UNIT 2 CENTRAL CONICOIDS

Structure

2.1 INTRODUCTION

In the previous unit you were introduced to certain surfaces in a three-dimensional system, which are called conicoids or quadric surfaces. There we discussed some general theory of conicoids and showed that a conicoid remains a conicoid under translation and rotation of axes. You also saw that some conicoids possess a centre and some don't. Based on this the conicoids are classified into two types – central and non-central conicoids. In this unit we shall concentrate only on central conicoids.

We first observe that a central conicoid can be reduced to a simpler form by an appropriate change of axes. Then we use these simpler form to discuss the geometrical properties of different types of central conicoids. You will see that there are four types of central conicoids — cone, imaginary ellipsoid, conicoid, hyperboloid of one sheet and hyperboloid of two sheets. You have already studies cones in detail in Unit 6. The remaining three real conicoids are the three-dimensional versions of the central conics that you studied in Block 1, namely, ellipses and hyperbolas.

Ancient mathematicians like Euclid, Archimedes and Appolonius were familiar with the above mentioned geometrical objects, though they did not study them analytically. The analytical study of these objects started much later, with the application of the three-dimensional coordinate system to geometry. The first mathematician to suggest the extension of the two-dimensional coordinate system to three dimensions was the Swiss mathematician John Bernoulli (1667-1748). But the actual application of space coordinates to geometry was done by another Swiss mathematician Jacob Hermann (1678-1733). He applied them to obtain the equations of several types of quadric surfaces.

Even though we are only concerned with central conicoids in this unit, in the first section we shall consider some necessary general theory of conicoids which we have not covered in the previous unit. We shall define a centre of a conicoid and obtain a characterisation for central conicoids. Then we shall discuss the four different types of central conicoids in separate sections. We will end this unit with a discussion on sections obtained by intersecting a central conicoid by a line or a plane. In this connection we also discuss tangents to a central conicoid.

In the next unit we will tackle non-central conicoids. That will be easier for you to grasp if you ensure that you have achieved the objectives given below.

Objectives

After studying this unit you should be able to :

- check whether a conicoid is central or not if you are given its equation;
- obtain standard forms of an ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets;
- trace the standard forms of the above mentioned three conicoids;
- obtain tangent lines and tangent planes at a point to a central conicoid;
- check whether a plane is a tangent plane to a conicoid or not;
- use the fact that a planar section of a central conicoid is a conic.

2.2 A CONICOID'S CENTRE

In the last unit you saw that a point P is called a centre of a conicoid F (x, y, z) = 0 if its coordinates satisfy a system of linear equations (see Equations (18) of Unit 7). In this unit we define a centre geometrically and then see the relationship between the geometrically and analytical defections. Let us consider the conicoid S given by $ax^2 + by^2 + cz^3 + d = 0$, $abc \neq 0$

Let $P(x_1, y_1, z_1)$ be a point on the conicoid. Then you can see that $P(-x_1, -y_1, -z_1)$ also lies on the conicoid. This means that S is symmetric about the origin O. Because of this property O is called the centre of the conicoid (see Fig. 1).

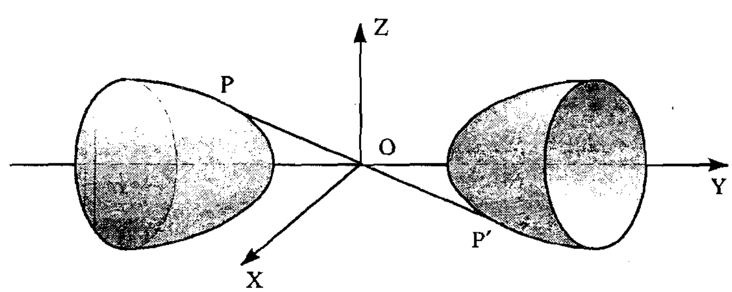

Fig. 1 : The pair of points P and P′ are symmetric about the centre O of the conicoid.

Definition : A conicoid S is called **symmetric with respect to a point** P if, when the origin is shifted to P, the transformed conicoid S is symmetric with respect to the origin.

Let us formally define a centre. To do so let us first see what we mean by symmetry.

Definition : A point P is called *a* **centre of a conicoid** *S* if S is symmetric with respect to P.

Using the definition above we can easily see that the origin O(0, 0, 0) is a centre of the spehre.

Let us consider an example now.

Example 1 : Show that the origin is the only centre of the cone
$ax^2 + by^2 + cz^2 = 0$, $abc \neq 0$

Solution : From the definition you can see that the origin is a centre of the sphere. Now let us take another point (x_0, y_0, z_0) which is non-zero. Shifting the origin to

(x_0, y_0, z_0), we get the relationship.

$x = x' + x_0$, $y = y_0$, $z = z'\ z_0$,

where (x', y', z') denote the coordinates in the new system. Substituting the equations above in the equation of the cone, we get

$$a(x' + x_0)^2 + b(y' + y_0)^2 + c(z' + z_0)^2 = 0$$
$$\Rightarrow ax'^2 + by'^2 + cz'^2 + 2[ax'x_0 + by'y_0 + cz'z_0] + ax_0^2 + by_0^2 + cz_0^2 = 0$$
$$\Rightarrow ax'^2 + by'^2 + cz'^2 + 2[ax'x_0 + by'y_0 + cz'z_0] = 0$$

This is the transformed equation of the cone. Because of the non-zero linear summand (inside brackets) of this equation, we see that (0, 0, 0) is the only point about which the transformed cone is symmetric. Hence the origin (0, 0, 0) is the only centre of the cone.

$ax^2 + by^2 + cz^2 = 0$, $abc \neq 0$.

We will discuss conditions under which you can easily judge if a point is centre of conicoid or not. But first try these exercises.

E1) Show that $(-u, -v, -w)$ is the only centre of the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

E2) Show that every point on the Z-axis is a centre of the cylinder $x^2 + y^2 - z^2 = 0$ (see Fig. 2).

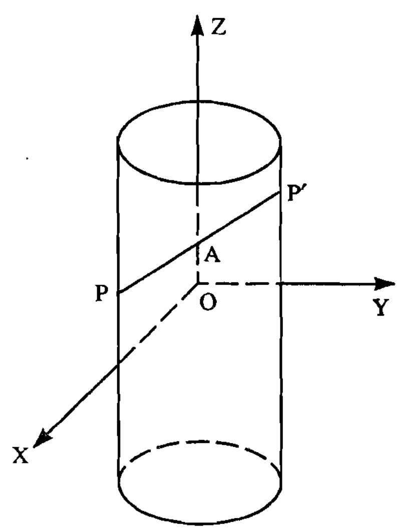

Fig. 2 : A cylinder with z-axis as axis.

Next, we shall prove a result which tells us something about the equation of a conicoid with the origin as a centre.

Theorem 1 : The origin O is a centre of the conicoid
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ ... (1)
If and only if $u = v = w = 0$.

Proof : Suppose that $u = v = w = 0$. Then the given equation of the conicoid takes the form
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0$.

Let $P(x_1, y_1, z_1)$ be any point on the conicoid. Then we have

$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 + d = 0.$

We can rewrite this equation is
$a(-x_1)^2 + b(-y_1)^2 + c(-z_1)^2 + 2f(-y_1)(-z_1) + 2g(-z_1)(-x_1) + 2h(-x_1)(-y_1) + d = 0.$

This shows that $(-x_1, -y_1, -z_1)$ lies on the given conicoid. Hence, O is the centre of the conicoid.

Conversely, suppose that O is a centre of the conicoid given by (1). Suppose P is and point (x_1, y_1, z_1) lying on the conicoid. Then the point P $(-x_1, -y_1, -z_1)$ also lies on the conicoid, since O is the centre. Then we have
$$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 + 2u_1x_1 + 2v_1y_1 + 2w_1z_1 + d = 0 \quad ...(2)$$

and

$$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 - 2u_1x_1 - 2v_1y_1 - 2w_1z_1 + d = 0 \quad ...(3)$$

Subtracting (3) from (2), we get
$ux_1 + vy_1 + wz_1 = 0$

This shows that (x_1, y_1, z_1) lies on the plane $ux + vy + wz = 0$. This is true for any point (x_1, y_1, z_1) on the conicoid. But how can every point on a conicoid lie on the plane $ux + vy + wz = 0$? This can happen only if $u = v = w = 0$.

Hence, the result.

Now, here is an exercise for you.

