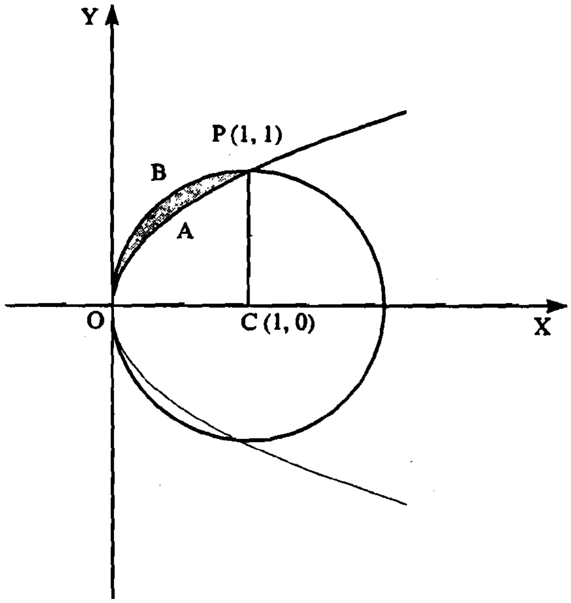


Fig. 8

From the figure we see that area of region OAPBO

= area of region OCPBO – area of region OCPAO

$$= \int_0^1 \sqrt{2x - x^2}\, dx - \int_0^1 \sqrt{x}\, dx$$

Now, $$\int_0^1 \sqrt{2x - x^2}\, dx = \int_0^1 \sqrt{1 - (1-x)^2}\, dx$$

$$= \int_{\pi/2}^{0} \cos\theta \;\; (-\cos\theta)\, d\theta, \text{ on putting } 1 - x = \sin\theta$$

$$= \int_{\pi/2}^{0} -\cos^2\theta\, d\theta = \int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi}{4}$$

Also, $\int_0^1 \sqrt{x}\, dx = \frac{2}{3}$

Therefore, the required area $= \left(\frac{\pi}{4} - \frac{2}{3}\right)$

Try to solve these exercises now.

E E1) Find the area under the curve $y = \sin x$ between $x = 0$ and $x = \pi$.

E E2) Find the area bounded by the x-axis, the curve $y = e^x$, and the ordinates $x = 1$ and $x = 2$.

E E3) Find the area of the region bounded by the curve $y = 5x - x^2$, $x = 0$, $x = 5$ and lying above the x-axis.

E E4) Find the area cut off from the parabola $y^2 = 4ax$ by its latus rectum, $x = a$.

E5) Find the area between the parabola $y^2 = 4ax$ and the chord $y = mx$.

In this sub-section we have derived a formula (Formula (1)) to find the area under a curve when the equation of the curve is given in the cartesian form. With slight modifications we can use this formula to find the area when the curve is described by a pair of parametric equations.

We shall take a look at curves given by parametric equations a little later. But first, let us consider the curves given by a polar equation.

2.2.2 Polar Equations

Sometimes the Cartesian equation of a curve is very complicated, but its polar equation is not so. Cardioids and spirals which you have encountered in Unit 9 are examples of such curves. For these curves it is much simpler to work with their polar equation rather than with the Cartesian ones. In this sub-section we shall see how to find the area under a curve if the equation of the curve is given in the polar form. Here we shall try to approximate the given area through the areas of a series of circular sectors. These circular sectors will perform the same function here as rectangles did in Cartesian coordinates.

Let $r = f(\theta)$ determine a continuous curve between the rays $\theta = \alpha$ and $\theta = \beta$, $(\beta - \alpha \leq 2\pi)$.

We want to find the area A(R) of the shaded region R in Fig. 9 (a).

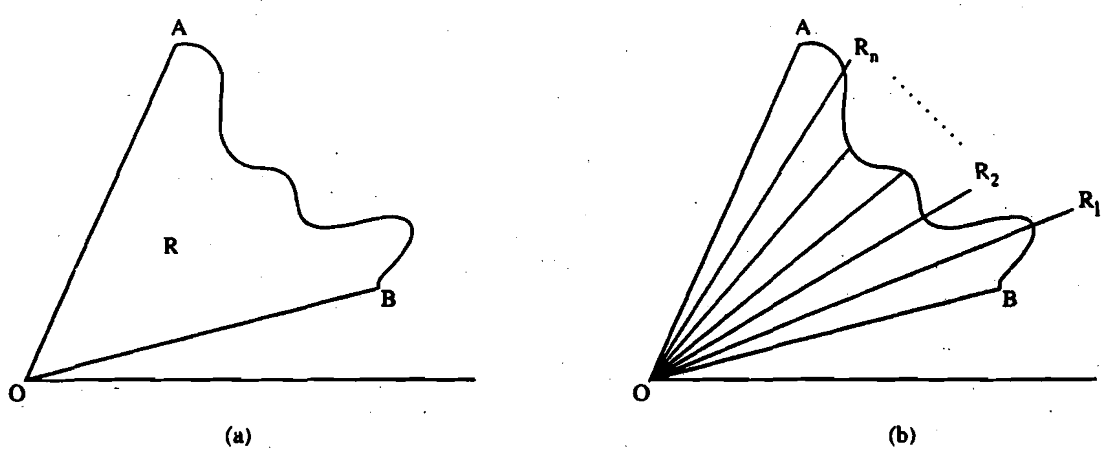


Fig. 9

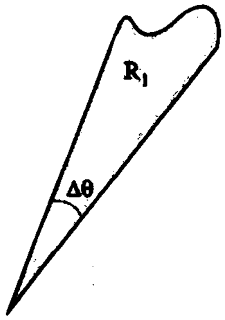


Fig. 10

The area of a sector of a circle of radius r and sectorial angle $\Delta\theta$ is $1/2\ r^2\ \Delta\theta$.

