Proiect Data Assimilation

1. Modelul dinamic

Sa consideram modelul dinamic unidimensional descris de ecuatia diferentiala de mai jos. Variabila ***t*** este variabila temporala (timpul) iar variabila ***x*** semnifica pozitia spatiala, la momentul de timp t, unidimensionala, a unei particule care pleaca din pozitia initiala $x(0) = x_0$.

$$\begin{cases} \dfrac{dx}{dt} = f(t,x) = -\cos t \cdot x + \sin(t) \quad \text{modelul dinamic} \\ x(0) = x_0 \quad \text{pozitia initiala} ----> \text{prior} \end{cases}$$

Traiectoria particulei la momentul de timp t este data de solutia $x(t)$ a ecuatiei diferentiale de mai sus. Solutia se va considera, in acest studiu, ca fiind solutia numerica data de metoda Euler imbunataţită (de mai jos)

$$x_i = x_{i-1} + \frac{h}{2}\left(g_1 + g_2\right)$$
$$g_1 = f(t_{i-1}, x_{i-1}), g_2 = f(t_{i-1} + h, x_{i-1} + hg_1)$$
$$x_i = x_{i-1} + \frac{h}{2}\left(f(t_{i-1}, x_{i-1}) + f(t_{i-1} + h, x_{i-1} + hf(t_{i-1}, x_{i-1}))\right)$$

Se va considera pasul de discretizare din metoda numerica $h = 0.03$, de unde rezultă discretizarea pe axa timpului

$$t_0 = 0, t_i = t_0 + ih .$$

Tot din rezolvarea numerică de mai sus reiese si modelul dinamic care ofera pozitia particulei intre doua intervale de timp consecutive şi anume

$$x_i = \underbrace{x_{i-1} + \frac{h}{2}\left(f(t_{i-1}, x_{i-1}) + f(t_{i-1} + h, x_{i-1} + hf(t_{i-1}, x_{i-1}))\right)}_{m(x_{i-1})} = m(x_{i-1}) + 0$$

Acea valoare 0 de la final înseamnă că în procesul de data assimilation, modelul este considerat perfect (i.e. nu are eroare)

2. Simularea observaţiilor

Vom considera ca model de referinţă ("truth") acel model pentru care pozitia initiala a particulei este $x_0\_truth = 0.1$ şi vom considera un număr de ***k=20*** de observaţii luate la un interval de timp **de** $h = 0.03$. Observaţiile vor fi considerate ca fiind solutia numerica x_truth a ecuatiei diferentiale de mai sus pentru valoarea $x_0\_truth = 0.1$

$$Obs = \left[x\_truth(h)\ x\_truth(2h) \ldots x\_truth(kh)\right]$$

Se va considera că eroarea în masurarea observatiilor urmeaza o distributie Gaussiana de medie 0 si deviatie standard 0.03.

$$\varepsilon \sim N(0; 0.03^2)$$

3. Incertitudinea iniţială

Incertitudinea modelului este data de necunoaşterea exactă a poziţiei iniţiale a particulei şi anume x_0. Din acest motiv o vom considera ca fiind o variabilă aleatoare cu distribuţia Gaussiană, $x_0 \sim N(0; 0.2^2)$.