E3) Which of the following conicoids has a centre at the origin?

a) $x^2 + y^2 + z^2 - 23x = 0.$

b) $2x^2 + 3y^2 - z^2 = 1.$

c) $14x^2 + 26y^2 + 2\sqrt{91}\, z^2 = 1.$

d) $41x^2 - 28y^2 = 0.$

In Unit 7 you saw that the existence of a centre is connected with the solvability of a system of equations. In the next theorem we establish the fact.

Theorem 2 : A conicoid S, given by the equation
$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$
has the point P (x_0, y_0, z_0) as a centre if and only if

$$\left.\begin{array}{l} ax_0 + hy_0 + gz_0 + u = 0 \\ hx_0 + by_0 + fz_0 + v = 0 \\ gx_0 + fy_0 + cz_0 + w = 0 \end{array}\right\} \quad ... (4)$$

Proof : Let us first assume that $P(x_0, y_0, z_0)$ is a centre of the given conicoid in the coordinate system XYZ. Now let us translate the origin from O to the centre P. Then from Unit 7 you know that the equation of the conicoid in the new coordinate system is given by

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' + 2u'x' + 2v'y' + 2w'z' + d' = 0 \quad ... (5)$$

where
$u' = ax_0 + hy_0 + gz_0 + u$
$v' = hx_0 + by_0 + fz_0 + v$
$w' = gx_0 + fy_0 + cz_0 + w$
and
$d' = ax_0^2 + by_0^2 + cz_0^2 + 2fy_0z_0 + 2gz_0x_0 + 2hx_0y_0 + 2ux_0 + 2vy_0 + 2wz_0 + d$

Now, this conicoid has a centre at the origin. Therefore, by theorem 1, we have $u' = v' = w' = 0$. This means that

$ax_0 + hy_0 + gz_0 + u = 0$
$hx_0 + by_0 + fz_0 + v = 0$
$gx_0 + fy_0 + cz_0 + w = 0$

Conversely, let us assume that (4) holds for some point $P(x_0, y_0, z_0)$. We shift the origin from O to P. Then we get an equation of the form (5). But since (4) holds for P, we see that
$u' = v' = w' = 0$.

Therefore the equation reduce to
$a'x'^2 + b'y'^2 + c'z'^2 + 2fy'z' + 2gz'x' + 2hx'y' + d' = 0$.

This equation does not have any first degree terms. Therefore, by Theorem 2, we see that the origin P is a centre.

Do you find any connection between Theorem 2 and Theorem 3? You might have noticed that Theorem 2 is a special case of Theorem 3 in the case when the point P is the origin.

Now let us go back to E1 and E2. From there you know that a conicoid may have a unique centre of infinitely many centres. In Unit 7 you have also seen examples of conicoids which have no centre. Mathematicians have divided all conicoids up into two types depending on whether they have a unique centre or not. We define these conicoids as follows.

Definition : A conicoid is called a **central conicoid** if it has a **unique centre**. A conicoid is called **non-central**, if it either has **no centre** or it has **infinitely many centres.**

Thus, a sphere is an example of a central conicoid and a cylinder is an example of a non-central conicoid.

Here is an exercise for you.

E4) Examine whether the conicoides given in E3 have a unique centre or not.

We will look at non-central conicoids in the next unit. In this unit we will only discuss central conicoids.

Let us now start this discussion.

2.3 CLASSIFICATION OF CENTRAL CONICOIDS

In this section we shall obtain the different forms of central conicoids. We shall also study the shape of these conicoids.

Let us consider a central conicoid. Then, from Corollary 1 in Sec. 7.4 of Unit 7 you know that by first shifting the origin to the cenre and then rotating the axes suitably about the centre, we can reduce the equation to its standard form

$$ax^2 + by^2 + cz^2 + d = 0 \qquad \text{... (6)}$$

Now, let's go back for a moment to Theorem 2. Over there you saw that a conicoid, $F(x, y, z) = 0$ has a unique centre iff the system of equations.

$ax + hy + gz + u = 0$
$hx + by + fz + v = 0$
$gx + fy + cz + w = 0$

has a unique solution. This means that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0.$$

Now, if the conicoid is in the form (6), then the condition reduces to

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \neq 0, \text{ that is}$$

$abc \neq 0$.

This means that $a \neq 0$, $b \neq 0$, $c \neq 0$.

Then, depending upon the signs of a, b, c and d, we have the following five possibilities:

Case 1 ($d = 0$): In this case the equation reduces to
$ax^2 + by^2 + cz^2 = 0$.

You know from Unit 6 that this represents a cone, irrespective of the signs of a, b and c.

Case 2 ($d \neq 0$ and a, b, c, dare of the same sign) : In this case there are no real values of (x, y, z) which satisfy (6). This is because for any $(x, y, z) \in \mathbf{R}^3$, the left hand side is either positive or negative, never zero. We call such a conicoid an **imaginary conicoid.** Infact it represents an imaginary ellipsoid.

Case 3 ($d \neq 0$ and the sign of the coefficients a, b, c are different from d): In this case we write (6) in the form
$ax^2 + by^2 + cz^2 = -d$,

$$\text{i.e., } \frac{x^2}{-\frac{d}{a}} + \frac{y^2}{-\frac{d}{b}} + \frac{z^2}{-\frac{d}{c}} = 1 \qquad \ldots (7)$$

Note that the numbers $-\frac{d}{a}, -\frac{d}{b}$ and $-\frac{d}{c}$ are positive.

Let $a_1 = \sqrt{-\frac{d}{a}}$, $b_1 = \sqrt{-\frac{d}{b}}$ and $c_1 = \sqrt{-\frac{d}{c}}$. Then (6) becomes

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \frac{z^2}{c_1^2} = 1.$$

This equation is the three-dimensional analogue of the equation of an ellipse. We call the conicoid represented by this equation an **ellipsoid.**

Case 4 ($d \neq 0$ and two of the four coefficient a, b, c and d are of the same sign). Let us assume that $a > 0$, $b > 0$, $c < 0$ and $d < 0$.

Then $-\frac{d}{a}, -\frac{d}{b}$ and $\frac{d}{c}$ are positive. We put $a_2 = \sqrt{-\frac{d}{a}}$,

$b_2 = \sqrt{-\frac{d}{b}}$, and $c_2 = \sqrt{\frac{d}{c}}$. Then (6) gives us

$$\frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} - \frac{z^2}{c_2^2} = 1.$$

The conicoid generated by this equation is called a **hyperboloid of one sheet**. (You will see why in Sec. 8.5).

Similarly, we can obtain the equations of hyperboloids of one sheet in cases $a < 0$, $b < 0$, $c > 0$, $d > 0$, and so on.

Case 5 ($d \neq 0$ and two of a, b and c have the same sign as d) : As in the other case we assume that $a > 0$, $b < 0$, $c < 0$ and $d < 0$. Then $-\frac{d}{a} > 0, \frac{d}{b} > 0, \frac{d}{c} > 0$.

Put $a_3 = \sqrt{-\frac{d}{a}}$, $b_3 = \sqrt{\frac{d}{b}}$ and $c_3 = \sqrt{\frac{d}{c}}$. Then, we have

$$\frac{x^2}{a_3^2} + \frac{y^2}{b_3^2} - \frac{z^2}{c_3^2} = 1.$$

The conicoid is called a **hyperboloid of two sheets.** (You will see why in Sec. 8.6.) The other forms of hyperboloids of two sheets can be similarly obtained.

Thus, we saw that central conicoids can be classified into 5 types namely : cone, imaginary conicoid, ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets. We have tabulated this fact in Table 1.

Table 1: Standard Forms of Central Conicoids

Type	Standard
Cone	$ax^2 + by^2 + cz^2 = 0$
Imaginary ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Hyperboloid of two sheets	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Why don't you do an exercise now?

E5) Identify the type of the conicoid from the following equations

a) $x^2 + 4y^2 - z^2 = 1$

b) $16z^2 = 4x^2 + y^2 + 16$

c) $\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$

d) $z^2 = 3x^2 + 3y^2$

e) $x^2 - y^2 - z^2 = 9$

If you look closely at the cases where $d \neq 0$, you will see that the equations in these four cases can be given by a single equation
$ax^2 + by^2 + cz^2 \neq 1$

This will represent

i) an ellipsoid if a, b, c are all positive;

ii) a hyperboloid of one sheet if two of a, b, c are positive and the third is negative;

iii) a hyperboloid of two sheets if two out of a, b, c are negative and the third is positive;

iv) an imaginary conicoid if all the a, b, c are negative,

We shall study the shapes of the real conicoids listed above one by one. Let us start with ellipsoids.

2.4 ELLIPSOID

Let us consider the ellipsoid given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ where } a, b, c > 0 \qquad \ldots (8)$$

Let S denote the surface generated by this equation.

