UNIT 3 NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

Structure **Page Nos.**

3.0 INTRODUCTION

In the previous two units, you have seen how a complicated or tabulated function can be replaced by an approximating polynomial so that the fundamental operations of calculus viz., differentiation and integration can be performed more easily. In this unit we shall solve a differential equation, that is, we shall find the unknown function which satisfies a combination of the independent variable, dependent variable and its derivatives. In physics, engineering, chemistry and many other disciplines it has become necessary to build mathematical models to represent complicated processes. Differential equations are one of the most important mathematical tools used in modeling problems in the engineering and physical sciences. As it is not always possible to obtain the analytical solution of differential equations recourse must necessarily be made to numerical methods for solving differential equations. In this unit, we shall introduce two such methods namely, Euler's method and Taylor series method to obtain numerical solution of ordinary differential equations (ODEs). To begin with, we shall recall few basic concepts from the theory of differential equations which we shall be referring to quite often.

3.1 OBJECTIVES

After studying this unit you should be able to:

- identify the initial value problem (IVP) for the first order ordinary differential equations;
- state the difference between the single step and multi-step methods of finding solution of IVP;
- obtain the solution of the initial value problems by using single-step methods viz., Taylor series method and Euler's method.

3.2 BASIC CONCEPTS

In this section we shall state a few definitions from the theory of differential equations and define some concepts involved in the numerical solution of differential equations.

Definition: An equation involving one or more unknown function (dependent variables) and its derivatives with respect to one or more known functions (independent variables) is called **differential equation.**
For example,

$$x\frac{dy}{dx} = 2y \qquad (1)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = 0 \qquad (2)$$

are differential equations.

Differential equations of the form (1), involving derivatives w.r.t. a single independent variable are called **ordinary differential equations** (ODEs) whereas, those involving derivatives w.r.t. two or more independent variables are **partial differential equations** (PDEs). Eqn. (2) is an example of PDE.

Definition: The **order** of a differential equation is the order of the highest order derivative appearing in the equation and its **degree** is the highest exponent of the highest order derivative after the equation has been rationalized i.e., after it has been expressed in the form free from radicals and any fractional power of the derivatives or negative power. For example equation

$$\left(\frac{d^3y}{dx^3}\right)^2 + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0 \qquad (3)$$

is of **third** order and **second** degree. Equation

$$y = x\frac{dy}{dx} + \frac{a}{dy/dx}$$

is of **first** order and **second** degree as it can be written in the form

$$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + a \qquad (4)$$

Definition: When the dependent variable and its derivatives occur in the first degree only and not as higher powers or products, the equation is said to be **linear,** otherwise it is **nonlinear.**

Equation $\frac{d^2y}{dx^2} + y = x^2$ is a linear ODE, whereas $(x+y)^2 \frac{dy}{dx} = 1$ is a nonlinear ODE.

Similarly, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\ \partial y}\right)^2 = 0$, is a nonlinear PDE.

In this unit we shall be concerned only with the ODEs.

The general form of a linear ODE of order n can be expressed in the form

$$L[y] \equiv a_0(t)\, y^{(n)}(t) + a_1(t)\, y^{(n-1)}(t) + \ldots + a_{n-1}(t)\, y'(t) + a_n(t)\, y(t) = r(t) \qquad (5)$$

where $r(t)$, $a_i(t)$, $i = 0,1,2, \ldots\ldots, n$ are known functions of t and

$$L = a_0(t)\frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \ldots\ldots + a_{n-1}(t)\frac{d}{dt} + a_n(t),$$

is the linear differential operator. The general nonlinear ODE of order n can be written as

$$F(t, y, y', y'', \ldots\ldots\ldots, y^{(n)}) = 0 \qquad (6)$$

or, $y^{(n)} = f(t, y, y', y'', \ldots\ldots\ldots, y^{(n-1)})$. (7)

Eqn. (7) is called a **canonical representation** of Eqn. (6). In such a form, the highest order derivatives is expressed in terms of lower order derivatives and the independent variable.

The **general solution** of an nth order ODE contains n arbitrary constants. In order to determine these arbitrary constants, we require n conditions. If all these conditions are given at one point, then these conditions are known as **initial conditions** and the differential equation together with the initial conditions is called an **initial value problem** (IVP). The nth order IVP alongwith associates initial conditions can be written as

$$y^{(n)}(t) = f(t, y, y', y'', \ldots\ldots, y^{(n-1)})$$
$$y^{(p)}(t_0) = y_0^{(p)}, \; p = 0, 1, 2, \ldots\ldots, n-1. \qquad (8)$$

We are required to find the solution y(t) for t > to

If the n conditions are prescribed at more than one point then these conditions are known as **boundary conditions.** These conditions are prescribed usually at two points, say t_0 and t_a and we are required to find the solution y(t) between to $<t<t_a$ .The differential equation together with the boundary conditions is then known as a **boundary value problem** (BVP).

