Segmentation

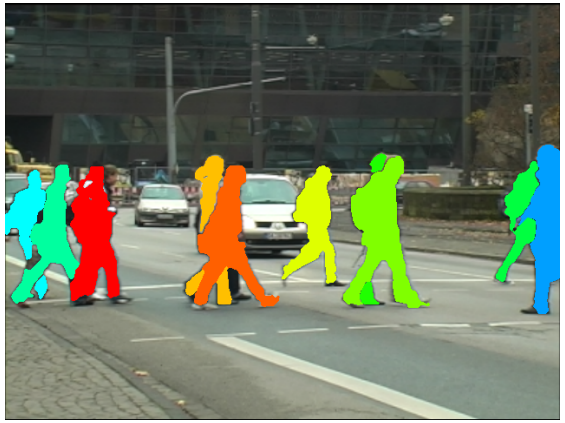
Segmentation : the problem

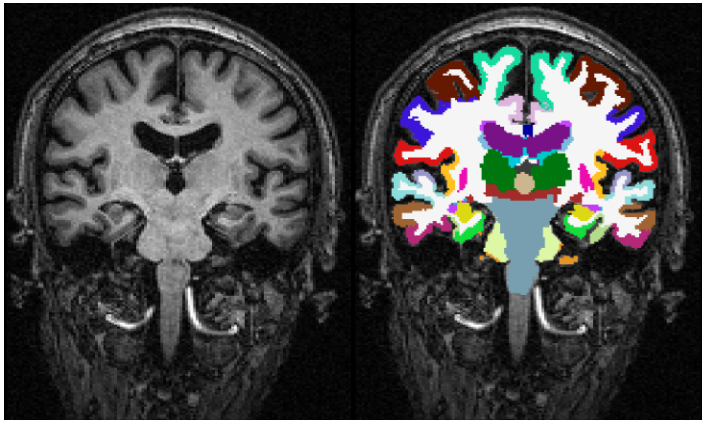
Identifying entities in the image, e.g. objects :

- grouping pixels into segments
- crucial and basically unsolved step

Segmentation : importance

- Understanding scenes
- Industrial inspection
- Disease diagnosis
- ….





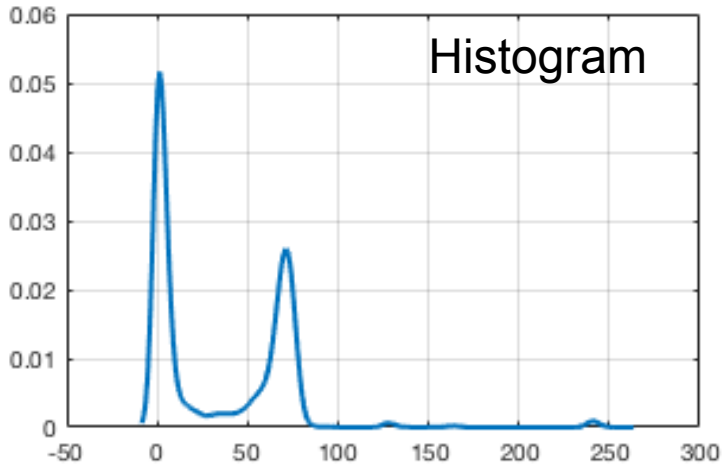
Segmentation : Outline

- Thresholding
- Edge based
- Region based
- Statistical Pattern Recognition based

Thresholding : basic idea

For high contrast between object(s) and background, determine intensity threshold that defines 2 pixel categories : object and background

Example image



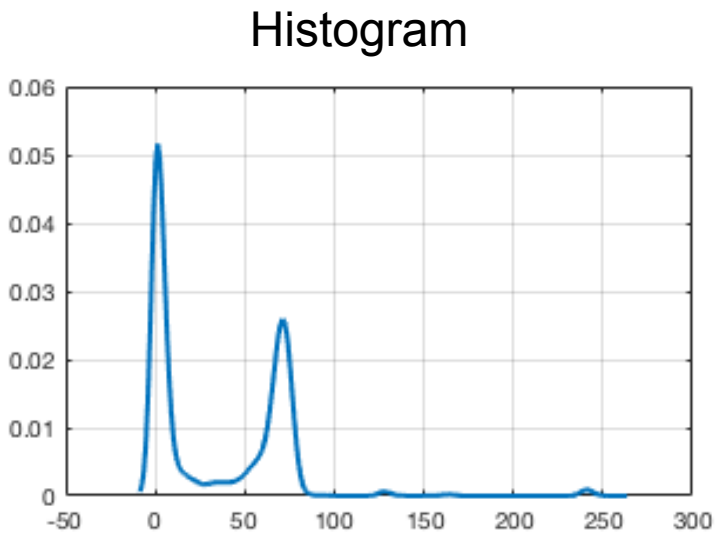
Threshold = 5

Threshold = 25

Threshold = 50

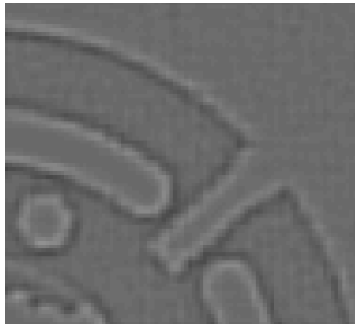
Threshold = 70

Thresholding : threshold selection



Alternatives:

1. If intensities of objects are known → easy
2. From histogram, take the minimum between two peaks
3. For known size (e.g. for industrial application), increase threshold until reaching a predefined area
4. Maximize sum of gradients at pixels with threshold intensity
5. Low gradient magnitude areas
6. Use regions with high response to Laplacian filter – points around the edge



Thresholding: Otsu criterion

We want large areas of low variance.
Hence, minimize within-group variance
weighted by group size *p*

Group 1
I > threshold

p_1, σ_1^2

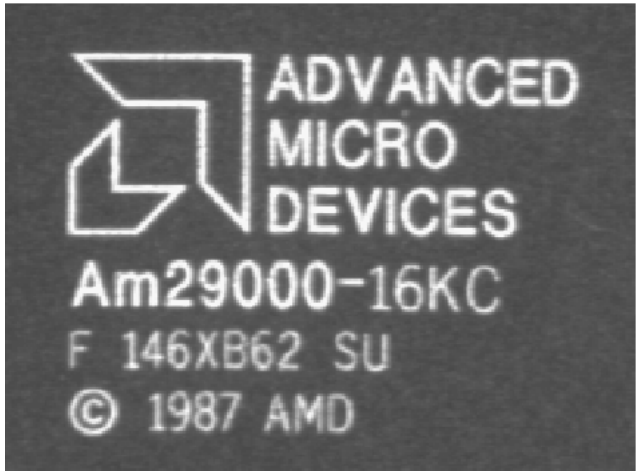
Group 2
I <= threshold

p_2, σ_2^2

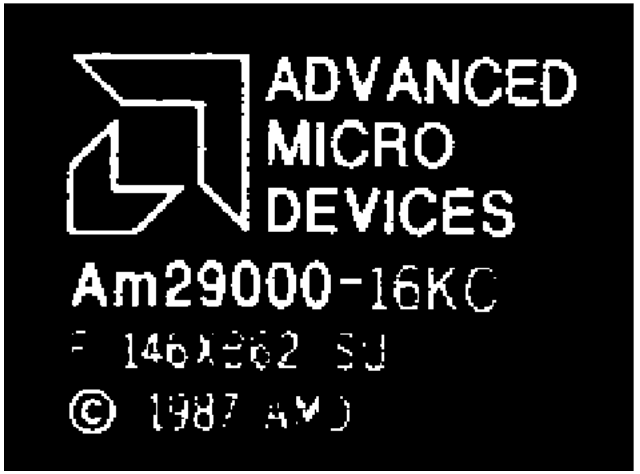
$$\text{determine threshold to minimize } p_1\sigma_1^2 + p_2\sigma_2^2$$

Otsu threshold = 35

Thresholding: Global vs. Local Otsu



Image



Fixed threshold

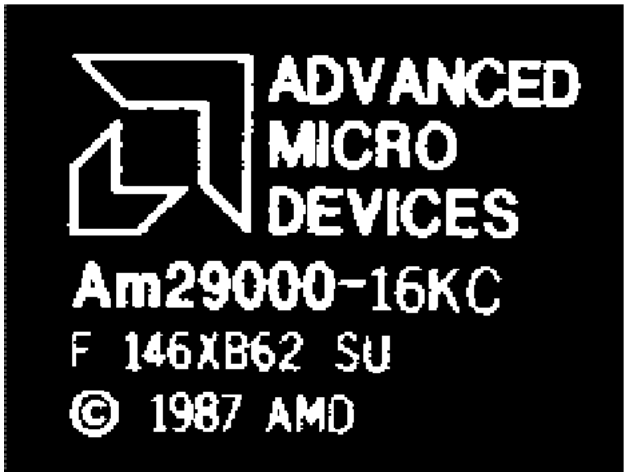
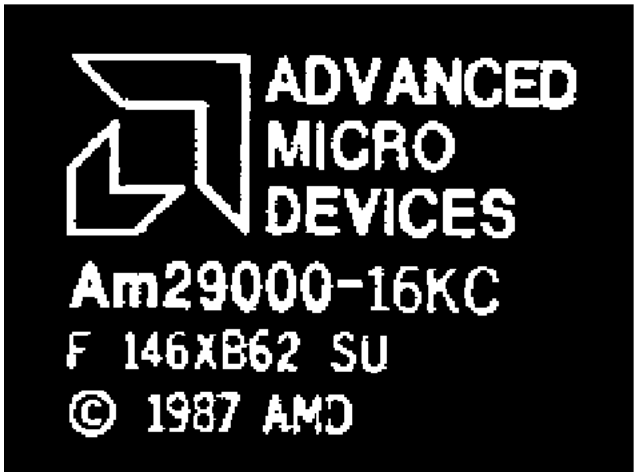


Image specific
Otsu threshold



Local Otsu
threshold

With noise

Pixels may fall on the wrong side of the threshold

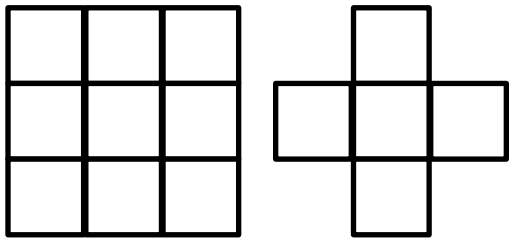
Threshold at 35:

Potential solution: Mathematical morphology
to enhance binary images,
e.g., remove isolated islands or holes

Mathematical Morphology: Basics

- Operations on binary images
- Shift-invariant
- Non-linear
- Based on pixel neighborhood defined by structural elements
- View binary image as a set
- Two main operations

Binary Image: A



Binary structural element: B

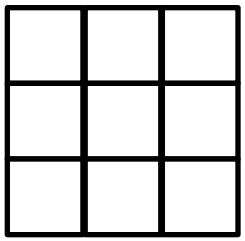
Dilation

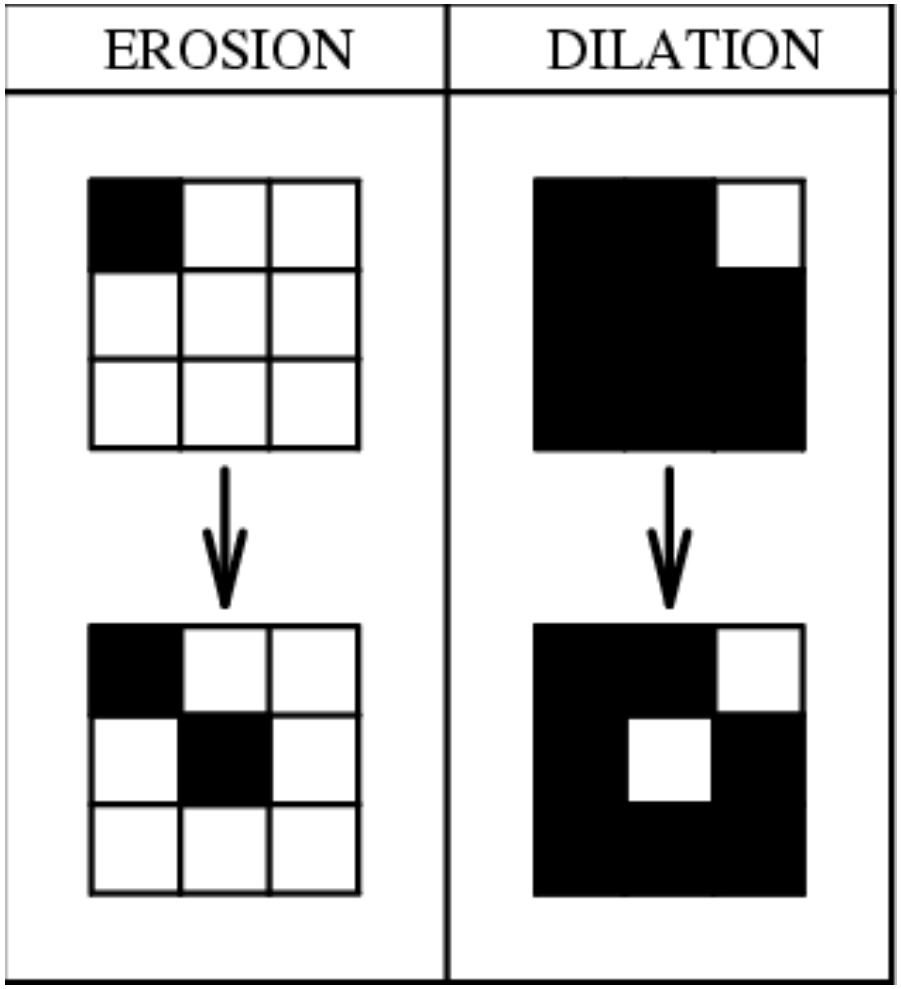
$$A \oplus B = \{x | B_x \cap A \neq \emptyset\}$$

Erosion

$$A \ominus B = \{x | B_x \subseteq A\}$$

Simple Illustration

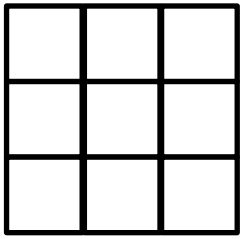




$$A \ominus B = \{x | B_x \subseteq A\} \quad A \oplus B = \{x | B_x \cap A \neq \emptyset\}$$

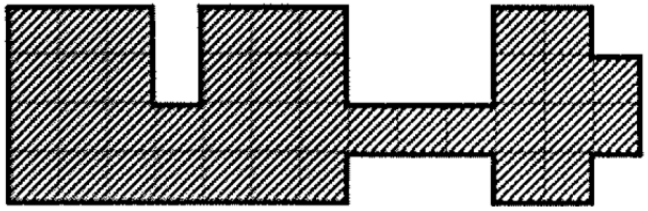
Erosion and Dilation: Example

Erosion

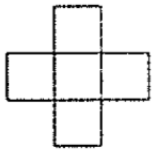


Dilation

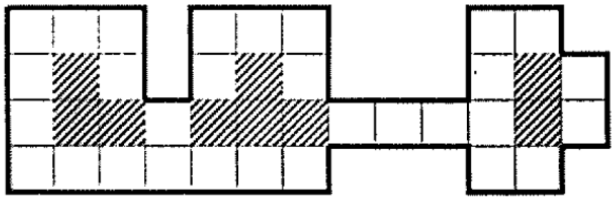
Concatenation of basic operations: Opening and closing



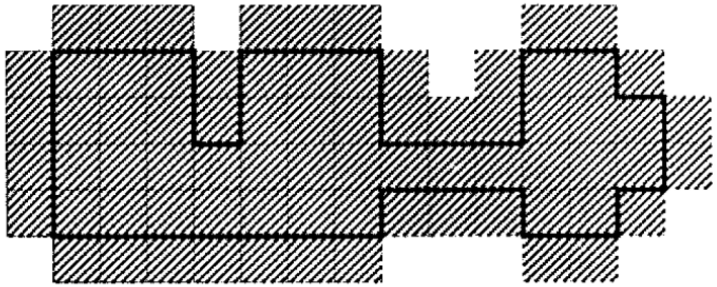
original image structure



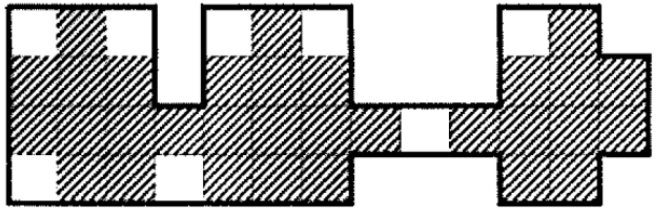
structuring element



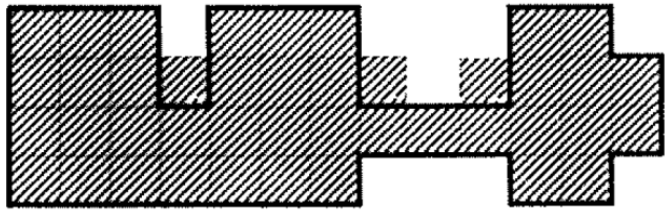
Erosion



Dilation

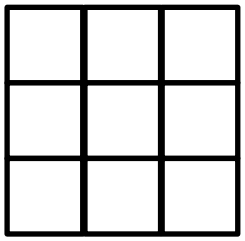


Opening $(A \ominus B) \oplus B$



Closing $(A \oplus B) \ominus B$

Opening and closing: Example



Opening
(mostly removes islands)

Closing
(mostly removes holes)

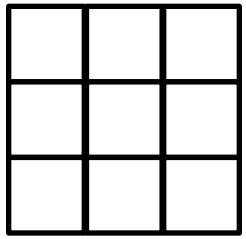
Binary enhancement as rank order operators

New intensity based on rank-ordered neighborhood values

Enables extension to gray level images

$$i_1 \leq i_2 \leq \cdots \leq i_N$$

$$i_t = f_t(i_1, \ldots, i_N)$$



Erosion	$i_t = i_1$
Dilation	$i_t = i_N$
Median filtering	$i_t = i_{N/2}$

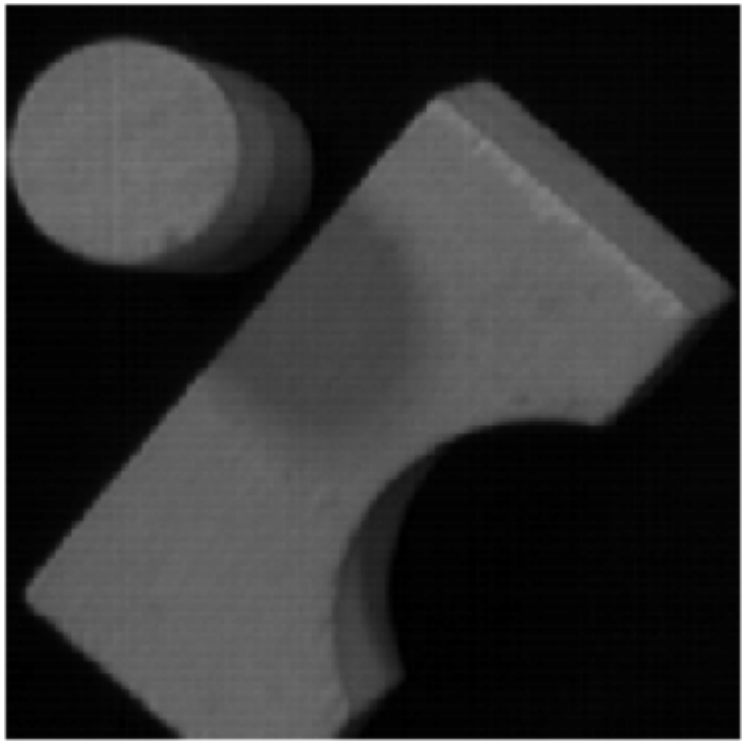
Important difference with convolution :

non-linearity

Binary enhancement : Remarks

- 1. Erosion + dilation (opening)
  Dilation + erosion (closing)
- 2. Use the same structural element for both steps
- 3. It is a post-processing approach
  (It has many alternatives that enforce neighborhood consistency during segmentation)
- 4. Reminder : median filtering very useful and commonly used as edge preserving smoothing

Multiple objects: Connected components



We would like to separate these objects

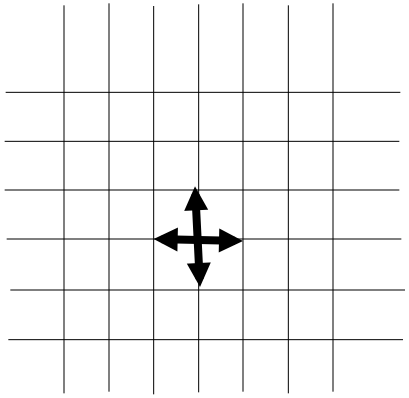
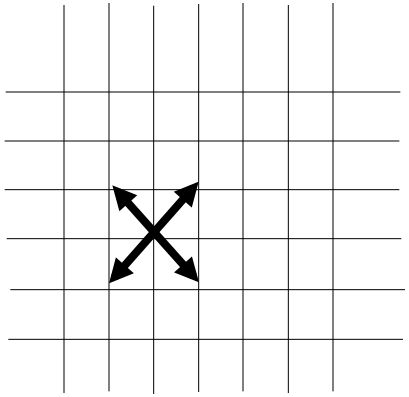
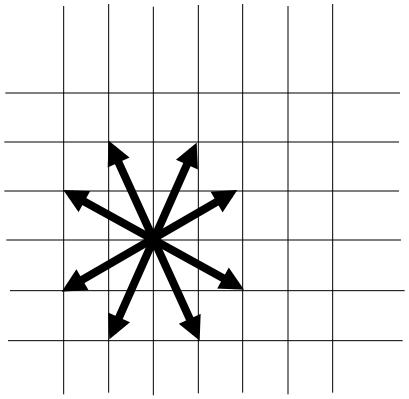
Connected component analysis

What does it mean to be a connected object?

