UNIT 4 INTEGRATION OF RATIONAL AND IRRATIONAL FUNCTIONS

Structure

4.1 INTRODUCTION

In the previous unit you have come across various methods of interation. This unit, which is the last one in this block, will complete the discussion of methods of integration in this course. Here we shall deal with the integration of rational functions in detail. The method. which we shall describe in Sec.2, depends upon partial fraction decomposition with which you might be already familiar.

Later on the unit we shall consider some simple types of irrational functions. But a full discussion of the integration of irrational functions is beyond the scope of this course. We end the unit by giving you a check list of points to be considered before deciding upon the method of integration for any given function. While going through this unit you will need to recall several standard forms like.

$\int \frac{dx}{\sqrt{x^2+a^2}}$. $\int \sqrt{x^2+a^2}\, dx$ etc. which we have already covered in Unit 13.

Objectives

After reading this unit you should be able to :

- recognise proper and improper rational functions
- integrate rational functions of a variable by using the method of partial fractions
- integrate certain types of rational functions of sinx and cosx
- evaluate the integrals of some specified types of irrational functions
- decide upon the method of integration to be used for integrating any given function.

4.2 INTEGRATION OF RATIONAL FUNCTIONS

We know by now that it is easy to integrate any polynomial function, that is, a function f given by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_n$. In this section we shall see how a rational function is integrated. But let us first define a rational function.

Definition 1 A function R is called a **rational function** if it is given by $R(x) = Q(x)/P(x)$, where $Q(x)$ and $P(x)$ are polynomials. It is defined for all x for which $P(x) \neq 0$.

If the degree of $Q(x)$ is less than the degree of $P(x)$, we say the $R(x)$ is a **proper rational function.** Otherwise, it is called an improper rational function, Thus,

$$f(x) = \frac{x+1}{x^2+x+2} \text{ is a proper rational function, and}$$

$$g(x) = \frac{x^3+x+5}{x-2} \text{ is an improper one.}$$

But g(x) can also be written as

$$g(x) = (x^2+3x+6) + \frac{17}{x-2} \text{ (by long division)}$$

$$\begin{array}{r|l} & x^2 + 3x + 6 \\ \hline x-2 & x^3 + x + 5 \\ & x^3 - 2x^2 \\ \hline & 3x^2 + 5 \\ & 3x^2 - 6x \\ \hline & 6x + 5 \\ & 6x - 12 \\ \hline & 17 \end{array}$$

Here we have expressed g(x), which is an improper rational function, as the sum of a polynomial and a proper rational function. This can be done for any improper rational function. Thus, we can always write

an improper rational function	=	a polynomial.	+	a proper rational function

As we have already observed, a polynomial can be easily integrated. This means that the problem of integrating an improper rational function is reduced to that of integrating a proper rational function. Therefore, it is enough to study the techniques of integrating proper rational functions. But first let's see whether you can identify proper rational functions.

E E1) Which of the following functions are proper rational functions? Write the improper ones as a sum of a polynomial and a proper rational functions.

a) $\frac{x^3+1}{x^4+x+1}$ b) $\frac{x^2+x-3}{x^2+1}$ c) $\frac{x+8}{x^2+5x+18}$

d) $\frac{x^4+x^3-5}{x-2}$

4.2.1 Some Simple Rational Functions

Now we shall consider some simple types of proper rational functions, like

$\frac{1}{x-a}$, $\frac{1}{(x-b)^k}$ and $\frac{x-m}{ax^2+bx+c}$. Later you will see that any proper rational function can be written as a sum of these simple types of functions.

We shall illustrate the method of integrating these functions through some examples.

Example 1 The simplest proper rational function is of the type $\frac{1}{(x-a)}$. From Unit 2, we already know that

$$\int \frac{1}{(x-a)}\, dx = \ln|x-a| + c.$$

Example 2 Consider the function $f(x) = \dfrac{1}{(x+2)^4}$

To integrate this function we shall use the method of substitution which we have studied in Unit 3. Thus, if we put

$u = x + 2 \quad \dfrac{du}{dx} = 1$ and we can write

$$\int \frac{1}{(x+2)^4}\,dx = \int \frac{1}{u^4}\,du = \int u^{-4}\,du$$

$$= \frac{u^{-3}}{-3} + c = \frac{1}{3(x+2)^3} + c$$

The next example is a little more complicated.

Example 3 Consider the function $f(x) = \dfrac{2x+3}{x^2 - 4x + 5}$

This has a quadratic polynomial in the denominator.

Now $\displaystyle\int \frac{2x+3}{x^2 - 4x + 5}\,dx$ can be written as $\displaystyle\int \frac{2x-4}{x^2 - 4x + 5}\,dx + \int \frac{7}{x^2 - 4x + 5}\,dx$

perthaps you are wondering why we have split the integral into two parts. The reason for this break up is that now the intègrand in the first integral on the right is of the form $\dfrac{g'(x)}{g(x)}$; and we know that

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + c.$$

Thus, $\displaystyle\int \frac{2x-4}{x^2-4x+5}\,dx = \ln|x^2 - 4x + 5| > c_1$

To evaluate the second integral on the right, we write

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{(x^2-4x+4)+1}\,dx = \int \frac{1}{(x-2)^2+1}\,dx$$

Now, if we put $x - 2 = u$, $\dfrac{du}{dx} = 1$ and

$$\int \frac{1}{x^2 - 4x + 5}\,dx = \int \frac{1}{u^2+1}\,du = \tan^{-1} u + c_2$$

$$= \tan^{-1}(x-2) + c_2$$

This implies,

$$\int \frac{2x+3}{x^2-4x+5}\,dx = \ln|x^2 - 4x + 5| + 7\tan^{-1}(x-2) + c$$

In the beginning of this sub-section we said that any proper rational function can be written as the sum of some functions of the type we considered in the three examples above. In the next sub-secton we shall see how this is done. But try to solve an exercise before reading the next section. It will give you some practice in evaluating integrals of the types mentioned in this sub-section.