As may be seen below, the nth order IVP (8) is equivalent to solving following system of n first order equations:

Setting $y = y_1$,

Similarly setting $y'_{i-1} = y_i$, we may write

$$\begin{array}{ll} y' = y'_1 = y_2 & y_1(t_0) = y_0 \\ y'_2 = y_3 & y_2(t_0) = y'_0 \\ \ldots\ldots\ldots & \ldots\ldots\ldots \\ y'_{n-1} = y_n & y_{n-1}(t_0) = y_0^{(n-2)} \\ y'_n = f(t, y_1, y_2, \ldots\ldots, y_n) & y_n(t_0) = y_0^{(n-1)}; \end{array}$$

In vector notation, this system can be written as a single equation as

$$\frac{dy}{dx} = f(t, y), \qquad y(t_0) = \alpha \qquad (9)$$

where $y = (y_1, y_2 \ldots\ldots, y_n)^T$, $f(t,y) = (y_2, y_3, \ldots\ldots, f(t, y_1, \ldots\ldots, y_n))^T$

$$\alpha\,(y_0, y'_0, \ldots\ldots, y_0^{(n-1)})^T.$$

Hence, it is sufficient to study numerical methods for the solution of the first order IVP.

$$y' = f(t, y), \; y(t_0) = y_0 \qquad (10)$$

The vector form of these methods can then be used to solve Eqn. (9). Before attempting to obtain numerical solutions to Eqn, (10), we must make sure that the problem has a unique solution. The following theorem ensures the existence and uniqueness of the solution to IVP (10).

Theorem 1: If f(t, y) satisfies the conditions

i) f(t, y) is a real function
ii) f(t, y) is bounded and continuous for $t \in [t_0,b]$, $y \in]-\infty, \infty[$
iii) there exists a constant L such that for any $t \in [t_0, b]$ and for any two numbers y_1 and y_2

$$|f(t, y_1) - f(t, y_2)| \le L\,|y_1 - y_2|$$

then for any y_0, the IVP (10) has a unique solution. This condition is called the **Lipschitz condition** and L is called the **Lipschitz constant.**

We assume the existence and uniqueness of the solution and also that f(t,y) has continuous partial derivatives w.r.t. t and y of as high order as we desire.

Let us assume that $[t_0, b]$ be an interval over which the solution of the IVP (10) is required. If we subdivide the interval $[t_0, b]$ into n subintervals using a step size $h = \left[\frac{t_n - t_o}{n}\right]$, where $t_n = b$, we obtain the **mesh points** or **grid points** $t_0, t_1, t_2, \ldots.., t_n$ as shown in Fig. 1.

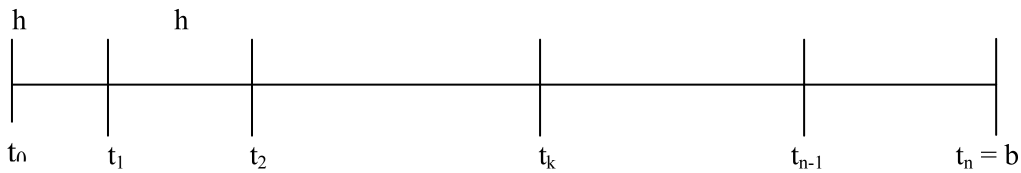

Fig. 1

We can then write $t_k = t_0 + kh$, $k = 0, 1, \ldots.., n$. A numerical method for the solution of the IVP (10), will produce approximate values y_k at the grid points t_k in a step by step manner i.e. values of $y_1, y_2, \ldots.$ etc in unit order.

Remember that the approximate values y_k may contain the truncation and round-off errors. We shall now discuss the construction of numerical methods and related basic concepts with reference to a simple ODE.

$$\frac{dy}{dt} = \lambda y, \quad t \varepsilon [a\ b] \tag{11}$$

$$y(t_0) = y_0,$$

where λ in a constant.

Let the domain [a, b] be subdivided into N intervals and

let the grid points be defined by,

$$t_j = t_0 + jh, \; j = 0, 1, \ldots\ldots\ldots\ldots.., N$$

where $t_0 = a$ and $t_N = t_0 + Nh = b$.

Separating the variables and integrating, we find that the exact solution of Eqn. (11) is

$$y(t) = y(t_0)\, e^{\lambda(t-t_0)}\,. \tag{12}$$

$y(t_0)\, e^{\lambda(t-t0)}$

In order to obtain a relation between the solutions at two successive permits, $t = t_n$ and t_{n+1} in Eqn. (12), we use,

$$y(t_n) = y(t_0)\ e^{\lambda(t_n - t_0)}$$

and

$$y(t_{n+1}) = y(t_0)\ e^{\lambda(t_{n+1} - t_0)}\ .$$

Dividing we get

$$\frac{y(t_{n+1})}{y(t_n)} = \frac{e^{\lambda t_{n+1}}}{e^{\lambda t_n}} = e^{\lambda(t_{n+1} - t_n)}\ .$$

Hence we have,

$$y(t_{n+1}) = e^{\lambda h}\ y\ (t_n),\ n = 0, 1, \ldots.., N\text{-}1. \tag{13}$$

Eqn. (13) gives the required relation between $y\ (t_n)$ and $y(t_{n+1})$.

Setting $n = 0, 1, 2, \ldots\ldots, N\text{-}1$, successively, we can find $y(t_1), y(t_2), \ldots\ldots.., y(t_N)$ from the given value $y(t_0)$.

An approximate method or a numerical method can be obtained by approximating $e^{\lambda h}$ in Eqn. (13). For example, we may use the following polynomial approximations,

$$e^{\lambda h} = 1 + \lambda h + 0\ (\left|\lambda h\right|^2) \tag{14}$$

$$e^{\lambda h} = 1 + \lambda h + \frac{\lambda^2 h^2}{2} + 0\ (\left|\lambda h\right|^3) \tag{15}$$

$$e^{\lambda h} = 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + 0\ (\left|\ \lambda h\right|^4) \tag{16}$$

and so on.

