UNIT 1 GENERAL THEORY OF CONICOIDS

Structure

1.1 INTRODUCTION

You have seen in Block 7 that the general equation of second degree in two variables x and y represents a conic. In analogy with this we can ask: what will a general second degree equation in three variables represent? In Block 8 you have studied some particular forms of second degree equations in three variables, namely, those representing spheres, cones and cylinders. In this unit we study the most general form of a second degree equation in three variables. The surface generated by these equations are called quadrics or conicoids. This name is apt because, as you will see in Unit 8, they can be formed by revolving a conic about a line called an axis.

Alexis Clairaut (1713-1765), a French mathematician, was one of the pioneers to study quadric surfaces. He specified that a surface, in general, can be represented by an equation in three variables. He presented his ideas in his book 'Recherche Sur Les Courbes a Double Courbure' in which he gave the equations of several conicoids like the sphere, cylinder, hyperboloid and ellipsoid.

We start this unit with a small section in which we define a conicoid. In the next section we discuss rigid body motions in a three-dimensional system. We shall consider two types of transformations: translation of axes and rotation of axes. You can see that a conicoid remains unchanged in shape and size under these transformations. Lastly, we shall discuss how to reduce the equation of conicoid into a more simple form.

Objectives

After studying this unit you should be able to :

- define a general conicoid;
- obtain the new coordinates when a given coordinate system is translated or rotated;
- use the fact that translation and rotation of axes are rigid body motions;
- check whether a given conicoid has a centre or not;
- prove and apply the fact that if a conicoid has a centre, then it can be reduced to standard form.

1.2 WHAT IS A CONICOID?

In his section we shall define surfaces in a three-dimensional coordinate system which are analogous to conic sections in a two-dimensional system.

Let us start with a definition.

Definition: A general second degree equation in three variables is an equation of the form
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$ ... (1)
where a, b, c, d, f, g, h, u, v, w are real numbers and at least one of a, b, c, d, f, g, h is non-zero.

Note that if we put either z = k, a constant, x = k or y = k, in (1), then the equation reduces to a general second degree equation in two variables, and therefore, represents a conic.

Now, we shall see what a general second degree equation in three variables represents. Let us first consider some particular cases of (1).

Case 1 : Suppose we put a = b = c = 1 and g = h = f = 0 in (1). Then we get the equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{... (2)}$$

Does this equation seem familiar to you? In Unit 4 of Block 8 you saw that if $u^2 + v^2 + w^2 - d > 0$, then (2) represents a sphere with centre (–f, –g, –h) and radius $\sqrt{u^2 + v^2 + w^2 - d}$.

Case 2 : Suppose we put u = v = w = d = 0 in (1), then we get
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$

What does this equation represent? You know from Unit 4 of Block 8 that this equation represents a cone.

Case 3 : If we put a = b = 1, h = 0 and z = k in (1), then it reduces to
$x^2 + y^2 + 2ux + 2vy + d = 0, z = k$ ...(3)

This represents a right circular cylinder (see Unit 4 of Block 8, Sec. 6.4).

Similarly, you can see that if we put x = k or y = k and a = b = 1, h = 0, then again (3) represents a cylinder.

We will discuss the surfaces represented by (1) in detail in the next unit.

The particular cases 1, 2, and 3 suggest that the points whose coordinates satisfy (1) lie on a surface in the three-dimensional system. Such a surface is called a conicoid or a quadric. Algebraically, we define a conicoid as follows :

Definition : A conicoid (or quadric surface) in the XYZ-coordinate system is the set S of points $(x, y, z) \in \mathbf{R}^3$ that satisfy a general second degree eqution in three variables.

So, for example, if
$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$
is the second degree equation, then
$S = \{(x, y, z) \in \mathbf{R}^3 \mid F(x, y, z) = 0\}$

Note that S can be empty. For example, if $F(x, y, z) \equiv x^2 + y^2 + z^2 + 1 = 0$, then
$S = \{(x, y, z) \mid F(x, y, z) = 0\} = \phi$. the empty set.

In such cases we call the conicoid an imaginary conicoid.

Since the above expression is very lengthy, for convenience we often denote this coniocid by F (x, y, z) = 0.

Note : In future, whenever we use the expression F (x, y, z) = 0, we will mean the equation (1).

In this unit you will see that we can always reduce F(x, y, z) to a much smaller quadratic polynomial. To do this we need to transform the axes suitably. Let us see what this means.

1.3 CHANGE OF AXES

In Block 7, you saw that a general second degree equation can be transformed into the standard form using a suitable change of axes. You also saw that these standard forms are very useful for studying the geometrical properties of the conic concerned. Here we shall show that in the case of the three-dimensional system also we can transform an equation F (x, y, z) = 0 into the corresponding standard form by means of an appropriate change of coordinate axes. As in the case of the two-dimensional system, the transformations that we apply are of two types: change of origin and change of direction of axes. Let us consider these one by one in the following sub-sections.

1.3.1 Translation of Axes

Here we shall discuss how the coordinates of a point in the three-dimensional system get affected by shifting the origin from one point to another point, without changing the direction of the axes. The procedure is the same as in the two-dimensional case.

Let OX, OY, OZ be the coordinate axes of a three-dimensional system. What happens when we shift the origin from O to another point O′ (see Fig. 1)?

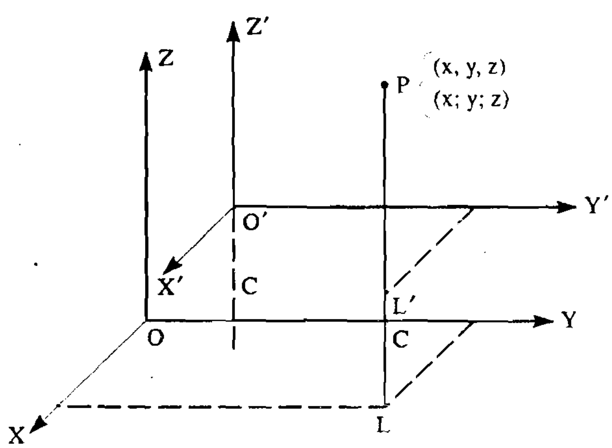


Fig. 1: Translation of axes through O′.

