UNIT 5 DERIVATIVES OF SOME STANDARD FUNCTIONS

Structure

5.1 INTRODUCTION

Exponential functions occupy an important place in pure and applied science. Laws of growth and decay are very often expressed in terms of these functions. In this unit we shall study the derivatives of exponential functions. The inverse function theorem which was stated in Unit 4 will then help us to differentiate this inverse, the logarithmic functions. In particular, you will find that the natural exponential function in its own derivative.

Further we shall introduce and differentiate hyperbolic functions and their inverses since they hold special significance for physical sciences. We shall demonstrate the method of finding derivatives by taking logarithms, and also that of differentiating implicit functions.

With this unit we come to the end of our quest for the derivatives of some standard, frequently used functions. In the next block we shall see the geometrical significance of derivatives and shall use them for sketching graphs of functions.

Objectives

After studying this unit you should be able to :

- find the derivatives of exponential and logarithmic functions
- define hyperbolic functions and discuss the existence of their inverses
- differentiate hyperbolic functions and inverse hyperbolic functions
- use the method of logarithmic differentiation for solving some problems
- differentiate implicit functions and also those functions which are defined with the help of a parameter.

5.2 EXPONENTIAL FUNCTIONS

Our main aim, here, is to find the derivatives of exponential functions. But let us first recall the definition of an exponential function.

5.2.1 Definition of an Exponential Function

A function f defined on **R** by $f(x) = a^x$, where $a > 0$, is known as an **exponential function.** Now to find the derivative of f, we shall have to take the limit:

$$\lim_{h\to 0} \frac{a^{x+b} - a^x}{h} = a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h} \qquad \dots (1)$$

a^x may not be defined for all x if a<0. For example, $(-2)^{1/2}$ is not defined in **R**.

So, if we put $\lim_{h\to 0} \frac{a^h - 1}{h} = k$, we get

$\frac{d}{dx} a^x = k \cdot a^x$. We can also interpret k as the derivative of a^x at $x = 0$. In Fig. 1 you can see the graphs of exponential functions for various values of a.

$$k = \lim_{h\to 0} \frac{a^h - 1}{h} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = f'(0) = \frac{d}{dx}(a^x) \mid x = 0$$

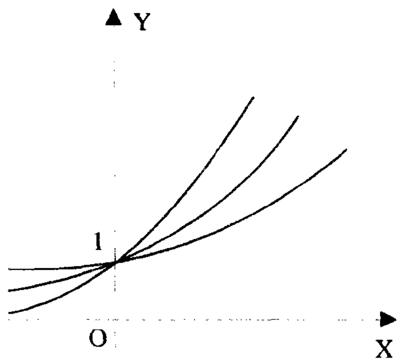


Fig. 1

All these curves pass through (0, 1) as $a^0 = 1$ for all a. Now from all these curves, we shall choose that one, whose tangent at (0, 1) has slope = 1. (We assume that such a curve exists). The value of a corresponding to this curve is then denoted by e. Thus, we have singled out the exponential function : $x \to e^x$, so that its derivative at $x = 0$ is 1. Thus,

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

This also means that

$$\frac{de^x}{dx} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

That is, the derivative of this function is the function itself.

This special exponential function is called the **natural exponential function.**

5.2.2 Derivative of an Exponential Function

In Unit 1, we compared the graphs of natural exponential function e^x and the natural logarithmic function lnx and found that they are reflections of each other w.r.t. the line $y = x$ (see Fig. 2). We concluded that = e^x and lnx are inverses of each other. This also means that $e^{\ln x} = x \quad \forall x > 0$.

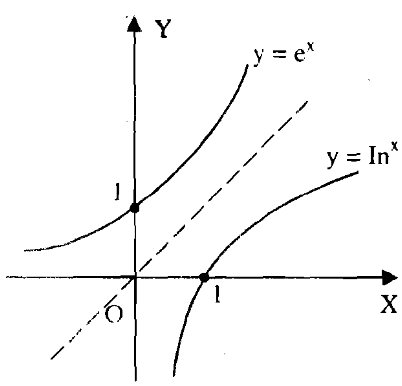


Fig. 2

From this we can write $a^x = e^{\ln a^x}$, or $a^x = e^{x\ln a}$, where $a > 0$.

$$\text{Thus } \frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} \qquad\qquad \ln a^b = b\ln a$$

$$= e^{x\ln a}\frac{d}{dx}(x\ln a) \text{ by chain rule}$$
$$= e^{x\ln a}\ln a$$
$$= a^x\ln a.$$

Remark 1 If we compare this result with (1) which we derived at the beginning of this section, we find that

$$\ln a = \lim_{h\to 0}\frac{a^h-1}{h}$$

Thus, we have

$$\frac{d}{dx}e^x = e^x, \text{ and}$$

$$\frac{d}{dx}a^x = a^x \ln a$$

Example 1 Let us use these formula to find the derivatives of

i) $e^{(x^2+2x)}$ ii) $\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$ iii) $a^{\sin^{-1}x}$

i) Let $y = e^{(x^2+2x)}$. Then, by chain rule

$$\frac{dy}{dx} = (2x+2)\, e^{(x^2+2x)}$$

Hence $\dfrac{d}{dx}\left(e^{(x^2+2x)}\right) = 2(x+1)\, e^{(x^2+2x)}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

ii) $$\frac{d}{dx}\frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{(e^x-e^{-x})\frac{d}{dx}(e^x+e^{-x}) - (e^x+e^{-x})\frac{d}{dx}(e^x-e^{-x})}{(e^x-e^{-x})^2}$$

$$= \frac{(e^x-e^{-x})(e^x-e^{-x}) - (e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}$$

$$= \frac{(e^x-e^{-x})^2 - (e^x+e^{-x})^2}{(e^x-e^{-x})^2}$$

$$= \frac{-4}{(e^x-e^{-x})^2}$$

iii) We apply the chain rule again to differentiate $a^{\sin^{-1}x}$

$$\frac{d}{dx}\left(a^{\sin^{-1}x}\right) = \ln a\; a^{\sin^{-1}x}\frac{d}{dx}(\sin^{-1}x)$$

$$= \ln a\,\frac{1}{\sqrt{1-x^2}}\,a^{\sin^{-1}x}$$

See, you can solve these exercises now.

