# UNIT 3 PARABOLOIDS

## Structure



## 3.1 INTRODUCTION

In the previous unit we discussed central conicoids. In this unit, which is the last unit of this course, we look at non-central conicoids. You are already familiar with one type of non-central conicoid, namely, a cylinder. Here, we discuss another such surface, called a paraboloid.

We shall start this unit with a discussion on the standard forms of a paraboloid. You will see that paraboloids can be divided into two types : elliptic and hyperbolic paraboloids. In Sec. 3.3, we discuss the shapes of the two types of paraboloids. The last section contains a brief discussion on the intersection of a paraboloid with a line and intersection with plane.

Like central conicoids, paraboloid s are also used in various fields. The most commonly found paraboloidal surfaces are dish antennas, which most of us are familiar with. You can see some more applications in the unit.

The way this unit unfolds is the same as the previous one, except for one difference. In this unit we assume that you have enough experience by now to bring out many properties of the surface by yourself. Accordingly, you will find that we have left most results to you to prove.

Now, please go through the following list of objectives. If you achieve these, then you can be sure that you have grasped the contents of this unit.

### Objectives

After studying this unit you should be able to

- check whether a given equation of a conicoid represents an elliptical paraboloid or a hyperbolic paraboloid;
- trace the standard elliptic or hyperbolic paraboloid;
- obtain the tangent lines and tangent planes to a standard paraboloid.

## 3.2 STANDARD EQUATIONS OF A PARABOLOID

In this section we shall obtain the standard equations of a non-central conicoid. Then we shall define a paraboloid and discuss its standard equations.

To begin with, let us go back to Theorem 4 in Unit 7 for a moment: According to this theorem any second degree equation can be reduced to an equation of the form
$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ ...(1)

Now let us assume that (1) represents a non-central conicoid. Since the conicoid has no centre, by Theorem 2 in Unit 8 we find that either

i) exactly two of the a, b and c are zero, or
ii) only one of the a, b and c is zero.

Let us look at these cases separately.

We first consider the case (i). Let us assume that $a = 0$, $b = 0$ and $c \neq 0$. (We can deal with the cases $a, c = 0, b \neq 0$; $b, c = 0, a \neq 0$ similarly.) In this case (i) becomes $cz^2 + 2ux + 2vy + 2wz + d = 0$

$$\Rightarrow c\left(z + \frac{w}{c}\right)^2 = -2ux - 2vy - d + \frac{w^2}{c}.$$

By shifting the origin to $\left(0, 0, -\frac{w}{c}\right)$, we see that the equation takes the form

$cZ^2 + 2uX + 2vY + d_0 = 0$,
where X, Y, Z, are the coordinates in the new system. What does this equation represent? Let's see.

If both u and v are zero, then the surface represents a pair of lines.

If one of the coefficients u and v is non-zero, say $v \neq 0$ and $u = 0$, then you can see that the surface is built up of a series of parabolas along a line parallel to the x-axis. Thus, it is a parabolic cylinder. In fact, even if both u, v are non-zero, the surface is a parabolic cylinder.

Let's now go to case (ii)
Here we assume that $a = 0$ and $b, c \neq 0$. (We can deal with the other two cases $b = 0, c, a \neq 0$; and $c = 0, a, b \neq 0$ similarly.)

In this case (1) reduces to the form
$by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$,

$$\text{i.e., } b\left(y + \frac{v}{b}\right)^2 + c\left(z + \frac{w}{c}\right)^2 = -2ux - d + \frac{v^2}{b} + \frac{w^2}{c}$$

$$= -2ux + d_1, \text{ where } d_1 = \frac{v^2}{b} + \frac{w^2}{c} - d$$

By shifting the origin to $\left(0, -\frac{v}{b}, -\frac{w}{c}\right)$, we see that the, above equation takes the form

$bY^2 + cZ^2 + 2ux + d_1 = 0$ ...(2)
where X, Y, Z are coordinates in the new system.

For example, $y^2 + 10z^2 = 2$ and $2y^2 + z^2 = 12x$ represent non-central conicoids. But is there a difference in the type of conicoid represented by them? Let's see.

Suppose that $u = 0$ in (2). In this case we get $by^2 + cz^2 + d = 0$. Do you recognize the surface given by this equation? It represents a cylinder or a pair of planes. We have already discussed these surfaces in detail in Block 2.

Now, let us assume that $u \neq 0$. Then we rewrite (2) in the form
$by^2 + cz^2 = -2ux - d$

i.e., $by^2 + cz^2 = 2u'\left(x - \frac{d}{2u'}\right)$ where $u' = -u$.

Now, by translating the origin to the point $\left(\frac{d}{2u'}, 0, 0\right)$, the equation reduces to

$$bY^2 + cZ^2 = 2u'X \qquad ...(3)$$

Do you agree that this equation is a three-dimensional version of the standard equation of a parabola? We call this surface a paraboloid.

So, for example, the equation $2y^2 + z^2 = 12x$ represents a paraboloid.

What are the other forms of an equation of a paraboloid? We leave this as an exercise for you (see E1).

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**E1)** Discuss what happens to (1) in the following cases:
a) $b = 0$, $a, c \neq 0$.
b) $c = 0$, $a, b \neq 0$.

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If you've done E1, you must have found that there are two more types of equations which represent a paraboloid, namely

$$ax^2 + by^2 = 2wz, \text{ and } ax^2 + cz^2 = 2vy \qquad ...(4)$$

Now let us look at the coefficients of these equations more closely, as we did in the case of central conicoids. Let us consider (4). We have the following two cases:

**Case 1** (a and b are of the same sign): Suppose a and b are positive. Let $a_1 = \sqrt{a}$ and $b_1 = \sqrt{b}$ then (4) becomes

$$\frac{x^2}{1/a_1^2} + \frac{y^2}{1/b_1^2} = 2wz$$

Similarly, if a and b are negative, we can write (4) in the above form.
Thus, when a and b are of the same sign, (4) reduces to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2wz \qquad ...(5)$$

The paraboloid represented by this equation is called an **elliptic paraboloid.**

**Case 2** (a and b are of opposite signs): In this case you can see that (4) reduces to the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2wz \qquad ...(6)$$

The surface represented by this equation is called **hyperbolic paraboloid.**

If you do E2, you will see why the adjectives 'elliptic' and 'hyperbolic' are appropriate.

