Mean –modified Value-at-Risk optimization With Hedge Funds

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Laurent Favre
Associate Researcher with EDHEC
Investment Solutions
UBS Wealth Management
laurent-za.favre@ubs.com

José-Antonio Galeano
Banque Cantonale Vaudoise
Switzerland
jose.antonio.galeano@bcv.ch

Abstract:

Based on the normal Value-at-Risk, we develop a new Value-at-Risk method called Modified Value-at-Risk. This Modified Value-at-Risk has the property to adjust the risk, measuring with the volatility only, with the skewness and the kurtosis of the distribution of returns. The Modified Value-at-Risk allows to measure first the risk of portfolio with assets non normally distributed like hedge funds or technology stocks and to compute optimal portfolio by minimizing the Modified Value-at-Risk at a given confidence level.

Both academic and practitioner research has questioned the use of the mean variance analysis as a central approach to evaluating the benefits of investing in hedge funds. In contrast, other researchers have considered another framework, that is using mean-Value-at-Risk setting in which the downside risk is incorporated into the asset allocation model. In short, the optimal portfolio is selected by maximizing the expected return over candidate portfolios so that some shortfall criterion is met. The literature expanded a lot on the Value at Risk subject in these recent years. Uryasev and Rockafellar [1999] propose to measure a Mean Shortfall or Conditional VaR which is the mean of the returns higher than the VaR. Flavin and Wickens [1998] use a Garch process to model asset returns. Artzner, Delbaen, Eber and Heath [1997] argue that their proposed coherent measures of risks have certain desirable properties that VaR lacks. Basak and Shapiro [1998] argue that VaR does not consider the magnitude of loss, which exceeds the threshold level. They propose an analytical formula to obtain the portfolio's weights of risky assets, assuming that they are log-normally distributed, by minimizing the losses over a threshold. Keating and Sahdwick [2002] show develop the Omega approach. They show that maximizing the ratio between the density higher than a threshold and the density lower than the threshold is the optimal approach. This Omega approach accounts for all the moments of the returns distribution.

In this paper, we first introduce the framework. It is based on the working paper by Huisman, Koedijk and Pownall [1999]. Then we use a Cornish-Fisher expansion in order to compute the Value-at-Risk for the left tail of the distribution. We did some empirical tests, not shown in these papers, which clearly show that this Modified VaR has a minimum and that the risk, measured with volatility only, is underestimated if the portfolio has negative skewness and/or positive excess kurtosis[1].

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[1] The skewness and kurtosis are priced in utility function (negative skewness negatively and positive excess kurtosis negatively if the investor has a Non Increasing Risk Aversion utility). See Jurczenko, Maillet [2001].

The normal Value-at-Risk approach

The risk of a portfolio composed of financial assets can be measured with the Value-at-Risk (VaR). The VaR, as a measure of risk, has some interesting advantages:

- It is recognized by practitioners,
- It measures the downside risk which is interesting for a risk averse investor like a pension fund,
- Many academic studies have been done on the subject,
- We can measure risk with just one easily understandable number.
- It can be used for non-normally distributed assets. We will adjust the Value-at-Risk method by using an empirical VaR and an analytical VaR, which takes the skewness and the kurtosis into account.

If we assume that the future distribution of returns can be accurately estimated with the normal distribution, then the standard deviation is the only risk factor influencing our downside risk measure. Remember that the Value-at-Risk corresponds to the amount of portfolio wealth that can be lost over a given period of time with a certain probability:

$$\text{Prob}(dW \leq -VaR) = 1 - a$$

with

$$VaR = n\boldsymbol{s}Wdt^{0.5} \tag{1}$$

n = number of standard deviation at $(1-\alpha)$[2]
α = probability
σ = yearly standard deviation
W = amount at risk or portfolio
dt = year fraction

But as mentioned by Wilmott [1998], the assumption of zero mean underlying the VaR concept is valid over short term horizons. For longer term horizons, the return is skewed to the right by an amount proportional to the time horizon. Thus, for longer time scales, equation (1) should be modified to account for the drift of the asset value. If the rate of the drift is μ, then equation (1) becomes:

$$VaR = W\left(\boldsymbol{m}dt - n\boldsymbol{s}\left(dt\right)^{0.5}\right)$$

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[2] With monthly returns, we obtain the monthly standard deviation of a portfolio or of a security. So, in order to get the annual one, the monthly standard deviation should be multiplied by the square root of the time.

Methodology

As shown in Arzac and Bawa [1977], a portfolio with returns derived with the VaR as the measure of risk is equal to a portfolio derived with the standard deviation as the measure of risk as long as the returns are normally distributed. In this case the VaR is only a multiple of standard deviation (ie. at 95% confidence interval, VaR is equal to - W*1.645σ). Minimizing σ or VaR for a given expected return leads to the same result.

Nevertheless, it is widely known that some financial assets are not normally distributed. This is also the case with the alternative investments we would like to include in a pension fund portfolio. Our approach incorporates the VaR in the computation of the optimal portfolio as first done by Arzac and Bawa [1977] and developed by Huisman, Koedjik and Pownall [1999]. They derive an optimal portfolio so that the maximum expected loss does not exceed a VaR limit for a chosen investment horizon at a given confidence level.

