UNIT 4 CENTRAL METHODS

Structure

4.0 INTRODUCTION

We have seen in the previous Units that problem solving is really the act of discovery. How does discovery come about? What is human creativity? Whence comes it? These are deep questions for which few answers are available; fortunately however some of the greatest minds of our times have examined the processes by which their own discoveries were made, and their recollections are included in the pages that follow. It transpires from these accounts that possibly two of the most important ingredients of the creative process are curiosity and hard work (without which of course nothing worthwhile is accomplished).

Much of mathematics and physics has had its genesis in plain inquisitiveness. Fermat was curious to know if there were any integers n greater than 2 such that $x^n + y^n = z^n$ where x, y and z are integral. He thought he had a proof (too long to be written in the margin of his copy of Diophantus' Arithmetica) that there were none, and the subsequent search for it led to the opening of new branches of mathematics. But unless curiosity engenders the right kinds of question it may not be of value, and the kind of curiosity that asks deep questions is a gift given only to a chosen few.

In the theory of equations mathematicians had found that there were general formulae for solving equations in which the unknown occurred to a power no greater than 4, and they did not doubt that formulae would be found also for equations of higher degree. But the young French genius Evariste Galois asked, Is it possible for such generalised formulae to exist? He approached the problem by studying general properties of the equations and their solutions, and created the subject of group theory, a subject that has important ramifications for quantum mechanics, solid state physics and particle physics. And further, Galois' ideas linked algebra to geometry, and helped settle three famous problems of classical times: squaring the circle, duplication of the cube, and the trisection of an angle using ruler and compasses.

For another example of the importance of asking the right question, consider the familiar Newtonian concepts of mass: there are two, which are quite distinct. One is the *inertial mass* of a body, which is a measure of the resistance it offers to a force applied to move it - the larger the inertial mass, declared Newton in his second law of motion, the larger the force one would need in order to move it with a given acceleration. And the second is *gravitational* mass, which for any body determines how strong a gravitational field it produces in its vicinity - the larger the gravitational mass, the stronger its gravity. Newton knew that the gravitational mass of a body is equal to its inertial mass; but why should they be equal, why should they have the same value for all objects in the universe? This was Einstein's question: and to answer it he built in 1915 the grand edifice of General Relativity.

Getting the right question is of course half the battle: but one does not then proceed like a lawyer or a detective, deducing conclusions from premises one after another: rather, one experiments endlessly, and follows leads that experience might suggest but which may often be false; and there is the risk that one may have achieved nothing after years of labour. As the molecular biologist Mark Ptashne said about his search for the repressor protein (presumed to exist from the work of the French biologists Jacques Monod and Francois Jacob), ". ..... how hard are you willing to work, are you willing to work with the possibility that you'll have nothing at all to show for it? You may work for two or three years, simply fail and look like a fool. If not a fool, at least empty-handed" But if one has worked hard, and is lucky, then she or he may be inspired by a sudden happy idea that seems to come from nowhere. Yet this much is certain: the brilliant idea or the burst of inspiration - the cry of "Eureka" from Archimedes when he discovered the principle of floating bodies - follows only after immense labour. When Newton was asked how he had discovered the law of gravity he replied, " By thinking on it continually."

Here is a problem that you will surely solve using the Newtonian formula (thinking on it continually):

When 4444^{7777} is expanded and written in a sequence of decimal digits, let the sum of these digits be A. Let the sum of the digits of A be B. Find the sum of digits of B.

4.1. OBJECTIVES

After reading this Unit you will learn:

how experimentation with an idea or a hypothesis may lead to a discovery
the method of induction for the confirmation of hypotheses
how ideas in one branch of knowledge may play a useful role in solving
problems in other areas.