Form your experience of Unit 2, can you note some geometrical properties of S from the above equation? Of course, if a = b = c, then the equation represents a sphere. And you have studied the geometry of a sphere in detail in Block 2.

So, let us look at a more general case. From the following exercises you can get some idea of the geometrical aspects of the ellipsoid.

E6) Show that the surface represented by (8) is symmetric about the YZ-plane, ZX-plane and XY-plane.

E7) Do all the coordinate planes, intersect the surface (8)? If so, find the sections obtained by the intersections.

E8) Check whether the coordinates axes intersect, the surface (8).

We say that a surface given by an equation $F(x, y, z) = 0$ is **symmetric with respect to the XY-plane,** if, when we replace z by $-z$ in $F(x, y, z)$, we get the same equation. Symmetry with respect to the YZ-Plane and XZ-plane is similarly defined.

If you have done the exercises, you will have noticed that the surface (8) intersects the coordinate axes in A (a, 0, 0) and A (–a, 0, 0), B (0, b, 0) and B′ (0, –b, 0), C(0, 0, c) and C′ (0, 0, –c).

Now let us consider the intersections of the surface by planes parallel to the coordinate planes. Here we assume that $a \neq b$. Let us first consider a plane parallel to the XY-plane, say z = k, a constant.

Putting z = k in (8), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2} \qquad \ldots (9)$$

If $\frac{k^2}{c^2} < 1$, i.e., $-c < k < c$, the equation represents an ellipse.

This is true for all values of k such that $|k| \leq c$. If we put $k = 0$, we get an ellipse whose semi-axes are a and b.

Thus, the surface can be considered as a family of ellipses placed one on top of another lying between the planes $z = c$ and $z = -c$. Now what if $k > c$ in (9)? The equation only has imaginary roots. Therefore no portion of the surface lies beyond the plane $|z| = c$. Similarly, we can show that no part of the surface lies below $z = -c$. You can also check that the surface lies between the planes $y = -b$ and $y = b$, as well as between the planes $x = -a$ and $x = a$. Consequently, the surface is a bounded surface formed by ellipses.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, z = k, \text{ where } |k| \leq |c|.$$

Collecting all this information about (8), we get a figure as shown in Fig. 3.

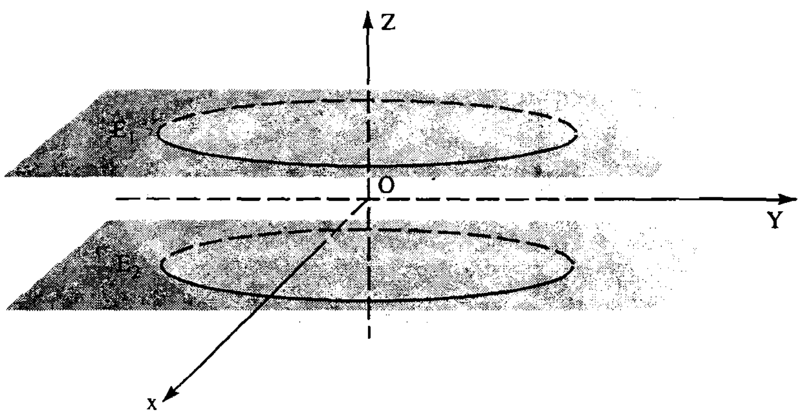

Fig. 3 : The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Its intersection with the planes $z = \frac{c}{2}$ and $z = -\frac{c}{2}$ are the ellipses E_1 and E_2.

We shall make some remarks here.

Remark : Suppose we have $a > b = c$ in (8). Then (8) can be rewritten as

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1.$$

The intersection of this ellipsoid with the plane $z = 0$ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If we revolve this ellipse about its major axis, i.e., the x-axis, then you can see that the surface formed is nothing but the given ellipsoid. This is the reason why the surface is known as an ellipsoid.

Now, what can you say about the ellipsoid in the case when $a = b > c$ in (8)? If $a = b > c$, the ellipsoid can be obtained by revolving the ellipse

$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, y = 0$ about its minor axis (i.e., the z-axis).

Let us consider an example now.

Example 3 : Trace the conicoid $\frac{x^2}{9} + \frac{y^2}{16} + z^2 = 1$ ...(10)

Solution : You know that the equation represents an ellipsoid. Let us try to trace it.

We first consider the intersection of the surface with the coordinate axes. From (10) we see that the x-axis intersects the surface in points (3, 0, 0) and (-3, 0, 0), the y-axis intersects the surface in (0, 4, 0) and (0, -4, 0) and the z-axis intersects the surface in two points (0, 0, 1) and (0, 0, -1). Now let us consider the intersection of the ellipsoid with planes parallel to coordinate planes.

Consider the plane $z = h$, a constant. Putting $z = h$ in (10) we get

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 - h^2 \qquad ...(11)$$

You know that this represents an ellipse for all ellipse for all h such that $|h| < 1$. If $h = 1$, we find that the intersection of surface with the plane $z = 1$ is the point (0, 0, 1). Similarly for $h = -1$, we get the point (0, 0, -1).

If $|h| > 1$, then there is no (x, y) satisfying (11). This shows that no portion of the surface lies above the plane $z = 1$ and below the plane $z = -1$.

Now we draw the ellipse corresponding to $z = 0$ (E_1 in Fig. 4).

Note that the major axis of the ellipse is 4 and the minor axis is 3. The figure shows some more ellipses corresponding to a few planes parallel to $z = 0$. Note that as h increases, the ellipses become smaller and smaller.

Likewise, if $x = h$, a constant, then again we get ellipses for $|h| < 3$. Since there is no (x, y) satisfying (10) for $|h| > 3$, we see that no portion of the surface lies to the right of the plane $x = 3$ and left of the plane $x = -3$. For $x = 0$, we get the ellipse, as shown in Fig. 4.

Similarly, you can see that the intersection with the planes $y = h$ are also ellipses for $|h| \leq 4$. and no portion of the surface lies to the right of the plane $y = 4$ and left of the plane $y = -4$. The ellipse corresponding to $y = 0$ is shown in Fig. 4.

Having got the intersections with the coordinate planes and the coordinate axes, we obtain the ellipse is shown in Fig. 4.

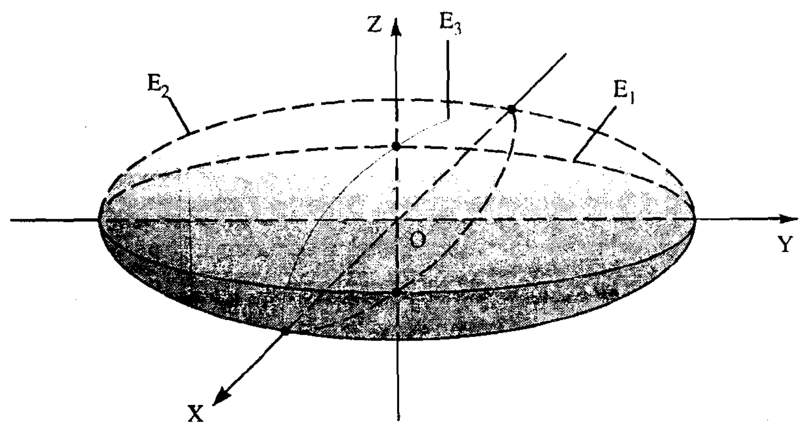

Fig. 4: The ellipses, E_1, E_2 and E_3 are the intersections of ellipsoid $\frac{x^2}{9} + \frac{y^2}{16} + z^2 = 1$ with the coordinate planes, $z = 0$, $x = 0$ and $y = 0$ respectively.

Why don't you trace an ellipsoid now?

E9) a) Trace the ellipsoid $x^2 + \left|\frac{y^2}{4}\right| + z^2 = 1$

b) Check whether the ellipsoid in (a) can be obtained by revolving an ellipse about any one of its axes.

So you have seen how to trace an ellipsoid in standard form. Actually, now you are in a position to trace any ellipsoid. How? Simply apply the transformations given in Sec. 1.3 of Unit 1, and reduce the given equation of ellipsoid to standard form! But we shall not go into such detail in this course.

Let us now consider an application of ellipsoids.

In Unit 2, you came across the reflecting property of an ellipse. This property is made use of in constructing whispering galleries. Whispering galleries are galleries with a rectangular base and ceiling in the form of an ellipsoidal surface. Because any vertical cross section of the ceiling is elliptical, the sound produced at one focus will be reflected at the other focus with little loss of intensity. This is called the **reflecting property** of an ellipsoid. This property is used by architects.