Imagine that the angle AOB is divided into n equal parts, each measuring $\Delta\theta$.

Then $\Delta\theta = \dfrac{\beta-\alpha}{n}$. This amounts to slicing R into n smaller regions,

$R_1, R_2, \ldots\ldots R_n$, as shown in Fig. 9 (b).

Then clearly

$$A(R) = A(R_1) + A(R_2) + \ldots\ldots + A(R_n),$$
$$= \sum_{i=1}^{n} A(R_i)$$

Now let us take the i^{th} slice R_i, and try to approximate its area. Look at Fig. 10.

Suppose f attains its minimum and maximum values on $[\theta_{i-1}, \theta_i]$ at u_i and v_i.

Then $\frac{1}{2}\ [f(u_i)]^2\ \Delta\theta \le A(R_i) \le \frac{1}{2}\ [f(v_i)]^2\ \Delta\theta$.

From this we get

$$\sum_{i=1}^{n} \frac{1}{2}[f(u_i)]^2\ \Delta\theta \le \sum_{i=1}^{n} A(R_i) \le \sum_{i=1}^{n} \frac{1}{2}[f(v_i)]^2\ \Delta\theta$$

The first and the third sums in this inequality are the lower and upper Riemann sums Block 3 for the same definite integral,

namely, $\displaystyle\int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2\ d\theta$.

Therefore, by applying the sandwich theorem as $\Delta\theta \to 0$, we get

$$A(R) = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\ d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta \qquad \ldots(4)$$

We shall illustrate the use of this formula through some examples. Study them carefully, so that you can do the exercises that follow later.

Example 4: Suppose we want to find the area enclosed by the cardioid $r = a(1-\cos\theta)$.

We have $r = 0$ for $\theta = 0$ and $r = 2a$ for $\theta = \pi$.

Since $\cos\theta = \cos(-\theta)$, the cardioid is symmetrical about the initial lines AOX (Fig. 11).

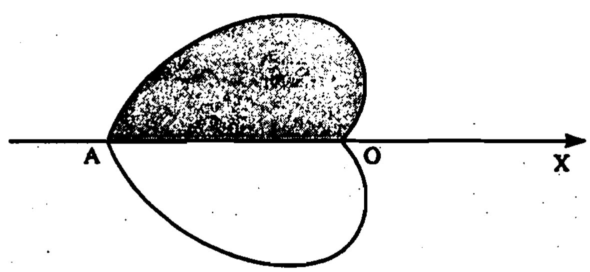


Fig. 11

Hence the required area A, which is twice the area of the shaded region in Fig. 11, is given by

$$A \quad = 2\int_0^{\pi}\frac{1}{2}r^2\ d\theta$$
$$= \int_0^{\pi} a^2\ (1-\cos\theta)^2\ d\theta$$
$$= 4a^2\int_0^{\pi}\sin^4\frac{\theta}{2}\ d\theta,\ \text{since}\ \cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= 8a^2 \int_0^{\pi/2} \sin^4 \phi \, d\phi, \text{ where } \phi = \frac{\theta}{2}$$

$$= 8a^2 \frac{3}{4} \frac{1}{2} \frac{\pi}{4} \text{ by applying the reduction formula from Section 3 of Block 3.}$$

$$= \frac{3}{2} a^2 \pi.$$

In the case of some Cartesian equations of higher degree it is often convenient to change the equation into polar form. The following example gives one such situation.

Example 5: To find the area of the loop of the curve.
$x^5 + y^5 = 5ax^2y^2$,

we change the given equation into a polar equation by the transformation $x = r \cos \theta$ and $y = r \sin \theta$. Thus, we obtain

$$r = \frac{5a \cos^2 \theta \sin^2 \theta}{\cos^5 \theta + \sin^5 \theta}$$

which yields $r = 0$ for $\theta = 0$ and $\theta = \pi/2$. Hence, area A of the loop is that of a sectorial area bounded by the curve and radius vectors $\theta = 0$ and $\theta = \pi/2$, that is, the area swept out by the radius vector as it moves from $\theta = 0$ to $\theta = \pi/2$. See Fig. 12.

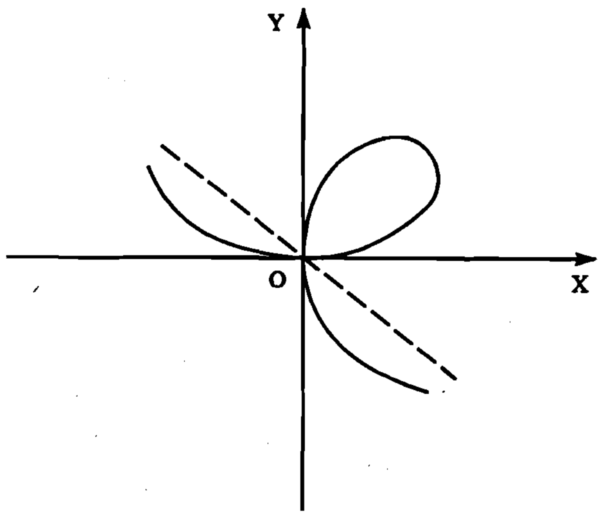


Fig. 12

Thus,

$$A = \frac{1}{2} \int_0^{\pi/2} \frac{25a^2 \cos^4 \theta \sin^4 \theta}{(\cos^5 \theta + \sin^5 \theta)^2} d\theta$$

$$= \frac{25}{2} a^2 \int_0^{\pi/2} \frac{\tan^4 \theta \sec^2 \theta}{(1 + \tan^5 \theta)^2} d\theta$$

$$= \frac{5}{2} a^2 \int_1^{\infty} \frac{dt}{t^2}, \text{ where } t = 1 + \tan^5 \theta.$$

$$= \frac{5}{2} a^2 [-1/t]_1^{\infty} = \frac{5}{2} a^2$$

Try to do these exercises now.

