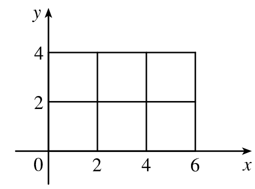
15 □ MULTIPLE INTEGRALS

15.1 Double Integrals over Rectangles

1. (a) The subrectangles are shown in the figure.



The surface is the graph of $f(x,y)=xy$ and $\Delta A=4$, so we estimate

$$V\approx \sum_{i=1}^{3}\sum_{j=1}^{2} f(x_i,y_j)\,\Delta A$$

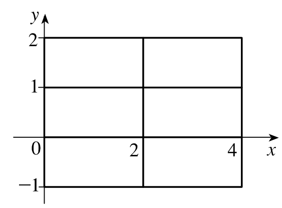
$$= f(2,2)\,\Delta A + f(2,4)\,\Delta A + f(4,2)\,\Delta A + f(4,4)\,\Delta A + f(6,2)\,\Delta A + f(6,4)\,\Delta A$$

$$= 4(4)+8(4)+8(4)+16(4)+12(4)+24(4)=288$$

(b) $V\approx \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A = f(1,1)\,\Delta A + f(1,3)\,\Delta A + f(3,1)\,\Delta A + f(3,3)\,\Delta A + f(5,1)\,\Delta A + f(5,3)\,\Delta A$

$$= 1(4)+3(4)+3(4)+9(4)+5(4)+15(4)=144$$

2. (a) The subrectangles are shown in the figure.



Here $\Delta A=2$ and we estimate

$$\iint_R \left(1-xy^2\right)dA \approx \sum_{i=1}^{2}\sum_{j=1}^{3} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A$$

$$= f(2,-1)\,\Delta A + f(2,0)\,\Delta A + f(2,1)\,\Delta A + f(4,-1)\,\Delta A + f(4,0)\,\Delta A + f(4,1)\,\Delta A$$

$$= (-1)(2)+1(2)+(-1)(2)+(-3)(2)+1(2)+(-3)(2)=-12$$

(b) $\iint_R \left(1-xy^2\right)dA \approx \sum_{i=1}^{2}\sum_{j=1}^{3} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A$

$$= f(0,0)\,\Delta A + f(0,1)\,\Delta A + f(0,2)\,\Delta A + f(2,0)\,\Delta A + f(2,1)\,\Delta A + f(2,2)\,\Delta A$$

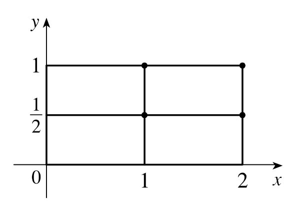
$$= 1(2)+1(2)+1(2)+1(2)+(-1)(2)+(-7)(2)=-8$$

3. (a) The subrectangles are shown in the figure. Since $\Delta A = 1\cdot\frac{1}{2}=\frac{1}{2}$, we estimate

$$\iint_R xe^{-xy}\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A$$

$$= f\left(1,\tfrac{1}{2}\right)\,\Delta A + f(1,1)\,\Delta A + f\left(2,\tfrac{1}{2}\right)\Delta A + f(2,1)\,\Delta A$$

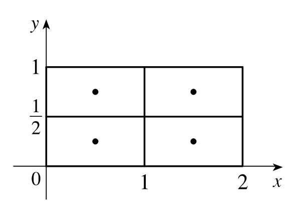
$$= e^{-1/2}\left(\tfrac{1}{2}\right)+e^{-1}\left(\tfrac{1}{2}\right)+2e^{-1}\left(\tfrac{1}{2}\right)+2e^{-2}\left(\tfrac{1}{2}\right)\approx 0.990$$



(b) $\iint_R xe^{-xy}\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A$

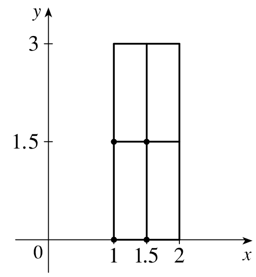
$$= f\left(\tfrac{1}{2},\tfrac{1}{4}\right)\,\Delta A + f\left(\tfrac{1}{2},\tfrac{3}{4}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{1}{4}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{3}{4}\right)\Delta A$$

$$= \tfrac{1}{2}e^{-1/8}\left(\tfrac{1}{2}\right)+\tfrac{1}{2}e^{-3/8}\left(\tfrac{1}{2}\right)+\tfrac{3}{2}e^{-3/8}\left(\tfrac{1}{2}\right)+\tfrac{3}{2}e^{-9/8}\left(\tfrac{1}{2}\right)\approx 1.151$$



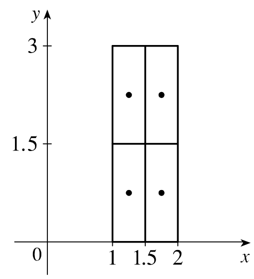
4. (a) The subrectangles are shown in the figure.

The surface is the graph of $f(x,y) = 1 + x^2 + 3y$ and $\Delta A = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$, so we estimate



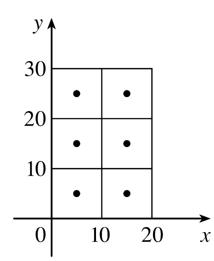
$$V = \iint_R (1 + x^2 + 3y)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$
$$= f(1,0)\,\Delta A + f\left(1, \tfrac{3}{2}\right) \Delta A + f\left(\tfrac{3}{2}, 0\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{3}{2}\right) \Delta A$$
$$= 2\left(\tfrac{3}{4}\right) + \tfrac{13}{2}\left(\tfrac{3}{4}\right) + \tfrac{13}{4}\left(\tfrac{3}{4}\right) + \tfrac{31}{4}\left(\tfrac{3}{4}\right) = \tfrac{39}{2}\left(\tfrac{3}{4}\right) = \tfrac{117}{8} = 14.625$$

(b) $$V = \iint_R (1 + x^2 + 3y)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\, \Delta A$$
$$= f\left(\tfrac{5}{4}, \tfrac{3}{4}\right) \Delta A + f\left(\tfrac{5}{4}, \tfrac{9}{4}\right) \Delta A + f\left(\tfrac{7}{4}, \tfrac{3}{4}\right) \Delta A + f\left(\tfrac{7}{4}, \tfrac{9}{4}\right) \Delta A$$
$$= \tfrac{77}{16}\left(\tfrac{3}{4}\right) + \tfrac{149}{16}\left(\tfrac{3}{4}\right) + \tfrac{101}{16}\left(\tfrac{3}{4}\right) + \tfrac{173}{16}\left(\tfrac{3}{4}\right) = \tfrac{375}{16} = 23.4375$$



5. The values of $f(x,y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)

6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x,y)$ to be the depth of the water at (x,y), so the volume of water in the pool is the volume of the solid that lies above the rectangle $R = [0, 20] \times [0, 30]$ and below the graph of $f(x,y)$. We can estimate this volume using the Midpoint Rule with $m = 2$ and $n = 3$, so $\Delta A = 100$. Each subrectangle with its midpoint is shown in the figure. Then



$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A = \Delta A[f(5,5) + f(5,15) + f(5,25) + f(15,5) + f(15,15) + f(15,25)]$$
$$= 100(3 + 7 + 10 + 3 + 5 + 8) = 3600$$

Thus, we estimate that the pool contains 3600 cubic feet of water.

Alternatively, we can approximate the volume with a Riemann sum where $m = 4$, $n = 6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A = 25$ and

$$V \approx \sum_{i=1}^{4} \sum_{j=1}^{6} f(x_i, y_j)\, \Delta A$$
$$= 25[3 + 4 + 7 + 8 + 10 + 8 + 4 + 6 + 8 + 10 + 12 + 10 + 3 + 4 + 5 + 6 + 8 + 7 + 2 + 2 + 2 + 3 + 4 + 4]$$
$$= 25(140) = 3500$$

So we estimate that the pool contains 3500 ft^3 of water.

7. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

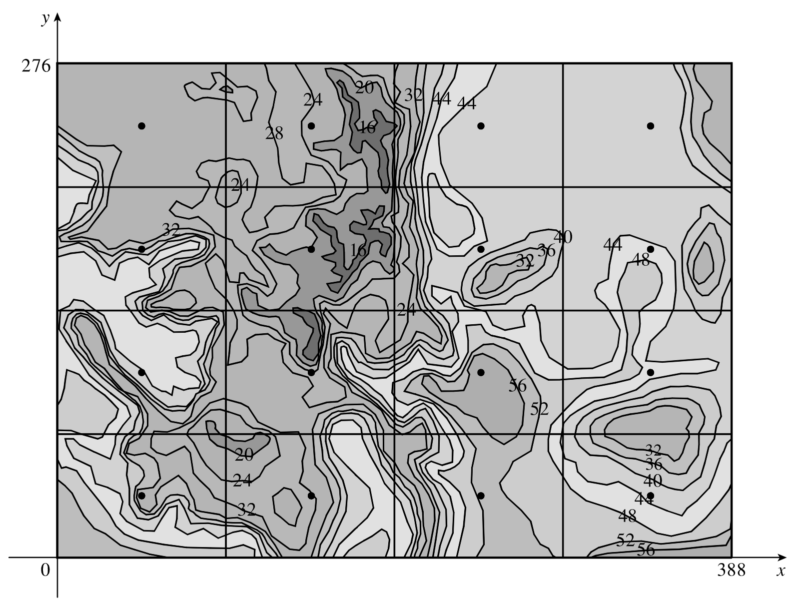
$$\iint_R f(x,y)\ dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\ \Delta A = \Delta A[f(1,1) + f(1,3) + f(3,1) + f(3,3)] \approx 4(27 + 4 + 14 + 17) = 248$$

(b) $f_{\text{avg}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA \approx \frac{1}{16}(248) = 15.5$

8. As in Example 9, we place the origin at the southwest corner of the state. Then $R = [0,388] \times [0,276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x,y)$ to be the temperature at the location (x,y). The average temperature is given by

$$f_{\text{avg}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA = \frac{1}{388 \cdot 276} \iint_R f(x,y)\,dA$$

To use the Midpoint Rule with $m = n = 4$, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated.



The area of each subrectangle is $\Delta A = \frac{388}{4} \cdot \frac{276}{4} = 6693$, so using the contour map to estimate the function values at each midpoint, we have

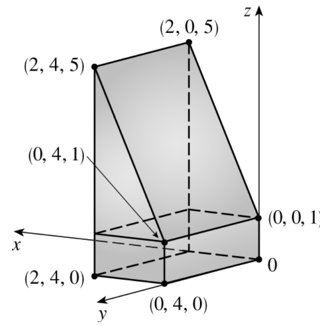
$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{4}\sum_{j=1}^{4} f(\overline{x}_i, \overline{y}_j)\ \Delta A \\ &\approx \Delta A\,[31 + 28 + 52 + 43 + 43 + 25 + 57 + 46 + 36 + 20 + 42 + 45 + 30 + 23 + 43 + 41] \\ &= 6693(605)\end{aligned}$$

Therefore, $f_{\text{avg}} \approx \dfrac{6693 \cdot 605}{388 \cdot 276} \approx 37.8$, so the average temperature in Colorado at 4:00 PM on a day in February was approximately 37.8°F.

9. $z = \sqrt{2} > 0$, so we can interpret the double integral as the volume of the solid S that lies below the plane $z = \sqrt{2}$ and above the rectangle $[2,6] \times [-1,5]$. S is a rectangular solid, so $\iint_R \sqrt{2}\,dA = 4 \cdot 6 \cdot \sqrt{2} = 24\sqrt{2}$.

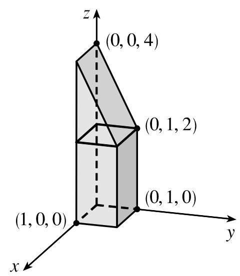
10. $z = 2x + 1 \geq 0$ for $0 \leq x \leq 2$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 2x + 1$ and above the rectangle $[0, 2] \times [0, 4]$. We can picture S as a rectangular solid (with height 1) surmounted by a triangular cylinder; thus

$$\iint_R (2x + 1)\, dA = (2)(4)(1) + \tfrac{1}{2}(2)(4)(4) = 24$$

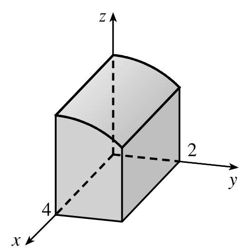


11. $z = 4 - 2y \geq 0$ for $0 \leq y \leq 1$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 4 - 2y$ and above the square $[0, 1] \times [0, 1]$. We can picture S as a rectangular solid (with height 2) surmounted by a triangular cylinder; thus

$$\iint_R (4 - 2y)\, dA = (1)(1)(2) + \tfrac{1}{2}(1)(1)(2) = 3$$



12. Here $z = \sqrt{9 - y^2}$, so $z^2 + y^2 = 9$, $z \geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2 + y^2 = 9$ that lies above the rectangle $[0, 4] \times [0, 2]$.



13. $\int_0^2 (x + 3x^2y^2)\, dx = \left[\dfrac{x^2}{2} + 3\,\dfrac{x^3}{3}\,y^2\right]_{x=0}^{x=2} = \left[\tfrac{1}{2}x^2 + x^3y^2\right]_{x=0}^{x=2} = \left[\tfrac{1}{2}(2)^2 + (2)^3y^2\right] - \left[\tfrac{1}{2}(0)^2 + (0)^3y^2\right] = 2 + 8y^2,$

$\int_0^3 (x + 3x^2y^2)\, dy = \left[xy + 3x^2\,\dfrac{y^3}{3}\right]_{y=0}^{y=3} = \left[xy + x^2y^3\right]_{y=0}^{y=3} = \left[x(3) + x^2(3)^3\right] - \left[x(0) + x^2(0)^3\right] = 3x + 27x^2$

14. $\int_0^2 y\sqrt{x + 2}\, dx = \left[y \cdot \tfrac{2}{3}(x + 2)^{3/2}\right]_{x=0}^{x=2} = \tfrac{2}{3}y(4)^{3/2} - \tfrac{2}{3}y(2)^{3/2} = \tfrac{16}{3}y - \tfrac{4}{3}\sqrt{2}\,y = \tfrac{4}{3}(4 - \sqrt{2}\,)\,y,$

$\int_0^3 y\sqrt{x + 2}\, dy = \left[\dfrac{y^2}{2}\sqrt{x + 2}\right]_{y=0}^{y=3} = \tfrac{1}{2}(3)^2\sqrt{x + 2} - \tfrac{1}{2}(0)^2\sqrt{x + 2} = \tfrac{9}{2}\sqrt{x + 2}$

15. $\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx = \int_1^4 \left[3x^2y^2 - 2xy\right]_{y=0}^{y=2} dx = \int_1^4 \left[(12x^2 - 4x) - (0 - 0)\right] dx$

$$= \int_1^4 (12x^2 - 4x)\, dx = \left[4x^3 - 2x^2\right]_1^4 = (256 - 32) - (4 - 2) = 222$$

16. $\int_0^1 \int_0^1 (x + y)^2\, dx\, dy = \int_0^1 \int_0^1 (x^2 + 2xy + y^2)\, dx\, dy = \int_0^1 \left[\tfrac{1}{3}x^3 + x^2y + xy^2\right]_{x=0}^{x=1} dy$

$$= \int_0^1 \left(\tfrac{1}{3} + y + y^2\right) dy = \left[\tfrac{1}{3}y + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3\right]_0^1 = \tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{3} - 0 = \tfrac{7}{6}$$

17. $\int_0^1 \int_1^2 (x + e^{-y})\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2 + xe^{-y}\right]_{x=1}^{x=2} dy = \int_0^1 \left[(2 + 2e^{-y}) - (\frac{1}{2} + e^{-y})\right] dy$

$= \int_0^1 (\frac{3}{2} + e^{-y})\,dy = \left[\frac{3}{2}y - e^{-y}\right]_0^1 = \left(\frac{3}{2} - e^{-1}\right) - (0 - 1) = \frac{5}{2} - e^{-1}$

18. $\int_{-3}^1 \int_1^2 \left(x^2 + y^{-2}\right)\,dy\,dx = \int_{-3}^1 \left[x^2 y - y^{-1}\right]_{y=1}^{y=2} dx = \int_{-3}^1 \left[\left(2x^2 - \frac{1}{2}\right) - (x^2 - 1)\right] dx$

$= \int_{-3}^1 \left(x^2 + \frac{1}{2}\right) dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x\right]_{-3}^1 = \left(\frac{1}{3} + \frac{1}{2}\right) - \left(-\frac{27}{3} - \frac{3}{2}\right) = \frac{34}{3}$

19. $\int_{-3}^3 \int_0^{\pi/2} (y + y^2 \cos x)\,dx\,dy = \int_{-3}^3 \left[xy + y^2 \sin x\right]_{x=0}^{x=\pi/2} dy = \int_{-3}^3 \left(\frac{\pi}{2}y + y^2\right) dy$

$= \left[\frac{\pi}{4}y^2 + \frac{1}{3}y^3\right]_{-3}^3 = \left[\left(\frac{9\pi}{4} + 9\right) - \left(\frac{9\pi}{4} - 9\right)\right] = 18$

20. $\displaystyle\int_1^3 \int_1^5 \frac{\ln y}{xy}\,dy\,dx = \int_1^3 \frac{1}{x}\,dx \int_1^5 \frac{\ln y}{y}\,dy$ [by Equation 11]

$= \Big[\ln|x|\Big]_1^3 \left[\frac{1}{2}(\ln y)^2\right]_1^5$ [substitute $u = \ln y \;\Rightarrow\; du = (1/y)\,dy$]

$= (\ln 3 - 0) \cdot \frac{1}{2}[(\ln 5)^2 - 0] = \frac{1}{2}(\ln 3)(\ln 5)^2$

21. $\displaystyle\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\,dx = \int_1^4 \left[x \ln|y| + \frac{1}{x} \cdot \frac{1}{2}\,y^2\right]_{y=1}^{y=2} dx = \int_1^4 \left(x \ln 2 + \frac{3}{2x}\right) dx = \left[\frac{1}{2}x^2 \ln 2 + \frac{3}{2}\ln|x|\right]_1^4$

$= \left(8 \ln 2 + \frac{3}{2}\ln 4\right) - \left(\frac{1}{2}\ln 2 + 0\right) = \frac{15}{2}\ln 2 + \frac{3}{2}\ln 4$ or $\frac{15}{2}\ln 2 + 3\ln(4^{1/2}) = \frac{21}{2}\ln 2$

22. $\int_0^1 \int_0^2 ye^{x-y}\,dx\,dy = \int_0^1 \int_0^2 ye^x e^{-y}\,dx\,dy = \int_0^2 e^x\,dx \int_0^1 ye^{-y}\,dy$ [by Equation 11]

$= [e^x]_0^2 \left[(-y - 1)e^{-y}\right]_0^1$ [by integrating by parts]

$= \left(e^2 - e^0\right)\left[-2e^{-1} - \left(-e^0\right)\right] = (e^2 - 1)(1 - 2e^{-1})$ or $e^2 - 2e + 2e^{-1} - 1$

23. $\int_0^3 \int_0^{\pi/2} t^2 \sin^3\phi\,d\phi\,dt = \int_0^{\pi/2} \sin^3\phi\,d\phi \int_0^3 t^2\,dt$ [by Equation 11] $= \int_0^{\pi/2} (1 - \cos^2\phi)\sin\phi\,d\phi \int_0^3 t^2\,dt$

$= \left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi/2} \left[\frac{1}{3}t^3\right]_0^3 = \left[(0 - 0) - \left(\frac{1}{3} - 1\right)\right] \cdot \frac{1}{3}(27 - 0) = \frac{2}{3}(9) = 6$

24. $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\,dy\,dx = \int_0^1 x\left[\frac{1}{3}(x^2 + y^2)^{3/2}\right]_{y=0}^{y=1} dx = \frac{1}{3}\int_0^1 x[(x^2 + 1)^{3/2} - x^3]\,dx = \frac{1}{3}\int_0^1 [x(x^2 + 1)^{3/2} - x^4]dx$

$= \frac{1}{3}\left[\frac{1}{5}(x^2 + 1)^{5/2} - \frac{1}{5}x^5\right]_0^1 = \frac{1}{15}\left[(2^{5/2} - 1) - (1 - 0)\right] = \frac{2}{15}\left(2\sqrt{2} - 1\right)$

25. $\int_0^1 \int_0^1 v(u + v^2)^4\,du\,dv = \int_0^1 \left[\frac{1}{5}v(u + v^2)^5\right]_{u=0}^{u=1} dv = \frac{1}{5}\int_0^1 v\left[(1 + v^2)^5 - (0 + v^2)^5\right] dv$

$= \frac{1}{5}\int_0^1 \left[v(1 + v^2)^5 - v^{11}\right] dv = \frac{1}{5}\left[\frac{1}{2} \cdot \frac{1}{6}(1 + v^2)^6 - \frac{1}{12}v^{12}\right]_0^1$

[substitute $t = 1 + v^2 \;\Rightarrow\; dt = 2v\,dv$ in the first term]

$= \frac{1}{60}\left[(2^6 - 1) - (1 - 0)\right] = \frac{1}{60}(63 - 1) = \frac{31}{30}$

26. $\int_0^1 \int_0^1 \sqrt{s + t}\,ds\,dt = \int_0^1 \left[\frac{2}{3}(s + t)^{3/2}\right]_{s=0}^{s=1} dt = \frac{2}{3}\int_0^1 [(1 + t)^{3/2} - t^{3/2}]\,dt = \frac{2}{3}\left[\frac{2}{5}(1 + t)^{5/2} - \frac{2}{5}t^{5/2}\right]_0^1$

$= \frac{4}{15}[(2^{5/2} - 1) - (1 - 0)] = \frac{4}{15}(2^{5/2} - 2)$ or $\frac{8}{15}\left(2\sqrt{2} - 1\right)$

27. $\iint_R x\sec^2 y\,dA = \int_0^2 \int_0^{\pi/4} x\sec^2 y\,dy\,dx = \int_0^2 x\,dx \int_0^{\pi/4} \sec^2 y\,dy = \left[\frac{1}{2}x^2\right]_0^2 \Big[\tan y\Big]_0^{\pi/4}$

$= (2 - 0)\left(\tan\frac{\pi}{4} - \tan 0\right) = 2(1 - 0) = 2$

28. $\iint_R (y + xy^{-2})\, dA = \int_1^2 \int_0^2 (y + xy^{-2})\, dx\, dy = \int_1^2 \left[xy + \frac{1}{2}x^2 y^{-2}\right]_{x=0}^{x=2} dy = \int_1^2 \left(2y + 2y^{-2}\right) dy$
$= \left[y^2 - 2y^{-1}\right]_1^2 = (4-1) - (1-2) = 4$

29. $\displaystyle \iint_R \frac{xy^2}{x^2+1}\, dA = \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx = \int_0^1 \frac{x}{x^2+1}\, dx \int_{-3}^3 y^2\, dy = \left[\tfrac{1}{2}\ln(x^2+1)\right]_0^1 \left[\tfrac{1}{3}y^3\right]_{-3}^3$
$= \frac{1}{2}(\ln 2 - \ln 1) \cdot \frac{1}{3}(27 + 27) = 9 \ln 2$

30. $\displaystyle \iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\, dA = \int_0^{1/2} \int_0^{\pi/3} \frac{\tan\theta}{\sqrt{1-t^2}}\, d\theta\, dt = \int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\, dt \int_0^{\pi/3} \tan\theta\, d\theta = \left[\sin^{-1} t\right]_0^{1/2} \left[\ln|\sec\theta|\right]_0^{\pi/3}$
$= \left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right)\left(\ln\left|\sec\frac{\pi}{3}\right| - \ln|\sec 0|\right) = \left(\frac{\pi}{6} - 0\right)(\ln 2 - \ln 1) = \frac{\pi}{6}\ln 2$

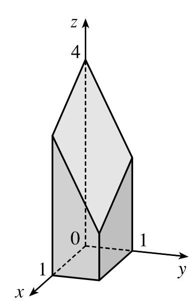
31. $\iint_R x \sin(x+y)\, dA = \int_0^{\pi/6} \int_0^{\pi/3} x \sin(x+y)\, dy\, dx$
$= \int_0^{\pi/6} \left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3} dx = \int_0^{\pi/6} \left[x\cos x - x\cos\left(x + \frac{\pi}{3}\right)\right] dx$
$= x\left[\sin x - \sin\left(x + \frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6} \left[\sin x - \sin\left(x + \frac{\pi}{3}\right)\right] dx$ [by integrating by parts separately for each term]
$= \frac{\pi}{6}\left[\frac{1}{2} - 1\right] - \left[-\cos x + \cos\left(x + \frac{\pi}{3}\right)\right]_0^{\pi/6} = -\frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2} + 0 - \left(-1 + \frac{1}{2}\right)\right] = \frac{\sqrt{3}-1}{2} - \frac{\pi}{12}$

32. $\displaystyle \iint_R \frac{x}{1+xy}\, dA = \int_0^1 \int_0^1 \frac{x}{1+xy}\, dy\, dx = \int_0^1 \left[\ln(1+xy)\right]_{y=0}^{y=1} dx = \int_0^1 \left[\ln(1+x) - \ln 1\right] dx$
$= \int_0^1 \ln(1+x)\, dx = \left[(1+x)\ln(1+x) - x\right]_0^1$ [by integrating by parts]
$= (2\ln 2 - 1) - (\ln 1 - 0) = 2\ln 2 - 1$

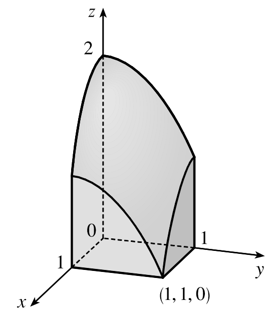
33. $\iint_R ye^{-xy}\, dA = \int_0^3 \int_0^2 ye^{-xy}\, dx\, dy = \int_0^3 \left[-e^{-xy}\right]_{x=0}^{x=2} dy = \int_0^3 (-e^{-2y} + 1)\, dy = \left[\frac{1}{2}e^{-2y} + y\right]_0^3$
$= \frac{1}{2}e^{-6} + 3 - \left(\frac{1}{2} + 0\right) = \frac{1}{2}e^{-6} + \frac{5}{2}$

34. $\displaystyle \iint_R \frac{1}{1+x+y}\, dA = \int_1^3 \int_1^2 \frac{1}{1+x+y}\, dy\, dx = \int_1^3 \left[\ln(1+x+y)\right]_{y=1}^{y=2} dx = \int_1^3 \left[\ln(x+3) - \ln(x+2)\right] dx$
$= \left[\left((x+3)\ln(x+3) - (x+3)\right) - \left((x+2)\ln(x+2) - (x+2)\right)\right]_1^3$
[by integrating by parts separately for each term]
$= (6\ln 6 - 6 - 5\ln 5 + 5) - (4\ln 4 - 4 - 3\ln 3 + 3) = 6\ln 6 - 5\ln 5 - 4\ln 4 + 3\ln 3$

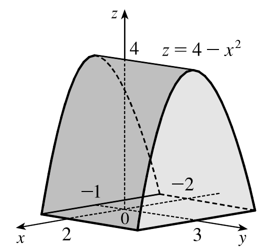
35. $z = f(x,y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0,1] \times [0,1]$.



36. $z = 2 - x^2 - y^2 \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z = 2 - x^2 - y^2$ and above $[0, 1] \times [0, 1]$.



37. $z = 4 - x^2 \geq 0$ for $-2 \leq x \leq 2$ and $-1 \leq y \leq 3$. So the solid is the region that lies below the parabolic cylinder $z = 4 - x^2$ and above $[-2, 2] \times [-1, 3]$.