Let us retain $(p + 1)$ terms in the expansion of $e^{\lambda h}$ and denote the approximation to $e^{\lambda h}$ by $E(\lambda h)$. The numerical method for obtaining the approximate values y_n of the exact solution $y(t_n)$ can then be written as

$$y_{n+1} = E(\lambda h)\ y_n,\ n = 0, 1, \ldots\ldots\ldots, N\text{-}1 \tag{17}$$

The truncation error (TE) of the method is defined by

$$TE = y\ (t_{n+1}) - y_{n+1}.$$

Since $(p + 1)$ terms are retained in the expansion of $e^{\lambda h}$, we have

$$TE = \left(1 + \lambda h + \ldots\ldots\ldots + \frac{(\lambda h)^p}{p!} + \frac{(\lambda h)^{p+1}}{(p+1)!} e^{\theta\lambda h}\right) - \left(1 + \lambda h + \ldots\ldots. + \frac{(\lambda h)^p}{p!}\right)$$

$$= \frac{(\lambda h)^{p+1}}{(p+1)!} e^{\theta\lambda h}, \qquad 0 < \theta < 1$$

The TE is of order p+1 and the numerical method is called of order p.

The concept of stability is very important in a numerical method.

We say that a numerical method is **stable** if the error at any stage, i.e. $y_n - y(t_n) = \epsilon_n$ remains bounded as $n \to \infty$. Let us examine the stability of the numerical method (17). Putting $y_{n+1} = y(t_{n+1}) + \epsilon_{n+1}$ and $y_n = y(t_n) + \epsilon_n$ in Eqn. (17), we get

$$y(t_{n+1}) + \epsilon_{n+1} = E(\lambda h)\,[y(t_n) + \epsilon_n]$$

$$\epsilon_{n+1} = E(\lambda h)\,[y(t_n) + \epsilon_n] - y(t_{n+1})$$

which on using eqn. (13) becomes

$$\epsilon_{n+1} = E(\lambda h)\,[y(t_n) + \epsilon_n] - e^{\lambda h} y(t_n)$$

$$\therefore \quad \epsilon_{n+1} = [E(\lambda h) - e^{\lambda h}]\, y(t_n) + E(\lambda h)\,\epsilon_n \qquad (18)$$

We **note** from Eqn. (18) that the error at t_{n+1} consists of two parts. The first part $E(\lambda h) - e^{\lambda h}$ is the **local truncation error** and can be made as small as we like by suitably determining $E|\lambda h|$. The second part $|E(\lambda h)|\,\epsilon_n$ is the **propagation error** from the previous step t_n to $t_{n+1}$and will not grow if $|E(\lambda h)| < 1$. If $|E(\lambda h)| < 1$, then as $n \to \infty$ the propagation error tends to zero and method is said to be absolutely stable. Further, a numerical method is said to be **relatively stable** if $|E(\lambda h)| \le e^{\lambda h}$, $\lambda > 0$.

The polynomial approximations (14), (15) and (16) always give relatively stable methods. Let us now find when the methods $y_{n+1} = E(\lambda h)\, y_n$ are absolutely stable where $E(\lambda h)$ is given by (14) (15) or (16).

These methods are given by

First order: $y_{n+1} = (1 + \lambda h)\, y_n$

Second order: $y_{n+1} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) y_n$ and

Third order: $y_{n+1} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6}\right) y_n$

Let us examine the conditions for absolute stability in various methods:

First order: $|1 + \lambda h| \le 1$
or $-1 \le 1 + \lambda h \le 1$
or $-2 \le \lambda h \le 0$

Second order: $\left|1 + \lambda h + \frac{\lambda^2 h^2}{2}\right| \le 1$

or $-1 \le 1 + \lambda h + \frac{\lambda^2 h^2}{2} \le 1$

The right inequality gives

$$\lambda h \left(1 + \frac{\lambda h}{2}\right) \le 0$$

i.e., $\lambda h \le 0$ and $1 + \frac{\lambda h}{2} \ge 0$.

The second condition gives $-2 \le \lambda h$. Hence the right inequality gives $-2 \le \lambda h \le 0$.

The left inequality gives

$$2 + \lambda h + \frac{\lambda^2 h^2}{2} \geq 0.$$

For $-2 \leq \lambda h \leq 0$, this equation is always satisfied. Hence the stability condition is $-2 \leq \lambda h \leq 0$.

Third order: $\left| 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} \right| \leq 1$

Using the right and left inequalities, we get

$$-2.5 \leq \lambda h \leq 0.$$

These intervals for λh are known as **stability intervals.**

Numerical methods for finding the solution of IVP given by Eqn. (10) may be broadly classified as,

i) Singlestep methods
ii) Multistep methods

Singlestep methods enable us to find y_{n+1}, an approximation to $y(t_{n+1})$, in terms of y_n, and y'_n.

Multistep methods enable us to find y_{n+1}, an approximation to $y(t_{n+1})$, in terms of y_i, y'_i, $i = n, n-1, \ldots n-m+1$ i.e. values of y and y′ at previous m points. Such methods are called m-step multistep methods.

In this course we shall be discussing about the singlestep methods only.

A singlestep method for the solution of the IVP

$$y' = f(t, y), \qquad y(t_0) = y_0, \; t \in (t_0, b)$$

is a recurrence relation of the form

$$y_{n+1} = y_n + h\,\phi\,(t_n, y_n, h) \tag{19}$$

where $\phi\,(t_n, y_n, h)$ is known as the **increment function.**

If y_{n+1} can be determined from Eqn. (19) by evaluating the right hand side, then the singlestep method is known as an **explicit method,** otherwise it is known as an **implicit method.** The local truncation error of the method (19) is defined by

$$TE = y\,(t_{n+1}) - y(t_n) - h\,\phi\,(t_n, y_n, h)\,. \tag{20}$$

The largest integer p such that

$$\left| h^{-1}\,TE \right| = O\,(h^p) \tag{21}$$

is called the **order** of the singlestep method.