Let the coordinates of O′ in the XYZ-system be (a, b, c). Let O′X′, O′Y′ and O′Z′ be the new axes which are parallel to the OX, OY and OZ axes. Suppose P is a point as given in Fig. 1. Let the coordinates of P in the XYZ-system be (x, y, z) and in the X′Y′Z′-system be (x′, y′, z′). We first find a relationship between z and z′. For this we draw a line through P, which is perpendicular to the XOY plane. Let it cut the planes XOY and X′O′Y′ in L and L′, respectively. Then we have PL = z and PL′ = z′.

From Fig. 1 we have
$PL = PL' + L'L$

Now $L'L$ is also the length of the perpendicular from the point O' to the XOY plane. Therefore $L'L = c$

Thus, we have $z = z' + c$.

Similarly, we get $x = x' + a$ and $y = y' + b$.

Hence if we shift the origin from O (0, 0, 0) to another point O' (a, b, c) without changing the direction of the axes, then the new coordinates of any point P (x, y, z) with respect to the origin O' will be
$x' = x - a$, $y' = y - b$ and $z' = z - c$ ...(4)

So, for example, what will the new coordinates (x', y', z') of a point P (x, y, z) be when we shift the origin to (2, –1, 1)? They will be $x' = x - 2$, $y' = y + 1$ and $z' = z - 1$.

When we transform the axes in such a way, we say we have **shifted the origin** to (2, –1, 1), or **translated the axes through** (2, –1, 1).

Now, what will the effect of such a transformation be on any equation in x, y, z? If, in an equation in the XYZ-system, we replace x, y, z by $x' + a$, $y' + b$, $z' + c$, then we get the new equation in $X'Y'Z'$-system. For example, when we shift the origin to (2, –1, 1), then the equation of the plane $3x + 2y - z = 5$ will be transformed into $3x' + 2y' - z' = 2$.

Note that the respective coefficients of x, y, z and of x', y', z' remain unchanged under a shift in origin. Thus, the direction ratios of the normal to a plane do not change when we shift the origin (Recall the definition of direction ratios from Unit 4). Can you guess why this happens? This is obviously because we are not shifting the direction of the coordinate planes, we are only shifting the origin.

Now, let us consider the effect of translation of axes on a general second degree equation.

Theorem 1 : Let the coordinates of a given surface S in a given coordinate system XYZ satisfy a second degree equation in three variables. Let us shift the origin from O to another ponit O' giving rise to a new coordinate system $X'Y'Z'$. Then S is still represented by a general second degree equation in three variables in the new coordinate system.

In (5) the group of terms having degree 2, namely $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ is called the **second degree** part of the equation and the group of the term $2ux + 2vy + 2wz$ is called the **linear part** of the equation.

Proof : Let the given surface S satisfy the equation
$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$

For convenience we write the equation in the form
$F(x, y, z) = \Sigma ax^2 + \Sigma 2fyz + 2\Sigma ux + d = 0$ ... (5)

Let (p, q, r) denote the coordinates of O' in the XYZ-system. Consider now the new system of coordinate axes $O'X'$, $O'Y'$, $O'Z'$ parallel to the given system with origin O'. You know that the relation between the coordinates in the original and new system are given by

$$\left.\begin{aligned} x &= x' + p \\ y &= y' + q \\ z &= z' + r \end{aligned}\right] \quad \text{... (6)}$$

Substituting these expressions for x, y, z in (5), we get
$\Sigma a(x + p)^2 + \Sigma 2f(y + q)(z + r) + \Sigma 2u(x + p) + d = 0$.

Now, we expand the above expression and simplify by collecting like terms. We get
$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' + 2u'x' + 2v'y' + 2w'z' + d = 0$, ... (7)

where

$$\left.\begin{aligned} u' &= (ap + hq + gr) + u \\ v' &= (hp + bq + fr) + v \\ w' &= (gp + fq + er) + w \\ \text{and } d' &= ap^2 + bq^2 + cr^2 + 2fqr + 2grp + 2hpq + 2up + 2vq + 2wr + d \end{aligned}\right] \quad \ldots (8)$$

Let G (x′, y′, z′) denote the expression in the left hand side of (7). Then we see that any point (x′, y′, z′) belonging to S satisfies the equation G (x′, y′, z′) = 0, which is again a general second degree equation.

Hence we have proved the result.

If you compare (5) and (7), then you will see that the second degree part of the equation remain unchanged whereas the linear part changes. Hence, we can conclude that

> under the transformation of shifting the origin of a coordinate system, the second degree part of a general second degree equation does not change.

Why don't you try some exercises now?

E1) a) What will the new equation of a right circular cone, with vertex O, axis OZ and semi-vertical angle α be, when we shift the origin to (–1, 1, 0)?
b) What does the new equation represent? Sketch the surface.

E2) Obtain the transformed equation of the following equations when the origin is shifted to (1, –3, 2).
a) $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$.
b) $x^2 - 2y^2 - 3z = 0$.

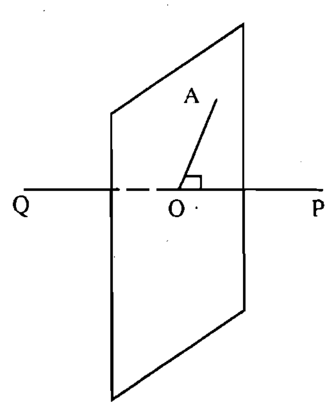
Let us now consider the transformation in which the direction of the axes is changed. For this we need to understand the concept of a projection. So, let us first see what a projection is.

1.3.2 Projection

In this section we shall talk about an important concept in geometry. This concept has even been used by artists through centuries for giving depth to their works of art. Let us define it.

Definition : Let A be a point in the XYZ coordinate system.
The **projection of A** on a line segment PQ is the foot of the perpendicular drawn from the point A to the line.

From Fig. 2 you can see that the projection of A on PQ will be the point O where the plane through A and perpendicular to PQ meets the line PQ.



Definition : The projection of a line segment AB on a line PQ is the segment A′B′ of the line PQ, where A′ and B′ are respectively the projections of the points A and B on the line PQ. (see Fig. 3).