38. (a) For any given value of $x = k$, $0 \leq k \leq 2$, we have the curve $z = k^2\sqrt{y}$, $x = k$, $1 \leq y \leq 4$.

The area below the curve and above xy-plane is given by $\displaystyle\int_1^4 k^2\sqrt{y}\,dy = \left[\frac{2k^2}{3}y^{3/2}\right]_{y=1}^{y=4} = \frac{2k^2}{3}(8-1) = \frac{14k^2}{3}$.

For $k = 1$, the area is $\dfrac{14(1^2)}{3} = \dfrac{14}{3}$. For $k = 2$, the area is $\dfrac{14(2^2)}{3} = \dfrac{56}{3}$.

(b) For any given value of $y = k$, $1 \leq k \leq 4$, we have the curve, $z = x^2\sqrt{k}$, $y = k$, $0 \leq x \leq 2$. The area below the curve and above xy-plane is given by $\displaystyle\int_0^2 x^2\sqrt{k}\,dx = \left[\frac{x^3}{3}\sqrt{k}\right]_{x=0}^{x=2} = \frac{8\sqrt{k}}{3}$. For $k = 1$, the area is $\dfrac{8\sqrt{1}}{3} = \dfrac{8}{3}$. For $k = 3$, the area is $\dfrac{8\sqrt{3}}{3}$.

(c) $\displaystyle\int_1^4\int_0^2 x^2\sqrt{y}\,dx\,dy = \int_1^4\left[\frac{x^3}{3}\sqrt{y}\right]_{x=0}^{x=2} dy = \int_1^4 \frac{8}{3}\sqrt{y}\,dy = \left[\frac{16}{9}y^{3/2}\right]_1^4 = \frac{16}{9}(8-1) = \frac{112}{9}$

39. (a) The volume under the surface $z = xy$ and over the square $R = [0, 2] \times [0, 2]$ is given by $\int_0^2\int_0^2 xy\,dx\,dy$.

(b) $\displaystyle\int_0^2\int_0^2 xy\,dx\,dy = \int_0^2\left[\frac{x^2}{2}\right]_{x=0}^{x=2} y\,dy = \int_0^2\left[\frac{2^2}{2} - \frac{0^2}{2}\right] y\,dy = \int_0^2 2y\,dy = \left[y^2\right]_0^2 = 4$

40. (a) The volume under the surface $z = \cos x \cos y$ and over the square $R = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is given by $\int_{-\pi/4}^{\pi/4}\int_{-\pi/4}^{\pi/4} \cos x \cos y\,dx\,dy$.

(b) $\int_{-\pi/4}^{\pi/4}\int_{-\pi/4}^{\pi/4} \cos x \cos y\,dx\,dy = \int_{-\pi/4}^{\pi/4} \cos x\,dx \int_{-\pi/4}^{\pi/4} \cos y\,dy$ [by Equation 11]

$$= \Big[\sin x\Big]_{x=-\pi/4}^{x=\pi/4}\Big[\sin y\Big]_{y=-\pi/4}^{y=\pi/4} = \left[\sin\tfrac{\pi}{4} - \sin\left(-\tfrac{\pi}{4}\right)\right]^2$$

$$= \left[\tfrac{\sqrt{2}}{2} - \left(-\tfrac{\sqrt{2}}{2}\right)\right]^2 = \left(\sqrt{2}\right)^2 = 2$$

41. (a) The volume under the surface $z = 1 + ye^{xy}$ and over the rectangle $R = [0,1] \times [1,2]$ is given by

$\int_1^2 \int_0^1 \left(1 + ye^{xy}\right)\, dx\, dy$.

(b) $$\int_1^2 \int_0^1 (1 + ye^{xy})\, dx\, dy = \int_1^2 \left[x + y \cdot \frac{1}{y} e^{xy}\right]_{x=0}^{x=1} dy = \int_1^2 \left[(1 + e^y) - (0 + 1)\right] dy = \int_1^2 e^y\, dy = e^2 - e$$

42. (a) The volume under the surface $z = x^2 + y^2$ and over the square $R = [1,2] \times [1,3]$ is given by $\int_1^3 \int_1^2 (x^2 + y^2)\, dx\, dy$.

(b) $$\int_1^3 \int_1^2 (x^2 + y^2)\, dx\, dy = \int_1^3 \left[\frac{x^3}{3} + xy^2\right]_{x=1}^{x=2} dy = \int_1^3 \left[\left(\frac{8}{3} + 2y^2\right) - \left(\frac{1}{3} + y^2\right)\right] dy = \int_1^3 \left(\frac{7}{3} + y^2\right) dy$$
$$= \left[\frac{7}{3}y + \frac{y^3}{3}\right]_1^3 = (7 + 9) - \left(\frac{7}{3} + \frac{1}{3}\right) = \frac{40}{3}$$

43. The solid lies under the plane $4x + 6y - 2z + 15 = 0$ or $z = 2x + 3y + \frac{15}{2}$ so

$$V = \iint_R (2x + 3y + \tfrac{15}{2})\, dA = \int_{-1}^1 \int_{-1}^2 (2x + 3y + \tfrac{15}{2})\, dx\, dy = \int_{-1}^1 \left[x^2 + 3xy + \tfrac{15}{2}x\right]_{x=-1}^{x=2} dy$$
$$= \int_{-1}^1 \left[(19 + 6y) - (-\tfrac{13}{2} - 3y)\right] dy = \int_{-1}^1 (\tfrac{51}{2} + 9y)\, dy = \left[\tfrac{51}{2}y + \tfrac{9}{2}y^2\right]_{-1}^1 = 30 - (-21) = 51$$

44. $$V = \iint_R (3y^2 - x^2 + 2)\, dA = \int_{-1}^1 \int_1^2 (3y^2 - x^2 + 2)\, dy\, dx = \int_{-1}^1 \left[y^3 - x^2 y + 2y\right]_{y=1}^{y=2} dx$$
$$= \int_{-1}^1 \left[(12 - 2x^2) - (3 - x^2)\right] dx = \int_{-1}^1 \left(9 - x^2 dx\right) = \left[9x - \tfrac{1}{3}x^3\right]_{-1}^1 = \tfrac{26}{3} + \tfrac{26}{3} = \tfrac{52}{3}$$

45. $$V = \int_{-2}^2 \int_{-1}^1 \left(1 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2\right) dx\, dy = 4 \int_0^2 \int_0^1 \left(1 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2\right) dx\, dy$$
$$= 4 \int_0^2 \left[x - \tfrac{1}{12}x^3 - \tfrac{1}{9}y^2 x\right]_{x=0}^{x=1} dy = 4 \int_0^2 \left(\tfrac{11}{12} - \tfrac{1}{9}y^2\right) dy = 4\left[\tfrac{11}{12}y - \tfrac{1}{27}y^3\right]_0^2 = 4 \cdot \tfrac{83}{54} = \tfrac{166}{27}$$

46. The solid lies under the surface $z = x^2 + xy^2$ and above the rectangle $R = [0,5] \times [-2,2]$, so its volume is

$$V = \iint_R (x^2 + xy^2)\, dA = \int_0^5 \int_{-2}^2 (x^2 + xy^2)\, dy\, dx = \int_0^5 \left[x^2 y + \tfrac{1}{3}xy^3\right]_{y=-2}^{y=2} dx$$
$$= \int_0^5 \left[\left(2x^2 + \tfrac{8}{3}x\right) - \left(-2x^2 - \tfrac{8}{3}x\right)\right] dx = \int_0^5 (4x^2 + \tfrac{16}{3}x)\, dx$$
$$= \left[\tfrac{4}{3}x^3 + \tfrac{8}{3}x^2\right]_0^5 = \tfrac{500}{3} + \tfrac{200}{3} - 0 = \tfrac{700}{3}$$

47. The solid lies under the surface $z = 1 + x^2 ye^y$ and above the rectangle $R = [-1,1] \times [0,1]$, so its volume is

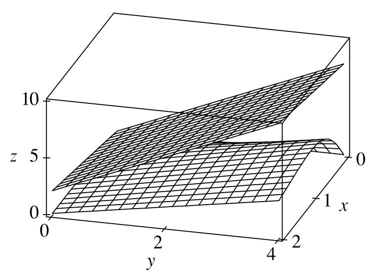
$$V = \iint_R (1 + x^2 ye^y)\, dA = \int_0^1 \int_{-1}^1 (1 + x^2 ye^y)\, dx\, dy = \int_0^1 \left[x + \tfrac{1}{3}x^3 ye^y\right]_{x=-1}^{x=1} dy$$
$$= \int_0^1 (2 + \tfrac{2}{3}ye^y)\, dy = \left[2y + \tfrac{2}{3}(y - 1)\, e^y\right]_0^1 \qquad \text{[by integrating by parts in the second term]}$$
$$= (2 + 0) - \left(0 - \tfrac{2}{3}e^0\right) = 2 + \tfrac{2}{3} = \tfrac{8}{3}$$

48. The cylinder $z = 16 - x^2$ intersects the xy-plane along the line $x = 4$, so in the first octant, the solid lies below the surface $z = 16 - x^2$ and above the rectangle $R = [0,4] \times [0,5]$ in the xy-plane.

$$V = \int_0^5 \int_0^4 (16 - x^2)\, dx\, dy = \int_0^4 (16 - x^2)\, dx\, \int_0^5 dy$$
$$= \left[16x - \tfrac{1}{3}x^3\right]_0^4 \left[\, y\, \right]_0^5 = (64 - \tfrac{64}{3} - 0)(5 - 0) = \tfrac{640}{3}$$

49. The solid lies below the surface $z = 2 + x^2 + (y-2)^2$ and above the plane $z = 1$ for $-1 \le x \le 1$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = 2 + x^2 + (y-2)^2$ over the rectangle $R = [-1, 1] \times [0, 4]$ and the solid that lies under $z = 1$ over R.

$$\begin{aligned}
V &= \textstyle\int_0^4 \int_{-1}^1 [2 + x^2 + (y-2)^2]\, dx\, dy - \int_0^4 \int_{-1}^1 (1)\, dx\, dy \\
&= \textstyle\int_0^4 \left[2x + \frac{1}{3}x^3 + x(y-2)^2\right]_{x=-1}^{x=1} dy - \int_{-1}^1 dx \int_0^4 dy \\
&= \textstyle\int_0^4 \left[\left(2 + \frac{1}{3} + (y-2)^2\right) - \left(-2 - \frac{1}{3} - (y-2)^2\right)\right] dy - [x]_{-1}^1\, [y]_0^4 \\
&= \textstyle\int_0^4 \left[\frac{14}{3} + 2(y-2)^2\right] dy - [1 - (-1)][4 - 0] = \left[\frac{14}{3}y + \frac{2}{3}(y-2)^3\right]_0^4 - (2)(4) \\
&= \left[\left(\frac{56}{3} + \frac{16}{3}\right) - \left(0 - \frac{16}{3}\right)\right] - 8 = \frac{88}{3} - 8 = \frac{64}{3}
\end{aligned}$$

50. 

The solid lies below the plane $z = x + 2y$ and above the surface $z = \dfrac{2xy}{x^2+1}$ for $0 \le x \le 2$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = x + 2y$ over the rectangle $R = [0, 2] \times [0, 4]$ and the solid that lies under $z = \dfrac{2xy}{x^2+1}$ over R.

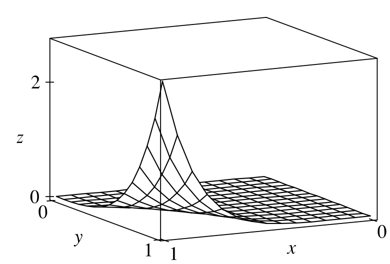
$$V = \int_0^2 \int_0^4 (x + 2y) dy\, dx - \int_0^2 \int_0^4 \frac{2xy}{x^2+1}\, dy\, dx = \int_0^2 \left[xy + y^2\right]_{y=0}^{y=4} dx - \int_0^2 \frac{2x}{x^2+1}\, dx \int_0^4 y\, dy$$

$$\begin{aligned}
&= \textstyle\int_0^2 [(4x + 16) - (0 + 0)]\, dx - \left[\ln\left|x^2+1\right|\right]_0^2 \left[\frac{1}{2}y^2\right]_0^4 = \left[2x^2 + 16x\right]_0^2 - (\ln 5 - \ln 1)(8 - 0) \\
&= (8 + 32 - 0) - 8 \ln 5 = 40 - 8 \ln 5
\end{aligned}$$

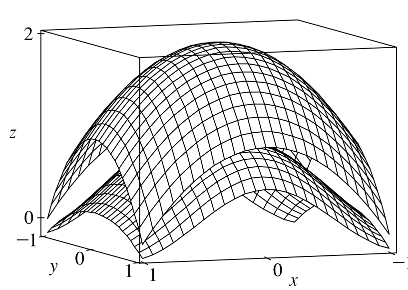
51. In Maple, we can calculate the integral by defining the integrand as `f` and then using the command `int(int(f,x=0..1),y=0..1);`. In Mathematica, we can use the command

`Integrate[f,{x,0,1},{y,0,1}]`

We find that $\iint_R x^5 y^3 e^{xy}\, dA = 21e - 57 \approx 0.0839$. We can use `plot3d` (in Maple) or `Plot3D` (in Mathematica) to graph the function.



52. In Maple, we can calculate the integral by defining `f:=exp(-x^2)*cos(x^2+y^2);` and `g:=2-x^2-y^2;` and then [since $2 - x^2 - y^2 > e^{-x^2}\cos(x^2 + y^2)$ for $-1 \le x \le 1$, $-1 \le y \le 1$] using the command `evalf(Int(Int(g-f,x=-1..1),y=-1..1));`. Using `Int` rather than `int` forces Maple to use purely numerical techniques in evaluating the integral.



In Mathematica, we can use the command `NIntegrate[g-f,{x,-1,1},{y,-1,1}]`. We find that

$\iint_R \left[(2-x^2-y^2) - \left(e^{-x^2}\cos(x^2+y^2)\right)\right] dA \approx 3.0271$. We can use the `plot3d` command (in Maple) or `Plot3D` (in Mathematica) to graph both functions on the same screen.

53. R is the rectangle $[-1,1]\times[0,5]$. Thus, $A(R) = 2\cdot 5 = 10$ and

$f_{\text{avg}} = \dfrac{1}{A(R)}\iint_R f(x,y)\,dA = \frac{1}{10}\int_0^5\int_{-1}^1 x^2y\,dx\,dy = \frac{1}{10}\int_0^5\left[\frac{1}{3}x^3y\right]_{x=-1}^{x=1} dy = \frac{1}{10}\int_0^5 \frac{2}{3}y\,dy = \frac{1}{10}\left[\frac{1}{3}y^2\right]_0^5 = \frac{5}{6}$.

54. $A(R) = 4\cdot 1 = 4$, so

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{A(R)}\iint_R f(x,y)\,dA = \frac{1}{4}\int_0^4\int_0^1 e^y\sqrt{x+e^y}\,dy\,dx = \frac{1}{4}\int_0^4\left[\frac{2}{3}(x+e^y)^{3/2}\right]_{y=0}^{y=1} dx \\ &= \frac{1}{4}\cdot\frac{2}{3}\int_0^4[(x+e)^{3/2}-(x+1)^{3/2}]\,dx = \frac{1}{6}\left[\frac{2}{5}(x+e)^{5/2}-\frac{2}{5}(x+1)^{5/2}\right]_0^4 \\ &= \frac{1}{6}\cdot\frac{2}{5}[(4+e)^{5/2}-5^{5/2}-e^{5/2}+1] = \frac{1}{15}[(4+e)^{5/2}-e^{5/2}-5^{5/2}+1] \approx 3.327 \end{aligned}$$

55. $\displaystyle\iint_R \frac{xy}{1+x^4}\,dA = \int_{-1}^1\int_0^1 \frac{xy}{1+x^4}\,dy\,dx = \int_{-1}^1 \frac{x}{1+x^4}\,dx\int_0^1 y\,dy$ [by Equation 11] but $f(x) = \dfrac{x}{1+x^4}$ is an odd function so $\displaystyle\int_{-1}^1 f(x)\,dx = 0$ (by Theorem 5.5.7). Thus $\displaystyle\iint_R \frac{xy}{1+x^4}\,dA = 0\cdot\int_0^1 y\,dy = 0$.

56.
$$\begin{aligned} \iint_R(1+x^2\sin y+y^2\sin x)\,dA &= \iint_R 1\,dA + \iint_R x^2\sin y\,dA + \iint_R y^2\sin x\,dA \\ &= A(R) + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^2\sin y\,dy\,dx + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} y^2\sin x\,dy\,dx \\ &= (2\pi)(2\pi) + \int_{-\pi}^{\pi}x^2dx\int_{-\pi}^{\pi}\sin y\,dy + \int_{-\pi}^{\pi}\sin x\,dx\int_{-\pi}^{\pi}y^2\,dy \end{aligned}$$

But $\sin x$ is an odd function, so $\int_{-\pi}^{\pi}\sin x\,dx = \int_{-\pi}^{\pi}\sin y\,dy = 0$ (by Theorem 5.5.7) and $\iint_R(1+x^2\sin y+y^2\sin x)\,dA = 4\pi^2+0+0 = 4\pi^2$.

57. Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1\int_0^1 f(x,y)\,dy\,dx = \frac{1}{2}$ and $\int_0^1\int_0^1 f(x,y)\,dx\,dy = -\frac{1}{2}$.

To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals that diverge at their lower limits of integration.

58. (a) Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second-order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).

(b) To find g_{xy}, we first hold y constant and use the single-variable Fundamental Theorem of Calculus, Part 1:

$g_x = \dfrac{d}{dx}g(x,y) = \dfrac{d}{dx}\displaystyle\int_a^x\left(\int_c^y f(s,t)\,dt\right)ds = \int_c^y f(x,t)\,dt$. Now we use the Fundamental Theorem again:

$g_{xy} = \dfrac{d}{dy}\displaystyle\int_c^y f(x,t)\,dt = f(x,y)$.

[continued]

To find g_{yx}, we first use Fubini's Theorem to find that $\int_a^x \int_c^y f(s,t)\,dt\,ds = \int_c^y \int_a^x f(s,t)\,dt\,ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx} = f(x,y)$. So $g_{xy} = g_{yx} = f(x,y)$.

15.2 Double Integrals over General Regions

1. $\int_1^5 \int_0^x (8x-2y)\,dy\,dx = \int_1^5 \left[8xy - y^2\right]_{y=0}^{y=x} dx = \int_1^5 [8x(x) - (x)^2 - 8x(0) + (0)^2]dx$

$= \int_1^5 7x^2\,dx = \frac{7}{3}x^3\Big]_1^5 = \frac{7}{3}(125-1) = \frac{868}{3}$

2. $\int_0^2 \int_0^{y^2} x^2 y\,dx\,dy = \int_0^2 \left[\frac{1}{3}x^3 y\right]_{x=0}^{x=y^2} dy = \int_0^2 \frac{1}{3}y\left[(y^2)^3 - (0)^3\right] dy$

$= \int_0^2 \frac{1}{3}y^7\,dy = \frac{1}{3}\left[\frac{1}{8}y^8\right]_0^2 = \frac{1}{3}(32-0) = \frac{32}{3}$

3. $\int_0^1 \int_0^y xe^{y^3}\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2 e^{y^3}\right]_{x=0}^{x=y} dy = \int_0^1 \frac{1}{2}e^{y^3}\left[(y)^2 - (0)^2\right] dy$

$= \frac{1}{2}\int_0^1 y^2 e^{y^3}\,dy = \frac{1}{2}\left[\frac{1}{3}e^{y^3}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{3}\left(e^1 - e^0\right) = \frac{1}{6}(e-1)$

4. $\int_0^{\pi/2} \int_0^x x\sin y\,dy\,dx = \int_0^{\pi/2} [x(-\cos y)]_{y=0}^{y=x} dx = \int_0^{\pi/2} (-x\cos x + x)\,dx = \int_0^{\pi/2} (x - x\cos x)\,dx$

$= \left[\frac{1}{2}x^2 - (x\sin x + \cos x)\right]_0^{\pi/2}$ (by integrating by parts in the second term)

$= \left(\frac{1}{2}\cdot\frac{\pi^2}{4} - \frac{\pi}{2} - 0\right) - (0 - 0 - 1) = \frac{\pi^2}{8} - \frac{\pi}{2} + 1$

5. $\int_0^1 \int_0^{s^2} \cos(s^3)\,dt\,ds = \int_0^1 \left[t\cos(s^3)\right]_{t=0}^{t=s^2} ds = \int_0^1 s^2\cos(s^3)\,ds = \frac{1}{3}\sin(s^3)\Big]_0^1 = \frac{1}{3}(\sin 1 - \sin 0) = \frac{1}{3}\sin 1$

6. $\int_0^1 \int_0^{e^v} \sqrt{1+e^v}\,dw\,dv = \int_0^1 \left[w\sqrt{1+e^v}\right]_{w=0}^{w=e^v} dv = \int_0^1 e^v\sqrt{1+e^v}\,dv = \frac{2}{3}(1+e^v)^{3/2}\Big]_0^1$

$= \frac{2}{3}(1+e)^{3/2} - \frac{2}{3}(1+1)^{3/2} = \frac{2}{3}(1+e)^{3/2} - \frac{4}{3}\sqrt{2}$

7. (a) We express the iterated integral as a Type I: $\displaystyle\int_0^2 \int_x^{3x-x^2} 2y\,dy\,dx$. A Type II would require the sum of two integrals.

(b) $$\int_0^2 \int_x^{3x-x^2} 2y\,dy\,dx = \int_0^2 \left[y^2\right]_{y=x}^{y=3x-x^2} dx = \int_0^2 \left[(3x-x^2)^2 - (x)^2\right] dx = \int_0^2 \left(8x^2 - 6x^3 + x^4\right) dx$$

$$= \left[\tfrac{8}{3}x^3 - \tfrac{3}{2}x^4 + \tfrac{1}{5}x^5\right]_0^2 = \tfrac{64}{3} - 24 + \tfrac{32}{5} = \tfrac{56}{15}$$

8. (a) We express the iterated integral as a Type II: $\int_0^1 \int_y^{2-y} (x+y)\,dx\,dy$. A Type I would require the sum of two integrals.

(b) $$\int_0^1 \int_y^{2-y} (x+y)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} + xy\right]_{x=y}^{x=2-y} dy = \int_0^1 \left[\left(\frac{(2-y)^2}{2} + (2-y)y\right) - \left(\frac{y^2}{2} + y^2\right)\right] dy$$

$$= \int_0^1 (2 - 2y^2)\,dy = \left[2y - \tfrac{2}{3}y^3\right]_0^1 = 2 - \tfrac{2}{3} = \tfrac{4}{3}$$

9. (a) We express the iterated integral as a Type II. A Type I would require the sum of two integrals. The curves intersect when $\sqrt{x} = x - 2 \;\Rightarrow\; x = x^2 - 4x + 4 \;\Leftrightarrow\; 0 = x^2 - 5x + 4 \;\Leftrightarrow\; (x-4)(x-1) = 0 \;\Leftrightarrow\; x = 1$ or $x = 4$. The point for $x = 1$ is not in D. Thus, the point of intersection of the curves is $(4, 2)$ and the integral is $\int_0^2 \int_{y^2}^{y+2} xy\,dx\,dy$.

(b) $\displaystyle\int_0^2 \int_{y^2}^{y+2} xy\,dx\,dy = \int_0^2 y\left[\frac{x^2}{2}\right]_{x=y^2}^{x=y+2} dy = \frac{1}{2}\int_0^2 y\left[(y+2)^2-(y^2)^2\right] dy = \frac{1}{2}\int_0^2 \left[y^3+4y^2+4y-y^5\right] dy$

$$= \tfrac{1}{2}\left[\tfrac{1}{4}y^4+\tfrac{4}{3}y^3+2y^2-\tfrac{1}{6}y^6\right]_0^2 = \tfrac{1}{2}\left(4+\tfrac{32}{3}+8-\tfrac{32}{3}\right)=6$$

10. (a) We express the iterated integral as a Type I. A Type II would require the sum of two integrals. The curves intersect when $6-x=x^2 \;\Leftrightarrow\; x^2+x-6=0 \;\Leftrightarrow\; (x+3)(x-2)=0 \;\Leftrightarrow\; x=-3$ or $x=2$. The point for $x=-3$ is not in D. Thus, the point of intersection of the two curves is $(2,4)$ and the integral is $\displaystyle\int_0^2 \int_{x^2}^{6-x} x\,dy\,dx$.

(b) $\displaystyle\int_0^2 \int_{x^2}^{6-x} x\,dy\,dx = \int_0^2 x\Big[y\Big]_{y=x^2}^{y=6-x} dx = \int_0^2 x[(6-x)-x^2]\,dx = \int_0^2 [6x-x^2-x^3]\,dx$

$$= \left[3x^2-\frac{x^3}{3}-\frac{x^4}{4}\right]_0^2 = 12-\frac{8}{3}-4=\frac{16}{3}$$

11. $\displaystyle\iint_D \frac{y}{x^2+1}\,dA = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2+1}\,dy\,dx = \int_0^4 \left[\frac{1}{x^2+1}\cdot\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^4 \frac{x}{x^2+1}\,dx$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}\ln\left|x^2+1\right|\right]_0^4 = \tfrac{1}{4}\left[\ln\left(x^2+1\right)\right]_0^4 = \tfrac{1}{4}(\ln 17-\ln 1)=\tfrac{1}{4}\ln 17$$

12. $\iint_D (2x+y)\,dA = \int_1^2 \int_{y-1}^1 (2x+y)\,dx\,dy = \int_1^2 \left[x^2+xy\right]_{x=y-1}^{x=1} dy = \int_1^2 \left[1+y-(y-1)^2-y(y-1)\right] dy$

$$= \textstyle\int_1^2 (-2y^2+4y)\,dy = \left[-\frac{2}{3}y^3+2y^2\right]_1^2 = \left(-\frac{16}{3}+8\right)-\left(-\frac{2}{3}+2\right)=\frac{4}{3}$$

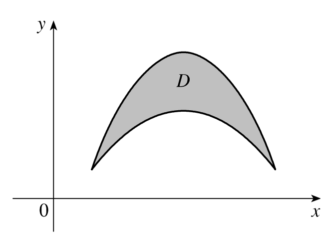
13. $\iint_D e^{-y^2}\,dA = \int_0^3 \int_0^y e^{-y^2}\,dx\,dy = \int_0^3 \left[xe^{-y^2}\right]_{x=0}^{x=y} dy = \int_0^3 \left(ye^{-y^2}-0\right) dy = \int_0^3 ye^{-y^2}\,dy$

$$= -\tfrac{1}{2}e^{-y^2}\Big]_0^3 = -\tfrac{1}{2}\left(e^{-9}-e^0\right)=\tfrac{1}{2}\left(1-e^{-9}\right)$$

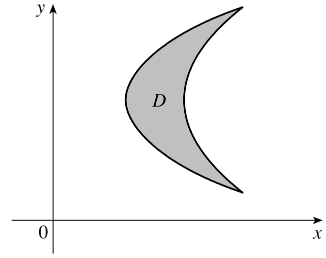
14. $\iint_D y\sqrt{x^2-y^2}\,dA = \int_0^2 \int_0^x y\sqrt{x^2-y^2}\,dy\,dx = \int_0^2 \left[-\frac{1}{3}(x^2-y^2)^{3/2}\right]_{y=0}^{y=x} dx = \int_0^2 \left[0+\frac{1}{3}(x^2)^{3/2}\right] dx$

$$= \textstyle\int_0^2 \frac{1}{3}x^3\,dx = \frac{1}{3}\cdot\frac{1}{4}x^4\Big]_0^2 = \frac{1}{12}(16-0)=\frac{4}{3}$$

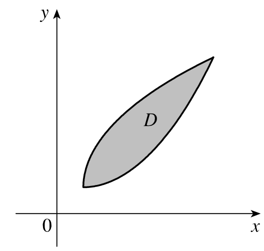
15. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) but not as lying between graphs of two continuous functions of y (a type II region). The regions shown in Figures 6 and 8 in the text are additional examples.