E E 1) Find the derivatives of :

a) $5e^{(x^2-2)x}$ b) $e^{(x+1)/x}$ c) $(x+2)\,e^{\sqrt{x}}$

d) $e^{-m}\tan^{-1}x$ e) 2^{2x} f) $7^{\cos x}$

E 2) How much faster is $f(x) = 2x$ increasing at $x = 1/2$ than at $x = 0$?

E

In this section we have defined e as that real number for which $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

Alternatively, e can also be defined as a limit:

$e = \lim_{h\to 0} (1 + 1/n)^n$, or as the sum of an infinite series: $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots$

But all these definitions give the same value, e = 2.718281828... e is an irrational number.

In many situations the rate of growth (of human beings, or bacteria or radioactive particles) is proportional to the present population. That is, if x(t) is the population at time t, then $\frac{dx}{dt} \propto x$. In such situations the exponential function is of great relevance since $\frac{d}{dt}(e^t) = e^t$.

Now let us turn our attention to logarithmic functions.

5.3 DERIVATIVES OF LOGARITHMIC FUNCTIONS

In Unit 4, we studied the inverse function theorem, (Theorem 1, Unit 4) and used it to find the derivatives of various functions such as $\sin^{-1}x$, $\cos^{-1}x$, and so on. Here, we shall, yet again, apply this theorem to calculate the derivative of the natural logarithmic function.

5.3.1 Differentiating the Natural Log Function

Consider the function $y = \ln x$. This is the inverse of the natural exponential function, that is, $y = \ln x$ if and only if $x = e^y$.

$\ln x$ is defined on $]0\ \infty[$.

From Fig. 2, you can see that the natural exponential function is a strictly increasing function. (You will be able to rigorously prove this result by the end of this course). Further, the derivative of the function $x = e^y$ is

$$\frac{dy}{dx} = \frac{d}{dx}(e^y) = e^y > 0 \text{ for all } y \in R.$$

Thus, all the conditions of the inverse function theorem are satisfied. This means we can conclude that the derivative of the natural logarithmic function (which is the inverse of the natural exponential function) exists, and

$$\frac{dy}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}$$

Thus, we have

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Let's see how we can use this result.

Example 2: Suppose we want to differentiate $y = \ln(x^2 - 2x + 2)$.

$$\frac{dy}{dx} = \frac{1}{x^2 - 2x + 2}\frac{dy}{dx}(x^2 - 2x + 2)$$

$$= \frac{2x - 2}{x^2 - 2x + 2}$$

Note that $x^2 - 2x + 2 = (x - 1)^2 + 1$ and hence, is non-zero for all x.

Therefore, $\ln(x^2 - 2x + 2)$ is well-defined.

Example 3: If we want to differentiate $y = \ln\left|\frac{1 + x^2}{1 - x^2}\right|$, $|x| \neq 1$, we will have to consider two cases: i) $|x| > 1$ and ii) $|x| < 1$

i) If $|x| > 1$, we get $\left|\frac{1 + x^2}{1 - x^2}\right| = \frac{1 + x^2}{-(1 - x^2)} = \frac{x^2 + 1}{x^2 - 1}$,

since $|x| > 1$ makes $1 - x^2$ negative. So in this case,

$$\frac{dy}{dx} = \frac{x^2 - 1}{x^2 + 1}\frac{d}{dx}\left(\frac{x^2 + 1}{x^2 - 1}\right)$$

$$= \frac{4x}{1 - x^4}, \text{ after simplification.}$$

ii) when $|x| < 1$, $\left|\frac{1 + x^2}{1 - x^2}\right| = \frac{1 + x^2}{1 - x^2}$ and so,

$$\frac{dy}{dx} = \frac{1 - x^2}{1 + x^2}\frac{d}{dx}\left(\frac{1 + x^2}{1 - x^2}\right)$$

$$= \frac{4x}{1 - x^4}$$

So, we see that $\frac{dy}{dx} = \frac{4x}{1 - x^4}$ for all x such that $|x| \neq 1$.

a^x is a constant function for $a = 1$. Hence, it does not have any inverse. The log functions are thus defined only for $a \neq 1$.

Now, let us turn our attention to logarithmic functions with arbitrary bases.

5.3.2 Differentiating the General Log Function

Let us consider any positive number $a \neq 1$. We say $\log_a x = y$ if and only if $x = a^y$. Obviously, the natural logarithmic function $\ln x$ can be written as $\log_e x$.

Further, we know that $\log_a x = \log_a e \cdot \ln x$. This rule gives a connection between the natural and general logarithmic functions. We shall use this relationship to find the derivative of $\log_a x$.