We shall now stop our discussion on ellipsoids and shift our attention to another central conicoid.

2.5 HYPERBOLOID OF ONE SHEET

In this section we shall study the shape of a hyperboloid of one sheet in detail and study some of its geometrical properties.

As in the case of ellipsoid, we shall restrict our attention to the standard forms.

Let us consider the hyperboloid of one sheet given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad ...(12)$$

Do you agree that this equation is represented by the surface in Fig. 5?

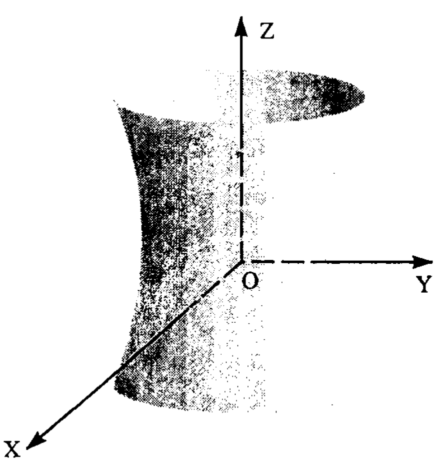

Fig. 5 : The hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$

You can obtain some geometrical properties of this surface by doing the following exercises.

E10) Show that the surface formed by (12) is symmetric about the coordinate planes.

E11) Check whether the coordinate axes intersect the surface formed by (12), if so, what are the intersections?

The next point to check is the intersection of (11) with the coordinate planes. Its intersection with the plane z = 0 is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Which is an ellipse. In fact, its intersection with any plane z = h will be an ellipse (see Fig. 6), and the size of the ellipse increase as h increases in both positive and negative directions. You may wonder why we don't call the surface an ellipsoid too.

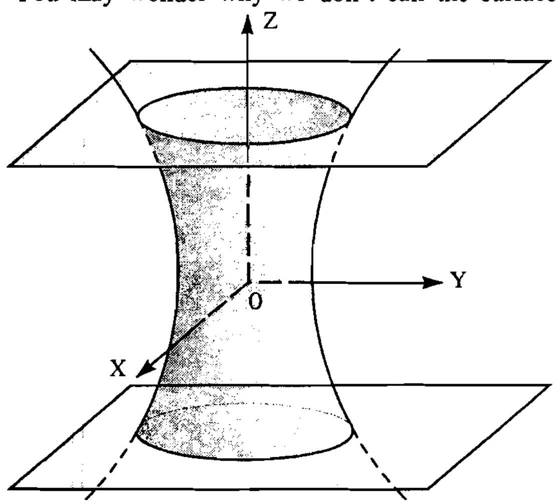

Fig. 6 : Sections of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. obtained of planes parallel to the XY-plane.

But, now look at what happens when we intersect the surface with the YZ-plane. The intersection is given by

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

which represents a hyperbola (see Fig. 7)

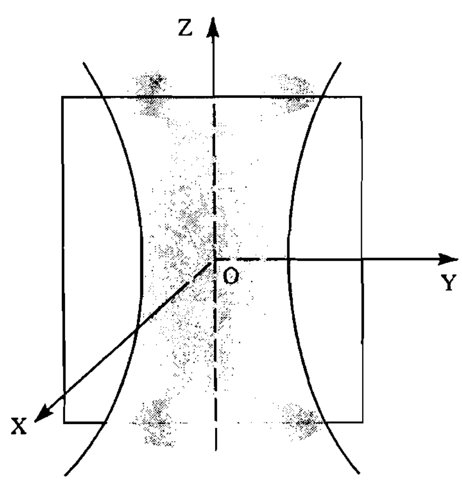

Fig. 7 : Intersection of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. by the XY-plane.

Similarly, you can see that the intersection of the surface with the plane y = 0 is the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ (see Fig. 8).

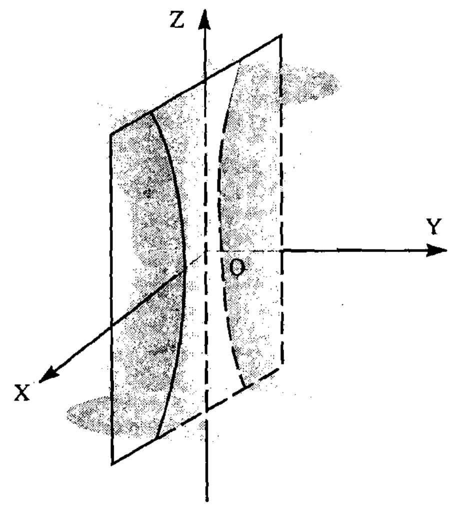

Fig. 8 : Intersection of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ by the ZX-plane.

From the above properties you surely agree with us that Fig. 5 represents (12).

Sometimes we say that the surface (12) is generated by the variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{h^2}{c^2},\ z=h,$$

which is parallel to the XY-plane and whose centre (0, 0, h) moves along the z-axis. This is because, as you can see, it is made up of these ellipses piled one top of the other.

Now, why do you think (12) is called a hyperboloid of one sheet? Firstly, it is of one sheet because it is a connected surface. This means that it is possible to travel from one point on it to any other point on it without leaving the surface. In the next section you will see that we also come across hyperboloids of two sheets.

To see why it is called a hyperboloid see what happens if, for example, a = b in (12). Then the equation reduces to the form

$$\frac{x^2+y^2}{b^2}-\frac{z^2}{c^2}=1. \qquad \text{...(13)}$$

If we put x = 0 in this, then we get the hyperbola

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ x=0,$$

whose conjugate axis is the z-axis. If we revolve this hyperbola about its conjugate axis, then we get the hyperboloid (13). Similarly, we can obtain certain such hyperboloids by revolving a hyperbola about its transverse axes.

So far we have been discussing only one standard form of a hyperboloid of one sheet. You know, from Table 1, that there are two more types of hyperboloid of one sheet. The following exercises are about them and the standard form (12).

E12) Consider the hyperboloid of one sheet given by $x^2 + y^2 - z^2 = 1$.
a) What are its horizontal cross-sections for $z = \pm 3, \pm 6$?
b) What are the vertical cross-sections for x = 0 or y = 0?
Describe the sections in (a) and (b) geometrically.

E13) a) Sketch the surface defined by the equation

$$-\frac{x^2}{16}+\frac{y^2}{9}+\frac{z^2}{9}=1 \qquad ...(14)$$

b) Sketch the curves of intersection of the surface given in (a) by the plane $z = k$, when $|k| = 3$, when $|k| < 3$ and when $|k| > 3$.

E14) a) Obtain the surface defined by the equation

$$\frac{x^2}{9}-\frac{y^2}{4}+\frac{z^2}{9}=1$$

b) What curve is formed by intersecting it with the plane $x = 1$?

Another interesting property of a hyperboloid of one sheet which makes it very useful in architecture is that it is a ruled surface. This means that the surface is composed of straight lines, and is therefore easy to make with a string (see Fig. 9).

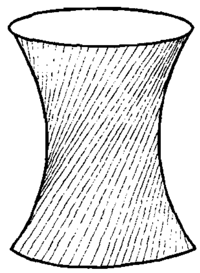

Fig. 9 : A model of a hyperboloid of one sheet.

Let us now discuss another type of hyperboloid.

2.6 HYPERBOLOID OF TWO SHEETS

In this section we shall concenrate on the geometrical features of a hyperboloid of two sheets. Its analytical properties are very similar to those of a hyperboloid of one sheet or an ellipsoid. So, it will be easy for you to bring out these properties yourself.

Let us start with a hyperboloid of two sheets given by the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad ...(14)$$

Note that this equation has two negative coefficients while the equation of a hyperboloid of one sheet has only 1 negative coefficient.

So let us see what (14) looks like. To start with, why don't you try the following exercises concerning (14)?

E15) Discuss the symmetry of the surface obtained by (14) with respect to the coordinate planes.

E16) Do all the coordinate axes and coordinate planes intersect the surface? Give reasons for your answer.

In E16 you must have observed that the XZ-plane and XY-plane intersect the surface in hyperbolas. What about the YZ-plane? You must have observed that the YZ plane does not intersect the surface.

So, now you know why this surface is called a hyperboloid. But, you may wonder why this surface is called a hyperboloid of two sheets. This is because of the following property.

Let us consider the intersection of the surface and the plane x = h, a constant. You can see that the curve of intersection is the ellipse

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{h^2}{a^2} - 1 \qquad \text{...(15)}$$

in the plane x = h.

This ellipse is real only if $\frac{h^2}{a^2} > 1$, i.e., h > a or h < – a.