E E2) Evaluate

a) $\displaystyle\int \frac{dx}{2x-3}$ b) $\displaystyle\int \frac{dt}{(t+5)^2}$ c) $\displaystyle\int \frac{2x+1}{x^2+8x+1}\,dx$ d) $\displaystyle\int \frac{4x+1}{x^2+x+2}\,dx$

4.2.2 Partial Fraction Decomposition

In school you must have studied the factorisation of polynomials. For example, we know that
$x^2 - 5x + 6 = (x-2)(x-3)$

Here $(x-2)$ and $(x-3)$ are two linear factors of $x^2 - 5x + 6$.

You must have also come across polynomials like $x^2 + x + 1$, which cannot be factorised into real linear factors. Thus, it is not always possible to factorise a given polynomial into linear factors. But any polynomial can, in principle, be factored into linear and quadratic factors. We shall not prove this statement here. It is a consequence of the Fundamental theorem of algebra which has been stated in Unit 3 of the Linear Algebra course. The actual factorisation of a polynomial may not be very easy to carry out. But, whenever we can factorise the denominator of a proper rational function we can integrate it by employing the method of partial fractions. The following examples will illustrate this method.

Example 4 Let us evaluate $\int \frac{5x-1}{x^2-1}\,dx$. Here the integrand $\frac{5x-1}{x^2-1}$ is a proper rational function.

Its denominator $x^2 - 1$ can be factored into linear factors as : $x^2 - 1 = (x-1)(x+1)$. This suggests that we can write the decomposition of $\frac{5x-1}{x^2-1}$ into partial fractions as :

$$\frac{5x-1}{x^2-1} = \frac{5x-1}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)}$$

If we multiply both sides by $(x-1)(x+1)$, we get

$5x - 1 = A(x+1) + B(x-1)$. That is,

$5x - 1 = (A+B)x + A - B$.

By equating the coefficients of x we get $A + B = 5$.

Equating the constant terms on both sides we get $A - B = -1$.

Solving these two equations in A and B we get $A = 2$ and $B = 3$

Thus $\frac{5x-1}{x^2-1} = \frac{2}{x-1} + \frac{3}{x+1}$

Integrating both sides of this equations, we obtain,

$$\int \frac{5x-1}{x^2-1}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{3}{x+1}\,dx$$

$2\ln|x-1| + 3\ln|x+1| + c$

As you have seen, the most important step in the evaluation of $\int \frac{5x - 1}{x^2 - 1} dx$ was the decomposition of the integrand into partial fractions. The procedure for finding the values of the two unknowns A and B, involved two simple simultaneous equations in two unknows. But the higher the degree of the denominator, the more will be the number of unknowns, and it might be very tedious to find them, What can we do in such cases? There is a very simple way out.

In the equation

$5x - 1 = A(x+1) + B(x-1)$, if we put $x = -1$, we get $-6 = -2B$, or $B = 3$. Similarly, if we put $x = 1$, we get $4 = 2A$ or $A = 2$. Isn't this a much simpler way of finding A and B ? Let's go on to our next example now.

Note that 1 and −1 are the zeros of the denominator $x^2 - 1$.

Example 5 Suppose we want to integrate $\frac{2x^2 + x - 4}{x^3 - x^2 - 2x}$.

We first observe that the denominator factors as $x(x+1)(x-2)$.

This means we can write

$$\frac{2x^2 + x - 4}{x^3 - x^2 - 2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{c}{x-2}$$

Multiplying by $x^3 - x^2 - 2x$ we get

$$2x^2 + x - 4 = (x+1)(x-2)A + Bx(x-2) + Cx(x+1)$$

Now, if we put $x = 0$ in this equation, we get

0 −1 and 2 are the zeros of $x^2 - 1$. $x^2 - 2x$.

$-4 = -2A$ or $A = 2$.

Putting $x = -1$ gives us $-3 = +3B$, or $B = -1$.

Putting $x = 2$, we get $6 = 6C$, or $C = 1$

$$\text{Thus, } \int \frac{2x^2 + x - 4}{x^3 - x^2 - 2x} dx = 2\int \frac{1}{x} dx - \int \frac{1}{x+1} dx + \int \frac{1}{x-2} dx$$

$$= 2\ln|x| - \ln|x+1| + \ln|x-2| + c.$$

our next example illustrates the use of this method when the denominator has repeated linear factors.

Example 6 Take a look at the denominator of the integrand in $\int \frac{x}{x^3 - 3x + 2} dx$.

It factors into $(x-1)^2(x+2)$. The linear factor $(x - 1)$ is repeated twice in the decomposition of $x^3 - 3x + 2$.

In this case we write

$$\frac{x}{x^3 - 3x + 2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

from this point we proceed as before to find A, B and C. We get

$x = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$

We put $x = 1$ and $x = -2$ and get $C = 1/3$ and $A = -2/9$

Then to find B, let us put any other convenient value, say $x = 0$

This give us $0 = A - 2B + 2C$

$$\text{or, } 0 = \frac{-2}{9} - 2B + \frac{2}{3}.$$

This implies $B = 2/9$. Thus,

$$\int \frac{x}{x^3 - 3x + 2} dx = \frac{-2}{9}\int \frac{1}{x+2} dx + \frac{2}{9}\int \frac{1}{x-1} dx + \frac{1}{3}\int \frac{1}{(x-1)^2} dx$$

$$= \frac{-2}{9}\ln|x+2| + \frac{2}{9}\ln|x-1| - \frac{1}{3}\left(\frac{1}{x-1}\right) + c$$

$$= \frac{2}{9}\ln\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c$$

In our next example, we shall consider the case when the denominator of the integrand contains an irreducible quadratic factor (i.e. a quadratic factor which cannot be further factored into linear factors).

Example 7 To evaluate

$$\int \frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x}\, dx$$

we factorise $x^4 - 2x^3 + x^2 - 2x$ as $x(x-2)(x^2+1)$. Then we write

$$\frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{x-2} + \frac{Cx+D}{x^2+1}$$

Thus,

$$6x^3 - 11x^3 + 5x - 4 = A(x-2)(x^2+1) + Bx(x^2+1) + (Cx+D)x(x-2)$$

Next, we substitute $x = 0$ and $x = 2$ to get $A = 2$ and $B = 1$.

