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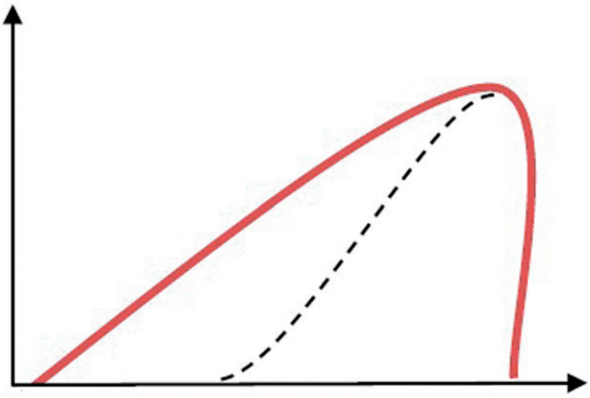

Effects of skewness and kurtosis on portfolio rankings

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1. Introduction

In this paper we discuss the issue of portfolio ranking and selection. We will concentrate on selecting one portfolio among a finite set of portfolios, where each portfolio is characterized by its own distribution of returns $p(x)$. This distribution may be inferred from past performance and assumed to be persistent, or it may be derived by some model of future performance.

We distinguish two main cases.

- **With risk-free asset:** An amount A must be invested and it can be distributed between a risk-free asset (that pays a risk-free rate r) and the selected portfolio.
- **Without risk-free asset:** An amount A must be invested exclusively in the selected portfolio.

The two problems are apparently similar, but they are conceptually different and therefore have different solutions. In this paper we clarify the difference between the two, discuss ranking schemes in both cases, and propose a theoretically sound ranking scheme for the second case in the presence of distributions that exhibit arbitrary skewness and kurtosis.

We conclude that, without a risk-free asset, and in the presence of skewness and kurtosis, for a distribution with finite momenta (distributions without fat tails), a rational investor using the CARA utility function should select that portfolio with a higher value of

$$\mu - \frac{m\sigma^2}{2} + \frac{m^2\sigma^3 S}{6} - \frac{m^3\sigma^4(K-3)}{720}, \tag{1}$$

where μ is the average return of the portfolio, σ the standard deviation, S the skewness, K the kurtosis, and m the CARA subjective risk-aversion parameter. Equation (1) extends a previous conclusion of Lévy and Markowitz (1979) by taking into considerations the effects of skewness and kurtosis. Our formula provides a simple practical way to select one out of many mutually exclusive portfolios. In this paper we discuss only the issue of portfolio selection, not portfolio construction. For a review, we refer the reader to the work of Ortobelli *et al.* (2005). In this analysis, we will always identify a portfolio by its distribution of returns p since we are not interested in the composition of the portfolio.

2. Portfolio selection

In this section we discuss the similarities and differences between the cases with and without a risk-free asset. The

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main similarity between the two cases is that they both require subjective choice. In Utility Theory (Neumann and Morgenstern 1947, Nash 1950), this is the choice of a utility function $U(x)$ and its parameters. $U(x)$ is a function that takes as input a possible return x from an investment and outputs a number that represents the investor's degree of satisfaction associated with return x. In neoclassical economics, an investor is defined 'rational' if:

- the investor has a utility function $U(x)$;
- the investor acts in order to maximize $U(x)$; and
- $U(x)$ is monotonic increasing (a higher return is preferred to a lower return).

A common choice for the utility function is $U_{\text{CARA}}(x) \equiv -\mathrm{e}^{-mx}$, which is known as the 'Constant Absolute Risk Averse' utility function. m is the risk-aversion parameter and it is of the order 1. We call a rational investor with a CARA utility function a 'rational risk-averse investor' or, more simply, 'investor'. The investor will rank a portfolio p by weighting the utility of a return x with the probability of that return $p(x)$, thus he would use the ranking function

$$R_U(p) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} U(x)\,p(x)\,\mathrm{d}x, \tag{2}$$

or an equivalent function. Two rankings function, R_1 and R_2, are equivalent if and only if they produce the same ranking, i.e. if there is a monotonic function h that maps $R_2(p)$ into $R_1(p)$ for every p. We will indicate the equivalence of two ranking functions as $R_1 \sim R_2$.