Therefore, it follows that those planes which are parallel to x = 0 and lie between the planes x = –a and x = a do not cut the surface. This means that no portion of the surface lies between the planes x = – a and x = a.

We note from (15) that the semi-axes of the ellipses are given by

$$b\sqrt{\frac{h^2}{a^2} - 1} \text{ and } c\sqrt{\frac{h^2}{a^2} - 1}.$$

and the semi-axes increase as h increases. Hence, we see that the surface has two branches: one on the left of the plane x = –a and one on the right of the plane x = a. Both these are generated by a variable ellipse. In fact, the shape of the surface is as shown in Fig. 10.

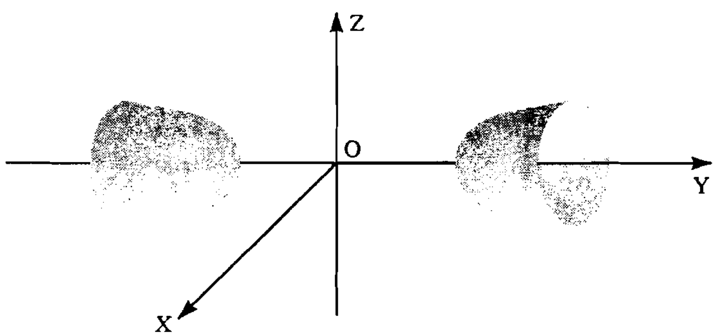

Fig. 10: The hyperboloid of two sheets $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

In the following exercises we ask you to trace the other two forms of hyperboloid of two sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ and}$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

E17) Check whether the coordinate planes intersect the surface

a) $z^2 - \frac{x^2}{4} - y^2 = 1$

b) $\frac{y^2}{4} - \frac{x^2}{9} - z^2 = 1.$

If so, what are the curves of intersection?

E18) Sketch the surfaces given in E17.

So, you have seen what the standard forms of the various central conicoids look like. Now let us see what their intersections with various lines and planes are.

2.7 INTERSECTION WITH A LINE OR A PLANE

In this section we shall first discuss the intersection of a line and a central conicoid. This will help us to derive conditions under which a line is a tangent to a central conicoid, and to obtain the tangent planes. Then we shall discuss the intersection of a plane and a central conicoid.

2.7.1 Line Intersection

Consider a central conicoid given by

$$ax^2 + by^2 + cz^2 = 1 \qquad \text{...(16)}$$

Let us prove the following result about the intersection of given line with this conicoid.

Theorem 3 : Any line intersects a conicoid in two points, which may be distinct real points, coincident real points or imaginary points.

Proof :

Let L be a given line with direction ratios α, β, γ that passes through a given point (x_0, y_0, z_0). Then the equation of the line L is

$$\frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta} = \frac{z - z_0}{\gamma} \qquad \text{...(17)}$$

If B(x, y, z) is another point on the line L, which is at a distance r from A, then the coordinates of B are given by $x = x_0 + \alpha r$; $y = y_0 + \beta r$, $z = z_0 + \gamma r$.

If B is a point of intersection of L with the conicoid (16) then its coordinates must satisfy (17). This means that

$$r^2 (a\alpha^2 + b\beta^2 + c\gamma^2) + 2r(ax_0\alpha + by_0\beta + cz_0\gamma) + ax_0^2 + by_0^2 + cz_0^2 - 1 = 0 \qquad \text{...(18)}$$

(18) is a quadratic in r. So it will give us two values of r each of these values will give us a point of intersection of L with the conicoid (16).

Thus, L, will meet (14) in two points, which may be real and distinct, coincident or imaginary.

This is true for any line. Hence, Theorem 1 is true for a central conicoid.

Now let us suppose that the point A (x_0, y_0, z_0) lies on the conicoid (16) itself. Then (18) becomes.

$$r^2 (a\alpha^2 + b\beta^2 + c\gamma^2) + 2r(ax_0\alpha + by_0\beta + cz_0\gamma) = 0 \qquad \text{...(19)}$$

The line L will be tangent to the conicoid at A if the points of intersection coincide with the point A (x_0, y_0, z_0), i.e., if (19) has coincident roots. As you can see, the condition for this is

$$ax_0\alpha + by_0\beta + cz_0\gamma = 0 \qquad \text{...(20)}$$

Thus, (20) gives the condition that the line

$$\frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta} = \frac{z - z_0}{\gamma}$$

is a tangent to the conicoid $ax^2 + by^2 + cz^2 = 1$ at A (x_0, y_0, z_0), For example, the line x = z, y = 4 is a tangent to the ellipsoid $\frac{x^2}{9}+\frac{y^2}{16}+z^2=1$ at (0, 4, 0) because $\alpha = 1 = \gamma$, $\beta = 0$. You can also check that the line through the point (0, 4, 0) and parallel to the x-axis is also tangent to this ellipsoid. In fact, there are infinitely many lines that are tangent to this ellipsoid at the point (0, 4, 0).

This means that at each point of the ellipsoid we can draw infinitely many tangents to the conicoid. This is not only true for this ellipsoid. It is true for any conicoid. Let us see what the set of all tangents at a point of a conicoid look like.

Let us eliminate α, β, γ from (17) and (20). Then we get
$ax_0(x-x_0)+by_0(y-y_0)+cz_0(z-z_0)=0.$

$\Leftrightarrow \quad ax_0x+by_0y+cz_0z-ax_0^2-by_0^2-cz_0^2=0$

$\Leftrightarrow \quad axx_0+byy_0+czz_0=1,$ ...(21)

since (x_0, y_0, z_0) lies on (17).

(20) is the equation of a plane. Thus, the set of all tangent lines to (16) is the plane (21).

Definition : The set of all tangent lines to a conicoid at a point on the conicoid is called the **tangent plane.**

So, let us assume that (17) is a tangent of (16) at (x_0, y_0, z_0).

For example, if the conicoid is an ellipsoid $\frac{x^2}{9}+\frac{y^2}{16}+z^2=1$, the equation of the tangent plane at any point (0, 4, 0) on it is y = 4.

Similarly, the equation of the tangent plane at the point (0, 4, 0) on the hyperboloid of one sheet $\frac{x^2}{9}+\frac{y^2}{16}-z^2=1$ is y = 4.

In Fig. 11, we have shown both these tangent planes.

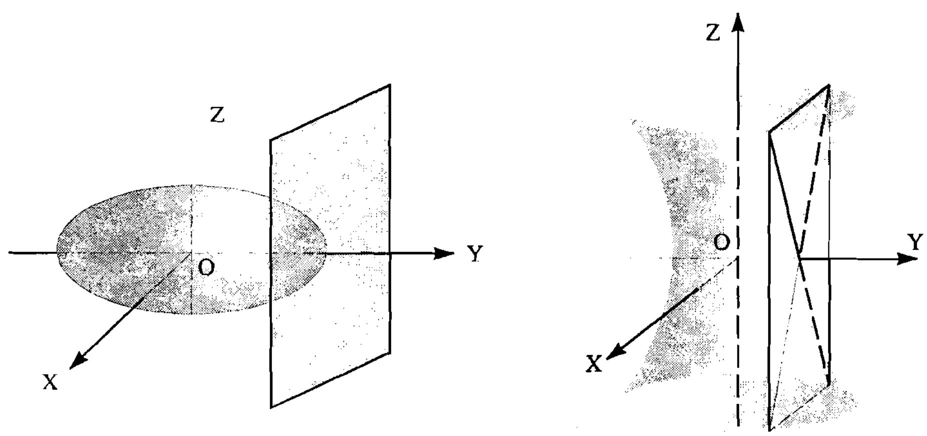

Fig. 11: π is a tangent to (a) an ellipsoid (b) a hyperboloid.

Note that the tangent plane y = 4 intersects the given ellipsoid in only one point on the other hand, this plane intersects the given hyperboloid along the two tangent lines $x = \pm 3z$, y = 4 at (0, 4, 0). The should not be surprising, since the plane is built up of tangent lines.

Now, suppose we are given a plane and a conicoid. Can we say when the plane will be tangent to the conicoid? Let us see. Let us consider the plane given by

$ux + vy + wz = p$ ...(22)

and the conicoid given by

$ax^2 + by^2 + cz^2 = 1.$

You know from (20) that the plane $ux + vy + wz = p$ will be a tangent plane to the given conicoid at some point (x_0, y_0, z_0) if and only if its equation is of the form

$ax x_0 + by y_0 + cz z_0 = 1$ ...(23)

Therefore, if (22) represents a tangent plane, the coefficients of (22) and (23) must be proportional, i.e.,

$$\frac{ax_0}{u} = \frac{by_0}{v} = \frac{cz_0}{w} = \frac{1}{p}, p \neq 0$$

$$\text{i.e., } x_0 = \frac{u}{ap}, y_0 = \frac{v}{bp}, z_0 = \frac{w}{cp} \qquad \text{...(24)}$$

(Remember that $a \neq 0$, $b \neq 0$ and $c \neq 0$.)