Discrete image space: Neighbourhood on Cartesian image raster

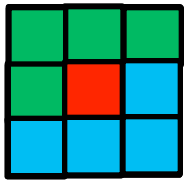
- Pixels connected through neighbourhood chain
- Connected component:
  if all its pixel pairs are connected through a chain of pixels all within the same component

- Defined by the pixel neighbourhood structure
- There is no unique definition
- 4- and 8-connectivity the most popular
- There are other possibilities

Single-pass connected-component labeling

- Scanning the image line-by-line (TV scan) enforces an (artificial) causality
- At every pixel (red) its neighbourhood is divided into past (green) and future (blue)



- Label red, considering all labels of past pixels. If
  - No label found: start a new label
  - 1 label found: copy it
  - >1 labels found: note their equivalence
- At the end, co-label equivalent components (connected but initially labeled as different)

Computer Vision

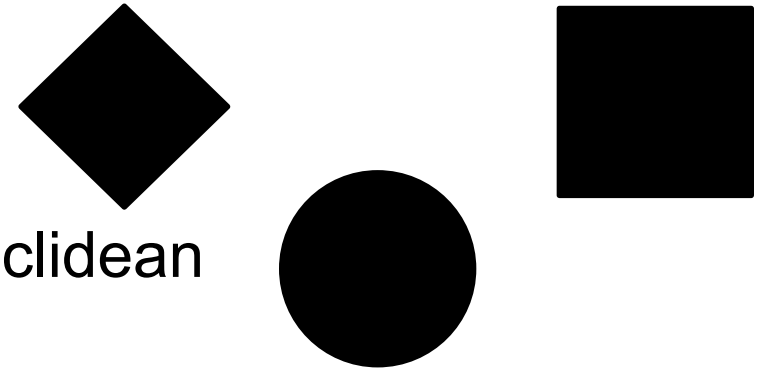
Topology induced (distance) metrics

- Depends on the chosen definition of image topology and neighborhood connectivity
- e.g., D_4 (Manhattan) and D_8 distances
- How to calculate between P(i,j) and Q(k,l)?

$$D_4(P,Q) = |i-k| + |j-l|$$

$$D_8(P,Q) = max(|i-k|, |j-l|)$$

- "Discs" (equidistant regions) in D_4 and D_8

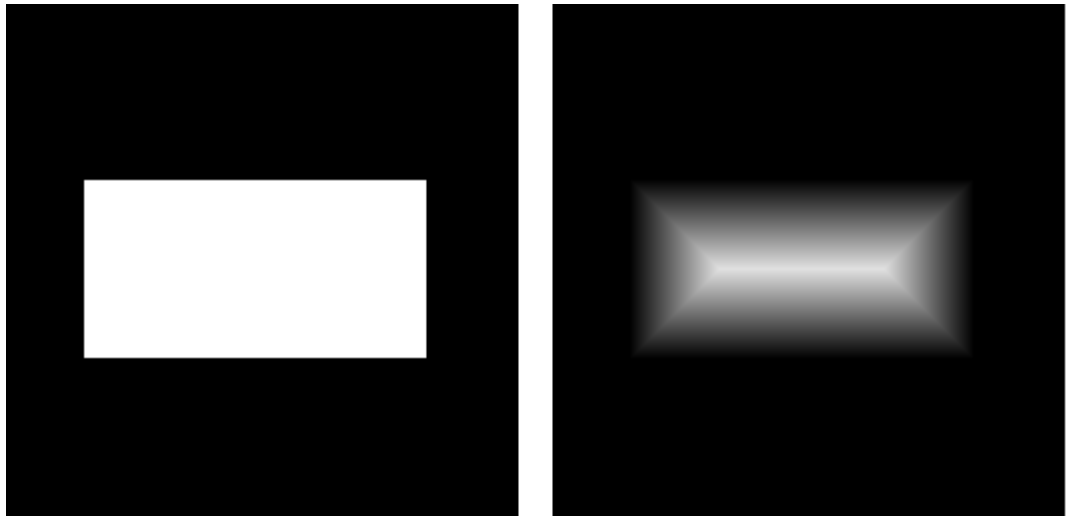


- Euclidean

does not conform with any discrete neighbourhood

Distance calculation

- Distance transformation: “distance map” based on distance propagation along neighbourhoods



- Euclidean distance map
  - True implementation is very cumbersome
  - Approximations are possible

Thresholding : remarks

- Threshold advantages:

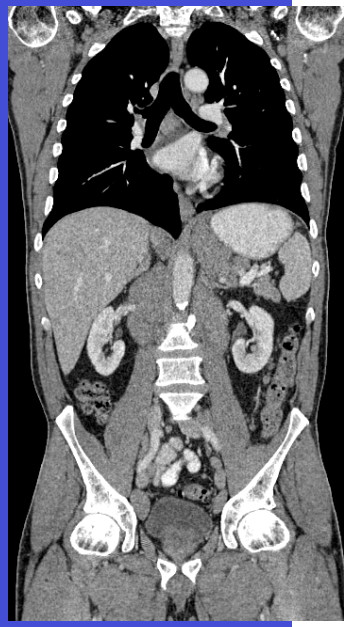
  1. Serious bandwidth reduction
  2. Simplification for further processing
  3. Availability of real-time hardware

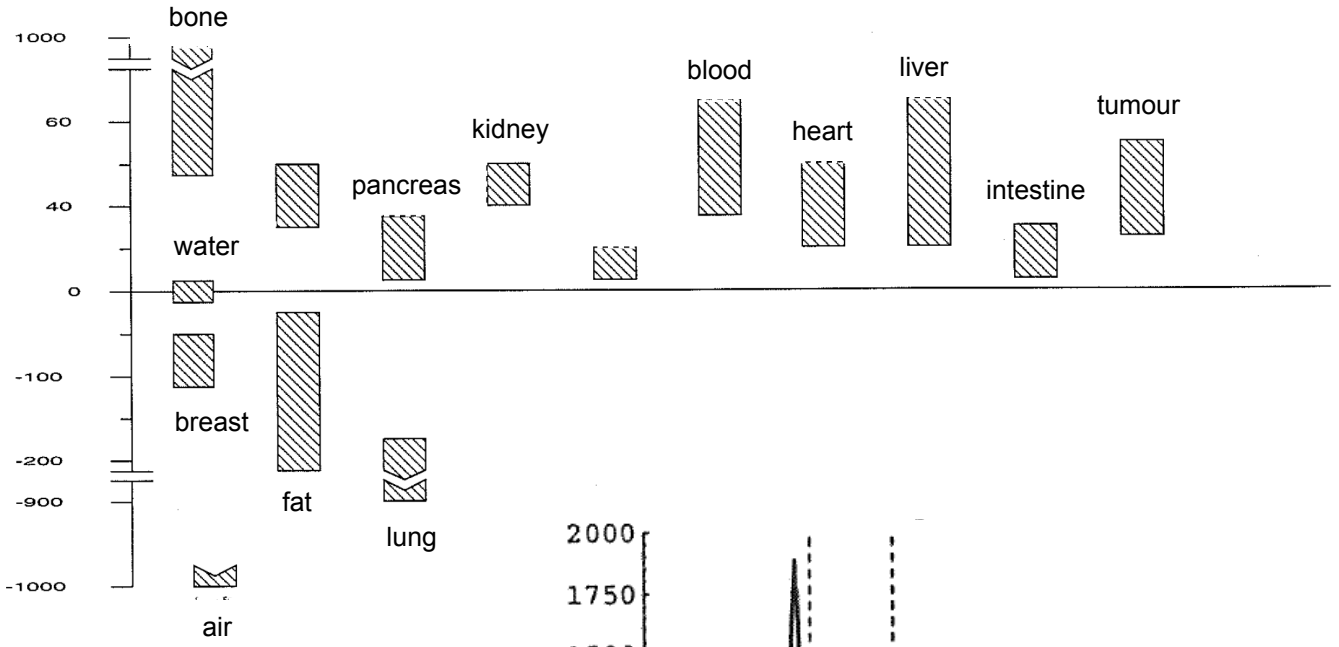
- Generally it won't provide a satisfying segmentation

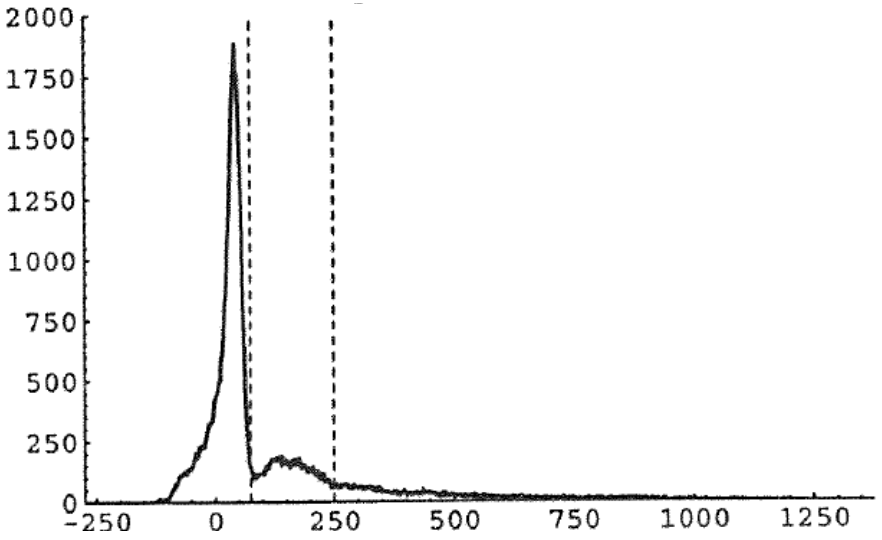
- Pixel-by-pixel decision
  - ignores neighbouring pixels
  - structural information lost

Thresholding has limitations

X-ray attenuation is tissue dependent







- strong overlap in intensity ranges
- thus, limited separability on histogram

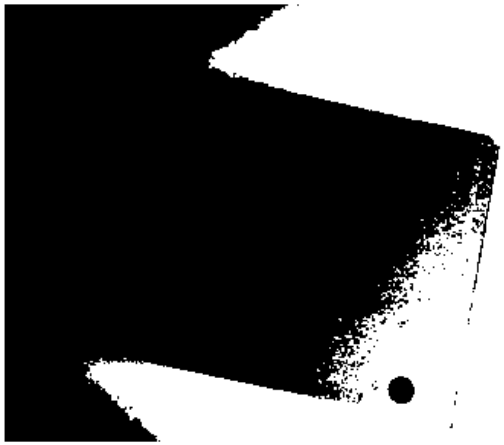
Segmentation : Outline

- Thresholding
- **Edge based**
- Region based
- Statistical Pattern Recognition based

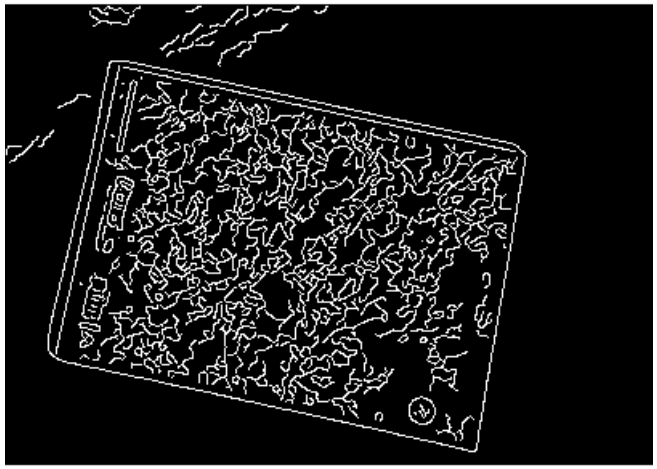
When thresholding is not enough, Edges can help

Identifying boundaries between different areas / objects

Image



After thresholding with Otsu criteria



Edge Detection with Canny

Edge linking techniques

- **Hough Transform**: for predefined shapes
- Elastically deformable contour models
  Snakes: generic shape priors
- Many other methods for grouping
  combination with user interaction

Hough transform: principle

Instead of testing every possible position and orientation for the given shapes, each pixel is visited while voting for all shapes it may belong to.

For this, Hough transform uses parametric shape models to extract objects in lower dimensional spaces.

The simplest example: straight lines
- the edge points vote for presence of a line model.

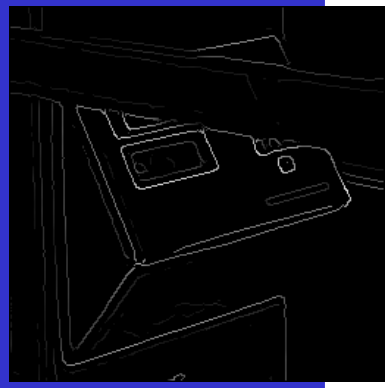
Many further possibilities, like circles, ellipses and generalizations to other shapes

Hough transform: straight lines

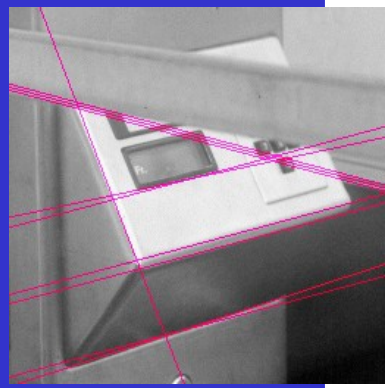
Given points that belong to a line, what is the line?

How many lines are there?

Which points belong to which lines?



Hough Transform is a voting technique that can be used to answer all of these questions.



Main idea:

1. Record vote for each possible line on which each edge point lies.
2. Look for lines that get many votes.

Slide credit: Kristen Grauman

Hough transform: straight lines

We write the equation of a straight line as

$$y = ax + b$$

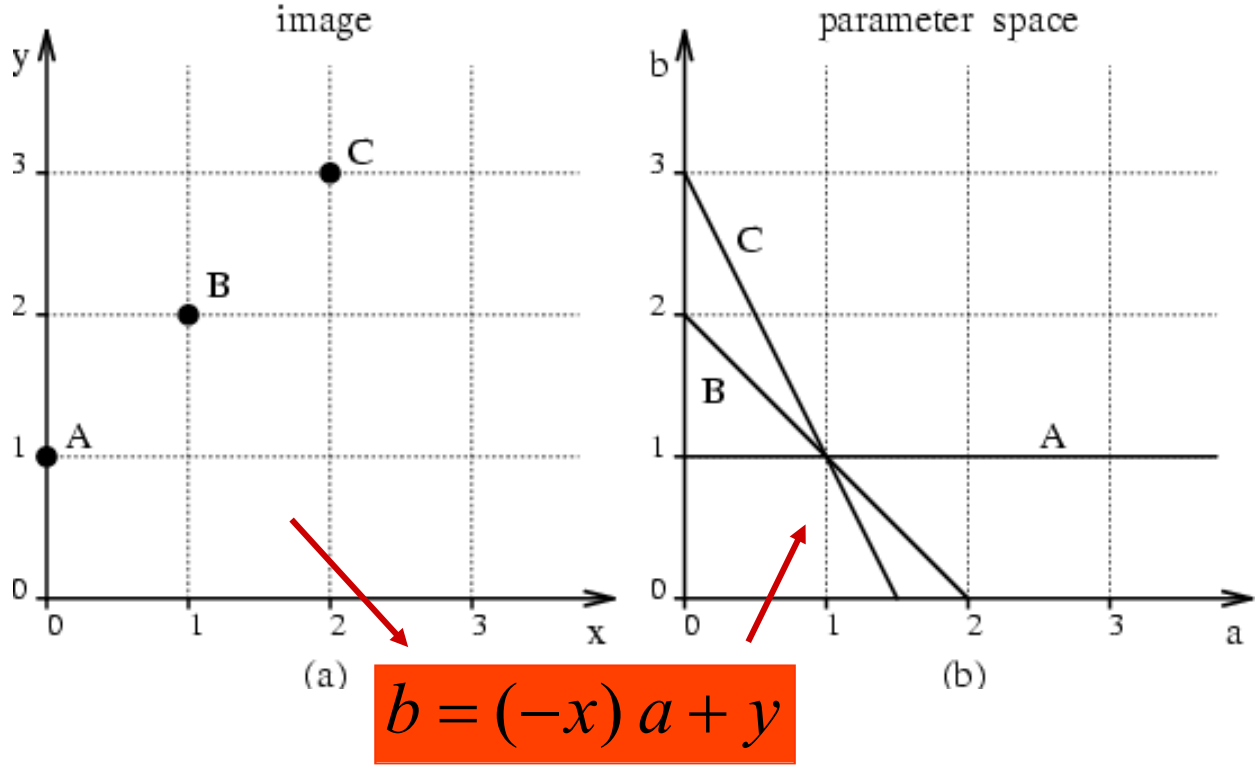
Fixing a point (x,y), all lines through the point :

$$b = (-x)\, a + y$$

The Hough transform:

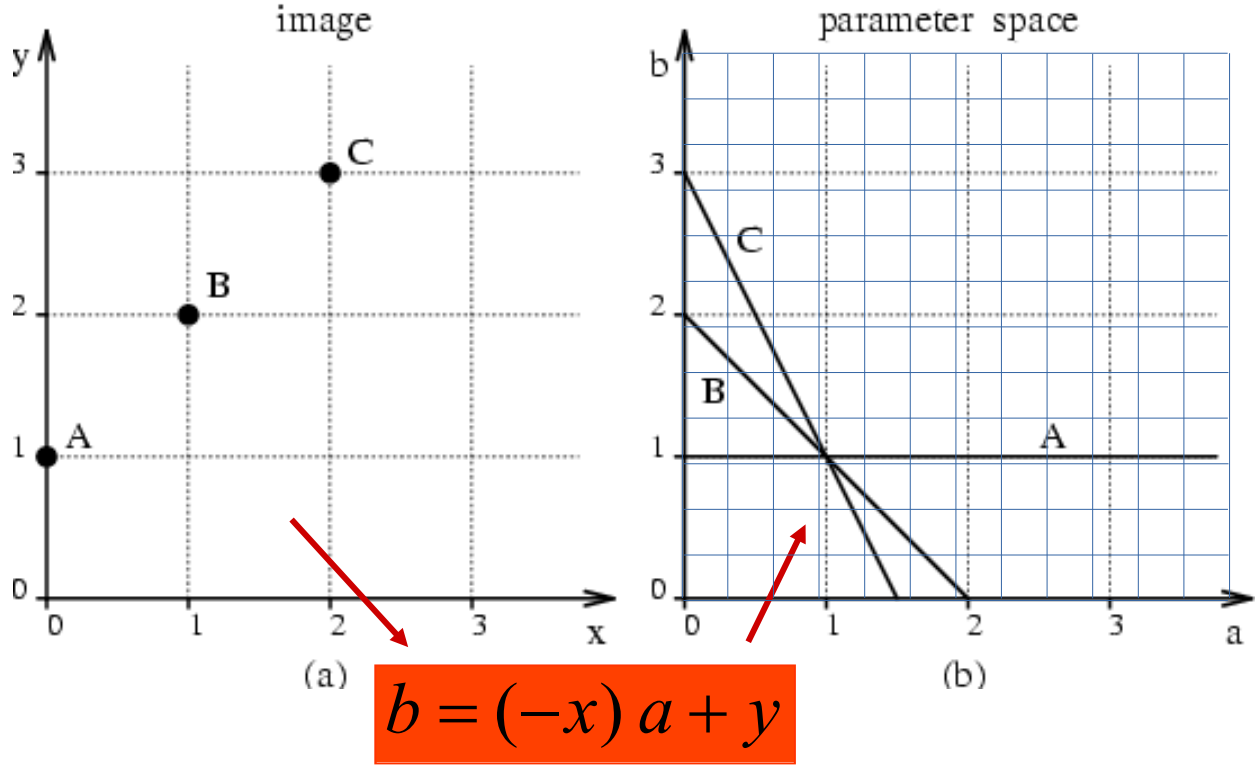
1. Inspect all (x,y) points of interest (edge points)
2. For each point (x,y) draws the parameter line (votes) in (a,b) parameter space

Hough transform: straight lines



a point in the image → a line in the parameter space
a line in the image → a point in the parameter space