Then we put $x = 1$ and $x = -1$ (some convenient values) to get $C = 3$ and $D = -1$

Thus $\displaystyle\int \frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x}\, dx = 2\int \frac{1}{x}\, dx + \int \frac{1}{x-2}\, dx + \int \frac{3x-1}{x^2+1}\, dx$

$$= 2\ \ln|x| + \ln|x-2| + \frac{3}{2}\int \frac{2x}{x^2+1}\, dx - \int \frac{dx}{x^2+1}$$

$$= 2\ \ln|x| + \ln|x-2| + \frac{3}{2}\ \ln|x^2+1| - \tan^{-1} x + c.$$

Thus, you see, once we decompose our integrand, which is a proper rational function, into partial fractions, then the given integral can be written as the sum of some integrals of the type discussed in examples 1,2 and 3.

All the functions which we integrated till now were proper rational functions. Now we shall take up an example of an improper rational function.

Example 8 Let us evaluate $\displaystyle\int \frac{x^3 + 2x}{x^2 - x - 2}\, dx$.

Since the integrand is an improper rational function, we shall first write it as the sum of a polynomial and a proper rational functions,

Thus, $\displaystyle\frac{x^3 + 2x}{x^2 - x - 2} = x + 1 + \frac{5x+2}{x^2 - x - 2}$

Therfore, $\displaystyle\int \frac{x^3 + 2x}{x^2 - x - 2}\, dx = \int x dx + \int dx + \int \frac{5x+2}{x^2 - x - 2}\, dx$

$$= \frac{x^2}{2} + x + \int \frac{5x+2}{x^2 - x - 2}\, dx$$

Now let us decompose $\dfrac{5x+2}{x^2 - x - 2}$ into partial fraction as

$$\frac{5x+2}{x^2 - x - 2} = \frac{5x+2}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$5x + 2 = A(x+1) + B(x-2)$

If $x = -1$, we get $-3 = -3B$, that is, $B = 1$

If $x = +2$, we get $12 = +3A$, that is $A = 4$

Therefore $\displaystyle\int \frac{5x+2}{x^2 - x - 2}\, dx = 4\int \frac{dx}{x-2} + \int \frac{dx}{x+1}$

$= 4 \ln|x - 2| + \ln|x + 1| + c$

Hence $\displaystyle\int \frac{x^3 + 2x}{x^2 - x - 2}\, dx = \frac{x^2}{2} + x + 4 \ln|x-2| + \ln|x+1| + c.$

Try to do the following exercise now. You will find that each integrand falls in one of the various types we have seen in Examples 1 to 8.

E3) Evaluate

a) $\displaystyle\int \frac{2}{x^2 + 2x}\, dx$ b) $\displaystyle\int \frac{x dx}{x^2 - 2x - 3}$ c) $\displaystyle\int \frac{3x - 13}{x^3 + 3x - 10}\, dx$

d) $\int \frac{6x^2 + 22x - 23}{(2x - 1)(x^2 + x - 6)}\, dx$ e) $\int \frac{3x^3}{x^2 + x - 2}\, dx$ f $\int \frac{x^2 + x - 1}{(x - 1)\left(x^2 - x + 1\right)}\, dx$

g) $\int \frac{x^3 - 4x}{(x^2 + 1)^2}\, dx$

4.2.3 Method of Substitution

The method of partial fraction decomposition which we studied in the last sub-section can be applied to all rational functions. We can say this because as we have mentioned earlier, the Fundametal Theorem of algebra guarantees the factrorisation of any polynomial into linear and quadratic factors. But the actual process of factorising a polynomial is sometimes not quite simple. In such cases it would be a good idea to critically examine the integrand to check if the method of substitution can be appplied. We will now give two examples to show how we can sometimes integrate a given rational function with the help of a suitable substitution.

Example 9 Suppose we want to integrate $\frac{1}{x(x^5+1)}$ with respect to x.

For this we write $\int \frac{dx}{x(x^5+1)} = \int \frac{x^4 dx}{x^5 (x^5+1)}$

Now let us write $x^5 = t$. Then $\frac{dt}{dx} = 5x^4$.

$$\int \frac{x^4 dx}{x^5(x^5+1)} = \frac{1}{5} \int \frac{dt}{t(t+1)}$$

$$= \frac{1}{5} \int [\frac{1}{t} - \frac{1}{t+1}] dt$$

$$= \frac{1}{5} \ln \left|\frac{t}{t+1}\right| + c$$

$$= \frac{1}{5} \ln \left|\frac{x^5}{x^5+1}\right| + c$$

Example 10 Let us integrate $\frac{x^2-1}{x^4+x^2+1}$ w.r.t. x

$$\int \frac{x^2-1}{x^4+x^2+1} dx = \int \frac{(1-1/x^2)}{x^2+1+1/x^2} dx \text{ (division by } x^2)$$

$$= \int \frac{(1-1/x^2)}{(x+1/x)^2-1} dx$$

$$= \int \frac{dt}{t^2-1} \text{ if we put } t = x + \frac{1}{x}$$

$$= \frac{1}{2} \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt$$

$$= \frac{1}{2} \ln \left|\frac{t-1}{t+1}\right| + c$$

$$= \frac{1}{2} \ln \left|\frac{x^2-x+1}{x^2+x+1}\right| + c.$$

In Examples 9 and 10 you must have noted that the denominators of the integrands were not easily factorisable. The method of substitution provided an easier alternative. See if you can solve this exercise now.

E E4) Integrate the following functions w.r.t.x

a) $\frac{x^2-1}{1+x^4}$ b) $\frac{1+x^2}{1+x^2+x^4}$

The exercises in this section have given you a fair amount of practice in integrating rational functions. In the next section we take up the case of rational trigonometric functions.