Now, the point (x_0, y_0, z_0) lies on the given conicoid. Therefore,

$$ax_0^2 + by_0^2 + cz_0^2 = 1 \Rightarrow a\frac{u^2}{a^2p^2} + b\frac{v^2}{b^2p^2} + c\frac{w^2}{c^2p^2} = 1$$

$$\Rightarrow \frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = p^2, \qquad \text{...(25)}$$

which is the required condition that a plane $ux + vy + wz = p$ touches the conicoid $ax^2 + by^2 + cz^2 = 1$. Also note that the point of contact of the plane and the conicoid is given by (24).

Let us consider some examples.

Example 4 : Show that the plane $3x + 12y - 6z = 17$ touches the hyperboloid $3x^2 - 6y^2 + 9z^2 + 17 = 0$, and find the point of contact.

Solution : We first rewrite the equation of the hyperboloid in the standard form as

$$\left(-\frac{3}{17}\right)x^2 + \frac{6}{17}y^2 + \left(-\frac{9}{17}\right)z^2 = 1$$

The condition that $3x + 12y - 6z = 17$ touches the conicoid is given by (25).

Here $a = \left(-\frac{3}{17}\right), b = \frac{6}{17}, c = \left(-\frac{9}{17}\right)$, $u = 3$, $v = 12$, $w = -6$, $p = 17$. Then

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = \frac{9 \times (-17)}{3} + \frac{144 \times 17}{6} + \frac{36 \times (-17)}{9}$$

$$= 17[-3 + 24 - 4] = 17^2 = p^2.$$

Thus the condition (25) is satisfied. Hence the plane touches the conicoid. The point of contact is given by

$$x_0 = \frac{3 \times (-17)}{3 \times 17} = -1$$

$$y_0 = \frac{12 \times 17}{6 \times 17} = 2$$

$$z_0 = \frac{(-6) \times (-17)}{9 \times 17} = \frac{2}{3}$$

Example 5 : Find equations of the tangent planes to the conicoid $7x^2 + 5y^2 + 3z^2 = 60$ which pass through the line $7x + 10y - 30 = 0$, $5y - 3z = 0$.

Solution : From Block 1 you know that any plane through the given line is of the form
$7x + 10y - 30 + \lambda (5y - 3z) = 0,$

where λ is a real number. Since this plane is a tangent plane to the given conicoid, we see that the equation of the plane must be of the form

$\frac{7xx_0}{60} + \frac{5yy_0}{60} + \frac{3yy_0}{60} = 1$, for some (x_0, y_0, z_0).

Comparing the coefficients, we get

$$\frac{7x_0}{7} = \frac{5y_0}{10+5\lambda} = \frac{3z_0}{-3\lambda} = \frac{60}{30}$$

i.e., $x_0 = 2,\ y_0 = 2\lambda + 4,\ z_0 = -2\lambda$

Since (x_0, y_0, z_0) lies on the given conicoid, we get
$7 \times 4 + 5 (2\lambda + 4)^2 + 12\lambda^2 = 60$
$\Rightarrow 28 + 20\lambda^2 + 80\lambda + 80 + 12\lambda^2 = 60$
$\Rightarrow 32\lambda^2 + 80\lambda + 48 = 0$
$\Rightarrow 2\lambda^2 + 5\lambda + 3 = 0.$

This is a quadratic equation in λ. Its roots are -1 and $-\frac{3}{2}$. For each of these values of λ we get a tangent plane. Therefore, there are two tangent planes passing through the given line. The required equations of the planes are
$7x + 3y + 3z = 30$ and $14x + 5y + 9z = 60$.

You can now try some exercises.

E19) Find the equation of the tangent plane to the hyperboloid
$x^2 + 3y^2 - 3z^2 = 1$ at the point $(1, -1, 1)$

E20) Find the equations of the tangent planes to the conicoid $2x^2 - 6y^2 + 3z^2 = 5$ which pass through the line $x + 9y - 3z = 0,\ 3x - 3y + 6z - 5 = 0$.

So far we have been discussing the tangent planes to a central conicoid. You have found that sometimes such planes intersect the conicoid in a point, and sometimes in a pair of lines. Does this give you a clue to what π and S will be where π is a plane and S is a central conicoid? We discuss this now.

8.7.2 Planar Intersections

In this unit you have seen that the section of a standard ellipsoid or hyperboloid by a plane parallel to the coordinate planes is either an ellipse, a hyperbola or their degenerate cases, i.e., the section is a conic. What do you expect in the case of the section by any plane which is not parallel to the coordinate planes? Will it still be a conic? Let's see.

Let us consider a central conicoid given by
$ax^2 + by^2 + cz^2 = 1,\ abc \neq 0.$

We want to find the section of this conicoid by a plane $ux + vy + wz = p$. The following result tells us about this. (We shall not prove it here. If you are interested in knowing the proof, refer to the miscellaneous exercises at the end of the block.)

Theorem 4 : The section of a central conicoid by a given plane is a conic section.

Further, if the conicoid is given by $ax^2 + by^2 + cz^2 = 1$ and the plane is given by $ux + vy + wz = p$, then the section will be a hyperbola, parabola or an ellipse according as

$bcu^2 + cav^2 + abw^2 < 0$, $bcu^2 + cav^2 + abw^2 = 0$ or $bcu^2 + cav^2 + abw^2 > 0$.

In case $abc > 0$, the condition reduces to

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} < 0, \frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = 0, \frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} > 0.$$

This theorem is not difficult to prove. We can obtain the condition by eliminating either x, y or z from the equations $ax^2 + by^2 + cz^2 = 1$ and $ux + vy + wz = p$. Since it is lengthy, we have not included it here.

From Theorem 4, you know that a planar section of central conicoid need not be a central conic. In Fig. 12 we illustrate this.

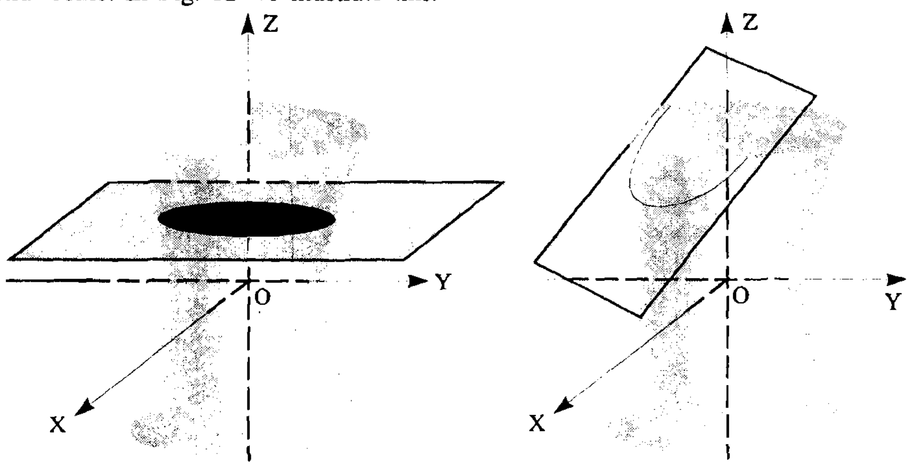

Fig. 12: A Planar section of the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. can be a) ellipse b) a parabola.

Let us consider an example.

Example 6 : Show that the section of the hyperboloid $9x^2 + 6y^2 - 14z^2 = 3$ by the plane $x + y + z = 1$ is a hyperbola.

Solution: Here $a = 3$, $b = 2$, $c = -\frac{14}{3}$ and $u = v = w = 1$

Therefore, $bcu^2 + cav^2 + abw^2 = 2 \times \left(-\frac{14}{3}\right) - 14 + 6 < 0$.

Hence by Theorem 4, the section is a hyperbola.
You can try some exercises now.

E21) Find the sections of the following conicoids by the plane given alongside.
a) $2x^2 + y^2 - z^2 = 1$; $3x + 4y + 5z = 0$.
b) $3x^2 + 3y^2 + 6z^2 = 10$; $x + y + z = 1$.

Let us now end our discussion on central conicoids by summarizing what we have covered in this unit.

2.8 SUMMARY

In this unit we have covered the following points :

1) The definition of a centre of a conicoid.