Hough transform: straight lines

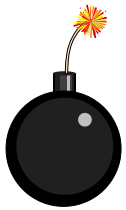


$$b = (-x)\,a + y$$

Implementation:

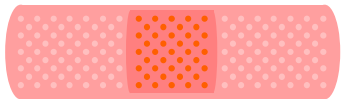
1. the parameter space is discretized (*aka* accumulator)
2. a counter is incremented at each bin/cell where the lines pass
3. peaks are detected → straight line solutions

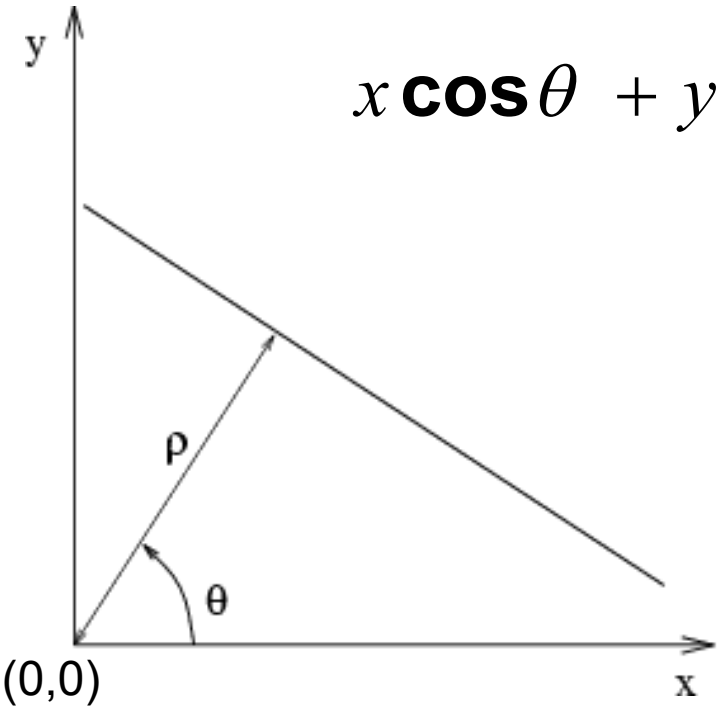
Hough transform: straight lines



Issues (a,b)-space: undefined for vertical lines, unbounded parameter domain

Solution: polar representation





$$x\cos\theta + y\sin\theta = \rho$$

ρ – perpendicular distance from origin to line

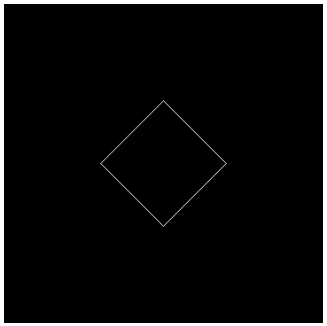
$\boldsymbol{\theta}$ - angle the perpendicular makes with x-axis

Each (x,y) point will add a sinusoidal function in the ($\boldsymbol{\theta}$,ρ) parameter space (accumulator)

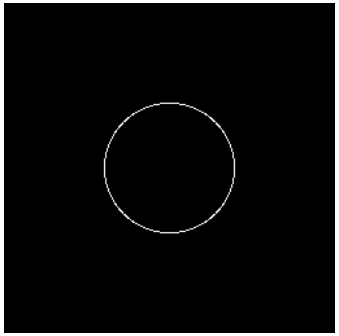
Computer Vision

Hough transform: straight lines

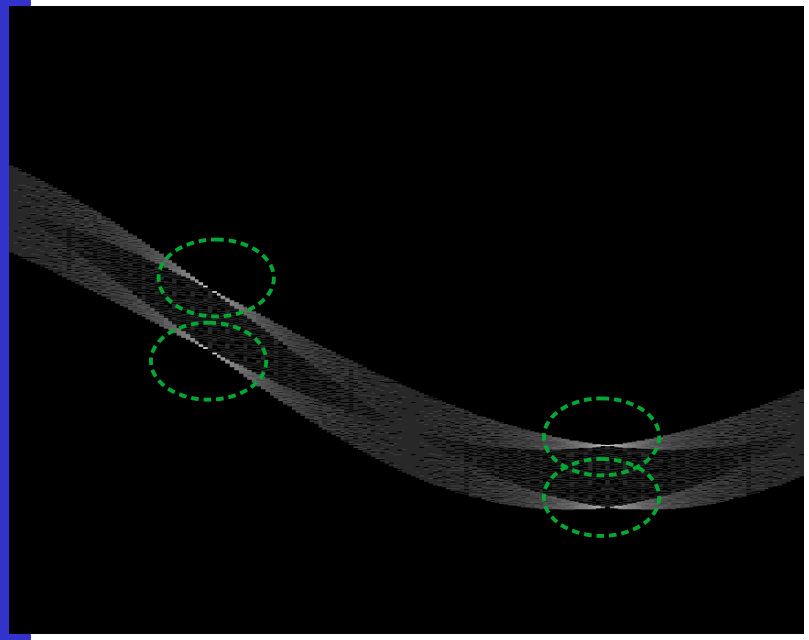
Square:



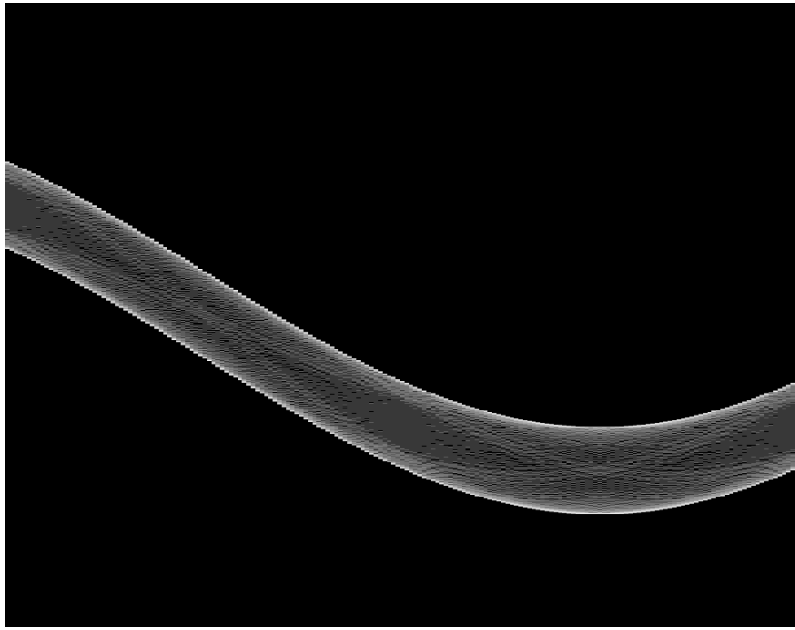
Circle:



4 peaks → 4 straight lines



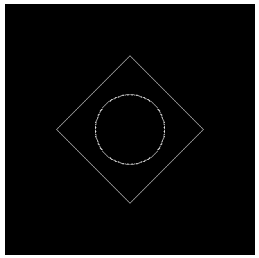
No clear peak

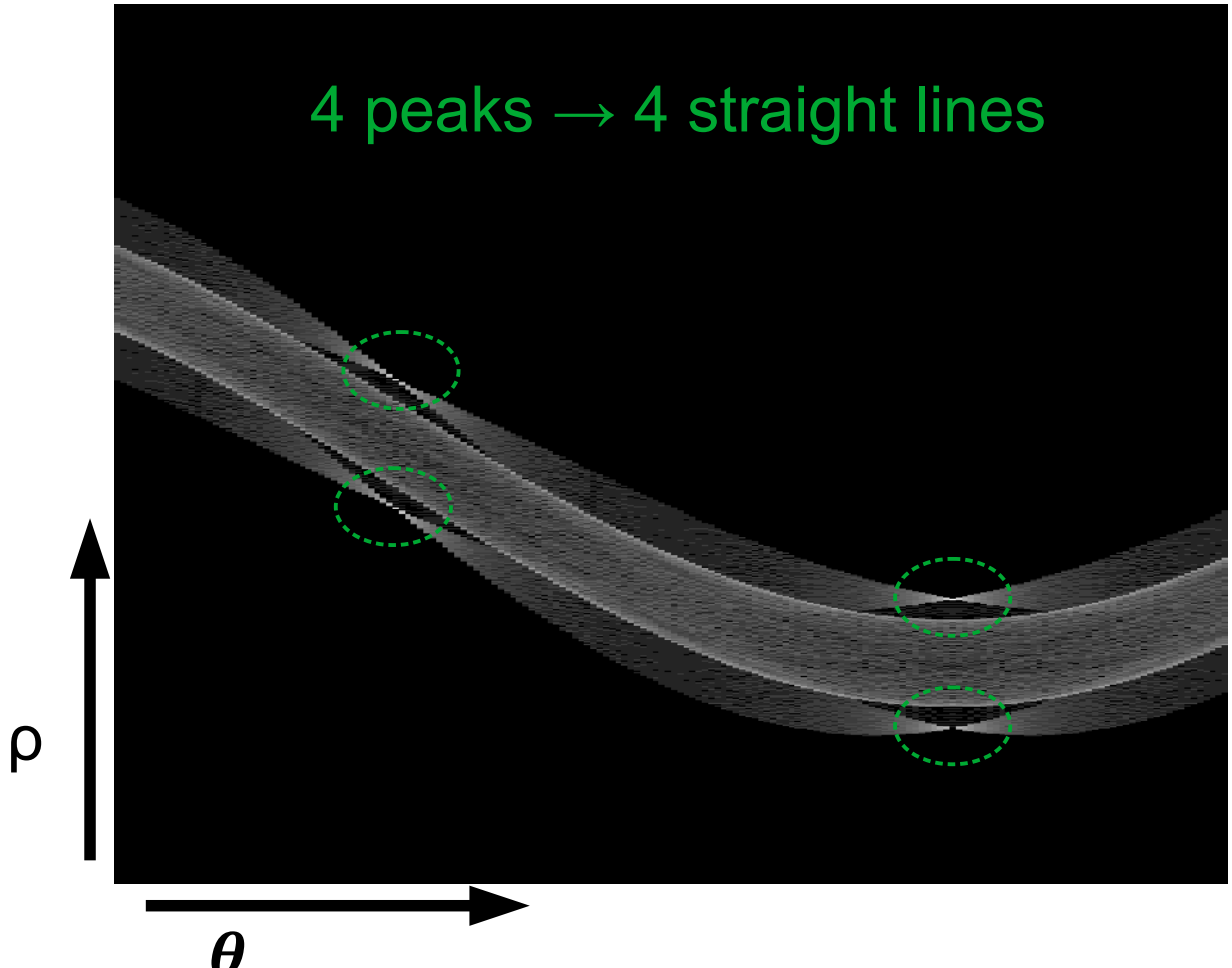


(normalized [0,1] values)

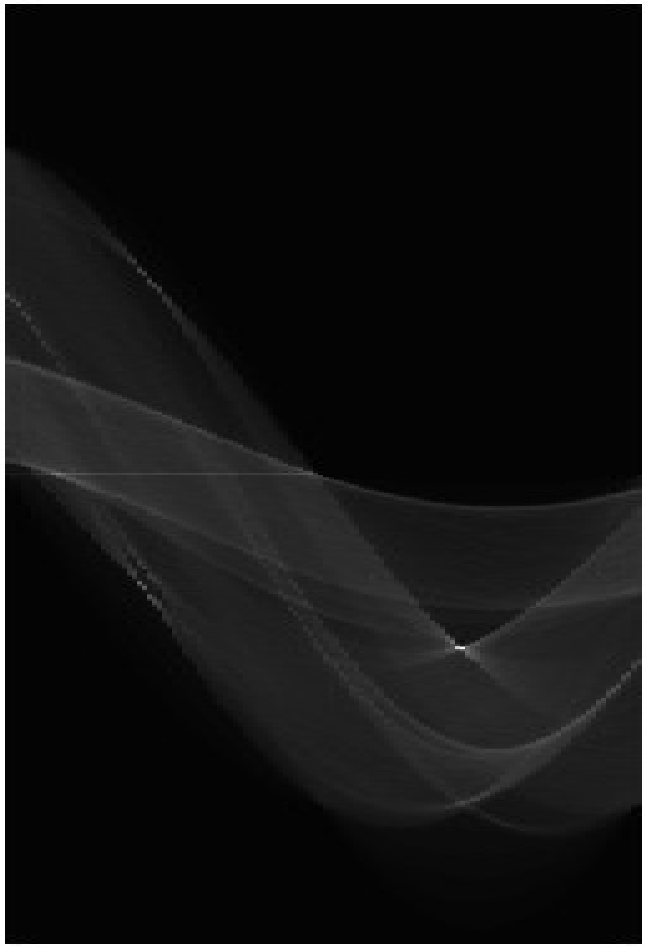
Hough transform: straight lines

Combined image:





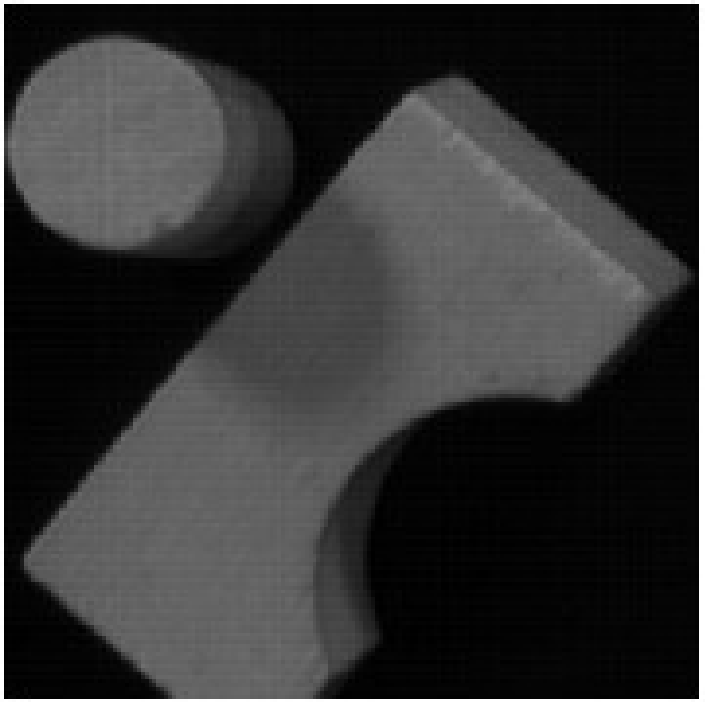
Hough transform: straight lines



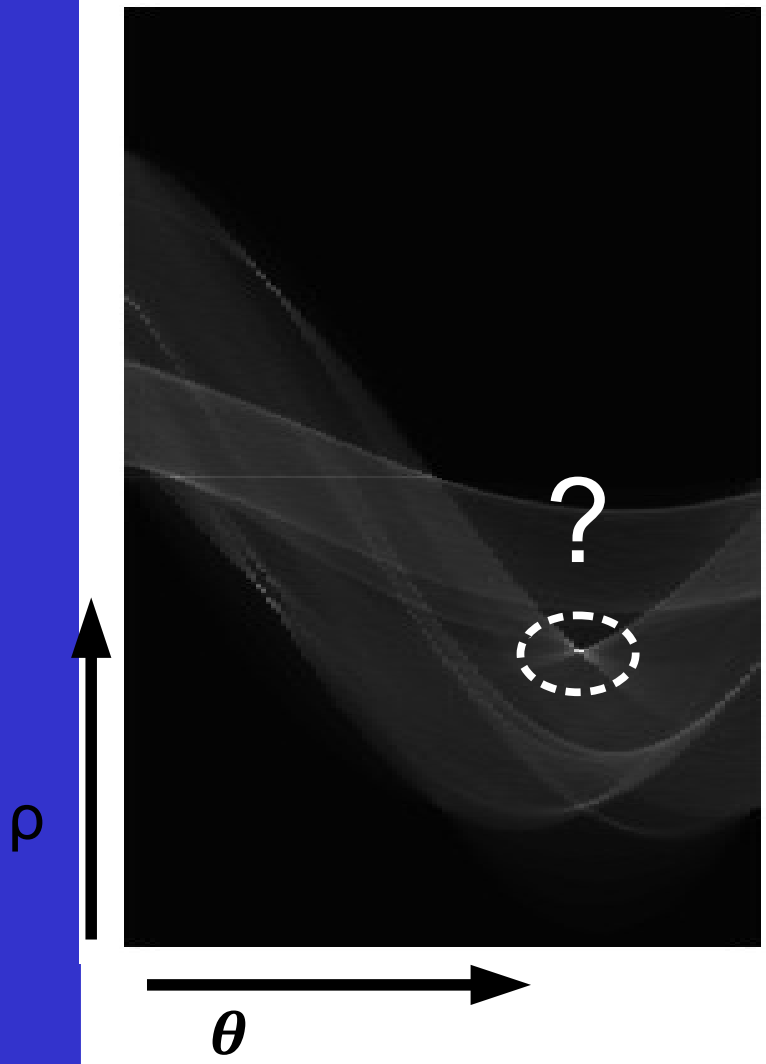
ρ

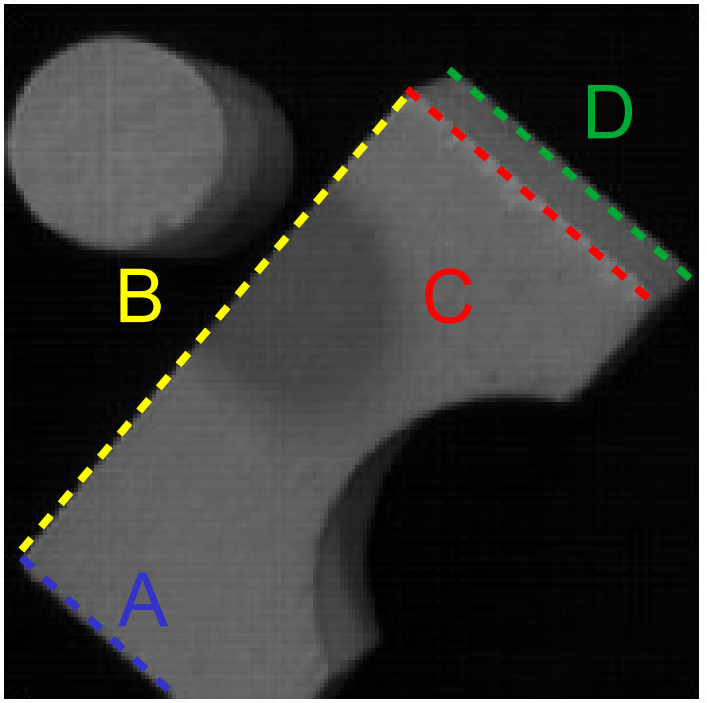
θ

(normalized [0,1] values)



Hough transform: straight lines

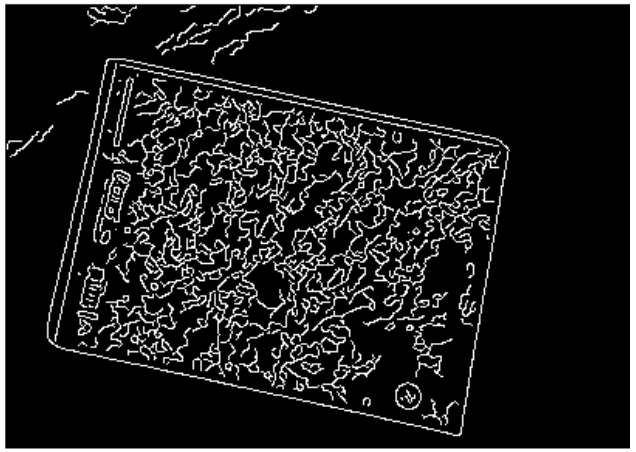


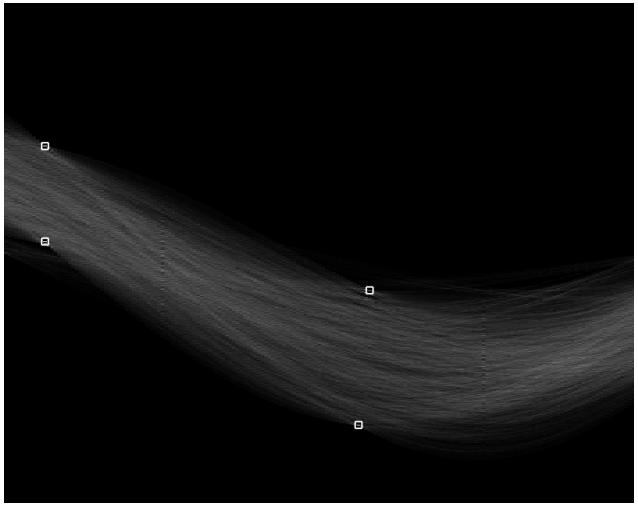


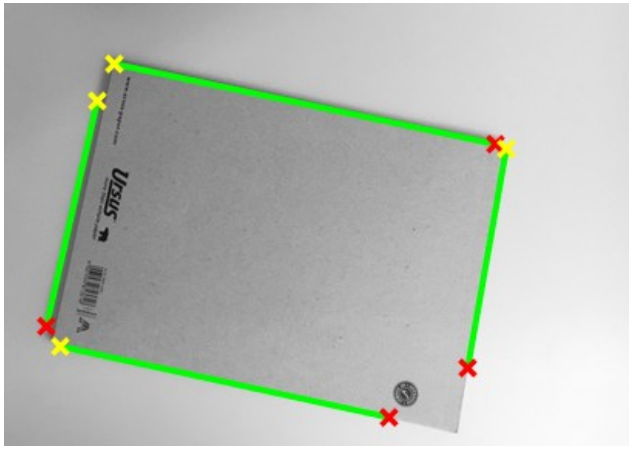
Which line generated the maximum peak?

