Stock Market Diversity

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Organization of the talk

1. Introduction: the classical case
2. The logarithmic representation
3. The market portfolio
4. Stock market diversity
5. Diversity-weighted portfolios
6. Manager performance and the diversity cycle

References

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[2] Fernholz, R. and R. Garvy (1999, May 17). Diversity changes effect relative performance. *Pensions & Investments*, 112.

[3] Fernholz, R., R. Garvy, and J. Hannon (1998, Winter). Diversity-weighted indexing. *Journal of Portfolio Management 24*, 74–82 .

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Introduction: the classical case

Suppose that $X(t)$ represents the price of a stock at time t. If we assume that stocks pay no dividends, etc., then the return on this stock over the time interval dt is

$$\frac{dX(t)}{X(t)} = \alpha(t)\,dt + \sigma(t)\,dW(t),$$

where $\alpha(t)$ represents the *rate of return*, $\sigma^2(t)$ represents the *variance (rate)*, and W is *Brownian motion*, the continuous-time version of a random walk.

Suppose we have a market of stocks represented by their price processes $X_1, \ldots, X_n$. In this market, a *portfolio* π is identified by its *proportions*, or *weights*, $\pi_1(t), \ldots, \pi_n(t)$, in each of the stocks. The portfolio weights must sum to one: $\pi_1(t) + \cdots + \pi_n(t) = 1$.

Let $Z_\pi(t)$ represent the *value* of the portfolio at time t. Then the portfolio return will satisfy

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dX_i(t)}{X_i(t)},$$

and it follows from this that the *portfolio rate of return* is

$$\alpha_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\alpha_i(t).$$

We also have the *portfolio variance (rate)*

$$\sigma_\pi^2(t) = \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t),$$

where $\sigma_{ij}(t)$ is the *covariance (rate)* for the stocks X_i and X_j. We shall assume that the covariance matrix $\big(\sigma_{ij}(t)\big)$ is nonsingular.

The logarithmic representation

It turns out that it is advantageous for us to use the *logarithmic return (log-return)*, sometimes called the *continuous return*, rather than the classical return. In this case

$$d \log X(t) = \gamma(t)\, dt + \sigma(t)\, dW(t),$$

where $\gamma(t)$ is the *growth rate* of X at time t. The growth rate is also sometimes called the *geometric rate of return*, the *logarithmic rate of return*, or the *continuous rate of return.* The relation between the rate of return and the growth rate is

$$\alpha(t) = \gamma(t) + \frac{1}{2}\sigma^2(t). \tag{1}$$

This follows from Itô's rule in stochastic calculus, which appeared at about the same time that Harry Markowitz published his paper on portfolio theory.

The log-return of a portfolio π is given by

$$d \log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\, d \log X_i(t) + \gamma_\pi^*(t)\, dt, \tag{2}$$

where

$$\gamma_\pi^*(t) = \frac{1}{2}\Big(\sum_{i=1}^{n} \pi_i(t)\sigma_i^2(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t) \Big)$$

is called the *excess growth rate*, and contains all the $\sigma^2(t)/2$ terms from equation (1). The portfolio growth rate is therefore

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t).$$

For an all-long portfolio with more than one stock, $\gamma_\pi^*(t) > 0$, and measures the amount by which the portfolio growth rate *exceeds* the weighted average of the stock growth rates.