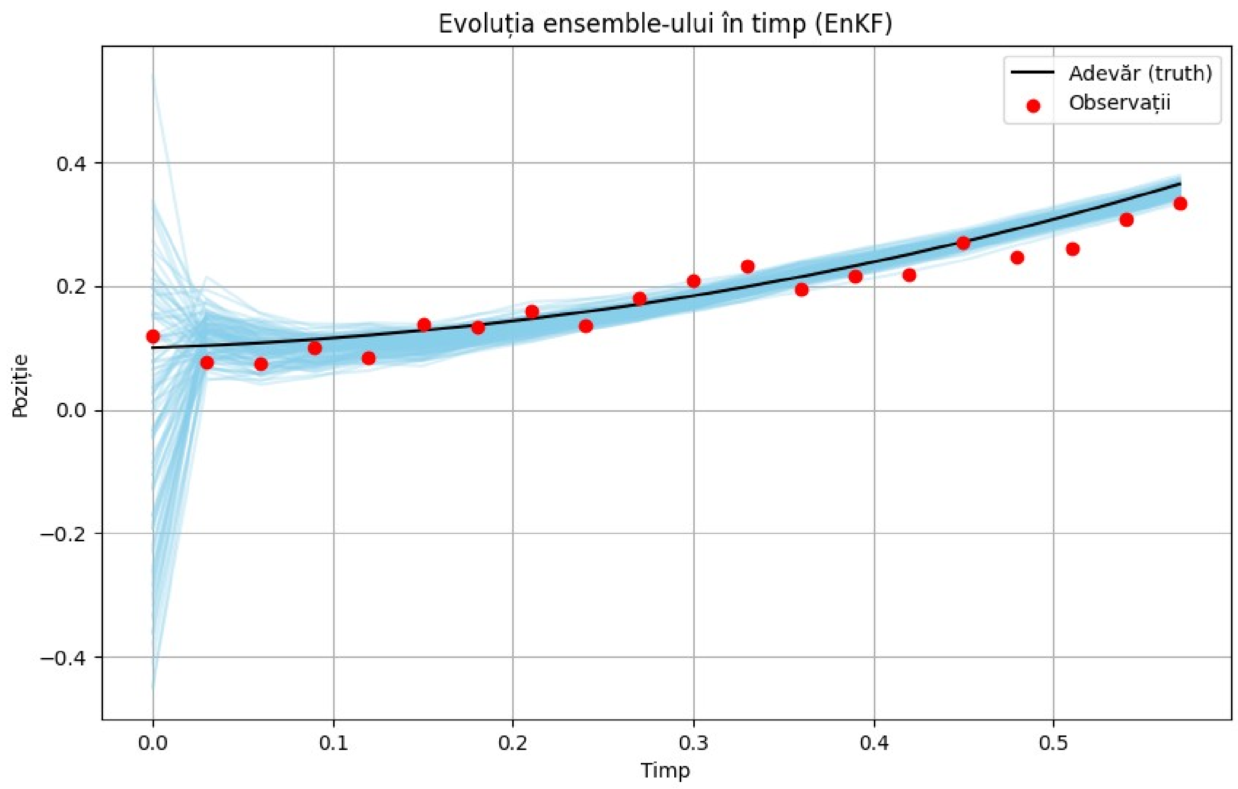
4. Cerinta modelului de data assimilation

Date fiind modelul dinamic, cele 20 de observatii ale pozitiei particulei, incertitudinile in observatii si in pozitia initiala a particulei