Computer Vision

Hough transform: detecting the book







Hough transform: practical tips

- Minimize irrelevant voting points
- Choose a good discretization / grid
- Vote for neighbors, also (smoothing in accumulator)
- Weight the votes (e.g. by intensity gradient magnitude)
- Use everything you know (e.g. direction of the edge)

A demo:
http://dersmon.github.io/HoughTransformationDemo/

Hough transform: pros and cons

Pros:

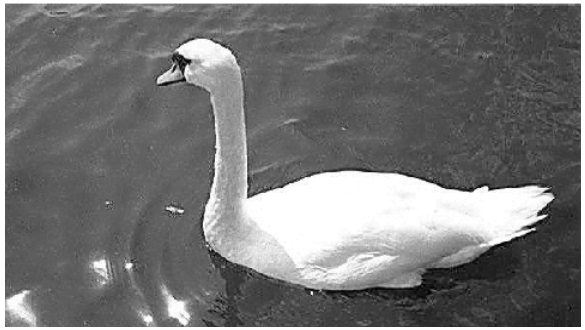
- All points are processed independently, so can cope with occlusions, gaps
- Some robustness to noise: noise points unlikely to affect the voting outcome
- Can detect multiple instances of a shape model in a single pass

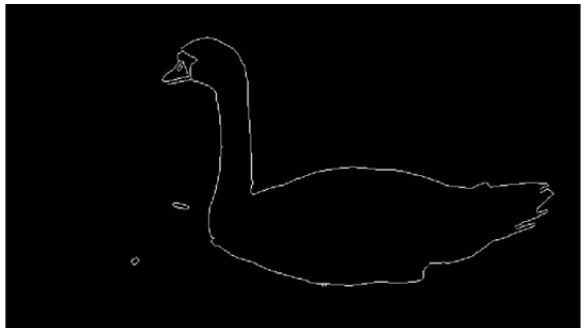
Cons:

- Time complexity exponential in number of model parameters
- Ambiguities are possible (spurious peaks) – if similar shapes are close by
- Discretization: can be tricky to pick a good grid size
- “Good” peak detection is nontrivial

Edges vs. boundaries

Edges are useful to infer shape and occlusion, e.g.





Here the raw edge output is not so bad

But, quite often the boundaries of interest are fragmented, and we have a set of “cluttered” edges

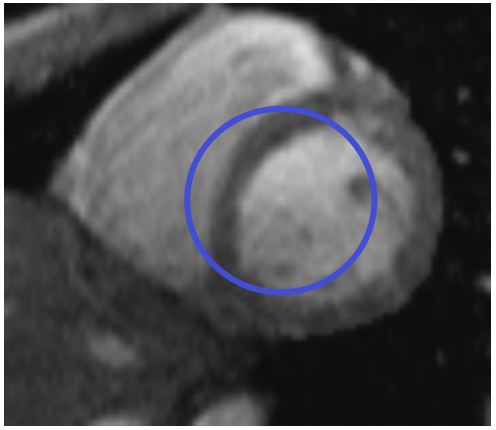
Images from D. Jacobs

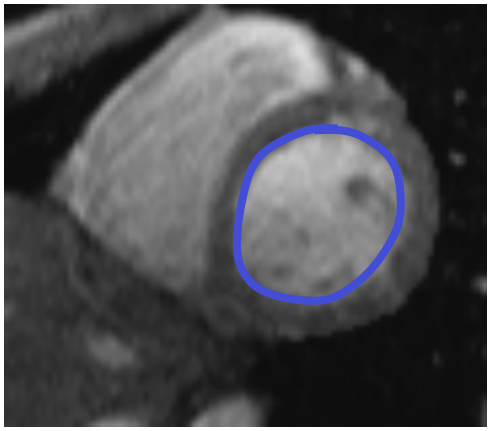
Active contour models: **Snakes**

[Snakes: Active contour models, Kass, Witkin, & Terzopoulos, ICCV1987]

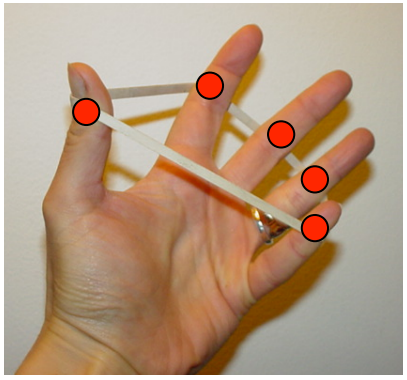
Given: initial contour (model) near desired object

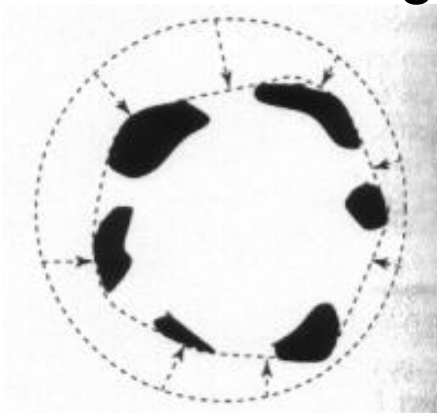
Goal: evolve the contour to fit the object boundary





Intuition: an elastic (rubber) band wrapping around structures to cover / fill-in missing parts





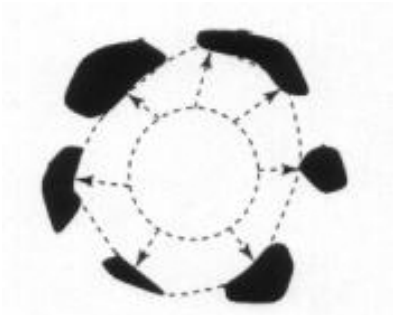
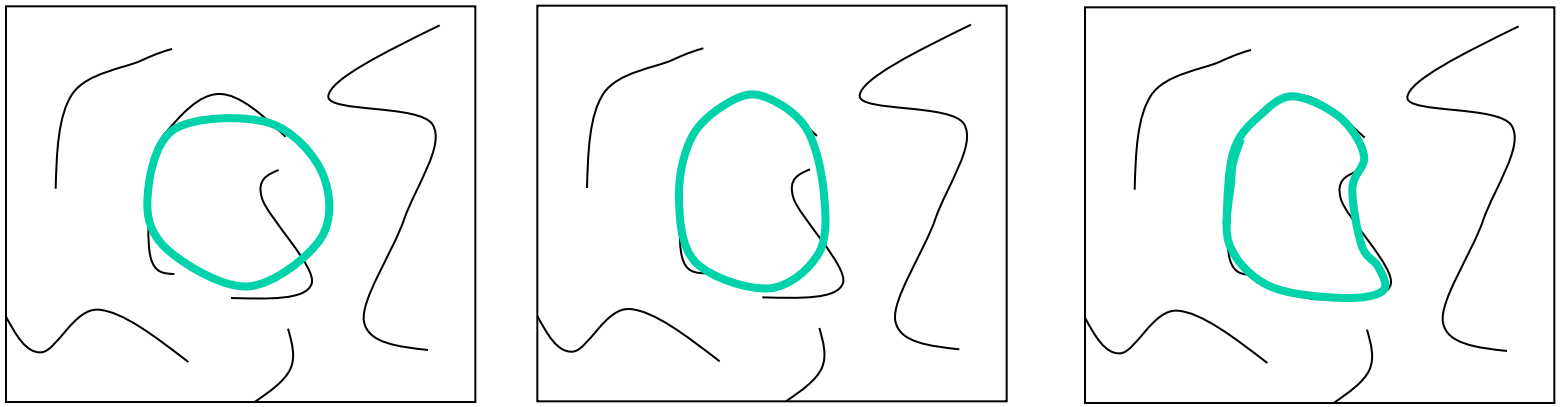


Image from http://www.healthline.com/

Shapiro & Stockman

Snakes: Idea

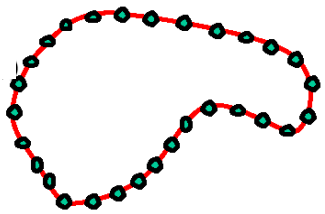


How to adjust (iteratively refine) an elastic band for a particular purpose, e.g., to fit high image gradients

- Define an energy function that says how good any configuration is
- Seek next configuration that minimizes energy

Snakes energy function

The total energy of the current snake is:



$$E_{total} = E_{internal} + E_{external}$$

Internal energy: encourage prior shape preferences: e.g. smoothness, elasticity, known shape prior

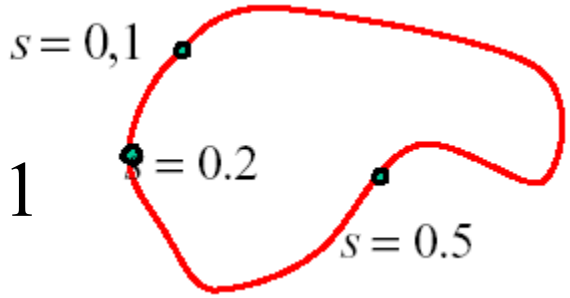
External energy (image energy): encourage contour to fit interesting image structures, e.g. edges

A **good** fit between the current snake and the target shape in the image will yield a **low** energy

Parametric curve representation

Continuous case:

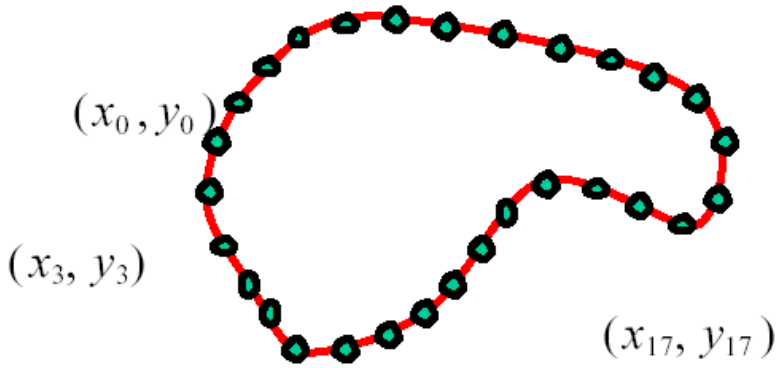
$$v(s) = (x(s), y(s)) \qquad 0 \le s \le 1$$



For numerical computation on the image

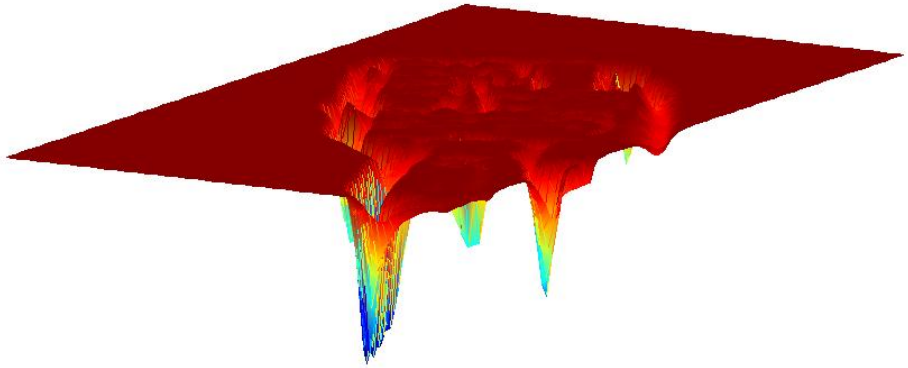
Discretization by a set of *n* points:

$$v_i = (x_i, y_i) \qquad i = 0 \ldots n-1$$



External (image) energy

- Measure how well the curve matches the image data
- Attracts the curve toward interesting image features
  - Edges, lines, etc.



- (Magnitude of gradient)

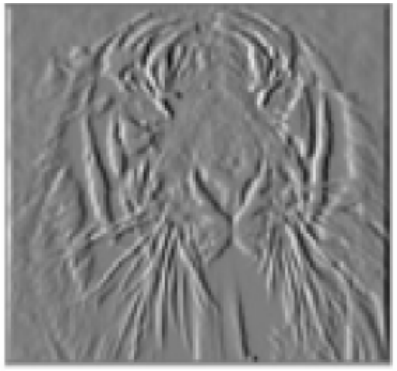
$$-\left(G_x(I)^2 + G_y(I)^2\right)$$

Defines how the image (edges) affect rubber band

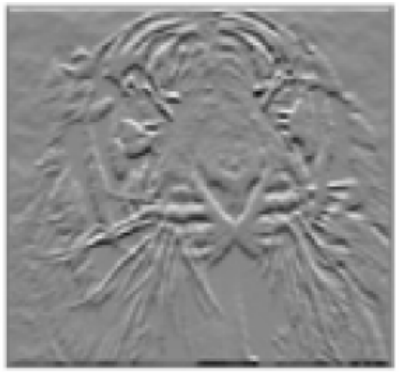
Think of it as gravitational pull towards regions of, e.g., high contrast

External (image) energy

- Image *I(x,y)*
- Directional derivatives



$G_x(x, y)$



$G_y(x, y)$

- External energy at a point *v(s)* on the curve:

$$E_{external}(v(s)) = -(|G_x(v(s))|^2 + |G_y(v(s))|^2)$$

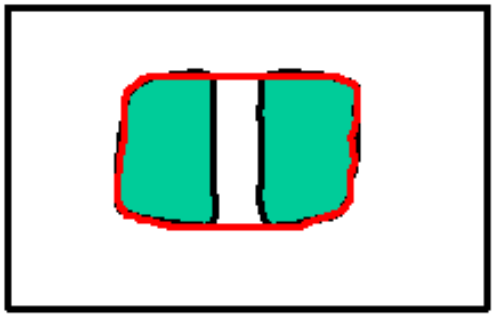
$$E_{external} = \int_0^1 E_{external}(v(s))\,ds$$

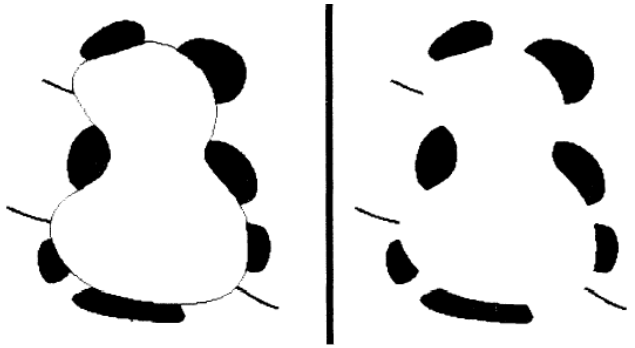
- External energy for the curve on discrete image

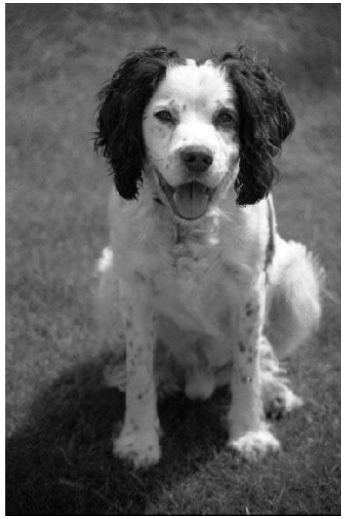
$$E_{external} = -\sum_{i=0}^{n-1} |G_x(x_i, y_i)|^2 + |G_y(x_i, y_i)|^2$$

Internal energy: intuition

Given that typical shapes smoothly continuous boundaries, we thus wish to favor smooth contours with low curvature (to fill-in missing info)







Internal energy

A common choice: Deformation Energy

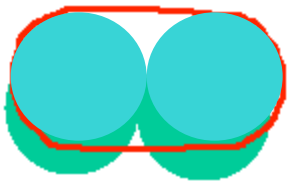
- The more stretch and bend, the larger this energy value is
- Weights α and β adjust influence of each component

$$E_{internal} = \int_0^1 E_{internal}(v(s))\,ds$$

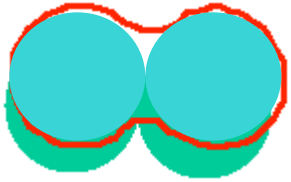
$$E_{internal}(v(s)) = \alpha \left| \frac{dv}{ds} \right|^2 + \beta \left| \frac{d^2 v}{d^2 s} \right|^2$$

Models elasticity
Penalizes tension
(inhibits stretch)

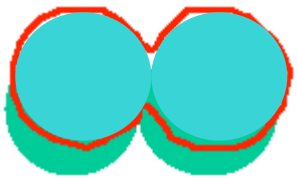
Models stiffness
Penalizes curvature
(inhibits bending)



β: large



medium



small

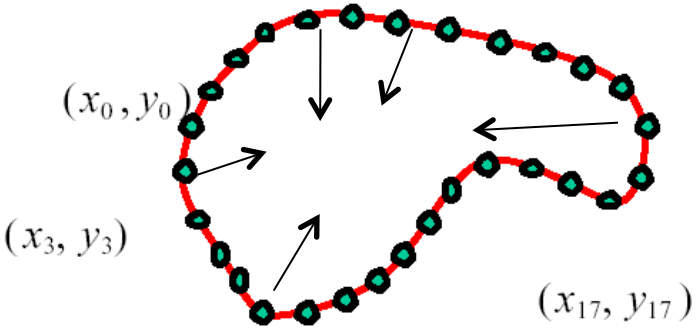
Internal energy: Discretization

$$\frac{dv}{ds} \approx v_{i+1} - v_i \qquad \frac{d^2v}{ds^2} \approx (v_{i+1} - v_i) - (v_i - v_{i-1}) = v_{i+1} - 2v_i + v_{i-1}$$

$$E_{internal} = \sum_{i=0}^{n-1} \alpha \left\| v_{i+1} - v_i \right\|^2 + \beta \left\| v_{i+1} - 2v_i + v_{i-1} \right\|^2$$

Elasticity, Tension

Stiffness, Curvature



First-term prefers shorter curves.
Problem: This encourages a closed curve to shrink, eventually, to a cluster of coincident points

Possible remedy: adjusting energy term

$$E_{internal} = \sum_{i=0}^{n-1} \alpha \left(\left\| v_{i+1} - v_i \right\| - 1 \right)^2 + \beta \left\| v_{i+1} - 2v_i + v_{i-1} \right\|^2$$

Encourages equal spacing but makes optimization harder

Energy minimization

Several methods proposed to fit snakes, including methods based on :

- Partial Differential Equations (PDEs)
- Greedy search
- Dynamic programming (for 2D snakes)

Energy minimization through PDEs

Energy to minimize in the continuous case:

$$\frac{1}{2}\int_0^1 \left(\alpha(s)\left\|\frac{\partial v}{\partial s}\right\|^2 + \beta(s)\left\|\frac{\partial^2 v}{\partial s^2}\right\|^2\right) ds - \int_0^1 P(v)ds$$

Deformation energy

Image energy (gradients)

Variational calculus → Euler-Lagrange differential equation

$$-\frac{\partial}{\partial s}\left(\alpha(s)\frac{\partial v}{\partial s}\right) + \frac{\partial^2}{\partial s^2}\left(\beta(s)\frac{\partial^2 v}{\partial s^2}\right) = -\vec{\nabla}P(v)$$

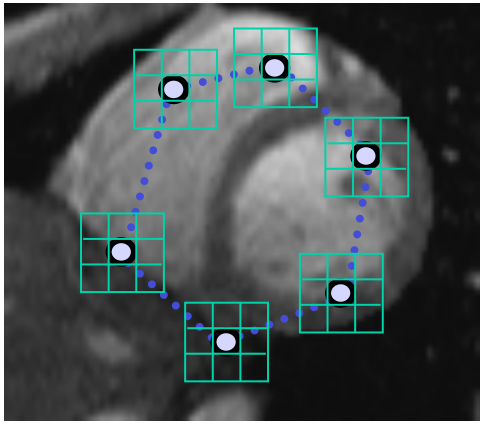
$$-\alpha v_{ss} + \beta v_{ssss} = -P_v$$

In an iterative scheme, use image gradients at curve positions in the previous iteration *t-1*, and hence solve:

$$-\alpha v_{ss}^t + \beta v_{ssss}^t = -P_v\Big|_{v=v^{t-1}} \quad \Rightarrow \quad \mathbf{K}v^t = -P_v\Big|_{v=v^{t-1}}$$

Greedy energy minimization

- For each point, place a search grid (e.g., 5x5) of discrete positions around it and pick the location where the energy function is minimal
- Keep applying this and circling around the curve
- Stop when a predefined number of points have not changed in the last iteration,
  or after max number of iterations
- Remarks:
  - Individual decisions are not globally optimal
  - Convergence not guaranteed



Energy minimization: dynamic programming

Snake energy can be rewritten as a sum of
* unary potentials for individual points and
* interaction potentials between pairs of points

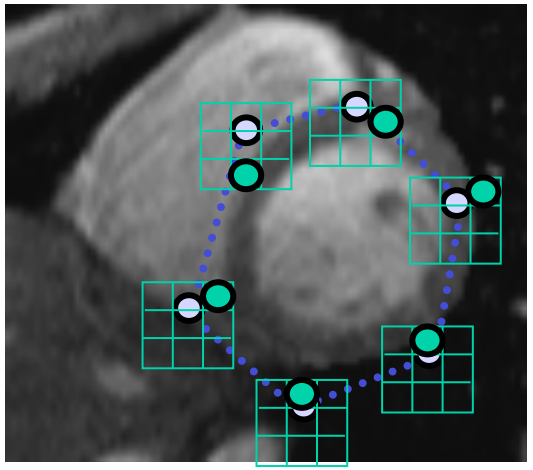
$$E_{total}(v_0,\ldots,v_{n-1}) = \sum_{i=0}^{n-1} U_i(v_i) + \sum_{i=0}^{n-2} P_i(v_i, v_{i+1})$$

Image energy (gradients)

Deformation energy (good for tension, but for bending?)