4.2 EXPERIMENTATION

Experiment 1 : Consider the problem of finding the sum of the first n cubes:

$S_n = 1^3 + 2^3 + 3^4 + 4^3 + ... + n^3$

We notice that if $n = 1$, then $S_1 = 1$; if $n = 2$, $S_2 = 9$; if n = 3, $S_3 = 36$; and if $n = 4$, $S_4 = 100$.This is exciting! Is the sum of the first n cubes always a perfect square ? We test for a few more values, and tabulate our results:

$S_1 = 1 = 1^2$

$S_2 = 9 = 3^2$ $\quad$ $S_5 = 225 = 15^2$

$S_3 = 36 = 6^2$ $\quad$ $S_6 = 441 = 21^2$

$S_4 = 100 = 10^2$ $\quad$ $S_7 = 784 = 28^2$

Wait Do we not find a pattern in the squares? When n increases from 1 to 2 me change from S_1 to S_2 is from 1^2 to 3^2. When n changes from 2 to 3 the change from S_2 to S_3, is that the number that is squared is larger by 3 from the preceding; in going from S_3 to S_4 it's larger by 4, and in going from S_4 to S_5 it's larger by 5. Then what is S_8 Surely $(28 + 8)^2$ or 36^2? Experiment! Is it? And S_9? Is it 45^2? (Verify.) There is a pattern here, certainly. Can we express S_n as a formula in terms of n ? A formula which is a perfect square expression in terms of n? Pause and think.

Experiment 2: Here is a theorem that Euler proved: Every positive whole number is the sum of not more than four squares. This is really a remarkable theorem, because the gaps between the squares increase rather rapidly: 1, 4, 9, 16, 25, 36, 49... Yet no more than four are required to be summed to produce any number! Experiment: Write down the squares of the first ten integers. How many numbers can You make using four or fewer of these squares? Which is the smallest number that requires squares different from those on your list?

Experiment 3: Recall Goldbach's conjecture, that every even number after 4 can be written as the sum of two odd primes. Prepare a list of the first 100 primes, and now answer the following question: In how many *different* ways can even numbers be expressed as the sum of two primes from your list? For example: 16 = 3 + 13 = 5 + 11, yielding 2 partitions; 60 = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 29 + 31 (6 partitions). Can you conjecture that as an even number N increases, the number of its possible partitions into primes also increases? Is there a pattern in the numbers of partitions, that is are there some N for which the number of partitions is larger dm for the neighbours of N? Do even factors of 15 have more partitions than their neighbours?

Experiment 4: A mathematician remarked once, " When primes are written in binary notation (i.e., in base 2, where the only digits are 0 and 1), one cannot expect them to have, asymptotically (i.e., in the large), the same number of 1s and 0's in their development, since the numbers divisible by 3 have an even number of 1s." Is it true that all numbers divisible by 3 have an even number of 1s in their binary representation?

Experiment. 5: Except for 5, all primes must end in one or other of the digits 1, 3, 7 or 9. Use your list of 100 primes (see Experiment 3 above) to find if some last digits occur more frequently than others. Are any conclusions possible?

Experiment 6: Lattice points are points that have integer co-ordinates. Count (by devising an accurate manual method, or a computer program) the numbers of lattice points within and on the boundaries of several circles. Clearly, these numbers should increase as the areas of the circles increase. Therefore, if we divide the number of lattice points within a circle of radius R by R^2 we should obtain an approximation to *pi*. And intuitively, this approximation should improve as the radius increases. Check if the approximation to *pi* improves when R increases from 100 to 200. (In doing this experiment you will be following in the footsteps of the great Gauss: his values, for R= 10, 20, 30, and R = 100, 200, 300, may be found in D. Hilbert and S. Cohn Vossen, *Geometry and the Imagination*., Chelsea, New York, 1952.)