The market portfolio

Let us assume that each stock has just one share outstanding, so $X_i(t)$ represents the total capitalization of the ith stock. The *market portfolio* μ has weights

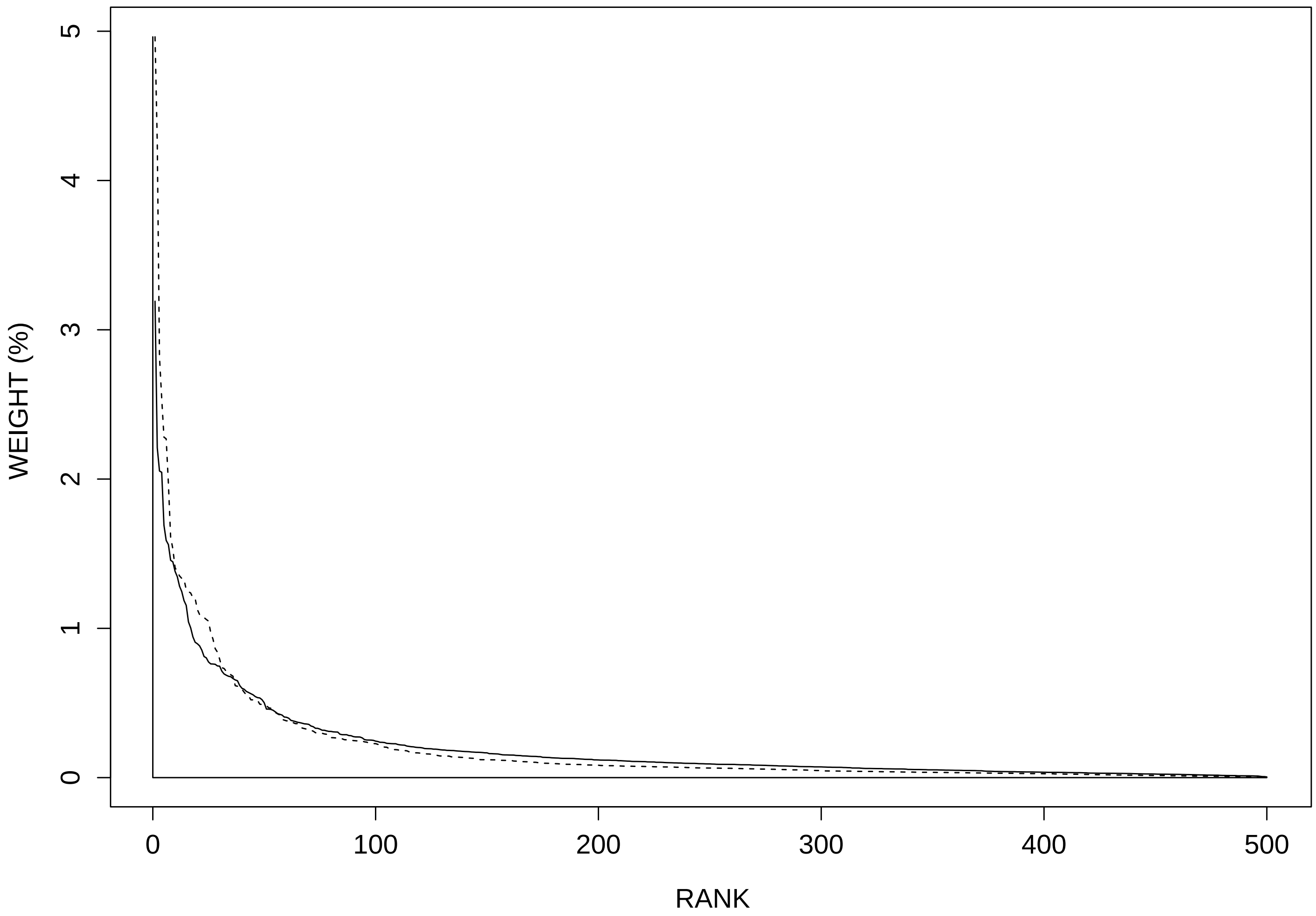
$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)},$$

and its value process is given by

$$Z_\mu(t) = X_1(t) + \cdots + X_n(t).$$

This represents the total capitalization of the market, and the weights μ_i are called *market weights*, or *cap weights*. The *capital distribution* of the market consists of the market weights arranged in decreasing order. Let us observe the capital distribution of the S&P 500.