So, if $y = \log_a x = \log_a e \ln x$,

$$\frac{d}{dx}(\log_a x) = \log_a e \cdot \frac{1}{x}$$

If we put a = 3 in this, we get our earlier result:

$$\frac{dy}{dx} = \log_a e \frac{d \ln x}{dx} = \log_a e \frac{1}{x}$$

Thus, we arrive at

$$\frac{d}{dx}(\log_a x) = \log_a e \cdot \frac{1}{x}$$

If we put a = e in this, we get our earlier result:

$$\frac{d}{dx} \ln x = \frac{1}{x}, \text{ since } \log_e e = 1$$

Example 4 Let us differentiate $y = \log_7 \tan^3 x$

$$\frac{dy}{dx} = \log_7 e \frac{1}{\tan^3 x} \frac{d}{dx}(\tan^3 x)$$

$$= \log_7 e \frac{1}{\tan^3 x} 3 \tan^2 x \sec^2 x$$

$$= 3 \log_7 e \frac{\sec^2 x}{\tan x}$$

If you have followed the solved examples in this section you should have no difficulty in solving these exercises.

E 3) Find the derivatives of :

a) $\log_2 2x$ b) $7\log_{11}(5x^2 + 2)$ c) $e^{-x}\ln x$

d) $\ln\left(\frac{1+x}{1-x}\right), |x| < 1$ e) $\ln(\sin^4 x)$

5.4 HYPERBOLIC FUNCTIONS

In applications of mathematics to other sciences, we, very often, come across certain combinations of e^x and e^{-x}. Because of their importance, these combinations are given special names, like the hyperbolic sine, the hyperbolic cosine etc. These names suggest that they have some similarity with the trigonometric functions. Let's look at their precise definitions and try to understand the points of similarity and dissimilarity between the hyperbolic and the trigonometric functions.

5.4.1 Definitions and Basic Properties

Definition 1 The hyperbolic sine function is defined by $\sinh x = \dfrac{e^x + e^{-x}}{2}$ for all $x \in \mathbf{R}$. The range of this function is also **R**.

Definition 2 The hyperbolic cosine function is defined by $\cosh x = \dfrac{e^x - e^{-x}}{2}$ for all $x \in \mathbf{R}$. The range of this function is $[1, \infty]$.

You will notice that

$$\sinh(-x) = \frac{e^x + e^{-(-x)}}{2} = \frac{e^x + e^{-x}}{2}$$
$$= -\sinh(x), \text{ and}$$

$$\cosh(-x) = \frac{e^x - e^{-(-x)}}{2} = -\frac{e^x - e^{-x}}{2} = \cosh x$$

In other words, the hyperbolic sine is an odd function, while the hyperbolic cosine is an even function. Fig. 3(a) and (b) show the graphs of these two functions.

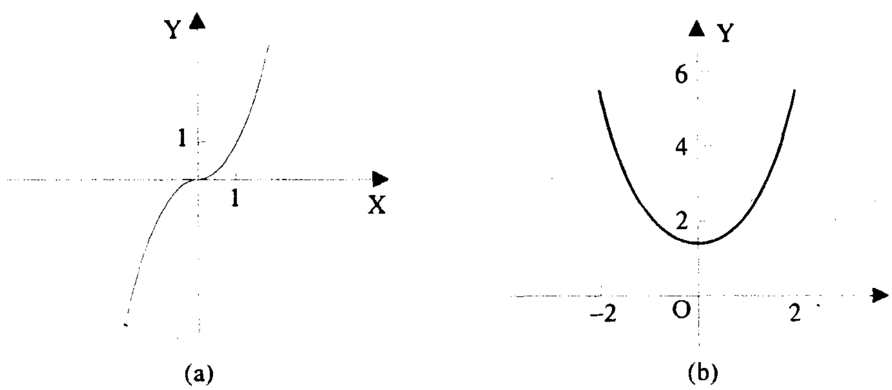


Fig. 3 : Graph of (a) sinh x (b) cosh x

We also define four other hyperbolic functions as :

$$\tanh x = \frac{e^x - e^{-(-x)}}{e^x + e^{-x}}, \quad \coth x = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}, \quad \operatorname{cosech} x = \frac{2}{e^x - e^{-x}}.$$

E E 4) Verify that a) $\cosh^2 x - \sinh^2 x = 1$

b) $\tanh x = \dfrac{\sinh x}{\cosh x}$

c) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

E 5) Derive an identity connecting coth x and cosech x.

You must have noticed that the identities involving these hyperbolic functions are similar to those involving trigonometric functions. It is possible to extend this analogy and get some more formulas:

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh x$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

Since we have seen that $\cosh^2 t - \sinh^2 t = 1$, it is obvious that a point with coordinates (cosh t, sinh t) lies on the unit hyperbola: $x^2 - y^2 = 1$. (Hence the name, hyperbolic functions). We have a similar situation in the case of trigonometric functions. The point (cost, sint) lies on the unit circle: $x^2 + y^2 = 1$. That is why trigonometric functions are also called circular functions.

There is one major point of difference between the hyperbolic and circular functions, though. While t in sint, cost, etc. is the measure of an angle, the t which appears in sinht, cosht, etc. cannot be interpreted as the measure of an angle. However, it is sometimes called the hypberbolic radian.

5.4.2 Derivatives of Hyperbolic Functions

Since the hyperbolic functions are defined in terms of the natural exponential function, whose derivative we already know, it is very easy to calculate their derivatives, For example,

$\sinh x = \frac{e^x - e^{-x}}{2}$. This means,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

Similarly, $\cosh x = \frac{e^x + e^{-x}}{2}$ gives us

$$\frac{d}{dx}(\cosh x) = \frac{e^x - e^{-x}}{2} = \sinh x$$

In the case of $\tanh x = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$, we get

$$\frac{d}{dx}(\tanh x) = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= 1 - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= 1 - \tanh^2 x = \operatorname{sech}^2 x$$

We can adopt the same method for finding the derivatives of coth x, sech x and cosech x. In Table 1 we have collected all these results.