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**E2)** Show that the intersection with any plane parallel to the XY-plane of the paraboloid
i) $x^2 + 2y^2 = 3z$ is an ellipse.
ii) $3x^2 - y^2 = 4x$ is a hyperbola.

**E3)** Check whether the following equations represent a paraboloid or not? for those that do, classify the paraboloids as elliptic or hyperbolic.
a) $4y^2 - 4z^2 - 2x - 14y - 22z + 33 = 0$
b) $x^2 + y^2 + z^2 - 2x + 4y = 1$
c) $4x^2 - y^2 - z^2 - 8x - 4y + 8z - 2 = 0$
d) $9x^2 + 4z^2 - 36 = 0$
e) $2x^2 + 20y^2 + 22x + 6y - 2z - 2 = 0$

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From E2 you may have already realised that the two types of paraboloids are geometrically different. Let us now see whether there are more differences.

## 3.3 TRACING PARABOLOIDS

In this section we shall discuss the geometry of the two types of paraboloids, and see how to trace their standard forms. We shall trace an elliptic paraboloid here and leave the tracing of a hyperbolic paraboloid as an exercise for you.

So, let us consider (5), the standard equation of an elliptic paraboloid. We can observe some geometrical poperties, similar to the properties you have seen in Unit 3 for an ellipsoid or hyperboloid.

In E4 we have asked you to obtain them, using the knowledge you have gained in previous units.

**E4)** Check whether the surface (5) is symmetrical about the coordinate planes.

**E5)** Do all the coordinates axes intersect the surface (5)? If so, what are their intersections?

**E6)** Obtain the intersections of the surface (5) with the XZ-and YZ-planes.

If you've done E6, then you must have realised why this surface is called a paraboloid. You already know why the surface is called elliptic from E2. You may be more convinced about this fact if you look at the following property.

Let us look at sections of the surface by the plane $z = k$, where k is a constant. It is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2wk \qquad ...(7)$$

The left hand side of (7) is positive for all values of x and y. Therefore, w and k must be of the same sign. So, if $w > 0$, then $k > 0$. In this case (7) represents an ellipse (or circle if $a = b$) with centre at (0, 0, k) on the positive direction of the z-axis. Note that the size of the ellipse increases as k increases.

If $k = 0$, the plane $z = 0$ just touches the surface at the point (0, 0, 0).

If $k < 0$, the plane $z = k$ does not intersect the surface. Therefore, no portion of the surface (5) lies below the plane $z = 0$. We have drawn the surface in Fig. 1(a).

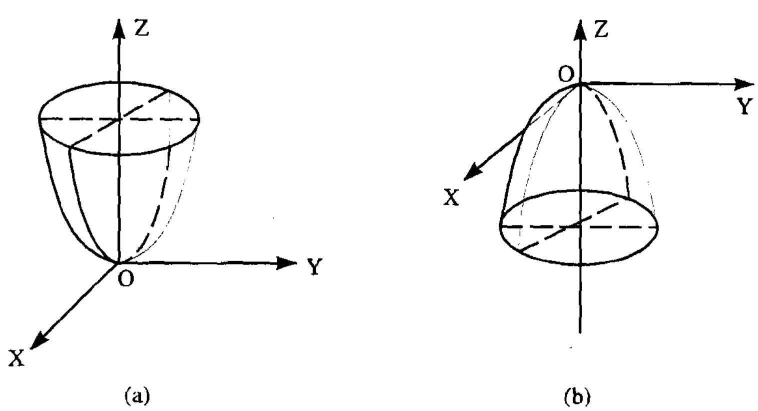


Fig. 1: The elliptic Paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2wk$ in case (a) $w > 0$, (b) $w < 0$

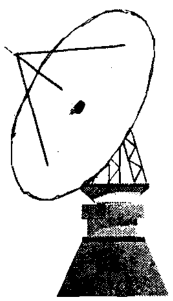

Fig. 2: Antenna in the shape of a circular paraboloid.

Now what if $w < 0$ in (7)? Then k has to be negative. In this case we get an ellipse whose centre lie on the negative direction of the z-axis. As above, you can see that no portion of (5) lies above the plane $z = 0$. We have drawn the surface in Fig. 1(b). Now you know geometrically why this surface is called an elliptic paraboloid.

Why don't you try to trace some paraboloids on your own now?

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**E7)** Trace the paraboloid given by

a) $x^2 + y^2 = z$; and describe its sections obtained by the planes $x = 0$ and $y = 0$.

b) $y^2 + 4z^2 = x$ and describe its sections by the planes $y = 0$ and $z = 0$.

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The paraboloid you got in E7 (a) is called a **circular paraboloid.** There you can see that the plane section of the surface by the plane $x = 0$ is a parabola with focus at the point (0, 0, 1/4). When we revolve this parabola about the z-axis, we get the surface you have traced in E7 (a). Therefore, we also call this surface a **paraboloid of revolution.**

Paraboloids of revolution have many applications. Circular paraboloids are used for dish antennas and antennas in radio telescopes (see Fig. 2). This is because of the property that of all the paraboloids having the same area, a circular paraboloid has he largest reflecting surface.

Circular paraboloids are also used for satellite trackers and microwave radio links.

Now let us consider the hyperbolic paraboloid given by (6), that is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2wz.$$

As in the case of the elliptic paraboloid, we have two cases; $w < 0$ and $w > 0$. We shall restrict our discussion to $w < 0$. (Exactly similar properties hold for the case $w > 0$.) In this case we ask you to find out the properties of the following exercises.

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**E8)** a) What are the properties of a hyperbolic paraboloid which are analogous to those obtained by you in E4 for an elliptic paraboloid?

b) What are the sections of the paraboloid (6) with $z = k$, $k < 0$, and $k > 0$.