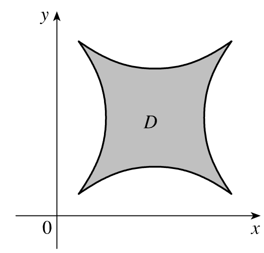
(b) Now we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of y but not as lying between graphs of two continuous functions of x. The first region shown in Figure 7 is another example.



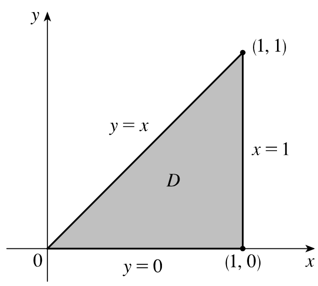
16. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) and also as lying between graphs of two continuous functions of y (a type II region). For additional examples see Figures 9–10 and 15–16 in the text.



(b) Now we sketch an example of a region D that can't be described as lying between the graphs of two continuous functions of x or between graphs of two continuous functions of y. The region shown in Figure 18 is another example.



17.

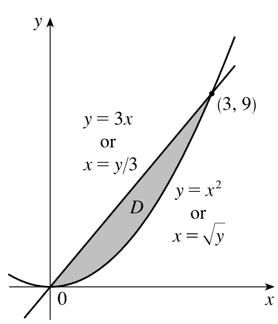


As a type I region, D lies between the lower boundary $y = 0$ and the upper boundary $y = x$ for $0 \le x \le 1$, so $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le x\}$. If we describe D as a type II region, D lies between the left boundary $x = y$ and the right boundary $x = 1$ for $0 \le y \le 1$, so $D = \{(x, y) \mid 0 \le y \le 1, y \le x \le 1\}$.

Thus $\iint_D x\,dA = \int_0^1 \int_0^x x\,dy\,dx = \int_0^1 \left[xy\right]_{y=0}^{y=x} dx = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}$ or

$\iint_D x\,dA = \int_0^1 \int_y^1 x\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2\right]_{x=y}^{x=1} dy = \frac{1}{2}\int_0^1 (1-y^2)\,dy = \frac{1}{2}\left[y - \frac{1}{3}y^3\right]_0^1 = \frac{1}{2}\left[\left(1-\frac{1}{3}\right) - 0\right] = \frac{1}{3}$.

18.



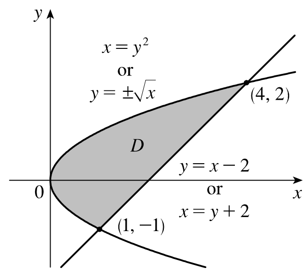
The curves $y = x^2$ and $y = 3x$ intersect at points $(0, 0)$, $(3, 9)$. As a type I region, D is enclosed by the lower boundary $y = x^2$ and the upper boundary $y = 3x$ for $0 \le x \le 3$, so $D = \left\{(x, y) \mid 0 \le x \le 3, x^2 \le y \le 3x\right\}$. If we describe D as a type II region, D is enclosed by the left boundary $x = y/3$ and the right boundary $x = \sqrt{y}$ for $0 \le y \le 9$, so $D = \left\{(x, y) \mid 0 \le y \le 9, y/3 \le x \le \sqrt{y}\right\}$. Thus

$$\begin{aligned}\iint_D xy\,dA &= \int_0^3 \int_{x^2}^{3x} xy\,dy\,dx = \int_0^3 \left[x \cdot \tfrac{1}{2}y^2\right]_{y=x^2}^{y=3x} dx = \tfrac{1}{2}\int_0^3 x(9x^2 - x^4)\,dx = \tfrac{1}{2}\int_0^3 (9x^3 - x^5)\,dx \\ &= \tfrac{1}{2}\left[9 \cdot \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^3 = \tfrac{1}{2}\left[\left(\tfrac{9}{4} \cdot 81 - \tfrac{1}{6} \cdot 729\right) - 0\right] = \tfrac{243}{8}\end{aligned}$$

or

$$\begin{aligned}\iint_D xy\,dA &= \int_0^9 \int_{y/3}^{\sqrt{y}} xy\,dx\,dy = \int_0^9 \left[\tfrac{1}{2}x^2 y\right]_{x=y/3}^{x=\sqrt{y}} dy = \tfrac{1}{2}\int_0^9 \left(y - \tfrac{1}{9}y^2\right) y\,dy = \tfrac{1}{2}\int_0^9 \left(y^2 - \tfrac{1}{9}y^3\right) dy \\ &= \tfrac{1}{2}\left[\tfrac{1}{3}y^3 - \tfrac{1}{9} \cdot \tfrac{1}{4}y^4\right]_0^9 = \tfrac{1}{2}\left[\left(\tfrac{1}{3} \cdot 729 - \tfrac{1}{36} \cdot 6561\right) - 0\right] = \tfrac{243}{8}\end{aligned}$$

19.



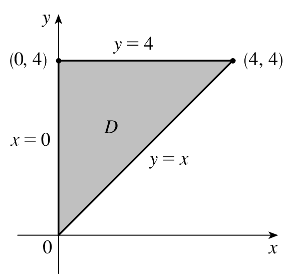
The curves $y = x - 2$ or $x = y + 2$ and $x = y^2$ intersect when $y + 2 = y^2 \Leftrightarrow y^2 - y - 2 = 0 \Leftrightarrow (y-2)(y+1) = 0 \Leftrightarrow y = -1, y = 2$, so the points of intersection are $(1, -1)$ and $(4, 2)$. If we describe D as a type I region, the upper boundary curve is $y = \sqrt{x}$ but the lower boundary curve consists of two parts, $y = -\sqrt{x}$ for $0 \le x \le 1$ and $y = x - 2$ for $1 \le x \le 4$.

Thus $D = \{(x, y) \mid 0 \le x \le 1, -\sqrt{x} \le y \le \sqrt{x}\} \cup \{(x, y) \mid 1 \le x \le 4, x - 2 \le y \le \sqrt{x}\}$ and $\iint_D y\,dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\,dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} y\,dy\,dx$. If we describe D as a type II region, D is enclosed by the left boundary $x = y^2$ and the right boundary $x = y + 2$ for $-1 \le y \le 2$, so $D = \{(x, y) \mid -1 \le y \le 2, y^2 \le x \le y + 2\}$ and $\iint_D y\,dA = \int_{-1}^2 \int_{y^2}^{y+2} y\,dx\,dy$. In either case, the resulting iterated integrals are not difficult to evaluate but the region D is more simply described as a type II region, giving one iterated integral rather than a sum of two, so we evaluate the latter integral:

$$\iint_D y\,dA = \int_{-1}^2 \int_{y^2}^{y+2} y\,dx\,dy = \int_{-1}^2 \left[xy\right]_{x=y^2}^{x=y+2} dy = \int_{-1}^2 (y + 2 - y^2)y\,dy = \int_{-1}^2 (y^2 + 2y - y^3)\,dy$$

$$= \left[\tfrac{1}{3}y^3 + y^2 - \tfrac{1}{4}y^4\right]_{-1}^2 = \left(\tfrac{8}{3} + 4 - 4\right) - \left(-\tfrac{1}{3} + 1 - \tfrac{1}{4}\right) = \tfrac{9}{4}$$

20.

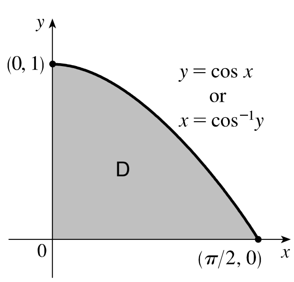


As a type I region, $D = \{(x, y) \mid 0 \le x \le 4, x \le y \le 4\}$ and $\iint_D y^2 e^{xy}\,dA = \int_0^4 \int_x^4 y^2 e^{xy}\,dy\,dx$. As a type II region, $D = \{(x, y) \mid 0 \le y \le 4, 0 \le x \le y\}$ and $\iint_D y^2 e^{xy}\,dA = \int_0^4 \int_0^y y^2 e^{xy}\,dx\,dy$. Evaluating $\int y^2 e^{xy}\,dy$ requires integration by parts whereas $\int y^2 e^{xy}\,dx$ does not, so the iterated integral corresponding to D as a type II region appears easier to evaluate.

$$\iint_D y^2 e^{xy}\,dA = \int_0^4 \int_0^y y^2 e^{xy}\,dx\,dy = \int_0^4 \left[ye^{xy}\right]_{x=0}^{x=y} dy = \int_0^4 \left(ye^{y^2} - y\right) dy$$

$$= \left[\tfrac{1}{2}e^{y^2} - \tfrac{1}{2}y^2\right]_0^4 = \left(\tfrac{1}{2}e^{16} - 8\right) - \left(\tfrac{1}{2} - 0\right) = \tfrac{1}{2}e^{16} - \tfrac{17}{2}$$

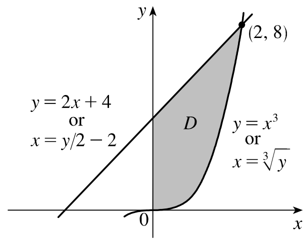
21.



If we describe D as a type I region, $D = \{(x, y) \mid 0 \le x \le \pi/2, 0 \le y \le \cos x\}$ and $\iint_D \sin^2 x\,dA = \int_0^{\pi/2} \int_0^{\cos x} \sin^2 x\,dy\,dx$. As a type II region, $D = \{(x, y) \mid 0 \le x \le \cos^{-1} y, 0 \le y \le 1\}$ and $\iint_D \sin^2 x\,dA = \int_0^1 \int_0^{\cos^{-1} y} \sin^2 x\,dx\,dy$. Evaluating $\int_0^{\cos^{-1} y} \sin^2 x\,dx$ will result in a very difficult integral. Therefore, we evaluate the iterated integral that describes D as a type I region because integrating $\sin^2 x$ with respect to y is easy.

$$\int_0^{\pi/2} \int_0^{\cos x} \sin^2 x\,dy\,dx = \int_0^{\pi/2} \sin^2 x \left[y\right]_{y=0}^{y=\cos x} dx = \int_0^{\pi/2} \cos x \sin^2 x\,dx$$

$$= \int_0^1 u^2\,du \quad \begin{bmatrix} u = \sin x, \\ du = \cos x\,dx \end{bmatrix} \quad = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

22. 

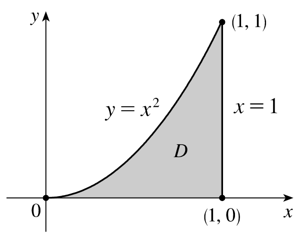
By inspection, the curves $y = 2x + 4$ and $y = x^3$ intersect when $x^3 = 2x + 4 \;\Leftrightarrow\; x = 2$, so the point of intersection is $(2, 8)$. If we describe D as a type 1 region, $D = \{(x, y) \mid 0 \le x \le 2, x^3 \le y \le 2x + 4\}$ and the integral is $\displaystyle\iint_D 6x^2\, dA = \int_0^2 \int_{x^3}^{2x+4} 6x^2\, dy\, dx.$

If we describe D as a type II region, the right boundary curve is $x = \sqrt[3]{y}$, but the left boundary curve consists of two parts, $x = 0$ for $0 \le y \le 4$ and $x = y/2 - 2$ for $4 \le y \le 8$.

In either case, the resulting iterated integrals are not difficult to evaluate, but the region D is more simply described as a type I region, giving one iterated integral rather than a sum of two, so we evaluate that integral:

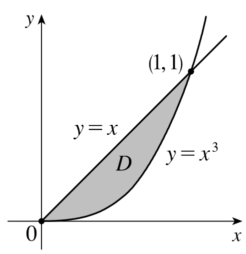
$$\int_0^2 \int_{x^3}^{2x+4} 6x^2\, dy\, dx = \int_0^2 \Big[6x^2 y\Big]_{y=x^3}^{y=2x+4} dx = \int_0^2 [6x^2(2x + 4 - x^3)]\, dx = \int_0^2 (12x^3 + 24x^2 - 6x^5)\, dx$$

$$= \big[3x^4 + 8x^3 - x^6\big]_0^2 = 48 + 64 - 64 = 48$$

23. 

$\int_0^1 \int_0^{x^2} x \cos y\, dy\, dx = \int_0^1 \big[x \sin y\big]_{y=0}^{y=x^2} dx = \int_0^1 x \sin x^2\, dx$

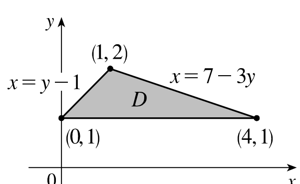
$= -\frac{1}{2} \cos x^2\big]_0^1 = -\frac{1}{2}(\cos 1 - \cos 0) = \frac{1}{2}(1 - \cos 1)$

24. 

$\iint_D (x^2 + 2y)\, dA = \int_0^1 \int_{x^3}^{x} (x^2 + 2y)\, dy\, dx = \int_0^1 \big[x^2 y + y^2\big]_{y=x^3}^{y=x} dx$

$= \int_0^1 (x^3 + x^2 - x^5 - x^6)\, dx = \big[\frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{6}x^6 - \frac{1}{7}x^7\big]_0^1$

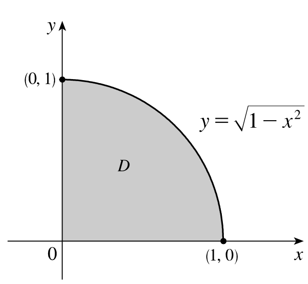
$= \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7} = \frac{23}{84}$

25. 

$\iint_D y^2\, dA = \int_1^2 \int_{y-1}^{7-3y} y^2\, dx\, dy = \int_1^2 \big[xy^2\big]_{x=y-1}^{x=7-3y} dy$

$= \int_1^2 [(7 - 3y) - (y - 1)]\, y^2\, dy = \int_1^2 (8y^2 - 4y^3)\, dy$

$= \big[\frac{8}{3}y^3 - y^4\big]_1^2 = \frac{64}{3} - 16 - \frac{8}{3} + 1 = \frac{11}{3}$

26. 

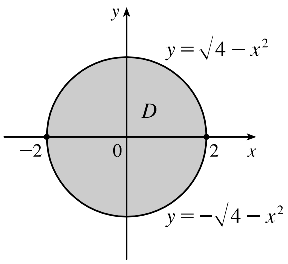
$\iint_D xy\, dA = \int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dy\, dx$

$= \int_0^1 \big[\frac{1}{2}xy^2\big]_{y=0}^{y=\sqrt{1-x^2}} dx = \int_0^1 \frac{1}{2}x(1 - x^2)\, dx$

$= \frac{1}{2}\int_0^1 (x - x^3)\, dx = \frac{1}{2}\big[\frac{1}{2}x^2 - \frac{1}{4}x^4\big]_0^1$

$= \frac{1}{2}\big(\frac{1}{2} - \frac{1}{4} - 0\big) = \frac{1}{8}$

27.



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(2x-y)\,dy\,dx$$

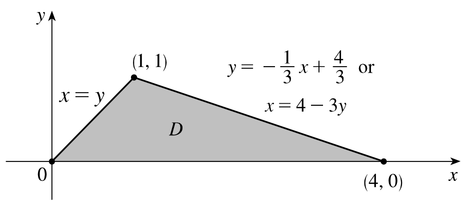
$$= \int_{-2}^{2}\left[2xy-\tfrac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}dx$$

$$= \int_{-2}^{2}\left[2x\sqrt{4-x^2}-\tfrac{1}{2}\left(4-x^2\right)+2x\sqrt{4-x^2}+\tfrac{1}{2}\left(4-x^2\right)\right]dx$$

$$= \int_{-2}^{2}4x\sqrt{4-x^2}\,dx = -\tfrac{4}{3}\left(4-x^2\right)^{3/2}\Big]_{-2}^{2} = 0$$

[Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^{2}4x\sqrt{4-x^2}\,dx=0$.]

28.

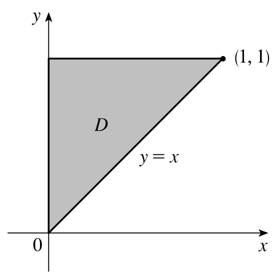


$$\iint_D y\,dA = \int_0^1\int_y^{4-3y} y\,dx\,dy$$

$$= \int_0^1 [xy]_{x=y}^{x=4-3y}\,dy = \int_0^1\left(4y-3y^2-y^2\right)dy$$

$$= \int_0^1\left(4y-4y^2\right)dy = \left[2y^2-\tfrac{4}{3}y^3\right]_0^1 = 2-\tfrac{4}{3}-0=\tfrac{2}{3}$$

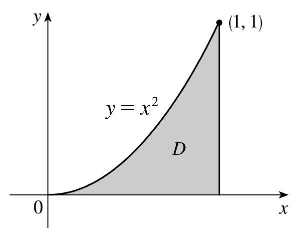
29. (a)



As a Type I region, $D=\{(x,y)\mid 0\le x\le 1,\ x\le y\le 1\}$ and the volume V of the solid that lies under the surface and above D is given by $V=\iint_D(1+xy)\,dA=\int_0^1\int_x^1(1+xy)\,dy\,dx$. As a Type II region, $D=\{(x,y)\mid 0\le y\le 1,\ 0\le x\le y\}$ and $V=\int_0^1\int_0^y(1+xy)\,dx\,dy$. Evaluate either integral in part (b).

(b) $$\int_0^1\int_0^y(1+xy)\,dx\,dy=\int_0^1\left[x+y\,\frac{x^2}{2}\right]_{x=0}^{x=y}dy=\int_0^1\left[\left(y+\frac{y^3}{2}\right)-0\right]dy=\left[\frac{y^2}{2}+\frac{y^4}{8}\right]_0^1=\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$$

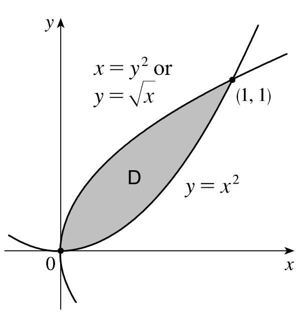
30. (a)



As a Type I region, $D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le x^2\}$ and the volume V of the solid that lies under the surface and above D is given by $V=\iint_D(x^2+y^2)\,dA=\int_0^1\int_0^{x^2}(x^2+y^2)\,dy\,dx$. As a Type II region, $D=\{(x,y)\mid 0\le y\le 1,\ \sqrt{y}\le x\le 1\}$ and $V=\int_0^1\int_{\sqrt{y}}^1(x^2+y^2)\,dx\,dy$.

(b) $$\int_0^1\int_0^{x^2}(x^2+y^2)\,dy\,dx=\int_0^1\left[x^2y+\frac{y^3}{3}\right]_{y=0}^{y=x^2}dx=\int_0^1\left[x^4+\frac{x^6}{3}\right]dx=\left[\frac{x^5}{5}+\frac{x^7}{21}\right]_0^1=\frac{1}{5}+\frac{1}{21}=\frac{26}{105}$$

31.

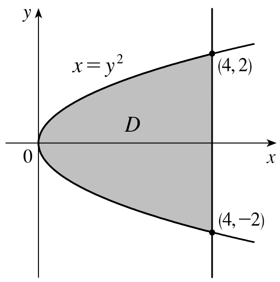


$$V=\int_0^1\int_{x^2}^{\sqrt{x}}(3x+2y)\,dy\,dx=\int_0^1\left[3xy+y^2\right]_{y=x^2}^{y=\sqrt{x}}dx$$

$$= \int_0^1\left[\left(3x\sqrt{x}+x\right)-\left(3x^3+x^4\right)\right]dx=\int_0^1\left(3x^{3/2}+x-3x^3-x^4\right)dx$$

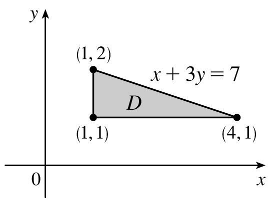
$$= \left[3\cdot\tfrac{2}{5}x^{5/2}+\tfrac{1}{2}x^2-\tfrac{3}{4}x^4-\tfrac{1}{5}x^5\right]_0^1=\tfrac{6}{5}+\tfrac{1}{2}-\tfrac{3}{4}-\tfrac{1}{5}-0=\tfrac{3}{4}$$

32.



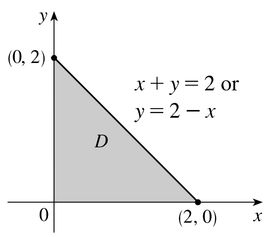
$$V = \int_{-2}^{2}\int_{y^2}^{4}(1 + x^2y^2)\,dx\,dy$$
$$= \int_{-2}^{2}\left[x + \tfrac{1}{3}x^3y^2\right]_{x=y^2}^{x=4}\,dy = \int_{-2}^{2}\left(4 + \tfrac{61}{3}y^2 - \tfrac{1}{3}y^8\right)dy$$
$$= \left[4y + \tfrac{61}{9}y^3 - \tfrac{1}{27}y^9\right]_{-2}^{2} = 8 + \tfrac{488}{9} - \tfrac{512}{27} + 8 + \tfrac{488}{9} - \tfrac{512}{27} = \tfrac{2336}{27}$$

33.



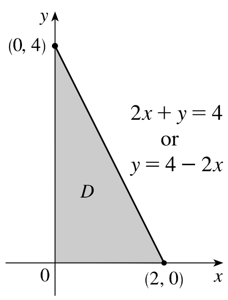
$$V = \int_1^2\int_1^{7-3y} xy\,dx\,dy = \int_1^2\left[\tfrac{1}{2}x^2y\right]_{x=1}^{x=7-3y}\,dy$$
$$= \tfrac{1}{2}\int_1^2 y\left[(7-3y)^2 - 1\right]dy = \tfrac{1}{2}\int_1^2(48y - 42y^2 + 9y^3)\,dy$$
$$= \tfrac{1}{2}\left[24y^2 - 14y^3 + \tfrac{9}{4}y^4\right]_1^2 = \tfrac{31}{8}$$

34.



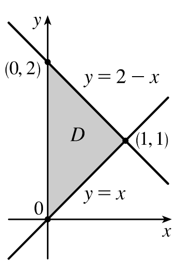
$$V = \int_0^2\int_0^{2-x}(x^2 + y^2 + 1)\,dy\,dx = \int_0^2\left[x^2y + \tfrac{1}{3}y^3 + y\right]_{y=0}^{y=2-x}\,dx$$
$$= \int_0^2\left[x^2(2-x) + \tfrac{1}{3}(2-x)^3 + (2-x) - 0\right]dx$$
$$= \int_0^2\left(-\tfrac{4}{3}x^3 + 4x^2 - 5x + \tfrac{14}{3}\right)dx = \left[-\tfrac{1}{3}x^4 + \tfrac{4}{3}x^3 - \tfrac{5}{2}x^2 + \tfrac{14}{3}x\right]_0^2$$
$$= -\tfrac{16}{3} + \tfrac{32}{3} - 10 + \tfrac{28}{3} - 0 = \tfrac{14}{3}$$

35.



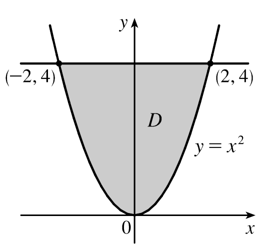
$$V = \int_0^2\int_0^{4-2x}(4 - 2x - y)\,dy\,dx = \int_0^2\left[4y - 2xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x}\,dx$$
$$= \int_0^2\left[4(4-2x) - 2x(4-2x) - \tfrac{1}{2}(4-2x)^2 - 0\right]dx$$
$$= \int_0^2\left(2x^2 - 8x + 8\right)dx = \left[\tfrac{2}{3}x^3 - 4x^2 + 8x\right]_0^2 = \tfrac{16}{3} - 16 + 16 - 0 = \tfrac{16}{3}$$

36.



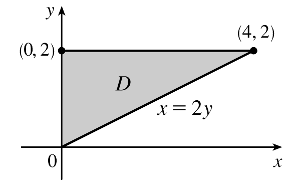
$$V = \int_0^1\int_x^{2-x} x\,dy\,dx$$
$$= \int_0^1\left[xy\right]_{y=x}^{y=2-x}\,dx = \int_0^1(2x - 2x^2)\,dx$$
$$= \left[x^2 - \tfrac{2}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

37.



$$V = \int_{-2}^{2}\int_{x^2}^{4} x^2\,dy\,dx$$
$$= \int_{-2}^{2}\left[x^2y\right]_{y=x^2}^{y=4}\,dx = \int_{-2}^{2}(4x^2 - x^4)\,dx$$
$$= \left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_{-2}^{2} = \tfrac{32}{3} - \tfrac{32}{5} + \tfrac{32}{3} - \tfrac{32}{5} = \tfrac{128}{15}$$

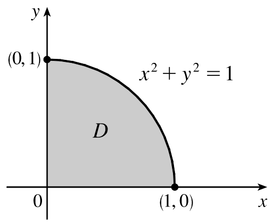
38.



$$V = \int_0^2 \int_0^{2y} \sqrt{4-y^2}\,dx\,dy = \int_0^2 \left[x\sqrt{4-y^2}\right]_{x=0}^{x=2y} dy$$

$$= \int_0^2 2y\sqrt{4-y^2}\,dy = \left[-\tfrac{2}{3}\left(4-y^2\right)^{3/2}\right]_0^2 = 0 + \tfrac{16}{3} = \tfrac{16}{3}$$

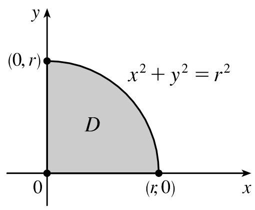
39.



$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1-x^2}{2}\,dx = \tfrac{1}{2}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

40.



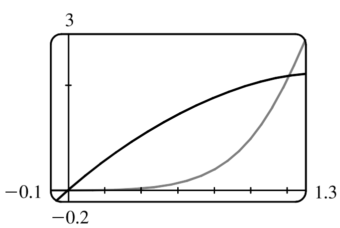
By symmetry, the desired volume V is 8 times the volume V_1 in the first octant. Now

$$V_1 = \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\,dx\,dy = \int_0^r \left[x\sqrt{r^2-y^2}\right]_{x=0}^{x=\sqrt{r^2-y^2}} dy$$

$$= \int_0^r \left(r^2-y^2\right)dy = \left[r^2y - \tfrac{1}{3}y^3\right]_0^r = \tfrac{2}{3}r^3$$

Thus $V = \frac{16}{3}r^3$.

41.



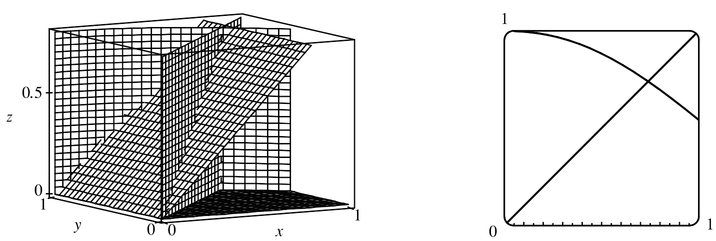
From the graph, it appears that the two curves intersect at $x = 0$ and at $x \approx 1.213$. Thus the desired integral is

$$\iint_D x\,dA \approx \int_0^{1.213}\int_{x^4}^{3x-x^2} x\,dy\,dx = \int_0^{1.213}\left[xy\right]_{y=x^4}^{y=3x-x^2} dx$$

$$= \int_0^{1.213}\left(3x^2 - x^3 - x^5\right)dx = \left[x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^{1.213}$$

$$\approx 0.713$$

42.