1. Sa se implementeze un model de EnKF (Ensemble Kalman Filter) cu 100 de membri, simulati aleator din distributia initiala, cu ajutorul caruia să se estimeze/calibreze la fiecare pas pozitia particulei.
2. Să se prezinte grafic variabilitatea modelului initial si variabilitatea modelului updatat.
3. Să se aproximeze din ansamblul updatat, media si deviatia standard a pozitiei finale a particulei.
4. Să se implemeteze acelasi model de data assimilation, insă numărul de observatii sa fie 10, luate la un interval de timp h=0.3, însa modelul dinamic (solutia numerica) sa se rezolve la un pas de discretizare de 0.03.

```python
import numpy as np
import math
import matplotlib.pyplot as plt

def f(x,t):
    return math.cos(t) * x + math.sin(t)


#functia euler imbunatatita
def heun_step(x, t, h):
    k1 = f(x, t)
    # k2 = f(x + h * k1, t + h)
    k2 = f(t+h, x + h * k1)
    return x + (h / 2) * (k1 + k2)

#returneaza intervalele de timp la care s-au facut
def integrate_heun(x0, t0, t_end, h):
    #times - intervalele de timp
    times = np.arange(t0, t_end + h, h)
    x = np.zeros_like(times)
    x[0] = x0
    for i in range(1, len(times)):
        x[i] = heun_step(x[i - 1], times[i - 1], h)
    return times, x

#x0 - valoarea initiala
#h - intervalul de esantionare
#k - nr de observatii
#noise_std - deviatia standard a observatiilor
#returneaza observatiile si adevarul
def generate_observations(x0, h, k, noise_std):
    _, truth = integrate_heun(x0, 0, h * (k-1), h)
    obs = truth + np.random.normal(0, noise_std, size=k)
    return obs, truth



def enkf(x0_mean, x0_std, obs, h, R, ensemble_size):
    k = len(obs)
    ensemble = np.random.normal(x0_mean, x0_std, size=ensemble_size)
    all_ensembles = []

    # itterate through k obs steps
    for step in range(k):

        #for each itteration we apply the heun step
        for i in range(ensemble_size):
            ## prediction step ##
            ensemble[i] = heun_step(ensemble[i], step * h, h)
        # store the ensebmle values after the prediction step and
before the update step
        all_ensembles.append(ensemble.copy())

        ## update step based on the observations ##

        x_mean = np.mean(ensemble)

        #calculate the variance of the model(std deviation squared)
        P = np.var(ensemble)

        # calculate the Kalman gain
        # number bewtween 0 and 1 that represents how much I trust the
observation in compaison to the model
        K = P / (P + R)
        print(f"Step {step}: Ensemble mean = {x_mean}, Ensemble
variance = {P}, Kalman gain = {K}")
        obs_noise = np.random.normal(0, np.sqrt(R),
size=ensemble_size)
        ensemble = ensemble + K * (obs[step] + obs_noise - ensemble)

    return np.array(all_ensembles)



# Configurare parametri
x0_true = 0.1
x0_std = 0.2
h = 0.03
k = 20
obs_noise_std = 0.03
R = obs_noise_std**2
ensemble_size = 100

# Simulări
obs, truth = generate_observations(x0_true, h, k, obs_noise_std)
all_ensembles = enkf(0, x0_std, obs, h, R, ensemble_size)

# Plot
plt.figure(figsize=(10, 6))
for i in range(ensemble_size):
    plt.plot(np.arange(k) * h, all_ensembles[:, i], color='skyblue',
alpha=0.3)
plt.plot(np.arange(k) * h, truth, color='black', label='Adevăr
(truth)')
plt.scatter(np.arange(k) * h, obs, color='red', label='Observații',
zorder=10)
plt.title("Evoluția ensemble-ului în timp (EnKF)")
plt.xlabel("Timp")
plt.ylabel("Poziție")
plt.legend()
plt.grid()
plt.show()

# Medie și dev. standard la pas final
final_ensemble = all_ensembles[-1]
print("Media la pasul final:", np.mean(final_ensemble))
print("Deviația standard la pasul final:", np.std(final_ensemble))
```