Table 1

Function	Derivative
sinh x	cosh x
cosh x	sinh x
tanh x	$\operatorname{sech}^2 x$
coth x	$-\operatorname{cosech}^2 x$
sech x	$-$ sech x tanh x
cosech x	$-$ cosech x coth x

Example 5: Suppose we want to find dy/dx when $y = \tanh(1 - x^2)$.

$$\frac{dy}{dx} = \operatorname{sech}^2(1 - x^2) \cdot \frac{d}{dx}(1 - x^2)$$
$$= -2x \operatorname{sech}^2(1 - x^2)$$

See if you can solve these exercises on your own.

E E 6) Find f′(x) when f(x) =

a) $\tanh \frac{4x + 1}{5}$ b) $\sinh e^{2x}$ c) coth (1/x)

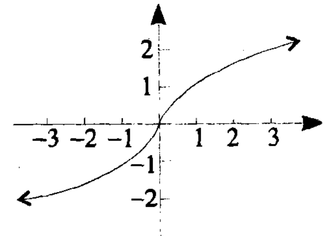
d) sech (ln x) e) $e^x \cosh x$

5.4.3 Derivatives of Inverse Hyperbolic Functions

We shall try to find the derivatives of the inverse hyperbolic functions now. Let us start with the inverse hyperbolic sine functions.

From Fig. 3(a) you can see that the hyperbolic sine is a strictly increasing function. This means that its inverse exists, and

$$y = \sinh^{-1} x \Leftrightarrow \sinh y = \frac{e^y - e^{-y}}{2}$$
$$\Leftrightarrow 2x = e^y - e^{-y}$$
$$\Leftrightarrow e^{2y} - 2xe^y - 1 = 0$$
$$\Leftrightarrow (e^y)^2 - 2xe^y - 1 = 0$$
$$\Leftrightarrow e^y = x + (\sqrt{1 + x^2})$$
$$\Leftrightarrow y = \ln (x + \sqrt{1 + x^2})$$



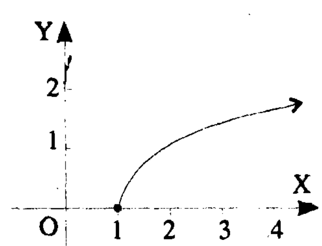
Fi.g 4

We have used the formula for finding the roots of a quadratic equation here. Note that if $e^y = x - \sqrt{1 + x^2}$, then $e^y < 0$, which is impossible. Therefore we ignore this root.

Thus $\sinh^{-1}x = \ln (x + \sqrt{1 + x^2})$, $x \in]-\infty, \infty[$. In Fig. 4, we have drawn the graph of $\sinh^{-1}x$. Now,

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{d}{dx}(\ln (x + \sqrt{1 + x^2}))$$
$$= \frac{1}{x + \sqrt{1 + x^2}} \frac{d}{dx}(x + \sqrt{1 + x^2})$$
$$= \frac{1}{x + \sqrt{1 + x^2}} \left(1 + \frac{x}{\sqrt{1 + x^2}}\right)$$
$$= \frac{1}{\sqrt{1 + x^2}}$$

Thus, $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}} = \frac{1}{\sqrt{x^2 + 1}}$



Fl.g 5

Again we ignore the root $e^y = x - \sqrt{x^2 - 1}$, because then $e^y < 1$, which is impossible since $y > 0$.

In the case of the hyperbolic cosine function, we see from Fig. 3 (b), that its inverse will exist if we restrict its domain to $[0, \infty[$. The domain of this inverse function will be $[1, \infty[$, and its range will be $[0, \infty[$.

Now $y = \cosh^{-1} x \Leftrightarrow x = \cosh y = \frac{e^y + e^{-y}}{2}$

$$\Leftrightarrow e^{2y} - 2xe^y + 1 = 0$$
$$\Leftrightarrow e^y = x + \sqrt{x^2 - 1})$$
$$\Leftrightarrow y = \ln (x + \sqrt{x^2 - 1})$$

Thus $\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}), x \geq 1$.

Fig. 5 shows the graph of $\cosh^{-1}x$.

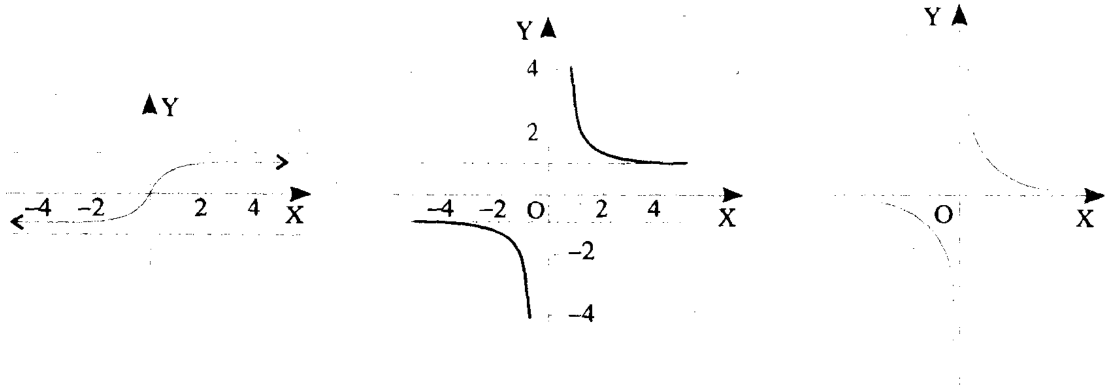
$$\text{Further } \frac{d}{dx} (\cosh^{-1} x) = \frac{1}{x + \sqrt{x^2 - 1}} \frac{d}{dx} (x + \sqrt{x^2 - 1})$$

$$= \frac{1}{\sqrt{x^2 - 1}}, x > 1$$

Note that the derivative of $\cosh^{-1} x$ does not exist at $x = 1$.