2.1. *With risk-free asset*

In this case, our investor can choose to invest part of the funds A in a risk-free asset, for example a US Treasury bill. This means that, given any two portfolios p_1 and p_2 characterized by an average return and risk (standard deviation) μ_1, σ_1 and μ_2, σ_2, if

$$\frac{\mu_1 - r}{\sigma_1} > \frac{\mu_2 - r}{\sigma_2}, \tag{3}$$

then the investor should never choose p_2 over p_1. This is because the investor can invest a fraction $\alpha = \sigma_2/\sigma_1$ of the total funds A in portfolio p_1 and a fraction $(1-\alpha)$ in the risk-free asset and obtain a new combined portfolio with the same risk as portfolio p_2, but a higher return, given by (Sharpe 1964)

$$\mu' = (1-\alpha)r + \alpha\mu_1 > \mu_2. \tag{4}$$

Hence, portfolio p_1 is always preferable to portfolio p_2. If one applies this argument to every portfolio in the set, one finds that our investor should invest part of the funds A in portfolio p with the largest value of

$$R_{\text{Sharpe}}(p) \stackrel{\text{def}}{=} \frac{\mu - r}{\sigma}. \tag{5}$$

This is the well-known Sharpe ratio or Sharpe ranking function (Markowitz 1952, Sharpe 1964). The value of α is then determined by maximizing the utility function U. If the expected returns of the portfolios have a Gaussian distribution, then

$$\alpha = \alpha \text{ that maximizes} \int U(\alpha x + (1-\alpha)r)\,p(x)\,\mathrm{d}x = \frac{\sqrt{\mu/r}}{m\sigma}. \tag{6}$$

If the return of the portfolios is not Gaussian-distributed, the first equality in equation (6) remains true, while the second equality is only an approximation since the integral must be performed numerically. Similarly, if one adopts a definition of risk other than the standard deviation of the returns, the argument that led to the R_{Sharpe} measure remains valid, but σ must be consistently replaced by the new measure of risk. For example, if one chooses to measure risk as the downside risk only

$$\sigma_n^- \stackrel{\text{def}}{=} \left[\int_{-\infty}^{r} p(x)(r-x)^n \mathrm{d}x\right]^{1/n}, \tag{7}$$

then σ is replaced by σ_n^-, and the Sharpe ranking function is replaced by the Kappa ranking function (Kaplan and Knowles 2004)

$$R_{\text{Kappa}-n}(p) \stackrel{\text{def}}{=} \frac{\mu - r}{\sigma_n^-}. \tag{8}$$

The Kappa ranking function above is the Sortino (Sortino and Van Der Meer 1991, Sortino and Price 1994, Sortino and Forsey 1996) ranking function when $n=2$ and it is equivalent to the Omega (Kazemi *et al.* 2003) ranking function when $n=1$. A different ranking scheme has been proposed by Stutzer (2000). For a theoretical analysis of 'good' vs. 'bad' risk measures we refer the reader to the work of Artzner *et al.* (2000). It was proven by Ortobelli *et al.* (2005) that all the above rankings are equivalent. In appendix A we provide exact mapping formulas from the above rankings to the Sharpe ratio. If returns are not Gaussian, Sharpe, Sortino, Kappa, and Omega are not equivalent because they follow from different definitions of risk.

2.2. *Without risk-free asset, the wrong way*

In this case, our investor has to choose a portfolio and invest the entire available funds in it, hence the argument presented at the beginning of the previous section does not apply. The reason behind the use of the Sharpe ranking (or the Sortino ranking) falls apart, as pointed out by Sharpe (1964). In this subsection we answer two questions.

- Is the use of the Sharpe ranking (or any of the other rankings) justified?
- If not, what is an appropriate ranking scheme that leads to the correct choice for a rational risk-averse investor?

Let us examine first the Sharpe ranking function and assume that portfolio returns are Gaussian-distributed.