We can minimize this form of energy function, using **dynamic programming**, with the *Viterbi* algorithm*:*

1. Forward-pass over nodes X positions to find accumulated optimal energies
2. Backtrack to find best positions

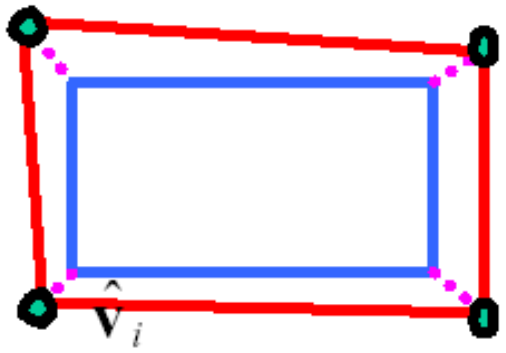


Complexity: $O(nh^2)$ vs brute force search ____?

Optional: specify shape prior

If object is some smooth variation of a known shape, we can add a term to penalize deviations from that shape:

$$\delta \sum_{i=0}^{n-1} (v_i - \hat{v}_i)^2$$

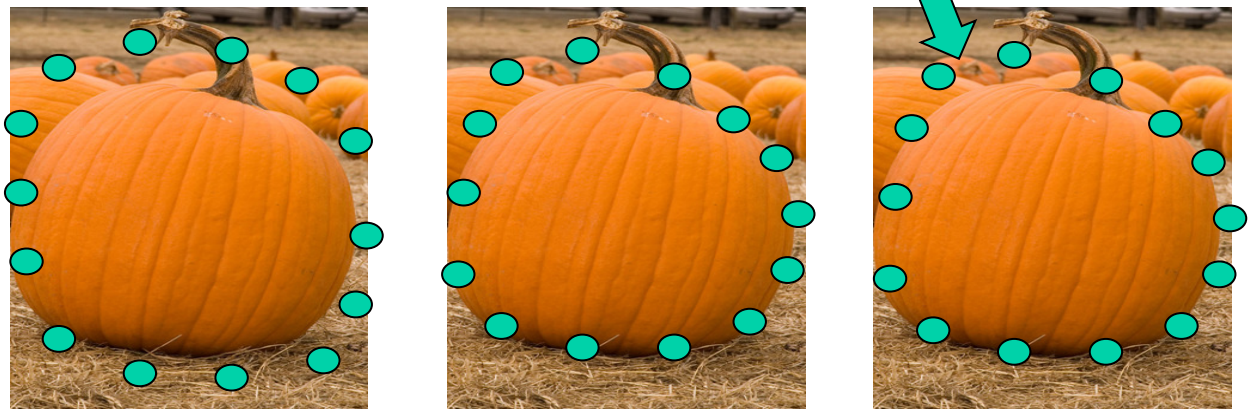


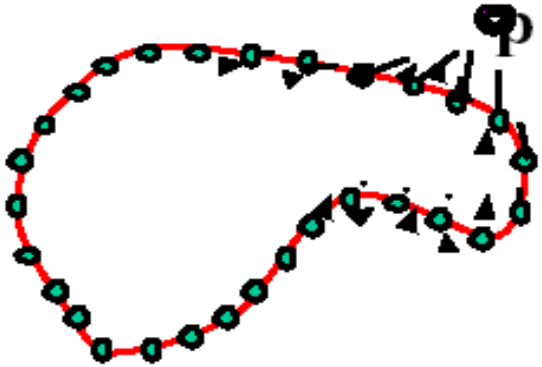
where $\{\hat{v}_i\}$ are the points of the known shape

Pressure or Interactive forces

Quite easy to define additional / alternative forces in the energy:

- Constant pressure can push the curve outside; aka. balloons
- Energy function can be altered online based on user input;
  e.g. push or pull the snake points with the mouse pointer
- Some heuristic force
  e.g. avoiding image borders, utilizing output of another algorithm



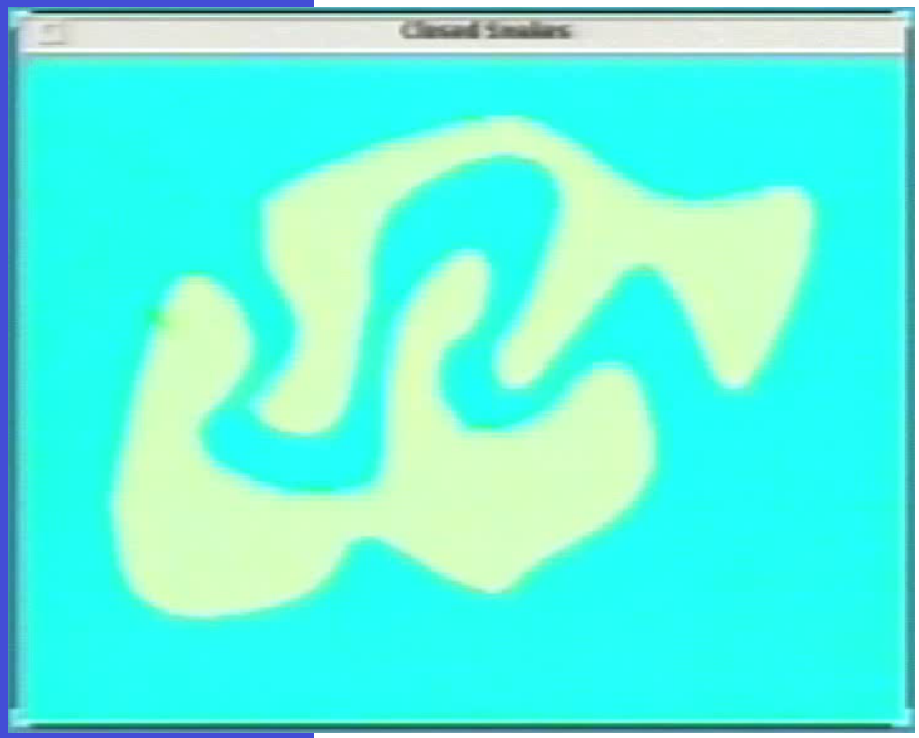


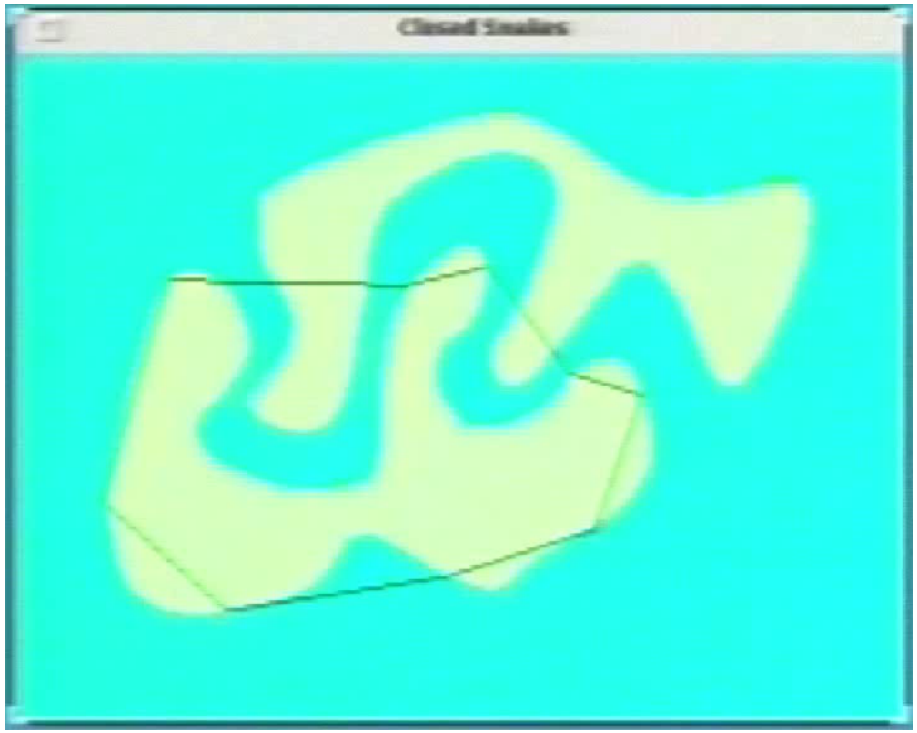
$$E_{push} = + \sum_{i=0}^{n-1} \frac{r^2}{|v_i - p|^2}$$

distance of snake points to pointer *p*

"Elastic springs" attached to snakes points push them away from the pointer (p) with nearby points pushed the hardest

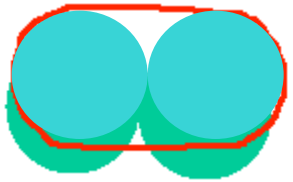
Snakes with Interactive Forces
Example video



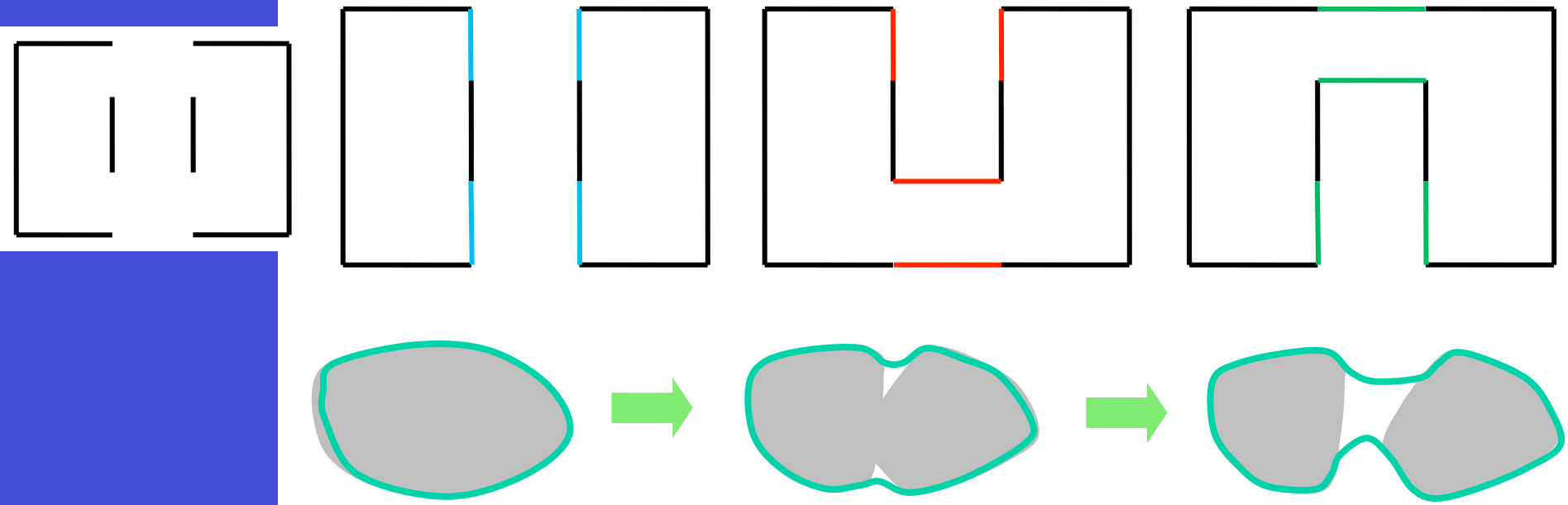


Limitations

- Smoothing choice is critical,
  not to over-smooth the boundary



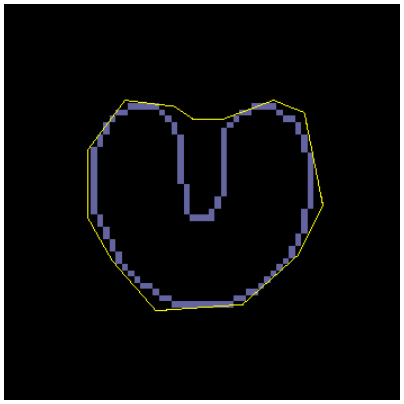
- Not robust to topological differences changes,
  e.g., “what is a gap to fill in?”, un/connected components

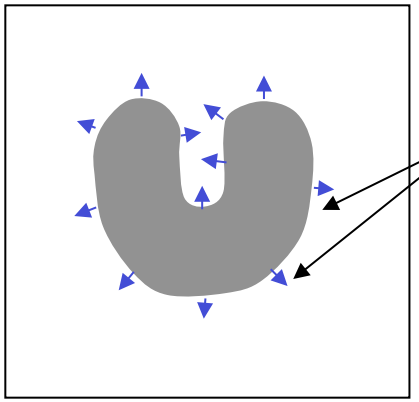


Limitations: Only locally optimal

Snakes only "see" nearby object boundaries

i.e. the external energy does not consider the edges far away from the curve (determined by gradient kernel, DP search box, etc)



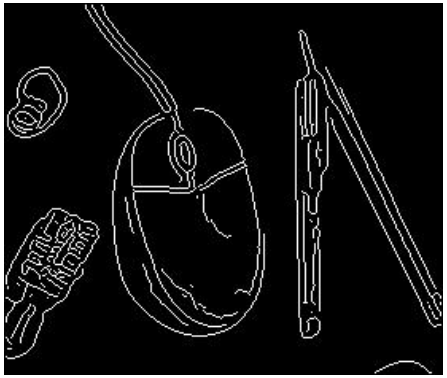


Potential remedy: External energy based on the **distance** from edges (makes snakes "farsighted")

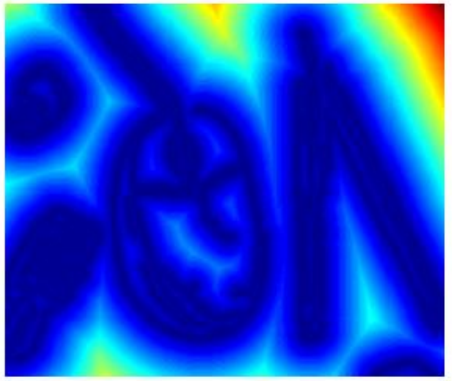
Values tell how far each location is from nearest edge

original

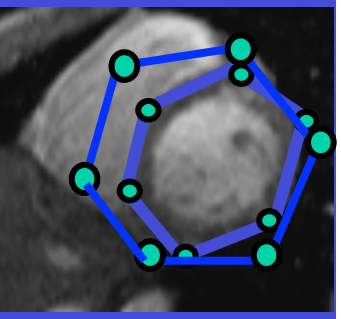
gradient



edges



distance map



Snakes: Summary

- Framework to fit deformable contours via optimization
- Define a curve as a set of *n* points, an internal deformation and an external image-based energy
- Initialize "near" object boundary, and iteratively optimize the curve points to minimize the total energy

Pros:

- Useful to fit non-rigid arbitrary prior shapes in images
- **Contour remains connected, i.e. topology is fixed**
- Possible to connect / fill in invisible contours
- **Flexibility in energy function definition,** i.e., allows other forces and interactive input

Cons:

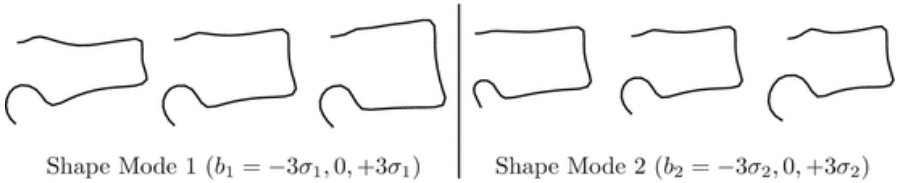
- Local optimization: may get stuck in local minimum
  Thus, needs good initialization near true boundary
- Susceptible to parameterization of energy function, must be set based on prior information, experience, etc.

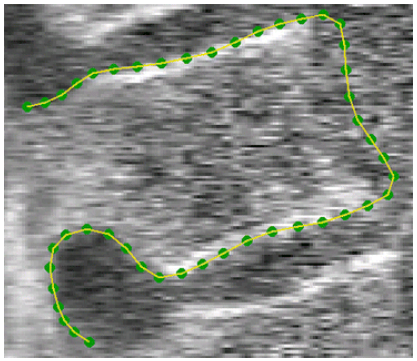
Computer Vision

Active Shape and Appearance Models

Shape model: Implicit energy via fitness to a statistical model

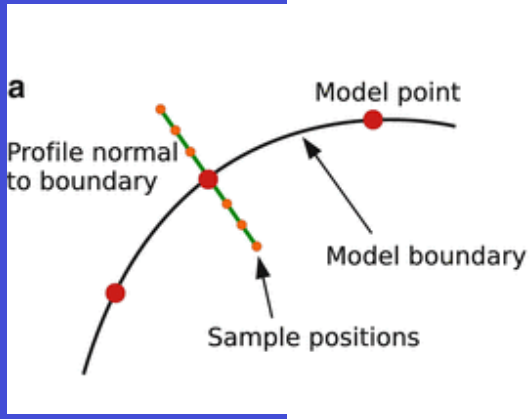
(remember PCA)

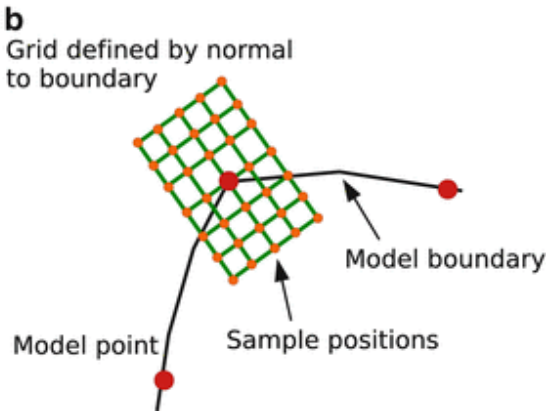


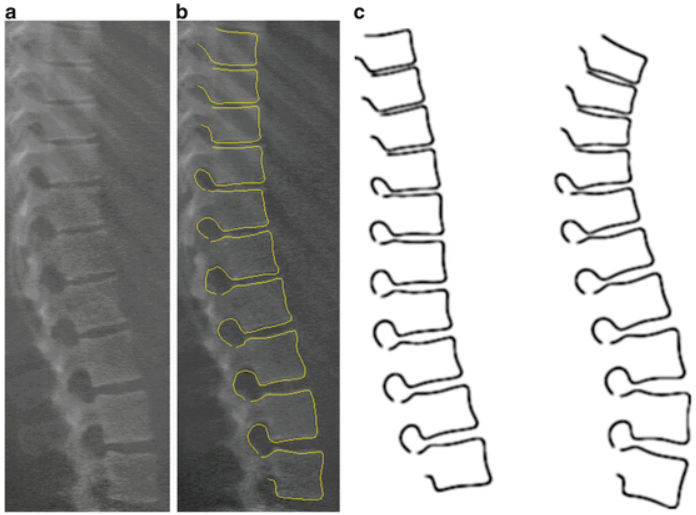


While updating the curve, project on model space to find closest shape model

Appearance model: Make a local intensity model of edge at point v_i



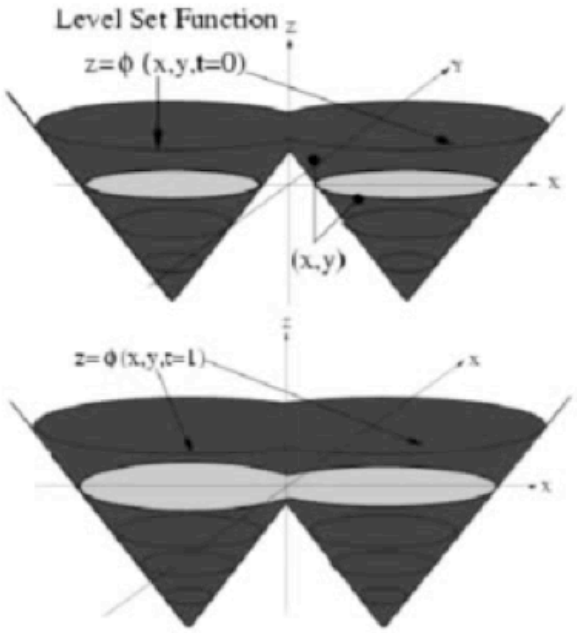




[Cootes et al., Handbook of Biomedical Imaging, 2015]