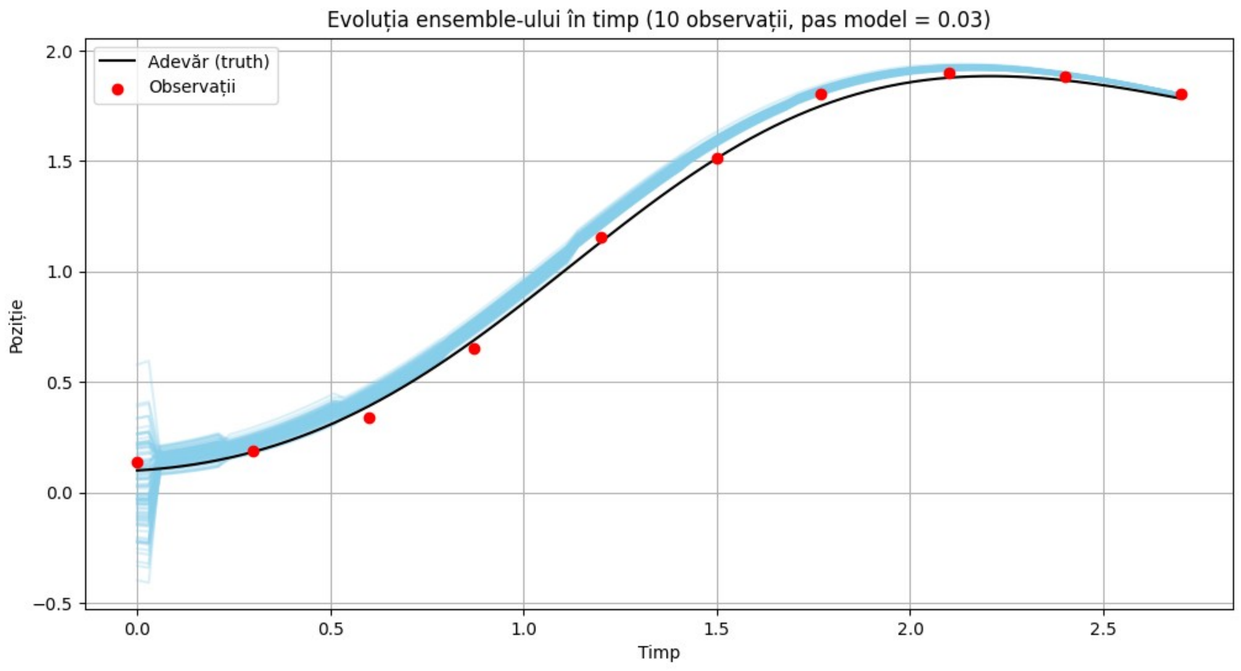
```
Step 0: Ensemble mean = -0.011797607074414787, Ensemble variance =
0.041165035990486584, Kalman gain = 0.9786045588977139
Step 1: Ensemble mean = 0.11871074423606304, Ensemble variance =
0.0010291294974089304, Kalman gain = 0.5334683331477664
Step 2: Ensemble mean = 0.09967576003802937, Ensemble variance =
0.0006696205759807673, Kalman gain = 0.4266130211515348
Step 3: Ensemble mean = 0.0960636443785855, Ensemble variance =
0.0003167907513037271, Kalman gain = 0.26034940762353964
Step 4: Ensemble mean = 0.10458360842291622, Ensemble variance =
0.0002469668158464821, Kalman gain = 0.2153216749032239
Step 5: Ensemble mean = 0.1071194329954196, Ensemble variance =
0.00020789452092437116, Kalman gain = 0.18764829773768937
Step 6: Ensemble mean = 0.122038924399994, Ensemble variance =
0.00015205464703728377, Kalman gain = 0.14453113007531357
Step 7: Ensemble mean = 0.13501360386826225, Ensemble variance =
0.0001464042941850845, Kalman gain = 0.13991178648507058
Step 8: Ensemble mean = 0.14976620694438603, Ensemble variance =
0.00011701045454297096, Kalman gain = 0.11505334485037685
Step 9: Ensemble mean = 0.161656174803717, Ensemble variance =
9.834062084727386e-05, Kalman gain = 0.09850407645820715
Step 10: Ensemble mean = 0.17829493033526803, Ensemble variance =
9.933700952690252e-05, Kalman gain = 0.09940291271102807
Step 11: Ensemble mean = 0.19720745483489296, Ensemble variance =
9.898091207660108e-05, Kalman gain = 0.09908188522926582
Step 12: Ensemble mean = 0.21778100420592736, Ensemble variance =
9.266986502723387e-05, Kalman gain = 0.09335416364703635
Step 13: Ensemble mean = 0.234126748285557, Ensemble variance =
9.679032586593355e-05, Kalman gain = 0.0971019916167923
Step 14: Ensemble mean = 0.25202712515308945, Ensemble variance =
9.437375860716168e-05, Kalman gain = 0.0949077324198024
Step 15: Ensemble mean = 0.2692431514844387, Ensemble variance =
8.53088440012272e-05, Kalman gain = 0.08658081627969326
Step 16: Ensemble mean = 0.2914721165327785, Ensemble variance =
7.917021728166997e-05, Kalman gain = 0.08085439679881083
Step 17: Ensemble mean = 0.311281366303509, Ensemble variance =
8.313966481297074e-05, Kalman gain = 0.08456546693066946
Step 18: Ensemble mean = 0.33160553627478045, Ensemble variance =
8.616841377425294e-05, Kalman gain = 0.08737697595126794
Step 19: Ensemble mean = 0.35560999904424484, Ensemble variance =
7.964578638680211e-05, Kalman gain = 0.08130059608642554
```



```
Media la pasul final: 0.35560999904424484
Deviația standard la pasul final: 0.008924448800166995
```

```python
import numpy as np
import matplotlib.pyplot as plt

def f(x,t):
    return math.cos(t) * x + math.sin(t)

def heun_step(x, t, h):
    k1 = f(x, t)
    k2 = f(t+h, x + h * k1)
    return x + (h / 2) * (k1 + k2)

def integrate_heun(x0, t0, t_end, h):
    times = np.arange(t0, t_end + h, h)
    x = np.zeros_like(times)
    x[0] = x0
    for i in range(1, len(times)):
        x[i] = heun_step(x[i-1], times[i-1], h)
    return times, x

def generate_observations(x0, h_obs, h_model, num_obs, noise_std):
    t_end = h_obs * (num_obs - 1)
    times_full, truth = integrate_heun(x0, 0, t_end, h_model)
    obs_indices = [int(i * h_obs / h_model) for i in range(num_obs)]
    obs_values = truth[obs_indices] + np.random.normal(0, noise_std,
size=num_obs)
    obs_times = times_full[obs_indices]
    return obs_times, obs_values, truth, times_full