An explicit computation shows that, for every Gaussian portfolio p,

$$R_{U_{\text{naive}}}(p) \sim R_{\text{Sharpe}}(p), \tag{9}$$

where

$$U_{\text{naive}}(x) \stackrel{\text{def}}{=} \begin{Bmatrix} -1 & (\text{if } x < 0) \\ +1 & (\text{if } x \geq 0) \end{Bmatrix}, \tag{10}$$

and the monotonic mapping function between the two rankings is

$$h(y) = \text{erf}(y/\sqrt{2}). \tag{11}$$

Therefore, an investor who ranks portfolios using the Sharpe function in the absence of a risk-free asset is implicitly adopting the utility function in equation (10). The problem here is that equation (10) is not a risk-averse utility function. This function states that a positive return $x > 0$ (gain) has utility $+1$ and a negative return $x < 0$ (loss) has utility -1. This investor does not believe that a 20% return is better than a 10% return, or that a loss of 100% is worse than a loss of 1%. The investor who uses the Sharpe ranking function in the absence of a risk-free asset is not a rational risk-averse investor.

Under the assumption of Gaussian returns, the Sortino, Omega, Stutzer and Kappa rankings are all equivalent (as proven by Ortobelli *et al.* 2005 and shown in appendix A), therefore the use of any of these rankings, in the case considered here, is not consistent with being risk-averse. Returns of real portfolios are, generally, non-Gaussian-distributed, but, if a ranking function works for a general distribution, it must also work for Gaussian returns. Since this is not true for Sharpe, Sortino, Omega, Stutzer, nor Kappa, these ranking schemes should not be used when the investor is not allowed to invest in a risk-free asset.

2.3. *Without risk-free asset, the right way*

Our investor, a rational risk-averse investor, would use the CARA utility function to make choices and would rank portfolios using equation (2) with U being U_{CARA}. In appendix B we prove that $R_{U_{\text{CARA}}}$ is equivalent to R_*, where

$$R_*(p) \stackrel{\text{def}}{=} -\log(-R_{U_{\text{CARA}}}(p))/m, \tag{12}$$

$$= \mu - \frac{m\sigma^2}{2} + \frac{m^2\sigma^3 S}{6} - \frac{m^3\sigma^4(K-3)}{720} + O(m^4\sigma^5), \tag{13}$$

and

- μ is the average return of portfolio p,
- σ is the standard deviation of portfolio p,
- S is the skewness,
- K is the kurtosis ($K-3$ is the reduced kurtosis),
- m is a parameter of order 1 that measures the risk-aversion of our investor, and
- $O(m^4\sigma^5)$ is the order of terms that are ignored.

Note that a positive skewness is good while a positive reduced kurtosis is bad because it results in fatter tails for fixed σ. It is also important to note that because of our choice of the CARA utility function, (13) does not converge unless all momenta of the distribution are finite. If this is not the case, for example for fat tail distributions, equation (13) does not produce a finite result and our approximate formula loses its value. For Gaussian-distributed returns equation (12) reduces to

$$R_*(p) = \mu - \frac{m\sigma^2}{2}, \tag{14}$$

and is exact. Note that the ranking (14) was originally proposed by Lévy and Markowitz (1979). Our approximated formula, equation (13), extends that result in the case of non-Gaussian returns.

We conclude that a rational risk-averse investor who has to choose one portfolio among many and has to invest all funds in the selected portfolio, should make their choice based on the ranking function in equation (12). This is a general result and it does not make any assumption concerning the distribution of the returns of the portfolios.

3. Practical considerations

Our conclusions have immediate practical applicability when the investor is not allowed to invest in a risk-free asset and has to choose one and only one mutually exclusive portfolio (or investment alternatives) according only to past performance. The following is an outline of the decision algorithm.