E E6) Find the area of a loop of the curve $r = a \sin 3\theta$.

E E7) Find the area enclosed by the curve $r = a \cos 2\theta$ and the radius vectors $\theta = 0, \theta = \pi/2$.

E E8) Find the area of the region outside the circle $r = 2$ and inside the lemniscate $r^2 = 8 \cos 2\theta$.

[*Hint:* First find the points of intersection. Then the required area = the area under the lemniscate – the area under the circle.]

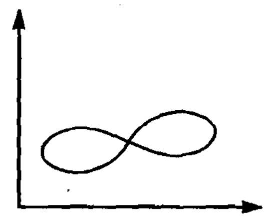
2.2.3 Area Bounded by a Closed Curve

Now we shall turn our attention to closed curves whose equations are given in the parametric form.

Let the parametric equations

$x = \phi(t), y = \psi(t), t \in [\alpha, \beta]$,
where $\phi(\alpha) = \phi(\beta)$, and $\psi(\alpha) = \psi(\beta)$, represent a plane closed curve (Fig. 13).

Here we shall consider curves which do not intersect themselves, e.g. of the form:



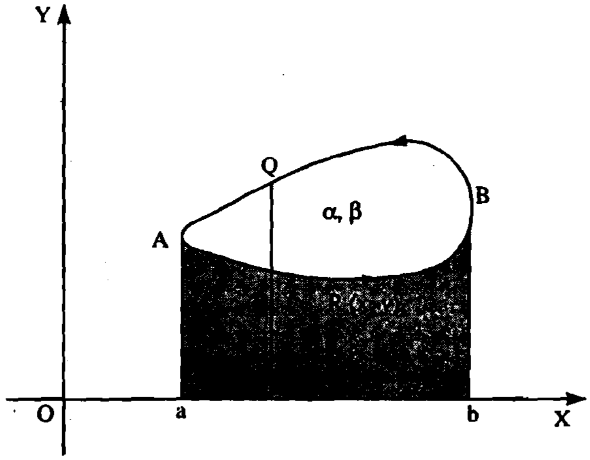


Fig. 13

This means that as the parameter t increases from a value α to a value β, the point P(x, y) describes the curve completely in the counter clockwise sense. Since the curve is closed, the points on it corresponding to the value β is the same as the point corresponding to the value α. This is reflected by the conditions $\phi(\alpha) = \phi(\beta)$ and $\psi(\alpha) = \psi(\beta)$.

Suppose further that the curve is cut at most in two points by every line drawn parallel to the x or y-axis. We also assume that the functions ϕ and ψ are differentiable, and that the derivatives ϕ' and ψ' do not vanish simultaneously. Let the point R on the closed curve correspond to the values α and β i.e., at R we have $\phi(\alpha) = \phi(\beta)$ and $\psi(\alpha) = \psi(\beta)$.

Now suppose A is a point on the curve which has the least x-coordinate, say a. Similarly, suppose B is a point on the curve which has the greatest x-coordinate, say b. Thus the lines x = a and x = b touch the curve in points A and B, respectively. Further let t_1 and t_2 be the values of t that correspond to A and B, respectively. Then,

$\alpha < t_2 < t_1 < \beta$.

Let a point Q correspond to $t = t_3$ such that $t_2 < t_3 < t_1$. The area of the region enclosed is $S = S_2 - S_1$, where S_2 and S_1 are the areas under the arcs AQB and ARB, respectively. The minus is because one is clockwise and other is anti-clockwise (see Fig. 13). Hence,

$$S_2 = \underset{(AQB)}{\int_a^b} y\,dx \text{ and } S_1 = \underset{(ARB)}{\int_a^b} y\,dx$$

Now, as a point P(x, y) moves from B to A along BQA, the value of the parameter increases from t_2 to t_1. Therefore,

$$\underset{(BQA)}{\int_b^a} y\,dx = \int_{t_2}^{t_1} y \frac{dx}{dt}\,dt$$

$$\text{Hence } S_2 = -\int_{t_2}^{t_1} y \frac{dx}{dt}\,dt$$

Note how we have modified formula (1) for a pair of parametric equations.

Remember the point R corresponds to $t = \alpha$ and also to $t = \beta$.

Now the movement of P from A to B along ARB, can be viewed in two parts: From A to R and from R to B. As P moves from A to R, the value of the parameter increases from t_1 to β, and as P moves from R to B, t increases from α to t_2.

$$\text{Therefore } S_1 = -\underset{(ARB)}{\int_a^b} y\,dx = \int_{t_1}^{\beta} y\frac{dx}{dt}\,dt + \int_{\alpha}^{t_2} y\frac{dx}{dt}\,dt$$

Thus, we have

$$S = \underset{(AQB)}{\int_a^b} ydx - \underset{(ARB)}{\int_a^b} ydx = S_2 - S_1$$

$$= -0\int_{t_2}^{t_1} y\frac{dx}{dt}\,dt - \int_{t_1}^{b} y\frac{dx}{dt}\,dt - \int_{a}^{t_2} y\frac{dx}{dt}\,dt = -\int_a^b y\frac{dx}{dt}\,dt \quad \text{...(i)}$$

Note that the negative sign is due to the direction in which we go round the curve as marked by arrows in Fig. 13.