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In E8 (b) you must have observed that the section of a hyperbolic paraboloid by the plane $z = k$ ($k \neq 0$) is the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2wk$. This hyperbola is real for all non-zero values of k ($\neq 0$), positive or negative.

If $k > 0$, it will have its transverse axis parallel to the x-axis, and if $k < 0$, its transverse axis will be parallel to the y-axis. You can see one branch of the hyperbola in Fig. 3(a).

In Fig. 3 (b) you can see the parabolas which are sections of the paraboloid by the planes $x = 0$ and $y = 0$.

You can also observe that for $k > 0$, the length of the semi-transverse axis of the hyperbola is $\sqrt{2\,k\,a}$, which increases as k increases. Similarly, for $k < 0$, the length increases as $|k|$ increases.

You can now try these exercises.

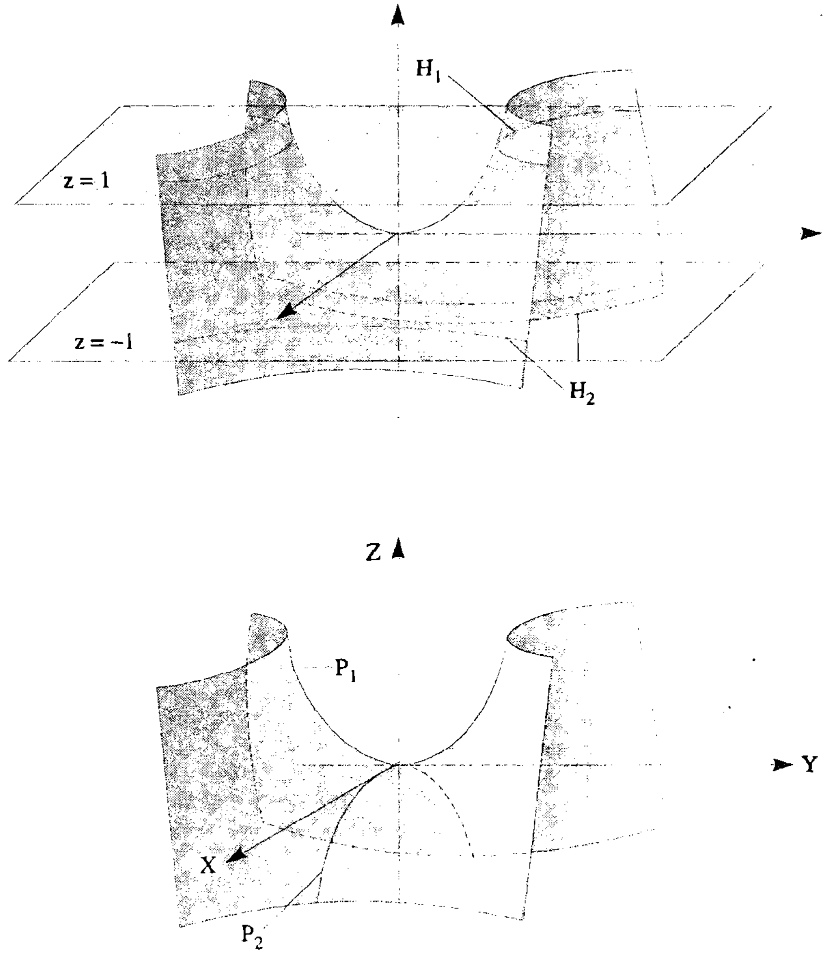


**Fig. 3: The planar section of the hyperbolic paraboloid** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2wz$ **by the planes (a) z = 1 and z = −1, are the hyperbolas** $H_1$ **and** $H_2$ **(b) x = 0 and y = 0, are the parabolas** $P_1$ **and** $P_2$.

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**E9)** Consider the hyperbolic paraboloid given by
$x^2 - 2y^2 = z$
a) What are its sections by the planes $x = 0$ and $y = 0$?
b) What are its sections by the planes $z = 0, \pm 1$?
c) Sketch the surface described by the given equation.

**E10)** Sketch the surface given by the equation $2z^2 - y^2 = x$.

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So far we have discussed how to trace the standard paraboloids. For this purpose we considered their intersection with planes parallel to the coordinate planes. Now let us discuss the intersection of the paraboloid with a general plane, as well as with a line.

## 3.4 INTERSECTION WITH A LINE OR A PLANE

In this section we shall discuss some results for paraboloid similar to those obtained for central conicoids in Sec. 2.7 of Unit 2. We shall present some of these result in the form of exercises for you to do.

Let us start with an exercise on the analogue of Theorem 1 in Unit 2 for the paraboloid given by

$$ax^2 + by^2 = 2z. \qquad ...(8)$$

**E11)** Prove that a line intersects a paraboloid at two points which may be real or imaginary.

What can you say about the intersection of a line parallel to the z-axis with the elliptic paraboloid (8) (when a and b are of the same sign)? Let us go back for a moment to Fig. 1. There you can see that there is only one real point of intersection on the paraboloid. Let us see what happens to the intersections with the x- and y-axes. Again from Fig. 1 and E7, we observe that the lines just **touch** the paraboloid, that is, the points of intersection are coincident. You know that such lines are called **tangent lines.** As in the case of central conicoids, the set of all tangent plane lines at a point of the surface is a plane, called the **tangent plane**. You should be able to write the equation of the tangent plane from the corresponding results on central conicoids. In fact, this is what the following exercises are about.

**E12)** Prove that the condition for a line with direction ratios $\alpha$, $\beta$, $\gamma$ through the point $(x_0, y_0, z_0)$ to be a tangent to the paraboloid (8) is $ax_0\alpha + by_0\beta - \gamma = 0$.

**E13)** Prove that the equation of the tangent plane at a point $(x_0, y_0, z_0)$ on the paraboloid (8) is
$axx_0 + byy_0 = (z + z_0)$.

**E14)** a) Prove that the plane $ux + vy + wz = p$ will be a tangent plane to the paraboloid (8) iff

$$\frac{u^2}{a} + \frac{v^2}{b} + 2pw = 0 \qquad ...(9)$$

b) Obtain the point of contact.