4.3 INTEGRATION OF RATIONAL TRIGONOMETRIC FUNCTIONS

You know that a polynomial in two variables x and y is an expression of the form

$$P(x, y) = \sum_{n=0}^{k} \sum_{m=0}^{p} a_{m,n} \ x^m y^n, \ a_{m,n} \in \mathbf{R}.$$

Accordingly, a polynomial in sinx and cosx is an expression of the form

$$P(\sin x, \cos x) = \sum_{n=0}^{k} \sum_{m=0}^{p} a_{m,n} \sin^m x \cos^n x, \ a_{m,n} \in \mathbf{R}.$$

The integration of P(sinx, cosx) can be carried out easily as we have already integrated $\sin^m x \cos^n x$ in Unit 12. An expression, which is the ratio of two polynomials, P(sinx,cosx) and Q(sinx, cosx) is called a **rational function of sinx and cosx.** In this section we shall discuss the integration of some simple rational functions in sin x and cos x. We shall first indicate a general method for integrating these functions.

Let f(sin x, cos x) be a rational function in sin x and cosx. The first step in the evaluation of the integral of f is to make the substitution $\tan \frac{x}{2} = t$.

Thus, $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1+t^2}{2}$

Since $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2t}{1+t^2}$,

and $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$

we get,

$$\int f(\sin x, \cos x)\, dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}\, dt$$

$$= \int F(t)\, dt,$$

where $F(t) = f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}$

is a rational function of t. Now we can use the method of partial fraction decomposition to integrate F(t). In principle then, we can integrate any rational function in sinx and cosx. But in actual practice we find that the rational function F(t) is often complicated, and it is not feasible to apply the method of partial fractions. In this unit, however, we shall restrict ourselves to a few simple rational functions only.

Example 11 Let us integrate $\frac{1}{a+b\cos x}$

Now $a + b\cos x = a(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}) + b(\cos^2\frac{x}{2} - \sin^2\frac{x}{2})$

$$= (a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}$$

Therefore, $\int \frac{dx}{a+b\cos x} = \int \frac{\sec^2\frac{x}{2}\, dx}{(a+b) + (a-b)\tan^2\frac{x}{2}}$

$$= \int \frac{\sec^2\frac{x}{2}\, dx}{(a-b)\left[\frac{a+b}{a-b} + \tan^2\frac{x}{2}\right]}$$

If we put $\tan\frac{x}{2} = t$, we get

$$\int \frac{dx}{a+b\cos x} = 2\int \frac{dt}{(a-b)\left(\frac{a+b}{a-b} + t^2\right)}$$

$$= \frac{2}{a-b}\int \frac{dt}{\frac{a+b}{a-b} + t^2}$$

If $a > b > 0$, then $\frac{a+b}{a-b} > 0$, and we get

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(t\sqrt{\frac{a-b}{a+b}}\right)$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

If $0 < a < b$, then $\frac{a+b}{a-b} < 0$, and

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{b^2-a^2}}\ln\frac{\sqrt{b+a}+\sqrt{b-a}\,t}{\sqrt{b+a}-\sqrt{b-a}\,t}$$

$$= \frac{1}{\sqrt{b^2-a^2}}\ln\frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}$$

$1 + \cos x = 2\cos^2 \frac{x}{2}$

$\sin x = 2\sin \frac{x}{2} \cos \frac{x}{2}$

Example 12 To evaluate $\int \frac{1+\sin x}{\sin x(1+\cos x)} dx$, we write

$$\int \frac{1+\sin x}{\sin x(1+\cos x)} dx = \int \frac{dx}{\sin x(1+\cos x)} + \int \frac{dx}{1+\cos x}$$

$$= \frac{1}{4} \int \frac{dx}{\sin \frac{x}{2} \cos^3 \frac{x}{2}} + \frac{1}{2} \int \frac{dx}{\cos^2 \frac{x}{2}}$$

$$= \frac{1}{4} \int \frac{\sec^4 \frac{x}{2}}{\tan \frac{x}{2}} dx + \frac{1}{2} \int \sec^2 \frac{x}{2} dx$$

$$= \frac{1}{2} \int \frac{1+t^2}{t} dt + \int dt \qquad (\tan \frac{x}{2} = t)$$

$$= \frac{1}{2} \left[\int \frac{1}{t} dt + \int t\, dt \right] + \int dt$$

$$= \frac{1}{2} \left[\ln|t| + \frac{t^2}{2} \right] + t + c$$

Thus, $\int \frac{1+\sin x}{\sin x(1+\cos x)} dx = \frac{1}{2} \ln|\tan x/2| + \frac{1}{4} \tan^2 \frac{x}{2} + \tan \frac{x}{2} + c$

Now proceeding exactly as in Examples 11 and 12, you can do these exercises.

E E5) Evaluate $\int \frac{dx}{a+b\sin x}$

E E6) Integrate a) $\frac{1}{4+5\cos x}$ b) $\frac{\cos x}{2-\cos x}$ w.r.t. x.

By now you have seen and applied many different methods of integration. The crux of the matter lies in choosing the appropriate method for integrating a given function. For example, suppose we ask you to integrate the function $\frac{\sin x \cos x}{1+\sin^2 x}$ Realising that this is a rational function in sinx and cosx, you may put $\tan \frac{x}{2} = t$ and proceed :

$$\int \frac{\sin x \cos x}{1+\sin^2 x} = 4 \int \frac{t(1-t^2)dt}{(1+t^2)(1+6t^2+t^4)} dt$$

Now $1+6t^2+t^4 = (3+\sqrt{8}+t^2)(3-\sqrt{8}+t^2)$

By this step you will realise that it is going to be a tough job. But don't worry. There is an easy way out.

In $\int \frac{\sin x \cos x}{1+\sin^2 x}$ dx, if we make the substitution $1 + \sin^2 x = t$, we get

$$\int \frac{\sin x \cos x dx}{1+\sin^2 x} = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln|t| + c$$

$$= \frac{1}{2} \ln(1+\sin^2 x) + c.$$

Thus, the choice of the method is very crucial. And only practice can help you make a good choice.

We shall now illustrate some techniques use in integrating irrational functions.

4.4 INTEGRATION OF IRRATIONAL FUNCTIONS

The task of integrating functions gets tougher if the given function is an irrational one, that is, it is not of the form $\frac{Q(x)}{P(x)}$. In this section we shall give you some tips for evaluating some particular types of irrational functions. In most cases our endeavour will be to arrive at a rational function through an appropriate substitution. This rational function can then be easily evaluated by using the techniques developed in Sec. 2.