The desired solid is shown in the first graph. From the second graph, we estimate that $y = \cos x$ intersects $y = x$ at $x \approx 0.7391$. Therefore the volume of the solid is

$$V \approx \int_0^{0.7391}\int_x^{\cos x} x\,dy\,dx = \int_0^{0.7391}\left[xy\right]_{y=x}^{y=\cos x} dx$$

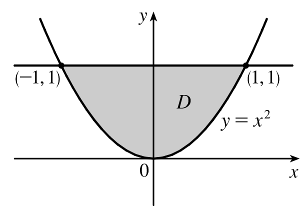
$$= \int_0^{0.7391}\left(x\cos x - x^2\right)dx = \left[\cos x + x\sin x - \tfrac{1}{3}x^3\right]_0^{0.7391} \approx 0.1024$$

Note: There is a different solid which can also be construed to satisfy the conditions stated in the exercise. This is the solid bounded by all of the given surfaces, as well as the plane $y = 0$. In case you calculated the volume of this solid and want to check your work, its volume is $V \approx \int_0^{0.7391}\int_0^x x\,dy\,dx + \int_{0.7391}^{\pi/2}\int_0^{\cos x} x\,dy\,dx \approx 0.4684$.

43. The region of integration is bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$ which intersect at $(\pm 1, 0)$ with $1 - x^2 \geq x^2 - 1$ on $[-1, 1]$. Within this region, the plane $z = 2x + 2y + 10$ is above the plane $z = 2 - x - y$, so

$$\begin{aligned}
V &= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2x + 2y + 10)\, dy\, dx - \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2 - x - y)\, dy\, dx \\
&= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2x + 2y + 10 - (2 - x - y))\, dy\, dx \\
&= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (3x + 3y + 8)\, dy\, dx = \int_{-1}^{1} \left[3xy + \tfrac{3}{2}y^2 + 8y\right]_{y=x^2-1}^{y=1-x^2} dx \\
&= \int_{-1}^{1} \left[3x(1 - x^2) + \tfrac{3}{2}(1 - x^2)^2 + 8(1 - x^2) - 3x(x^2 - 1) - \tfrac{3}{2}(x^2 - 1)^2 - 8(x^2 - 1)\right] dx \\
&= \int_{-1}^{1} (-6x^3 - 16x^2 + 6x + 16)\, dx = \left[-\tfrac{3}{2}x^4 - \tfrac{16}{3}x^3 + 3x^2 + 16x\right]_{-1}^{1} \\
&= -\tfrac{3}{2} - \tfrac{16}{3} + 3 + 16 + \tfrac{3}{2} - \tfrac{16}{3} - 3 + 16 = \tfrac{64}{3}
\end{aligned}$$

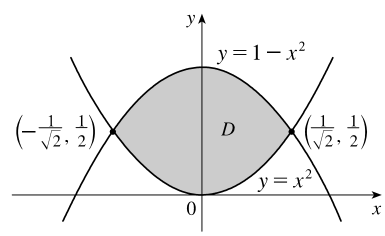
44. The two planes intersect in the line $y = 1$, $z = 3$, so the region of integration is the plane region enclosed by the parabola $y = x^2$ and the line $y = 1$. We have $2 + y \geq 3y$ for $0 \leq y \leq 1$, so the solid region is bounded above by $z = 2 + y$ and bounded below by $z = 3y$.



$$\begin{aligned}
V &= \int_{-1}^{1} \int_{x^2}^{1} (2 + y)\, dy\, dx - \int_{-1}^{1} \int_{x^2}^{1} (3y)\, dy\, dx = \int_{-1}^{1} \int_{x^2}^{1} (2 + y - 3y)\, dy\, dx \\
&= \int_{-1}^{1} \int_{x^2}^{1} (2 - 2y)\, dy\, dx = \int_{-1}^{1} \left[2y - y^2\right]_{y=x^2}^{y=1} dx \\
&= \int_{-1}^{1} (1 - 2x^2 + x^4)\, dx = x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\Big]_{-1}^{1} = \tfrac{16}{15}
\end{aligned}$$

45. The region of integration is bounded by the curves $y = x^2$ and $y = 1 - x^2$ which intersect at $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

The solid lies under the graph of $z = 3$ and above the graph of $z = y$, so its volume is



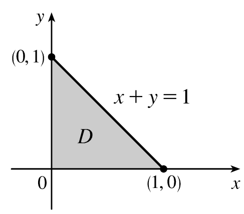
$$\begin{aligned}
V &= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2}^{1-x^2} 3\, dy\, dx - \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2}^{1-x^2} y\, dy\, dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2}^{1-x^2} (3 - y)\, dy\, dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[3y - \tfrac{1}{2}y^2\right]_{y=x^2}^{y=1-x^2} dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[\left(3(1 - x^2) - \tfrac{1}{2}(1 - x^2)^2\right) - \left(3x^2 - \tfrac{1}{2}(x^2)^2\right)\right] dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\tfrac{5}{2} - 5x^2\right) dx = \left[\tfrac{5}{2}x - \tfrac{5}{3}x^3\right]_{-1/\sqrt{2}}^{1/\sqrt{2}} = \left(\tfrac{5}{2\sqrt{2}} - \tfrac{5}{6\sqrt{2}}\right) - \left(-\tfrac{5}{2\sqrt{2}} + \tfrac{5}{6\sqrt{2}}\right) \\
&= \tfrac{10}{3\sqrt{2}} \text{ or } \tfrac{5\sqrt{2}}{3}
\end{aligned}$$

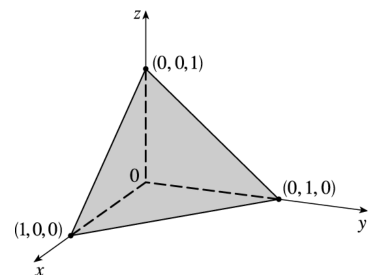
46. The region of integration is the portion of the first quadrant bounded by the axes and the curve $y = \sqrt{4 - x^2}$. The solid lies under the graph of $z = x + y$ and above the graph of $z = xy$, so its volume is

$$\begin{aligned}
V &= \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y)\,dy\,dx - \int_0^2 \int_0^{\sqrt{4-x^2}} xy\,dy\,dx = \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y-xy)\,dy\,dx \\
&= \int_0^2 \left[xy + \tfrac{1}{2}y^2 - \tfrac{1}{2}xy^2 \right]_{y=0}^{y=\sqrt{4-x^2}} dx = \int_0^2 \left[x\sqrt{4-x^2} + \tfrac{1}{2}(4-x^2) - \tfrac{1}{2}x(4-x^2) - 0 \right] dx \\
&= \int_0^2 \left(x\sqrt{4-x^2} + 2 - \tfrac{1}{2}x^2 - 2x + \tfrac{1}{2}x^3 \right) dx = \left[-\tfrac{1}{3}(4-x^2)^{3/2} + 2x - \tfrac{1}{6}x^3 - x^2 + \tfrac{1}{8}x^4 \right]_0^2 \\
&= \left(4 - \tfrac{4}{3} - 4 + 2\right) - \left(-\tfrac{1}{3} \cdot 4^{3/2}\right) = \tfrac{2}{3} + \tfrac{8}{3} = \tfrac{10}{3}
\end{aligned}$$

47. $\int_0^1 \int_0^{1-x} (1 - x - y)\,dy\,dx$.

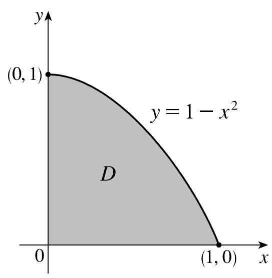
The solid lies below the plane $z = 1 - x - y$ or $x + y + z = 1$ and above the region $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$ in the xy-plane. The solid is a tetrahedron.

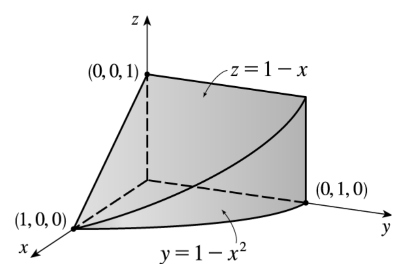




48. $\displaystyle\int_0^1 \int_0^{1-x^2} (1 - x)\,dy\,dx$.

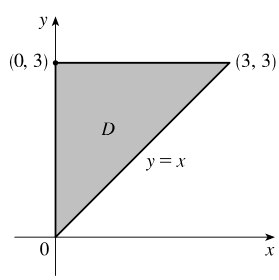
The solid lies below the plane $z = 1 - x$ and above the region $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x^2\}$ in the xy-plane.

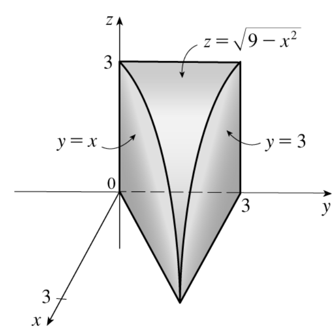




49. $\int_0^3 \int_0^y \sqrt{9 - x^2}\,dx\,dy$.

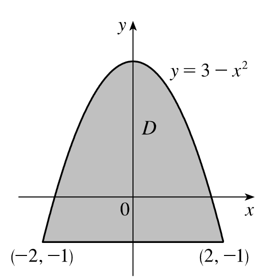
The solid lies under the top half of the cylinder $x^2 + z^2 = 9$; that is, $z = \sqrt{9 - x^2}$, and above the region $D = \{(x,y) \mid 0 \le x \le y, 0 \le y \le 3\}$.

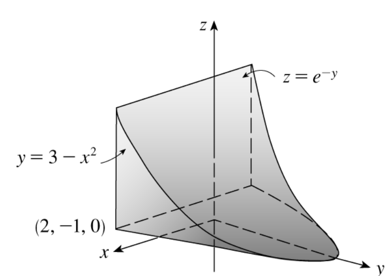




50. $\displaystyle\int_{-2}^2 \int_{-1}^{3-x^2} e^{-y}\,dy\,dx$.

The solid lies below the surface $z = e^{-y}$ and above the region $D = \{(x,y) \mid -2 \le x \le 2, -1 \le y \le 3 - x^2\}$.





51. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume of the solid is

$$V = \int_0^2 \int_{x^3 - x}^{x^2 + x} (x^3y^4 + xy^2)\, dy\, dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

52. For $|x| \le 1$ and $|y| \le 1$, $2x^2 + y^2 < 8 - x^2 - 2y^2$. Also, the cylinder is described by the inequalities $-1 \le x \le 1$, $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$. So the volume is given by

$$V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[(8 - x^2 - 2y^2) - (2x^2 + y^2)\right] dy\, dx = \frac{13\pi}{2} \qquad \text{[using a CAS]}$$

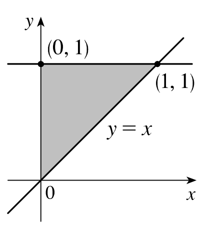
53. The two surfaces intersect in the circle $x^2 + y^2 = 1$, $z = 0$ and the region of integration is the disk D: $x^2 + y^2 \le 1$.

Using a CAS, the volume is $\iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \dfrac{\pi}{2}$.

54. The projection onto the xy-plane of the intersection of the two surfaces is the circle $x^2 + y^2 = 2y \Rightarrow$ $x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y - 1)^2 = 1$, so the region of integration is given by $-1 \le x \le 1$, $1 - \sqrt{1 - x^2} \le y \le 1 + \sqrt{1 - x^2}$. In this region, $2y \ge x^2 + y^2$ so, using a CAS, the volume is

$$V = \int_{-1}^1 \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} [2y - (x^2 + y^2)]\, dy\, dx = \frac{\pi}{2}$$

55.

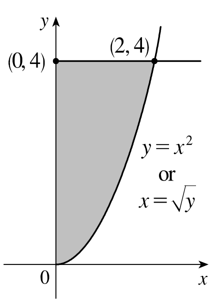


Because the region of integration is

$D = \{(x, y) \mid 0 \le x \le y, 0 \le y \le 1\} = \{(x, y) \mid x \le y \le 1, 0 \le x \le 1\}$

we have $\int_0^1 \int_0^y f(x, y)\, dx\, dy = \iint_D f(x, y)\, dA = \int_0^1 \int_x^1 f(x, y)\, dy\, dx$.

56.

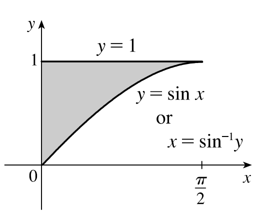


Because the region of integration is

$$D = \{(x, y) \mid x^2 \le y \le 4, 0 \le x \le 2\}$$
$$= \{(x, y) \mid 0 \le x \le \sqrt{y}, 0 \le y \le 4\}$$

we have $\int_0^2 \int_{x^2}^4 f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA = \int_0^4 \int_0^{\sqrt{y}} f(x, y)\, dx\, dy$.

57.



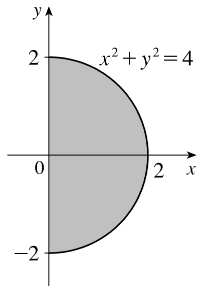
Because the region of integration is

$$D = \{(x, y) \mid 0 \le x \le \pi/2,\ \sin x \le y \le 1\}$$
$$= \{(x, y) \mid 0 \le x \le \sin^{-1} y, 0 \le y \le 1\}$$

we have

$\int_0^{\pi/2} \int_{\sin x}^1 f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA = \int_0^1 \int_0^{\sin^{-1} y} f(x, y)\, dx\, dy$

58.



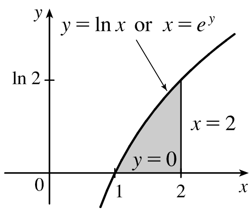
Because the region of integration is

$$D = \left\{(x,y) \mid 0 \le x \le \sqrt{4-y^2}, -2 \le y \le 2\right\}$$
$$= \left\{(x,y) \mid -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, 0 \le x \le 2\right\}$$

we have

$$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dA = \int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx.$$

59.



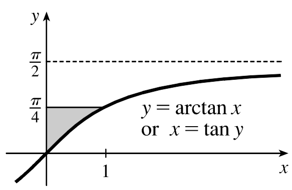
Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le \ln x, 1 \le x \le 2\} = \{(x,y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$$

we have

$$\int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA = \int_0^{\ln 2}\int_{e^y}^2 f(x,y)\,dx\,dy$$

60.



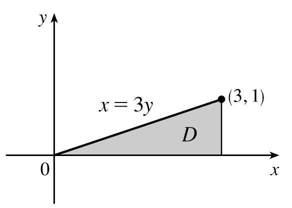
Because the region of integration is

$$D = \left\{(x,y) \mid \arctan x \le y \le \tfrac{\pi}{4}, 0 \le x \le 1\right\}$$
$$= \left\{(x,y) \mid 0 \le x \le \tan y, 0 \le y \le \tfrac{\pi}{4}\right\}$$

we have

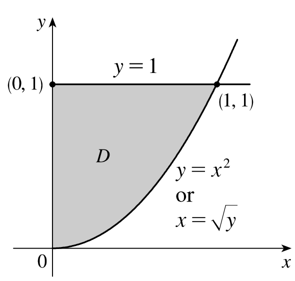
$$\int_0^1\int_{\arctan x}^{\pi/4} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA = \int_0^{\pi/4}\int_0^{\tan y} f(x,y)\,dx\,dy$$

61.



$$\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3\int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 \left[e^{x^2}y\right]_{y=0}^{y=x/3} dx$$
$$= \int_0^3 \left(\frac{x}{3}\right)e^{x^2}\,dx = \tfrac{1}{6}\,e^{x^2}\Big]_0^3 = \frac{e^9-1}{6}$$

62.

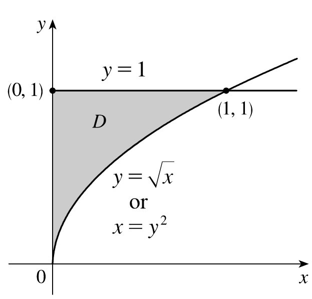


$$\int_0^1\int_{x^2}^1 \sqrt{y}\,\sin y\,dy\,dx = \int_0^1\int_0^{\sqrt{y}} \sqrt{y}\,\sin y\,dx\,dy = \int_0^1 \sqrt{y}\,\sin y\,[x]_{x=0}^{x=\sqrt{y}}\,dy$$
$$= \int_0^1 \left(\sqrt{y}\,\sin y\right)\left(\sqrt{y}-0\right)dy = \int_0^1 y\,\sin y\,dy$$
$$= -y\cos y\big]_0^1 + \int_0^1 \cos y\,dy$$

[by integrating by parts with $u = y$, $dv = \sin y\,dy$]

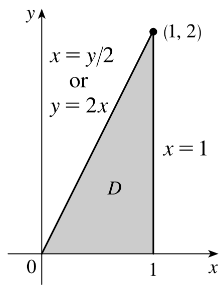
$$= [-y\cos y + \sin y]_0^1 = -\cos 1 + \sin 1 - 0 = \sin 1 - \cos 1$$

63.



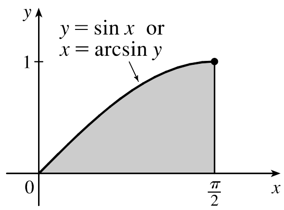
$$\int_0^1\int_{\sqrt{x}}^1 \sqrt{y^3+1}\,dy\,dx = \int_0^1\int_0^{y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^1 \sqrt{y^3+1}\,[x]_{x=0}^{x=y^2}\,dy$$
$$= \int_0^1 y^2\sqrt{y^3+1}\,dy = \tfrac{2}{9}\left(y^3+1\right)^{3/2}\Big]_0^1$$
$$= \tfrac{2}{9}\left(2^{3/2}-1^{3/2}\right) = \tfrac{2}{9}\left(2\sqrt{2}-1\right)$$

64.



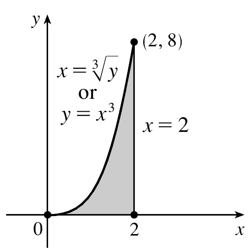
$$\int_0^2 \int_{y/2}^1 y\cos(x^3-1)\,dx\,dy = \int_0^1 \int_0^{2x} y\cos(x^3-1)\,dy\,dx$$
$$= \int_0^1 \cos(x^3-1)\,\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=2x}\,dx$$
$$= \int_0^1 2x^2\cos(x^3-1)\,dx = \tfrac{2}{3}\sin(x^3-1)\Big]_0^1$$
$$= \tfrac{2}{3}\left[0-\sin(-1)\right] = -\tfrac{2}{3}\sin(-1) = \tfrac{2}{3}\sin 1$$

65.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2}\int_0^{\sin x} \cos x\,\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\left[\,y\,\right]_{y=0}^{y=\sin x}\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\sin x\,dx \qquad \begin{bmatrix}\text{Let } u=\cos x,\, du=-\sin x\,dx,\\ dx = du/(-\sin x)\end{bmatrix}$$
$$= \int_1^0 -u\sqrt{1+u^2}\,du = -\tfrac{1}{3}\left(1+u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac{1}{3}\left(\sqrt{8}-1\right) = \tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

66.



$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$
$$= \int_0^2 e^{x^4}\left[\,y\,\right]_{y=0}^{y=x^3}\,dx = \int_0^2 x^3 e^{x^4}\,dx$$
$$= \tfrac{1}{4}e^{x^4}\Big]_0^2 = \tfrac{1}{4}(e^{16}-1)$$

67. $D = \{(x,y) \mid 0 \le x \le 1,\ -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0,\ x+1 \le y \le 1\}$
$\cup \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x-1\}$, all type I.

$$\iint_D x^2\,dA = \int_0^1\int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0\int_{x+1}^1 x^2\,dy\,dx + \int_0^1\int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0\int_{-1}^{-x-1} x^2\,dy\,dx$$
$$= 4\int_0^1\int_{1-x}^1 x^2\,dy\,dx \qquad \text{[by symmetry of the regions and because } f(x,y)=x^2\ge 0]$$
$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

68. $D = \left\{(x,y) \mid -1 \le y \le 0,\ -1 \le x \le y-y^3\right\} \cup \left\{(x,y) \mid 0 \le y \le 1,\ \sqrt{y}-1 \le x \le y-y^3\right\}$, both type II.

$$\iint_D y\,dA = \int_{-1}^0\int_{-1}^{y-y^3} y\,dx\,dy + \int_0^1\int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy = \int_{-1}^0 \left[xy\right]_{x=-1}^{x=y-y^3}dy + \int_0^1 \left[xy\right]_{x=\sqrt{y}-1}^{x=y-y^3}dy$$
$$= \int_{-1}^0 (y^2-y^4+y)\,dy + \int_0^1 (y^2-y^4-y^{3/2}+y)\,dy$$
$$= \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 + \tfrac{1}{2}y^2\right]_{-1}^0 + \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 - \tfrac{2}{5}y^{5/2} + \tfrac{1}{2}y^2\right]_0^1$$
$$= \left(0-\tfrac{11}{30}\right) + \left(\tfrac{7}{30}-0\right) = -\tfrac{2}{15}$$

69. Since $x^2 + y^2 \le 1$ on S, we must have $0 \le x^2 \le 1$ and $0 \le y^2 \le 1$, so $0 \le x^2y^2 \le 1 \quad \Rightarrow$

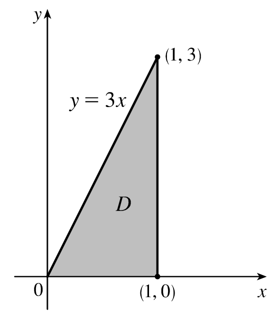
$3 \le 4 - x^2y^2 \le 4 \quad \Rightarrow \quad \sqrt{3} \le \sqrt{4 - x^2y^2} \le 2$. Here we have $A(S) = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$, so by Property 10,

$\sqrt{3}A(S) \le \iint_S \sqrt{4 - x^2y^2}\,dA \le 2A(S) \quad \Rightarrow \quad \frac{\sqrt{3}}{2}\pi \le \iint_S \sqrt{4 - x^2y^2}\,dA \le \pi$ or we can say

$2.720 < \iint_S \sqrt{4 - x^2y^2}\,dA < 3.142$. (We have rounded the lower bound down and the upper bound up to preserve the inequalities.)

70. T is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ so $A(T) = \frac{1}{2}(1)(2) = 1$. We have $0 \le \sin^4(x + y) \le 1$ for all x, y, and Property 10 gives $0 \cdot A(T) \le \iint_T \sin^4(x + y)dA \le 1 \cdot A(T) \quad \Rightarrow \quad 0 \le \iint_T \sin^4(x + y)dA \le 1$.

71. The average value of a function f of two variables defined on a rectangle R was defined in Section 15.1 as $f_{\text{avg}} = \frac{1}{A(R)}\iint_R f(x,y)dA$. Extending this definition to general regions D, we have $f_{\text{avg}} = \frac{1}{A(D)}\iint_D f(x,y)dA$.

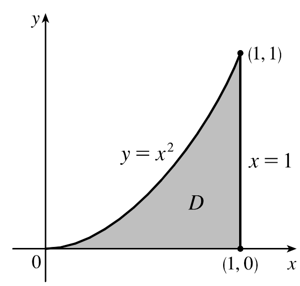


Here $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 3x\}$, so $A(D) = \frac{1}{2}(1)(3) = \frac{3}{2}$ and

$$f_{\text{avg}} = \frac{1}{A(D)}\iint_D f(x,y)dA = \frac{1}{3/2}\int_0^1\int_0^{3x} xy\,dy\,dx$$
$$= \frac{2}{3}\int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=3x} dx = \frac{1}{3}\int_0^1 9x^3\,dx = \left.\tfrac{3}{4}x^4\right]_0^1 = \frac{3}{4}$$

72. Here $D = \left\{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\right\}$, so

$A(D) = \int_0^1 x^2\,dx = \left.\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$ and

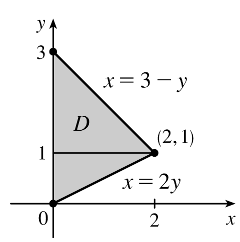


$$f_{\text{avg}} = \frac{1}{A(D)}\iint_D f(x,y)dA = \frac{1}{1/3}\int_0^1\int_0^{x^2} x\sin y\,dy\,dx$$
$$= 3\int_0^1 \left[-x\cos y\right]_{y=0}^{y=x^2} dx$$
$$= 3\int_0^1 \left[x - x\cos(x^2)\right] dx = 3\left[\tfrac{1}{2}x^2 - \tfrac{1}{2}\sin(x^2)\right]_0^1$$
$$= 3\left(\tfrac{1}{2} - \tfrac{1}{2}\sin 1 - 0\right) = \tfrac{3}{2}(1 - \sin 1)$$

73. Since $m \le f(x,y) \le M$, $\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$ by (7) $\quad \Rightarrow$

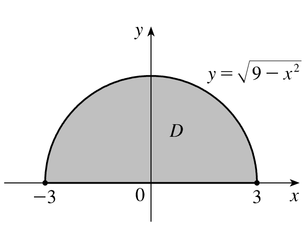
$m\iint_D 1\,dA \le \iint_D f(x,y)\,dA \le M\iint_D 1\,dA$ by (6) $\quad \Rightarrow \quad m \cdot A(D) \le \iint_D f(x,y)\,dA \le M \cdot A(D)$ by (9).

74.



$$\iint_D f(x,y)\,dA = \int_0^1\int_0^{2y} f(x,y)\,dx\,dy + \int_1^3\int_0^{3-y} f(x,y)\,dx\,dy$$
$$= \int_0^2\int_{x/2}^{3-x} f(x,y)\,dy\,dx$$

75.

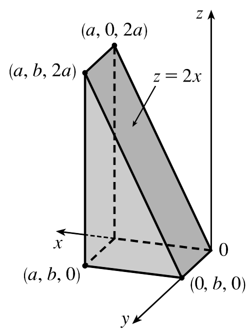


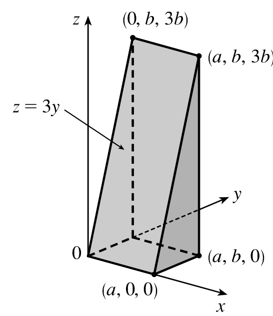
First we can write $\iint_D (x + 2)\,dA = \iint_D x\,dA + \iint_D 2\,dA$. But $f(x,y) = x$ is an odd function with respect to x [that is, $f(-x,y) = -f(x,y)$] and D is symmetric with respect to x. Consequently, the volume above D and below the graph of f is the same as the volume below D and above the graph of f, so

$\iint_D x\,dA = 0$. Also, $\iint_D 2\,dA = 2 \cdot A(D) = 2 \cdot \frac{1}{2}\pi(3)^2 = 9\pi$ since D is a half disk of radius 3. Thus $\iint_D (x+2)\,dA = 0 + 9\pi = 9\pi$.

76. The graph of $f(x,y) = \sqrt{R^2 - x^2 - y^2}$ is the top half of the sphere $x^2 + y^2 + z^2 = R^2$, centered at the origin with radius R, and D is the disk in the xy-plane also centered at the origin with radius R. Thus $\iint_D \sqrt{R^2 - x^2 - y^2}\,dA$ represents the volume of a half ball of radius R which is $\frac{1}{2} \cdot \frac{4}{3}\pi R^3 = \frac{2}{3}\pi R^3$.