Capital distribution of the S&P 500 Index



12/30/1997 (solid line), 12/29/1999 (broken line)

Stock market diversity

Stock market diversity measures how evenly capital is distributed among the stocks in the market. When market diversity is high, capital is more evenly spread among the stocks, when it is low, capital is more concentrated. The chart we just saw showed that the diversity of the S&P 500 declined from 1997 to 1999.

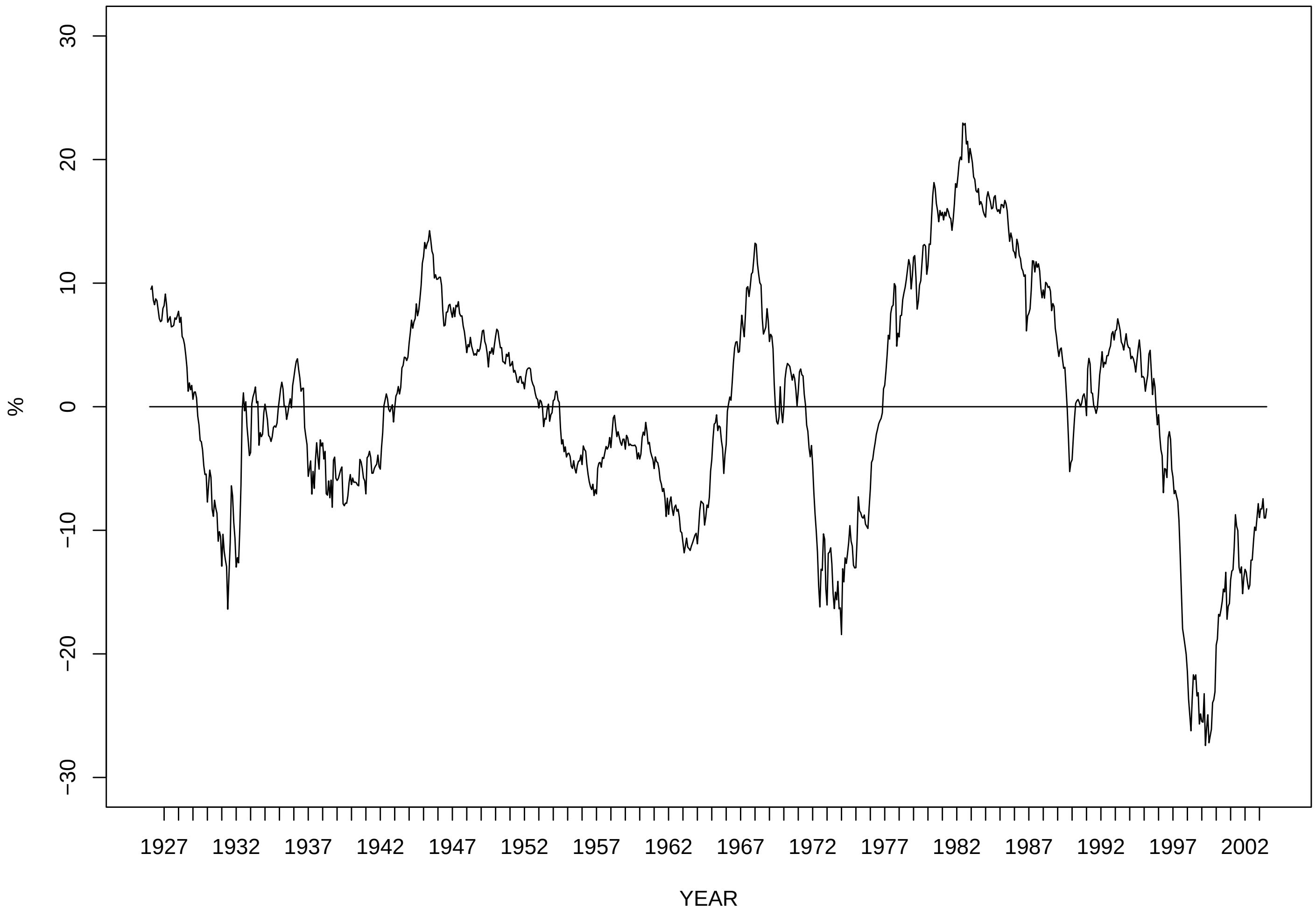
We can measure market diversity using the function