- For each portfolio k, collect the historical returns r_{kt} (return for portfolio k at time t) and measure

$$\mu_k = (1/N)\sum_t r_{kt},$$
$$\sigma_k = (1/N)\sum_t (r_{kt} - \mu_k)^2,$$
$$S_k = (1/N)\sum_t (r_{kt} - \mu_k)^3/\sigma_k^3,$$
$$K_k = (1/N)\sum_t (r_{kt} - \mu_k)^4/\sigma_k^4$$

(where N is the number of available data points).

- For each portfolio, compute

$$Rank_k = \mu_k - \frac{m\sigma_k^2}{2} + \frac{m^2\sigma_k^3 S_k}{6} - \frac{m^3\sigma_k^4(K_k-3)}{720}. \tag{15}$$

- Sort the portfolios according to $Rank_k$ and select the one with the highest rank.

If $S_k = 0$ and $K_k = 0$, our rank is equivalent to the formula proposed by Lévy and Markowitz (1979). Note that the rank depends on the time-scale of our analysis via our choice of returns r, which can be daily returns, weekly returns, monthly returns, etc. For a fixed skewness and kurtosis, their relative contribution to the rank increases with the size of the time-scale. In any case, because of the 1/720 factor, the relative contribution of kurtosis is generally very small.

4. Conclusions

In this paper we discuss the issue of portfolio selection for a rational risk-averse investor. We consider two cases: the investor is free to distribute funds between the selected portfolio and a risk-free asset, and the case when the investor has to invest all funds in the selected portfolio. In the first case we find that Sharpe, Omega, Sortino, and Kappa provide valid ranking schemes, although they follow from different definitions of risk, σ, σ_1^-, σ_2^- and σ_n^-, respectively. We also find that, in the Gaussian case, the Sharpe, Omega, Sortino, Kappa, and Stutzer rankings are all equivalent. In the non-Gaussian case, these rankings are not equivalent (Artzner *et al.* 2000, Ortobelli *et al.* 2005).

In the second case we find that, contrary to what is sometimes claimed, the use of any of the above ranking schemes does not correspond to being a rational risk-averse investor. In fact, we prove that a rational risk-averse investor (compatible with the choice of the CARA utility function) would choose a portfolio according to the following ranking function:

$$R_*(p) \stackrel{\text{def}}{=} -\log(-R_{U_{\text{CARA}}}(p))/m, \tag{16}$$

$$= \mu - \frac{m\sigma^2}{2} + \frac{m^2\sigma^3 S}{6} - \frac{m^3\sigma^4(K-3)}{720} + O(m^4\sigma^5), \tag{17}$$

which correctly takes into account the skewness of the portfolio, S, and its kurtosis, K, for distributions with finite momenta. Here, m is a coefficient of order 1 that measures the risk-aversion of the investor. Our approximated formula, equation (13), extends a result originally due to Lévy and Markowitz (1979) in the case of non-Gaussian returns. Our results do not apply to fat tail distributions, since some of their momenta are not finite and the integral in equation (13) does not converge. To take this case into account it is necessary to move beyond the CARA utility function. We will consider this case in a subsequent paper.

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Appendix A:

It was shown by Ortobelli *et al.* (2005) that if $p(x)$ is a Gaussian distribution with mean μ and standard deviation σ, then the Sharpe, Sortino, Omega, and Stutzer ranking schemes are equivalent. In this appendix we provide explicit mapping formulas from these ranking schemes, and from a general Kappa, into the Sharpe ratio. Two rankings, R_1 and R_2, are equivalent if and only if, for any two portfolios p_1 and p_2, $R_1(p_1) < R_1(p_2)$ implies $R_2(p_1) < R_2(p_2)$ and *vice versa*. This can only occur if there is a monotonic increasing function h such that, for any portfolio p, $R_2(p) = h(R_1(p))$.