77. We can write $\iint_D (2x + 3y)\,dA = \iint_D 2x\,dA + \iint_D 3y\,dA$. $\iint_D 2x\,dA$ represents the volume of the solid lying under the plane $z = 2x$ and above the rectangle D. This solid region is a triangular cylinder with length b and whose cross-section is a triangle with width a and height $2a$. (See the first figure.)

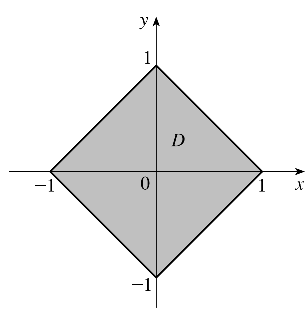




Thus its volume is $\frac{1}{2} \cdot a \cdot 2a \cdot b = a^2 b$. Similarly, $\iint_D 3y\,dA$ represents the volume of a triangular cylinder with length a, triangular cross-section with width b and height $3b$, and volume $\frac{1}{2} \cdot b \cdot 3b \cdot a = \frac{3}{2}ab^2$. (See the second figure.) Thus

$$\iint_D (2x + 3y)\,dA = a^2 b + \tfrac{3}{2}ab^2$$

78.



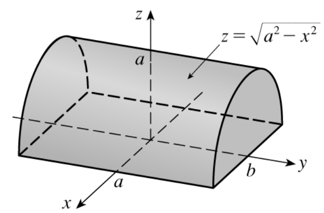
In the first quadrant, x and y are positive and the boundary of D is $x + y = 1$. But D is symmetric with respect to both axes because of the absolute values, so the region of integration is the square shown at the left. To evaluate the double integral, we first write $\iint_D (2 + x^2y^3 - y^2 \sin x)\,dA = \iint_D 2\,dA + \iint_D x^2y^3\,dA - \iint_D y^2 \sin x\,dA$. Now $f(x,y) = x^2y^3$ is odd with respect to y [that is, $f(x,-y) = -f(x,y)$] and D is symmetric with respect to y, so $\iint_D x^2y^3\,dA = 0$.

Similarly, $g(x,y) = y^2 \sin x$ is odd with respect to x [since $g(-x,y) = -g(x,y)$] and D is symmetric with respect to x, so $\iint_D y^2 \sin x\,dA = 0$. D is a square with side length $\sqrt{2}$, so $\iint_D 2\,dA = 2 \cdot A(D) = 2\left(\sqrt{2}\right)^2 = 4$, and $\iint_D (2 + x^2y^3 - y^2 \sin x)\,dA = 4 + 0 + 0 = 4$.

79. $\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right)dA = \iint_D ax^3\,dA + \iint_D by^3\,dA + \iint_D \sqrt{a^2 - x^2}\,dA$. Now ax^3 is odd with respect to x and by^3 is odd with respect to y, and the region of integration is symmetric with respect to both x and y, so $\iint_D ax^3\,dA = \iint_D by^3\,dA = 0$.

[continued]

$\iint_D \sqrt{a^2 - x^2}\, dA$ represents the volume of the solid region under the graph of $z = \sqrt{a^2 - x^2}$ and above the rectangle D, namely a half circular cylinder with radius a and length $2b$ (see the figure) whose volume is $\frac{1}{2} \cdot \pi r^2 h = \frac{1}{2}\pi a^2 (2b) = \pi a^2 b$. Thus

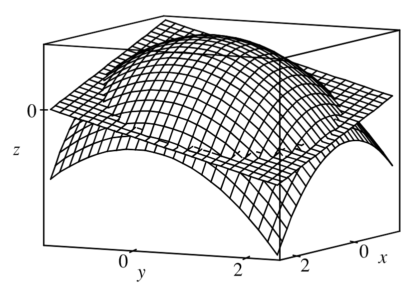


$\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA = 0 + 0 + \pi a^2 b = \pi a^2 b$.

80. By the Extreme Value Theorem (14.7.8), f has an absolute minimum value m and an absolute maximum value M in D. Then by Property 15.2.10, $mA(D) \leq \iint_D f(x, y)\, dA \leq MA(D)$. Dividing through by the positive number $A(D)$, we get $m \leq \dfrac{1}{A(D)} \displaystyle\iint_D f(x, y)\, dA \leq M$. This says that the average value of f over D lies between m and M. But f is continuous on D and takes on the values m and M, and so by the Intermediate Value Theorem must take on all values between m and M. Specifically, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \dfrac{1}{A(D)} \displaystyle\iint_D f(x, y)\, dA$ or equivalently $\iint_D f(x, y)\, dA = f(x_0, y_0)\, A(D)$.

81. For each r such that D_r lies within the domain, $A(D_r) = \pi r^2$, and by the Mean Value Theorem for double integrals there exists (x_r, y_r) in D_r such that $f(x_r, y_r) = \dfrac{1}{\pi r^2} \displaystyle\iint_{D_r} f(x, y)\, dA$. But $\lim\limits_{r \to 0^+} (x_r, y_r) = (a, b)$, so $\lim\limits_{r \to 0^+} \dfrac{1}{\pi r^2} \displaystyle\iint_{D_r} f(x, y)\, dA = \lim\limits_{r \to 0^+} f(x_r, y_r) = f(a, b)$ by the continuity of f.

82. To find the equations of the boundary curves, we require that the z-values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y,y);` and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y,y]`. We find that the curves have equations $y = \dfrac{1 \pm \sqrt{13 + 4x - 4x^2}}{2}$. To find the two points of intersection of these curves, we use the CAS to solve $13 + 4x - 4x^2 = 0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is



$$V = \int_{(1 - \sqrt{14})/2}^{(1 + \sqrt{14})/2} \int_{\left(1 - \sqrt{13 + 4x - 4x^2}\right)/2}^{\left(1 + \sqrt{13 + 4x - 4x^2}\right)/2} \left[(4 - x^2 - y^2) - (1 - x - y)\right] dy\, dx = \frac{49\pi}{8}$$

15.3 Double Integrals in Polar Coordinates

1. The region R is more easily described with polar coordinates: $R = \{(r, \theta) \mid 0 \leq r \leq 4,\ 0 \leq \theta \leq 3\pi/2\}$. Thus, $\iint_R f(x, y)\, dA = \int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \{(x, y) \mid -1 \leq x \leq 1,\ -x \leq y \leq 1\}$. Thus, $\iint_R f(x, y)\, dA = \int_{-1}^1 \int_{-x}^1 f(x, y)\, dy\, dx$.

3. The region R is more easily described with polar coordinates: $R = \{(r, \theta) \mid 1 \le r \le 3, 0 \le \theta \le \pi\}$.

Thus, $\iint_R f(x, y)\,dA = \int_0^\pi \int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

4. The region R is more easily described by polar coordinates: $R = \left\{(r, \theta) \mid 0 \le r \le 3, -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\right\}$.

Thus, $\iint_R f(x, y)\,dA = \int_{-\pi/4}^{3\pi/4} \int_0^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

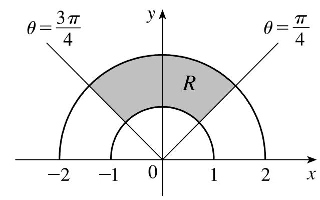
5. The region R is more easily described with rectangular coordinates: $R = \{(x, y) \mid 2y - 2 \le x \le -2y + 2, 0 \le y \le 1\}$.

Thus, $\iint_R f(x, y)\,dA = \int_0^1 \int_{2y-2}^{-2y+2} f(x, y)\,dx\,dy$.

6. The region R is more easily described with polar coordinates: $R = \{(r, \theta) \mid 8 \le r \le 10, 0 \le \theta \le 2\pi\}$.

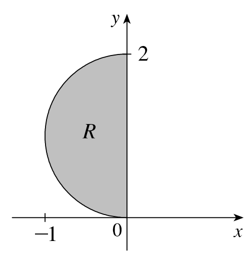
Thus, $\iint_R f(x, y)\,dA = \int_0^{2\pi} \int_8^{10} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

7. The integral $\int_{\pi/4}^{3\pi/4} \int_1^2 r\,dr\,d\theta$ represents the area of the region $R = \{(r, \theta) \mid 1 \le r \le 2, \pi/4 \le \theta \le 3\pi/4\}$, the top quarter portion of a ring (annulus).



$$\int_{\pi/4}^{3\pi/4} \int_1^2 r\,dr\,d\theta = \left(\int_{\pi/4}^{3\pi/4} d\theta\right)\left(\int_1^2 r\,dr\right)$$
$$= \left[\,\theta\,\right]_{\pi/4}^{3\pi/4} \left[\tfrac{1}{2}r^2\right]_1^2 = \left(\tfrac{3\pi}{4} - \tfrac{\pi}{4}\right) \cdot \tfrac{1}{2}\,(4 - 1) = \tfrac{\pi}{2} \cdot \tfrac{3}{2} = \tfrac{3\pi}{4}$$

8. The integral $\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\,dr\,d\theta$ represents the area of the region $R = \{(r, \theta) \mid 0 \le r \le 2\sin\theta, \pi/2 \le \theta \le \pi\}$. Since $r = 2\sin\theta \;\Rightarrow\; r^2 = 2r\sin\theta \;\Leftrightarrow\; x^2 + y^2 = 2y \;\Leftrightarrow\; x^2 + (y-1)^2 = 1$, R is the portion in the second quadrant of a disk of radius 1 with center $(0, 1)$.



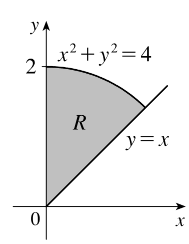
$$\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\,dr\,d\theta = \int_{\pi/2}^{\pi} \left[\tfrac{1}{2}r^2\right]_{r=0}^{r=2\sin\theta} d\theta = \int_{\pi/2}^{\pi} 2\sin^2\theta\,d\theta$$
$$= \int_{\pi/2}^{\pi} 2 \cdot \tfrac{1}{2}(1 - \cos 2\theta)\,d\theta = \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/2}^{\pi}$$
$$= \pi - 0 - \tfrac{\pi}{2} + 0 = \tfrac{\pi}{2}$$

9. The half-disk D can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \le r \le 5, 0 \le \theta \le \pi\}$. Then

$$\iint_D x^2 y\,dA = \int_0^\pi \int_0^5 (r\cos\theta)^2 (r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^\pi \cos^2\theta\sin\theta\,d\theta\right)\left(\int_0^5 r^4\,dr\right)$$
$$= \left[-\tfrac{1}{3}\cos^3\theta\right]_0^\pi \left[\tfrac{1}{5}r^5\right]_0^5 = -\tfrac{1}{3}(-1-1)\cdot 625 = \tfrac{1250}{3}$$

10. The region R is $\frac{1}{8}$ of a disk, as shown in the figure, and can be described by $R = \{(r, \theta) \mid 0 \le r \le 2, \pi/4 \le \theta \le \pi/2\}$. Thus

$$\iint_R (2x - y)\,dA = \int_{\pi/4}^{\pi/2} \int_0^2 (2r\cos\theta - r\sin\theta)\,r\,dr\,d\theta$$
$$= \int_{\pi/4}^{\pi/2} (2\cos\theta - \sin\theta)\,d\theta \int_0^2 r^2\,dr$$
$$= \left[2\sin\theta + \cos\theta\right]_{\pi/4}^{\pi/2} \left[\tfrac{1}{3}r^3\right]_0^2$$
$$= \left(2 + 0 - \sqrt{2} - \tfrac{\sqrt{2}}{2}\right)\left(\tfrac{8}{3}\right) = \tfrac{16}{3} - 4\sqrt{2}$$



11. $\iint_R \sin(x^2+y^2)\,dA = \int_0^{\pi/2}\int_1^3 \sin(r^2)\,r\,dr\,d\theta = \int_0^{\pi/2} d\theta\,\int_1^3 r\sin(r^2)\,dr = \left[\theta\right]_0^{\pi/2}\left[-\frac{1}{2}\cos(r^2)\right]_1^3$

$$= \left(\tfrac{\pi}{2}\right)\left[-\tfrac{1}{2}(\cos 9 - \cos 1)\right] = \tfrac{\pi}{4}(\cos 1 - \cos 9)$$

12. $\displaystyle\iint_R \frac{y^2}{x^2+y^2}\,dA = \int_0^{2\pi}\int_a^b \frac{(r\sin\theta)^2}{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta\int_a^b r\,dr = \int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\,d\theta\int_a^b r\,dr$

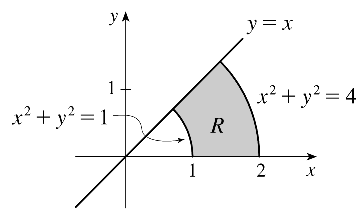
$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{2}r^2\right]_a^b = \tfrac{1}{2}(2\pi - 0 - 0)\cdot\tfrac{1}{2}\left(b^2-a^2\right) = \tfrac{\pi}{2}(b^2-a^2)$$

13. $\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta\,\int_0^2 re^{-r^2}\,dr$

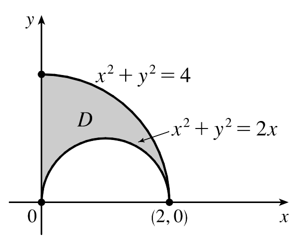
$$= \left[\,\theta\,\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4}-e^0) = \tfrac{\pi}{2}(1-e^{-4})$$

14. $\iint_D \cos\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2 \cos\sqrt{r^2}\;r\,dr\,d\theta = \int_0^{2\pi} d\theta\,\int_0^2 r\cos r\,dr$. For the second integral, integrate by parts with $u = r$, $dv = \cos r\,dr$. Then $\iint_D \cos\sqrt{x^2+y^2}\,dA = \left[\,\theta\,\right]_0^{2\pi}\left[r\sin r + \cos r\right]_0^2 = 2\pi(2\sin 2 + \cos 2 - 1)$.

15. R is the region shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le \theta \le \pi/4, 1 \le r \le 2\}$. Thus $\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$ since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $0 \le \theta \le \pi/4$, so the integral becomes $\int_0^{\pi/4}\int_1^2 \theta\,r\,dr\,d\theta = \int_0^{\pi/4}\theta\,d\theta\,\int_1^2 r\,dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4}\left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32}\cdot\frac{3}{2} = \frac{3}{64}\pi^2$.



16.



$$\iint\limits_D x\,dA = \iint\limits_{\substack{x^2+y^2\le 4\\ x\ge 0,\,y\ge 0}} x\,dA - \iint\limits_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\tfrac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \tfrac{8}{3}[\sin\theta]_0^{\pi/2} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2}$$

$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0 + \tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \tfrac{16-3\pi}{6}$$

17. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant enclosed by the cardiod $r = 1 - \cos\theta$ (see the figure). Here $D = \{(r,\theta) \mid 0 \le r \le 1 - \cos\theta, 0 \le \theta \le \pi/2\}$, so the total area is

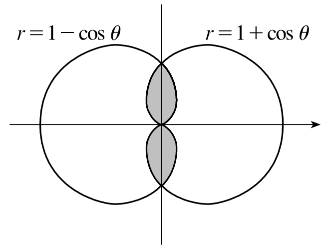
$$4A(D) = 4\iint_D dA = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 4\int_0^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=1-\cos\theta}d\theta$$

$$= 2\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = 2\int_0^{\pi/2}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/2}\left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= 2\left[\theta - 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$= 2\left(\tfrac{\pi}{2} - 2 + \tfrac{\pi}{4}\right) = \tfrac{3\pi}{2} - 4$$

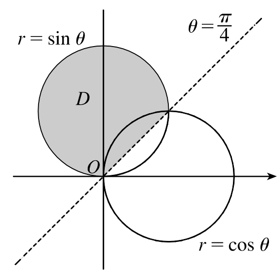


18. The region D is described by $D = \{(r, \theta) \mid 0 \le r \le \sqrt{\theta}, 0 \le \theta \le 2\pi\}$, so the area is

$$A(D) = \int_0^{2\pi} \int_0^{\sqrt{\theta}} r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{\theta}} d\theta = \int_0^{2\pi} \frac{\theta}{2}\,d\theta = \left[\frac{\theta^2}{4}\right]_0^{2\pi} = \pi^2.$$

19. By symmetry, the total area is twice the area defined by

$D = \{(r, \theta) \mid 0 \le r \le \sin\theta, \pi/4 \le \theta \le \pi\}$ (see the figure).



The total area is

$$2A(D) = 2\int_{\pi/4}^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta = 2\cdot\tfrac{1}{2}\int_{\pi/4}^{\pi}\left[r^2\right]_{r=0}^{r=\sin\theta} d\theta = \int_{\pi/4}^{\pi}\sin^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi}$$

$$= \tfrac{1}{2}(\pi - 0) - \tfrac{1}{2}\left(\tfrac{\pi}{4} - \tfrac{1}{2}\right) = \tfrac{3\pi}{8} + \tfrac{1}{4}$$

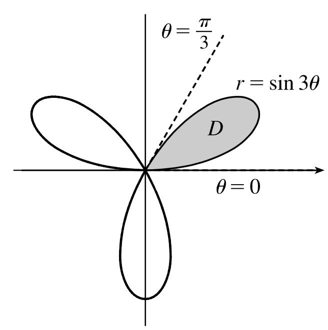
20. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant between the circle $r = 1/\sqrt{2}$ and the curve $r^2 = \cos 2\theta \;\Rightarrow\; r = \sqrt{\cos 2\theta}$. The curves intersect in the first quadrant when $\cos 2\theta = \left(\frac{1}{\sqrt{2}}\right)^2 \;\Rightarrow$ $\cos 2\theta = \frac{1}{2} \;\Rightarrow\; 2\theta = \frac{\pi}{3} \;\Rightarrow\; \theta = \frac{\pi}{6}$. Thus, $D = \{(r, \theta) \mid 1/\sqrt{2} \le r \le \sqrt{\cos 2\theta}, 0 \le \theta \le \pi/6\}$, so the total area is

$$4A(D) = 4\int_0^{\pi/6}\int_{1/\sqrt{2}}^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = 4\cdot\frac{1}{2}\int_0^{\pi/6}\left[r^2\right]_{r=1/\sqrt{2}}^{r=\sqrt{\cos 2\theta}} d\theta = 2\int_0^{\pi/6}\left[\cos 2\theta - \frac{1}{2}\right] d\theta$$

$$= 2\left[\tfrac{1}{2}\sin 2\theta - \tfrac{\theta}{2}\right]_0^{\pi/6} = \tfrac{\sqrt{3}}{2} - \tfrac{\pi}{6}$$

21. One loop is given by the region

$D = \{(r, \theta) \mid 0 \le r \le \sin 3\theta, 0 \le \theta \le \pi/3\}$, so the area is

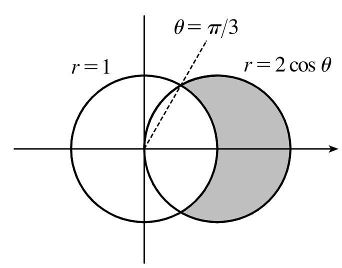


$$\iint_D dA = \int_0^{\pi/3}\int_0^{\sin 3\theta} r\,dr\,d\theta = \tfrac{1}{2}\int_0^{\pi/3}\left[r^2\right]_{r=0}^{r=\sin 3\theta} d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi/3}\sin^2 3\theta\,d\theta = \tfrac{1}{2}\int_0^{\pi/3}\tfrac{1}{2}(1-\cos 6\theta)\,d\theta$$

$$= \tfrac{1}{4}\left[\theta - \tfrac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \tfrac{\pi}{12}$$

22. In polar coordinates the circle $(x-1)^2 + y^2 = 1 \;\Leftrightarrow\; x^2 + y^2 = 2x$ is $r^2 = 2r\cos\theta \;\Rightarrow\; r = 2\cos\theta$, and the circle $x^2 + y^2 = 1$ is $r = 1$. The curves intersect in the first quadrant when $2\cos\theta = 1 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \pi/3$, so the portion of the region in the first quadrant is given by $D = \{(r, \theta) \mid 1 \le r \le 2\cos\theta, 0 \le \theta \le \pi/3\}$. By symmetry, the total area is twice the area of D:



$$2A(D) = 2\iint_D dA = 2\int_0^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3}\left[\tfrac{1}{2}r^2\right]_{r=1}^{r=2\cos\theta} d\theta$$

$$= \int_0^{\pi/3}\left(4\cos^2\theta - 1\right) d\theta = \int_0^{\pi/3}\left[4\cdot\tfrac{1}{2}(1+\cos 2\theta) - 1\right] d\theta$$

$$= \int_0^{\pi/3}(1 + 2\cos 2\theta)\,d\theta = \left[\theta + \sin 2\theta\right]_0^{\pi/3} = \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}$$

23. (a) $V = \iint_D (1 + xy)\,dA$, where D is the portion of the circle $x^2 + y^2 = 4$ in the first quadrant. Thus,

$V = \iint_D (1 + xy)\,dA = \int_0^{\pi/2} \int_0^2 (1 + r^2 \cos\theta \sin\theta) r\,dr\,d\theta$.

(b) $$\int_0^{\pi/2} \int_0^2 (1 + r^2 \cos\theta \sin\theta) r\,dr\,d\theta = \int_0^{\pi/2} \int_0^2 \left(r + r^3 \cos\theta \sin\theta\right)\,dr\,d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2} + \frac{r^4}{4} \cos\theta \sin\theta\right]_{r=0}^{r=2} d\theta$$
$$= \int_0^{\pi/2} (2 + 4\cos\theta \sin\theta)\,d\theta = \left[2\theta + 2\sin^2\theta\right]_0^{\pi/2} \qquad [u = \sin\theta,\, du = \cos\theta\,d\theta]$$
$$= \pi + 2$$

24. (a) $V = \iint_D (x^2 + y^2)\,dA$, where D is the region on or between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Thus,

$V = \iint_{1 \le x^2 + y^2 \le 4} (x^2 + y^2)\,dA = \int_0^{2\pi} \int_1^2 r^2 r\,dr\,d\theta$.

(b) $$\int_0^{2\pi} \int_1^2 r^3\,dr\,d\theta = \int_0^{2\pi} d\theta \int_1^2 r^3\,dr = \Big[\theta\Big]_{\theta=0}^{\theta=2\pi} \cdot \frac{1}{4}\Big[r^4\Big]_{r=1}^{r=2} = 2\pi \cdot \frac{1}{4}(16 - 1) = \frac{15\pi}{2}$$

25. (a) $V = \iint_D y\,dA$, where D is the portion of the circle $x^2 + y^2 = 9$ in quadrants I–III. Thus,

$V = \iint_D y\,dA = \int_0^{3\pi/2} \int_0^3 (r\sin\theta)\,r\,dr\,d\theta$.

(b) $$\int_0^{3\pi/2} \int_0^3 r^2 \sin\theta\,dr\,d\theta = \int_0^{3\pi/2} \sin\theta\,d\theta \int_0^3 r^2\,dr = \Big[-\cos\theta\Big]_{\theta=0}^{\theta=3\pi/2} \left[\frac{r^3}{3}\right]_{r=0}^{r=3} = [0 - (-1)](9 - 0) = 9$$

26. (a) $V = \iint_D xy^2\,dA$, where D is the region on or between the circles $r = 2$ and $r = 3$ in the first quadrant. Thus,

$V = \iint_D xy^2\,dA = \int_0^{\pi/2} \int_2^3 (r\cos\theta)(r\sin\theta)^2\,r\,dr\,d\theta$.

(b) $$\int_0^{\pi/2} \int_2^3 r^4 \cos\theta \sin^2\theta\,dr\,d\theta = \int_0^{\pi/2} \cos\theta \sin^2\theta\,d\theta \int_2^3 r^4\,dr = \left[\frac{\sin^3\theta}{3}\right]_{\theta=0}^{\theta=\pi/2} \left[\frac{r^5}{5}\right]_{r=2}^{r=3}$$
$$= \left(\tfrac{1}{3} - 0\right)\left(\tfrac{243}{5} - \tfrac{32}{5}\right) = \tfrac{211}{15}$$

27. (a) The region is described by $D = \{(r, \theta) \mid 0 \le r \le \sin\theta,\, 0 \le \theta \le \pi/2\}$, so $\iint_D x\,dA = \int_0^{\pi/2} \int_0^{\sin\theta} (r\cos\theta)\,r\,dr\,d\theta$.

(b) $$\int_0^{\pi/2} \int_0^{\sin\theta} r^2 \cos\theta\,dr\,d\theta = \int_0^{\pi/2} \cos\theta \left[\frac{r^3}{3}\right]_{r=0}^{r=\sin\theta} d\theta = \frac{1}{3}\int_0^{\pi/2} \cos\theta \sin^3\theta\,d\theta$$
$$= \tfrac{1}{3}\int_0^1 u^3\,du \qquad [u = \sin\theta,\, du = \cos\theta\,d\theta]$$
$$= \tfrac{1}{3}\left[\tfrac{1}{4}u^4\right]_0^1 = \tfrac{1}{12}$$

28. (a) The region is described by $D = \{(r, \theta) \mid 0 \le r \le 1 + \cos\theta,\, 0 \le \theta \le \pi\}$, so $\iint_D dA = \int_0^{\pi} \int_0^{1+\cos\theta} 1 \cdot r\,dr\,d\theta$.