If you have done E14, you know that the point of contact is that situation will be $\left(\frac{-u}{aw}, \frac{-v}{bw}, \frac{-p}{w}\right)$

Let us consider an example.

**Example 1:** Show that the plane $8x - 6y - z - 5 = 0$ touches the paraboloid $\frac{x^2}{2} - \frac{y^2}{3} = z$, and find the point of contact.

**Solution:** Let us check the condition given in (9). Here $a = 1$, $b = -\frac{2}{3}$, $u = 8$, $v = -6$, $w = -1$, $p = 5$.

Substituting in (9), we get
$64 - 54 - 10 = 0$, which is true.
Therefore, the plane touches the paraboloid.
The point of contact is (8, 9, 5).

In the following exercises we ask you to apply what you have proved in E13 and E14.

**E15)** Find the equation of tangent plane to the given conicoid at the indicated point
a) $x^2 + y^2 = 4z$, (2, –4, 5)
b) $x^2 - 3y^2 = z$, (3, 2, –3).

**E16)** Show that the plane $2x - 4y - z + 3 = 0$ touches the paraboloid $x^2 - 2y^2 = 3z$, and find the point of contact.

Let us now see what the intersection of a paraboloid with a general plane is. Consider the following theorem, which is analogous to Theorem 4 in Unit 2. We will not be doing its proof here, but leave it as an exercise for you to do if you are interested in proving it. (See Miscellaneous Exercises.)

**Theorem 4**

a) The section of a paraboloid $ax^2 + by^2 = 2z$ by a plane $ux + vy + wz = p$ is a conic.

b) If $w = 0$, the section is always a parabola.

c) If $w \neq 0$, then the section is
   i) a hyperbola if a and b are of opposite signs,
   ii) a parabola if at least one of a and b is zero,
   iii) an ellipse if a and b are of the same sign.

In Fig. 4 we have diagrammatically illustrated some particular cases.

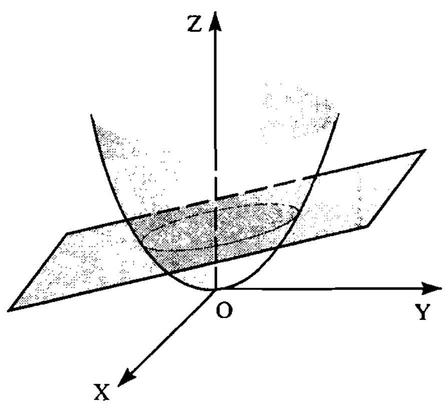


**Fig. 4: Planar section of $ax^2 + by^2 = 2z$ by the plane $us + vy + wz = p$ when $u, v, w, p \neq 0$.**

In Fig. 4 you can see the elliptical section.

Why don't you try an exercise now?

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**E17)** Sketch the sections of the following conicoids by a plane perpendicular to the XY-plane.
   a) $3x^2 - y^2 = z$.
   b) $2x^2 + y^2 = z$.

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We shall end this unit now with a brief review of what we have covered in it.

## 9.5 SUMMARY

In this unit we have discussed the following points.

1) The standard form of a non-central conicoid is $ax^2 + by^2 + 2wz + d = 0$. If $w = 0$, the equation represents a cylinder or a pair of straight lines. If $w \neq 0$, the surface represented by the equation is called a paraboloid.

2) The standard equation of a paraboloid is
   $ax^2 + by^2 = 2wz,\ w \neq 0$.
   There are two types of paraboloids.
   When a and b are of the same signs, we get an elliptic paraboloid.
   When a and b are of opposite signs, we get a hyperbolic paraboloid.

3) How to trace an elliptic paraboloid and a hyperbolic paraboloid.

4) The condition for a line with direction ratios $\alpha$, $\beta$, $\gamma$ to be a tangent to the central conicoid $ax^2 + by^2 = 2z$ at $(x_0, y_0, z_0)$ is $ax_0\alpha + by_0\beta = \gamma$.

5) The equation of the tangent plane to the paraboloid $ax^2 + by^2 = 2z$ at a point $(x_0, y_0, z_0)$ is $axx_0 + byy_0 = (z + z_0)$.

6) The condition that the plane $ux + vy + wz = 0$ is a tangent plane to the paraboloid $ax^2 + by^2 = 2z$ is

$$\frac{u^2}{a} + \frac{v^2}{b} + 2wp = 0$$

7) The planar section of a paraboloid is a conic section.

Now you may like to go back to Sec. 3.1 to see if you've achieved the objectives listed there. You must have solved the exercises as you came to them in the unit. In the next section we have given our answers to the exercises. You may like to have a look at them.

## 3.6 SOLUTIONS/ANSWERS

E1) a) Putting $b = 0$, $a, c \neq 0$ in (1) we get
$ax^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$.

$$\text{i.e., } a\left(x + \frac{u}{a}\right)^2 + c\left(z + \frac{w}{c}\right)^2 = -2vy - d + \frac{u^2}{a} + \frac{w^2}{c}$$

$$= -2vy + d_2, \text{ where } d_2 = \frac{u^2}{a} + \frac{w^2}{c} - d$$

By shifting the origin to $\left(-\frac{u}{a}, 0, -\frac{w}{c}\right)$, the above equation reduces to the form
$aX^2 + cZ^2 + 2vY + d = 0$.
where X, Y, Z denote the coordinates in the new system.

b) Similarly, putting $c = 0$, $a, b \neq 0$, in (1) we get that the equation reduces to the form
$aX^2 + bY^2 + 2wZ + d = 0$,
where, X, Y, Z denote the coordinate in the system obtained by shifting the origin to $\left(-\frac{u}{a}, -\frac{u}{b}, 0\right)$

E2) Any plane parallel to the XY-plane is of the form $z = k$, where k is a constant, $k \neq 0$.

i) The given ellipsoid is $x^2 + 2y^2 = 3z$.
Putting $z = k$ in this equation, we get

$$\frac{x^2}{3k} + \frac{y^2}{2k} = 1,$$

which represents an ellipse.

ii) Similarly, putting $z = k$ in the equation of the hyperboloid, we get

$$\frac{x^2}{4/3k} - \frac{y^2}{4k} = 1,$$

which is a hyperbola

E3) (a) and (e) represent a paraboloid. (b), (c) and (d) do not represent a paraboloid. (a) represents a hyperbolic paraboloid, whereas (e) represents an elliptic paraboloid.