Assume that the portfolio manager invests his wealth W(0) in n assets and lends or borrows an amount B. Therefore, ω_i denotes a fraction invested in the risky asset i. P_i is the price of that risky asset. Hence, the initial value of the portfolio is given by its budget constraint:

$$W(0) + B = \sum_{i=1}^{m} w_i P_i \qquad (2)$$

If the investor is a pension fund, short selling is not allowed. We have the additional constraints:

$$w_i \geq 0$$
$$\sum_{i=1}^{m} w_i = 1, \text{for } i = 1 \text{ to } m \text{ assets} \qquad (3)$$

Furthermore, the portfolio manager of the pension fund knows from the risk management department that he has a limit of VaR, which we will call VaR*, that he should not exceed. He has the following downside risk constraint:

$$\Pr\{W(0) - W(T) \geq VaR^*\} \leq (1-c) \qquad (4)$$

Assume that the investor is able to borrow or lend at the risk free rate r_f. Moreover, the manager is concerned with a maximum loss[3]; that is, he wants to manage the downside risk of his portfolio. Therefore, the portfolio manager wants to allocate his portfolio taking into account a desired level (or limit) of Value-at-Risk as VaR*. By this way, his risk aversion is reflected by the VaR limit (VaR*) and the confidence interval of his VaR.

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[3] The second derivative of his utility function is negative.

The expected wealth at the end of the time horizon is:

$$E(W_T) = (W(0) + B)(1 + r_p) - \underbrace{B(1 + r_f)}_{borrowing} \tag{5}$$

with r_P = expected total return on the portfolio.

Substituting B as given in equation (2) in (5), we are able to express the final wealth in terms of the risk free rate of return and the expected portfolio risk premium:

$$E(W_T) = \sum_{i=1}^{m} w_i P_i * (1 + r_p) - (\sum_{i=1}^{m} w_i P_i - W) * (1 + r_f) = \sum_{i=1}^{m} w_i P_i * (r_p - r_f) + W * (1 + r_f) \tag{6}$$

With equation (6), we can observe that as long as $r_P > r_f$, a risk averse investor will always invest in the risky assets.

In order to determine the optimal portfolio that maximizes the expected final wealth subject to the VaR constraint, we take equation (1) and adding the initial portfolio to both sides of the inequality equals:

$$\text{Prob}\left(\frac{dW + W}{W_T} \leq W - Va\check{R} \right) = 1 - a \tag{7}$$

Substituting (6) in (7) after some manipulations yields:

$$\text{Prob}\left(r_p \leq r_f - \frac{Va\check{R} + W * r_f}{w_i P_i} \right) = 1 - a \tag{8}$$

The right-hand-side of the inequality is the quintile $q(\alpha,p)$ of the distribution which corresponds to the cumulative probability density function of the portfolio at $(1-\alpha)$ confidence interval level. It is also the maximum loss-amount that the investor wants to bear at a $(1-\alpha)$ level. So we can rewrite (8) as:

$$q(a, p) - r_f = -\frac{VaR^* + W * r_f}{\sum_{i=1}^{m} w_i P_i} \tag{9}$$

Substituting the denominator of equation (9) in (6),

$$E(W_T) = -\frac{VaR^* + Wr_f}{q(a, p) - r_f}(r_p - r_f) + W(1 + r_f)$$

Dividing by W,

$$
(10)
$$

$$
E\left(\frac{W_T}{W(0)}\right) = \frac{r_p - r_f}{\underbrace{W(0)*(q(\boldsymbol{a},p)-r_f)}_{S(p)}}(VaR^* + W(0)*r_f) + (1+r_f)
\qquad (11)
$$

Remember that VaR* is the Value at Risk limit that the investor wants to bear. This term can be seen as an additional constraint for the investor. The objective of the investor, concerned by the downside risk, is to maximize the expected return of his portfolio. Hence, he wants to maximize equation (11), which is equivalent to maximizing the ratio S(p):

$$
\max\_S(p) = \frac{r_p - r_f}{W(0)*r_f - W(0)*q(\boldsymbol{a},p)} = \frac{r_P - r_f}{W(0)*r_f - VaR}
\qquad (12)
$$

So, S(p) equals the expected excess return on portfolio p divided by the expected loss on portfolio p that is incurred with probability (1-α). Note that the asset allocation process is thus independent of wealth. VaR is the Value at Risk of the portfolio p. Remember that this VaR is the Value at Risk of the optimal portfolio, which may not be the VaR limit that the investor has. As the risk is measured with the VaR, the denominator can be seen as a measure of regret since it measures the potential loss of investing in risky assets.

S(p) is a measure of performance like the Sharpe ratio. The advantage of this measure is that it does not rely on any distribution assumptions and is therefore able to incorporate non-normalities in the portfolio allocation, through the VaR term. The existence of non-normalities may lead to the choice of different portfolios. If we assume that returns are normally distributed and that the risk free is equal to zero, then, S(p) collapses to the classical Sharpe ratio.