Let us now take up an example to understand how the singlestep method works.

Example 1: find the solution of the IVP $y' = \lambda y$, $y(0) = 1$ in $0 < t \leq 0.5$, using the first order method

$$y_{n+1} = (1 + \lambda h)\,y_n \text{ with } h = 0.1 \text{ and } \lambda = \pm 1.$$

Solution: Here the number of intervals are N = $\frac{0.5}{h} = \frac{0.5}{0.1} = 5$

We have $y_0 = 1$

$y_1 = (1 + \lambda h)\, y_0 = (1 + \lambda h) = (1 + 0.1\lambda)$

$y_2 = (1 + \lambda h)\, y_1 = (1 + \lambda h)^2 = (1 + 0.1\lambda)^2$

$y_5 = (1 + \lambda h)^5 = (1 + 0.1\lambda)^5$

The exact solution is $y(t) = e^{\lambda t}$.

We now give in Table 1 the values of y_n for $\lambda = \pm 1$ together with exact values.

Table 1

Solution of $y' = \lambda y$, $y(0) = 1$, $0 \le t \le 0.5$ with h = 01.

	$\lambda = 1$		$\lambda = -1$	
t	First Order method	Exact Solution	First Order method	Exact Solution
0	1	1	1	1
0.1	1.1	1.10517	0.9	0.90484
0.2	1.21000	1.22140	0.81	0.81873
0.3	1.33100	1.34986	0.729	0.74082
0.4	1.46410	1.49182	0.6561	0.67032
0.5	1.61051	1.64872	0.59049	0.60653

Similarly you can obtain the solution using the second order method and compare the results obtained in the two cases.

Ex 1) Find the solution of the IVP

$$y' = \lambda y,\ y(0) = 1$$

in $0 \le t \le 0.5$ using the second order method

$$y_{n+1} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) y_n \text{ with } h = 0.1 \text{ and } \lambda = 1.$$

We are now prepared to consider numerical methods for integrating differential equations. The first method we discuss is the Taylor series method. It is not strictly a numerical method, but it is the most fundamental method to which every numerical method must compare.

3.3 TAYLOR SERIES METHOD

Let us consider the IVP given by Eqn. (10), i.e.,

$$y' = f(t,y),\ y(t_0) = y_0, \qquad t \in [t_0, b]$$

The function f may be linear or nonlinear, but we assume that f is sufficiently differentiable w.r.t. both t and y.

The Taylor series expansion of y(t) about any point t_k is given by

$$y(t) = y(t_k) + (t-t_k)\, y'(t_k) + \frac{(t-t_k)^2}{2!} y''(t_k) + \ldots\ldots + \frac{(t-t_k)^p}{p!} y^{(p)}(t_k) + \ldots\ldots \qquad (22)$$

Substituting $t = t_{k+1}$ in Eqn. (22), we have

$$y(t_{k+1}) = y(t_k) + hy'(t_k) + \frac{h^2 y''(t_k)}{2!} + \ldots\ldots.. + \frac{h^p y^{(p)}(t_k)}{p!} + \ldots.. \qquad (23)$$

where $t_{k+1} = t_k + h$. Neglecting the terms of order h^{p+1} and higher order terms, we have the approximation

$$y_{k+1} = y_k + hy'_k + \frac{h^2}{2!} y''_k + \ldots.. + \frac{h^p}{p!} y_k^{(p)} \qquad (24)$$

$$= y_k + h\,\phi\,(t_k, y_k, h)$$

where $\phi\,(t_k, y_k, h) = y'_k + \frac{h}{2!} y''_k + \ldots.. + \frac{h^{p-1}}{p!} y_k^{(p)}$

This is called the Taylor Series method of order p. The truncation error of the method is given by

$$TE = y\,(t_{k+1}) - y(t_k) - h\phi\,(t_k, y(t_k), h) \qquad (25)$$

$$= \frac{h^{p+1}}{(p+1)!} y^{(p+1)}\ (t_k + \theta h), 0 < \theta < 1$$

when p = 1, we get from Eqn. (24)

$$y_{k+1} = y_k + hy'_k \qquad (26)$$

which is the Taylor series method of order one.

To apply (24), we must know $y(t_k)$, $y'(t_k)$, $y''(t_k)$………., $y^{(p)}\,(t_k)$.

However, $y(t_k)$ is known to us and if f is sufficiently differentiable, then higher order derivatives can be obtained by calculating the total derivative of the given differential equation w.r.t. t, keeping in mind that y is itself a function of t. Thus we obtain for the first few derivatives as:

$$y' = f(t,y)$$
$$y'' = f_t + f\,f_y$$
$$y''' = f_{tt} + 2\,f\,f_{ty} + f^2\,f_{yy} + f_y\,(f_t + ff_y) \text{ etc.}$$

where $f_t = \partial f / \partial t, f_{tt} = \partial^2 f / \partial t^2$, $f_{ty} = \frac{\partial^2 f}{\partial t 2y}$ etc.

The number of terms to be included in the method depends on the accuracy requirements.