Experiment 1 Again: *Read this section only after you have tacklde Experiment 1.*

Check that the sum S_n, of the first *n* cubes obeys the formula :

$S_n = (1/2\ n\ (n + 1))^2$

We prove this formula by the method of induction as follows. The formula is certainly true for n = 1 called the base case, since

$1^3 = (1/2\ 1\ (1 + 1)^2$

We now establish the inductive case, namely that if the formula is true for *n = k*, then it is true also for *n = k + 1*:

Assume that for some fixed *k*,

$$S_k = (1/2\ k\ (k + 1))^2$$

Then

$$
\begin{aligned}
S_{k+1}. \quad &= (1/2\, k\, (k+1)^2 + (k+1)^3 \\
&= (k+1)^2\, (k+1+(1/2k))^2 \\
&= (k+1)^2\, (1/2(k+2))^2 \\
&= (1/2(k+1)(k+2))^2
\end{aligned}
$$

Thus, if the formula is true for some *k* then it is true for *k + 1*. In particular, if the formula is true for *k = 1*, it is necessarily true for *k = 2*. But we know by direct testing that the formula does hold for *k = 1*. Therefore it holds for *k = 2*. Therefore it holds for *k =3*. Therefore for *k = 4*. And similarly for any *k*. Such is the *method of induction.*

Check Your Progress 1

1. For all n, prove that 2^n > = 2n.
2. For all n prove that 2^n >= 3n - 2.
3. For all n >= 4 prove that 2^n > =4n.
4. Prove that for all n, 2^n >= n^2 - 1.
5. Prove that for all n > = 4, 2^n > n^2.
6. Prove that for all n > =10, 2^n > =100n.
7. Prove that the sum of the squares of -the first *n* integers is 1/24 (2n) (2n + 1)(2n + 2).
8. Prove that the sum of the fourth powers of the first n integers is 1/30 n (n + 1)(2n + 1)($3n^2$ + 3n - 1),

4.3 FIVE SAILORS, A MONKEY AND MANY COCONUTS

Five sailors and a monkey were once stranded on a desert island. The seamen had nothing very much to do, so they whiled away their time gathering coconuts; by the time they had made a reasonably-sized pile it became dark, so they decided to defer dividing the fruit between themselves for the following day. However, one of the sailors got up in the middle of the night and thought it prudent to take his share while the others were still asleep. He found that he could divide the number of coconuts evenly by five, with but one left over, which he threw at the monkey. Then he hid a fifth of the pile for himself, and went back to sleep. A little later another man woke up, and it occurred to *him* too that it might be a good thing to take his share of the coconuts before the rest were up. And coincidentally he also found that dividing the number of coconuts in the pile by five left one coconut over, which, quite considerately, he tossed to the monkey, quite as his predecessor had done. Then *he* dug a hole in the ground, hid his share of the coconuts, and went back to sleep. The same thing happened with the other three seamen. Each woke up, found the others asleep, and decided there was no point in waiting for the morrow to do something that had to be done anyway; and every sailor found that on taking his fifth from the pile that was left, one coconut remained. True to the adage that great men think alike, each threw it to the obliging monkey before going back to sleep again.

When they woke up the next morning they naturally discovered that the pile of coconuts was much smaller than they had left the night before. They all knew what had happened, but each was guilty of the pilferage, so none complained... They decided to share the remaining fruit, and found that on dividing them into five heaps, one was again left over. Following their earlier precedent, they hurled it at the waiting monkey. The question is: what is the smallest number of coconuts that satisfies the conditions of the problem?

This problem - somewhat more complicated than the similar question introduced in Unit 2 - has an interesting history: it was doing the rounds at Cambridge University in the late twenties, when according to legend the pioneering physicist P.A.M. Dirac - the architect of Relativistic Quantum Mechanics, and the creator of the fundamental concept of anti-particles - chanced upon it. He solved the problem by an elegant trick involving the creation of anti-coconuts, and thus, the story goes, arrived upon a natural interpretation of anti-particles.