(b) $$\int_0^{\pi} \int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=1+\cos\theta} d\theta = \frac{1}{2}\int_0^{\pi} (1 + \cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right)\,d\theta$$
$$= \tfrac{1}{2}\int_0^{\pi} \left[1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right]\,d\theta = \tfrac{1}{2}\int_0^{\pi} \left[\tfrac{3}{2} + 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right]\,d\theta$$
$$= \tfrac{1}{2}\left[\tfrac{3\theta}{2} + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi} = \tfrac{3\pi}{4}$$

29. $V = \iint_{x^2 + y^2 \le 25} \left(x^2 + y^2\right) dA = \int_0^{2\pi} \int_0^5 r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^5 r^3\,dr = \Big[\theta\Big]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^5 = 2\pi\left(\tfrac{625}{4}\right) = \tfrac{625}{2}\pi$

30. $V = \iint_{1 \le x^2+y^2 \le 4} \sqrt{x^2+y^2}\, dA = \int_0^{2\pi}\int_1^2 \sqrt{r^2}\; r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_1^2 r^2\,dr = \big[\,\theta\,\big]_0^{2\pi}\left[\frac{1}{3}r^3\right]_1^2 = 2\pi\left(\frac{8}{3}-\frac{1}{3}\right) = \frac{14}{3}\pi$

31. $2x+y+z=4 \quad\Leftrightarrow\quad z=4-2x-y$, so the volume of the solid is

$$\begin{aligned} V &= \iint_{x^2+y^2\le 1}(4-2x-y)\,dA = \int_0^{2\pi}\int_0^1 (4-2r\cos\theta - r\sin\theta)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \left[4r - r^2\,(2\cos\theta+\sin\theta)\right]dr\,d\theta = \int_0^{2\pi}\left[2r^2 - \frac{1}{3}r^3\,(2\cos\theta+\sin\theta)\right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi}\left[2-\frac{1}{3}\,(2\cos\theta+\sin\theta)\right]d\theta = \left[2\theta - \frac{1}{3}\,(2\sin\theta-\cos\theta)\right]_0^{2\pi} = 4\pi + \frac{1}{3} - 0 - \frac{1}{3} = 4\pi \end{aligned}$$

32. The sphere $x^2+y^2+z^2=16$ intersects the xy-plane in the circle $x^2+y^2=16$, so

$$\begin{aligned} V &= 2\iint\limits_{4\le x^2+y^2\le 16}\sqrt{16-x^2-y^2}\,dA \quad \text{[by symmetry]} \quad = 2\int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\;r\,dr\,d\theta \\ &= 2\int_0^{2\pi} d\theta \int_2^4 r(16-r^2)^{1/2}dr = 2\big[\,\theta\,\big]_0^{2\pi}\left[-\frac{1}{3}(16-r^2)^{3/2}\right]_2^4 \\ &= -\frac{2}{3}(2\pi)(0-12^{3/2}) = \frac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi \end{aligned}$$

33. By symmetry,

$$\begin{aligned} V &= 2\iint\limits_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\;r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_0^a r\sqrt{a^2-r^2}\,dr \\ &= 2\big[\,\theta\,\big]_0^{2\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0+\frac{1}{3}a^3\right) = \frac{4}{3}\pi a^3 \end{aligned}$$

34. The paraboloid $z=1+2x^2+2y^2$ intersects the plane $z=7$ when $7=1+2x^2+2y^2$ or $x^2+y^2=3$ and we are restricted to the first octant, so

$$\begin{aligned} V &= \iint\limits_{\substack{x^2+y^2\le 3,\\ x\ge 0,\, y\ge 0}}\left[7-\left(1+2x^2+2y^2\right)\right]dA = \int_0^{\pi/2}\int_0^{\sqrt{3}}\left[7-(1+2r^2)\right]r\,dr\,d\theta \\ &= \int_0^{\pi/2} d\theta\int_0^{\sqrt{3}}\left(6r-2r^3\right)dr = \big[\,\theta\,\big]_0^{\pi/2}\left[3r^2-\frac{1}{2}r^4\right]_0^{\sqrt{3}} = \frac{\pi}{2}\cdot\frac{9}{2} = \frac{9}{4}\pi \end{aligned}$$

35. The cone $z=\sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2=1$ when $x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=1$ or $x^2+y^2=\frac{1}{2}$. So

$$\begin{aligned} V &= \iint\limits_{x^2+y^2\le 1/2}\left(\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\right)dA = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\left(\sqrt{1-r^2}-r\right)r\,dr\,d\theta \\ &= \int_0^{2\pi}d\theta\int_0^{1/\sqrt{2}}\left(r\sqrt{1-r^2}-r^2\right)dr = \big[\,\theta\,\big]_0^{2\pi}\left[-\frac{1}{3}(1-r^2)^{3/2}-\frac{1}{3}r^3\right]_0^{1/\sqrt{2}} = 2\pi\left(-\frac{1}{3}\right)\left(\frac{1}{\sqrt{2}}-1\right) = \frac{\pi}{3}\left(2-\sqrt{2}\right) \end{aligned}$$

36. The two paraboloids intersect when $6-x^2-y^2=2x^2+2y^2$ or $x^2+y^2=2$. For $x^2+y^2\le 2$, the paraboloid $z=6-x^2-y^2$ is above $z=2x^2+2y^2$ so

$$\begin{aligned} V &= \iint\limits_{x^2+y^2\le 2}\left[(6-x^2-y^2)-(2x^2+2y^2)\right]dA = \iint\limits_{x^2+y^2\le 2}\left[6-3(x^2+y^2)\right]dA = \int_0^{2\pi}\int_0^{\sqrt{2}}(6-3r^2)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}d\theta\int_0^{\sqrt{2}}(6r-3r^3)\,dr = \big[\,\theta\,\big]_0^{2\pi}\left[3r^2-\frac{3}{4}r^4\right]_0^{\sqrt{2}} = 2\pi\,(6-3) = 6\pi \end{aligned}$$

37. The given solid is the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = \sqrt{64 - 4x^2 - 4y^2}$ and $z = -\sqrt{64 - 4x^2 - 4y^2}$. So

$$V = \iint\limits_{x^2+y^2 \le 4} \left[\sqrt{64 - 4x^2 - 4y^2} - \left(-\sqrt{64 - 4x^2 - 4y^2}\right)\right] dA = \iint\limits_{x^2+y^2 \le 4} 2 \cdot 2\sqrt{16 - x^2 - y^2}\, dA$$

$$= 4\int_0^{2\pi}\int_0^2 \sqrt{16 - r^2}\, r\, dr\, d\theta = 4\int_0^{2\pi} d\theta \int_0^2 r\sqrt{16 - r^2}\, dr = 4\left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac{1}{3}(16 - r^2)^{3/2}\right]_0^2$$

$$= 8\pi\left(-\tfrac{1}{3}\right)(12^{3/2} - 16^{2/3}) = \tfrac{8\pi}{3}\left(64 - 24\sqrt{3}\right)$$

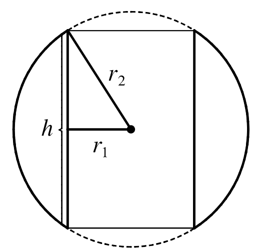
38. (a) Here the region in the xy-plane is the annular region $r_1^2 \le x^2 + y^2 \le r_2^2$ and the desired volume is twice that above the xy-plane. Hence

$$V = 2\iint\limits_{r_1^2 \le x^2+y^2 \le r_2^2} \sqrt{r_2^2 - x^2 - y^2}\, dA = 2\int_0^{2\pi}\int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\, r\, dr\, d\theta = 2\int_0^{2\pi} d\theta \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\, r\, dr$$

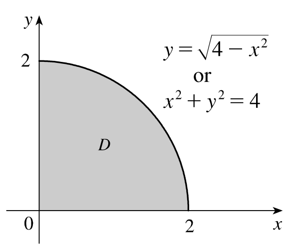
$$= 2\,(2\pi)\left[-\tfrac{1}{3}(r_2^2 - r^2)^{3/2}\right]_{r_1}^{r_2} = \tfrac{4\pi}{3}(r_2^2 - r_1^2)^{3/2}$$

(b) A cross-sectional cut is shown in the figure. So $r_2^2 = \left(\frac{1}{2}h\right)^2 + r_1^2$ or $\frac{1}{4}h^2 = r_2^2 - r_1^2$.

Thus the volume in terms of h is $V = \frac{4\pi}{3}\left(\frac{1}{4}h^2\right)^{3/2} = \frac{\pi}{6}h^3$.



39.

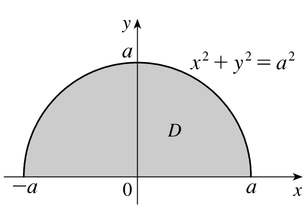


$$\int_0^2\int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx = \int_0^{\pi/2}\int_0^2 e^{-r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^2 re^{-r^2}\, dr = \left[\,\theta\,\right]_0^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2$$

$$= \tfrac{\pi}{2}\left[-\tfrac{1}{2}\left(e^{-4} - 1\right)\right] = \tfrac{\pi}{4}(1 - e^{-4})$$

40.



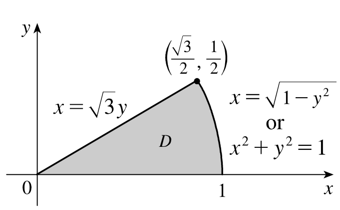
$$\int_0^a\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x + y)\, dx\, dy = \int_0^{\pi}\int_0^a (2r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi} (2\cos\theta + \sin\theta)\, d\theta \int_0^a r^2\, dr$$

$$= \left[2\sin\theta - \cos\theta\right]_0^{\pi}\left[\tfrac{1}{3}r^3\right]_0^a$$

$$= \left[(0 + 1) - (0 - 1)\right] \cdot \tfrac{1}{3}(a^3 - 0) = \tfrac{2}{3}a^3$$

41. The region D of integration is shown in the figure. In polar coordinates the line $x = \sqrt{3}\,y$ is $\theta = \pi/6$, so



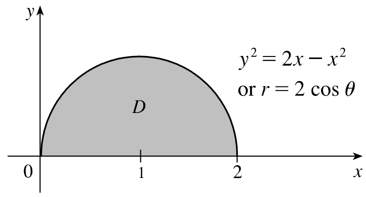
$$\int_0^{1/2}\int_{\sqrt{3}\,y}^{\sqrt{1-y^2}} xy^2\, dx\, dy = \int_0^{\pi/6}\int_0^1 (r\cos\theta)(r\sin\theta)^2\, r\, dr\, d\theta$$

$$= \int_0^{\pi/6} \sin^2\theta\cos\theta\, d\theta \int_0^1 r^4\, dr$$

$$= \left[\tfrac{1}{3}\sin^3\theta\right]_0^{\pi/6}\left[\tfrac{1}{5}r^5\right]_0^1$$

$$= \left[\tfrac{1}{3}\left(\tfrac{1}{2}\right)^3 - 0\right]\left[\tfrac{1}{5} - 0\right] = \tfrac{1}{120}$$

42.



$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r \cdot r\,dr\,d\theta$$
$$= \int_0^{\pi/2} \left[\tfrac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta} d\theta$$
$$= \int_0^{\pi/2} \left(\tfrac{8}{3}\cos^3\theta\right) d\theta$$
$$= \tfrac{8}{3}\int_0^{\pi/2} (1-\sin^2\theta)\cos\theta\,d\theta$$
$$= \tfrac{8}{3}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \tfrac{16}{9}$$

43. $D = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$, so
$\iint_D e^{(x^2+y^2)^2}\,dA = \int_0^{2\pi}\int_0^1 e^{(r^2)^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 re^{r^4}\,dr = 2\pi\int_0^1 re^{r^4}\,dr$. Using a calculator, we estimate $2\pi\int_0^1 re^{r^4}\,dr \approx 4.5951$.

44. $D = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le \pi/2\}$, so

$$\iint_D xy\sqrt{1+x^2+y^2}\,dA = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)\sqrt{1+r^2}\;r\,dr\,d\theta$$
$$= \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \int_0^1 r^3\sqrt{1+r^2}\,dr = \left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} \int_0^1 r^3\sqrt{1+r^2}\,dr$$
$$= \tfrac{1}{2}\int_0^1 r^3\sqrt{1+r^2}\,dr \approx 0.1609$$

45. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y), then the volume of water in the pool is the volume of the solid that lies above $D = \left\{(x,y) \mid x^2+y^2 \le 400\right\}$ and below the graph of $f(x,y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0,-20) = 2$ to $f(0,20) = 7$. The trace in the yz-plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y-20) \;\Rightarrow\; z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x,y)$ is independent of x, $f(x,y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r,\theta) \mid 0 \le r \le 20, 0 \le \theta \le 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right) r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta = \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right) d\theta$$
$$= \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi} = 1800\pi$$

Thus the pool contains $1800\pi \approx 5655$ ft^3 of water.

46. (a) If $R \le 100$, the total amount of water supplied each hour to the region within R feet of the sprinkler is

$$V = \int_0^{2\pi}\int_0^R e^{-r}r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^R re^{-r}\,dr = \left[\,\theta\,\right]_0^{2\pi}\left[-re^{-r} - e^{-r}\right]_0^R$$
$$= 2\pi[-Re^{-R} - e^{-R} + 0 + 1] = 2\pi(1 - Re^{-R} - e^{-R})\text{ ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is

$\dfrac{V}{\text{area of region}} = \dfrac{V}{\pi R^2} = \dfrac{2\left(1 - Re^{-R} - e^{-R}\right)}{R^2}$ ft^3 (per hour per square foot). See the definition of the average value of a function following Example 15.1.8.

47. As in Exercise 15.2.71, $f_{\text{avg}} = \frac{1}{A(D)} \iint_D f(x,y)\,dA$. Here $D = \{(r,\theta) \mid a \le r \le b, 0 \le \theta \le 2\pi\}$,

so $A(D) = \pi b^2 - \pi a^2 = \pi(b^2 - a^2)$ and

$$f_{\text{avg}} = \frac{1}{A(D)} \iint_D \frac{1}{\sqrt{x^2+y^2}}\,dA = \frac{1}{\pi(b^2-a^2)} \int_0^{2\pi} \int_a^b \frac{1}{\sqrt{r^2}}\,r\,dr\,d\theta = \frac{1}{\pi(b^2-a^2)} \int_0^{2\pi} d\theta \int_a^b dr$$

$$= \frac{1}{\pi(b^2-a^2)} \big[\theta\big]_0^{2\pi} \big[r\big]_a^b = \frac{1}{\pi(b^2-a^2)} (2\pi)(b-a) = \frac{2(b-a)}{(b+a)(b-a)} = \frac{2}{a+b}$$

48. The distance from a point (x,y) to the origin is $f(x,y) = \sqrt{x^2+y^2}$, so the average distance from points in D to the origin is

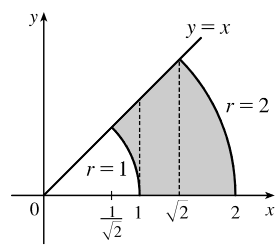
$$f_{\text{avg}} = \frac{1}{A(D)} \iint_D \sqrt{x^2+y^2}\,dA = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \sqrt{r^2}\,r\,dr\,d\theta$$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} d\theta \int_0^a r^2\,dr = \frac{1}{\pi a^2} \big[\theta\big]_0^{2\pi} \left[\tfrac{1}{3}r^3\right]_0^a = \frac{1}{\pi a^2} \cdot 2\pi \cdot \tfrac{1}{3}a^3 = \tfrac{2}{3}a$$

49. $$\int_{1/\sqrt{2}}^1 \int_{\sqrt{1-x^2}}^x xy\,dy\,dx + \int_1^{\sqrt{2}} \int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$

$$= \int_0^{\pi/4} \int_1^2 r^3 \cos\theta \sin\theta\,dr\,d\theta = \int_0^{\pi/4} \left[\frac{r^4}{4} \cos\theta \sin\theta\right]_{r=1}^{r=2} d\theta$$

$$= \frac{15}{4} \int_0^{\pi/4} \sin\theta \cos\theta\,d\theta = \frac{15}{4} \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16}$$



50. (a) $\iint_{D_a} e^{-(x^2+y^2)} dA = \int_0^{2\pi} \int_0^a r e^{-r^2}\,dr\,d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi\left(1 - e^{-a^2}\right)$ for each a. Then $\lim\limits_{a\to\infty} \pi\left(1 - e^{-a^2}\right) = \pi$

since $e^{-a^2} \to 0$ as $a \to \infty$. Hence $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$.

(b) $\iint_{S_a} e^{-(x^2+y^2)}\,dA = \int_{-a}^a \int_{-a}^a e^{-x^2} e^{-y^2}\,dx\,dy = \left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right)$ for each a.

From part (a), $\pi = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA$, so then

$\pi = \lim\limits_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\,dA = \lim\limits_{a\to\infty} \left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$, which is what we wish to show.

To evaluate $\lim\limits_{a\to\infty} \left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right)$, we are using the fact that these integrals are bounded. This is true since on $[-1,1]$, $0 < e^{-x^2} \le 1$ while on $(-\infty,-1)$, $0 < e^{-x^2} \le e^x$ and on $(1,\infty)$, $0 < e^{-x^2} < e^{-x}$. Hence $0 \le \int_{-\infty}^{\infty} e^{-x^2}\,dx \le \int_{-\infty}^{-1} e^x\,dx + \int_{-1}^1 dx + \int_1^{\infty} e^{-x}\,dx = 2(e^{-1}+1)$.

(c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \pi$ and y can be replaced by x, $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$ implies that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \pm\sqrt{\pi}$. But $e^{-x^2} \ge 0$ for all x, so $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

(d) Letting $t = \sqrt{2}\,x$, $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\left(e^{-t^2/2}\right) dt$, so that $\sqrt{\pi} = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2/2}\,dt$ or $\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$.

51. (a) We integrate by parts with $u = x$ and $dv = xe^{-x^2}\,dx$. Then $du = dx$ and $v = -\frac{1}{2}e^{-x^2}$, so

$$\int_0^\infty x^2 e^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t x^2 e^{-x^2}\,dx = \lim_{t\to\infty}\left(\left[-\tfrac{1}{2}xe^{-x^2}\right]_0^t + \int_0^t \tfrac{1}{2}e^{-x^2}\,dx\right)$$

$$= \lim_{t\to\infty}\left(-\tfrac{1}{2}te^{-t^2}\right) + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx = 0 + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx \qquad \text{[by l'Hospital's Rule]}$$

$$= \tfrac{1}{4}\int_{-\infty}^\infty e^{-x^2}\,dx \qquad [\text{since } e^{-x^2} \text{ is an even function}]$$

$$= \tfrac{1}{4}\sqrt{\pi} \qquad \text{[by Exercise 50(c)]}$$

(b) Let $u = \sqrt{x}$. Then $u^2 = x \quad\Rightarrow\quad dx = 2u\,du \quad\Rightarrow$

$$\int_0^\infty \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^t \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du = 2\int_0^\infty u^2e^{-u^2}\,du = 2\left(\tfrac{1}{4}\sqrt{\pi}\right) \quad \text{[by part(a)]} \;=\tfrac{1}{2}\sqrt{\pi}.$$

15.4 Applications of Double Integrals

1. $Q = \iint_D \sigma(x,y)\,dA = \int_0^5\int_2^5 (2x+4y)\,dy\,dx = \int_0^5 \left[2xy + 2y^2\right]_{y=2}^{y=5}\,dx$

$= \int_0^5 (10x + 50 - 4x - 8)\,dx = \int_0^5 (6x+42)\,dx = \left[3x^2 + 42x\right]_0^5 = 75 + 210 = 285$ C

2. $Q = \iint_D \sigma(x,y)\,dA = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2}\;r\,dr\,d\theta$

$= \int_0^{2\pi} d\theta\;\int_0^1 r^2\,dr = [\theta]_0^{2\pi}\;\left[\frac{1}{3}r^3\right]_0^1 = 2\pi\cdot\frac{1}{3} = \frac{2\pi}{3}$ C

3. Since the density of the lamina is higher as $x \to 1$, we might estimate $\overline{x} = 0.7$. There is no vertical change in the density, so we would be confident that $\overline{y} = 0.5$.

$m = \iint_D \rho(x,y)\,dA = \int_0^1\int_0^1 x^2\,dy\,dx = \int_0^1 dy\;\int_0^1 x^2\,dx = \frac{1}{3}$

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^1\int_0^1 x^3\,dy\,dx = \int_0^1 dy\;\int_0^1 x^3\,dx = \frac{1}{4}$

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^1\int_0^1 x^2y\,dy\,dx = \int_0^1 y\,dy\;\int_0^1 x^2\,dx = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$

Hence, $(\overline{x},\overline{y}) = (M_y/m, M_x/m) = \left(\frac{1/4}{1/3}, \frac{1/6}{1/3}\right) = \left(\frac{3}{4}, \frac{1}{2}\right)$.

4. Since the density of the lamina increases in a uniform fashion as $(x,y)\to(1,1)$, we might estimate $(x,y) = (0.7, 0.7)$.

$m = \iint_D \rho(x,y)\,dA = \int_0^1\int_0^1 xy\,dy\,dx = \int_0^1 y\,dy\;\int_0^1 x\,dx = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^1\int_0^1 x^2y\,dy\,dx = \int_0^1 y\,dy\;\int_0^1 x^2\,dx = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^1\int_0^1 xy^2\,dy\,dx = \int_0^1 y^2\,dy\;\int_0^1 x\,dx = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$

Hence, $(\overline{x},\overline{y}) = (M_y/m, M_x/m) = \left(\frac{1/6}{1/4}, \frac{1/6}{1/4}\right) = \left(\frac{2}{3}, \frac{2}{3}\right)$.

5. $m = \iint_D \rho(x,y)\,dA = \int_1^3\int_1^4 ky^2\,dy\,dx = k\int_1^3\,dx\;\int_1^4 y^2\,dy = k\,[x]_1^3\;\left[\frac{1}{3}y^3\right]_1^4 = k(2)(21) = 42k$,

$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{1}{42k}\int_1^3\int_1^4 kxy^2\,dy\,dx = \frac{1}{42}\int_1^3 x\,dx\;\int_1^4 y^2\,dy = \frac{1}{42}\left[\frac{1}{2}x^2\right]_1^3\;\left[\frac{1}{3}y^3\right]_1^4 = \frac{1}{42}(4)(21) = 2$,

$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{1}{42k}\int_1^3\int_1^4 ky^3\,dy\,dx = \frac{1}{42}\int_1^3 dx\;\int_1^4 y^3\,dy = \frac{1}{42}\,[x]_1^3\;\left[\frac{1}{4}y^4\right]_1^4 = \frac{1}{42}(2)\left(\frac{255}{4}\right) = \frac{85}{28}$

Hence, $(\overline{x},\overline{y}) = \left(2, \frac{85}{28}\right)$.

6. $m = \iint_D \rho(x,y)\,dA = \int_0^a \int_0^b (1+x^2+y^2)\,dy\,dx = \int_0^a \left[y + x^2y + \frac{1}{3}y^3\right]_{y=0}^{y=b} dx = \int_0^a \left(b + bx^2 + \frac{1}{3}b^3\right) dx$

$= \left[bx + \frac{1}{3}bx^3 + \frac{1}{3}b^3x\right]_0^a = ab + \frac{1}{3}a^3b + \frac{1}{3}ab^3 = \frac{1}{3}ab(3+a^2+b^2),$

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^a \int_0^b (x + x^3 + xy^2)\,dy\,dx = \int_0^a \left[xy + x^3y + \frac{1}{3}xy^3\right]_{y=0}^{y=b} dx$

$= \int_0^a \left(bx + bx^3 + \frac{1}{3}b^3x\right) dx = \left[\frac{1}{2}bx^2 + \frac{1}{4}bx^4 + \frac{1}{6}b^3x^2\right]_0^a = \frac{1}{2}a^2b + \frac{1}{4}a^4b + \frac{1}{6}a^2b^3$

$= \frac{1}{12}a^2b(6+3a^2+2b^2)$, and

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^a \int_0^b (y + x^2y + y^3)\,dy\,dx = \int_0^a \left[\frac{1}{2}y^2 + \frac{1}{2}x^2y^2 + \frac{1}{4}y^4\right]_{y=0}^{y=b} dx$

$= \int_0^a \left(\frac{1}{2}b^2 + \frac{1}{2}b^2x^2 + \frac{1}{4}b^4\right) dx = \left[\frac{1}{2}b^2x + \frac{1}{6}b^2x^3 + \frac{1}{4}b^4x\right]_0^a = \frac{1}{2}ab^2 + \frac{1}{6}a^3b^2 + \frac{1}{4}ab^4$

$= \frac{1}{12}ab^2(6+2a^2+3b^2).$

Hence, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\dfrac{\frac{1}{12}a^2b(6+3a^2+2b^2)}{\frac{1}{3}ab(3+a^2+b^2)}, \dfrac{\frac{1}{12}ab^2(6+2a^2+3b^2)}{\frac{1}{3}ab(3+a^2+b^2)}\right)$

$= \left(\dfrac{a(6+3a^2+2b^2)}{4(3+a^2+b^2)}, \dfrac{b(6+2a^2+3b^2)}{4(3+a^2+b^2)}\right).$

7. $m = \int_0^2 \int_{x/2}^{3-x} (x+y)\,dy\,dx = \int_0^2 \left[xy + \frac{1}{2}y^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left[x(3-x) + \frac{1}{2}(3-x)^2 - \frac{1}{2}x^2 - \frac{1}{8}x^2\right] dx$

$= \int_0^2 \left(-\frac{9}{8}x^2 + \frac{9}{2}\right) dx = \left[-\frac{9}{8}\left(\frac{1}{3}x^3\right) + \frac{9}{2}x\right]_0^2 = 6,$

$M_y = \int_0^2 \int_{x/2}^{3-x} (x^2+xy)\,dy\,dx = \int_0^2 \left[x^2y + \frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(\frac{9}{2}x - \frac{9}{8}x^3\right) dx = \frac{9}{2},$

$M_x = \int_0^2 \int_{x/2}^{3-y} (xy+y^2)\,dy\,dx = \int_0^2 \left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(9 - \frac{9}{2}x\right) dx = 9.$

Hence, $m = 6$, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\dfrac{3}{4}, \dfrac{3}{2}\right).$

8. Here $D = \left\{(x,y) \mid 0 \le y \le \frac{2}{5},\ \ y/2 \le x \le 1-2y\right\}.$

$m = \int_0^{2/5} \int_{y/2}^{1-2y} x\,dx\,dy = \int_0^{2/5} \left[\frac{1}{2}x^2\right]_{x=y/2}^{x=1-2y} dy = \frac{1}{2}\int_0^{2/5} \left[(1-2y)^2 - \left(\frac{1}{2}y\right)^2\right] dy$

$= \frac{1}{2}\int_0^{2/5} \left(\frac{15}{4}y^2 - 4y + 1\right) dy = \frac{1}{2}\left[\frac{5}{4}y^3 - 2y^2 + y\right]_0^{2/5} = \frac{1}{2}\left[\frac{2}{25} - \frac{8}{25} + \frac{2}{5}\right] = \frac{2}{25},$

$M_y = \int_0^{2/5} \int_{y/2}^{1-2y} x \cdot x\,dx\,dy = \int_0^{2/5} \left[\frac{1}{3}x^3\right]_{x=y/2}^{x=1-2y} dy = \frac{1}{3}\int_0^{2/5} \left[(1-2y)^3 - \left(\frac{1}{2}y\right)^3\right] dy$

$= \frac{1}{3}\int_0^{2/5} \left(-\frac{65}{8}y^3 + 12y^2 - 6y + 1\right) dy = \frac{1}{3}\left[-\frac{65}{32}y^4 + 4y^3 - 3y^2 + y\right]_0^{2/5} = \frac{1}{3}\left[-\frac{13}{250} + \frac{32}{125} - \frac{12}{25} + \frac{2}{5}\right] = \frac{31}{750},$

$M_x = \int_0^{2/5} \int_{y/2}^{1-2y} y \cdot x\,dx\,dy = \int_0^{2/5} y\left[\frac{1}{2}x^2\right]_{x=y/2}^{x=1-2y} dy = \frac{1}{2}\int_0^{2/5} y\left(\frac{15}{4}y^2 - 4y + 1\right) dy$

$= \frac{1}{2}\int_0^{2/5} \left(\frac{15}{4}y^3 - 4y^2 + y\right) dy = \frac{1}{2}\left[\frac{15}{16}y^4 - \frac{4}{3}y^3 + \frac{1}{2}y^2\right]_0^{2/5} = \frac{1}{2}\left[\frac{3}{125} - \frac{32}{375} + \frac{2}{25}\right] = \frac{7}{750}.$

Hence, $m = \frac{2}{25}$, $(\overline{x},\overline{y}) = \left(\frac{31/750}{2/25}, \frac{7/750}{2/25}\right) = \left(\frac{31}{60}, \frac{7}{60}\right).$

9. $m = \int_{-1}^1 \int_0^{1-x^2} ky\,dy\,dx = k\int_{-1}^1 \left[\frac{1}{2}y^2\right]_{y=0}^{y=1-x^2} dx = \frac{1}{2}k\int_{-1}^1 (1-x^2)^2\,dx = \frac{1}{2}k\int_{-1}^1 (1-2x^2+x^4)\,dx$

$= \frac{1}{2}k\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^1 = \frac{1}{2}k\left(1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{8}{15}k,$

[continued]

$$M_y = \int_{-1}^{1}\int_0^{1-x^2} kxy\,dy\,dx = k\int_{-1}^{1}\left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=1-x^2} dx = \tfrac{1}{2}k\int_{-1}^{1} x\,(1-x^2)^2\,dx = \tfrac{1}{2}k\int_{-1}^{1}(x-2x^3+x^5)\,dx$$
$$= \tfrac{1}{2}k\left[\tfrac{1}{2}x^2 - \tfrac{1}{2}x^4 + \tfrac{1}{6}x^6\right]_{-1}^{1} = \tfrac{1}{2}k\left(\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{6}-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{6}\right) = 0,$$

$$M_x = \int_{-1}^{1}\int_0^{1-x^2} ky^2\,dy\,dx = k\int_{-1}^{1}\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=1-x^2} dx = \tfrac{1}{3}k\int_{-1}^{1}(1-x^2)^3\,dx = \tfrac{1}{3}k\int_{-1}^{1}(1-3x^2+3x^4-x^6)\,dx$$
$$= \tfrac{1}{3}k\left[x - x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7\right]_{-1}^{1} = \tfrac{1}{3}k\left(1-1+\tfrac{3}{5}-\tfrac{1}{7}+1-1+\tfrac{3}{5}-\tfrac{1}{7}\right) = \tfrac{32}{105}k.$$

Hence, $m = \frac{8}{15}k$, $(\overline{x},\overline{y}) = \left(0, \frac{32k/105}{8k/15}\right) = \left(0, \frac{4}{7}\right)$.