Implicit curve definitions: Level-sets

- Instead of representing the contour explicitly as a set of points
- Implicit models - the contour is the *level set* of a higher dimensional function



The level set function:

$$z = \Phi(x,y,t)$$

Contour at time *t:*

$$0 = \Phi(x,y,t)$$

This allows the topology of the curve to change

Segmentation : Outline

- Thresholding
- Edge based
- **Region based**
- Statistical Pattern Recognition based

Region growing : principle

On the basis of segment homogeneity rather than inhomogeneity around edges

start with detection of “homogeneous” regions (e.g. low intensity variance) as the “seeds”

These are grown as long as homogeneity criterion is satisfied

Choice of appropriate homogeneity criteria is not straightforward

Region growing : example

Seeds of low intensity variance are grown, keeping intensity between two slowly floating thresholds and merging overlapping segments

Region growing : ex. cont'd

* Often does not perform well
* Also, risk of leakage through low contrast edges

Region growing : remarks

region growing pure is a one-way process :
if seeds are wrong, errors cannot be corrected

solution : split-and-merge procedures

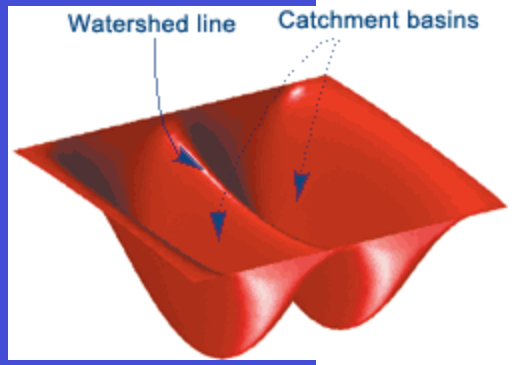
merges of similar regions
splitting more difficult : many ways to do it

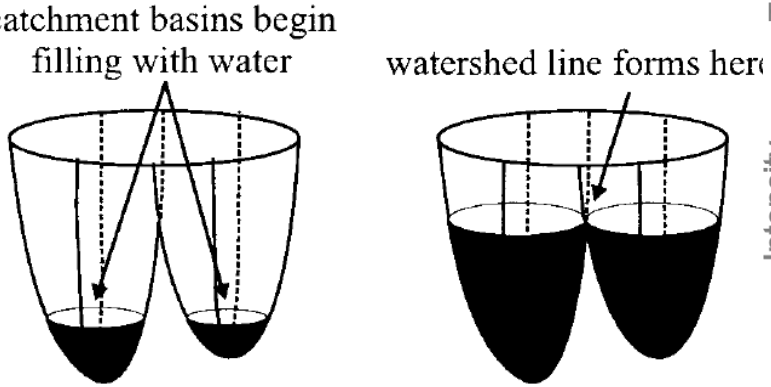
Splitting calls for special decompositions of segments, e.g. “quadtrees”

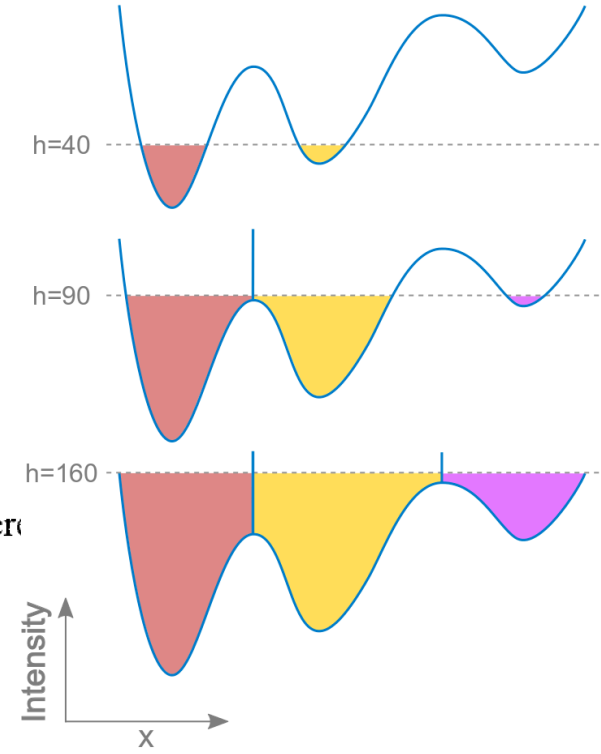
Region and edge based methods can be combined: hybrid approaches

Watershed algorithm

Finding catchment basins
in the image intensity graph



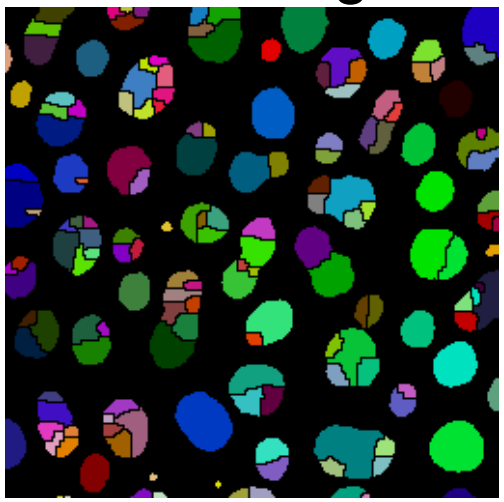




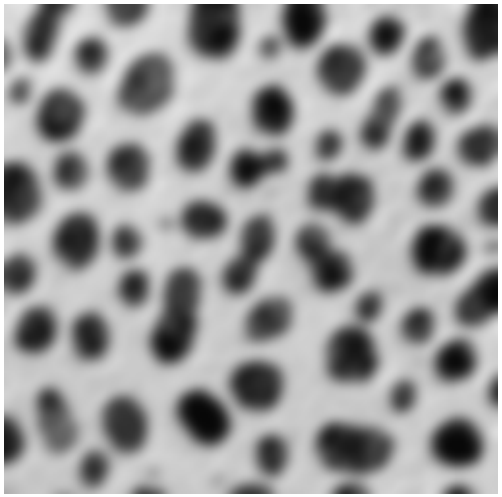
- Starting with local minima
- Different implementations of water filling
  (usually based on some type of region growing)
  Stop propagation at detected ridges (lines)
- Usually strong over-segmentation

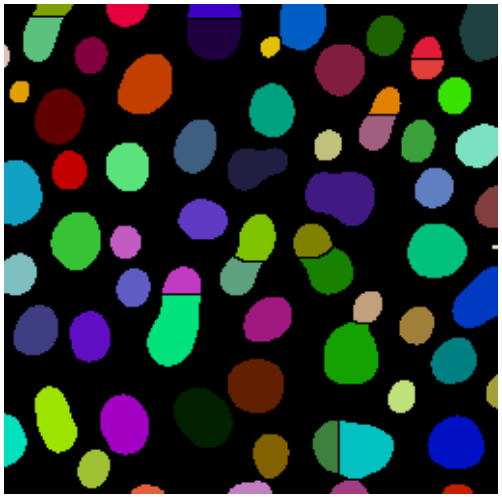
Watershed segmentation: Example

Watershed on original image



Watershed on Gaussian blurred image



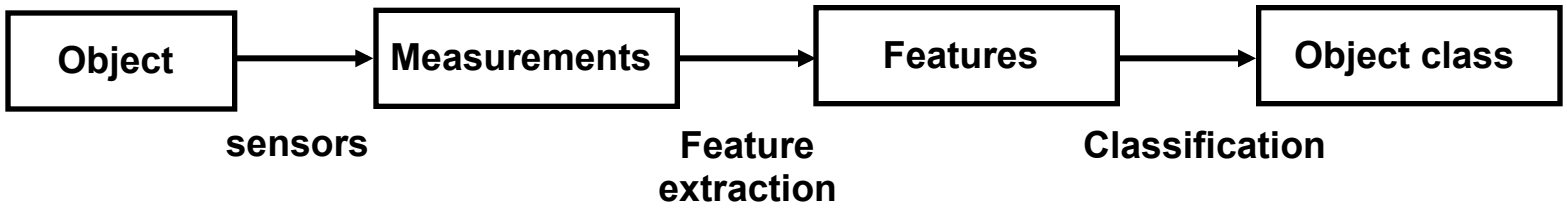


Segmentation : Outline

- Thresholding
- Edge based
- Region based
- **Statistical Pattern Recognition based**

Statistical pattern recognition

- General scheme
- Feature based
- Probabilistic and learning based formulations

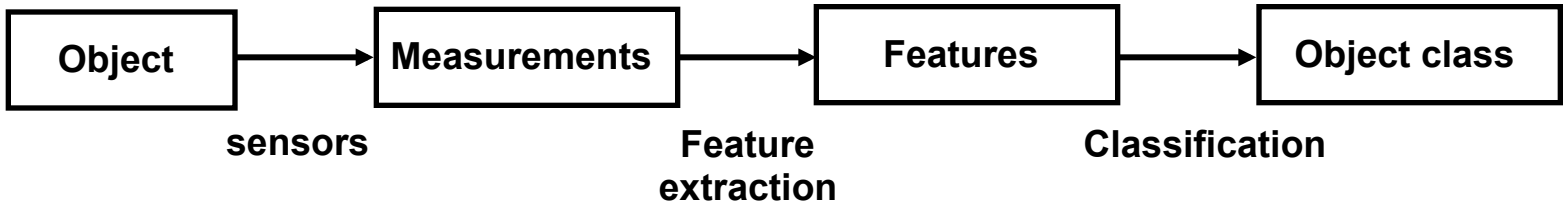


Three alternatives:

1. Unsupervised clustering
2. Supervised generative modeling
3. Supervised discriminative modeling

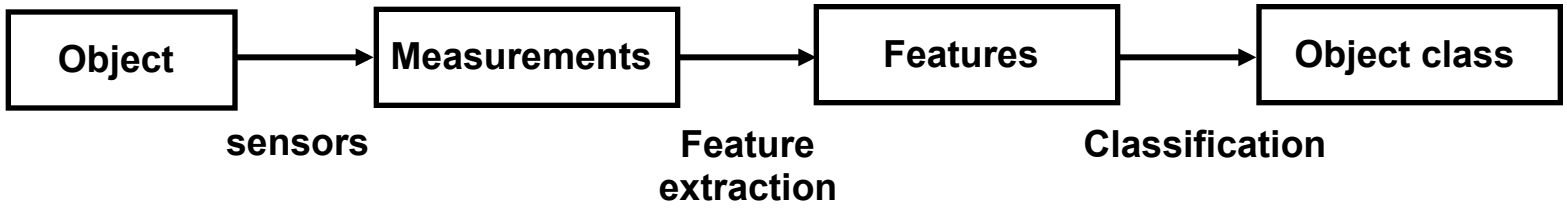
Additional variations and formulations exist.
Each of these can go very deep

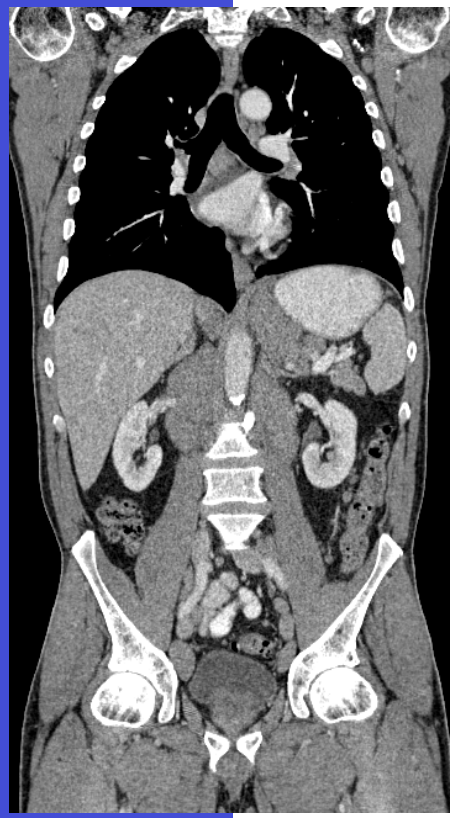
Example application : 1



Objects: Frog
Sensors: Camera
Measurements: Pixels intensity
Features: Color
Object classes: Foreground / Background

Example application : 2





Objects: Human body
Sensors: X-ray Computed Tomography
Measurements: X-ray attenuation (Hounsfield unit)
Features: Hounsfield unit
Object classes: Different organs and outside

In general, the features are essential and have major influence on the results

Basic Notation

The set of all possible classes:

$$\Omega = \{w_1, w_2, \ldots, w_K\}$$

For *M* measurements, and
n features extracted for each measurement :

$$\{\vec{v}_j\}_{j=1}^{M} \qquad \vec{v} = \{v_1, v_2, \ldots, v_n\} \in \mathbb{R}^n$$

Examples of features:

- Color – 3 dimensional
- X-ray attenuation – 1 dimensional
- Pixel location – 2 dimensional
- Combinations…

Unsupervised clustering: principles

We want to distribute measurements to classes

Goal:

- Homogeneity within classes
- Reducing variance over features
- Can take into account multiple features

Available information:

- We only know the features
- No information on the classes
  (to be found as a result of this "unsupervised" process)

Unsupervised clustering: K-means Algorithm

One of the most popular and widely used algorithm

$$\{\vec{v}_j\}_{j=1}^{M} \qquad \vec{v} = \{v_1, v_2, \ldots, v_n\} \in \mathbb{R}^n$$

Choose K centers / means

$$m_i \in \mathbb{R}^n, \;\; i = 1, \ldots, K$$

Repeat until centers (m's) do not change:

For all measurements *j*, assign to nearest center *i*

$$c_j = \arg_i \min \|\vec{v}_j - m_i\|^2$$

For all centers *i*, update it to center-of-mass

$$m_i = \sum_j \delta_{i=c_j} \vec{v}_j \Big/ \sum_j \delta_{i=c_j}$$

$$\delta_{i=c_j} = \begin{cases} 0, & i \neq c_j \\ 1, & i = c_j \end{cases}$$

K-means analysis

(for K classes)

K=2

K=3

K=4

K-means remarks

Extremely easy to implement

Useful initial analysis

Choice of K has a major influence
Methods exist to choose it automatically:
- Heuristic methods based on variance
- Non-parametric Bayesian

Initialization is important
K-means can get stuck in local minima
- Multiple initializations
- Multi-scale methods

Generic probabilistic formulation

Extracted features are different for each measurement : random / non-deterministic

Joint probability distribution of features and classes

$$p(\vec{v}, w_i)$$

Marginal distributions

- Probability of class occurrence (a priori probability)

$$P(w_i) = \int p(\vec{v}, w_i) d\vec{v}$$

- Probability of observing a specific feature

$$p(\vec{v}) = \sum_{i=1}^{s} p(\vec{v}, w_i)$$

$P \equiv p$

Generic probabilistic formulation

Conditional distributions

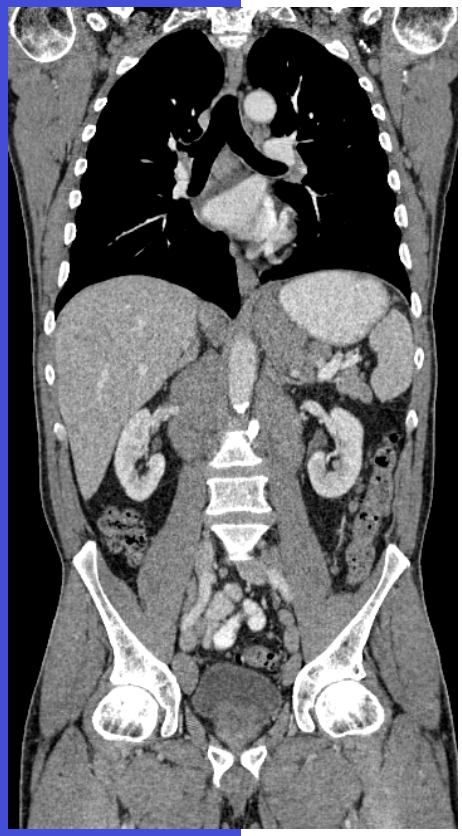
- Given class information (measurement likelihood)

$$p(\vec{v}|w_i) = \frac{p(\vec{v}, w_i)}{p(w_i)}$$

- Given measurements (class posterior) → Bayes' theorem

$$P(w_i|\vec{v}) = \frac{p(\vec{v}, w_i)}{p(\vec{v})} = \frac{p(\vec{v}|w_i)P(w_i)}{p(\vec{v})}$$

Generic probabilistic formulation: examples



$\Omega = \{\text{bone, nervous tissue, muscle, air}\}$

$\vec{v}$: X-ray attenuation (HU) at a given pixel

$P(\text{bone})$: Probability that any pixel is bone

$P(\vec{v} = 250)$: Probability that a pixels value is 250 HU

$P(\vec{v} = 250|\text{bone})$: Probability that a bone pixel is 250 HU

$P(\text{bone}|\vec{v} = 250)$: Probability of the pixel being bone given its measurement of 250 HU

Supervised Generative Models : principle

Assumes existing examples where we can learn distributions for measurements and classes:

$$p(\vec{v}|w_i) \qquad p(w_i)$$

Make use of this information to segment images, e.g.

$\vec{v}$: grey level intensity

$$p(\vec{v}|w_{bg}) = \mathcal{N}(250, 10)$$

$$p(\vec{v}|w_{fg}) = \mathcal{N}(150, 50)$$

$$p(w_{fg}) = p(w_{bg}) = 0.5$$

a-posteriori distribution

Compute posterior distributions via Bayes theorem:

Can be learnt Prior

$$P(w_i|\vec{v}) = \frac{p(\vec{v}, w_i)}{p(\vec{v})} = \frac{p(\vec{v}|w_i)P(w_i)}{p(\vec{v})}$$

Constant

Final segmentation by maximum a-posterior (MAP)

$$\arg_i \max(p(w_i|\vec{v}))$$

$p(w_{bg}|\vec{v})$

$p(w_{fg}|\vec{v})$

Importance of distributions

Learning the right distribution is crucial for the accuracy of the segmentation

e.g., what would happen with a different choice:

$$p(\vec{v}|w_{fg}) = \mathcal{N}(150, 50)$$

$$p(\vec{v}|w_{bg}) = \mathcal{N}(200, 50)$$

$p(w_{bg}|\vec{v})$

$p(w_{fg}|\vec{v})$

Likelihood instead of posterior

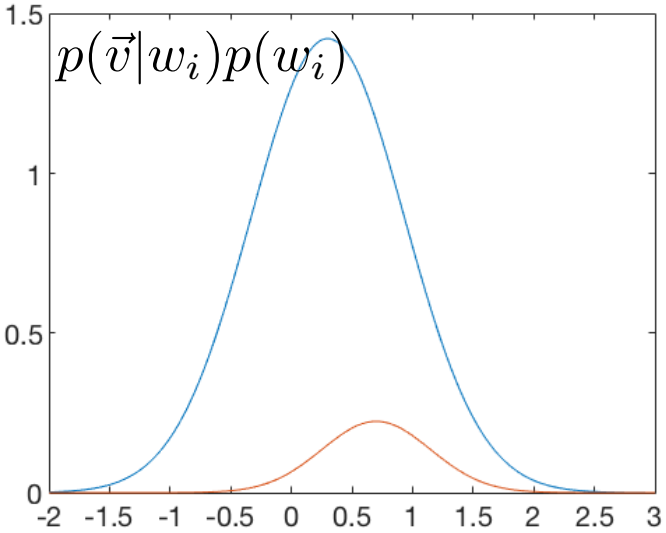
$$P(w_i|\vec{v}) = \frac{p(\vec{v}, w_i)}{p(\vec{v})} = \frac{p(\vec{v}|w_i)P(w_i)}{p(\vec{v})}$$

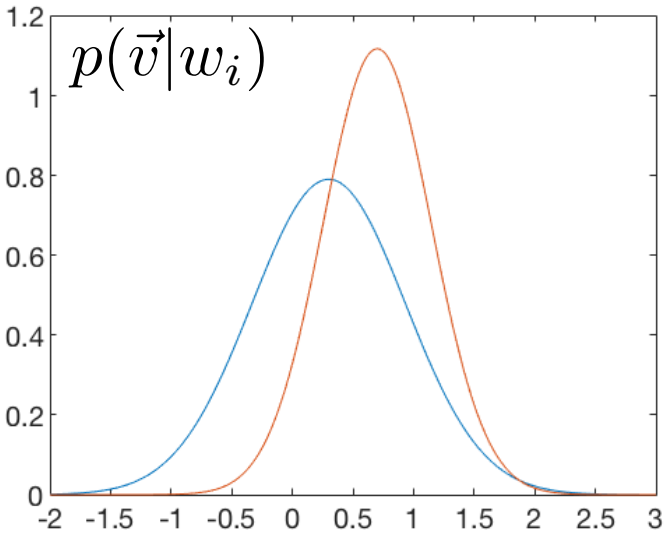
Posterior distribution can be difficult to compute

- How to get class priors?
  in accuracy, class imbalance, etc.

$$p(\vec{v}|w_{bg}) = \mathcal{N}(0.3, 0.4) \qquad p(\vec{v}|w_{fg}) = \mathcal{N}(0.7, 0.2)$$

$$p(w_{bg}) = 0.9, \; p(w_{fg}) = 0.1$$





Alternatively: Maximize the likelihood function $\arg_i \max p(\vec{v}|w_i)$

Outline

- **Supervised generative learning:**
  From individual pixels to combinations
  Markov Random Fields
  Gibbs sampling
  Graph-cuts

- **Supervised discriminative learning for segmentation**
  KNN
  Random Forests

From individual pixels to combinations:

- Earlier probabilistic segmentation model considers each pixel independently (similarly to simple thresholding)
- Independence between pixels
- Mathematical morphology for thresholding results
- Is there a way to do this with graphical models?