Let p = 2. Then the Taylor Series method of $O(h^2)$ is

$$y_{k+1} = y_k + hy'_k + \frac{h^2 y''_k}{2} \qquad (27)$$

with the TE = $\frac{h^3}{6} y'''(\alpha), t_n < \alpha < t_{n+1}$

The Taylor series method of $O(h^3)$, (p = 3) is

$$y_{k+1} = y_k + hy'_k + \frac{h^2 y''_k}{2} + \frac{h^3}{6} y'''_k \qquad (28)$$

with the TE = $\frac{h^4}{24} y^{(IV)}\ (\alpha),\ t_k \le \alpha \le t_{k+1}$

Let us consider the following examples.

Example 2: Using the third order Taylor series method find the solution of the differential equation.

$xy' = x-y$, $y = 2$ at $x = 2$, taking $h = 1$.

Solution: We have the derivatives and their values at x=2, y=2 as follows:

y(2) = 2 is given. Further, $xy' = x-y$ can be written as

$$y' = 1 - \frac{y}{x}$$

$$y'(2) = 1 - \frac{2}{2} = 0.$$

Differentiating w.r.t. x, we get

$$y'' = 0 - \frac{y'}{x} + \frac{y}{x^2}$$

$$y''(2) = -\frac{0}{2} + \frac{2}{4} = \frac{1}{2}.$$

$$y' = 1 - \frac{y}{k} \qquad = 1 - \frac{2}{2} = 1 - 1$$

Similarly,

$$y''' = -\frac{y''}{x} + \frac{2y'}{x^2} - \frac{2y}{x^3}, \qquad y'''(2) = -3/4$$

$$y'''(2) = -\frac{1}{4} + \frac{2 \times 0}{4} - \frac{2 \text{ x } 2}{8}$$

$$= \frac{3}{4}$$

Using taylor series method of $O(h^3)$ given by Eqn. (28), we obtain

$$y(2 + .1) = y(2) + 01 \text{ x } y'(2) + \frac{(.1)^2}{2} y''(2) + \frac{(.1)^3}{6} y'''(2)$$

or

$$y(2.1) = 2 + .1 \text{ x } 0 + .005 \text{ x } .5 + .001 \text{ x } \frac{1}{6} \times (-.75)$$

$$= 2+0.0025-0.000125 = 2.002375.$$

Example 3: Solve the equation $x^2y' = 1 - xy - x^2y^2$, $y(1) = -1$ from x=1 to x=2 by using Taylor series method of $O(h^2)$ with h = 1/3 and ¼ and find the actual error at x=2 if the exact solution is $y = -1/x$.

Solution: From the given equation, we have $y' = \frac{1}{x^2} - \frac{y}{x} - y^2$

Differentiating it w.r.t. x, we get

$$y'' = \frac{-2}{x^3} - \frac{y'}{x} + \frac{y}{x^2} - 2yy'$$

Using the second order method (27),

$$y_{k+1} = y_k + hy'_k + \frac{h^2}{2} y''_k$$

We have the following results

$h = \frac{1}{3}$,	$y(1) = -1$,	$y'(1) = 1$,	$y''(1) = -2$.
$x_1 = \frac{4}{3}$,	$y(x_1) = -0.7778$,	$y'(x_1) = 0.5409$,	$y''(x_1) = -0.8455$.
$x_2 = \frac{5}{3}$,	$y(x_2) = -0.6445$,	$y'(x_2) = 0.3313$,	$y''(x_2) = -0.4358$.
$x_3 = 2$,	$y(x_3) = -0.5583$	$= y(2)$	
$h = \frac{1}{4}$			
$x_1 = \frac{5}{4}$,	$y(x_1) = -0.8125$,	$y'(x_1) = 0.6298$,	$y''(x_1) = -1.0244$.
$x_2 = \frac{3}{2}$,	$y(x_2) = -0.6871$,	$y'(x_2) = 0.4304$,	$y''(x_2) = -0.5934$.
$x_3 = \frac{7}{4}$,	$y(x_3) = -0.5980$,	$y'(x_3) = 0.3106$,	$y''(x_3) = -0.3745$.
$x_4 = 2$,	$y(x_4) = -0.5321$	$= y(2)$	

Since the exact value is $y(2) = -0.5$, we have the actual errors as

$$e_1 = 0.0583 \text{ with } h = \frac{1}{3}$$

$$e_2 = 0.0321 \text{ with } h = \frac{1}{4}$$

Note that error is small when the step size h is small.
You may now try the following exercise.

Write the Taylor series method of order four and solve the IVPs E2) and E3).

E2) $y' = x - y^2$, $y(0) = 1$. Find $y(0.1)$ taking $h = 0.1$.

E3) $y' = x^2 + y^2$, $y(0) = 0.5$. Find $y(0.4)$ taking $h = 0.2$.

E4) Using second order Taylor series method solve the IVP
$y' = 3x + \frac{y}{2}$, $y(0) = 1$. Find $y(0.6)$ taking $h = 0.2$ and $h = 0.1$.

Find the actual error at $x = 0.6$ if the exact solution is $y = -6x - 12$.

Notice that though the Taylor series method of order p give us results of desired accuracy in a few number of steps, it requires evaluation of the higher order derivatives and becomes tedious to apply if the various derivatives are complicated. Also, it is difficult to determine the error in such cases. We now consider a method, called **Euler's method** which can be regarded as Taylor series method of order one and avoids these difficulties.

3.4 EULER'S METHOD

Let the given IVP be

$$y' = f(t, y),\ y(t_0) = y_0.$$

Let $[t_0, b]$ be the interval over which the solution of the given IVP is to be determined. Let h be the steplength. Then the nodal points are defined by $t_k = t_0 + kh$, $k = 0, 1, 2, \ldots\ldots\ldots\ldots, N$ with $t_N = t_0 + Nh = b$.