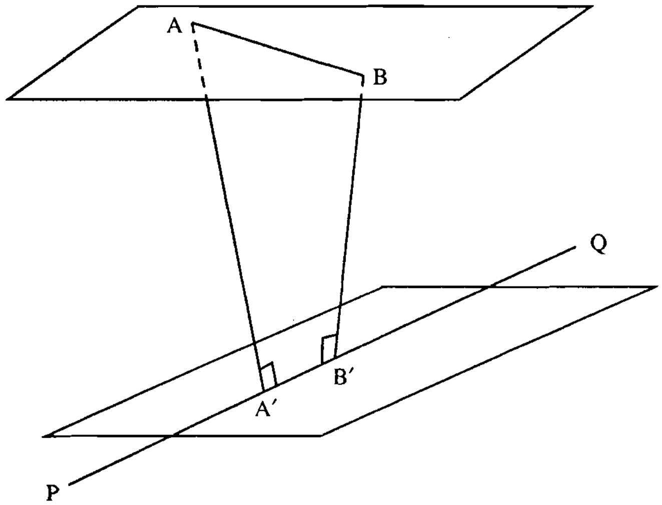


Fig. 3 : The Projection of the line segment AB on the line PQ is the line segment A′B′.

Remark : From Fig. 4 you can see that the length of A′B′ = | AB | cos θ, where θ is the angle between AB and A′B′. We also call this number the projection of AB on PQ.

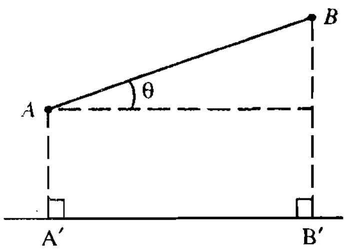


Fig. 4

So what will the projection of BA be? It will be | BA | cos (π + θ) that is – AB cos θ. This shows that the projection can be positive or negative depending upon the direction of the line segment.

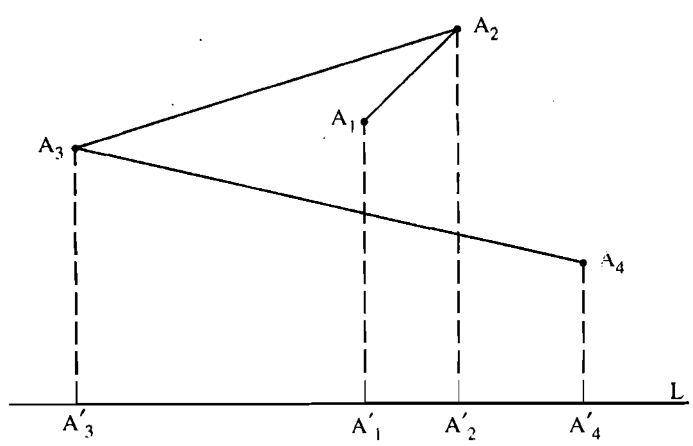
Whenever you come across the term projection, from the context it will be clear whether we are referring to a line segment or a number.

We shall now state a simple result which we shall need while discussing rotation of axes.

Theorem 2 : Suppose $A_1, A_2, \ldots, A_n$ are n points in space. Then the algebraic sum of the projections of $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ on a line is equal to the projection of $A_1 A_n$ on that line.

We will not prove this result in general here; but shall give you a proof in a particular case only. The proof, in any situation, is on the same lines.

Proof : Consider the situation in Fig. 5 concerning 4 points A_1, A_2, A_3, A_4 and their projections A'_1, A'_2, A'_3, A'_4 on a given line L.



Then

$A'_1A'_2 + A'_2A'_3 + A'_3A'_4 = A'_1A'_2 - (A'_1A'_1 + A'_1A'_2) + (A'_3A'_1 + A'_1A'_2 + A'_2A'_4)$
$= A'_1A'_4$

This shows that the sum of the projections of the lien segments A_1A_2, A_2A_3, A_3A_4, is equal to the projection of A_1A_4. So, we have proved the result for the situations in Fig. 5.

Now, consider another useful result involving projections.

Theorem 3 : Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be two points in the XYZ coordinate system. Then the projection of the segment PQ on a line with direction cosines l, m, n is given by
$(x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$.

Proof : In Unit 4, sec. 4.3.2, you have seen that the direction ratios of the line joining P and Q are $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$.

Let | PQ | denote the distance between P and Q, i.e.,

$$| PQ | = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Then the direction cosines of the line segment PQ are

$$\frac{x_2 - x_1}{|PQ|}, \frac{y_2 - y_1}{|PQ|}, \frac{z_2 - z_1}{|PQ|}$$

Let θ be the angle between the lines PQ and L. Then the projection of the line segment PQ on the line L is | PQ | cos θ. But, from Unit 4, Sec. 4.3.3, we have

$$\cos\theta = \frac{x_2 - x_1}{|PQ|} l + \frac{y_2 - y_1}{|PQ|} m + \frac{z_2 - z_1}{|PQ|} n$$

Therefore, the required **projection** $= (x_2 - x_1)l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$

For example, what will the projection of the line segment joining O (0, 0, 0,) to the point P (5, 2, 4) on the line having 2, –3, 6 as its direction ratios be? We know that the direction cosines of the line with direction ratios 2, –3, 6 is.

$$\frac{2}{\sqrt{(2)^2 + (-3)^2 + (6)^2}} \quad \frac{-3}{\sqrt{(2)^2 + (-3)^2 + (6)^2}}$$
$$\frac{6}{\sqrt{(2)^2 + (-3)^2 + (6)^2}}$$

i.e., $\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}$.

Thus, the projection of OP $= 5 \times \frac{2}{7} + 2 \times \left(-\frac{3}{7}\right) + 4 \times \frac{6}{7} = 4.$

Now here is an exercise for you.

E3) Let P(6, 3, 2), Q (5, 1, 4), R (3, –4, 7) and S (0, 2, 5) be four points in space. Find the projection of the line segment PQ on RS.

Now we are in a position to discuss how the coordinates in space are affected by changing the direction of the axes without changing the origin.

1.3.3 Rotation of Axes

Let us now consider the transformation of coordinates when the coordinate system is rotated about the origin through an angle θ. Let the original system be OXYZ.