I) Integration of functions containing only fractional powers of x:

In this case we put $x = t^n$, where n is the lowest common multiple (l.c.m.) of the denominators of powers of x. This substitution reduces the function to a rational function of t.

Look at the following example.

Example 13 Let us evaluate $\int \frac{2x^{1/2}+3x^{1/3}}{1+x^{1/3}}\,dx$.

We put $x = t^6$, as 6 is the l.c.m. of 2 and 3. We get

$$\int \frac{2x^{1/2} + 3x^{1/3}}{1+x^{1/3}}\,dx = 6 \int \frac{2t^3+3t^2}{1+t^2}\,t^5\,dt$$

$$= 6 \int \frac{2t^8+3t^7}{1+t^2}\,dt = 6\int\left[2t^6 + 3t^5 - 2t^4 - 3t^3 + 2t^2 + 3t - 2 - \frac{3t-2}{1+t^2}\right]dt$$

$$= 6\left[\frac{2}{7}t^7 + \frac{1}{2}t^6 - \frac{2}{5}t^5 - \frac{3}{4}t^4 + \frac{2}{3}t^3 + \frac{3}{2}t^2 - 2t - \frac{3}{2}\ln(1+t^2) + 2\tan^{-1}t\right] + c$$

$$= \frac{12}{7}x^{7/6} + 3x - \frac{12}{5}x^{5/6} - \frac{9}{2}x^{2/3} + 4x^{1/2} + 9x^{1/3} - 12x^{1/6} - 9\ln|1+x^{1/3}| +$$

$$12\tan^{-1}x^{1/6} + c$$

II) Integral of the type $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Here we shall have to consider two cases : (i) $a > 0$ and (ii) $a < 0$.
In each case we will try to put the given integrand in a form which we have already seen how to integrate.

i) $a > 0$ $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{x^2+bx/a+c/a}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{(a+b/2a)^2+c/a-b^2/4a^2}}$

If we put $t = x + b/2a$, we get

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{t^2+(c/a-b^2/4a^2)}}.$$

This is one of the standard types of integrals listed in Table 3 in Unit 3, and can thus be evaluated.

ii) $a < 0$: If we put $-a = d$, then $d > 0$, and we cane write

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{d}}\int \frac{dx}{\sqrt{(c/d+b^2/4d^2)-(x-b/2d)^2}}$$

$$= \frac{1}{\sqrt{d}}\int \frac{dt}{\sqrt{(c/d+b^2/4d^2)-t^2}}, \text{ if } t = x - b/2d$$

This is again in one of the standard forms.

III) Integration of $\frac{1}{(fx+e)\sqrt{ax^2+bx+c}}$.

We will illustrate the method through an example.

Example 14 Suppose we want to evaluate $\int \frac{dx}{(x+1)\sqrt{x^2+4x+2}}$

Let us put $x + 1 = 1/y$. Then $\frac{-1}{y^2}\frac{dy}{dx} = 1$.

Now we will try to express $x^2 + 4x + 2$ in terms of y.

For this we write

$$x^2 + 4x + 2 = (x+1)^2 + 2(x+1) - 1$$

$$= \frac{1}{y^2} + \frac{2}{y} - 1 = \frac{1+2y-y^2}{y^2}$$

Therefore,

$$\int \frac{dx}{(x+1)\sqrt{x^2+4x+2}} = \int \frac{\frac{-1}{y^2}\,dy}{\frac{1}{y}\sqrt{\frac{1+2y-y^2}{y^2}}} = -\int \frac{dy}{\sqrt{1+2y-y^2}}$$

$$= -\int \frac{dy}{\sqrt{2-(y-1)^2}} = \cos^{-1}\left(\frac{y-1}{\sqrt{2}}\right)$$

$$= \cos^{-1}\left[\frac{-x}{(x+1)\sqrt{2}}\right] + c.$$

This example suggests that in integrating $\frac{1}{(fx+e)\sqrt{ax^2+bx+c}}$, we should make the substitution $fx + e = \frac{1}{y}$, and then simplify the expression.

Let us move over to the next type now.

IV) Integration of $\frac{(Ax+b)}{\sqrt{ax^2+bx+c}}$

We break Ax + B into two parts such that the first part is a constant multiple of the differential coefficient of $ax^2 + bx + c$, that is, $2ax + b$, and the second part is independent of x. Thus,

$$Ax + B = \frac{A}{2a}(2ax+b) + B - \frac{Ab}{2a} \text{ and}$$

$$\int \frac{(Ax+B)\,dx}{\sqrt{ax^2+bx+c}} = \frac{A}{2a}\int \frac{(2ax+b)\,dx}{\sqrt{ax^2+bx+c}} + \frac{(2a\,B-Ab)}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$= \frac{A}{a}\sqrt{ax^2+bx+c} + \frac{(2aB-Ab)}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

Evaluation of the last integral has already been discussed in II)

V) Integration of $(Ax+B)\sqrt{ax^2+bx+c}$

We break Ax + B as we did in IV) and obtain

$$\int (Ax+B)\sqrt{ax^2\ bx+c}\,dx = \frac{A}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + \frac{B2a-Ab}{2?}\int \sqrt{ax^2+bx+c}\,dx$$

$$= \frac{A}{3a}(ax^2+bx+c)^{3/2} + \frac{2aB-Ab}{2a}\int \sqrt{ax^2+bx+c}\,dx.$$

We have already seen how to evaluate the integral on the right hand side (see Sec. 4. Unit 3). Let us use these methods to solve some examples now.