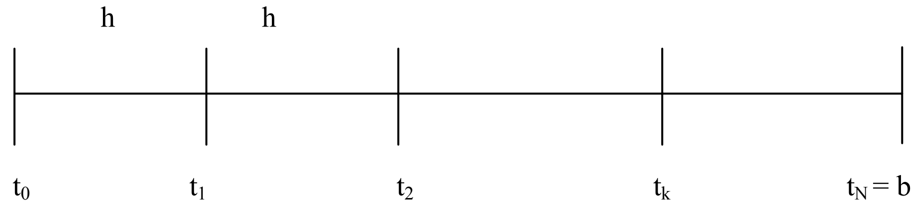

Fig. 1

The exact solution y(t) at $t = t_{k+1}$ can be written by Taylor series as

$$y(t_k + h) = y(t_k) + hy'(t_k) + \left(\frac{h^2}{2}\right) y''(t_k) + \ldots.. \tag{29}$$

Neglecting the term of $O(h^2)$ and higher order terms, we get

$$y_{k+1} = y_k + hy'_k \tag{30}$$

$$\text{with TE} = \left(\frac{h^2}{2}\right) y''(\alpha), t_k < \alpha < t_{k+1} \tag{31}$$

From the given IVP, $y'(t_k) = f(t_k, y_k) = f_k$

We can rewrite Eqn. (30) as

$$y_{k+1} = y_k + h\,f_k$$

for $k = 0, 1, \ldots\ldots.., N-1$. (32)

Eqn. (32) is known as the Euler's method and it calculates successively the solution at the nodal points t_k, $k = 1, \ldots\ldots\ldots, N$.

Since the truncation error (31) is of order h^2, Euler's method is of first order. It is also called an O(h) method.

Let us now see the geometrical representation of the Euler's method.

Geometrical Interpretation

Let y(t) be the solution of the given IVP. Integrating $\frac{dy}{dt}$ = (t, y) from t_k to t_{k+1}, we get

$$\int_{t_k}^{t_{k+1}} \frac{dy}{dt} dt = \int_{t_k}^{t_{k+1}} f(t, y)\, dt = y(t_{k+1}) - y(t_k). \tag{33}$$

We know that geometrically f(t, y) represents the slope of the curve y(t). Let us approximate the slope of the curve between t_k and t_{k+1} by the slope at t_k only. If we approximate $y(t_{k+1})$ and $y(t_k)$ by y_{k+1} and y_k respectively, then we have

$$y_{k+1} - y_k = f(t_k, y_k) \int_{t_k}^{t_{k+1}} dt \tag{34}$$

$$= (t_{k+1} - t_k)\, f(t_k, y_k)$$

$$= hf\,(t_k, y_k)$$

$$\therefore\ y_{k+1} = y_k + \ \ hf(t_k, y_k),\ k = 0, 1, 2, \ldots, N\text{-}1.$$

Thus in Euler's method the actual curve is approximated by a sequence of the segments and the area under the curve is approximated by the area of the quadrilateral.

Let us now consider the following examples.

Example 4: Use Euler method to find the solution of $y' = t + y$, given $y(0) = 1$. Find the solution on [0, 0.8] with h = 0.2.

Solution: We have

$$y_{n+1} = y_n + hf_n$$

$$y(0.2) \approx y_1 = y_0 + (0.2)\, f_0$$
$$= 1 + (0.2)\,[0 + 1] = 1.2$$

$$y(0.4) \approx y_2 = y_1 + (0.2)\, f_1$$
$$= 1.2 + (0.2)\,[0.2 + 1.2]$$
$$= 1.48$$

$$y(0.6) \approx y_3 = y_2 + (0.2)\, f_2$$
$$= 1.48 + (0.2)\,[0.4 + 1.48]$$
$$= 1.856$$

$$y(0.8) \approx y_4 = y_3 + (0.2)\, f_3$$
$$= 1.856 + (0.2)\,[0.6 + 1.856]$$
$$= 2.3472$$

Example 5: Solve the differential equation $y' = t+y$, $y(0) = 1$. $t \in [0, 1]$ by Euler's method using h = 0.1. If the exact value is $y(1) = 3.436564$, find the exact error.

Solution: Euler's method is

$$y_{n+1} = y_n + hy'_n$$

For the given problem, we have

$$y_{n+1} = y_n + h\,[t_n + y_n]$$
$$(1 + h)\, y_n + ht_n.$$
$$h = 0.1,\ y(0) = 1,$$
$$y_1 = y_0 = (1+0.1) + (0.1)\,(0) = 1.1$$
$$y_2 = (1.1)\,(1.1) + (0.1)\,(0.1) = 1.22,\ y_3 = 1.362$$
$$y_4 = 1.5282,\ y_5 = 1.72102,\ y_6 = 1.943122,$$
$$y_7 = 2.197434,\ y_8 = 2.487178,\ y_9 = 2.815895$$
$$y_{10} = 3.187485 \approx y(1)$$
$$\text{exact error} = 3.436564 - 3.187485 = .249079$$

Example 6: Using the Euler's method tabulate the solution of the IVP

$$y' = -2ty^2,\ y(0) = 1$$

in the interval [0,1] taking h = 0.2, 0.1.

Solution: Euler's method gives

$$y_{k+1} = y_k + h\, f_k \text{ where } t_k = -2t_k\ y_k^2$$
$$= y_k - 2h\, t_k\ y_k^2.$$

Starting with $t_0 = 0$, $y_0 = 1$, we obtain the following table of values for h = 0.2.