$$\mathbf{D}_p(\mu(t)) = \Big(\sum_{i=1}^{n} \mu_i^p(t)\Big)^{1/p},$$

where $0 < p < 1$. We see that $1 \leq \mathbf{D}_p(\mu(t)) \leq n^{(1-p)/p}$.

- Since early in the last century, U.S. stock market diversity, measured by $\mathbf{D}_p$, has been *mean-reverting* with intermediate-term trends.

Cumulative change in market diversity, 1927–2004.



$\mathbf{D}_p(\mu(t))$, with $p = 1/2$.

Diversity-weighted portfolios

The *relative log-return* of the ith stock (i.e., relative to the market) is

$$d \log X_i(t) - d \log Z_\mu(t) = d \log \big(X_i(t)/Z_\mu(t)\big) = d \log \mu_i(t).$$

With this and equation (2), the equation for portfolio log-return, we can represent the relative log-return of a portfolio π by

$$d \log \big(Z_\pi(t)/Z_\mu(t)\big) = \sum_{i=1}^{n} \pi_i(t)\, d \log \mu_i(t) + \gamma_\pi^*(t)\, dt.$$

Here is another representation of the relative log-return: We say that a function $\mathbf{S}$ of the market weights *generates* a portfolio π if

$$d \log \big(Z_\pi(t)/Z_\mu(t)\big) = d \log \mathbf{S}(\mu(t)) + d\Theta(t),$$

where the *drift process* Θ contains no Brownian motion terms.

It can be shown that the measure of diversity $\mathbf{D}_p$ generates the *diversity-weighted* portfolio π with weights