Example 15 To evaluate $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx.$

we note that $x+2 = \frac{1}{2}(2x+2)+1$ and write $\int \frac{(x+2)dx}{\sqrt{x^2+2x+3}} = 1/2\int \frac{(2x+2)dx}{\sqrt{x^2+2x+3}} + \int \frac{dx}{\sqrt{x^2+2x+3}}$

$$= \sqrt{x^2+2x+3} + \int \frac{dx}{\sqrt{x^2+2x+3}}$$

$$= \sqrt{x^2+2x+3} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c.$$

$$\frac{1}{\sqrt{x^2+2x+3}} = \frac{1}{\sqrt{(x+1)^2+2}}$$

Example 16 To evaluate $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx$

we note that $x^2+2x+3 = x^2+x+1+x+2 = x^2+x+1+\frac{1}{2}(2x+1)+\frac{3}{2}$

Hence $\int \frac{(x^2+2x+3)}{\sqrt{x^2+x+1}}\,dx = \int \sqrt{x^2+x+1}\,dx + \frac{1}{2}\int \frac{(2x+1)}{\sqrt{x^2+x+1}}\,dx$

$+\frac{3}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$

$= \int \sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx + \sqrt{x^2+x+1}\,dx$

$+\frac{3}{2}\ln\frac{2}{\sqrt{3}}(x+\frac{1}{2}+\sqrt{x^2+x+1}+c$

We have used two results from Unit 3 here.

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+c$

and

$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+c$

$= \frac{(x+\frac{1}{2})}{2}\sqrt{x^2+x+1} + \frac{3}{8}\ln\frac{2}{\sqrt{3}}(x+\frac{1}{2}+\sqrt{x^2+x+1})$

$+\sqrt{x^2+x+1} + \frac{3}{2}\ln\frac{2}{\sqrt{3}}(x+\frac{1}{2}+\sqrt{x^2+x+1}) + c$

$= \frac{1}{4}(2x+5)\sqrt{x^2+x+1} + \frac{15}{8}\ln\frac{2}{\sqrt{3}}(x+\frac{1}{2}+\sqrt{x^2+x+1}) + c.$

See, if you can solve this exercise.

E E7) Integrate the following :

a) $\frac{\sqrt{x}}{1+\sqrt[4]{x}}$ b) $\frac{1}{(2-x)\sqrt{1-2x+3x^2}}$

When you are faced with a new integrand, the following suggestions furnish a thread through the labyrinth-of methods.

(1) Check the integrand to see if it fits one of the patterns

$$\int u^n du \text{ or } \int \frac{du}{u}$$

(2) See if the integrand fits any one of the patterns obtained by the reversal of differentiation formulas (We have considered these in Unit 3).

(3) If none of these patterns is appropriate, and if the integrand is a rational functions, then our theory of partial fraction enables us to integrate it.

(4) If the integrand is a rational function of sinx and cosx, and simpler methods of previous units fail, the substitution $t = \tan\frac{x}{2}$ will make the integrand into a rational function of t, which can then be evaluated.

(5) If the integrand is a radical of one of the forms $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$. then the trigonometric substitutions $x = a\sin\theta$, $x = a\cos\theta$ or $x = a\sec\theta$ will reduce the integrand to a rational functions of $\sin\theta$ and $\cos\theta$. If the radical is of the form $\sqrt{ax^2 + bx + c}$, a square completion $\sqrt{a(x + b/2a)^2 + c - b^2/4a}$ will reduce it essentially to one of the above radicals.

(6) If the integrand is an irrational function of x, try to express it as a rational function or an integrable radical through appropriate substitutions.

(7) Inspect the integrand to see if it will yield to integration by parts.

Finally, we would like to remind you again that a lot of practice is essential if you want to master the various techniques of integration. We have already mentioned that a proper choice of the method of integration is the key to the correct evaluation of any integral. Now let us briefly recall what we have covered in this unit.

4.5 SUMMARY

In this unit we have covered the following points :

1 A rational function f of x is given by $f(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials in x. It is called proper if the degree of $P(x)$ is less than the degree of $Q(x)$. Otherwise it is called improper.

2 A proper rational expression can be resolved into partial fractions with linear or quadratic denominators.

3 A rational function can be integrated by the method of partial fractions.

4 Integration of a rational function of sinx and cosx can be done by putting $t = \tan\frac{x}{2}$.

5 Integration of irrational functions of the following types is discussed.

i) integrand contains fractional power of x.

ii) $\dfrac{1}{\sqrt{ax^2 + bx + c}}$ iii) $\dfrac{1}{(fx + e)\sqrt{ax^2 + bx + c}}$

iv) $\dfrac{Ax + B}{\sqrt{ax^2 + bx + c}}$ v) $(Ax + B)\sqrt{ax^2 + bx + c}$

6 A check list of points to be considered while evaluating any integral is given.

4.6 SOLUTIONS AND ANSWERS

E1) a) and c) are proper.

b) $\dfrac{x^2 + x - 3}{x^2 + 1} = 1 + \dfrac{x - 4}{x^2 + 1}$

d) $\dfrac{x^4 + x^3 - 5}{x - 2} = x^3 + 3x^2 + 6x + 12 + \dfrac{19}{x - 2}$

E2) a) $\int \frac{dx}{2x-3} = \frac{1}{2}\int \frac{2dx}{2x-3} = \frac{1}{2}\ln|2x-3|+c$

b) $\int \frac{dt}{(t+5)^2} = \frac{-1}{t+5} + c$

c) $\int \frac{2x+1}{x^2+8x+1}dx = \int \frac{2x+8}{x^2+8x+1}dx - 7\int \frac{dx}{x^2+8x+1}$

$$= \ln|x^2+8x+1| - 7\int \frac{dx}{(x+4)^2-15}$$

$$= \ln|x^2+8x+1| - 7\int \frac{du}{u^2-15}, \text{ if } u = x+4$$

$$= \ln|x^2+8x+1| - \frac{7}{2\sqrt{15}}\ln\left|\frac{u-\sqrt{15}}{u+\sqrt{15}}\right| + c$$

$$= \ln|x^2+8x+1| - \frac{7}{2\sqrt{15}}\ln\left|\frac{x+4-\sqrt{15}}{x+4+\sqrt{15}}\right| + c$$

d) $\int \frac{4x+1}{x^2+x+2}dx = \int \frac{2(2x+1)-1}{x^2+x+2}dx$

$$= 2\int \frac{2x+1}{x^2+x+2}dx - \int \frac{dx}{x^2+x+1}$$

$$= 2\ln|x^2+x+2| - \int \frac{dx}{(x+\frac{1}{2})^2+\frac{7}{4}}$$

$$= 2\ln|x^2+x+2| - \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{x+1/2}{\sqrt{7}/2}\right) + c$$

$$= 2\ln|x^2+x+2| - \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + c$$

E3) a) $\frac{2}{x^2+2x} = \frac{2}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$\Rightarrow 2 = A(x+2) + Bx$