Table 2: h = 0.2

T	y(t)
0.2	0.92
0.4	
0.6	0.78458
0.8	0.63684
1.0	0.50706

Thus y(1.0) = 0.50706 with h = 0.2
Similarly, starting with $t_0 = 0$, $y_0 = 1$, we obtain the following table of values for h = 0.1.

Table 3: h = 0.1

t	y(t)	T	y(t)
0.1	1.0	0.6	0.75715
0.2	0.98	0.7	0.68835
0.3	0.94158	0.8	0.62202
0.4	0.88839	0.9	0.56011
0.5	0.82525	1.0	0.50364

y(1.0) = 0.50364 with h = 0.1.

Remark: Since the Euler's method is of O(h), it requires h to be very small to obtain the desired accuracy. Hence, very often, the number of steps to be carried out becomes very large. In such cases, we need higher order methods to obtain the required accuracy in a limited number of steps.

Euler's method constructs $y_k \approx y(t_k)$ for each k = 1, 2, …., N,
where

$$y_{k+1} = y_k + hf(t_k, y_k).$$

This equation is called the **difference equation** associated with Euler's method. A difference equation of order N is a relation involving $y_n, y_{n+1}, \ldots\ldots, y_{n+N}$. Some simple difference equations are

$$\left.\begin{aligned} y_{n+1} - y_n &= 1 \\ y_{n+1} - y_n &= n \\ y_{n+1} - (n+1)\, y_n &= 0 \end{aligned}\right] \qquad (35)$$

where n is an integer.

A difference equation is said to be **linear** if the unknown function y_{n+k} (k = 0, 1, ……., N) appear linearly in the difference equation. The general form of a linear non-homogeneous difference equation of order N is

$$y_{n+N} + a_{N-1}\, y_{n+N-1} + \ldots\ldots\ldots\ldots + a_0\, y_n = b \qquad (36)$$

where the coefficients $a_{N-1}, a_{N-2}, \ldots\ldots\ldots, a_0$ and b may be functions of n but not of y. All the Eqns. (35) are linear. It is easy to solve the difference Eqn. (36), when the coefficients are constant or a linear or a quadratic function of n.
The general solution of Eqn. (36) can be written in the form

$$y_n = y_n(c) + y_n^{(p)}$$

where y_n (c) is the complementary solution of the homogenous equation associated with Eqn. (36) and y_n(p) is a particular solution of Eqn. (36). To obtain the complementary solution of the homogeneous equations, we start with a solution in the form $y_n = \beta^n$ and substitute it in the given equation. This gives us a polynomial of degree N. We assume that its roots $\beta_1, \beta_2, \ldots\ldots\ldots, \beta_N$ are all real and distinct.

Therefore, the general solution of the given problem is

$$y_k = C\,(1+3h)^k - \frac{5}{3}$$

Using the condition y(0) = 1, we obtain C = 8/3.

Thus

$$y_k = \frac{8}{3}(1+3h)^k - \frac{5}{3}.$$

Eqn. (41) gives the formula for obtaining $y_k \; \forall$ k.

$$y_6 = y(0.6) = \frac{8}{3}(1+3\times 0.1)^6 - \frac{5}{3}$$

$$= 11.204824.$$

Now Euler's method is

$$y_{k+1} = (1 + 3h)\, y_k + 5h$$

and we get for h = 0.1

$$y_1 = 1.8,\ y_2 = 2.84,\ y_3 = 4.192,\ y_4 = 5.9496,\ y_5 = 8.23448,\ y_6 = 11.204824.$$

You may now try the following exercises

Solve the following IVPs using Euler's method

E5) $y' = 1 - 2xy$, $y(0.2) = 0.1948$. Find y(0.4) with h = 0.2

E6) $y' = \dfrac{1}{x^2 - 4y}$, y(4) = 4. Find y(4.1) taking h = 0.1

E7) $y' = \dfrac{y - x}{y + x}$, y(0) = 1. Find y(0.1) with h = 0.1

E8) $y' = 1 + y^2$, y(0) = 1. Find y(0.6) taking h = 0.2 and h = 0.1

E9) Use Euler's method to solve numerically the initial value problem $y' = t + y$, y(0) = 1 with h = 0.2, 0.1 and 0.05 in the interval [0.06].

3.5 SUMMARY

In this unit, we have covered the following

1) $y' = f(t, y)$, $y(t_0) = y_0$, $t \in [t_0, b]$ is the initial value problem (IVP) for the first order ordinary differential equation.
2) Singlestep methods for finding the solution of IVP enables us to find y_{n+1}, if y_n, y'_n and h are known.
3) Multistep method for IVP enables as to find y_{n+1}, if y_i, y'_i, i = n, n-1, ….., n-m+1 and h are known and are called m-step multistep methods.

4) Taylor series method of order p for the solution of the IVP is given by
$y_{k+1} = y_k + h\,\phi\,[t_k, y_k, h]$
where ϕ [tk, yk, h] = $y'k + \frac{h}{2!} y''_k + ... + \frac{h^{p-1}}{p!} y_k^{(p)}$ and $t_k = t_0 + kh, k = 0, 1, 2,$ ………….N-1, $t_N = b$. The error of approximation is given by
$TE = \frac{h^{p+1}}{(p+1)!} y^{(p+1)} (t_k + \theta h),\ 0 < \theta < 1.$

5) Euler's method is the Taylor series method of order one. The steps involved in solving the IVP given by (10) by Euler's method are as follows:

Step 1: Evaluate $f(t_0, y_0)$
Step 2: Find $y_1 = y_0 + h\,f(t_0, y_0)$
Step 3: If $t_0 < b$, change t_0 to $t_0 + h$ and y_0 to y_1 and repeat steps 1 and 2
Step 4 : If $t_0 = b$, write the value of y_1.