To solve the question we observe that if there were N coconuts initially, them must have been 4/5 (N - 1) left over after the first seaman had removed his share. The second sailor threw away one from these for the monkey, took a fifth for himself, leaving behind 4/5 (4/5 (N - 1) - 1). Likewise the third sailor left a pile of 4/5 (4/5 (4/5 (N - 1) - 1) - 1) coconuts, the fourth of 4/5 (4/5 (4/5 (4/5 (N - 1) - 1) - 1) - 1), and the fifth of 4/5 (4/5 (4/5 (4/5 (415 (N - 1) - 1) - 1) - 1) - 1) coconuts. The remaining number of coconuts, which the seamen divided amongst themselves in the morning, was one larger dm a multiple of 5, i.e. 5M + 1.

The Diophantine equation that we must solve is then to determine the smallest positive integers M and N such that

$$4/5\,(4/5\,(4/5\,(4/5\,(4/5\,(N - 1) - 1) - 1) - 1) - 1) = 5M + 1$$

which becomes quite a task. (Verify.)

The trick to solving it lies in observing two facts: first, that after the simplification in this equation have been performed the coefficient of M must be 5^6 i.e. 15625. *Therefore, if n is any solution for N, the initial number of coconuts, then so also is n + 15625.* And second, *whatever its physical interpretation may be, - 4 is a possible solution the number N of coconuts.*

For, beginning with -4 (minus four) coconuts (or, in the language of Relativistic Quantum Mechanics, 4 anti-coconuts!), if the first sailor "creates" a positive coconut somehow and throws it to the monkey, he then has -5 left, a number evenly divisible by 5. He takes. a fifth of these, i.e. -1 coconuts, leaves -4 in the pile, and goes to sleep. The second sailor finds -4 in the pile, creates one as before for the monkey, again has -5 left, from which he keeps -1, leaving - 4 for the third sailor. The division thus continues. The conditions of the problem are satisfied by N = - 4 for the number of coconuts! Therefore 15625 + (-4) = 15621 is another solution, the smallest positive solution (verify).

It would be nice to believe that this elegant solution to the problem of the coconuts inspired Dirac to interpret negative energy solutions of his relativistic wave equation as anti-electrons (or positrons): but the truth is different. Martin Gardner (who for many years edited the Mathematical Games section of *Scientific American*) asked Dirac if this was how he made his great discovery, but learnt that it wasn't! Sorry, folks.

4.4 THE TWELVE COINS PROBLEM

Let's now consider Problem 3, Check your Progress 3, Unit 1 of this block. If you have twelve coins, one of which is *different* - heavier or lighter- from the rest, how can you determine it in no more than three weighings? Obviously, since nothing but an equal-arm balance is provided, one can only compare weights of sets of coins. If six coins are placed in the left-arm pan, and six in the right, and one of the pans - say the left pan - goes down, it may do so due to two reasons: the defective coin is lighter, and is in this pan, or it's heavier, and in the right-hand pan. One is not thus brought nearer the solution: the defective coin is not isolated; so this method of approach cannot succeed.

Progress is possible if the coins are divided into three groups of four coins each, say A, B and C, and A is weighed against B. One of three possibilities may occur: the scale balances, in which case the set C contains the defective coin and a subsequent weighing of C against A or B will determine whether it is heavier or lighter than the rest.

But if in the first weighing the pan containing A goes down, or the pan containing B, the defective coin is then in one of these pans. In either of these eventualities, one set of coins, say A, is weighed against C. If the pans balance, the defective coin is in the set B, and one knows whether it is heavier or lighter from result of the first weighing. However if the pan containing A goes down against C, then the defective coin is heavier, and in A; and if A goes up against C, then the defective coin is lighter, and in A.

Thus two weighings can decide if the defective coin is heavier or lighter, and isolate it to within four coins. It seems impossible now to find it within just one more weighing; for we divide the four coins into groups of two, weigh one against the other, thus pinning it down to the two coins in one or other of the pans. The defective coin can then be picked out in the fourth and final weighing. But the problem states that it must be done in three weighings. There lies the rub.