E4) Surface (5) is symmetric about YZ and ZX-planes. It is not symmetric about the XY-plane.

E5) Yes. When we put y = 0 and z = 0 in (5), we see that x = 0. Thus, (0, 0, 0) is the only point of intersection of the x-axis with (5). Similarly, we can show, that (0, 0, 0) is the only point of intersection with the y and z-axes.

E6) a) Putting z = 0 in (5), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0.$$

The only point which satisfies the above equation is (0, 0, 0). Therefore the XY-plane intersects the surface in the point (0, 0, 0).

b) Putting y = 0 in (5) we get

$$\frac{x^2}{a^2} = 2wz,$$

i.e., $x^2 = 2a^2wz$,
which represents a parabola.

c) Similarly, the YZ-plane intersects the surface in a parabola.

E7) a)

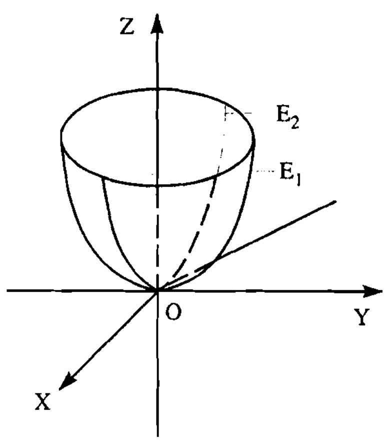



Fig. 5: The parabolas $E_1$ and $E_2$ are the sections obtained by intersecting the elliptic paraboloid with the planes x = 0 and y = 0, respectively.

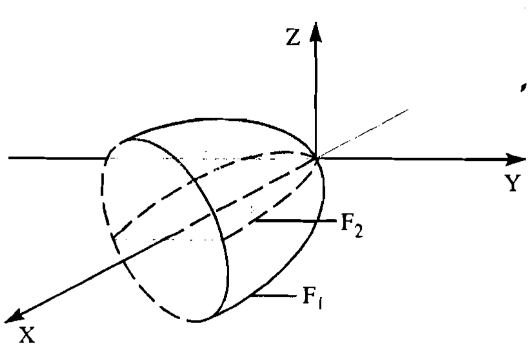



Fig. 6: $F_1$ and $F_2$ are the sections obtained by intersecting the elliptic paraboloid with the planes y = 0 and z = 0, respectively.

E8) The surface has the following properties.

i) It is symmetrical about the XZ-plane and the YZ-plane.

ii) The coordinate axes intersects the surface in the point (0, 0, 0).

iii) By putting $z = 0$ in (6), we see that the XY-plane intersects the surface in two lines.

$$y = \pm \frac{b}{a} x.$$

Similarly, by putting $x = 0$ in (6), we get
$y_2 = -2b^2wz$,
which represents a parabola, i.e., the intersection with the YZ-plane is a parabola.

The intersection of (6) with the ZX-plane also is a parabola given by the equation
$x^2 = 2a^2wz$.

b) Putting $z = k$ in (6), we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2wk$$

When $k < 0$ and $k > 0$, this represents a hyperbola.

When $k = 0$, this represents a pair of lines $y = \pm \frac{b}{a} x$.

E9) a) The section by the plane $x = 0$ is the parabola

$$y^2 = -\frac{1}{2} z \text{ (see Fig. 7)}$$

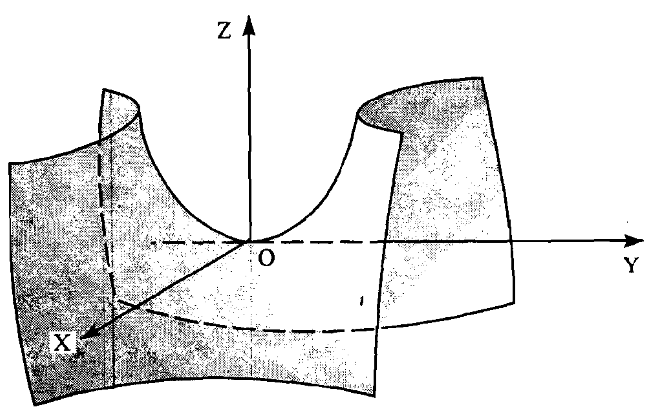
The section by the plane $y = 0$ is the parabola
$x^2 = z$ (see Fig. 7)

b) The section by the plane $z = 0$ is the pair of lines

$$y = \pm \frac{1}{2} x.$$

The section by the plane $z = 1$ is the hyperbola
$x^2 - 2y^2 = 1$.
and the section by the plane $z = -1$ is the hyperbola
$2y^2 - x^2 = 1$.

This shows that the set of all tangent lines i.e., tangent plane is given by the equation.
$axx_0 + byy_0 = z + z_0$.



E10) The figure is similar to the figure in E9 with a change of the coordinate axes.

E11) The equation of a line through $(x_0, y_0, z_0)$ with direction cosines $\alpha$, $\beta$, $\gamma$ is

$$\frac{x-x_0}{\alpha}=\frac{y-y_0}{\beta}=\frac{z-z_0}{\gamma}$$

Any point on this line is of the form $(\alpha r + x_0, \beta r + y_0, \gamma r + z_0)$ for some r.
When this line meets the paraboloid $ax^2 + by^2 = 2z$, we have
$a(\alpha r + x_0)^2 + b(\beta r + y_0)^2 = 2\alpha(\gamma r + z_0)$,
i.e., $(a\alpha^2 + b\beta^2)r^2 + 2r(a\alpha x_0 + b\beta y_0 - \gamma) + ax_0{}^2 + by_0{}^2 - 2z_0 = 0$ ...(10)

This is a quadratic equation in r which gives two values of r, which may be real or imaginary. Hence the result.