2) The necessary and sufficient condition for the conicoid to $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ to have a centre is $\begin{vmatrix} a & h & b \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0.$

3) Conicoids are divided into two groups — those with a unique centre (that is, central conicoids) and those which have either to centre or infinitely many centres (that is, non-central conicoids).

4) The standard form of a central conicoid is
$ax^2 + by^2 + cz^2 + d = 0$, $abc \neq 0$.
If $d \neq 0$, then there are four categories, as given in the table below:

Table 1: Standard Form of central conicoids

Type	**Standard form**
Cone	$ax^2 + by^2 + cz^2 = 0$
Imaginary ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1.$
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1.$
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Hyperboloid of two sheets	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

n $d = 0$, then the equation represents a cone.

5) How to trace the standard forms of an ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets.

6) The condition for a line to be a tangent to the central conicoid $ax^2 + by^2 + cz^2 + d = 0$ at (x_0, y_0, z_0) is $ax_0\alpha + by_0\beta + cz_0\gamma = 0$,
where α, β, γ are the direction ratios of the line.

7) The equation of the tangent plane to a central conicoid $ax^2 + by^2 + cz^2 = 1$ at a point (x_0, y_0, z_0) is $axx_0 + byy_0, czz_0 = 1$.

8) The condition that the plane $ux + vy + wz = p$ is a tangent to the central conicoid $ax^2 + by^2 + cz^2 = 1$ is

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = p^2.$$

9) A planar section of a central conicoid is a conic section.

Now you may like to go back to Sec. 3.1 to see if you have achieved the **objeectives** listed there. You must have solved the exercises in the unit as you came to them. In the next section we have given our answers to the exercises. You may like to have to look at them.

2.9 SOLUTIONS/ANSWERS

E1) We first show that (–u, v, –w) is a centre of the sphere
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$
You can check that by shifting the origin to (–u, –v, –w), we get the transformed equation as
$x'^2 + y'^2 + z'^2 = u^2 + v^2 + w^2 - d$
where x′, y′, z′ denote the coordinates in the new system. Certainly (0, 0, 0) to (x_0, y_0, z_0). Then, we get the transformed equation as
$x'^2 + y'^2 + z'^2 + 2x'(u + x_0) + 2y'(v + y_0) + 2z'(w + z_0) + [2ux_0 + 2vy_0 + 2wz_0 + d] = 0$
This is the transformed equation of the sphere. Since at least one of the u, v, w are different from (x_0, y_0, z_0) respectively, the linear summand is non-zero, therefore the equation is not symmetric with respect to the origin (0, 0, 0). Hence, (–u, –v, –w) is the only centre of the sphere.

E2) Let us take any point (0, 0, z) on the z-axis. If we transform the origin to this point, we get the same equation is symmetric with respect to the origin. This shows that any point on the z-axis a centre of the sphere.

E3) a) Origin is not a centre.
b) Origin is a centre.
c) Origin is a centre.
d) Origin is not a centre.

E4) a) The conicoid is $x^2 + y^2 + z^2 - 23x = 0$
$x_0 - 23 = 0$, $y_0 = 0$ and $z_0 = 0$
a = 1, b = 1, c = 1, u = –23. So we get
The system has a unique solution. Thus the conicoid has a unique centre.
b) Unique centre at (0, 0, 0)
c) Unique Centre at (0, 0, 0)
d) Infinitely many centres-every point on the z-axis.

E5) a) Hyperboloid of one sheet.
b) Hyperboloid of one sheet.
c) Ellipsoid.
d) Cone.
e) Hyperboloid of two sheets.

E6) If we change x to –x, there is no change in the equation. This implies that the surface (8) is symmetric about the YZ-plane. Similarly, the surface is symmetric about the XZ-plane and XY-planes.

E7) Yes. All the coordinate planes intersect the surface.
Let the equation of the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The equation of the YZ-plane is x = 0. To find the intersection, we put x = 0 in the equation of the ellipsoid. Then we get

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

which represents an ellipse (or a circle).

Similarly, substituting y = 0 or z = 0, we get an ellipse (or a circle) in the XZ-plane or the XY-plane.

E8) The equation of the ellipsoid is (8). We first check whether the x-axis intersects the surface. Any point on the x-axis is of the form (r, 0, 0). So, to find the intersection of the x-axis, we substitute (r, 0 0) in (8). We get

$$\frac{r^2}{a^2} = 1, \text{ i.e., } r = \pm a.$$

Thus, the x-axis intersects the surface in the two points (a, 0, 0) and (–a, 0, 0). Similarly, the y-axis intersects it in two points (0, b, 0) and (0, –b, 0) and the z-axis intersects it in the points (0, 0, c) and (0, 0, –c).

E9) a) The given ellipsoid is $x^2 + \frac{y^2}{4} + z^2 = 1$.

We first find the intersection of the ellipsoid with the coordinate plane. Let z = h, a constant. Then, we et

$$x^2 + \frac{y^2}{4} = 1 - h^2.$$

If $|h| < 1$, this is an ellipse centred at the origin.
Likewise if x = h, a constant, we get

$$\frac{y^2}{4} + z^2 = 1 - h^2.$$

This again represents an ellipse if $|h| < 1$.
Putting y = h, we get circles given by

$$x^2 + z^2 = 1 - \frac{h^2}{4}$$

if $|h| \leq 4$.

Let us now draw the ellipses and the circle for z = 0, x = 0, y = 0 respectively (see Fig. 13).

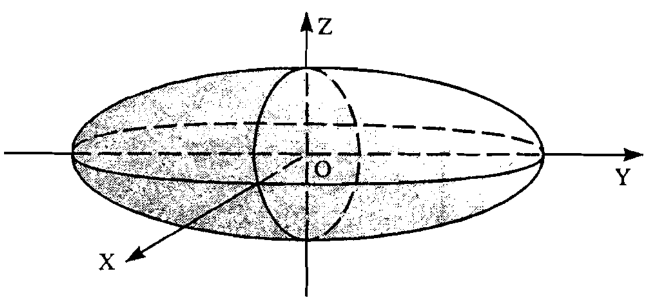

Fig. 13: The ellipsoid $x^2 + \frac{y^2}{4} + z^2 = 1$.

The shaded portion in Fig. 13 shows the ellipsoid.

b) Suppose we revolve the circle $x^2 + z^2 = 4$ around the y-axis. We get the ellipsoid.

E10) Let the equation of the surface be $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Changing x to –x, y to –y and z to –z, there is no changing in the equation. Therefore, the surface is symmetric about the XY, YZ and ZX planes.

E11) Let the equation of the surface be $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

Then, we see that the x-axis intersects the surface at the points (a, 0, 0) and (–a, 0, 0).

Similarly, the y-axis intersects the surface at the points (0, b, 0) and (0, – b, 0). Next we put z = 0 in the given equation of the conicoid, to get $z^2 = -c^2$.

This shows that the points of intersection are imaginary.
That is, the z-axis does not intersect the surface.

E12) a) The given equation is $x^2 + y^2 - z^2 = 1$.
For $z = \pm 3, \pm 6$, the horizontal cross-sections are circles with centres on the z-axis and radius 1 and 2 respectively (see Fig. 14).

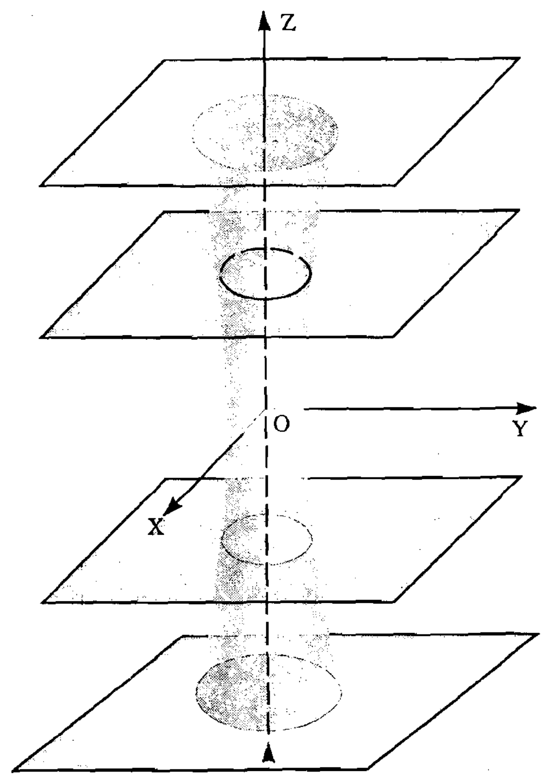

Fig. 14: Circles obtained by intersecting the planes $z = \pm 1, \pm 2$ with the hyperboloid of one sheet $x^2 + y^2 + z^2 = 1$.

b) When x = 0, the equation is $y^2 - z^2 = 1$,
Which represents a hyperbola (similar to Fig. 7) in the plane x = 0.
When y = 0, the equation is $x^2 - z^2 = 1$ or $x = \pm z$, which again represents a hyperbola.