From individual pixels to combinations: general formulation

Class for each pixel $\mathbf{c} = \{c_j\}_{j=1}^{M},\ c_j \in \{w_1, \ldots, w_K\}$

Joint distribution for an individual pixel:

$$\begin{aligned} p(c_j = w_i, \vec{v}_j) &= p(\vec{v}_j | w_i) P(w_i) \\ &= p(\vec{v}_j | c_j) P(c_j) \end{aligned}$$

Joint distribution of all pixels when independent

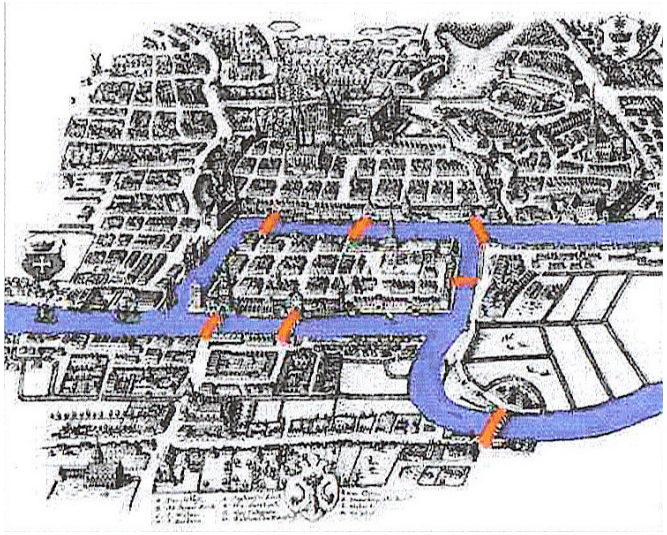
$$p(\mathbf{c}, \vec{\mathbf{v}}) = \prod_{j=1}^{M} p(\vec{v}_j | c_j) P(c_j)$$

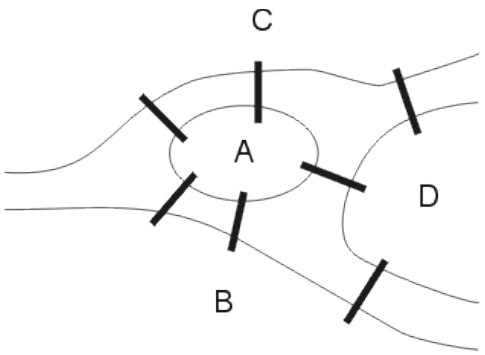
Joint distribution when pixels are not independent

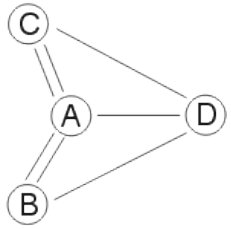
$$p(\mathbf{c}, \vec{\mathbf{v}}) = \prod_{j=1}^{M} p(\vec{v}_j | c_j) P(\mathbf{c})$$

Topology

- The problem of the bridges of Königsberg Prusia (Today's Russia - Kaliningrad)
- Solution: Euler, 1736
- The birth of graph theory
- Independent of distances



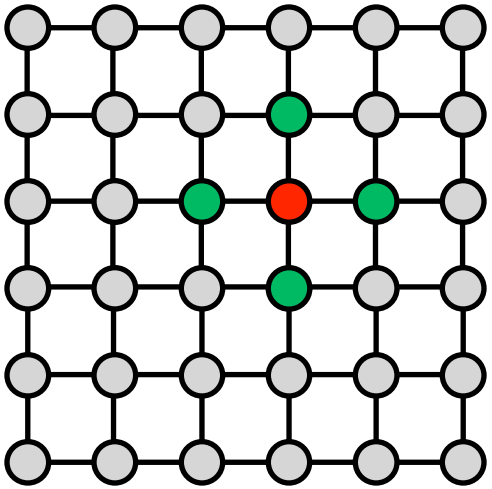




Markov Random Fields: principle

$$p(\mathbf{c}, \vec{\mathbf{v}}) = \prod_{j=1}^{M} p(\vec{v}_j | c_j) P(\mathbf{c})$$

- MRF sets up the prior distribution based on the **Markovian property**
- Depends on the neighborhood structure



$$G_j = \text{neighbors of j}$$

$$P(c_j | \mathbf{c}_{/j}) = P(c_j | \mathbf{c}_{G_j})$$

- Probability of c_j only depends on its neighbors if all others are given

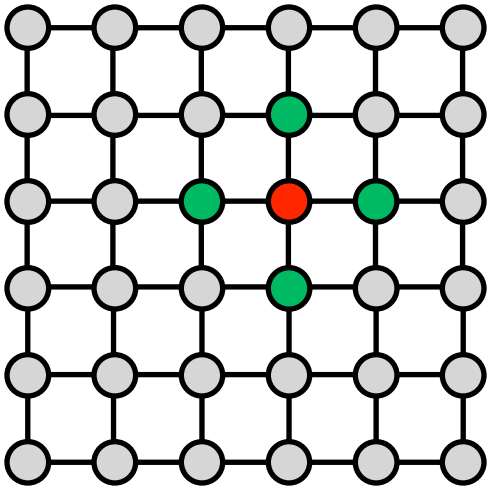
Markov Random Fields: Energies and Gibbs distributions

$$P(c_j|\mathbf{c}_{/j}) = P(c_j|\mathbf{c}_{G_j})$$

Common way to define it is through defining an **energy**

$$E(\mathbf{c}) = \sum_{j=1}^{M} \sum_{k \in G_j} d(c_j, c_k)$$

$$d(c_j, c_k) : \text{distance between } c_j \text{ and } c_k$$



From energies we can define a probability distribution
Gibbs Distribution
(Boltzmann Distribution)

$$P(\mathbf{c}) = \frac{1}{Z} \exp(-E(\mathbf{c}))$$

Distance to enforce consistency!

Markov Random Fields: Posterior Maximization

Segmentation through posterior maximization:

$$\begin{aligned}\arg_{\mathbf{c}} \max P(\mathbf{c}|\vec{\mathbf{v}}) &= \arg_{\mathbf{c}} \max p(\vec{\mathbf{v}}|\mathbf{c})P(\mathbf{c}) \\ &= \arg_{\mathbf{c}} \max \prod_{j=1}^{M} p(\vec{v}_j|c_j) \exp(-E(\mathbf{c}))\end{aligned}$$

If the data model is also exponential of an energy

$$p(\vec{v}_j|c_j) \propto \exp(-f(\vec{v}_j, \theta_{c_j}))$$

Energy model for the prior distribution – **Ising / Potts Model**

$$d(c_j, c_k) = \lambda\delta(c_j \neq c_k) = \begin{cases} 0, & c_j = c_k \\ \lambda, & c_j \neq c_k \end{cases}$$

Optimization

$$\arg_{\mathbf{c}} \max p(\mathbf{c}|\vec{\mathbf{v}}) \quad = \quad \arg_{\mathbf{c}} \max p(\vec{\mathbf{v}}|\mathbf{c})P(\mathbf{c})$$

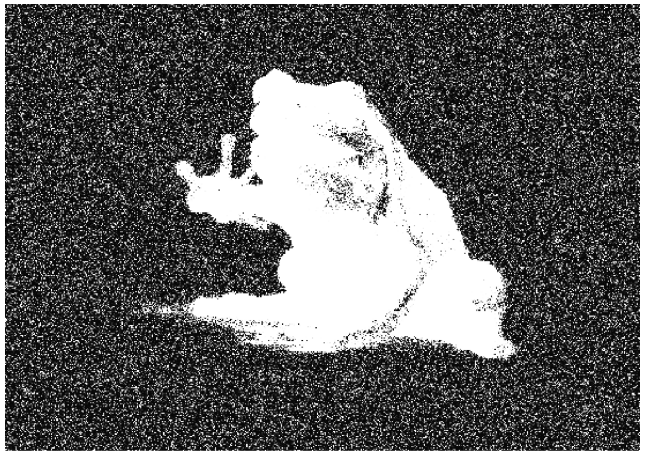
Equivalently

$$\arg_{\mathbf{c}} \min \sum_{j=1}^{M} (\vec{v}_j - \mu_{c_j})^T \Sigma_{c_j}^{-1} (\vec{v}_j - \mu_{c_j}) + \lambda \sum_{j=1}^{M} \sum_{k \in G_j} \delta(c_j \neq c_k)$$

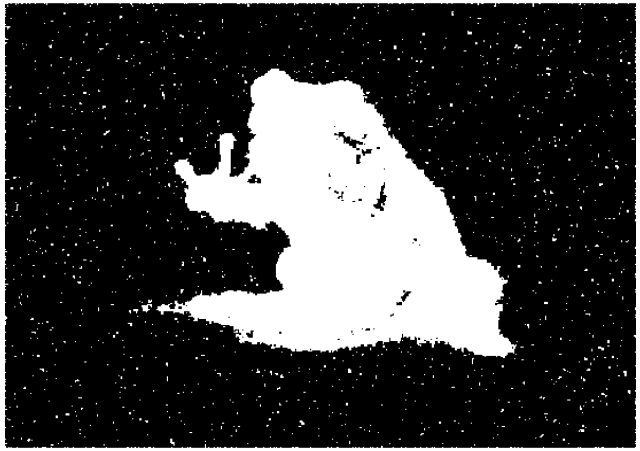
- Very difficult to compute the posterior distribution
- Difficult to solve the optimization exactly in 2D and higher
- NP-hard [Boykov, Veksler and Zabih 2001 TPAMI]
- Approximate energy minimization
  Gibbs sampling [Geman & Geman 1984]
  Iterated conditional modes (ICM) [Ferrari et al. 1995]
  via Graph Cuts [Boykov, Veksler & Zabih 2001]
  …

Stochastic Relaxation / Gibbs Sampling: in action

Noisy image



Initial segmentation



After 1 iteration of Gibbs

After 25 iterations

Stochastic Relaxation / Gibbs Sampling: analysis with respect to λ

Noisy image

$\lambda = 0$

$\lambda = 5$

$\lambda = 20$

Stochastic Relaxation / Gibbs Sampling: remarks

- Very simple implementation and effective
- Stochastic in nature
- Solution depends on the lambda parameter
- Solution depends on initial condition
- Not very efficient – convergence is slow
- Does not always converge to a pleasing result

Remarks on Generative Supervised Models

- Mathematically sound
- Flexible and generic
  Model specifications can change
- Can be extended to various features
- Wealth of research
- Links to Bayesian methods
- Features are very important

- Optimization and inference can be hard
- For high dimensional features
  estimation of distributions can be problematic
  Latent variables / factors

Discriminative learning: principles

For mapping from features to classes

$$\{\vec{v}_j\}_{j=1}^{M} \longrightarrow \{w_i\}_{i=1}^{K}$$

A procedure that takes the features as input and predicts the segmentation label / class assignment

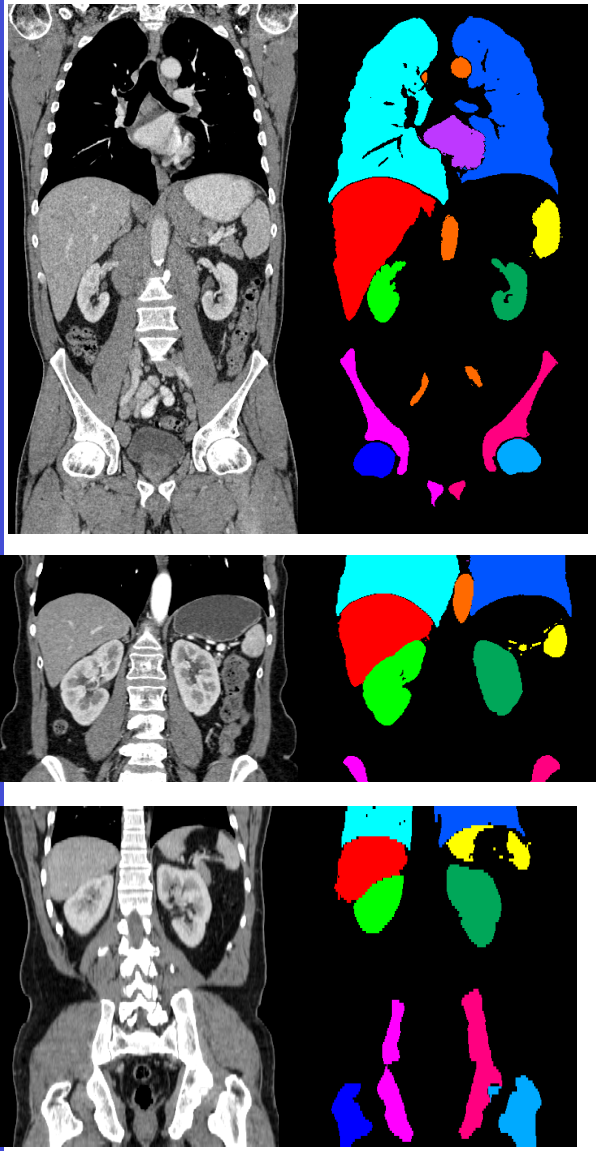
$$f(\vec{v}_j) = c_j$$

In terms of probabilities, this is similar to directly modeling posterior or its maximum

$$f(\vec{v}_j) \sim p(c_j|\vec{v}_j) \qquad f(\vec{v}_j) \sim \arg_{c_j} \max p(c_j|\vec{v}_j)$$

”Learn” the “mapping” from “examples”

Discriminative learning: examples - Training Data



- Composed of images and the corresponding segmentations of the objects you are interested in
- Application dependent
- The “ground-truth” segmentations are often annotated manually or with a semi-automatic algorithm
- Can be VERY expensive
- and VERY valuable
- Think of other applications…

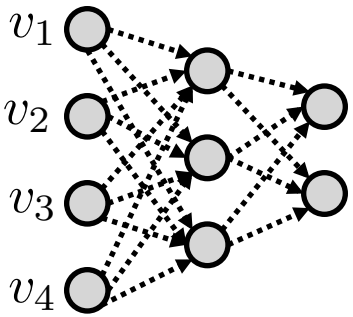
Discriminative learning: Mapping

$$f(\vec{v}_j) = c_j$$

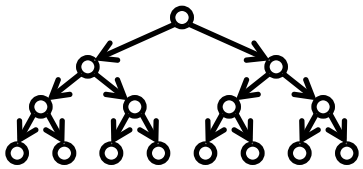
- Mapping is a parametric model
- KNN – K-nearest neighbors
- Logistic regression (of binary class)

$$c_j = \frac{1}{1 + \exp\left(-\beta_0 - \beta^T \vec{v}\right)}, \quad \beta^T \vec{v} = \sum_{i=1}^{d} \beta_i v_i$$

- Neural networks: $c_j = \sigma\left(\beta_2^T \sigma\left(\beta_1^T \sigma(\beta_0^T \vec{v})\right)\right)$

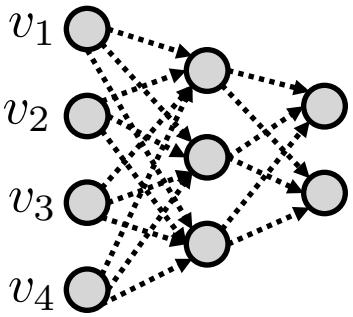


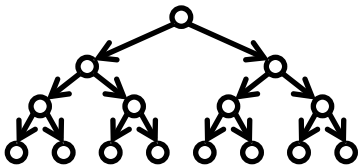
- Decision trees - random forests
- …



Discriminative learning: Learning / Training Phase

$$c_j = \frac{1}{1 + \exp\left(-\beta_0 - \beta^T \vec{v}\right)}$$





$$c_j = \sigma\left(\beta_2^T \sigma\left(\beta_1^T \sigma(\beta_0^T \vec{v})\right)\right)$$

- Estimating the parameters of the model, in order to obtain the best possible segmentation in the training examples

Training data: $\{(\vec{v}_n, c_n)\}_{n=1}^{N}$

- Optimization by minimizing the discrepancy between algorithm segmentation and ground truth

Discriminative learning: Segmentation Testing phase

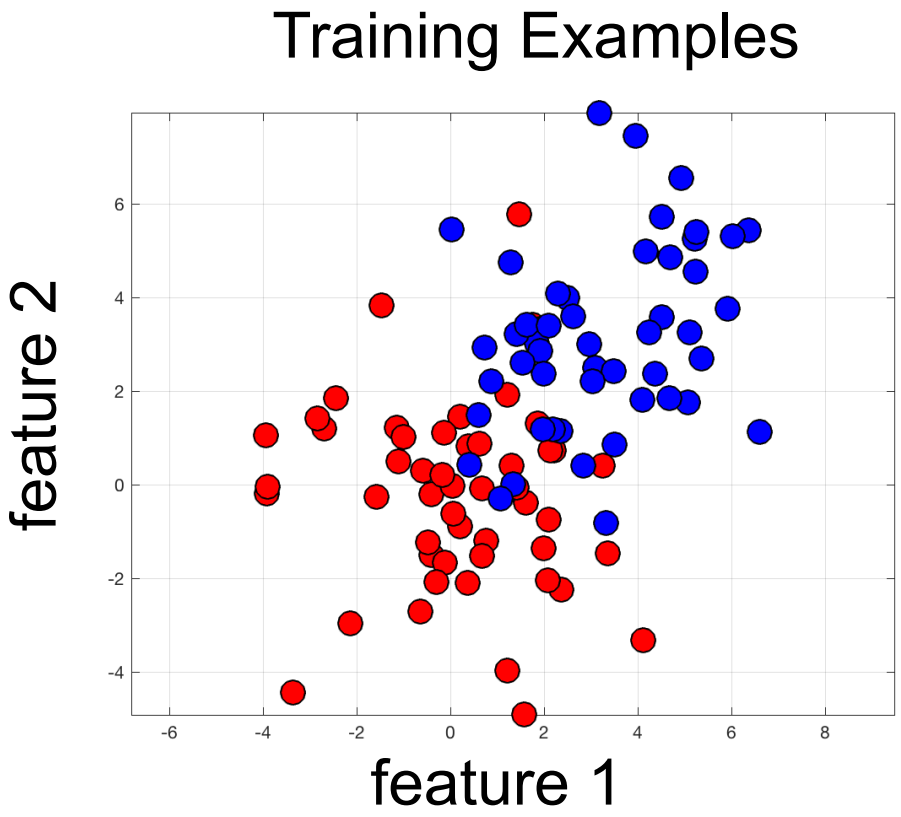
$$\{\vec{v}_j\}_{j=1}^{M}$$

$$f(\vec{v}_j) = c_j$$

- Extract the features used during training
- Use the mapping learned before

K-nearest neighbors - KNN

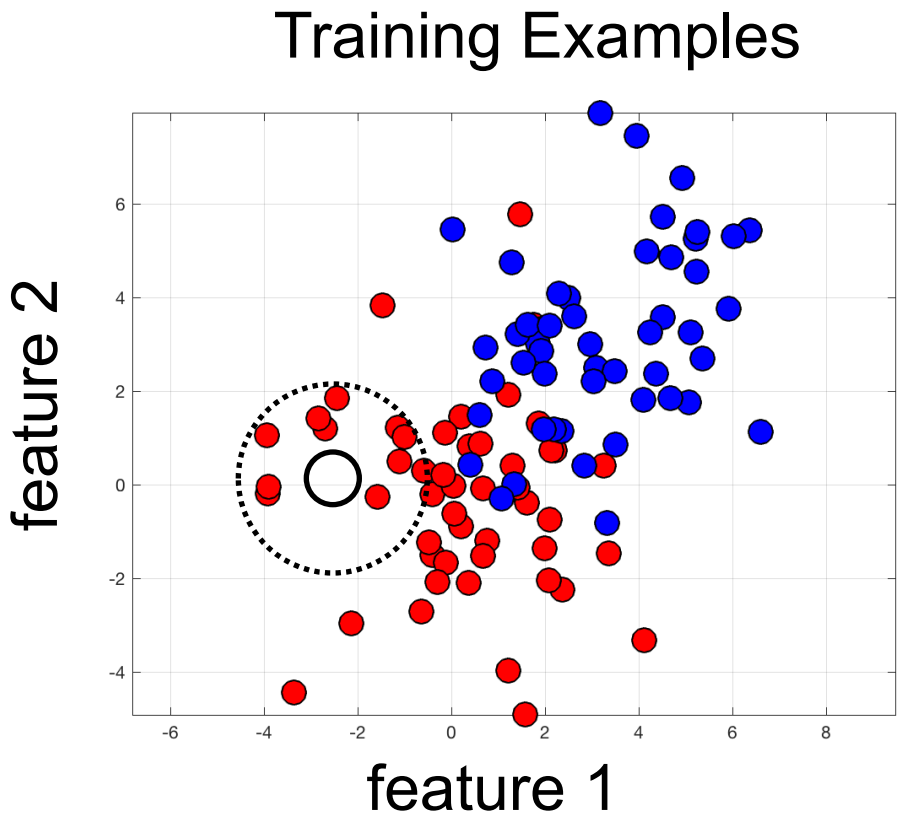
- Find the K nearest neighbors within the training dataset.
- For training examples we know the features and the labels