Similarly, by drawing tangents to the curve that are parallel to the x-axis, it can be shown that

$$S = \int_{\alpha}^{\beta} x\frac{dy}{dt}\,dt \quad \text{...(ii)}$$

From (i) and (ii), we get

$$2S = \int_{\alpha}^{\beta}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

Hence, the area enclosed is

$$S = \frac{1}{2}\int_{\alpha}^{\beta}(xdy - ydx) \quad \text{...(5)}$$

We can use any of the formulas (i), (ii) and (5) above for calculating S. But in many cases you will find that formula (5) is more convenient because of its symmetry.

Example 6: Let us find the area of the astroid

$x = a\cos^3 t,\ y = b\sin^3 t,\ 0 \le t \le 2\pi.$

The region bounded by the astroid is shown in Fig. 14.

The area A of the region is given by

$$A = \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \frac{1}{2}\int_{\alpha}^{2\pi} a\cos^3 t\ (3b\sin^2 t\cos t) - b\sin^3 t(-3a\cos^2 t\sin t)\,dt$$

$$= \frac{3ab}{2}\int_0^{2\pi}\cos^2 t\sin^2 t\,dt$$

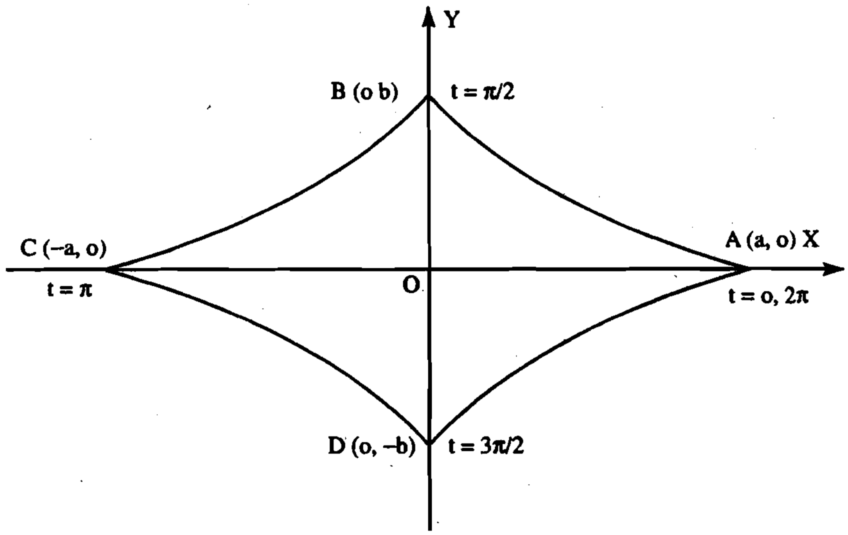


Fig. 14

We have seen in Section 3 of Unit 11 that

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$$

$$= 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x).$$

Here $\cos^2(2\pi - t)\sin^2(2\pi - t) = \cos^2 t \sin^2 t$.

Hence $\int_0^{2\pi} \cos^2 t \ \sin^2 t\,dt = 2\int_0^{\pi} \cos^2 t \ \sin^2 t\,dt$.

Therefore, $A = 3ab \int_0^{\pi} \cos^2 t \ \sin^2 t\,dt$.

Now, by a similar argument we can say that

$$A = 6\,ab \int_0^{\pi/2} \cos^2 t \ \sin^2 t\,dt,$$

$= \dfrac{3\pi\,ab}{8}$, by using the reduction formula from Section 4 of Unit 12.

You can solve these exercises now.

E E9) Find the area of the curve
$x = a(3\sin\theta - \sin^3\theta)$, $y = a\cos^3\theta$, $0 \le \theta \le 2\pi$.

E E 10) Find the area enclosed by the curve
$x = a\cos\theta + b\sin\theta + c$
$y = a'\cos\theta + b'\sin\theta + c'$, where $0 \le \theta \le 2\pi$

E E 11) Find the area of one of the loops of the curve $x = a\sin 2t$, $y = a\sin t$.

(*Hint:* first find two values of t which give the same values of x and y, and take these as the limits of integration.)

2.3 NUMERICAL INTEGRATION

In many practical problems, the value of the integrand, that is, the function whose integral is required, are known only at some chosen points. For example,

x	x_0	x_1	x_2	x_3	x_4	x_5
y = f(x)	y_0	y_1	y_2	y_3	y_4	y_5

In some cases, no simple integral is known for the given integrand. For example, the function $\frac{\sin x}{x}$ does not have a simple indefinite integral. In such a situation the integral

$\int_a^b f(x)\,dx$ cannot be evaluated exactly. But we can find an approximate value of the integral

by considering the sum of the areas of inscribed (inner) or circumscribed (outer) rectangles, as we have seen at the beginning of this unit. In this section, we shall describe two more methods for approximating the value of an integral:

i) Trapezoidal Rule, and ii) Simpson's Rule.