$x = 0 \Rightarrow 2 = 2A \Rightarrow A = 1$

$x = -2 \Rightarrow 2 = -2B \Rightarrow B = -1$

$\therefore \frac{2}{x^2+2x} = \frac{1}{x} - \frac{1}{x+2}$

$\therefore \int \frac{2}{x^2+2x}dx = \int \frac{1}{x}dx - \int \frac{1}{x+2}dx$

$= \ln|x| - \ln|x+2| + c = \ln\left|\frac{x}{x+2}\right| + c$

b) $\frac{x}{x^2-2x-3} = \frac{x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$x = A(x+1) + B(x-3)$

$x = 3 \Rightarrow 3 = 4A \Rightarrow A = \frac{3}{4}$

$x = -1 \Rightarrow -1 = -4B \Rightarrow B = \frac{1}{4}$

$\therefore \int \frac{x}{x^2-2x-3}dx = \int \frac{3x}{4(x-3)} + \int \frac{dx}{4(x+1)}$

$= \frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + c$

c) $\dfrac{3x-13}{x^2+3x-10} = \dfrac{3x-13}{(x+5)(x-2)} = \dfrac{A}{x+5} + \dfrac{B}{x-2}$

$\therefore 3x-13 = A(x-2) + B(x+5)$

$x=2 \Rightarrow -7 = 7B \Rightarrow B = -1$

$x\ -5 \Rightarrow -28 = -7A \Rightarrow A=4$

$\Rightarrow \int \dfrac{3x-13}{x^2+3x-10}\,dx = 4\int \dfrac{dx}{x+5} - \int \dfrac{dx}{x-2}$

$= 4\ \ln|x+5| - \ln|x-2| + c$

d) $\dfrac{6x^2+22x-23}{(2x-1)(x^2+x-6)} = \dfrac{6x^2+22x-23}{(2x-1)(x+3)(x-2)} = \dfrac{A}{2x-1} + \dfrac{B}{x+3} + \dfrac{C}{x-2}$

$6x^2 + 22x - 23 = A(x+3)(x-2) + B(x-2)(2x-1) + C(2x-1)(x+3)$

$x=2 \Rightarrow 45 = 15C \Rightarrow C=3$

$x=-3 \Rightarrow -35 = 35B \Rightarrow B=-1$

$x = 1/2 \Rightarrow \dfrac{-21}{2} = \dfrac{-21}{4}A \Rightarrow A = 2$

$\therefore \int \dfrac{6x^2+22x-23}{(2x-1)(x^2+x-6)}\,dx = \dfrac{1}{2}\ln|2x-1| - \ln|x+3| + 3\ln|x-2| + C$

e) $\dfrac{3x^2}{x^2+x-2} = 3x - 3 + \dfrac{9x-6}{x^2+x-2}$

$\therefore \int \dfrac{3x^3}{x^2+x-2}\,dx = \int (3x-3)\,dx + 3\int \dfrac{3x-2}{x^2+x-2}\,dx$

$= \dfrac{3x^2}{2} - 3x + 8\ln|x+2| + \ln|x-1| + c$

f) $\dfrac{x^2+x-1}{(x-1)(x^2-x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2-x+1}$

$\therefore x^2+x-1 = A(x^2-x+1) + (Bx+c)(x-1)$

$x = 1 \Rightarrow 1 = A$

we have

$x^2+x-1 = x^2-x+1+Bx^2+(C-B)x-C$

Thus $1 = 1 + B$ (Coefficients of x^2)

$\therefore B = 0$

Also $-1 = 1 - C$ (Constant terms)

$\therefore C = 2$

$\therefore \int \dfrac{x^2+x-1}{(x-1)(x^2-x+1)}\,dx = \int \dfrac{dx}{x-1} + 2\int \dfrac{dx}{x^2-x+1}$

$= \ln|x-1| + \dfrac{4}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}} + c$

g) $\dfrac{x^3-4x}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

$\therefore x^3 - 4x = (Ax+B)(x^2+1) + (Cx+D)$

$\therefore x^3 - 4x = Ax^3 + Bx^2 + (A+C)x + (B+D)$

$\therefore A = 1, B = 0, C = -5, D = 0$

$\therefore \int \dfrac{x^3-4x}{(x^2+1)^2}\,dx = \int \dfrac{x}{x^2+1}\,dx - 5\int \dfrac{x}{(x^2+1)^2}\,dx$

$$= \frac{1}{2} \ln (x^2+1) + \frac{5}{2} \frac{1}{(x^2+1)} + c$$

E4) a) $\int \frac{x^2-1}{1+x^4} dx = \int \frac{1-\frac{1}{x^2}}{\frac{1}{x^2}+x^2} dx$

$$= \int \frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2 - 2} dx$$

$$= \int \frac{dt}{t^2-2} \text{ if } t = x + \frac{1}{x}$$

$$= \frac{1}{2\sqrt{2}} \ln \left| \frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}} \right| + c$$

b) $\int \frac{1+x^2}{1+x^2+x^4} dx = \int \frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+1+x^2} dx$

$$= \int \frac{\frac{1}{x^2}+1}{(x-\frac{1}{x})^2+3} dx$$

$$= \int \frac{dt}{t^2+3}, \text{ if } t = x - \frac{1}{x}, \frac{dt}{dx} = 1 + \frac{1}{x^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left\{ \frac{1}{\sqrt{3}} (x - \frac{1}{x}) \right\} + c$$

$$= \frac{1}{3} \tan^{-1} \left(\frac{x^2-1}{\sqrt{3}x} \right) + c$$

E5) $\int \frac{dx}{a+b\sin x} = \int \frac{2dt}{a(1+t^2)+2bt}$, if $t = \tan x/2$

$$= 2\int \frac{2dt}{at^2+2bt+a} = 2\int \frac{dt}{(\sqrt{a}t + \frac{b}{\sqrt{a}})^2 + (\frac{a^2-b^2}{a})}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \left(\frac{at+b}{\sqrt{a^2-b^2}} \right) + c$$

$$= \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \left(\frac{a \tan \frac{x}{2} + c}{\sqrt{a^2-b^2}} \right) + c$$