Suppose we rotate the coordinate axes through an angle θ in the anti-clockwise direction. Let OX′, OY′, OZ′ denote the new coordinate axes (see Fig. 6). Suppose the direction cosines of OX′, OY′ and OZ′ be $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3\ n_3$, respectively.

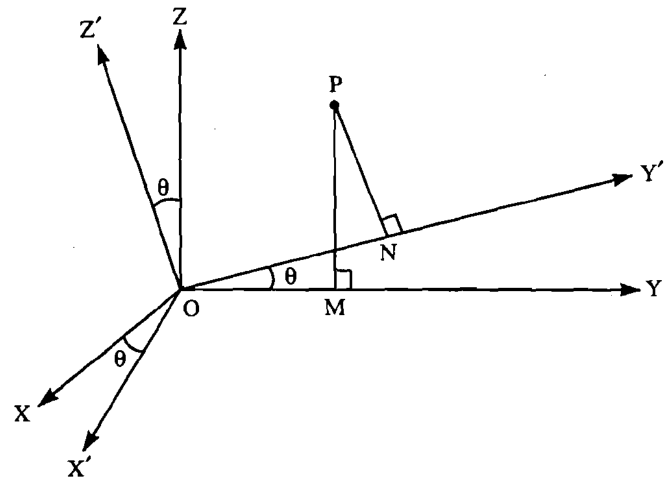


Fig 6: The axes OX′, OY′ and OZ′ are obtained by rotating the axes OX, OY and OZ through an angle θ.

Let, P be any point in space having coordinates (x, y, z) and (x′, y′, z′) with respect to the old and new coordinate systems. Let PN be the perpendicular from P on OY′. Then

$OY = y'$

The line segment ON is also the projection of OP on the line OY′ with direction cosines l_2, m_2, n_2. Therefore, by Theorem 2, we have

$ON = (x - 0)l_2 + (y - 0)m_2 + (z - 0)n_2.$

Hence we get

$y' = xl_2 + ym_2 + zn_2$ ... (9)

Similarly, we can show that

$x' = xl_1 + ym_1 + zn_1$ ... (10)

and

$z' = xl_3 + ym_3 + zn_3$ ... (11)

Therefore, given (x, y, z) and the direction cosines of the new coordinate axes, we can get the new coordinates (x′, y′, z′) using equations (9), (10) and (11).

Now how can we find x, y, z in terms of x′, y′, z′? For this we draw PM perpendicular to OY (see Fig. 6). Then,

$OM = y.$

OM is also equal to the projection of OP on OY. Now let us see what the direction cosines of OY with respect to the new coordinate axes OX′, OY′, OZ′ are. We know that the direction cosines of OX′, OY′, OZ′ with respect to OY are m_1, m_2 and m_3. Do you agree that the direction cosines of OY with respect to OX′, OY′, OZ′ are also m_1, m_2, m_3? (We leave this as an exercise for you to verify.) Therefore, by Theorem 2, we get

$y = OM = (x' - 0)\ m_1 + (y' - 0)m_2 + (z - 0)m_3,$

i.e., $y = m_1x' + m_2y' + m_3z'$ ...(12)

Similarly, we get

$x = l_1x' + l_2y' + l_3z'$ ... (13)

and

$z = n_1x' + n_2y' + n_3z'$ ... (14)

Hence, (12), (13) and (14) give the coordinates of x, y, z in terms of x′, y′ and z′. You may find that these equations are not easy to remember. For easy reference, we arrange the equations in a table, as shown in Table 1.

	x	y	z
x′	l_1	m_1	n_1
y′	l_2	m_2	n_2
z′	l_3	m_3	n_3

Note that for finding x, y z we make use of the elements in the respective columns, and to find x′, y′ and z′ we make use of the elements ini the respective rows.

Let us consider an example.

Example 1 : Find the new equation of the conicoid
$3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy = 1$
when the coordinate system is transformed into a new system with the same origin and with the coordinate axes having direction ratios –1, 0, 1; 1, –1, 1; 1, 2, 1 with respect to the old system.

Solution : The given surface is

$$3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy = 1 \qquad \text{... (15)}$$

Let OX′ Y′ Z′ be the new coordinate system. Then the direction cosines of OX′, OY′ and OZ′ with respect to the original axes are

$$\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \text{ respectively.}$$

We form the transformation table :

	x	y	z
x′	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$
y′	$\frac{1}{\sqrt{3}}$	$\frac{-1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
z′	$\frac{1}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$

From the table we have

$$x = \frac{1}{\sqrt{2}} \times x' + \frac{1}{\sqrt{3}} \times y' + \frac{1}{\sqrt{6}} \times z'$$

$$= -\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}$$

$$\text{and } y = 0 \times x' + \left(\frac{-1}{\sqrt{3}}\right) \times y' + \left(\frac{-2}{\sqrt{6}}\right) \times z'$$

$$= -\frac{y'}{\sqrt{3}} + \frac{2y'}{\sqrt{6}}$$

$$\text{and } z = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}$$

To find the new equation, we substitute the expressions of x, y, z in (15). Then we get

$$3\left(-\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}\right)^2 + 5\left(-\frac{y}{\sqrt{3}} + \frac{2z'}{\sqrt{6}}\right)^2 + 3\left(-\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}\right)^2$$

$$+2\left(-\frac{y'}{\sqrt{3}}+\frac{2z'}{\sqrt{6}}\right)\left(\frac{x'}{\sqrt{2}}+\frac{y'}{\sqrt{3}}+\frac{z'}{\sqrt{6}}\right)+\left(\frac{x'}{\sqrt{2}}+\frac{y'}{\sqrt{3}}+\frac{z'}{\sqrt{6}}\right)$$

$$\left(\frac{x'}{\sqrt{2}}+\frac{y'}{\sqrt{3}}+\frac{z'}{\sqrt{6}}\right)+2\left(-\frac{x'}{\sqrt{2}}+\frac{y'}{\sqrt{3}}+\frac{z'}{\sqrt{6}}\right)\left(-\frac{y}{\sqrt{3}}+\frac{2z'}{\sqrt{6}}\right)=1$$

Simplifying each term of the above expression and collecting the coefficients of like terms, we get
$2x'^2 + 3y'^2 + 6z'^2 = 1.$
This is the new equation of the conicoid.