2.3.1 Trapezoidal Rule

We know that a given definite integral can be approximated by inner and outer rectangles. A better method is given by the trapezoidal rule in which we approximate the area of each strip by the area of a trapezium. Such an approximation is also called a **linear** approximation since the portions of the curve in each strip are approximated by line segments. As before, we divide the interval [a, b] into n subintervals, each of length

$\Delta x = \frac{(b-a)}{n}$, by using the points $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, ...,

$x_{n-1} = a + (n-1)\Delta x$ between $x_0 = a$ and $x_n = b$. Then

$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \ldots + \int_{x_{n-1}}^{b} f(x)\,dx$$

$$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\,dx.$$

Now, we approximate the first integral on the RHS by the area of the trapezium $aP_0P_1x_1$ (Fig. 15), the second by $x_1P_1P_2x_2$ and so on, thus getting

$$\int_a^b f(x)\,dx = \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \ldots + \frac{1}{2}(y_{n-1} + y_n)\Delta x.$$

$$= \left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right)\Delta x \qquad \ldots(6)$$

The area of a trapezium = $\frac{1}{2}$ (the sum of its parallel sides) × height.

where $y_0 = f(x_0)$, $y_1 = f(x_1)$, ..., $y_n = f(x_n)$

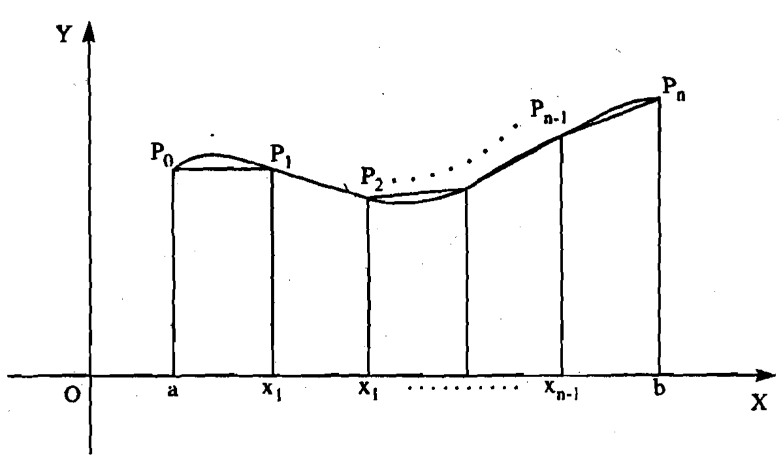


Fig. 15

The formula (6) is known as the trapezoidal rule for approximating the value of a definite integral.

See if you can solve this exercise now.

E E 12) Use the trapezoidal rule to estimate the following integrals with the given value of n.

a) $\int_1^2 x^2 \, dx, n = 4$ b) $\int_1^4 \sqrt{x^2 + 4} \, dx, n = 6.$

2.3.2 Simpson's Rule

In this method, instead of approximating the given curve by line segments, we approximate it by segments of a simple curve such as a parabola.

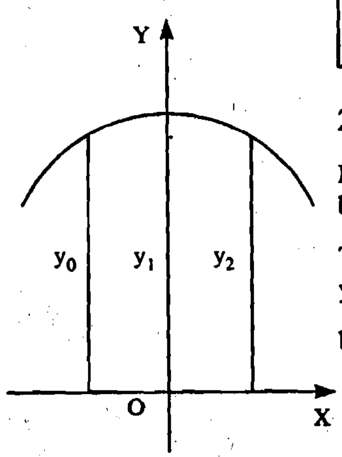


Fig. 16

The area under the arc of the parabola
$y = Ax^2 + Bx + C$

between $x = -h$ and $x = h$ (Fig. 16) is given by

$$A_1 = \int_{-h}^{h} (Ax^2 + Bx + C) \, dx$$

$$= \frac{2\,Ah^3}{3} + 2\,C.h$$

Since the curve passes through the points, $(-h, y_0)$, $(0, y_1)$ and (h, y_2), substituting the coordinates of these points in the equation of the parabola, we get

$$y_0 = Ah^2 - Bh + C$$
$$y_1 = C$$
$$y_2 = Ah^2 + Bh + C,$$

from which we get

$$C = y_1$$
$$Ah^2 - Bh = y_0 - y_1$$
$$Ah^2 + Bh = y_2 - y_1$$
$$\Rightarrow 2\,Ah^2 = y_0 + y_2 - 2y_1$$

Hence, on substitution we get

$$A_1 = \frac{h}{3}\,[2Ah^2 + 6C] = \frac{h}{3}\,[(y_0 + y_2 - 2y_1) + 6y_1]$$

or $$A_1 = \frac{h}{3}\,[y_0 + 4y_1 + y_2] \qquad \ldots(7)$$

To obtain Simpson's rule, we apply the above result to successive pieces of the curve $y = f(x)$ between $x = a$ and $x = b$. For this we divide $[a, b]$ into n sub-intervals, each of width h.

Then we approximate the portion of the curve in each pair of sub-intervals by an arc of a parabola passing through the end points of that portion of the curve and the point corresponding to the common point (Fig. 17) of those sub-intervals.

Now consider the first two sub-intervals $[a, x_1]$ and $[x_1, x_2]$. The points on the curve corresponding to a, x_1 and x_2 are P_0, P_1 and P_2, respectively. Let us draw a parabola passing through P_0, P_1 and P_2. We will assume that the portion of the curve passing through P_0, P_1 and P_2 coincides with this parabola. See Fig.17.