Kappa

Proof:

$$R_{\text{Kappa}-n}(p) \stackrel{\text{def}}{=} \frac{\mu - r}{\left[\int_{-\infty}^{r} (r-x)^n p(x)\,\mathrm{d}x\right]^{1/n}} = h(R_{\text{Sharpe}}(p)) \tag{A1}$$

and

$$h(y) = \frac{\pi^{1/2n} 2^{(2-n)/2n} \mathrm{e}^{y^2/2n} y}{\left\{ \begin{array}{l} \left[\Gamma((1+n)/2)\,{}_1F_1((1+n)/2, 1/2, y^2/2)\right. \\ \left. -\sqrt{2}\Gamma(1+(n/2))\,{}_1F_1(1+(n/2), 3/2, y^2/2)\right]^{1/n} \end{array} \right\}}$$

(Γ is Euler's Gamma function and ${}_1F_1$ is a hypergeometric function). □

Omega

Proof:

$$R_{\text{Omega}}(p) \stackrel{\text{def}}{=} \frac{\int_r^{\infty}(1 - F_p(x))\,\mathrm{d}x}{\int_{-\infty}^{r} F_p(x)\,\mathrm{d}x} = R_{\text{Kappa}-1}(p) \qquad \text{(A2)}$$

(here F_p is the cumulative distribution function associated with p). □

Sortino

Proof:

$$R_{\text{Sortino}}(p) \stackrel{\text{def}}{=} \frac{\mu - r}{[\int_{-\infty}^{r}(r - x)^2 p(x)\,\mathrm{d}x]^{1/2}} = R_{\text{Kappa}-2}(p). \qquad \text{(A3)}$$

□

Stutzer

Proof:

$$R_{\text{Stutzer}}(p) \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{-\log F_p(rT)}{T} = h(R_{\text{Sharpe}}(p)), \qquad \text{(A4)}$$

where R_{Stutzer} is well-defined only for portfolios with positive Sharpe ratio, and

$$h(y) = y^2/2 \quad (\text{for } y > 0 \text{ only}). \qquad \text{(A5)}$$

□

Appendix B:

In this section we prove that $R_{U_{\text{CARA}}}$ is equivalent to R_*. Consider a portfolio characterized by a distribution of returns p. Let μ, σ, S and K be the average, standard deviation, skewness and kurtosis of p,

$$R_{U_{\text{CARA}}}(p) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} -\mathrm{e}^{-mx} p(x)\mathrm{d}x = \int_{-\infty}^{\infty} -\mathrm{e}^{-mx} \widetilde{p}((x-\mu)/\sigma)\mathrm{d}x,$$

where $\widetilde{p}(y) = p(\sigma y + \mu)\sigma$. With the change of variable $y = (x-\mu)/\sigma$ and a Taylor series expansion in y of the exponential, we obtain

$$\begin{aligned} R_{U_{\text{CARA}}}(p_i) &= -\mathrm{e}^{-m\mu} \sum_{i=0} \frac{(-m\sigma)^i}{i!} \int_{-\infty}^{\infty} y^i \widetilde{p}(y)\mathrm{d}y \\ &= -\mathrm{e}^{-m\mu} \mathrm{e}^{\log(1 + \frac{m^2\sigma^2}{2} - \frac{m^3\sigma^3 S}{3!} + \frac{m^4\sigma^4(K-3)}{6!} + O(m^4\sigma^5))} \\ &= h(R_*(p)), \end{aligned} \qquad \text{(B1)}$$

where $h(y) = -\mathrm{e}^{-my}$ is a monotonic increasing function in y, and

$$R_*(p) \stackrel{\text{def}}{=} -\log(-R_{U_{\text{CARA}}}(p))/m \qquad \text{(B2)}$$

$$= \mu - \frac{m\sigma^2}{2} + \frac{m^2\sigma^3 S}{6} - \frac{m^3\sigma^4(K-3)}{720} + O(m^4\sigma^5). \qquad \text{(B3)}$$

Hence the two ranking functions $R_*(p)$ and $R_{U_{\text{CARA}}}(p)$ are equivalent. This is a general result and no assumption concerning the distribution p has been made. In the special case of p Gaussian, we are able to perform the integration analytically without the need for a Taylor expansion and we find that the following exact relation holds (Lévy and Markowitz 1979):

$$R_*(p) = \mu - \frac{m\sigma^2}{2}. \qquad \text{(B4)}$$