You can do the following excercises on the same lines.

E4) Find the new equation of the following conicoids when the coordinates system is changed into a new system with the same origin and direction ratios 1, 2, 3; 1, −2, 1; 4, 1, −2; with respect to the old system.
a) $x^2 - 5y^2 + z^2 = 1.$
b) $x^2 + y^2 + z^2 - 2xy - 2yz + 2zx + x - y + z = 0$

E5) For the conicoid in Example 1 and E4, calculate the sum of coefficients of the square terms in the original equation and in the new equation. Can you infer anything from the outcome?

E6) What will the new equation of the plane $x + y + z = 0$ be when the coordinate system XYZ is transformed into another coordinate system X′Y′Z′ by the following equations?

$$x=\frac{x'}{\sqrt{6}}+\frac{y'}{\sqrt{2}}+\frac{z'}{\sqrt{3}}$$

$$y=-\frac{2}{\sqrt{6}}x'+\frac{z'}{\sqrt{3}}$$

$$z=\frac{x'}{\sqrt{6}}-\frac{y'}{\sqrt{2}}+\frac{z'}{\sqrt{3}}$$

E7) Does a cone remain a cone under rotation of axes? Give reasons for your answer.

Now let us consider the effect of rotation of axes on $F(x, y, z) = 0$.

Theorem 4 : Let S be a conicoid satisfying a second degree equation in a coordinate system XYZ is transformed into another coordinate system X′Y′Z′ by the direction of axes, without changing the origin, S will still be represented by a second degree equation.

Proof : Suppose S is represented by the second degree equation
$\Sigma ax^2 + 2\Sigma fyz + \Sigma 2ux + d = 0$ in the XYZ-systems.

Let $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ be the direction cosines of the new coordinate axes OX′, OY′, OZ′ respectively. Then you know that the coordinates (x, y, z), (x′, y′, z′) of a point in the old and new system respectively, satisfy the following relationship.

$$x = l_1x' + l_2y' + l_3z'$$
$$y = m_1x' + m_2y' + m_3z'$$
$$z = n_1x' + n_2y' + n_3z'$$

Let us substitute these expressions in the given equation. We consider the second degree parts and linear parts of the equation separately.

The second degree part is $\Sigma ax^2 + 2\Sigma fyz$. When we substitute the expressions for x, y, z in this part, we get

$\{\Sigma a\,(l_1x' + l_2y' + l_3z')^l + 2f\,(m_1x' + m_2y' + m_3z')\,(n_1x' + n_2y' + n_3z)\}$.

The coefficient of x'^2 in the above expression is
$(al_1^2 + bm_1^2 + cn_1^2 + 2fm_2n_2 + 2gn_1l_1 + 2hl_1m_1)$,

Similarly the coefficient of y'^2 in the above expression is
$(al_2^2 + bm_2^2 + cn_2^2 + 2fm_2n_2 + 2gn_2l_2 + 2hl_2m_2)$,
and that of z'^2 is
$(al_3^2 + bm_3^2 + cn_3^2 + 2fm_3n_3 + 2gh_3l_3 + 2hl_3m_3)$.

Similarly we collect the coefficients of $y'z'$, $z'x'$ and $x'y'$. Then we get an expression of the form

$$a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y' \qquad \dots (16)$$

You can also see that the linear part becomes

$$u'x + v'y + w'z \qquad \dots (17)$$

Where
$u' = ul_1 + vm_1 + wn_1$
$v' = ul_2 + vm_2 + wn_2$, and
$w' = ul_3 + vm_3 + wn_3$

From (16) and (17) we see that the transformed equation is a general second degree equation.

Now looking at expression (17), can you say anything about the change in the constant term? It remains unchanged under rotation of axes. Another interesting fact that you may have observed in the proof above is given in the following exercise.

E8) Suppose that the second degree part $\Sigma ax^2 + 2\Sigma fyz$ of a general second degree equation transforms into $\Sigma a'x'^2 + 2\Sigma f'y'z'$ under rotation of axes. Show that $a + b + c = a' + b' + c'$.

Well, let us see what we can gather from Theorems 1 and 4. They say that if a conicoid S is represented by a second degree equation $F(x,y, z) = 0$ in one coordinate system, then it is still represented by a second degree equation in any other coordinate system obtained by a translation or rotation of axes.
Thus

a conicoid remains a conicoid under translation or rotation of axes.

In fact, every geometrical figure remains unchanged in shape and size under translation or rotation of axes. Therefore these transformations are called **rigid body motions.**

In this section we have discussed two important transformations of a three-dimensional coordinate system. We also said that the importance of these transformations lies in the fact that we can use them to reduce any general second degree equation in variables into a simple form. Let us see how this happens.

1.4 REDUCTION TO STANDARD FORM

In this section we shall show that by suitably applying the transformations that we have discussed in the previous section, we can write the general equation of a conicoid in a simpler form.

Let us consider a conicoid given by the equation
$F(x, y, z) = \Sigma ax^2 + 2\Sigma fyz + \Sigma 2ux + d = 0$

Let us assume that there exists a new Cartesian coordinate system, obtained by translating the origin, in which the linear part of $F(x, y, z) = 0$ vanishes. You will see that this is possible only for a particular type of conicoids.

Let the coordinates of O′ be (x_0, y_0, z_0) in the new system. Then we know that in the transformed equation, the second degree part is unchanged and the linear part becomes.

$u'x' + v'y' + w'z'$
where u′, v′ and w′ are as in (8). We have assumed that the linear part vanishes.

Therefore $u' = v' = w' = 0$. This means that we should have
$ax_0 + hy_0 + gz_0 + u = 0$
$hx_0 + by_0 + fz_0 + v = 0$
$gx_0 + fy_0 + cz_0 + w = 0$

In other words, (x_0, y_0, z_0) is a solution of the system of equations

$$\left.\begin{array}{l} ax + hy + gz + u = 0 \\ hx + by + fz + v = 0 \\ gz + fy + cz + w = 0 \end{array}\right] \qquad \ldots (18)$$

If the sysem of equations (18) has a solution for $(x_0, y_0, z_0) \in \mathbf{R}^3$, then the point, (x_0, y_0, z_0), is called a centre of the given conicoid and we say that the conicoid has a centre at (x_0, y_0, z_0). You will understand why this is called a centre later in Unit 8.