We proceed by marking the coins from 1 to 12 in the decimal as well as in the ternary scales of enumeration. In the *ternary* system the fundamental digits are 0, 1 and 2. Ternary numbers are written using just these three digits, and no others. A number larger than 2 is built by using digits from this fundamental set: thus (decimal) 3, the successor of 2, is ternary 10, i.e., 3 plays the role of "10" in this system; decimal 4 is ternary 11, decimal 5 is ternary 12, and decimal 6 is ternary 20. (To distinguish decimal 20; from ternary 20 we use subscripts: 20_d, 20_t, etc.)

In a number represented in the ternary system, each digit is "weighted" by some power of 3. The particular power of 3 associated with a digit corresponds to its position in the number. This is similar to the way in which a number is represented in the decimal system, where each digit carries a weight of a power of 10. Thus the decimal number 4295 is 4 thousands, 2 hundreds, 9 tens and 5 ones:

$$4295_d = 4 \times 10^3 + 2 \times 10^2 + 9 \times 10^1 + 5 \times 10^0$$

Similarly the ternary number 12021 is:

$$12021_t = 1 \times 3^4 + 2 \times 3^3 + 0 \times 3^2 + 2 \times 3^1 + 1 \times 3^0 = 142_d$$

The twelve coins are labelled as shown in Fig. 1. In each label, the first line contains a decimal number from 1 through 12, the second line the corresponding ternary number, and in the third line we have the number obtained by changing every 0 in the ternary number to 2, and every 2 to 0. The third entry in

the label of each coin is required for a reason which will shortly emerge.

001_d	002_d	003_d	004_d	005_d	006_d	007_d	008_d	009_d	010_d	011_d	012_d
001_t	002_t	010_t	011_t	012_t	020t	021_t	022_t	100_t	101_t	102_t	110_t
221_t	220_t	212_t	211_t	210_t	202_t	201_t	200_t	122_t	121_t	120_t	112_t

Figure 1: The twelve coins labelled by decimal and ternary number

One further categorisation of the twelve coins is possible: for note that each ternary label in the middle row of Fig. 1 is either *clockwise or anti-clockwise*. In a clockwise label, the *first change in ternary digits* may be from 0 to 1, or from 1 to 2, or from 2 to 0. Thus coins with decimal numbers 1, 3, 4 and 5 carry clockwise ternary labels. In Fig. 2, in the ternary numbers of the second and third rows, the clockwise labels are marked with a C and the anti-clockwise labels are marked with an A:

001_d	002_d	003_d	004_d	005_d	006_d	007_d	008_d	009_d	010_d	011_d	012_d
$C001_t$	$A002_t$	$C010_t$	$C011_t$	$C012_t$	$A020_t$	$A021_t$	$A022_t$	$A100_t$	$A101_t$	$A102_t$	$A110_t$
$A221_t$	$C220_t$	$A212_t$	$A211_t$	$A210_t$	$C202_t$	$C201_t$	$C200_t$	$C122_t$	$C121_t$	$C120_t$	$C112_t$

Figure 2: The twelve coins labelled by clockwise and anti-clockwise ternary numbers

Note that of the 24 ternary labels, 12 are clockwise and 12 anti-clockwise. Note also that the label associated with any coin is unique: none of the labels which occurs in the second line is repeated in the third line, Thus if we are given the label $C202_t$, we realise at once that this clockwise label goes with the anti-clockwise label $A020_t$, and belongs to the coin whose decimal number is 6. These 24 ternary labels can be made to correspond to the 24 situations that may arise with 12 coins, in which any one may be heavier or lighter than the rest.

Three weighings are allowed by the problem: each is performed. The first yields the first (ternary) digit which labels the defective coin, the second weighing provides the second digit and the third weighing gives the third digit of the coin's label. The order of the digits, clockwise or anti-clockwise, tells if the coin is overweight or underweight.

With every coin thus marked, we proceed by dividing the twelve coins into three sets of four coins.