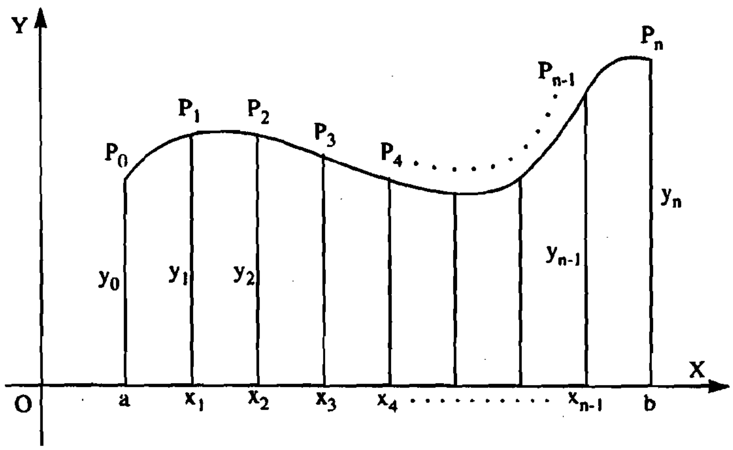


Fig. 17

Similarly we can approximate the portion of the curve in the interval $[x_2, x_4]$ by a parabola passing through the points P_2, P_3 and P_4. We shall repeat this process for the remaining pairs of intervals. Now the area under the parabola $P_0 P_1 P_2$ is given by

$A_1 = \frac{h}{3}\,(y_0 + 4y_1 + y_2)$. (On using formula (7)).

Similarly, the area under the parabola passing through the points P_2, P_3, P_4 is given by

$$A_3 = \frac{h}{3}\,[y_2 + 4y_3 + y_4]$$

Next, we use formula (7) for the parabola passing through the points P_4, P_5, P_6 and get the area

$A_5 = \frac{h}{3}\,[y_4 + 4y_5 + y_6]$, and so on.

Note, that to approximate the whole area under the given curve in this manner, the number n of the subdivisions of the interval [a, b] must be even. Summing all the areas we obtain

$$A = \frac{h}{3}\,[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n]$$

as the total area. The above formula can also be written in the form

$$A = \frac{h}{3}\,[y_0 + y_{2m} + 2(y_2 + y_4 + \ldots + y_{2m-2}) + 4(y_1 + y_3 + \ldots + y_{2m-1})],$$

where we have taken n =2m. Note that $y_0, y_1, y_2, \ldots, y_{2m}$, are the values of the function, f, for x = a, a + h, a + 2h, ..., a + 2mh = b, respectively. Hence by Simpson's rule, we have

$$\int_a^b f(x)\,dx = \frac{h}{3}\,[(y_0 + y_{2m}) + 2(y_2 + y_4 + \ldots + y_{2m-2}) + 4(y_1 + y_3 + \ldots + y_{2m-1})]$$

The following example will help us compare the accuracy of the trapezoidal and Simpson's rules. You will find that Simpson's rule is more accurate than the trapezoidal rule.

Example 7: Taking four subdivisions of the interval [1, 3], let us find the approximate value of $\int_1^3 x^2 dx$ by the trapezoidal rule and also by Simpson's rule.

Division by 4 gives 0.5 as the width of each sub-interval of [1, 3]. The values of the integrand at these points of subdivision are given in the following table:

x	1	1.5	2	2.5	3
y = f (x)	1 (y_0)	2.25 (y_1)	4 (y_2)	6.25 (y_3)	9 (y_4)

Using the formula for the trapezoidal rule, we obtain

$$\int_1^3 x^2 dx = \left(\frac{1}{2}y_0 + y_1 + y_2 + y_3 + \frac{1}{2}y_4\right)\Delta x$$

$$= \left(\frac{1}{2} + 2.25 + 4 + 6.25 + \frac{9}{2}\right) 0.5 = 8.75 \qquad \ldots(a)$$

Using the formula for Simpson's rule, we get

$$\int_1^3 x^2 dx = \frac{0.5}{3}\,[1 + 9 + 2(4) + 4(2.25 + 6.25)]$$

$$= \frac{1}{6}\,[10 + 8 + 34] = 8.66\ldots.. 6\ldots\ldots\ldots\ldots.. \text{ or } 8.67 \qquad \ldots(b)$$

the actual value of the definite integral

$$\int_1^3 x^2 dx = x^3/3 = 8.66\ldots. \cong 8.67 \qquad \ldots(c)$$

On comparing (a) with the actual value in (c), we observe that the value given by the trapezoidal rule has an error of + 0.08. Comparison of (b) with (c) shows that the error in the value given by Simpson's rule is zero in this case.

Example 8: Let us use the trapezoidal rule with six sub divisions to evaluate

$\int_0^{\pi} \sqrt{\sin x}\, dx$. We shall also find the value of the integral by using Simpson's rule.

We have

x	0	π/6	π/3	π/2	2π/3	5π/6	π
$y = \sqrt{\sin x}$	0	$\sqrt{1/2}$	$\sqrt{\sqrt{3}/2}$	1	$\sqrt{\sqrt{3}/2}$	$\sqrt{1/2}$	0

Using trapezoidal rule, we obtain

$$\int_a^b \sqrt{\sin x}\, dx = \left[\frac{1}{2} \times 0 + \sqrt{1/2} + \sqrt{\sqrt{3}/2} + 1 + \sqrt{\sqrt{3}/2} + \frac{1}{2} \times 0\right]\frac{\pi}{6}$$

$$= \left(\sqrt{2} + 1 + 2\sqrt{\sqrt{3}/2}\right)\frac{\pi}{6} = 2.23$$

Using Simpson' rule, we have

$$\int_0^\pi \sqrt{\sin x}\, dx \equiv \left(\frac{1}{3} \times \frac{\pi}{6}\right)[(0+0) + 2(\sqrt{\sqrt{3}/2} + \sqrt{\sqrt{3}/2}) + 4(\sqrt{1/2} + 1 + \sqrt{1/2})]$$

$$= \frac{4\pi}{18}(\sqrt{\sqrt{3}/2} + 1 + \sqrt{2}) = 2.33$$

Exmaple 9: A river is 80 m wide. The depth d in metres at a distance x meters from one bank is given by the following table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Let us find, approximately, the area of cross-section. Applying Simpson's rule with n = 8, we obtain the area of cross-section

$$A = \frac{10}{3}[(0+3) + 2(7+12+14) + 4(4+9+15+8)]$$

$$= \frac{10}{3}[3 + 66 + 144] = 710 \text{ sq. m.}$$

As we have seen in this example, Simpson's rule is very useful in approximating the area of irregular figures like the corss-sections of lakes and rivers.