E13) a) By renaming the axes, we can obtain the surface as shown in Fig. 15.

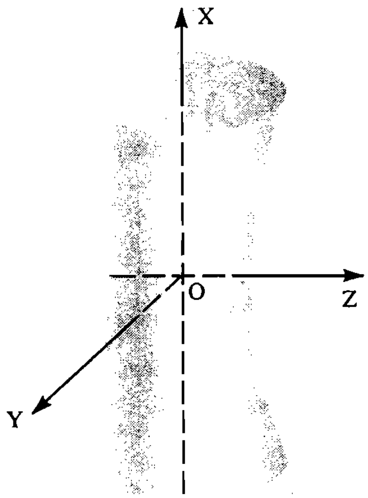

Fig. 15: The hyperboloid of one sheet $-\frac{x^2}{16}+\frac{y^2}{9}+\frac{z^2}{9}=1.$

b) When $|k| = 3$, we get

$$\frac{y^2}{9}+\frac{x^2}{16}=0$$

$$\Rightarrow\left(\frac{1}{3}y+\frac{1}{4}x\right)\left(\frac{1}{3}y-\frac{1}{4}x\right)=0.$$

which represents a pair of lines.
When $|k| < 3$, we get

$$\frac{y^2}{9}-\frac{x^2}{16}=1-\frac{k^2}{9}$$

which represents a hyperbola with transverse axis parallel to the y-axis.
When $|k| > 3$, we get

$$\frac{y^2}{9}-\frac{x^2}{16}=1-\frac{k^2}{9}$$

i.e. $\frac{x^2}{16}-\frac{y^2}{9}=\frac{k^2}{9}-1$

which represents a hyperbola with transverse axis parallel to the x-axis.
The cross-sections are shown in Fig. 15.

E14) a)

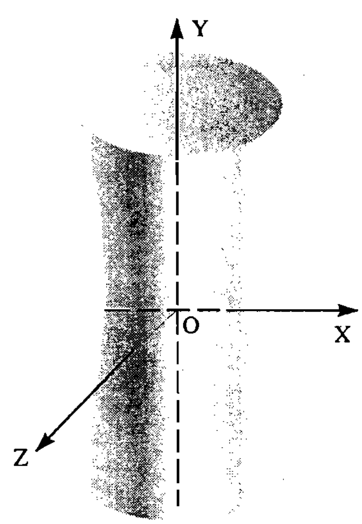

Fig. 16: The hyperboloid of one sheet $\frac{x^2}{9}-\frac{y^2}{4}+\frac{z^2}{9}=1.$

b) The hyperbola $\frac{z^2}{8} - \frac{y^2}{32/9} = 1$ in the YZ-plane.

E15) The surface is symmetric about the coordinate planes.

E16) The coordinate planes y = 0 and z = 0 intersects the surface in hyperbolas

$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ respectively.

The plane x = 0 does not intersect the surface. The x-axis meets the surface in points (a, 0, 0) and (–a, 0, 0). The y-axis and the z-axis do not meet the surface.

E17) a) The XY-plane does not intersect the surface. The YZ-plane and the XZ-plane intersect the surface in hyperbolas

$$z^2 - y^2 = 1 \text{ and } z^2 - \frac{x^2}{4} = 1.$$

b) The XZ-plane does not intersect the surface. The XY-plane and the YZ-plane intersect the surface in hyperbolas

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

and

$$\frac{y^2}{4} - z^2 = 1.$$

E18)

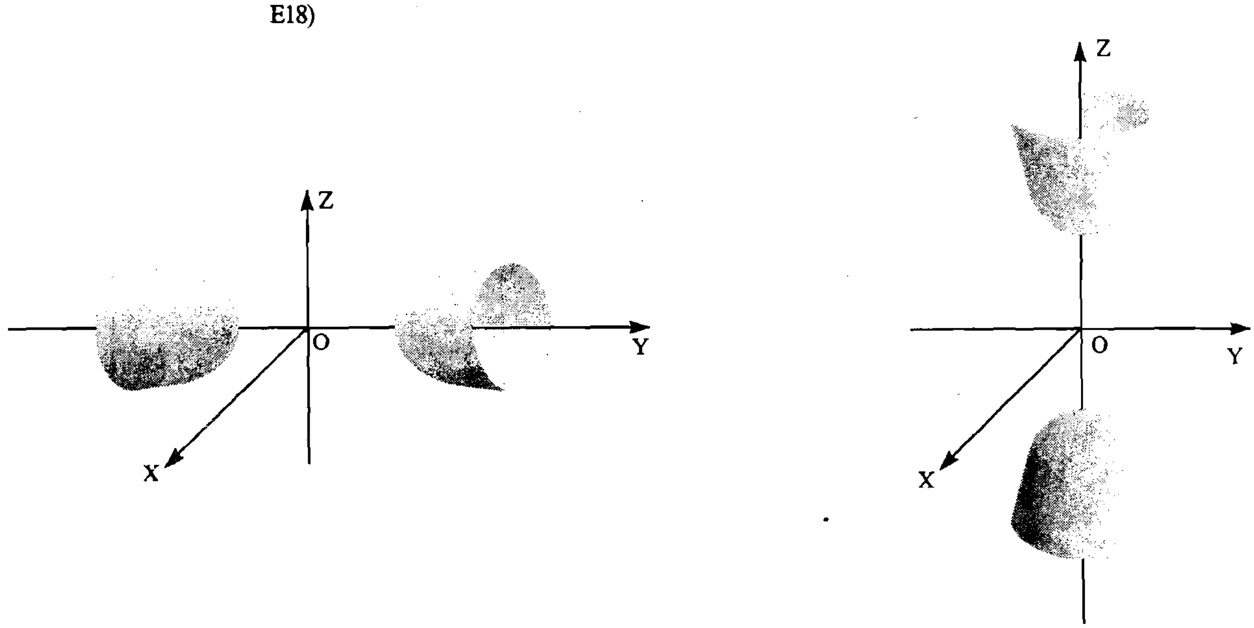

Fig. 17: The hyperboloid of two sheets a) $z^2 - \frac{x^2}{4} - y^2 = 1$. b) $\frac{y^2}{4} - \frac{x^2}{9} - z^2 = 1$.

E19) The equation of the tangent plane at (x_0, y_0, z_0) is $axx_0 + byy_0 + czz_0 = 1$.
The given equation is $x^2 + 3y^2 - 3z^2 = 1$.
So, here a = 1, b = 3, c = –3, $x_0 = 1$, $y_0 = -1$ and $z_0 = 1$.
Thus the required plane is x – 3y – 3z = 1.

E20) Any plane through the line x + 9y – 3z = 0, 3x – 3y + 6z – 5 = 0 is
x + 9y – 3z + λ (3x – 3y + 6z – 5) = 0
where λ is a real number.

Now, we know that this plane touches the given conicoid.
Therefore, we must have

$\frac{2xx_0}{5} - \frac{6yy_0}{5} + \frac{3zz_0}{5} = 1$ for some (x_0, y_0, z_0). Then we get

$$\frac{2x_0}{1+3\lambda} = \frac{6y_0}{9-3\lambda} = \frac{3z_0}{-3+6\lambda} = \frac{1}{\lambda}$$

$$\Rightarrow x_0 = \frac{1+3\lambda}{2\lambda}, y_0 = \frac{9-3\lambda}{6\lambda}, z_0 = \frac{-3+6\lambda}{3\lambda}$$

$$\Rightarrow x_0 = \frac{1+3\lambda}{2\lambda}, y_0 = \frac{3-\lambda}{2\lambda}, z_0 = \frac{-1+2\lambda}{\lambda}$$

Substituting this in the equation $2x^2 - 6y^2 + 3z^2 = 5$, we see that $\lambda = 1$ and $\lambda = -1$.
Hence, there are two tangent planes given by
$4x + 6y + 3z = 5$ and $2x - 12y + 9z = 5$.

E21) a) Here $a = 2$, $b = 1$, $c = -1$, $u = 3$, $v = 4$, $w = 5$.
Then $bcu^2 + cav^2 + abw^2 = -9 - 32 + 50 > 0$.
Therefore the section is an ellipse.

b) The section is an ellipse.