E12) The equation of the line L passing through $(x_0, y_0, z_0)$, having direction ratios $\alpha$, $\beta$, $\gamma$ is

$$\frac{x-x_0}{\alpha}=\frac{y-y_0}{\beta}=\frac{z-z_0}{\gamma}$$

In E13 we saw that this line L meets the conicoid in two points, which may be real and distinct, real and coincident or imaginary.

If the line is a tangent to the conicoid at the point $(x_0, y_0, z_0)$, then the points of intersection conicide. That is, (0), in (E11) has real coincident roots.

The condition for this is
$a\alpha x_0 + b\alpha y_0 - \gamma = 0.$ ...(11)
Note that since $(x_0, y_0, z_0)$ lies on the conicoid, we have $ax_0{}^2 + by_0{}^2 - 2z_0 = 0$.

E13) The conicoid is $ax^2 + by^2 = 2z$ ...(12)

We know that the tangent plane at $(x_0, y_0, z_0)$ is the set of all tangent lines at $(x_0, y_0, z_0)$. Let us assume that the line

$$\frac{x-x_0}{\alpha}=\frac{y-y_0}{\beta}=\frac{z-z_0}{\gamma}$$

where $\alpha$, $\beta$, $\gamma$ are the direction ratios of the line, is a tangent to the conicoid (11).

Eliminating $\alpha$, $\beta$, $\gamma$ between (11) and (12), we get
$ax_0(x - x_0) + by_0(y - y_0) - (z - z_0) = 0$
$axx_0 + byy_0 - (ax_0{}^2 + by_0{}^2) = z - z_0$
$axx_0 + byy_0 - 2z_0 = z - z_0$, since $ax_0{}^2 + by_0{}^2 = 2z_0$
$axx_0 + byy_0 = z + z_0$.

E14) By E13, the plane $ux + vy + wz = p$ will be a tangent plane to the paraboloid $ax^2 + by^2 = 2z$ if it is of the form
$axx_0 + byy_0 = z + z_0$
for some point $(x_0, y_0, z_0)$ on the paraboloid. That means, we have

$$\frac{ax_0}{u}=\frac{by_0}{v}=\frac{-1}{w}=\frac{z_0}{p}$$

$$\Rightarrow x_0=-\frac{u}{aw},\ y_0=-\frac{v}{bw},\ z_0=-\frac{p}{w},$$

Since $(x_0, y_0, z_0)$ lies on the conicoid $ax^2 + by^2 = 2z$, we get

$$a\left(\frac{u^2}{a^2w}\right)+b\left(\frac{v^2}{b^2w}\right)=-\frac{2p}{w}$$

$$\text{i.e., } \frac{u^2}{aw}+\frac{v^2}{bw}=-2p$$

i.e., $\frac{u^2}{a} + \frac{v^2}{b} + 2pw = 0$ ...(13)

b) The point of contact is given by $\left(\frac{-u}{aw}, \frac{-v}{pw}, \frac{-p}{w}\right)$

E15) Equation of the tangent plane is $axx_0 + byy_0 = z + z_0$ ...(14)

a) The given equation can be written as

$$\frac{x^2}{2} + \frac{y^2}{2} = 2z$$

So, in this case $a = \frac{1}{2} = b$ and $x_0 = 2$, $y_0 = -4$ and $z_0 = 5$.

Then, we have from (14)

$$\frac{1}{2} \times x \times 2 + \frac{1}{2} \times y \times (-4) = z + 5$$

i.e., $x - 2y - z = 5$.

b) $6x - 12y - z + 3 = 0$.

E16) The given conicoid can be rewritten as

$$\frac{2x^2}{3} - \frac{4y^2}{3} = 2z.$$

The given plane is $2x - 4y - z = -3$. The condition that this plane touches the paraboloid is given by (13) in E14. In this case $u = 2$, $v = -4$, $w = -1$,

$p = -3$, $a = \frac{2}{3}$, $b = \frac{4}{3}$c. Then, we have

$$\frac{u^2}{a} - \frac{v^2}{b} + 2pw = b - 12 + 6 = 0.$$

This shows that the plane touches the paraboloid.
The point of contact is given by (3, 3, –3).

E17) a)

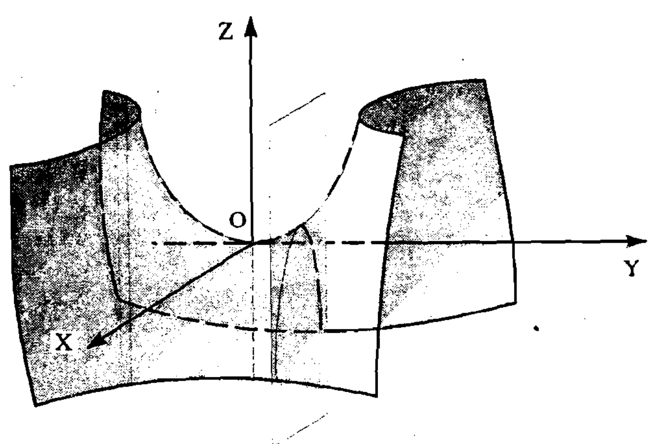



**Fig. 8: F is the section of the paraboloid $3x^2 - y^2 = z$ by a plane perpendicular to the XY-plane.**

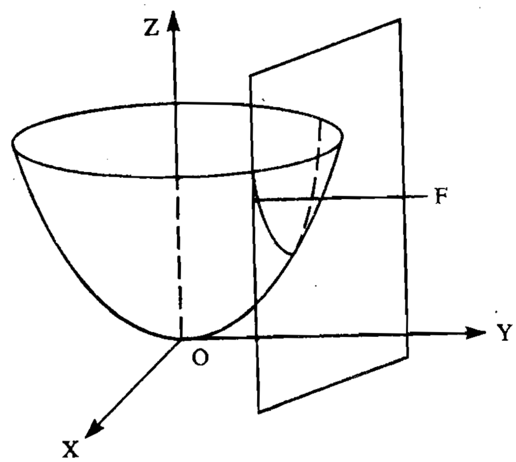



**Fig. 9: F is the section of the paraboloid $2x^2 + y^2 = z$ by a plane perpendicular to the XY-plane.**

# MISCELLANEOUS EXERCISES

**(This section is optional)**

In this section we have gathered some problems related to the contents of this block. You may like to do them to get a better understanding of these contents. Our solutions to the questions follow the list of problems, in case you'd like to counter-check your answers.