Fig. 6 (a), (b) and (c) show the graphs of tanh x, coth x and cosech x. You can see that each of these functions is one-one and strictly monotonic. Thus, we can talk about the inverse in each case.

Arguing as for $\sinh^{-1}$ and $\cosh^{-1}x$, we get



(a) (b) (c)

Fig. 6

$$y = \tanh^{-1} x \Leftrightarrow x = \tanh y \Leftrightarrow y = \frac{1}{2} \ln \left(\frac{1 + x}{1 - x}\right), |x| < 1$$

$$y = \coth^{-1}x \Leftrightarrow x = \coth y \Leftrightarrow y = \frac{1}{2} \ln \left(\frac{x - 1}{x + 1}\right), |x| < 1$$

$$y = \operatorname{cosech}^{-1}x \Leftrightarrow x = \operatorname{cosech} y \Leftrightarrow y = \ln \left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|}\right), x \neq 0$$

Since $\operatorname{sech} x = \frac{1}{\cosh x}$, we shall have to restrict the domain of sech x to $[0, \infty[$ before talking about its inverse, as we did for coshx. $\operatorname{Sech}^{-1}x$ is defined for all $x \in]0, 1]$, and we can write $\operatorname{sech}^{-1} x = \ln \left(\frac{1 + \sqrt{1 + x^2}}{|x|}\right), 0 < x \leq 1$

Now, we can find the derivatives of each of these inverse hyperbolic functions. We proceed exactly as we did for the inverse hyperbolic sine and cosine funcitons and get

$$\frac{d}{dx} (\tanh^{-1} x) = \frac{1}{1 - x^2}, |x| < 1$$

$$\frac{d}{dx} (\coth^{-1} x) = \frac{1}{1 - x^2}, |x| > 1$$

$$\frac{d}{dx} (\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1 - x^2}}, 0 < x < 1$$

$$\frac{d}{dx}(\text{cosech}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}, \ x \neq 0$$

Let us use these results to solve some problems now.

Example 6 Suppose we want to find the derivatives of
(a) $f(x) = \sinh^{-1}(\tan x)$, and
(b) $g(x) = \tanh^{-1}(\cos e^x)$.

Let's start with $f(x) = \sinh^{-1}(\tan x)$.

$$f'(x) = \frac{-1}{\sqrt{\tan^2 x + 1}} \frac{d}{dx}(\tan x).$$

$$= \frac{1}{|\sec x|} \sec^2 x = |\sec x|$$

Now if $g(x) = \tanh^{-1}(\cos e^x)$, this means that

$$g'(x) = \frac{1}{1-\cos^2 e^x} \frac{d}{dx}(\cos e^x)$$

$$= \frac{1}{\sin^2 e^x} (-\sin e^x) . e^x$$

$$= \frac{-e^x}{\sin e^x} = - e^x \text{ cosec } e^x$$

We are now listing some functions for you to differentiate.

E E7) Differentiate the following functions on their respective domains.

a) $\text{cosech}^{-1}(5\sqrt{x})$ b) $[\text{sech}^{-1}(\cos^2 x)]^{1/3}$

c) $\coth^{-1} e^{(x^2+5x-6)}$ d) $\tanh^{-1}(\coth x) + \coth^{-1}(2x)$

e) $\sinh^{-1}\sqrt{x} + \cosh^{-1}(2x^2)$

5.5 METHODS OF DIFFERENTIATION

In this section, we shall study different methods of finding derivatives. We shall also see that the problem of differentiating some functions is greatly simplified by using these methods. Some of the results we derived in the earlier sections will be useful to us here.

5.5.1 Derivative of X^r

In Unit 4 we have seen that $\frac{d}{dx}(x^r) = rx^{r-1}$ when r is a rational number. Now, we are in a position to extend this result to the case when r is any real number. So if $y = x^r$, where $x > 0$ and $r \in \mathbf{R}$, we can write this as

As we have mentioned in Unit 4, if $x < 0$, x^r may not be a real number. For example, $-3^{1/2} = \sqrt{-3} \in R$

$y = e^{\ln x^r} = e^{r \ln x}$, since the natural exponential and logarithmic functions are inverses of each other.

$$\text{Thus} \quad \frac{dy}{dx} = \frac{d}{dx}(e^{r.\ln x}) = e^{r.\ln x}\frac{d}{dx}(r \ln x)$$

$$= re^{r.\ln x}\frac{1}{x} = \frac{rx^r}{x} = rx^{r-1}$$

This proves that

$$\frac{d}{dx}(x^r) = rx^{r-1} \qquad \text{for } x > 0, \quad r \in \mathbf{R}.$$

We are sure, you will be able to solve this exercise now.

E E 8) Differentiate

a) $x^{\sqrt{2}}$ b) x^e

5.5.2 Logarithmic Differentiation

Sometimes we find that the process of taking derivatives becomes simple if we take logarithms before differentiating. In this section we shall illustrate this point through some examples. But to take the logarithm of any quantity we have to be sure that it is non-negative. To overcome this difficulty, let us first try to find the derivative of ln (| x |).