10. The boundary curves intersect when $x+2 = x^2 \iff x^2 - x - 2 = 0 \iff x=-1, x=2$. Thus here $D = \{(x,y) \mid -1 \le x \le 2,\ x^2 \le y \le x+2\}$.

$$m = \int_{-1}^{2}\int_{x^2}^{x+2} kx^2\,dy\,dx = k\int_{-1}^{2} x^2\left[y\right]_{y=x^2}^{y=x+2} dx = k\int_{-1}^{2}(x^3+2x^2-x^4)\,dx$$
$$= k\left[\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 - \tfrac{1}{5}x^5\right]_{-1}^{2} = k\left(\tfrac{44}{15}+\tfrac{13}{60}\right) = \tfrac{63}{20}k,$$

$$M_y = \int_{-1}^{2}\int_{x^2}^{x+2} kx^3\,dy\,dx = k\int_{-1}^{2} x^3\left[y\right]_{y=x^2}^{y=x+2} dx = k\int_{-1}^{2}(x^4+2x^3-x^5)\,dx$$
$$= k\left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 - \tfrac{1}{6}x^6\right]_{-1}^{2} = k\left(\tfrac{56}{15}-\tfrac{2}{15}\right) = \tfrac{18}{5}k,$$

$$M_x = \int_{-1}^{2}\int_{x^2}^{x+2} kx^2y\,dy\,dx = k\int_{-1}^{2} x^2\left[\tfrac{1}{2}y^2\right]_{y=x^2}^{y=x+2} dx = \tfrac{1}{2}k\int_{-1}^{2} x^2\,(x^2+4x+4-x^4)\,dx$$
$$= \tfrac{1}{2}k\int_{-1}^{2}(x^4+4x^3+4x^2-x^6)\,dx = \tfrac{1}{2}k\left[\tfrac{1}{5}x^5 + x^4 + \tfrac{4}{3}x^3 - \tfrac{1}{7}x^7\right]_{-1}^{2} = \tfrac{1}{2}k\left(\tfrac{1552}{105}+\tfrac{41}{105}\right) = \tfrac{531}{70}k.$$

Hence, $m = \frac{63}{20}k$, $(\overline{x},\overline{y}) = \left(\frac{18k/5}{63k/20}, \frac{531k/70}{63k/20}\right) = \left(\frac{8}{7}, \frac{118}{49}\right)$.

11. $$m = \int_0^1\int_0^{e^{-x}} xy\,dy\,dx = \int_0^1 x\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=e^{-x}} dx = \tfrac{1}{2}\int_0^1 x\left(e^{-x}\right)^2 dx = \tfrac{1}{2}\int_0^1 xe^{-2x}\,dx \quad \left[\begin{array}{c}\text{integrate by parts with}\\ u=x,\ dv=e^{-2x}\,dx\end{array}\right]$$
$$= \tfrac{1}{2}\left[-\tfrac{1}{4}(2x+1)e^{-2x}\right]_0^1 = -\tfrac{1}{8}\left(3e^{-2}-1\right) = \tfrac{1}{8} - \tfrac{3}{8}e^{-2},$$

$$M_y = \int_0^1\int_0^{e^{-x}} x^2y\,dy\,dx = \int_0^1 x^2\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=e^{-x}} dx = \tfrac{1}{2}\int_0^1 x^2e^{-2x}\,dx \quad [\text{integrate by parts twice}]$$
$$= \tfrac{1}{2}\left[-\tfrac{1}{4}\left(2x^2+2x+1\right)e^{-2x}\right]_0^1 = -\tfrac{1}{8}\left(5e^{-2}-1\right) = \tfrac{1}{8} - \tfrac{5}{8}e^{-2},$$

$$M_x = \int_0^1\int_0^{e^{-x}} xy^2\,dy\,dx = \int_0^1 x\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=e^{-x}} dx = \tfrac{1}{3}\int_0^1 xe^{-3x}\,dx$$
$$= \tfrac{1}{3}\left[-\tfrac{1}{9}(3x+1)e^{-3x}\right]_0^1 = -\tfrac{1}{27}\left(4e^{-3}-1\right) = \tfrac{1}{27} - \tfrac{4}{27}e^{-3}.$$

Hence, $m = \frac{1}{8}\left(1-3e^{-2}\right)$, $(\overline{x},\overline{y}) = \left(\dfrac{\frac{1}{8}\left(1-5e^{-2}\right)}{\frac{1}{8}(1-3e^{-2})}, \dfrac{\frac{1}{27}\left(1-4e^{-3}\right)}{\frac{1}{8}(1-3e^{-2})}\right) = \left(\dfrac{e^2-5}{e^2-3}, \dfrac{8\left(e^3-4\right)}{27(e^3-3e)}\right)$.

12. Note that $\cos x \ge 0$ for $-\pi/2 \le x \le \pi/2$.

$$m = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} y\,dy\,dx = \int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\cos x} dx = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^2 x\,dx = \tfrac{1}{2}\left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_{-\pi/2}^{\pi/2} = \tfrac{\pi}{4},$$

$$M_y = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} xy\,dy\,dx = \int_{-\pi/2}^{\pi/2} x\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\cos x} dx = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} x\cos^2 x\,dx \quad \left[\begin{array}{c}\text{integrate by parts with}\\ u=x,\ dv=\cos^2 x\,dx\end{array}\right]$$
$$= \tfrac{1}{2}\left[x\left(\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right)\Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2}\left(\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right)dx\right]$$
$$= \tfrac{1}{2}\left(\tfrac{1}{8}\pi^2 - \tfrac{1}{8}\pi^2 - \left[\tfrac{1}{4}x^2 - \tfrac{1}{8}\cos 2x\right]_{-\pi/2}^{\pi/2}\right) = \tfrac{1}{2}\left(0 - \left[\tfrac{1}{16}\pi^2 + \tfrac{1}{8} - \tfrac{1}{16}\pi^2 - \tfrac{1}{8}\right]\right) = 0,$$ [continued]

$$M_x = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y^2\, dy\, dx = \int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{3}y^3\right]_{y=0}^{y=\cos x} dx = \tfrac{1}{3}\int_{-\pi/2}^{\pi/2} \cos^3 x\, dx = \tfrac{1}{3}\int_{-\pi/2}^{\pi/2} (1-\sin^2 x)\cos x\, dx$$

[substitute $u = \sin x \quad \Rightarrow \quad du = \cos x\, dx$]

$$= \tfrac{1}{3}\left[\sin x - \tfrac{1}{3}\sin^3 x\right]_{-\pi/2}^{\pi/2} = \tfrac{1}{3}\left(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right) = \tfrac{4}{9}.$$

Hence, $m = \frac{\pi}{4}$, $(\overline{x}, \overline{y}) = \left(0, \frac{4/9}{\pi/4}\right) = \left(0, \frac{16}{9\pi}\right)$.

13. "The density at any point is proportional to its distance from the x-axis" $\Rightarrow \quad \rho(x,y) = ky$.

$$m = \iint_D ky\, dA = \int_0^{\pi/2}\int_0^1 k(r\sin\theta)\, r\, dr\, d\theta = k\int_0^{\pi/2} \sin\theta\, d\theta \int_0^1 r^2\, dr$$

$$= k\left[-\cos\theta\right]_0^{\pi/2} \left[\tfrac{1}{3}r^3\right]_0^1 = k(1)\left(\tfrac{1}{3}\right) = \tfrac{1}{3}k,$$

$$M_y = \iint_D x\cdot ky\, dA = \int_0^{\pi/2}\int_0^1 k(r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta = k\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^1 r^3\, dr$$

$$= k\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} \left[\tfrac{1}{4}r^4\right]_0^1 = k\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) = \tfrac{1}{8}k,$$

$$M_x = \iint_D y\cdot ky\, dA = \int_0^{\pi/2}\int_0^1 k(r\sin\theta)^2\, r\, dr\, d\theta = k\int_0^{\pi/2} \sin^2\theta\, d\theta \int_0^1 r^3\, dr$$

$$= k\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\tfrac{1}{4}r^4\right]_0^1 = k\left(\tfrac{\pi}{4}\right)\left(\tfrac{1}{4}\right) = \tfrac{\pi}{16}k.$$

Hence, $(\overline{x}, \overline{y}) = \left(\frac{k/8}{k/3}, \frac{k\pi/16}{k/3}\right) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

14. "The density at any point is proportional to the square of its distance from the origin" $\Rightarrow$

$\rho(x,y) = k\left(\sqrt{x^2+y^2}\right)^2 = k(x^2+y^2) = kr^2$.

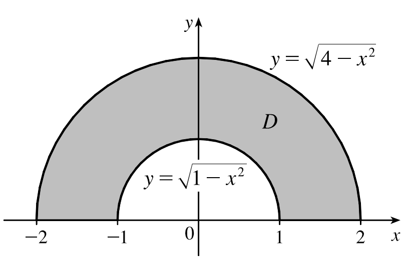
$m = \int_0^{\pi/2}\int_0^1 kr^3\, dr\, d\theta = \frac{\pi}{8}k$,

$M_y = \int_0^{\pi/2}\int_0^1 kr^4\cos\theta\, dr\, d\theta = \frac{1}{5}k\int_0^{\pi/2}\cos\theta\, d\theta = \frac{1}{5}k\left[\sin\theta\right]_0^{\pi/2} = \frac{1}{5}k$,

$M_x = \int_0^{\pi/2}\int_0^1 kr^4\sin\theta\, dr\, d\theta = \frac{1}{5}k\int_0^{\pi/2}\sin\theta\, d\theta = \frac{1}{5}k\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{5}k$.

Hence, $(\overline{x}, \overline{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right)$.

15.



$\rho(x,y) = k\sqrt{x^2+y^2} = kr$,

$$m = \iint_D \rho(x,y)dA = \int_0^{\pi}\int_1^2 kr\cdot r\, dr\, d\theta$$

$$= k\int_0^{\pi} d\theta \int_1^2 r^2\, dr = k(\pi)\left[\tfrac{1}{3}r^3\right]_1^2 = \tfrac{7}{3}\pi k,$$

$$M_y = \iint_D x\rho(x,y)dA = \int_0^{\pi}\int_1^2 (r\cos\theta)(kr)\, r\, dr\, d\theta = k\int_0^{\pi}\cos\theta\, d\theta \int_1^2 r^3\, dr$$

$$= k\left[\sin\theta\right]_0^{\pi} \left[\tfrac{1}{4}r^4\right]_1^2 = k(0)\left(\tfrac{15}{4}\right) = 0$$

[this is to be expected as the region and density function are symmetric about the y-axis]

[continued]

$$M_x = \iint_D y\rho(x,y)\,dA = \int_0^\pi \int_1^2 (r\sin\theta)(kr)\,r\,dr\,d\theta = k\int_0^\pi \sin\theta\,d\theta\,\int_1^2 r^3\,dr$$
$$= k\left[-\cos\theta\right]_0^\pi \left[\tfrac{1}{4}r^4\right]_1^2 = k(1+1)\left(\tfrac{15}{4}\right) = \tfrac{15}{2}k.$$

Hence, $(\overline{x},\overline{y}) = \left(0, \frac{15k/2}{7\pi k/3}\right) = \left(0, \frac{45}{14\pi}\right)$.

16. Now $\rho(x,y) = k\,/\sqrt{x^2+y^2} = k/r$, so

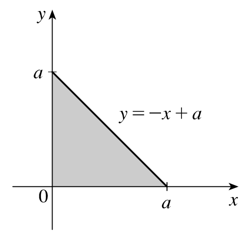
$$m = \iint_D \rho(x,y)dA = \int_0^\pi \int_1^2 (k/r)\,r\,dr\,d\theta = k\int_0^\pi d\theta\,\int_1^2 dr = k(\pi)(1) = \pi k,$$

$$M_y = \iint_D x\rho(x,y)dA = \int_0^\pi \int_1^2 (r\cos\theta)(k/r)\,r\,dr\,d\theta = k\int_0^\pi \cos\theta\,d\theta\,\int_1^2 r\,dr$$
$$= k\left[\sin\theta\right]_0^\pi \left[\tfrac{1}{2}r^2\right]_1^2 = k(0)\left(\tfrac{3}{2}\right) = 0,$$

$$M_x = \iint_D y\rho(x,y)dA = \int_0^\pi \int_1^2 (r\sin\theta)(k/r)\,r\,dr\,d\theta = k\int_0^\pi \sin\theta\,d\theta\,\int_1^2 r\,dr$$
$$= k\left[-\cos\theta\right]_0^\pi \left[\tfrac{1}{2}r^2\right]_1^2 = k(1+1)\left(\tfrac{3}{2}\right) = 3k.$$

Hence, $(\overline{x},\overline{y}) = \left(0, \frac{3k}{\pi k}\right) = \left(0, \frac{3}{\pi}\right)$.

17. Placing the vertex opposite the hypotenuse at $(0,0)$ as in the figure,



$\rho(x,y) = k(x^2+y^2)$. Then

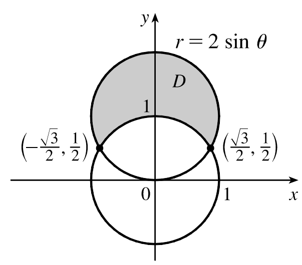
$$m = \int_0^a \int_0^{a-x} k\left(x^2+y^2\right)dy\,dx$$
$$= k\int_0^a \left[ax^2 - x^3 + \tfrac{1}{3}(a-x)^3\right]dx$$
$$= k\left[\tfrac{1}{3}ax^3 - \tfrac{1}{4}x^4 - \tfrac{1}{12}(a-x)^4\right]_0^a = \tfrac{1}{6}ka^4$$

By symmetry, $\quad M_y = M_x = \int_0^a \int_0^{a-x} ky(x^2+y^2)\,dy\,dx = k\int_0^a \left[\tfrac{1}{2}(a-x)^2x^2 + \tfrac{1}{4}(a-x)^4\right]dx$

$$= k\left[\tfrac{1}{6}a^2x^3 - \tfrac{1}{4}ax^4 + \tfrac{1}{10}x^5 - \tfrac{1}{20}(a-x)^5\right]_0^a = \tfrac{1}{15}ka^5$$

Hence, $(\overline{x},\overline{y}) = \left(\frac{2}{5}a, \frac{2}{5}a\right)$.

18. $\rho(x,y) = k/\sqrt{x^2+y^2} = k/r$.



$$m = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} \frac{k}{r}\,r\,dr\,d\theta = k\int_{\pi/6}^{5\pi/6} \left[(2\sin\theta) - 1\right]d\theta$$
$$= k\left[-2\cos\theta - \theta\right]_{\pi/6}^{5\pi/6} = 2k\left(\sqrt{3} - \tfrac{\pi}{3}\right)$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} kr\sin\theta\,dr\,d\theta = \tfrac{1}{2}k\int_{\pi/6}^{5\pi/6} (4\sin^3\theta - \sin\theta)\,d\theta$$
$$= \tfrac{1}{2}k\left[-3\cos\theta + \tfrac{4}{3}\cos^3\theta\right]_{\pi/6}^{5\pi/6} = \sqrt{3}\,k$$

Hence, $(\overline{x},\overline{y}) = \left(0, \frac{3\sqrt{3}}{2\left(3\sqrt{3}-\pi\right)}\right)$.

19. $I_x = \iint_D y^2\rho(x,y)dA = \int_1^3 \int_1^4 y^2 \cdot ky^2\,dy\,dx = k\int_1^3 dx\,\int_1^4 y^4\,dy = k\left[x\right]_1^3 \left[\tfrac{1}{5}y^5\right]_1^4 = k(2)\left(\tfrac{1023}{5}\right) = 409.2k,$

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_1^3 \int_1^4 x^2 \cdot ky^2\,dy\,dx = k\int_1^3 x^2\,dx\,\int_1^4 y^2\,dy = k\left[\tfrac{1}{3}x^3\right]_1^3 \left[\tfrac{1}{3}y^3\right]_1^4 = k\left(\tfrac{26}{3}\right)(21) = 182k,$

and $I_0 = I_x + I_y = 409.2k + 182k = 591.2k$.

20. $I_x = \iint_D y^2\rho(x,y)dA = \int_0^{2/5}\int_{y/2}^{1-2y} y^2 \cdot x\,dx\,dy = \int_0^{2/5} y^2\left[\frac{1}{2}x^2\right]_{x=y/2}^{x=1-2y} dy = \frac{1}{2}\int_0^{2/5} y^2\left(\frac{15}{4}y^2 - 4y + 1\right)dy$

$= \frac{1}{2}\int_0^{2/5}\left(\frac{15}{4}y^4 - 4y^3 + y^2\right)dy = \frac{1}{2}\left[\frac{3}{4}y^5 - y^4 + \frac{1}{3}y^3\right]_0^{2/5} = \frac{16}{9375},$

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^{2/5}\int_{y/2}^{1-2y} x^2 \cdot x\,dx\,dy = \int_0^{2/5}\left[\frac{1}{4}x^4\right]_{x=y/2}^{x=1-2y} dy = \frac{1}{4}\int_0^{2/5}\left[(1-2y)^4 - \frac{1}{16}y^4\right]dy$

$= \frac{1}{4}\int_0^{2/5}\left(\frac{255}{16}y^4 - 32y^3 + 24y^2 - 8y + 1\right)dy = \frac{1}{4}\left[\frac{51}{16}y^5 - 8y^4 + 8y^3 - 4y^2 + y\right]_0^{2/5} = \frac{78}{3125},$

and $I_0 = I_x + I_y = \frac{16}{9375} + \frac{78}{3125} = \frac{2}{75}$.

21. As in Exercise 17, we place the vertex opposite the hypotenuse at $(0,0)$ and the equal sides along the positive axes.

$I_x = \int_0^a\int_0^{a-x} y^2 k(x^2+y^2)\,dy\,dx = k\int_0^a\int_0^{a-x}(x^2y^2 + y^4)\,dy\,dx = k\int_0^a\left[\frac{1}{3}x^2y^3 + \frac{1}{5}y^5\right]_{y=0}^{y=a-x}\,dx$

$= k\int_0^a\left[\frac{1}{3}x^2(a-x)^3 + \frac{1}{5}(a-x)^5\right]dx = k\left[\frac{1}{3}\left(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6\right) - \frac{1}{30}(a-x)^6\right]_0^a = \frac{7}{180}ka^6,$

$I_y = \int_0^a\int_0^{a-x} x^2 k(x^2+y^2)\,dy\,dx = k\int_0^a\int_0^{a-x}(x^4 + x^2y^2)\,dy\,dx = k\int_0^a\left[x^4y + \frac{1}{3}x^2y^3\right]_{y=0}^{y=a-x}\,dx$

$= k\int_0^a\left[x^4(a-x) + \frac{1}{3}x^2(a-x)^3\right]dx = k\left[\frac{1}{5}ax^5 - \frac{1}{6}x^6 + \frac{1}{3}\left(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6\right)\right]_0^a = \frac{7}{180}ka^6,$

and $I_0 = I_x + I_y = \frac{7}{90}ka^6$.

22. If we find the moments of inertia about the x- and y-axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the x-axis is given by

$$I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^2\int_0^2 y^2(1+0.1x)\,dy\,dx = \int_0^2(1+0.1x)\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=2}\,dx$$
$$= \tfrac{8}{3}\int_0^2(1+0.1x)\,dx = \tfrac{8}{3}\left[x + 0.1\cdot\tfrac{1}{2}x^2\right]_0^2 = \tfrac{8}{3}(2.2) \approx 5.87$$

Similarly, the moment of inertia about the y-axis is given by

$$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^2\int_0^2 x^2(1+0.1x)\,dy\,dx = \int_0^2 x^2(1+0.1x)\left[\,y\,\right]_{y=0}^{y=2}\,dx$$
$$= 2\int_0^2(x^2+0.1x^3)\,dx = 2\left[\tfrac{1}{3}x^3 + 0.1\cdot\tfrac{1}{4}x^4\right]_0^2 = 2\left(\tfrac{8}{3}+0.4\right) \approx 6.13$$

Since $I_y > I_x$, more force is required to rotate the fan blade about the y-axis.

23. $I_x = \iint_D y^2\rho(x,y)dA = \int_0^h\int_0^b \rho y^2\,dx\,dy = \rho\int_0^b dx\int_0^h y^2\,dy = \rho\left[x\right]_0^b\left[\frac{1}{3}y^3\right]_0^h = \rho b\left(\frac{1}{3}h^3\right) = \frac{1}{3}\rho bh^3,$

$I_y = \iint_D x^2\rho(x,y)dA = \int_0^h\int_0^b \rho x^2\,dx\,dy = \rho\int_0^b x^2\,dx\int_0^h dy = \rho\left[\frac{1}{3}x^3\right]_0^b\left[y\right]_0^h = \frac{1}{3}\rho b^3h,$

and $m = \rho\,(\text{area of rectangle}) = \rho bh$ since the lamina is homogeneous. Hence $\bar{\bar{x}}^2 = \dfrac{I_y}{m} = \dfrac{\frac{1}{3}\rho b^3h}{\rho bh} = \dfrac{b^2}{3} \quad\Rightarrow\quad \bar{\bar{x}} = \dfrac{b}{\sqrt{3}}$

and $\bar{\bar{y}}^2 = \dfrac{I_x}{m} = \dfrac{\frac{1}{3}\rho bh^3}{\rho bh} = \dfrac{h^2}{3} \quad\Rightarrow\quad \bar{\bar{y}} = \dfrac{h}{\sqrt{3}}$.

24. Here we assume $b>0$, $h>0$ but note that we arrive at the same results if $b<0$ or $h<0$. We have $D = \left\{(x,y)\mid 0\le x\le b, 0\le y\le h - \frac{h}{b}x\right\}$, so

$I_x = \int_0^b\int_0^{h-hx/b} y^2\rho\,dy\,dx = \rho\int_0^b\left[\frac{1}{3}y^3\right]_{y=0}^{y=h-hx/b} dx = \frac{1}{3}\rho\int_0^b\left(h - \frac{h}{b}x\right)^3 dx$

$= \frac{1}{3}\rho\left[-\frac{b}{h}\left(\frac{1}{4}\right)\left(h - \frac{h}{b}x\right)^4\right]_0^b = -\frac{b}{12h}\rho(0 - h^4) = \frac{1}{12}\rho bh^3,$

$I_y = \int_0^b \int_0^{h-hx/b} x^2 \rho\, dy\, dx = \rho \int_0^b x^2 \left(h - \frac{h}{b}x\right) dx = \rho \int_0^b \left(hx^2 - \frac{h}{b}x^3\right) dx$

$= \rho \left[\frac{h}{3}x^3 - \frac{h}{4b}x^4\right]_0^b = \rho\left(\frac{hb^3}{3} - \frac{hb^3}{4}\right) = \frac{1}{12}\rho b^3 h,$

and $m = \int_0^b \int_0^{h-hx/b} \rho\, dy\, dx = \rho \int_0^b \left(h - \frac{h}{b}x\right) dx = \rho\left[hx - \frac{h}{2b}x^2\right]_0^b = \frac{1}{2}\rho bh$. Hence $\overline{\overline{x}}^2 = \dfrac{I_y}{m} = \dfrac{\frac{1}{12}\rho b^3 h}{\frac{1}{2}\rho bh} = \dfrac{b^2}{6} \Rightarrow$

$\overline{\overline{x}} = \dfrac{b}{\sqrt{6}}$ and $\overline{\overline{y}}^2 = \dfrac{I_x}{m} = \dfrac{\frac{1}{12}\rho bh^3}{\frac{1}{2}\rho bh} = \dfrac{h^2}{6} \Rightarrow \overline{\overline{y}} = \dfrac{h}{\sqrt{6}}$.

25. In polar coordinates, the region is $D = \left\{(r,\theta) \mid 0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}\right\}$, so

$I_x = \iint_D y^2 \rho\, dA = \int_0^{\pi/2} \int_0^a \rho(r\sin\theta)^2\, r\, dr\, d\theta = \rho \int_0^{\pi/2} \sin^2 d\theta \int_0^a r^3\, dr$

$= \rho\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\frac{1}{4}r^4\right]_0^a = \rho\left(\frac{\pi}{4}\right)\left(\frac{1}{4}a^4\right) = \frac{1}{16}\rho a^4 \pi,$

$I_y = \iint_D x^2 \rho\, dA = \int_0^{\pi/2} \int_0^a \rho(r\cos\theta)^2\, r\, dr\, d\theta = \rho \int_0^{\pi/2} \cos^2 d\theta \int_0^a r^3\, dr$

$= \rho\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\frac{1}{4}r^4\right]_0^a = \rho\left(\frac{\pi}{4}\right)\left(\frac{1}{4}a^4\right) = \frac{1}{16}\rho a^4 \pi,$

and $m = \rho \cdot A(D) = \rho \cdot \frac{1}{4}\pi a^2$ since the lamina is homogeneous. Hence $\overline{\overline{x}}^2 = \overline{\overline{y}}^2 = \dfrac{\frac{1}{16}\rho a^4 \pi}{\frac{1}{4}\rho a^2 \pi} = \dfrac{a^2}{4} \Rightarrow \overline{\overline{x}} = \overline{\overline{y}} = \dfrac{a}{2}$.

26. $m = \int_0^\pi \int_0^{\sin x} \rho\, dy\, dx = \rho \int_0^\pi \sin x\, dx = \rho\left[-\cos x\right]_0^\pi = 2\rho,$

$I_x = \int_0^\pi \int_0^{\sin x} \rho y^2\, dy\, dx = \frac{1}{3}\rho \int_0^\pi \sin^3 x\, dx = \frac{1}{3}\rho \int_0^\pi (1 - \cos^2 x)\sin x\, dx = \frac{1}{3}\rho\left[-\cos x + \frac{1}{3}\cos^3 x\right]_0^\pi = \frac{4}{9}\rho,$

$I_y = \int_0^\pi \int_0^{\sin x} \rho x^2\, dy\, dx = \rho \int_0^\pi x^2 \sin x\, dx = \rho\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^\pi$ [by integrating by parts twice]

$= \rho(\pi^2 - 4).$

Then $\overline{\overline{y}}^2 = \dfrac{I_x}{m} = \dfrac{2}{9}$, so $\overline{\overline{y}} = \dfrac{\sqrt{2}}{3}$ and $\overline{\overline{x}}^2 = \dfrac{I_y}{m} = \dfrac{\pi^2 - 4}{2}$, so $\overline{\overline{x}} = \sqrt{\dfrac{\pi^2 - 4}{2}}$.