See if you can do these exercises now.

E E 13) a) Use Simpson's rule to evaluate the following, taking the given value of n

$$\int_0^\pi \frac{\sin x}{x}\, dx, n = 4$$

b) From the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2},$$ calculate π, using Simpson's rule with h = 0.1

E E 14) A curve is drawn through the points (1, 2), (1.5, 2.4), (2, 2.27), (2.5, 2.8), (3, 3), (3.5, 2.6) and (4, 2.1). Estimate the area between the x-axis and the ordinates x = 1, x = 4.

E E 15) A river has width 30 metres. If the depth y metres at distance x metres from one bank be given by the table.

x	0	5	10	15	20	25	30
y	0	1.2	2.1	2.4	1.6	0.6	0

Find the approximate area of cross-section.

E E 16) The velocity of a train, which starts from rest, is given by the following table, the time being reckoned in minutes from the start and the speed in kms/hr.

Min	2	4	6	8	10	12	14	16	18	20
kms/hr	10	18	25	29	32	20	11	5	2	0

Estimate approximately the total distance run in 20 minutes.

Now let us quickly recall what we have done in this unit.

2.4 SUMMARY

In this unit we have covered the following points:

1) The knowledge of integration is helpful in finding areas enclosed by plane curves when their equations are known in

   a) Cartesian form: $A = \int_a^b y\,dx$

   b) Polar form: $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta.$

2) The area bounded by a closed curve given by parametric equation is

$$A = \int_\alpha^\beta y\frac{dx}{dt} = \frac{1}{2}\int_\alpha^\beta \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt.$$

3) When the integral cannot be exactly evaluated, we can use the method of numerical integration. The two methods given here are:

   a) Trapezoidal rule:

$$\int_a^b f(x)\,dx = \left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right)\Delta x$$

   where n is the number of sub-divisions of [a, b], and Δx the length of each sub-interval.

   b) Simpson's rule:

$$\int_a^b f(x)\,dx = \frac{h}{3}\,[(y_0 + y_{2m}) + 2(y_2 + y_4 + \ldots + y_{2m-2}) + 4(y_1 + y_3 + \ldots + y_{2m-1})]$$

   when [a, b] is divided into 2m sub-intervals of length h.

2.5 SOLUTIONS AND ANSWERS

E 1) $\int_0^\pi \sin x\,dx = -\cos x]_0^\pi$

$= -\cos\pi + \cos 0 = 1 + 1 = 2.$

E2) $\int_1^2 e^x \, dx = e^x]_1^2 = e^2 - e.$

E3) $\int_0^5 (5x - x^2] \, dx = \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5 = \frac{125}{6}.$

E4) $y = 2\sqrt{ax}$

$$A = 2\int_0^a 2\sqrt{ax} \, dx = 4\sqrt{a}\, \frac{x^{3/2}}{3/2}\Bigg]_0^a = 4\sqrt{a}\,\frac{a^{3/2}}{3/2} = \frac{8a^2}{3}.$$

E5) Points of intersection of $y^2 = 4ax$ and $y = mx$ are $(0, 0)$ and $(4a/m^2, 4a/m)$.

$$\therefore\ A = \int_0^{4a/m^2} 2\sqrt{ax} \, dx - \int_0^{4a/m^2} mx \, dx = 2\sqrt{a}\, \frac{x^{3/2}}{3/2}\Bigg]_0^{4a/m^2} - \frac{mx^2}{2}\Bigg]_0^{4a/m^2}$$

$$= \frac{4\sqrt{a}}{3}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2 = \frac{8a^2}{3m^3}.$$

E6) For a loop, we should find two distinct, consecutive values of θ, for which we get the same value of r.

If $r_1 = a\sin 3\theta_1$, and $r_2 = a \sin 3\theta_2$,

$r_2 = r_2 \Rightarrow \sin 3\theta_1 - \sin 3\theta_2 = 0.$

$$\text{or } 2\cos\frac{3(\theta_1 + \theta_2)}{2}\sin\frac{3(\theta_1 - \theta_2)}{2} = 0$$

$$\therefore\ \frac{3(\theta_1 + \theta_2)}{2} = \frac{\pi}{2} \text{ or } \frac{3(\theta_1 - \theta_2)}{2} = 0$$

$$\therefore\ \theta_1 + \theta_2 = \frac{\pi}{3} \text{ or } \theta_1 = \theta_2.$$

Thus, $\theta_1 = 0$ and $\theta_2 = \pi/3$ will give the same value of r.