Now you can check easily that $|x| = \sqrt{x^2}$.

Therefore, $\ln(|x|) = \ln\sqrt{x^2}$, and

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln\sqrt{x^2} = \frac{1}{\sqrt{x^2}}\frac{d}{dx}(\sqrt{x^2})$$

$$= \frac{1}{\sqrt{x^2}}\cdot\frac{x}{\sqrt{x^2}} = \frac{x}{x^2} = \frac{1}{x}$$

We get,

$$\frac{d}{dx}\ln(|x|) = \frac{1}{x}$$

Using chain rule we can now conclude that if u is any function of x, then $\frac{d}{dx}\ln(|u|)$ $= \frac{1}{u}\cdot\frac{du}{dx}$

Let us see how this result helps us in simplifying the differentiation of some functions.

Example 7 To differentiate $\frac{(x^2+1)^9(x-3)^{3/4}}{(x-5)^{2/3}(x^2+2x+1)^{-1/3}}$

we start by taking $y = \frac{(x^2+1)^9(x-3)^{3/4}}{(x-5)^{2/3}(x^2+2x+1)^{-1/3}}$

Thus, $|y| = \frac{|x^2+1|^9\,|x-3|^{3/4}}{|x-5|^{2/3}\,|x^2+2x+1|^{-1/3}}$

Then taking logarithms of both sides, we get

$$\ln|y| = \ln(|x^2+1|^9\,|x-3|^{3/4}) - \ln(|x-5|^{2/3}\,|x^2+2x+1|^{-1/3})$$

$$= \ln(|x^2+1|^9) + \ln(|x-3|^{3/4}) - \ln(|x-5|^{2/3}) - \ln(|x^2+2x+11|^{-1/3})$$

$$= 9\ln|x^2+1| + \frac{3}{4}\ln|x-3| - \frac{2}{3}\ln|x-5| + \frac{1}{3}\ln|x^2+2x+1|$$

Differentiating throughout we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{9}{x^2+1}2x + \frac{3}{4(x-3)} - \frac{2}{3(x-5)} + \frac{2x+2}{3(x^2+2x+1)}$$

$$= \frac{18x}{x^2+1} + \frac{3}{4(x-3)} - \frac{2}{3(x-5)} + \frac{2(x+1)}{3(x+1)^2}$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{18x}{x^2+1} + \frac{3}{4(x-3)} - \frac{2}{3(x-5)} + \frac{2}{3(x+1)}\right]$$

$$= \frac{(x^2+1)^9(x-3)^{3/4}}{(x-5)^{2/3}(x^2+2x+1)^{-1/3}}\left[\frac{18x}{x^2+1} + \frac{3}{4(x-3)} - \frac{2}{3(x-5)} + \frac{2}{3(x+1)}\right]$$

Example 8 Suppose we want to differentiate $x^{\sin x}$, $x > 0$.

Let us write $y = x^{\sin x}$. Then $y > 0$ and so we can take logarithms of both sides to the base e and write

$$\ln y = \ln x^{\sin x} = \sin x.\ln x$$

Differentiating throughout, we get,

$$\frac{1}{y}\frac{dy}{dx} = \sin x\,\frac{1}{x} + \cos x\ln x$$

$$= \frac{\sin x}{x} + \cos x\ln x$$

Therefore $\frac{dy}{dx} = y\left(\frac{\sin x}{x} + \cos x \ln x\right)$

or $\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right)$

Example 9 To differentiate $x^{\cos x} + (\cos x)^x$ let $f(x) = x^{\cos x}$ and $g(x) = \cos x^x$. To ensure that f(x) and g(x) are well defined, let us restrict their domain to $[0, \pi/2]$.

$y = x^{\cos x} + (\cos x)^x = f(x) + g(x) > 0$ for $x \in [0, \pi/2]$

Let us differentiate both f(x) and g(x) by taking logarithms. We have,

$f(x) = x^{\cos x}$

Therefore $\ln f(x) = \cos x \ln x$.

Thus, $\frac{1}{f(x)} f'(x) = -\sin x \ln x + \cos x \frac{1}{x}$

$$\begin{aligned}
\text{That is, } f'(x) &= f(x)\left(-\sin x \ln x + \frac{\cos x}{x}\right) \\
&= x^{\cos x}\left(\frac{-\sin x \ln x + \cos x}{x}\right) \\
&= x^{\cos x - 1}(\cos x - \sin x \ln x)
\end{aligned}$$

Similarly, $g(x) = (\cos x)^x$ and so $\ln g(x) = x \ln \cos x$

Then $\frac{1}{g(x)} g'(x) = \ln \cos x + \frac{x}{\cos x}(-\sin x)$

$$\begin{aligned}
\Rightarrow g'(x) &= (\cos x)^x\left(\frac{\cos x \ln \cos x - x \sin x}{\cos x}\right) \\
&= (\cos x)^{x-1}(\cos x \ln \cos x - x \sin x)
\end{aligned}$$

Hence, $\frac{dy}{dx} = f'(x) + g'(x)$

$= x^{\cos x - 1}(\cos x - x \sin x \ln x) + \cos x^{x-1}(\cos x \ln \cos x - x \sin x)$

If you have followed these examples you should have no difficulty in solving these exercises by the same method.