1) Which type of conicoids do the following equations represent?
   i) $x^2 - 16z^2 = 4y^2$
   ii) $5x^2 + 2y^2 - 6z^2 = 10$
   iii) $4y^2 = x$
   iv) $x^2 + 4y^2 + 16z^2 = 12$
   v) $2y^2 + x^2 = 4z$
   vi) $4x^2 - 3y^2 - 6z^2 = 10$
   vii) $x^2 + y^2 + z^2 = 4$
   viii) $2z^2 + x = y^2$
   ix) $4y^2 + 9x^2 - 36z^2 + 36 = 0$
   x) $25x^2 - 9y^2 = 225$

2) a) The hyperbola $\frac{z^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a > b)$ is rotated about the z-axis. What is the surface formed in this situation? Obtain the equation of the surface.

   b) Find the value of $x_0$ such that the plane $x = x_0$ intersects the surface obtained in (a) in a pair of straight dines.

3) a) A normal at a point P of a conicoid is the line through P which is perpendicular to the tangent plane at that point. Find the equation of a normal to a central conicoid $ax^2 + by^2 + cz^2 = 1$ at a point $(x_0, y_0, z_0)$.

   b) Using (a) obtain a normal at the point $\left(1, 1, \frac{1}{\sqrt{2}}\right)$ to the ellipsoid

   $$\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1.$$

4) Find the equation of a normal to a paraboloid $ax^2 + by^2 = 2z$ at any point $(x_0, y_0, z_0)$ on it.

5) Suppose that the XYZ-coordinate system is transformed into another coordinate system with the same origin and with the coordinate axes having direction cosines $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$ and $(0, 0, 1)$ with respect to the old system. What does the equation $xy = z$ represents in the new system?

6) Find the equations to the tangent plane of the given surfaces at the indicated points

   a) $x^2 + 2y^2 + 2z^2 = 5$ ; $(1, 1, 1)$

   b) $9x^2 + 4y^2 - 36z = 0$; $(2, -3, 2)$

   c) $x^2 + 4y^2 - 4z^2 - 4 = 0$ $(2, 1, 1)$

7) Prove that the section of a central conicoid by a given plane is a conic section. Further, if the conicoid is $ax^2 + by^2 + cz^2 = 1$ and the plane is $ux + vy + wz = p$, then prove that the section will be

i) an ellipse if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}>0$.

ii) a hyperbola if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}<0$.

iii) a parabola if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=0$.

8) a) Prove that the section of a paraboloid $ax^2 + by^2 = 2z$ by a plane $ux + vy + wz = 0$ is a conic section.

b) If $w = 0$, show that the section is a always a parabola.

c) If $w \neq 0$, then show that the section is

i) a hyperbola if a and b are of opposite signs

ii) a parabola if at least one of a and b is zero

iii) an ellipse if a and b are of the same sign.

9) a) Show that the intersection of the plane $y = 2$ and the ellipsoid $\frac{x^2}{9}+\frac{y^2}{9}+\frac{z^2}{16}=1$ is an ellipse.

b) Find the lengths of the semi-major axis and semi-minor axis, the coordinates of the centre, and the coordinates of the focii of the ellipse obtained in (a).

10) A tangent plane to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the coordinate axes in the points A, B, C. Prove that the centroid of the triangle ABC lies on the surface

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=9$. (A centroid of a triangle is the point of intersection of the medians of the triangle.)

**Answers**

1) i) Cone ii) hyperboloid of one sheet iii) cylinder iv) ellipsoid v) elliptic paraboloid vi) hyperboloid of two sheets vii) sphere viii) hyperbolic paraboloid ix) hyperboloid of two sheets x) cylinder.

2) a) The surface formed is a hyperboloid of one sheet. It's equation is given by

$$\frac{z^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2}{b^2}=1.$$

b) Putting $z = z_0$ in the above equation, we get

$$\frac{z_0^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2}{b^2}=1.$$

The equation represents a pair of straight lines only if $z_0 \pm a$.

3) a) The equation of the tangent plane at $(x_0, y_0, z_0)$ is
$axx_0 + byy_0 + czz_0 = 1$
where $ax_0$, $by_0$ and $xz_0$ are the direction ratios of the normal to the plane. This means that the direction ratios of any line perpendicular to the plane are $ax_0$, $by_0$ and $cz_0$. Since a normal to a conicoid at $(x_0, y_0, z_0)$ is a line perpendicular to the tangent plane and passes through $(x_0, y_0, z_0)$, its equation is given by

$$\frac{x-x_0}{ax_0}=\frac{y-y_0}{by_0}=\frac{z-z_0}{cz_0}$$

b) The equation of the normal to the ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at $(x_0, y_0, z_0)$ is

$$\frac{x-x_0}{\frac{x_0}{a^2}}=\frac{y-y_0}{\frac{y_0}{b^2}}=\frac{z-z_0}{\frac{z_0}{c^2}}$$

Here $x_0 = 1$, $y_0 = 1$, $z_0 = \frac{1}{\sqrt{2}}$, $a = 2$, $b = 2$, $c = 1$.

Then, we have

$$\frac{x-1}{1/4}=\frac{y-1}{1/4}=\frac{z-1/\sqrt{1}}{\frac{1}{\sqrt{2}}}$$

This is the equation of a normal at the point

$$\left(1, 1, \frac{1}{\sqrt{2}}\right)$$

4) $\frac{x-x_0}{ax_0}=\frac{y-y_0}{by_0}=\frac{z-z_0}{-1}$

5) The equations of the transformations are

$$x=\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'$$

$$y=\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'$$

$$z = z'$$

Substituting for x, y and z in the equation $xy = z$, we get

$$\left(\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'\right)\left(\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'\right)=z'$$

i.e., $\frac{x'^2}{2}-\frac{y'^2}{2}=z'$

which represents a hyperbolic paraboloid.