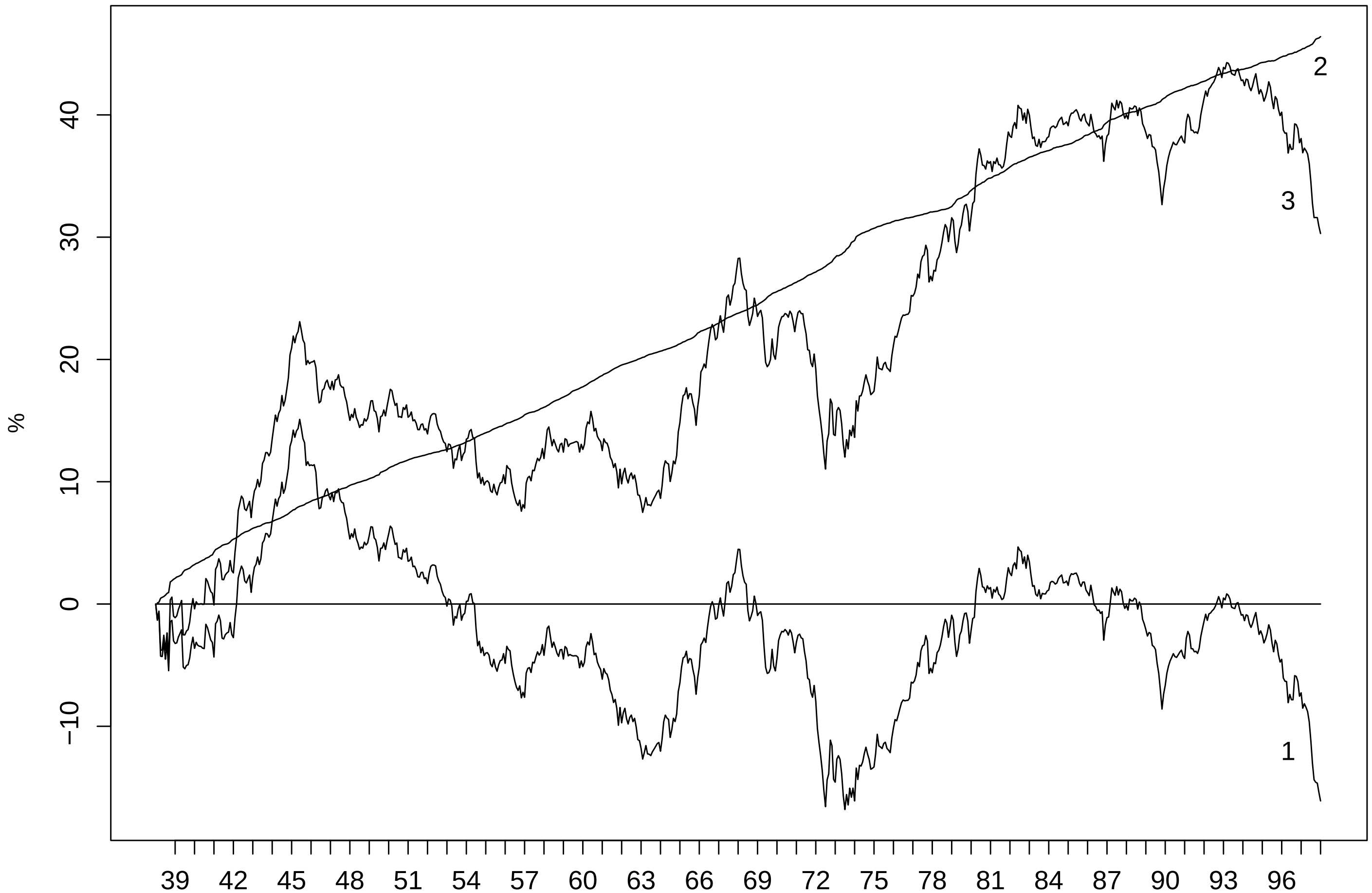
$$\pi_i(t) = \frac{\mu_i^p(t)}{\sum_{j=1}^n \mu_j^p(t)},$$

and relative log-return

$$d\log\big(Z_\pi(t)/Z_\mu(t)\big) = d\log\mathbf{D}_p(\mu(t)) + (1-p)\gamma_\pi^*(t)\,dt.$$

- Compared to the market portfolio, π underweights the larger stocks and overweights the smaller stocks.
- The diversity-weighted portfolio is likely to outperform the market, because diversity is mean-reverting and the drift process is increasing.
- The portfolio variance of π and μ will be about the same over intervals of length greater than or equal to the relaxation time of the change in diversity.

Top 1000 stocks: Diversity-weighted vs. cap-weighted, $p = 1/2$.