E E 9) Differentiate.

a) $(x^2 - 1)(x^2 + 2)^6 (x^3 - 1)^5$ b) $\frac{1}{(x-1)^5 (x-2)^6 (x-3)^7}$

c) $(\sin x)^x + (\cos x)^{\tan x}$ d) $(x^x)^x + x^{(x^x)}$

e) $(\sin x)^{\ln x} + x^x$

5.5.3 Derivatives of Functions Defined in Terms of a Parameter

Till now we were concerned with functions which were expressed as $y = f(x)$. We called x an independent variable, and y, a dependent one. But sometimes the relationship between two variables x and y may be expressed in terms of another variable, say t. That is, we may have a pair of equations $x = \phi(t)$, $y = \psi(t)$, where the functions ϕ and ψ have a common domain. For example, we know that the circle $x^2 + y^2 = a^2$ is also described by the pair of equations, $x = a \cos t$, $y = a \sin t$, $0 \le t \le 2\pi$.

In such cases the auxiliary variable t is called a parameter and the equations $x = \phi(t)$. $y = \psi(t)$ are called parametric equations. Now, suppose a function is defined in terms of a parameter. To obtain its derivative, we need only differentiate the relations in x and y separately. The following examples illustrates this method.

Example 10 Let us try to find $\frac{dy}{dx}$ if $x = a \cos \theta$ and $y = b \sin \theta$

(Here the parameter is θ)
We differentiate the given equations w.r.t. θ, and get

$$\frac{dy}{d\theta} = b \cos \theta, \text{ and } \frac{dx}{d\theta} = a \sin \theta$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b \cos \theta}{-a \sin \theta} = -\frac{b}{a} \cot \theta$$

Try to apply this method now.

E 10) Find $\frac{dy}{dx}$ if

a) $a \cos \theta$, $y = a \sin \theta$
b) at^2, $y = 2at$
c) $x = a\cos^3 \theta$, $y = b\sin^3 \theta$
d) $x = a(\theta - \sin \theta)$, $= a(1 - \cos \theta)$

5.5.4 Derivatives of Implicit Functions

It is not always necessary to express y explicitly in terms of x (as in $y = f(x)$) to find its derivative. We shall now see how to differentiate a function defined implicitly by a relation in x and y (such as, $g(x, y) = 0$).

Example 11 Let us find $\frac{dy}{dx}$ if x and y are related by
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Differentiating throughout with respect to x, we get

$$2ax + 2h \,.\, 1 \,.\, y + 2hx \,.\, \frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

or $\frac{dy}{dx}(2hx + 2by + 2f) = -2ax - 2hy - 2g$

or $\frac{dy}{dx} = \frac{-(ax + hy + g)}{(hx + by + f)}$

See if you can find $\frac{dy}{dx}$ for the following implicit functions.

E E 11) Find $\frac{dy}{dx}$ if x and y are related as follows:

a) $x^2 + y^2 = 1$
b) $y^2 = 4ax$
c) $x^3y^3 + x^2y^2 + xy + 1 = 0$
d) $\cos x \cos y - y^2 \sin^{-1}x + 2x^2 \tan x = 0$

5.6 SUMMARY

In this unit we have

1. obtained derivatives of the exponential and logarithmic functions, hyperbolic functions and their inverses. We give them in the following table.

Function	Derivative
e^x	e^x
$\ln x$	$\frac{1}{x}$
a^x	$a^x \ln a$
$\log_a x$	$\frac{1}{x}\log_a e$
sinh x	cosh x
cosh x	sinh x
tanh x	$\text{sech}^2 x$
coth x	$-\text{cosech}^2 x$
sech x	–sech x tanh x
cosech x	–cosech x coth x

Function	Derivative
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$, $x > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$, $\|x\| < 1$
$\coth^{-1} x$	$\frac{1}{1-x^2}$, $\|x\| > 1$
$\text{sech}^{-1} x$	$\frac{1}{x\sqrt{1-x^2}}$, $0 < x < 1$
$\text{cosech}^{-1} x$	$\frac{1}{\|x\| x\sqrt{1-x^2}}$, $x \neq 0$.

2. extended the result $\frac{d}{dx}(x^r) = rx^{r-1}$ to all $x \in \mathbf{R}$ and $x > 0$.

3. illustrated
   logarithmic differentiation,
   differentiation of functions involving parameters and
   differentiation of functions given by implicit relations.

5.7 SOLUTIONS AND ANSWERS

E 1) a) $5e^{(x^2-2)}(2x)$ b) $e^{(x+1)/x}(-1/x^2)$

c) $(x+2)\,e^{\sqrt{x}}\,\frac{1}{2}x^{-1/2}+e^{\sqrt{x}}$

d) $e^{-m\tan^{-1}x}\left(\frac{-m}{1+x^2}\right)$

e) $2^{2x+1}\ln 2$

f) $7^{\cos x}(-\sin x)\ln 7$

E 2) $f'(x) = 2^x \ln 2$ $f'(0) = \ln 2$

$f'(1/2) = 2^{1/2}\ln 2 = \sqrt{2}\ln 2$

Hence f increases $\sqrt{2}$ times faster at $x = 1/2$ than at $x = 0$.