6) i) The equation represents an ellipse. Therefore the tangent plane is
$x + 2y + 2z = 5$.

ii) The equation represents an elliptic paraboloid. The equation of the tangent plane is
$3x - 2y - 3z = 4$.

iii) The equation represents an hyperboloid of one sheet.
The equation of the tangent plane is
$x + 2y - 2z = 0$.

7) Let the equation of the central conicoid be
$ax^2 + by^2 + cz^2 = 1$, $abc \neq 0$.

Suppose that the plane $ux + vy + wz - p = 0$ $(u \neq 0)$ intersects the conicoid.

$$ux + vy + wz - p = 0 \Rightarrow x = \frac{-vy - wz + p}{u}$$

Substituting for x in the given equation of the conicoid, we get

$$\frac{a}{u^2}(-vy - wz + p)^2 + by^2 + cz^2 = 1$$

$$\left(\frac{av^2}{u^2} + b\right)y^2 + \left(\frac{aw^2}{u^2} + c\right)z^2 + \frac{2aw^2}{u^2}yz - \frac{2avp}{u^2}y$$

$$-\frac{2awp}{u^2}z - \frac{a}{u^2}p^2 - 1 = 0.$$

This is a general second degree equation and therefore represents a conic section. Hence the result.

Let us now find the nature of the conic section,

i) You know from Block 1 Unit 3 that the above equation represents an ellipse if

$$\frac{av^2 + bu^2}{u^2} - \frac{aw^2 + cu^2}{u^2} - \frac{a^2v^2w^2}{u^4} > 0$$

i.e., $(av^2 + bu^2)(aw^2 + cu^2) - a^2v^2w^2 > 0$

i.e., $a^2v^2w^2 + acv^2u^2 + abu^2w^2 + bcu^4 - a^2v^2w^2 > 0$

i.e. $acv^2 + abw^2 + bcu^2 > 0$

since $abc \neq 0$, we can divide throughout the left hand side by abc. Then we get

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} > 0.$$

ii) Similarly, we can show that the section will be an hyperbola if

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} < 0.$$

iii) The section will be a parabola if

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = 0.$$

8) a) The given paraboloid is $ax^2 + by^2 = 2z$. The given plane is $ux + vy + wz = p$. Now $ux + vy + wz = p \Rightarrow wz = -4x - vy + p$.

Substituting for wz in the given equation of the conicoid, we get
$ax^2 + by^2 + 2(ux + vy - p) = 0$
i.e., $wax^2 + wby^2 + 2ux + 2vy - 2p = 0$.

This is a general second degree equation and therefore represents a conic section. The section will be a hyperbola, a parabola or an ellipse according as $w^2ab < 0$, $w^2ab = 0$, $w^2ab > 0$ respectively.

b) If $w = 0$, we get $w^2ab = 0$ and therefore in the case the section is always a parabola.

c) If $w \neq 0$, then the condition in (a) reduces to
$ab < 0$, $ab = 0$ or $ab > 0$

Thus, the section in this case will be

i) a hyperbola if $ab < 0$ i.e., a and b are of opposite signs.

ii) a parabola if $ab = 0$ i.e, at least one of **a and b is zero.**

iii) an ellipse if $ab > 0$, i.e., a and b are of the same sign.

9) a) Substituting for $y = 2$ in the given equation of the ellipsoid, we get

$$\frac{x^2}{9}+\frac{4}{9}+\frac{z^2}{16}=1$$

i.e., $\frac{x^2}{9}+\frac{z^2}{16}=1-\frac{4}{9}=\frac{5}{9}$

$$\Rightarrow \frac{x^2}{5}+\frac{9}{80}z^2=1$$

$$\Rightarrow \frac{x^2}{5}+\frac{z^2}{80/9}=1.$$

This represents an ellipse.

b) The length of the semi-major aixs = $\sqrt{80/9}$.

The length of the semi-minor axis = $\sqrt{5}$.

(0, 0) is the centre.

The focii are given by $(0, \pm c)$ where $c = \sqrt{\left(\frac{80}{9}\right)^2 - 5^2} = \frac{25}{9}\sqrt{7}$.

i.e., $\left(9, \frac{25}{9}\sqrt{7}\right)$ and $\left(0, -\frac{25}{9}\sqrt{7}\right)$

10) Suppose $ux + vy + wz = p$ is a tangent plane to the given ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Then, we have the relation

$$p^2 = a^2u^2 + b^2v^2 + c^2$$

Now, we find the intersection of the plane and the coordinate axes. The plane meets the x-axis, y-axis and z-axis at the point.

$A\left(\frac{p}{u},0,0\right)$, $B\left(0,\frac{p}{v},0\right)$, $C\left(0,0,\frac{p}{w}\right)$ respectively.

Let $(x_0, y_0, z_0)$ be the centroid of the $\Delta ABC$. Then

$$x_0 = \frac{p/u+0+0}{3} \text{ i.e., } u = \frac{p}{3x_0}$$

$$y_0 = \frac{0+p/v+0}{3} \text{ i.e., } v = \frac{p}{3y_0}$$

$$z_0 = \frac{0+0+p/w}{3} \text{ i.e., } w = \frac{p}{3z_0}$$

Substituting for u, v and w in (*) we get

$$p^2 = \frac{a^2p^2}{9x_0^2}+\frac{b^2p^2}{9y_0^2}+\frac{c^2p^2}{9z_0^2}$$

That is, $\frac{a^2}{x_0^2}+\frac{b^2}{y_0^2}+\frac{c^2}{z_0^2}=9.$

This shows that the centroid $(x_0, y_0, z_0)$ lies on the surface

represented by the equation $\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}=9.$