3.6 SOLUTIONS/ANSWERS

E1) We have $y_0 = 1, \lambda = 1, h = 0.1$

$$y_1 = \left(1 + 0.1 + \frac{(0.1)^2}{2}\right)$$

$y_2 = (1.105)^2$

$y_5 = (1.105)^5$
Table giving the values of y_n together with exact values is

Table 4

t	Second order method	Exact solution
0	1	1
0.1	1.105	1.10517
0.2	1.22103	1.22140
0.3	1.34923	1.34986
0.4	1.49090	1.49182
0.5	1.64745	1.64872

E2) Taylor series method of $0(h^4)$ to solve $y' = x - y^2$, $y(0) = 1$ is

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n + \frac{h^3}{6} y'''_n + \frac{h^4}{24} y_n^{iv}$$

$y' = x - y^2$ $\quad$ $y'(0) = -1$
$y'' = 1 - 2xy'$ $\quad$ $y''(0) = 3$
$y''' = -2xy'' - 2(y')^2$ $\quad$ $y'''(0) = -8$
$y^{iv} = -2yy'' - 6y'\,y''$ $\quad$ $y^{iv}(0) = 34$

Substituting

$$y(0.1) = 1 - (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34)$$
$$= 0.9138083$$

E3) Taylor series method

$y' = x^2 + y^2$, $\quad$ $y(0) = 0.5$, $\quad$ $y'(0) = 0.25$, $\quad$ $y'(0.2) = 0.35175$

$y'' = 2x + 2yy'$ $\quad y''(0) = 0.25, \quad y''(0.2) = 0.79280$
$y''' = 2 + 2yy'' + 2(y')^2$ $\quad y'''(0) = 2.375, \quad y'''(0.2) = 3.13278$
$y^{iv} = 2yy''' + 6y'y''$ $\quad y^{iv}(0) = 2.75, \quad y^{iv}(0.2) = 5.17158$

E4) Second order Taylor's method is

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n.$$

h = 0.2

$y(0) = 1.$ $\quad y'(0) = 0.5,$ $\quad y''(0) = 3.25$
$y(0.2) = 1.165,$ $\quad y'(0.2) = 1.1825,$ $\quad y''(0.2) = 3.59125$
$y(0.4) = 1.47333,$ $\quad y'(0.4) = L93667,$ $\quad y''(0.4) = 3.96833$
$y(0.6) = 1.94003$

h = 0.1

$y(0.1) = 1.06625,$ $\quad y'(0.1) = 0.83313,$ $\quad y''(0.1) = 3.41656$
$y(0.2) = 1.16665,$ $\quad y'(0.2) = 1.18332,$ $\quad y''(0.2) = 3.59167$
$y(0.3) = 1.46457,$ $\quad y'(0.3) = 1.63228,$ $\quad y''(0.3) = 3.81614$
$y(0.4) = 1.64688,$ $\quad y''(0.4) = 2.02344,$ $\quad y''(0.4) = 4.01172$
$y(0.5) = 1.86928,$ $\quad y'(0.5) = 2.43464,$ $\quad y''(0.5) = 4.21732$
$y(0.6) = 2.13383$

E5) Euler's method is $y_{k+1} = y_k + hf_k = y_k + h(1-2x_k y_k)$
$y(0.4) = 0.1948 + (0.2)(1-2 \text{ x } 0.2 \text{ x } 0.1948)$
$= 0.379216.$

E6) $y' = \dfrac{1}{x^2 + y}, \quad y(4) = 4, \quad y'(4) = 0.05$

$y(4.1) = y(4) + hy'(4)$
$= 4 + (0.1)(0.05) = 4.005.$

E7) Euler's method $y' = (y-x)/(y+x)$, $y(0) = 1$, $y'(0) = 1$
$y(0.1) = 1 + (0.1)(1) = 1.1$

E8) Euler's method is
$y_{k+1} = h + y_k + hy_k^2$

Starting with $t_0 = 0$ and $y_0 = 1$, we have the following tables of values

Table 5: h = 0.2

T	Y(t)
0.2	1.4
0.4	1.992
0.6	2.986

$\therefore\ y(0.6) = 2.9856$

Table 6: h = 0.1

T	Y(t)
0.1	1.2
0.2	1.444
0.3	1.7525
0.4	2.1596
0.5	2.7260
0.6	3.5691

$\therefore\ y(0.6) = 3.5691$

E9) $y_{k+1} = y_k + h\ (t_k + y_k) = (1 + h)\ y_k + ht_k$

Starting with $t_0 = 0$, $y_0 = 1$, we obtain the following of values.

Table 7: h = 0.2

T	Y(t)
0.2	1.2
0.4	1.48
0.6	1.856

$\therefore$ y(0.6) = 1.856 with h = 0.2

Table 8: h = 0.1

t	y(t)
0.1	1.1
0.2	1.22
0.3	1.362
0.4	1.5282
0.5	1.72102
0.6	1.943122

$\therefore$ y(0.6) = 1.943122 with h = 0.1

Table 9: h = 0.05

t	y(t)	T	y(t)
0.05	1.05	0.35	1.46420
0.10	1.105	0.40	1.55491
0.15	1.16525	0.45	1.65266
0.20	1.23101	0.50	1.75779
0.25	1.30256	0.55	1.87068
0.30	1.38019	0.60	1.99171

$\therefore$ y(0.6) = 1.99171 with h = 0.05.