def enkf_variable_update(x0_mean, x0_std, obs_times, obs_values,
h_model, R, ensemble_size, t_total):
    n_steps = int(t_total / h_model) + 1
    times = np.arange(0, t_total + h_model, h_model)
    ensemble = np.random.normal(x0_mean, x0_std, size=ensemble_size)
    all_ensembles = np.zeros((n_steps, ensemble_size))
    all_ensembles[0] = ensemble

    obs_pointer = 0
    for step in range(1, n_steps):
        t = times[step-1]

        for i in range(ensemble_size):
            ensemble[i] = heun_step(ensemble[i], t, h_model)
        all_ensembles[step] = ensemble

        if obs_pointer < len(obs_times) and np.isclose(times[step],
obs_times[obs_pointer], atol=1e-1):
            x_mean = np.mean(ensemble)
            P = np.var(ensemble)
            K = P / (P + R)
            print(f"Step {step}: Ensemble mean = {x_mean}, Ensemble
variance = {P}, Kalman gain = {K}")
            obs_noise = np.random.normal(0, np.sqrt(R),
size=ensemble_size)
            ensemble = ensemble + K * (obs_values[obs_pointer] +
obs_noise - ensemble)
            obs_pointer += 1

    print(obs_pointer)
    return times, all_ensembles

x0_true = 0.1
x0_std = 0.2
h_model = 0.03
h_obs = 0.3
num_obs = 10
obs_noise_std = 0.03
R = obs_noise_std ** 2
ensemble_size = 100

obs_times, obs_values, truth, times_full =
generate_observations(x0_true, h_obs, h_model, num_obs, obs_noise_std)

times, all_ensembles = enkf_variable_update(0, x0_std, obs_times,
obs_values, h_model, R, ensemble_size, t_total=times_full[-1])

plt.figure(figsize=(12, 6))
for i in range(ensemble_size):
    plt.plot(times, all_ensembles[:, i], color='skyblue', alpha=0.3)
plt.plot(times_full, truth[:len(times_full)], color='black',
label='Adevăr (truth)')
plt.scatter(obs_times, obs_values, color='red', label='Observații',
zorder=10)
plt.title("Evoluția ensemble-ului în timp (10 observații, pas model =
0.03)")
plt.xlabel("Timp")
plt.ylabel("Poziție")
plt.legend()
plt.grid()
plt.show()

final_ensemble = all_ensembles[-1]
print("Media la pasul final:", np.mean(final_ensemble))
print("Deviația standard la pasul final:", np.std(final_ensemble))
```

```
Step 1: Ensemble mean = 0.03314919373559643, Ensemble variance =
0.03296182390855807, Kalman gain = 0.9734213962475737
Step 7: Ensemble mean = 0.19256141504229898, Ensemble variance =
0.0012017232390409186, Kalman gain = 0.5717799645158332
Step 17: Ensemble mean = 0.3709728500335692, Ensemble variance =
0.0006703682565321154, Kalman gain = 0.42688602099771605
Step 26: Ensemble mean = 0.6280418083677612, Ensemble variance =
0.0005434668401596908, Kalman gain = 0.3765010910119467
Step 37: Ensemble mean = 1.0749820317042358, Ensemble variance =
0.0004190988573686683, Kalman gain = 0.31771603396328046
Step 47: Ensemble mean = 1.4917790775120414, Ensemble variance =
0.0001983439800009125, Kalman gain = 0.18058457424307794
Step 56: Ensemble mean = 1.7548094007768384, Ensemble variance =
0.00010334151158436192, Kalman gain = 0.10299734476367559
Step 67: Ensemble mean = 1.910855931351045, Ensemble variance =
4.464125635956359e-05, Kalman gain = 0.047257364696944346
Step 77: Ensemble mean = 1.9114677722007274, Ensemble variance =
1.8159361915774456e-05, Kalman gain = 0.0197780065955917
Step 87: Ensemble mean = 1.8314574259566774, Ensemble variance =
6.304309508228416e-06, Kalman gain = 0.006956062596291978
10
```



```
Media la pasul final: 1.7992768395735133
Deviația standard la pasul final: 0.0021235275850068984
```