Now let us assume that the given conicoid S has a centre. We transfer the origin to the centre (x_0, y_0, z_0). Then the transformed equation becomes
$\Sigma ax^2 + 2\Sigma fyz + 2u'x + 2v'y + 2w'z + d' = 0$.
(Recall that the second degree part does not change by shifting the origin.)

Since (x_0, y_0, z_0) is a solution to (18), we see that $u' = v' = w' = 0$. Therefore, the above equation reduces to
$(\Sigma ax^2 + 2\Sigma fyz) + d' = 0$.
This equation does not have any linear part.

We have just proved the following result.

Theorem 5 : Suppose that S is a conicoid which is represented by a general second degree equation $F(x, y, z) = 0$ in a coordinate system XYZ. Suppose that S has a centre O′ (i.e., the system of equations (18) has a solution (x_0, y_0, z_0). Then by shifting the origin to the centre O′, the equation assumes the form
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d' = 0$
in the new coordinate sysem X′Y′Z′.

See Block 6, CS-60 for simultaneous linear equations

Let us consider an example.

Example 2 : Consider the conicoid given by the equation
$2x^2 + 3y^2 + 4z^2 - 4x - 12y - 24z + 49 = 0$.
Does it have a centre? If so, find it.

Solution : The given equation is
$F(x, y, z) = 2x^2 + 3y^2 + 4z^2 - 4x - 12y - 24z + 49 = 0$

We shall first check whether the given equation has a centre or not, that is, if the system of equations (18) has a solution or not. Here $a = 2$, $b = 3$, $c = 4$, $u = -2$, $v = -6$, $w = -12$. Then we have
$2x - 2 = 0$
$3y - 6 = 0$
$4z - 12 = 0$

This set of equations has a solution, namely, (1, 2, 3). Hence, (1, 2, 3) is a centre of S.

Note : Let us go back to (18) for a moment. There we saw that if the system of equations has a solution (x_0, y_0, z_0), then (x_0, y_0, z_0) is a centre of the conic. In Unit 5 of the course 'Elementary Algebra', you have seen that the system of equations has a solution if

$$\Delta = \begin{vmatrix} a & h & b \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$$

In fact if $\Delta \neq 0$, then there exists a unique solution.

Why don't you try some exercise now?

E9) Check whether the following conicoids have a centre or not
a) $3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy + 4x - 12y + 4z + 1 = 0$
b) $x^2 + y^2 + z^2 - 2yz + 2zx - 2xy - 2x + 2y - 2z - 3 = 0.$
c) $5x^2 + 6y^2 - 2x = 0$

E10) Find a centre of the conicoid
$14x^2 + 14y^2 + 8z^2 - 4yz - 4zx - 8xy + 18x - 18y + 5 = 0$. What will its new equation be if the origin is shifted to this centre?

We shall now state a theorem without proving it. To prove this we need some advanced techniques in calculus which are beyond the scope of this course.

Theorem 6 : Suppose S is a given conicoid whose equation is given by $F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$
with respect to a given system of coordinates XYZ. Then there exists a new Cartesian coordinate system X'Y'Z' obtained by rotating the axes of the given system XYZ, without shifting the origin, such that the representation of S in the new system takes the form
$G(x', y', z') = a'x'^2 + b'y'^2 + c'z'^2 + 2u'x' + 2v'y' + 2w'z' + d' = 0.$

That is, the new equation does not contain the product terms, yz, zx and xy.

Combining Theorems 5 and 6 we have the following result.

Corollary 1 : Let S be a conicoid given by the equation $F(x, y, z) = 0$, which has a centre O' in a coordinate system XYZ. There exists a new coordinate system obtained by shifting the origin from O to O' and then rotating the system about O', in which the equation takes the simpler form
$$a'x'^2 + b'y'^2 + c'z'^2 + d' = 0 \qquad \ldots (19)$$

If $d' \neq 0$, then we can divide throughout by d', and we get the form
$ax^2 + by^2 + cz^2 = 1,$

where $a = -\dfrac{a'}{d'}$, $b = -\dfrac{b'}{d'}$ and $c = -\dfrac{c'}{d'}$.

(19) is called the standard equation of a conicoid.

Recall that in the case of the two-dimensional system also we have seen that we can reduce any second degree equation into a simple form.

Let us now consider some examples.

Example 3 : Show that the conicoid given by $x^2 + 2yz - 4x + 6y + 2z = 0$ has a centre. Reduce it to standard form by shifting the origin to the centre, and then rotating the axes to get a new system in which the direction ratios of the new axes are given by

$$0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}; 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}; 1, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$$

with respect to the original coordinate system.

Solution : Here a = 1, b = 0, c = 0, f = 1, g = 0, h = 0, u = –2, v = 3, w = 2.

We first check whether the conicoid has a centre or not. Using (18) we see that (2, 0, 0) is a centre of the given conicoid.

Shifting the origin from (0, 0, 0) to (2, 0, 0), we get the new equation as
$x^2 + 2yz - 4 = 0$

Now, we apply a rotation of axes to the new equation. We note that the direction cosines of the new axes are

$$0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}; \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{4}}, -\frac{1}{\sqrt{4}}$$

Form Sec. 7,3,3 we have

$$x = -\frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{2}}$$

$$y = -\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{4}}y' - \frac{1}{\sqrt{4}}z'$$

$$z = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{4}}y' - \frac{1}{\sqrt{4}}z'$$

Substituting these equations in the given equation of the conicoid, we see that

$$\left(\frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{2}}\right)^2 + 2\left(-\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{4}}y' - \frac{1}{\sqrt{4}}z'\right)\left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{4}}y' - \frac{1}{\sqrt{4}}z'\right) - 4 = 0.$$

i.e., $\frac{y'^2}{2} + \frac{z'^2}{2} - x'^2 - 4 = 0.$

i.e., $y'^2 + z'^2 - 2x'^2 = 8.$

which is in the standard form.

Now here is an exercise for you.