$$\therefore\ A = \frac{1}{2}\int_0^{\pi/3} r^2 \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/3} \sin^2 3\theta \, d\theta$$

$$= \frac{a^2}{6}\int_0^{\pi} \sin^2 u \, du \quad \text{if } u = 3\theta.$$

$$= \frac{a^2\pi}{12}.$$

E7) $$A = \frac{1}{2}\int_0^{\pi/2} a^2\cos^2 2\theta \, d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi} \cos^2\phi \, d\phi, \quad \phi = 2\theta$$

$$= \frac{a^2}{4}\left(\int_0^{\pi/2} \cos^2 \phi \, d\phi + \int_0^{\pi/2} \cos^2 (\pi - \phi)\, d\phi\right)$$

$$= \frac{a^2}{4}\left(\int_0^{\pi/2} \cos^2 \phi \, d\phi + \int_0^{\pi/2} \cos^2 \phi \, d\phi\right)$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \cos^2 \phi \, d\phi = \pi a^2/8.$$

E8) Points of intersection are given by
$8\cos 2\theta = 4$ or $\cos 2\theta = 1/2$, i.e., $\cos^2\theta = 3/4$

$$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Because of symmetry w.r.t. the initial line, the required area A = 2 (the area under the lemniscate above the initial line from $\theta = 0$ to $\theta = \pi/6$ and from $\theta = 5\pi/6$ to $\theta = \pi$ minus the area under the wide from $\theta = 0$ to $\theta = \pi/6$ and from $\theta = 5\pi/6$ to $\theta = \pi$.

$$= 2\left[\frac{1}{2}\int_0^{\pi/6} 64\cos^2 2\theta\, d\theta \; \frac{1}{2}\int_{5\pi/6}^{\pi/6} 64\cos^2 2\theta\, d\theta - \frac{1}{2}\int_{5\pi/6}^{\pi} 4\,d\theta\right.$$

$$= 8\sqrt{3} + 28\pi/3$$

E9) $x = a(3\sin\theta - \sin^3\theta)$, $y = a\cos^3\theta$

$$\frac{dx}{d\theta} = 3a(\cos\theta - \sin^2\theta\cos\theta) = 3a\cos^3\theta$$

$$\therefore \int_0^{2\pi} y\frac{dx}{d\theta}\, d\theta = 3a^2 \int_0^{2\pi} \cos^6\theta\, d\theta = 12a^2 \int_0^{\pi/2} \cos^6\theta\, d\theta$$

$$= 12a^2\,\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad \text{(reduction formula)}$$

$$= \frac{15a^2\pi}{8}$$

E 10) $\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta$, $\frac{dy}{d\theta} = -a'\sin\theta + b'\cos\theta$

$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = (ab' - ba') + (ac' - ca')\sin\theta + (cb' - bc')\cos\theta$$

$$\therefore \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right)d\theta = (ab' - ba')\pi$$

E 11) A loop of this curve lies between $t = 0$ and $t = \pi$

$$\int_0^{\pi} x\frac{dy}{dt} = a^2\int_0^{\pi} \sin 2t\cos t\, dt = 2a^2\int_0^{\pi} \sin t\cos^2 t\, dt$$

$$= 4a^2\int_0^{\pi/2} \sin t\cos^2 t\, dt$$

$$= 4a^2\left.\frac{\cos^3 t}{3}\right]_0^{\pi/2} = \frac{4a^2}{3}$$

E 12) a)

x	0	1/2	1	3/2	2
x^2	0	1/4	1	9/4	4

$$A = \left(\frac{1}{2}\times 0 + 1/4 + 1 + 9/4 + \frac{1}{2}\times 4\right)\frac{1}{2}$$

$= 2.75.$

b)

x	1	1.5	2	2.5	3	3.5	4
$\sqrt{x^2+4}$	$\sqrt{5}$	2.5	$2\sqrt{2}$	$\sqrt{10.25}$	$\sqrt{13}$	$\sqrt{16.25}$	$2\sqrt{5}$

$$A \cong \left(\frac{1}{2}\times\sqrt{5} + 2.5 + 2\sqrt{2} + \sqrt{10.25} + \sqrt{13} + \sqrt{16.25} + \sqrt{5}\right)\frac{1}{2}$$

$= 9.76.$

E 13) a)

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$\frac{\sin x}{x}$	1	$4/\sqrt{2}\pi$	$2/\pi$	$4/\sqrt{2}\pi$	0

$$\therefore \int_0^{\pi}\frac{\sin x}{x}dx \cong \frac{\pi}{12}\left[1+0+2(2/\pi)+4(4/\sqrt{2}\pi+4/\sqrt{2}\,\pi)\right]$$

$$= \frac{\pi}{12}\left[1+\frac{4+16\sqrt{2}}{\pi}\right]$$

$$= \frac{\pi}{12}+\frac{1+4\sqrt{2}}{3}$$

$= 1.0665$

b)

x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\frac{1}{1+x^2}$	1	.99	.96	.92	.86	.8	.73	.67	.61	.55	.5

$$\therefore \int_0^1 \frac{1}{1+x^2}dx \cong \frac{.1}{3}\left[\frac{3}{2}+2(.96+.86+.73+.61)+4(.99+.92+.8+.67+.55)\right]$$

$= 0.7846$

Now $\pi/4 \cong 0.7846 \Rightarrow \pi \cong 3.1384.$

E 14)

1	1.5	2	2.5	3	3.5	4
2	2.4	2.27	2.8	3	2.6	2.1

$A = 0.5\,(1+2.4+2.8+2.27+3+2.6+1.05)$

$= 0.5\,(5.12) = 7.56$ (trapezoidal rule).

E 15) $A \cong \frac{5}{3}\,[0+2(2.1+1.6)+4\,(1.2+2.4+0.6)]$

$= \frac{5}{3}\,(7.4+16.8) = \frac{5}{3}\,(24.2) = \frac{121}{3}$ (Simpson's rule).

E 16) $S = 2(5+18+25+29+32+20+11+5+2)$

$= 2(147) = 294.$