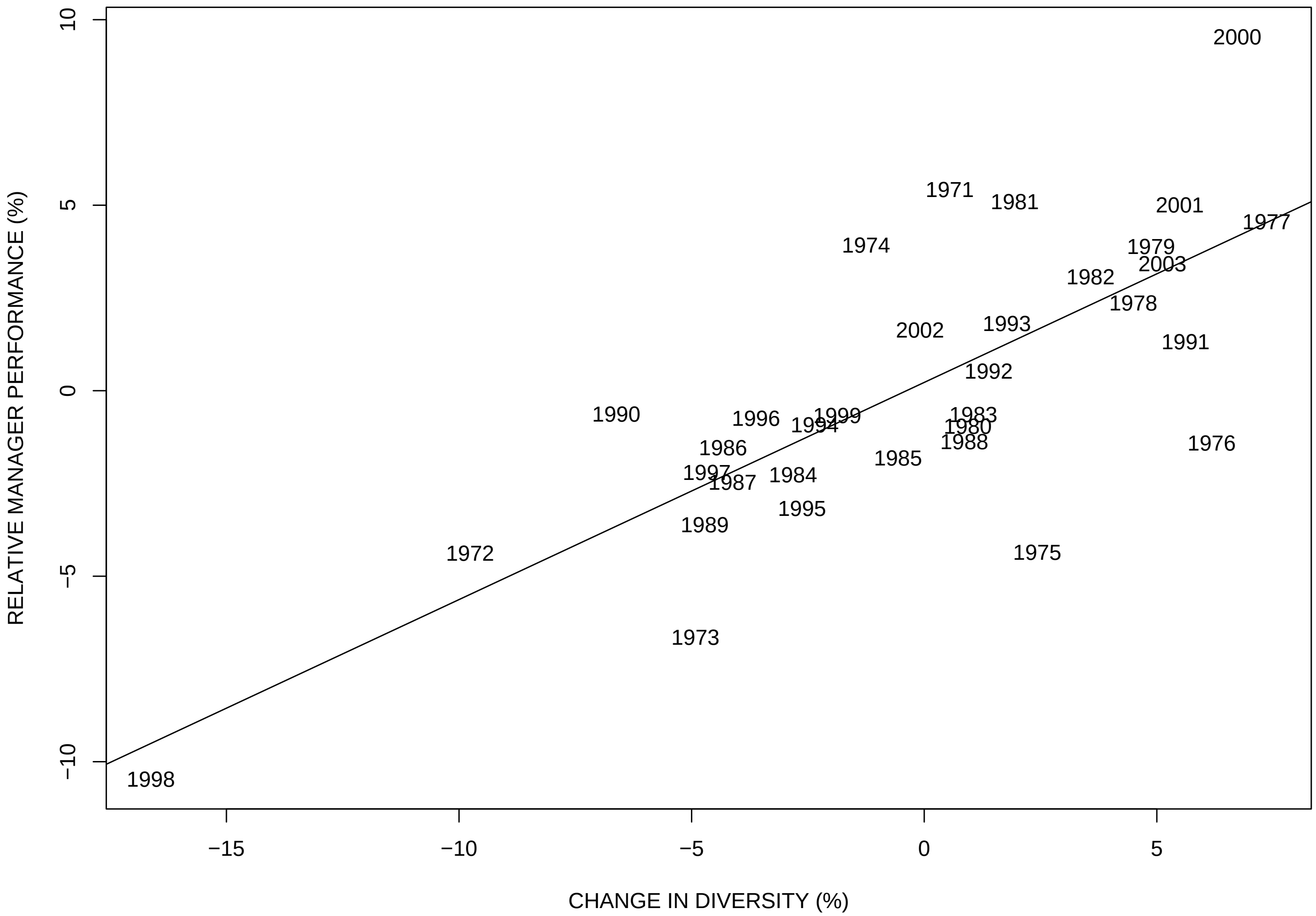


1: $\log \mathbf{D}_p(\mu(t))$; 2: $\Theta(t)$; 3: $\log\big(Z_\pi(t)/Z_\mu(t)\big)$

Manager performance and the diversity cycle

- Lowering the concentration of capital in the largest stocks allows the diversity-weighted portfolio to outperform the market, without increasing the intermediate-term risk.
- Active managers may take advantage of this phenomenon, perhaps unintentionally.
- If so, the average holdings of active managers would be closer to diversity weighting than to cap weighting.
- In that case, active managers' relative returns would be positively correlated with changes in market diversity.
- If trends in market diversity can be predicted, this could be helpful in the allocation of assets between active and passive managers.

Annual relative manager log-return vs. change in diversity.



$$R^2 = .58$$

(Median manager return data from Callan.)

Summary:

- The capital distribution of the market varies over time.
- Stock market diversity measures the ebb and flow of capital between large and small stocks.
- Over time, market diversity has been mean-reverting with intermediate-term trends.
- Diversity weighting is likely to outperform cap weighting over the long term.
- Forecasting the diversity cycle could be useful in asset allocation between active and passive equity management.