E11) Find the standard equation of the following conicoids.

a) $x^2 + y^2 + z^2 - 2x - 2y - 2z - 1 = 0$, by shifting the origin to the centre.

b) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy + 2x + 12y + 10z + 20 = 0$, by shifting the origin to the centre and then rotating the system so that the direction ratios of the new, axes are
–1, 0, 1; 1, 1, 1; 1, –2, 1.

We will stop our discussion on general theory of conicoids for now, though we shall refer to them off and on the following units. In the next unit we will discuss the surfaces formed by (19).

Let us now do a quick review of what we have covered in this unit.

1.5 SUMMARY

In this unit we have covered the following points :

1) A general second degree equation in three variables
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$
represents a conicoid.

2) Translation of axes: the transformation of a coordinate system in which the

origin is shifted to another point without changing the direction of the axes. The equation of transformations are given by
$x = x' + a$
$y = y' + b$
$z = z' + c$

3) Rotation of axes: the transformation of a coordinate system in which the direction of axes is changed without shifting the origin. The equations of transformation are given by the following table:

	x	y	z
x' y' z'	l_1 l_2 l_3	m_1 m_2 m_3	n_1 n_2 n_3

where l_i, m_i, n_i, i = 1, 2, 3 are the direction cosines of the axes.

4) A conicoid remains a conicoid under a translation or rotation of axes.

5) There is a Cartesian coordinate system in which the equation of a conicoid with a centre takes the standard form
$a'x'^2 + b'y'^2 + c'z'^2 + d' = 0$.

And now you may like to check whether you have achieved the **objectives** of this unit (see Sec. 7.1). If you would like to see our solutions to the exercises in this unit, we have given them in the following section.

1.6 SOLUTIONS/ANSWERS

E1) a) In unit 6 of Block 2 you have seen that the equation of a right circular cone with vertex O, axis OZ and semi-vertical angle α is
$x^2 + y^2 = z^2 \tan^2 \alpha$.
When we shift the origin to (–1, 1, 0), then the coordinates in the new system are given by
$x' = x + 1,\quad y' = y - 1,\quad z' = z,$
i.e., $x = x' - 1\quad y = y' + 1,\quad z = z'$
Substituting for x, y, z in the given equation of the cone, we get
$(x' - 1)^2 + (y' + 1)^2 = z'^2 \tan^2 \alpha$.

b) This equation represents a right circular cone with vertex at the point (–1, 1, 0) axis along the line parallel to the z-axis through the vertex and semi-vertical angle α (see E1 and Sec 6.2 of Unit 6, Block 8).

E2) The equations of transformation are given by
$x = x' + 1,\ y = y' - 3,\ z = z' + 2$

a) the given equation is
$x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$
Substituting for x, y, z, in the above equation, we get
$(x' + 1)^2 + (y' - 3)^2 + (z' + 2)^2 - 4(x' + 1) + 6(y' - 3) - 2(z' + 2) + 5 = 0$
Simplifying, we get
$x'^2 + y'^2 + z'^2 - 2x' + 2z' - 7 = 0$
which represents a sphere

b) The transformed equation is
$x'^2 - 2y'^2 + 2x' - 4y' - 3z' - 7 = 0$.

E3) The direction ratios of RS are –3, 6, –2.

Therefore, the direction cosines of RS are $-\frac{3}{7}, \frac{6}{7}, -\frac{2}{7}$.

Hence the projection of PQ on RS is

$$(5-6)\times\left(-\frac{3}{7}\right)+(1-3)\times\frac{6}{7}+(4-2)\times\left(-\frac{2}{7}\right)=\frac{3}{7}-\frac{12}{7}-\frac{4}{7}=-\frac{13}{7}.$$

E4) a) The given equation is
$x^2 - 5y^2 + z^2 = 1$
The direction ratios of the axes are given by
1, 2, 3; 1, –2, 1; 4, 1, –2.
Therefore, the direction cosines of the axes are given by

$$\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}};\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}};\frac{4}{\sqrt{21}},\frac{1}{21},-\frac{2}{\sqrt{21}}$$

The transformation Table is

	x	y	z
x'	$\frac{1}{\sqrt{14}}$	$\frac{2}{\sqrt{14}}$	$\frac{3}{\sqrt{14}}$
y'	$\frac{1}{\sqrt{6}}$	$\frac{-2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$
z'	$\frac{4}{\sqrt{21}}$	$\frac{1}{\sqrt{21}}$	$\frac{-2}{\sqrt{21}}$

Then we have

$$x=\frac{1}{\sqrt{14}}x'+\frac{1}{\sqrt{6}}y'+\frac{4}{\sqrt{21}}z'$$

$$y=\frac{2}{\sqrt{14}}x'-\frac{2}{\sqrt{6}}y'-\frac{1}{\sqrt{21}}z'$$

$$z=\frac{3}{\sqrt{14}}x'+\frac{1}{\sqrt{6}}y'-\frac{2}{\sqrt{21}}z'$$

Substituting this in the given equation, we get

$$\left(\frac{1}{\sqrt{14}}x'+\frac{1}{\sqrt{6}}y'+\frac{4}{\sqrt{21}}z'\right)^2-5\left(\frac{2}{\sqrt{14}}x'-\frac{2}{\sqrt{6}}y'-\frac{1}{\sqrt{21}}z'\right)$$

$$+\left(\frac{3}{\sqrt{14}}x'+\frac{1}{\sqrt{6}}y'-\frac{2}{\sqrt{21}}z'\right)^2=1$$

Simplifying we get

$$\frac{10}{14}x'^2-3y'^2+\frac{15}{21}z'^2+\frac{48}{\sqrt{14}\sqrt{6}}x'y'+\frac{24}{\sqrt{6}\sqrt{21}}y'z'-\frac{24}{\sqrt{21}\sqrt{14}}x'z'=1$$

b) The new equation is $x'^2 + y'^2 + z'^2 + \frac{8}{\sqrt{6}\sqrt{21}}y'z' - \frac{8}{\sqrt{21}\sqrt{14}}x'z' = 1.$

E5) From E4 (a) we get
$a + b + c = -5 + 2 = -3$
and $a' + b' + c' = \frac{10}{4}-\frac{18}{6}+\frac{15}{11}=-3$

This shows that under rotation the sum of coefficients of the square terms remains unaltered in value, i.e., $a + b + c = a' + b' + c'$.
Similarly, you can observe the same for E4 (b) and Example 1 also.