The first set, $\mathbf{CB_0}$, contains those coins whose **C**lockwise ternary labels **B**egin with **0**:

$\mathbf{CB_0}$ = {1, 3, 4, 5)

The second set, $\mathbf{CB_1}$**,** contains those coins whose **C**lockwise ternary labels **B**egin with **1**:

$\mathbf{CB_1}$ = {9,10, ,11,12}

The third set, $\mathbf{CB_2}$**,** Contains those coins whose **C**lockwise ternary labels **B**egin with **2**:

$\mathbf{CB_2}$= {2, 6, 7, 8}

We begin by weighing the coins of $\mathbf{CB_0}$ in the left-hand pan against the coins of $\mathbf{CB_2}$ in the right-hand pan. This first weighing yields the first (i.e. the **B**eginning) digit of the set of three ternary digits which label the defective coin. It is 0 if the left-hand pan goes down, 1 if the scale balances, and 2 if the right- hand pan goes down.

We now divide the coins into three different sets of four coins.

The first set, $\mathbf{CM_0}$, contains those coins whose Clockwise ternary label has the **M**iddle digit 0:

$\mathbf{CM_0}$ = {1, 6, 7, 8}

The second set, $\mathbf{CM_1}$ contains those coins whose Clockwise ternary label has the **M**iddle digit 1:

$\mathbf{CM_1}$ = {3,4,5,12}

The third set, $\mathbf{CM_2}$, contains those coins whose Clockwise ternary label has the **M**iddle digit 2:

$\mathbf{CM_2}$ = {2,9, 10, 1 1}

We now weigh the coins $\mathbf{CM_0}$ in the left-hand pan against the digits of $\mathbf{CM_2}$ in the right-hand pan. This second weighing yields the second digit of the ternary label of the defective coin. If the left-hand pan goes down, this digit is given the value 0, if the right-hand pan goes down this second digit is assigned the value 2, while if the scale balances this digit is set the value 1.

These first two weighings yield the first two digits of the ternary number corresponding to the defective coin. The third digit of the ternary label is obtained from the third weighing, for which we again divide the twelve coins into three sets of four coins.

The first set, $\mathbf{CE_0}$**,** contains those coins whose Clockwise ternary labels **E**nd with 0:

$$\mathbf{CE_0} = \{2, 3, 8, 11\}$$

The second set, $\mathbf{CE_1}$, contains those coins whose Clockwise ternary labels **E**nd with 1:

$$\mathbf{CE_1} = \{1, 4, 7, 10\}$$

The third set, $\mathbf{CE_2}$, contains those coins whose Clockwise ternary labels **E**nd with 2:

$$\mathbf{CE_2} = \{5,6,9,12\}$$

We weigh the coins in $\mathbf{CE_0}$ in the left-hand pan against the coins of $\mathbf{CE_2}$ in the right-hand pan. As before, if the left-hand pan goes down, the third digit is given the value 0, if the right-hand pan goes down it is assigned the value 2, while if the scale balances it gets the value 1.

We now have the three ternary digits which identify the defective coin. If the digits are in clockwise order, the coin is overweight, and if they are in anti-clockwise order the coin is underweight.(This method was suggested by F. J. Dyson and R. C. Lyness in *Math. Gazette*, Vol. 30, Oct. 1946)

Check Your Progress 2

1. Verify that if coin number 9_d is heavier than the rest, it is picked out and appropriately distinguished by the weighing process.

4.5 POINCARE ON THE PSYCHOLOGY OF INVENTION

In the early years of this century Henri Poincare, one of the greatest mathematicians of all time, gave a lecture at the Psychological Society in Paris in an attempt to describe the processes by which mathematical discoveries are made. A few excerpts from his lecture, translated and published in *scientific American*, Aug. 1948, are reproduced as follows.

"As for myself, I must confess I am absolutely incapable even of adding without mistakes ... My memory s not bad, but it would be insufficient to make me a good chess-player. Why then does it not fail me in a difficult piece of mathematical reasoning where most chess players would lose themselves? Evidently because it is guided by the general march of reasoning. A mathematical demonstration is not a simple juxtaposition of syllogisms placed in a certain order, and the order in which these elements are placed is much more important than the elements themselves.