The optimal portfolio, which maximizes S(p), is independent from the desired level of VaR* (see equation (4)) since the measure VaR in (12) represents the optimal portfolio Value at Risk. The investor first allocates the risky assets by maximizing S(p). Then, he decides the amount of wealth to lend (or borrow) depending on how much of the portfolio's VaR is higher (or lower) compared to the VaR* limit. This is exactly the same as moving on the CML in a mean-variance setting[4]. The amount to lend or borrow in order to obtain the desired level of VaR* is obtained by substituting (2) in (9)

$$
(13)
$$

[4] In a mean-variance setting, the investor computes the market portfolio $w = \frac{1}{\boldsymbol{g}}(\boldsymbol{m} - r_f)\Omega^{-1}$ and then according to his desired level of volatility borrows at r_f and places the proceed in the market portfolio if he wants to increase his risk (ie. he will move on the right on the Capital Market Line) or lends at r_f by decreasing his exposure to risky assets if he wants to decrease his risk (ie. he will move on the left on the Capital Market Line)

$$q(\boldsymbol{a}, p) - r_f = -\frac{VaR^* + W * r_f}{W + B}$$

Rearranging and extracting B becomes

$$B = \frac{Va\dot{R} + W * r_f}{r_f - q(a, p)} - W = \frac{W * (Va\dot{R} + W * r_f) - W^2 * (r_f - q(a, p))}{W * (r_f - q(a, p))} \qquad (14)$$

$$B = \frac{W * (Va\dot{R} - Va\dot{R})}{W * r_f + VaR} \qquad (15)$$

with   VaR : optimal Value at Risk[5]
        VaR*: Value at Risk limit for the investor

When the desired level of risk (ie. VaR*) is lower than the VaR of the optimal portfolio, the numerator will be lower than zero, B will be negative, so the investor will lend the amount B.

The objective now consists in drawing the efficient frontiers based on this new framework, that is in the mean-Value-at-Risk theory. The only problem is to compute the Value-at-Risk without doing any assumption on the underlying distribution. We have to estimate the VaR analytically, that is the VaR is derived with the parameters characterizing the distribution of the returns. We use a Cornish-Fisher [1937] expansion to compute the VaR analytically[6]. It adjusts the traditional VaR[7] with the skewness and kurtosis of the distribution:

$$z_{CF} = z_c + \frac{1}{6}(z_c^2 - 1)S + \frac{1}{24}(z_c^3 - 3z_c)K - \frac{1}{36}(2z_c^3 - 5z_c)S^2 \qquad (16)$$

with:
Z_c: critical value for probability (1-α)
S: skewness
K: excess kurtosis
The VaR is equal to:

$$VaR = W(\boldsymbol{m} - z_{CF}\boldsymbol{s}) \qquad (17)$$

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[5] VaR=Wq(α,p), with q(α,p)<0. So in the calculus we invert the sign of VaR in the numerator and at the denominator.

[6] David X Li [1999] derives also an analytical formula for confidence interval by using estimating functions. But with his formula, it was not possible to find a realistic one-side confidence level for negative returns.

[7] Mina and Ulmer [1999] provide four methods to compute the VaR for non-normally distributed assets: Johnson transformations, Cornish-Fisher expansion, Fourier method, partial Monte-Carlo. They found that Cornish-Fisher is fast and tractable, but sometimes not accurate with extremely sharp distributions.

Then, the modified VaR developed in this paper is equal to[8]

$$VaR = W\left[\mu - \left(z_c + \frac{1}{6}(z_c^2 - 1)S + \frac{1}{24}(z_c^3 - 3z_c)K - \frac{1}{36}(2z_c^3 - 5z_c)S^2\right)\sigma\right] \quad (18)$$

In formula (18), z_c is equal to –2.33 for a 99% probability or to –1.96 for a 95% probability. The modified VaR allows us to compute the Value-at-Risk for distributions with asymmetry (positive or negative skewness) and fat tails (positive excess kurtosis). Note that if the distribution is normal, S and K are equal to zero, which makes z_{CF} to be equal to z_c. We are, therefore, back to the normal case where the risk is only measured with volatility.

Conclusion

We have shown that it is possible to use the volatility, the skewness and the kurtosis in a new measure called "Modified VaR". If a financial asset (especially hedge funds, private equity, technology stocks, emerging markets stocks) has negative skewness and/or a positive excess kurtosis, the Modified VaR will be higher than a normal Value-at-Risk or the risk measured only with volatility will be lower than the risk measured with volatility, skewness and kurtosis.

Many investors are risk averse against extreme negative returns[9]. By using the Modified VaR at 99%, for investors risk averse against negative exterme returns, a numerical optimization will deliver the portfolio which has the lowest probability to loose more than the Modified VaR with at the 99% confidence.

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[8] This formula has been applied in a portfolio context with equities, bonds and hedge funds by Signer and Favre [2002].

[9] If the investor belongs to the Non Increasing Absolute Risk Aversion (Jurczenko, Maillet 2002).

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