E6) From the given equations of transformations we see that the coordinate system is changed into another coordinate system with the same origin and the direction cosines of the new axes, with respect to the old system, are given by

$$\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}; \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

To get the new equation, we substitute the values of x, y, z in the equation $x + y + z = 0$. Then we get

$$\frac{x'}{\sqrt{6}} + \frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{3}} - \frac{2}{\sqrt{6}} + \frac{z}{\sqrt{3}} + \frac{x'}{\sqrt{6}} - \frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{3}} = 0.$$

Therefore, under the transformation the plane $x + y + z = 0$ becomes the plane $z' = 0$.

E7) Yes. This is because with vertex at the origin is represented by a homogeneous second degree equation.
$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$.
Now, from the proof of Theorem 2, we see that under rotation of axes, the above equation becomes
$a'x'^2 + b'y'^2 + c'z'^2 + 2h'x'y' + 2f'y'z' + 2g'z'x' = 0$,
which is again a homogeneous second degree equation.
Therefore, it repreents a cone.

E8) From the proof of theorem 2, we have
$a' = al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gn_1l_1 \ x + 2hl_1m_1$
$b' = al_2^2 + bm_2^2 + cn_2^2 + 2fm_2n_2 + 2gn_2l_2 \ x + 2hl_2m_2$
$c' = al_3^2 + bm_3^2 + cn_3^2 + 2fm_3n_3 + 2gn_3l_3 \ x + 2hl_3m_3$
then $a' + b' + c' = a\,(l_1^2 + l_2^2 + l_3^2) + b\,(m_1^2 + m_2^2 + m_3^2)$
$+ c\,(n_1^2 + n_2^2 + n_3^2) + 2f\,(m_1 n_1 + m_2 n_2 + m_3 n_3) + 2g(n_1l_1 + n_2l_2 + n_3l_3)$
$+ 2h\,(l_1m_1 + l_2m_2 + l_3m_3)$.

We know that

$$\sum_{i=1}^{3} l_i^2 = 1 = \sum_{i=1}^{3} m_i^2 = \sum_{i=1}^{3} n_i^2$$

(see Unit 4, Block 8).

Also, since the axes are mutually perpendicular, by condition for perpendicularity given in Unit 4, Block, 8, we get

$$\sum_{i=1}^{3} m_i n_i = 0 = \sum_{i=1}^{3} n_i l_i = \sum_{i=1}^{3} l_i m_i$$

Therefore, we get $a' + b' + c' = a + b + c$

E9) a) We have to see whether the following system of linear equations is consistent or not.
$ax + hy + gz + u = 0$
$hx + by + fz + v = 0$
$gx + fy + cz + w = 0$
Here $a = 3$, $b = 7$, $c = 3$, $f = 5$, $g = -1$, $h = 5$, $u = 2$, $v = -6$, $w = -2$. So we have
$3x + 5y - z + 2 = 0$
$5x + 7y + 5z - 6 = 0$
$-x + 5y + 3z - 2 = 0$

On solving this system of equations, we find that it has a unique solution given by $x = \frac{1}{3}$, $y = -\frac{1}{3}$ and $z = \frac{4}{3}$. Therefore, the given conicoid has a unique centre at $\left(\frac{1}{3}, -\frac{1}{3}, \frac{4}{3}\right)$.

b) This conicoid has infinitely many centres.

c) This conicoid has no centre since the system of equations (18) is inconsistent in this case.

E10) The conicoid has a centre at $\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$. Now we shift the origin from (0, 0, 0) to $\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$. The equations of transformation are

$$x = x' - \frac{1}{2},\ y = y' + \frac{1}{2},\ z = z'$$

Substituting for x, y, z in the given equation, we get the new equation as $14x^2 + 14y^2 + 8z^2 - 4yz - 4zx - 8xy = 4$.

E11) a) We know that the given equation represents a sphere with centre at (1, 1, 1) (see Unit 5). Then by shifting the origin to the centre we get the standard equation as
$x'^2 + y'^2 + z'^2 = 4$.

b) The given equation is
$3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy + 2x + 12y + 10z + 20 = 0$.
We first check whether the conicoid represented by this equation has a centre.

Here a = 3, b = 5, c = 3, h = –1, g = 1, f = –1, u = 1, v = 0, w = 5 and d = 20. The system of equations for transformations are given by
$3x - y + z + 1 = 0$
$-x + 5y - z + 6 = 0$
$x - y + 3z + 5 = 0$
Solving these equations, we get $x = -\frac{1}{6},\ y = -\frac{5}{3}$ and $z = -\frac{13}{6}$. Hence a centre is $\left(-\frac{1}{6}, -\frac{5}{3}, -\frac{13}{6}\right)$.

Now we shift the origiin to the centre. Then we get the new equation as
$3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy + d' = 0$,
Where

$$\text{Where } d' = 3\left(-\frac{1}{6}\right)^2 + 5\left(-\frac{5}{3}\right)^2 + 3\left(-\frac{13}{6}\right)^2 - 2\left(-\frac{5}{3}\right)\left(-\frac{13}{6}\right) + 2\left(-\frac{13}{6}\right)\left(-\frac{1}{6}\right)$$
$$-2\left(-\frac{1}{6}\right)\left(-\frac{5}{3}\right) + 2\left(-\frac{1}{6}\right) + 12\left(-\frac{5}{3}\right) + 10\left(-\frac{13}{6}\right) + 20.$$

Now we apply the rotation of axes. The equations of transformation are
$x = -x' + y' + z'$
$y = y' - 2z'$
$z = x' + y' + z'$

Substituting for x, y, z in the given equation of the conicoid, we get
$3(-x' + y' + z')^2 + 5(y' - 2z')^2 + 3(x' + y' + z')^2 - 2(y' - 2z')(x' + y' + z') + 2(x' + y' + z')(-x' + y' + z') - 2(-x' + y' + z')(y' - 2z') + d' = 0$
i.e., $4x'^2 + 9y'^2 + 36z'^2 + d' = 0$
This is the standard form of the given conicoid.