E 3) a) $\frac{1}{x}\log_2 e$

b) $7\log_{11} e\left(\frac{10x}{5x^2+2}\right)$

c) $e^{-x}(1/x) - e^{-x}\ln x$

d) $\frac{1-x}{1+x}\left[\frac{(1-x)+(1+x)}{(1-x)^2}\right]$

$= \frac{2}{(1-x)(1+x)}$

e) $\frac{1}{\sin^4 x}4\sin^3 x\cos x$

E 4) a) $\cosh^2 x = \frac{e^{2x}+e^{-2x}+2}{4}$, $\sinh^2 x = \frac{e^{2x}+e^{-2x}-2}{4}$

$\cosh^2 x - \sinh^2 x = 1$

E 5) $\coth^2 x - 1 = \frac{e^{2x}+e^{-2x}+2}{e^{2x}+e^{-2x}-2} - 1 = \frac{4}{e^{2x}+e^{-2x}-2} = \text{cosech}^2 x$

E 6) a) $\frac{4}{5}\text{sec h}^2\left[\frac{4x+1}{5}\right]$ b) $2e^{2x}\cosh e^{2x}$

c) $\frac{1}{x^2}\text{cosec h}^2\left(\frac{1}{x}\right)$ d) $-\text{sec h}\ln x\tanh\ln x\cdot\frac{1}{x}$

e) $e^x(\sinh x+\cosh x)$

E 7) a) $\frac{-1}{5\sqrt{x}\sqrt{1+25x}}\left(\frac{5}{2\sqrt{x}}\right) = \frac{-1}{2x\sqrt{1+25x}}$

b) $\frac{1}{3}[\text{sec h}^{-1}(\cos^2 x)]^{-2/3}\left(\frac{1}{\cos^2 x\sqrt{1-\cos^4 x}}\right).2\cos x\sin x$

c) $\frac{-1}{e^{2(x^2+5x-6)}-1}e^{(x^2+5x-6)}(2x+5)$

d) $\frac{-\text{cosech}^2 x}{1-\coth^2 x} - \frac{2}{4x^2-1}$

e) $\frac{1}{2\sqrt{x}\sqrt{1+x}} - \frac{4}{\sqrt{4x^4-1}}$

E 8) a) $\sqrt{2}x^{\sqrt{2}-1}$ b) ex^{e-1}

E 9) a) $\ln|y| = \ln|x^2-1| + 6\ln|x^2+2| + 5\ln|x^3-1|$

$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2-1}+\frac{12x}{x^2+2}+\frac{5x^2}{x^3-1}$

$\frac{dy}{dx} = (x^2-1)(x^2+2)^6(x^3-1)^5\left[\frac{2x}{x^2-1}+\frac{12x}{x^2+2}+\frac{5x^2}{x^3-1}\right]$

b) $\ln|y| = -5\ln|x-1| - 6\ln|x-2| - 7\ln|x-3|$

$\frac{dy}{dx} = \frac{-1}{(x-1)^5(x-2)^6(x-3)^7}\left(\frac{5}{x-1}+\frac{6}{x-2}+\frac{7}{x-3}\right)$

c) Then $f'(x) = \sin x^x (\ln \sin x + x \cot x)$ and $g'(x) = \cos x^{\tan x}$

$(\sec^2 x \ln \cos x - \tan^2 x)\frac{dy}{dx} = f'(x) + g'(x)$

d) Let $f(x) = (x^x)^x$, $g(x) = x^{(x^x)}$, $x > 0$
If $y = x^x$, $\ln y = x\ln$

$\Rightarrow \frac{dy}{dx} = x^x (1 + \ln x)$

$\ln f(x) = x \ln x^x$

$\Rightarrow \frac{1}{f(x)} f'(x) \ln x^x + x(1 + \ln x)$

$\Rightarrow f'(x) = (x^x)^x [\ln x^x + x(1 + \ln x)]$

$\ln g(x) = x^x \ln x$

$\Rightarrow \frac{1}{g(x)} g'(x) = \frac{x^x}{x} + \ln x\, x^x (1 + \ln x)$

$\Rightarrow g'(x) = x^{(x)} [x^{x-1} + x^x \ln x (1 + \ln x)]$

Answer $= f'(x) + g'(x)$

$= (x^x)^x [\ln x^x + x(1 + \ln x)] + x^{(xx)} [x^{x-1} + x^{x-1} \ln x (1 + \ln x)]$

e) $\frac{d}{dx}(\sin x)^{\ln x} = (\sin x)^{\ln x}\left(\ln x \cot x + \frac{\ln \sin x}{x}\right)$

$\frac{d}{dx}(x^x) = x^x (1 + \ln x)$

Answer $= (\sin x)^{\ln x}\left(\ln x \cot x + \frac{\ln \sin x}{x}\right) + x^x (1 + \ln x)$

E 10) a) $\frac{dx}{d\theta} = -a \sin\theta$, $\frac{dx}{d\theta} = a \cos\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\cot\theta$

b) $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

c) $\frac{dy}{dx} = \frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{b}{a}\tan\theta$

d) $\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{(1-\cos\theta)}$

E 11) a) $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

b) $2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

c) $3x^3y^2\frac{dy}{dx} + 3x^2y^3 + 2x^2y\frac{dy}{dx} + 2xy^2 + x\frac{dy}{dx} + y = 0$

$\Rightarrow (3x^3y^2 + 2x^2y + x)\frac{dy}{dx} = -(3x^2y^3 + 2xy^2 + y)$

$\frac{dy}{dx} = -\frac{(3x^2y^3 + 2xy^2 + y)}{(3x^3y^2 + 2x^2y + x)}$

d) $-\cos x \sin y\frac{dy}{dx} - \sin x \cos y - 2y\frac{dy}{dx}\sin^{-1}x - \frac{y^2}{\sqrt{(1-x^2)}} + 4x\tan x$
$+ 2x^2\sec^2 x = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x\cos y + \frac{y^2}{\sqrt{(1-x^2)}} - 4x\tan x - 2x^2\sec^2 x}{-\left(\cos x\sin y + 2y\sin^{-1}x\right)}$$