K-nearest neighbors - KNN

- The mapping is defined through the labels of the K-nearest neighbors

$$f(\vec{v}) = c$$



Find the K training examples with minimum distance

$$d(\vec{v}, \vec{v}_n)$$

Mapping is the function of the corresponding labels

$$f(\vec{v}) = f(c_1, \ldots, c_K)$$

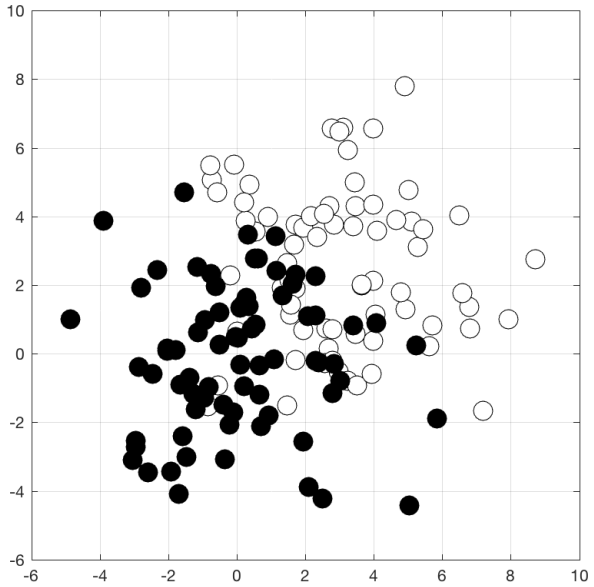
K-nearest neighbors – KNN: parameterization

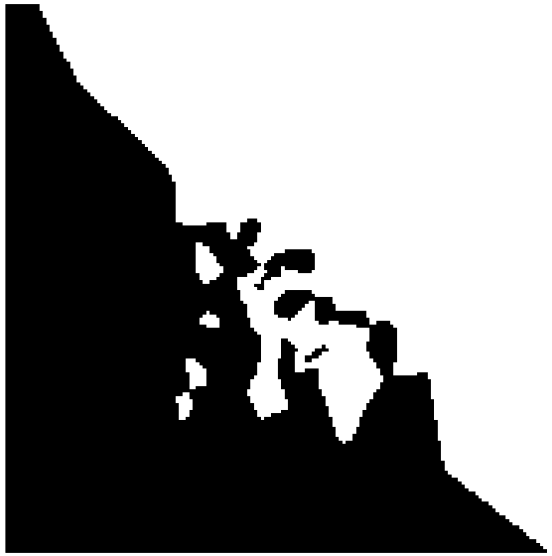
- Define the term “nearest” → distance

- Define the mapping $f(\vec{v}) = f(c_1, \ldots, c_K)$
  - Majority voting
  - Weighted majority voting
  - Probabilities with uncertainties

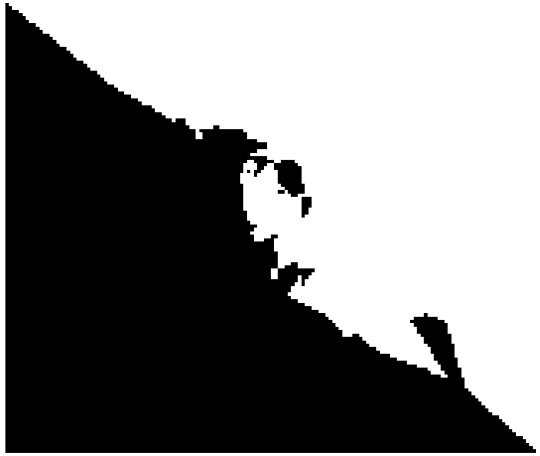
- With an additional training,
  the parameters of the distance and mapping can also be estimated if there are any…

KNN defines a partitioning – majority voting

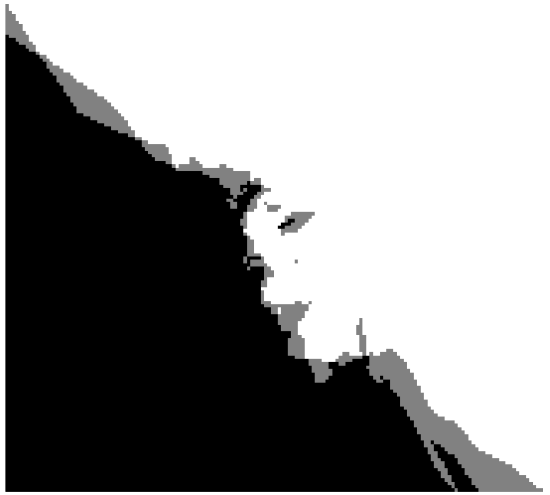




K=1

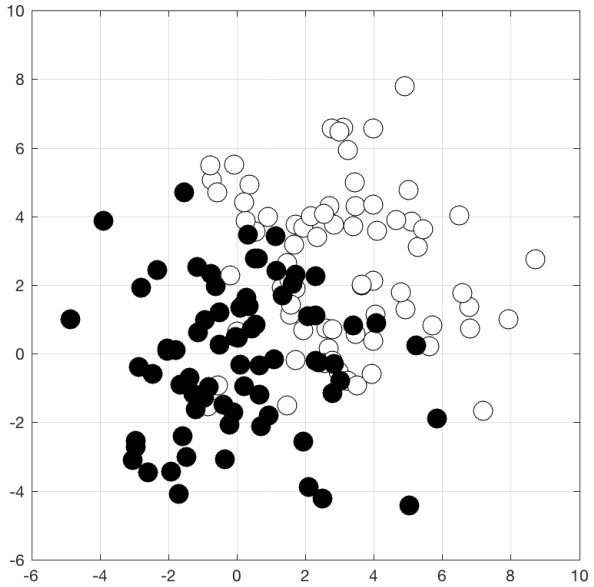


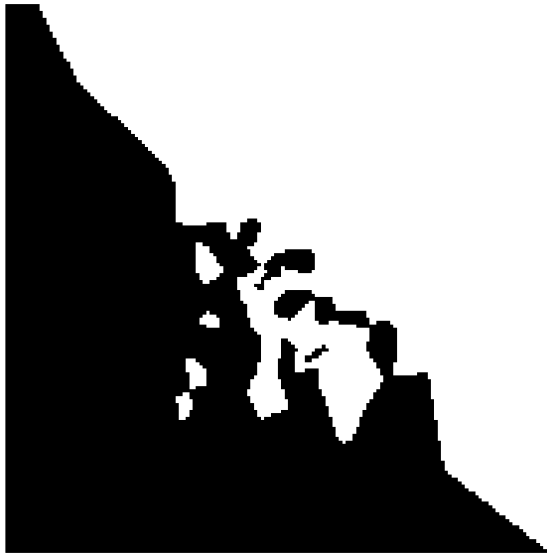
K=5



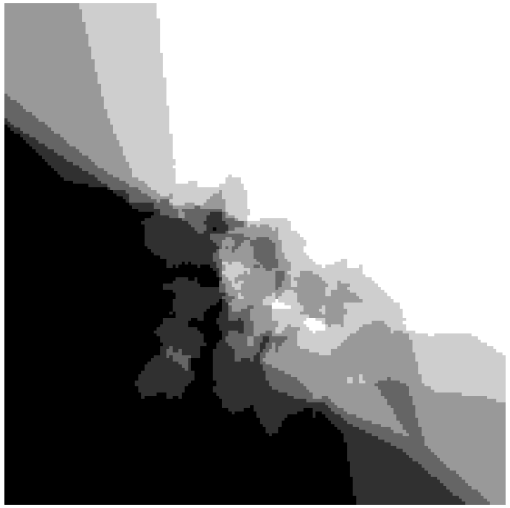
K=10

Probabilities via neighboorhood proportions

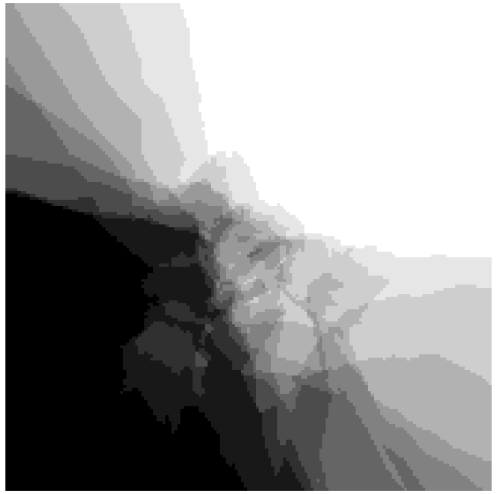




K=1



K=5



K=10

Remarks on KNN

Pros:

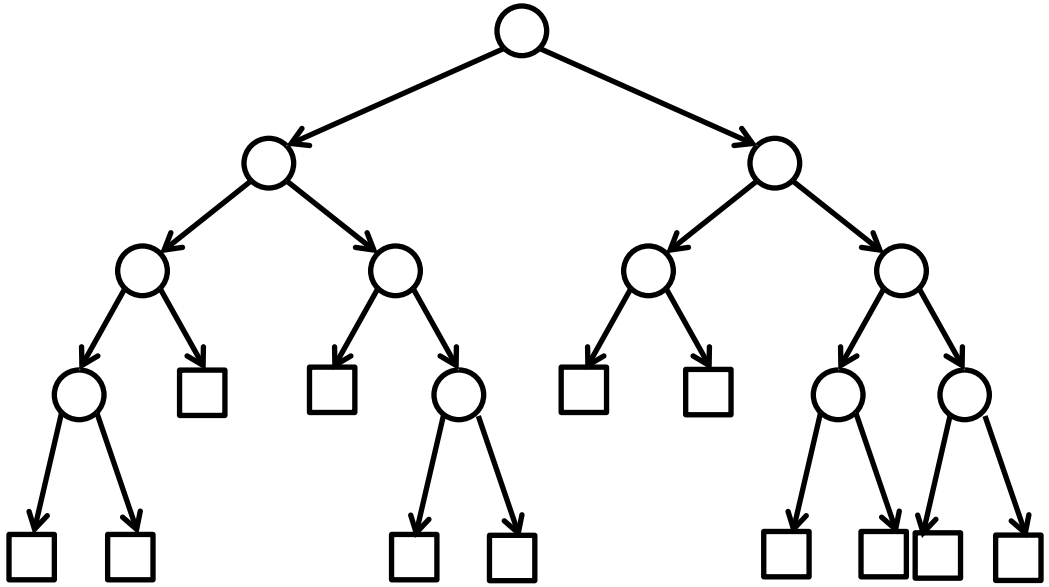
- Very simple to implement
- Very simple to understand
- Efficient implementations possible
  approximate nearest neighbors,…
- Distance definition is flexible

Cons:

- Highly depends on the definitions and K
- Need to keep the entire data in memory
  for distance computations
- For high dimensional problems (with high d)
  need a LOT of training samples for accuracy
  (use alternatives instead, e.g. NNs, RFs, …)

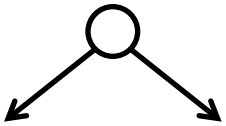
Discriminative learning: Random Forests

- Binary decision trees

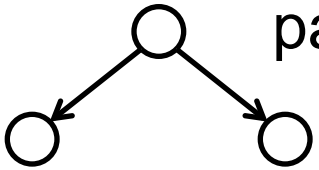


: internal nodes

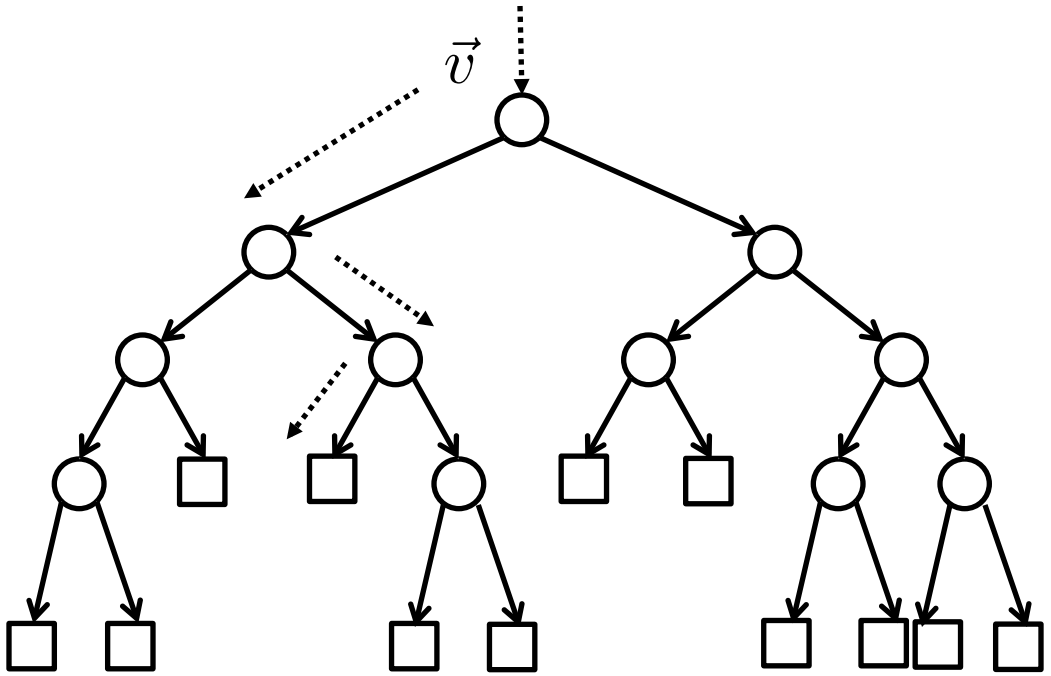
: leaf nodes



: splits



Discriminative learning: Decision Trees



At each internal node n there is a binary question on the features

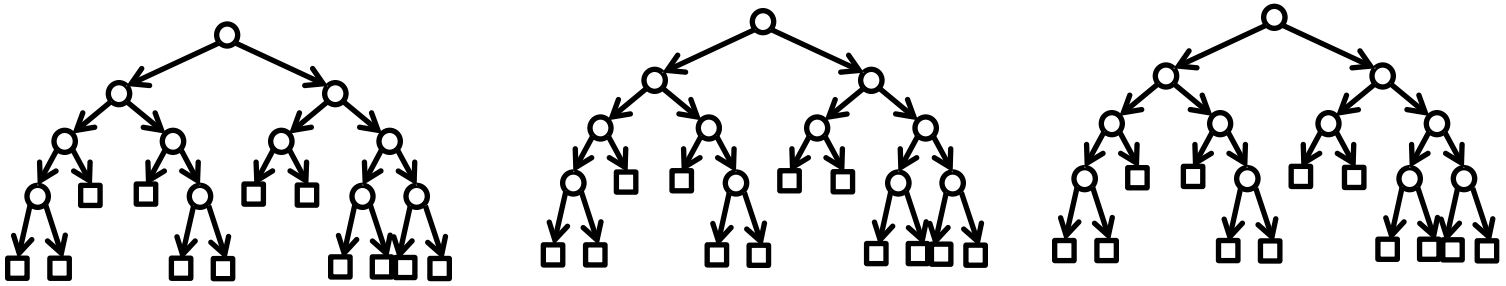
$$f_n(\vec{v}) = \begin{cases} 0 \rightarrow & \text{go left} \\ 1 \rightarrow & \text{go right} \end{cases}$$

At each leaf node there is a prediction and it changes from leaf node to leaf node

$$f_l(\vec{v}) = c$$

Discriminative learning: Random Forest

- Ensemble of decision trees



- Each tree is different than others
- Focus on a different set of features

$$\vec{v} = \{v_1, \ldots, v_d\}$$

$$\vec{v}_i = \{v_{i_1}, v_{i_2}, \ldots, v_{i_D}\} \quad d >> D$$

- An approach to deal with high dimensionality

Random Forests: parameterization

Two levels of parametrization:

- What are the tests?

$$f_n(\vec{v}) = \begin{cases} 0 \rightarrow & \text{go left} \\ 1 \rightarrow & \text{go right} \end{cases}$$

- How are the predictions done?

$$f_l(\vec{v}) = c$$

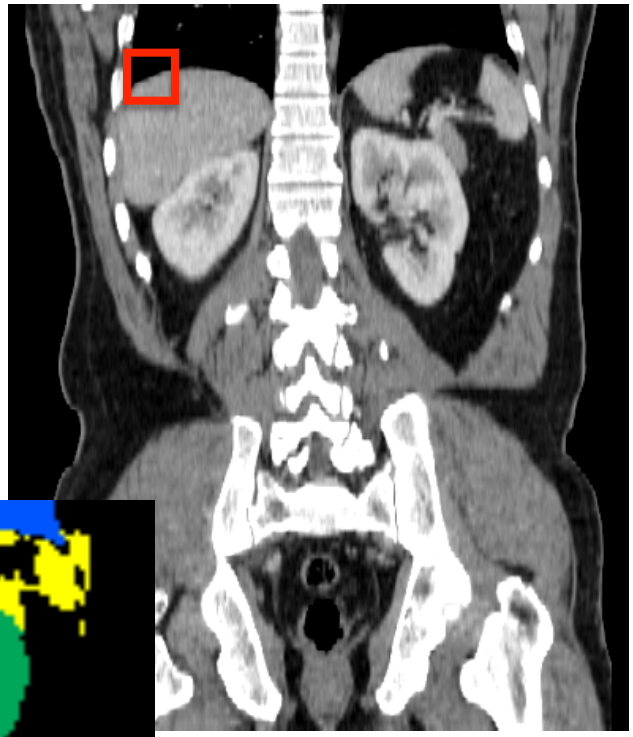
Both of these are learned during training

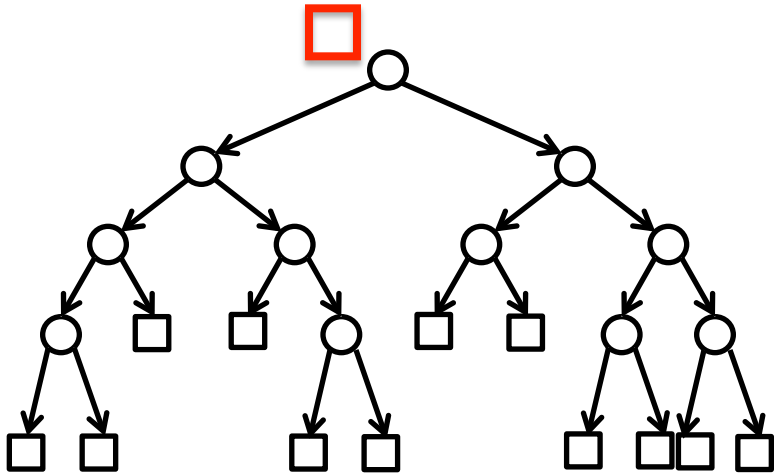
Various alternatives for both

Computer Vision

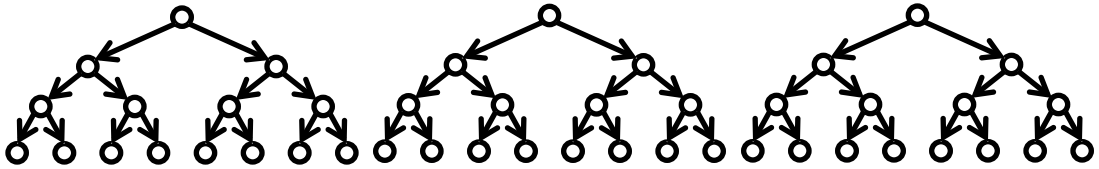
Random forests: During Segmentation

$= \vec{v} = \{v_1, \ldots, v_d\}$





$$f_n(\vec{v}) = \begin{cases} 0 \rightarrow & \text{go left} \\ 1 \rightarrow & \text{go right} \end{cases} \quad f_l(\vec{v}) = c$$



Random Forests: remarks

- Easy to implement
- VERY efficient
- Technology behind Kinect (earlier version)

- Lots of parametric choices
- Needs large number of data
- Training can take time