"In fact, what is mathematical creation? It does not consist in making new combinations with mathematical entities already known. Anyone could do that, but the combinations so made would be infinite in number and most of them absolutely without interest. To create consists precisely in not making useless combinations and in making those which are useful and which are a small minority. Invention is discernment, choice.

"It is time to penetrate deeper and to see what goes on in the very soul of the mathematician. For this, I believe, I can do best by recalling memories of my own. But I shall limit myself to telling how I wrote my first memoir on Fuchsia functions. I beg the reader's pardon; I am about to use some technical expressions, but they need not frighten him, for he is not obliged to understand them. I shall say, for example, that I have found the demonstration of such a theorem under such circumstances. This theorem will have a barbarous name, unfamiliar to many, but that is unimportant: what is of interest to the psychologist is not the theorem but the circumstances.

"For fifteen days I strove to prove that there could not be any functions like those I have called Fuchsian functions. I was then very ignorant; every day I seated myself at my work table, stayed an hour or two, tried a great number of combinations and reached no results. One evening, contrary to my custom, I drank black coffee and could not sleep. Ideas rose in crowds: I felt them collide until pairs interlocked, so to speak, making a stable combination. By the next morning I had established the existence of a class of Fuchsian functions, those which come from the hypergeometric series; I had only to write out the results, which took but a few hours.

'Then I wanted to represent these functions by the quotient of two series; this idea was perfectly conscious and deliberate, the analogy with elliptic functions guided me. I asked myself what properties these series must have if they existed, and I succeeded without difficulty in forming the series I have called theta-Fuchsian.

"Just at this time I left Caen, where I was then living, to go on a geologic excursion under the auspices of the school of mines. The changes of travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step the idea came to me, without anything in my former thoughts having paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had time, as, upon taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake, I verified the result at my leisure....

"Most striking at first is this appearance of sudden illumination, a manifest sign of long, unconscious prior work. The role of this unconscious work in mathematical invention appears to me incontestable, and traces of it would be found in other cases where it is less evident. Often when one Works at a hard question, nothing good is accomplished at the first attack. Then one takes a rest, longer or shorter, and sits down anew to the work. During the first half-hour, as before, nothing is found, and then all of a sudden the decisive idea presents itself to the mind..."

Christopher Zeeman, the British knot theorist who gained fame for his applications of catastrophe theory to a variety of unconventional situations (including the behaviour of a slowly provoked dog) tried unsuccessfully for many years to prove a theorem in knot theory, that it is possible in five dimensions to tie a sphere in a knot. Here he relates how he eventually succeeded:

"I sat down one Saturday morning and I thought, "Well, I'll have another crack at this damn problem." And to and behold, I suddenly found to my surprise, that I had proved the opposite ... and I was so excited that I spent the whole weekend writing this paper up, about twenty pages. And then, late that night, I confess, I went and sat on the lavatory and while I was there the real flash of inspiration struck me like a bomb. I suddenly saw how to reduce the proof from twenty pages to ten lines.

4.6 SUMMARY

It is an inescapable fact of life that it takes a long time to solve a worthwhile problem. Part of the reason for this is of course that it is necessary to assimilate current knowledge in the field, to learn what others have done, before one can fruitfully apply oneself. Second, passion and discipline are necessary ingredients. Gifted scientists are men and women of stamina, devotion and intellectual courage, and they work very hard at their problems. Though we may be inspired by the cry of *Eureka!* over the centuries of scientific or mathematical discovery, we must not forget the years of labour that must have been devoted to the formulation of the problem, to "thinking on it continually", and to false attempts at solution. Does luck play a role? Listen to what

Pasteur said, when asked whether many of his great discoveries depended upon luck: "...fortune only favours the prepared mind", which a later scientist thus modified, "Fortune only favours the busy mind."