E6) a) $\int \frac{dx}{4+6\cos x} = 2\int \frac{dt}{\left\{4+5\left(\frac{1-t^2}{1+t^2}\right)\right\}(1+t^2)}$

$$= 2\int \frac{dt}{4+4t^2+5-5t^2} = 2\int \frac{dt}{9-t^2}$$

$$= \frac{1}{3} \ln \left| \frac{3+t}{3-t} \right| + c$$

b) $\int \frac{\cos x}{2-\cos x} dx = 2\int \frac{\left(\frac{1-t^2}{1+t^2}\right)}{\left\{2-\left(\frac{1-t^2}{1+t^2}\right)\right\}} \frac{dt}{(1+t^2)}$

$$= 2\int \frac{1-t^2}{2(1+t^2)^2 - 1 + t^4}\, dt$$

$$= 2 \int \frac{1-t^2}{(t^2+1)(3t^2+1)}\, dx$$

If we write $\frac{1-t^2}{(t^2+1)(3t^2+1)} = \frac{At+B}{t^2+1} + \frac{Ct+D}{3t^2+1}$,

then $1-t^2 = (At+B)(3t^2+1) + (Ct + D)(t^2+1)$

$\therefore$ $1 = B + D$ (constants)

$0 = A + C$ (coefficient of t)

$-1 = 3B + D$ (constants of t^2)

$0 = 3A + C$ (coefficient of t^3)

$\therefore A = C = 0, B = -1, D = 2$

$\therefore$ Answer $= -2 \int \frac{dt}{t^2+1} + 4 \int \frac{dt}{3t^2 + 1}$

$$= -2\tan^{-1}(t) + \frac{4}{\sqrt{3}} \tan^{-1}(\sqrt{3}\, t) + c$$

$$= -2\, \frac{x}{2} + \frac{4}{\sqrt{3}} \tan^{-1}\left(\sqrt{3} \tan \frac{x}{2}\right) + c$$

$$= -x + \frac{4}{\sqrt{3}} \tan^{-1}\left(\sqrt{3} \tan \frac{x}{2}\right) + c$$

E7) a) $\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\, dx = \int \frac{t^2}{1+t}\, 4t^3\, dt$ if $t = \sqrt[4]{x}$

$$= 4\int \frac{t^5}{1+t}\, dt$$

$$= 4\int \left[t^4 - t^3 + t^2 - t + 1 - \frac{1}{t+1}\right] dt$$

$$= 4\left[\frac{t^5}{5} - \frac{t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right] + c$$

$$= 4\left[\frac{x^{5/4}}{5} - \frac{x}{4} + \frac{x^{3/4}}{3} - \frac{x^{1/2}}{2} + x^{1/4} - \ln|x^{1/4} + 1|\right] + c$$

b) ln $\int \frac{dx}{(2-x)\sqrt{1-2x+3x^2}}$ put $2-x = \frac{1}{t}$. Then $\frac{dx}{dt} = \frac{1}{t^2}$

$$\int \frac{dx}{(2-x)\sqrt{1-2x+3x^2}} = \int \frac{dx}{(2-x)\sqrt{3(2-x)^2 - 10(2-x) + 9}}$$

$$= \int \frac{t}{\sqrt{\frac{3}{t^2} - \frac{10}{t} + 9}}\, \frac{1}{t^2}\, dt$$

$$= \int \frac{dt}{\sqrt{9t^2 - 10t + 3}} = \int \frac{dt}{\sqrt{(3t - \frac{5}{3})^2 + \frac{2}{9}}}$$

$$= \frac{1}{3} \int \frac{dt}{\sqrt{(t - \frac{5}{9})^2 + (\frac{\sqrt{2}}{9})^2}} = \frac{1}{3} \sinh^{-1}\left(\frac{t - 5/9}{\sqrt{2}/9}\right) + c$$

$$= \frac{1}{3} \sinh^{-1} \frac{9}{\sqrt{2}} \left(\frac{1}{2-x} - \frac{5}{9} \right) + c$$

$$= \frac{1}{3} \sinh^{-1} \left(\frac{5x-1}{\sqrt{2}\,(2-x)} \right) + c$$

Solution of E8, Unit 3

i) $\sin^{-1}\left(\frac{x}{3}\right) + c$

ii) $\cosh^{-1}\left(\frac{u}{2}\right) + c$

iii) Putting $2x = -\ t$ we get $\frac{1}{2} \int \frac{dt}{1+t^2} = \frac{1}{2} \tan^{-1} t + c$

iv) $= \frac{1}{2} \tan^{-1} (2x) + c$

$\frac{1}{\sqrt{10}} \tan^{-1} x \sqrt{\frac{2}{5}} + c$

v) Put $x^2 = y$, then $\int \frac{x}{\sqrt{x^4 - 1}} dx = \frac{1}{2} \int \frac{dy}{\sqrt{y^2 - 1}} = \frac{1}{2} \cosh^{-1} y + c$

$= \frac{1}{2} \cosh^{-1} (x^2) + c$

vi) $\frac{1}{12} \tan^{-1} \left(\frac{t^3}{4} \right) + c$

vii) $\frac{1}{3} \sin^{-1} \left(\frac{u^3}{2} \right) + c$

viii) $\sin^{-1} (x - 1) + c$ (as in Example 13).

ix) $$\frac{1}{\sqrt{1+x+x^2}} = \frac{1}{\sqrt{\frac{3}{4} + \left(x + \frac{1}{2}\right)^2}}$$

Let $x + \frac{1}{2} = y$, Then the given integral is

$$\sinh^{-1} \left(\frac{2y}{\sqrt{3}} \right) + c = \sinh^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + c$$

x) $\cosh^{-1} \left(\frac{y+3}{2} \right) + c$

xi) $x - \tan^{-1} x + c$