27. The right loop of the curve is given by $D = \{(r,\theta) \mid 0 \le r \le \cos 2\theta,\ -\pi/4 \le \theta \le \pi/4\}$. Using a CAS, we find $m = \iint_D \rho(x,y)\, dA = \iint_D (x^2 + y^2)\, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^2\, r\, dr\, d\theta = \dfrac{3\pi}{64}$. Then

$$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\, dA = \frac{64}{3\pi}\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} (r\cos\theta)\, r^2\, r\, dr\, d\theta = \frac{64}{3\pi}\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r^4 \cos\theta\, dr\, d\theta = \frac{16384\sqrt{2}}{10395\pi} \text{ and}$$

$$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\, dA = \frac{64}{3\pi}\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} (r\sin\theta)\, r^2\, r\, dr\, d\theta = \frac{64}{3\pi}\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r^4 \sin\theta\, dr\, d\theta = 0, \text{ so}$$

$(\overline{x}, \overline{y}) = \left(\dfrac{16384\sqrt{2}}{10395\pi}, 0\right)$.

The moments of inertia are

$I_x = \iint_D y^2 \rho(x,y)\, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (r\sin\theta)^2\, r^2\, r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^5 \sin^2\theta\, dr\, d\theta = \dfrac{5\pi}{384} - \dfrac{4}{105},$

$I_y = \iint_D x^2 \rho(x,y)\, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (r\cos\theta)^2\, r^2\, r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^5 \cos^2\theta\, dr\, d\theta = \dfrac{5\pi}{384} + \dfrac{4}{105}$, and

$I_0 = I_x + I_y = \dfrac{5\pi}{192}$.

28. Using a CAS, we find $m = \iint_D \rho(x,y)\,dA = \int_0^2 \int_0^{xe^{-x}} x^2y^2\,dy\,dx = \frac{8}{729}(5 - 899e^{-6})$. Then

$$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{729}{8(5-899e^{-6})}\int_0^2\int_0^{xe^{-x}} x^3y^2\,dy\,dx = \frac{2(5e^6-1223)}{5e^6-899} \text{ and}$$

$$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{729}{8(5-899e^{-6})}\int_0^2\int_0^{xe^{-x}} x^2y^3\,dy\,dx = \frac{729(45e^6-42037e^{-2})}{32768(5e^6-899)}, \text{ so}$$

$$(\overline{x},\overline{y}) = \left(\frac{2(5e^6-1223)}{5e^6-899}, \frac{729(45e^6-42037e^{-2})}{32768(5e^6-899)}\right).$$

The moments of inertia are $I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^2\int_0^{xe^{-x}} x^2y^4\,dy\,dx = \frac{16}{390625}(63 - 305593e^{-10})$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^2\int_0^{xe^{-x}} x^4y^2\,dy\,dx = \frac{80}{2187}(7 - 2101e^{-6})$, and

$I_0 = I_x + I_y = \frac{16}{854296875}(13809656 - 4103515625e^{-6} - 668331891e^{-10})$.

29. (a) $f(x,y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,1]\times[0,2]$, we can say

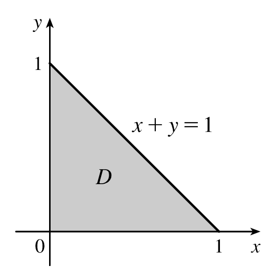
$$\begin{aligned}\iint_{\mathbb{R}^2} f(x,y)\,dA &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^2 Cx(1+y)\,dy\,dx \\ &= C\int_0^1 x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=2}\,dx = C\int_0^1 4x\,dx = C\left[2x^2\right]_0^1 = 2C\end{aligned}$$

Then $2C = 1 \;\Rightarrow\; C = \frac{1}{2}$.

(b) $P(X \le 1, Y \le 1) = \int_{-\infty}^1\int_{-\infty}^1 f(x,y)\,dy\,dx = \int_0^1\int_0^1 \frac{1}{2}x(1+y)\,dy\,dx$

$$= \int_0^1 \tfrac{1}{2}x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=1}\,dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{3}{2}\right)dx = \tfrac{3}{4}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{3}{8} \text{ or } 0.375$$

(c) $P(X + Y \le 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$\begin{aligned}P(X+Y\le 1) &= \iint_D f(x,y)\,dA = \int_0^1\int_0^{1-x} \tfrac{1}{2}x(1+y)\,dy\,dx \\ &= \int_0^1 \tfrac{1}{2}x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=1-x}\,dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{1}{2}x^2 - 2x + \tfrac{3}{2}\right)dx \\ &= \tfrac{1}{4}\int_0^1 \left(x^3 - 4x^2 + 3x\right)dx = \tfrac{1}{4}\left[\tfrac{x^4}{4} - 4\tfrac{x^3}{3} + 3\tfrac{x^2}{2}\right]_0^1 \\ &= \tfrac{5}{48} \approx 0.1042\end{aligned}$$

30. (a) $f(x,y) \ge 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Here, $f(x,y) = 0$ outside the square $[0,1]\times[0,1]$, so $\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_0^1\int_0^1 4xy\,dy\,dx = \int_0^1 \left[2xy^2\right]_{y=0}^{y=1}\,dx = \int_0^1 2x\,dx = \left.x^2\right]_0^1 = 1$.

Thus, $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on Y, so

$$P\left(X \ge \tfrac{1}{2}\right) = \int_{1/2}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_{1/2}^1\int_0^1 4xy\,dy\,dx = \int_{1/2}^1 \left[2xy^2\right]_{y=0}^{y=1}\,dx = \int_{1/2}^1 2x\,dx = \left.x^2\right]_{1/2}^1 = \tfrac{3}{4}.$$

(ii) $P\left(X \ge \frac{1}{2}, Y \le \frac{1}{2}\right) = \int_{1/2}^{\infty}\int_{-\infty}^{1/2} f(x,y)\,dy\,dx = \int_{1/2}^1\int_0^{1/2} 4xy\,dy\,dx$

$$= \int_{1/2}^1 \left[2xy^2\right]_{y=0}^{y=1/2}\,dx = \int_{1/2}^1 \tfrac{1}{2}x\,dx = \tfrac{1}{2}\cdot\left.\tfrac{1}{2}x^2\right]_{1/2}^1 = \tfrac{3}{16}$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x\, f(x,y)\, dA = \int_0^1 \int_0^1 x(4xy)\, dy\, dx = \int_0^1 2x^2 \left[y^2\right]_{y=0}^{y=1} dx = 2\int_0^1 x^2\, dx = 2\left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3}$$

The expected value of Y is

$$\mu_2 = \iint_{\mathbb{R}^2} y\, f(x,y)\, dA = \int_0^1 \int_0^1 y(4xy)\, dy\, dx = \int_0^1 4x\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=1} dx = \tfrac{4}{3}\int_0^1 x\, dx = \tfrac{4}{3}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{2}{3}$$

31. (a) $f(x,y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$. Here, $f(x,y) = 0$ outside the first quadrant, so

$$\begin{aligned}
\iint_{\mathbb{R}^2} f(x,y)\, dA &= \int_0^\infty \int_0^\infty 0.1e^{-(0.5x+0.2y)}\, dy\, dx = 0.1\int_0^\infty \int_0^\infty e^{-0.5x}e^{-0.2y}\, dy\, dx \\
&= 0.1\int_0^\infty e^{-0.5x}\, dx \int_0^\infty e^{-0.2y}\, dy = 0.1 \lim_{t\to\infty}\int_0^t e^{-0.5x}\, dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\, dy \\
&= 0.1 \lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t = 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right] \\
&= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-1) = 1
\end{aligned}$$

Thus $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on X, so

$$\begin{aligned}
P(Y \geq 1) &= \int_{-\infty}^\infty \int_1^\infty f(x,y)\, dy\, dx = \int_0^\infty \int_1^\infty 0.1e^{-(0.5x+0.2y)}\, dy\, dx \\
&= 0.1\int_0^\infty e^{-0.5x}\, dx \int_1^\infty e^{-0.2y}\, dy = 0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}\, dx \lim_{t\to\infty}\int_1^t e^{-0.2y}\, dy \\
&= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_1^t = 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-e^{-0.2})\right] \\
&(0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2}) = e^{-0.2} \approx 0.8187
\end{aligned}$$

(ii)
$$\begin{aligned}
P(X \leq 2, Y \leq 4) &= \int_{-\infty}^2 \int_{-\infty}^4 f(x,y)\, dy\, dx = \int_0^2\int_0^4 0.1e^{-(0.5x+0.2y)}\, dy\, dx \\
&= 0.1\int_0^2 e^{-0.5x}\, dx \int_0^4 e^{-0.2y}\, dy = 0.1\left[-2e^{-0.5x}\right]_0^2 \left[-5e^{-0.2y}\right]_0^4 \\
&= (0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1) \\
&= (e^{-1}-1)(e^{-0.8}-1) = 1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481
\end{aligned}$$

(c) The expected value of X is given by

$$\begin{aligned}
\mu_1 &= \iint_{\mathbb{R}^2} x\, f(x,y)\, dA = \int_0^\infty \int_0^\infty x\left[0.1e^{-(0.5x+0.2y)}\right] dy\, dx \\
&= 0.1\int_0^\infty xe^{-0.5x}\, dx \int_0^\infty e^{-0.2y}\, dy = 0.1\lim_{t\to\infty}\int_0^t xe^{-0.5x}\, dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\, dy
\end{aligned}$$

To evaluate the first integral, we integrate by parts with $u = x$ and $dv = e^{-0.5x}\, dx$ (or we can use Formula 96 in the Table of Integrals): $\int xe^{-0.5x}\, dx = -2xe^{-0.5x} - \int -2e^{-0.5x}\, dx = -2xe^{-0.5x} - 4e^{-0.5x} = -2(x+2)e^{-0.5x}$.

Thus

$$\begin{aligned}
\mu_1 &= 0.1\lim_{t\to\infty}\left[-2(x+2)e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t \\
&= 0.1\lim_{t\to\infty}(-2)\left[(t+2)e^{-0.5t}-2\right]\lim_{t\to\infty}(-5)\left[e^{-0.2t}-1\right] \\
&= 0.1(-2)\left(\lim_{t\to\infty}\frac{t+2}{e^{0.5t}} - 2\right)(-5)(-1) = 2 \qquad \text{[by l'Hospital's Rule]}
\end{aligned}$$

[continued]

The expected value of Y is given by

$$\mu_2 = \iint_{\mathbb{R}^2} y\, f(x,y)\, dA = \int_0^\infty \int_0^\infty y\left[0.1e^{-(0.5\,+0.2y)}\right] dy\, dx$$
$$= 0.1 \int_0^\infty e^{-0.5x}\, dx \int_0^\infty ye^{-0.2y}\, dy = 0.1 \lim_{t\to\infty} \int_0^t e^{-0.5x}\, dx \lim_{t\to\infty} \int_0^t ye^{-0.2y}\, dy$$

To evaluate the second integral, we integrate by parts with $u = y$ and $dv = e^{-0.2y}\, dy$ (or again we can use Formula 96 in the Table of Integrals) which gives $\int ye^{-0.2y}\, dy = -5ye^{-0.2y} + \int 5e^{-0.2y}\, dy = -5(y+5)e^{-0.2y}$. Then

$$\mu_2 = 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5(y+5)e^{-0.2y}\right]_0^t$$
$$= 0.1 \lim_{t\to\infty} \left[-2(e^{-0.5t} - 1)\right] \lim_{t\to\infty} \left(-5\left[(t+5)e^{-0.2t} - 5\right]\right)$$
$$= 0.1(-2)(-1)\cdot(-5)\left(\lim_{t\to\infty} \frac{t+5}{e^{0.2t}} - 5\right) = 5 \qquad \text{[by l'Hospital's Rule]}$$

32. (a) The lifetime of each bulb has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

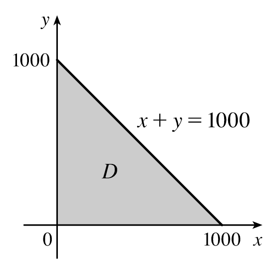
If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x,y) = \begin{cases} 10^{-6}e^{-(x+y)/1000} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$P\left(X \leq 1000, Y \leq 1000\right) = \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x,y)\, dy\, dx = \int_0^{1000} \int_0^{1000} 10^{-6}e^{-(x+y)/1000}\, dy\, dx$$
$$= 10^{-6} \int_0^{1000} e^{-x/1000}\, dx \int_0^{1000} e^{-y/1000}\, dy$$
$$= 10^{-6} \left[-1000e^{-x/1000}\right]_0^{1000} \left[-1000e^{-y/1000}\right]_0^{1000}$$
$$= \left(e^{-1} - 1\right)^2 \approx 0.3996$$

(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X + Y \leq 1000)$, or equivalently $P((X,Y) \in D)$ where D is the triangular region shown in the figure. Then



$$P(X + Y \leq 1000) = \iint_D f\,(x,y)\ dA$$
$$= \int_0^{1000} \int_0^{1000-x} 10^{-6}e^{-(x+y)/1000}\, dy\, dx$$
$$= 10^{-6} \int_0^{1000} \left[-1000e^{-(x+y)/1000}\right]_{y=0}^{y=1000-x} dx = -10^{-3} \int_0^{1000} \left(e^{-1} - e^{-x/1000}\right) dx$$
$$= -10^{-3} \left[e^{-1}x + 1000e^{-x/1000}\right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642$$

33. (a) The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$.

The individual density functions for X and Y, then, are $f_1(x) = \dfrac{1}{0.5\sqrt{2\pi}}\, e^{-(x-45)^2/0.5}$ and

$f_2(y) = \dfrac{1}{0.1\sqrt{2\pi}}\, e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product

$f(x,y) = f_1(x) f_2(y) = \dfrac{1}{0.5\sqrt{2\pi}}\, e^{-(x-45)^2/0.5} \dfrac{1}{0.1\sqrt{2\pi}}\, e^{-(y-20)^2/0.02} = \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2}$.

Then $P(40 \le X \le 50,\, 20 \le Y \le 25) = \int_{40}^{50}\int_{20}^{25} f(x,y)\,dy\,dx = \frac{10}{\pi}\int_{40}^{50}\int_{20}^{25} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx$.

Using a CAS or calculator to evaluate the integral, we get $P(40 \le X \le 50,\, 20 \le Y \le 25) \approx 0.500$.

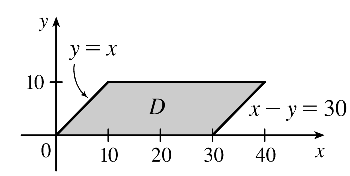
(b) $P(4(X-45)^2 + 100(Y-20)^2 \le 2) = \iint_D \frac{10}{\pi} e^{-2(x-45)^2-50(y-20)^2}\,dA$, where D is the region enclosed by the ellipse $4(x-45)^2 + 100(y-20)^2 = 2$. Solving for y gives $y = 20 \pm \frac{1}{10}\sqrt{2 - 4(x-45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y = 20$ [since the ellipse is centered at $(45, 20)$] $\Rightarrow$ $4(x-45)^2 = 2$ $\Rightarrow$ $x = 45 \pm \frac{1}{\sqrt{2}}$. Thus

$$\iint_D \tfrac{10}{\pi} e^{-2(x-45)^2-50(y-20)^2}\,dA = \frac{10}{\pi}\int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}} \int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx.$$

Using a CAS or calculator to evaluate the integral, we get $P(4(X-45)^2 + 100(Y-20)^2 \le 2) \approx 0.632$.

34. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y) = f_1(x) f_2(y) = \begin{cases} \frac{1}{50} e^{-x} y & \text{if } x \ge 0,\, 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$



Since Xavier won't wait for Yolanda, they won't meet unless $X \ge Y$. Additionally, Yolanda will wait up to half an hour but no longer, so they won't meet unless $X - Y \le 30$. Thus the probability that they meet is $P((X,Y) \in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so

$$\begin{aligned} P((X,Y) \in D) &= \iint_D f(x,y)\,dx\,dy = \int_0^{10}\int_y^{y+30} \tfrac{1}{50} e^{-x} y\,dx\,dy \\ &= \tfrac{1}{50}\int_0^{10} y\big[-e^{-x}\big]_{x=y}^{x=y+30}\,dy = \tfrac{1}{50}\int_0^{10} y\big(-e^{-(y+30)} + e^{-y}\big)\,dy \\ &= \tfrac{1}{50}\big(1 - e^{-30}\big)\int_0^{10} y e^{-y}\,dy \end{aligned}$$

By integration by parts (or Formula 96 in the Table of Integrals), this is

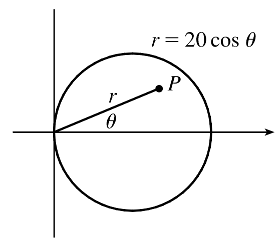
$\frac{1}{50}(1 - e^{-30})\big[-(y+1)e^{-y}\big]_0^{10} = \frac{1}{50}(1 - e^{-30})(1 - 11e^{-10}) \approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

35. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P, and $k\,dA$ is the number of infected individuals in an element of area dA, then $f(P, A)k\,dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA. Integration over D gives the number of infections of the person at A due to all the infected people in D. In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

$$E = \iint_D kf(P, A)\,dA = k\iint_D \tfrac{1}{20}\,[20 - d(P, A)]\;dA = k\iint_D \left[1 - \tfrac{1}{20}\sqrt{(x - x_0)^2 + (y - y_0)^2}\right] dA$$

(b) If $A = (0, 0)$, then

$$\begin{aligned} E &= k\iint_D \left[1 - \tfrac{1}{20}\sqrt{x^2 + y^2}\right] dA \\ &= k\int_0^{2\pi}\int_0^{10}\left(1 - \tfrac{1}{20}r\right) r\,dr\,d\theta = 2\pi k\left[\tfrac{1}{2}r^2 - \tfrac{1}{60}r^3\right]_0^{10} \\ &= 2\pi k\left(50 - \tfrac{50}{3}\right) = \tfrac{200}{3}\pi k \approx 209k \end{aligned}$$



For A at the edge of the city, it is convenient to use a polar coordinate system centered at A. Then the polar equation for the circular boundary of the city becomes $r = 20\cos\theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$\begin{aligned} E &= k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta}\left(1 - \tfrac{1}{20}r\right) r\,dr\,d\theta = k\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2}r^2 - \tfrac{1}{60}r^3\right]_{r=0}^{r=20\cos\theta} d\theta \\ &= k\int_{-\pi/2}^{\pi/2}\left(200\cos^2\theta - \tfrac{400}{3}\cos^3\theta\right) d\theta = 200k\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta - \tfrac{2}{3}\left(1 - \sin^2\theta\right)\cos\theta\right] d\theta \\ &= 200k\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{2}{3}\sin\theta + \tfrac{2}{3}\cdot\tfrac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} = 200k\left[\tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9} + \tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9}\right] \\ &= 200k\left(\tfrac{\pi}{2} - \tfrac{8}{9}\right) \approx 136k \end{aligned}$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

15.5 Surface Area

1. Here $z = f(x, y) = 10 + x + y^2$ and D is the triangle with vertices $(0, 0)$, $(0, -2)$, and $(2, -2)$. By Formula 2, the area of the surface is

$$\begin{aligned} A(S) &= \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\,dA = \int_{-2}^{0}\int_0^{-y}\sqrt{1^2 + (2y)^2 + 1}\,dx\,dy \\ &= \int_{-2}^{0}\int_0^{-y}\sqrt{2 + 4y^2}\,dx\,dy = \int_{-2}^{0}\sqrt{2 + 4y^2}\,[x]_{x=0}^{x=-y}\,dy = -\int_{-2}^{0} y\sqrt{2 + 4y^2}\,dy \\ &= -\frac{1}{8}\cdot\left[\frac{2}{3}(2 + 4y^2)^{3/2}\right]_{-2}^{0} = \frac{18^{3/2} - 2^{3/2}}{12} = \frac{54\sqrt{2} - 2\sqrt{2}}{12} = \frac{13\sqrt{2}}{3} \end{aligned}$$

2. Here $z = f(x, y) = 3 + xy$ and D is the circle $x^2 + y^2 \le 1$. By Formula 2, the area of the surface is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\,dA = \iint_{x^2+y^2\le 1}\sqrt{y^2 + x^2 + 1}\,dA \qquad \text{[Switch to polar coordinates]}$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{r^2 + 1}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_0^1 \sqrt{r^2 + 1}\,r\,dr = 2\pi\cdot\tfrac{1}{2}\left[\tfrac{2}{3}(r^2 + 1)^{3/2}\right]_{r=0}^{r=1} = \tfrac{2\pi}{3}\left(2^{3/2} - 1\right)$$

3. Here $z = f(x, y) = 5x + 3y + 6$ and D is the rectangle $[1, 4] \times [2, 6]$. By Formula 2, the area of the surface is

$$\begin{aligned} A(S) &= \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA = \iint_D \sqrt{5^2 + 3^2 + 1}\, dA = \sqrt{35} \iint_D dA \\ &= \sqrt{35}\, A(D) = \sqrt{35}\,(4-1)(6-2) = 12\sqrt{35} \end{aligned}$$

4. Here $z = f(x, y) = \frac{1}{2} - 3x - 2y$ and D is the disk $x^2 + y^2 \le 25$. By Formula 2, the area of the surface is

$$A(S) = \iint_D \sqrt{(-3)^2 + (-2)^2 + 1}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D) = \sqrt{14}\,(\pi \cdot 5^2) = 25\sqrt{14}\,\pi$$

5. The surface S is given by $z = f(x, y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\left\{(x, y) \mid 0 \le x \le 2,\, 0 \le y \le 3 - \frac{3}{2}x\right\}$. By Formula 2, the surface area of S is

$$A(S) = \iint_D \sqrt{(-3)^2 + (-2)^2 + 1}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D) = \sqrt{14}\left(\tfrac{1}{2} \cdot 2 \cdot 3\right) = 3\sqrt{14}$$

6. $z = f(x, y) = \frac{1}{4}x^2 - \frac{1}{2}y + \frac{5}{4}$, and D is the triangular region given by $\{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 2x\}$. By Formula 2,

$$\begin{aligned} A(S) &= \iint_D \sqrt{\left(\tfrac{1}{2}x\right)^2 + \left(-\tfrac{1}{2}\right)^2 + 1}\, dA = \int_0^2 \int_0^{2x} \sqrt{\tfrac{1}{4}x^2 + \tfrac{5}{4}}\, dy\, dx = \int_0^2 \tfrac{1}{2}\sqrt{x^2+5}\, \Big[\, y \,\Big]_{y=0}^{y=2x} dx \\ &= \tfrac{1}{2}\int_0^2 2x\sqrt{x^2+5}\, dx = \tfrac{1}{2} \cdot \tfrac{2}{3}(x^2+5)^{3/2}\Big]_0^2 = \tfrac{1}{3}(9^{3/2} - 5^{3/2}) = 9 - \tfrac{5}{3}\sqrt{5} \end{aligned}$$

7. The paraboloid intersects the plane $z = -2$ when $1 - x^2 - y^2 = -2 \quad\Leftrightarrow\quad x^2 + y^2 = 3$, so $D = \left\{(x, y) \mid x^2 + y^2 \le 3\right\}$. Here $z = f(x, y) = 1 - x^2 - y^2 \quad\Rightarrow\quad f_x = -2x,\ f_y = -2y$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\, dA = \iint_D \sqrt{4(x^2+y^2)+1}\, dA = \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{4r^2+1}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} r\sqrt{4r^2+1}\, dr = \Big[\, \theta \,\Big]_0^{2\pi} \left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_0^{\sqrt{3}} = 2\pi \cdot \tfrac{1}{12}\left(13^{3/2} - 1\right) = \tfrac{\pi}{6}\left(13\sqrt{13} - 1\right) \end{aligned}$$

8. $x^2 + z^2 = 4 \quad\Rightarrow\quad z = \sqrt{4 - x^2}$ (since $z \ge 0$), so $f_x = -x(4 - x^2)^{-1/2}$, $f_y = 0$ and

$$\begin{aligned} A(S) &= \int_0^1 \int_0^1 \sqrt{[-x(4-x^2)^{-1/2}]^2 + 0^2 + 1}\; dy\, dx = \int_0^1 \int_0^1 \sqrt{\frac{x^2}{4-x^2} + 1}\; dy\, dx \\ &= \int_0^1 \frac{2}{\sqrt{4-x^2}}\, dx \int_0^1 dy = \left[2\sin^{-1}\frac{x}{2}\right]_0^1 \Big[\, y \,\Big]_0^1 = \left(2 \cdot \tfrac{\pi}{6} - 0\right)(1) = \tfrac{\pi}{3} \end{aligned}$$

9. $z = f(x, y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$\begin{aligned} A(S) &= \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi}\int_1^2 \sqrt{4r^2+1}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{4r^2+1}\, dr \\ &= \Big[\, \theta \,\Big]_0^{2\pi} \left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \end{aligned}$$

10. $z = f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 + 1}\, dA = \int_0^1 \int_0^1 \sqrt{x+y+1}\, dy\, dx = \int_0^1 \left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{y=0}^{y=1} dx \\ &= \tfrac{2}{3}\int_0^1 \left[(x+2)^{3/2} - (x+1)^{3/2}\right] dx = \tfrac{2}{3}\left[\tfrac{2}{5}(x+2)^{5/2} - \tfrac{2}{5}(x+1)^{5/2}\right]_0^1 \\ &= \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1) = \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1) \end{aligned}$$

11. $z = f(x, y) = xy$ with $x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x \quad \Rightarrow$

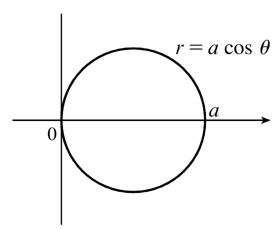
$$A(S) = \iint_D \sqrt{y^2 + x^2 + 1}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}(r^2 + 1)^{3/2} \right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi} \tfrac{1}{3}\left(2\sqrt{2} - 1\right) d\theta = \tfrac{2\pi}{3}\left(2\sqrt{2} - 1\right)$$

12. Given the sphere $x^2 + y^2 + z^2 = 4$, when $z = 1$, we get $x^2 + y^2 = 3$ so $D = \left\{(x, y) \mid x^2 + y^2 \le 3\right\}$ and $z = f(x, y) = \sqrt{4 - x^2 - y^2}$. Thus

$$A(S) = \iint_D \sqrt{[(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4 - r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2 + 4 - r^2}{4 - r^2}}\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}}\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[-2(4 - r^2)^{1/2} \right]_{r=0}^{r=\sqrt{3}} d\theta = \int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

13. $z = \sqrt{a^2 - x^2 - y^2}$, $z_x = -x(a^2 - x^2 - y^2)^{-1/2}$, $z_y = -y(a^2 - x^2 - y^2)^{-1/2}$,

$$A\,(S) = \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dA$$
$$= \int_{-\pi/2}^{\pi/2} \int_0^{a \cos\theta} \sqrt{\frac{r^2}{a^2 - r^2} + 1}\, r\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \int_0^{a \cos\theta} \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \left[-a\sqrt{a^2 - r^2} \right]_{r=0}^{r=a\cos\theta} d\theta$$
$$= \int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2 - a^2\cos^2\theta} - a \right) d\theta = 2a^2 \int_0^{\pi/2} \left(1 - \sqrt{1 - \cos^2\theta} \right) d\theta$$
$$= 2a^2 \int_0^{\pi/2} d\theta - 2a^2 \int_0^{\pi/2} \sqrt{\sin^2\theta}\, d\theta = a^2\pi - 2a^2 \int_0^{\pi/2} \sin\theta\, d\theta = a^2(\pi - 2)$$



14. To find the region D: $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z - 2)^2 = 4$, so $z = 3$ intersects the upper hemisphere. Thus $(z - 2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3 - 2)^2 = 4$, that is, $D = \left\{(x, y) \mid x^2 + y^2 \le 3\right\}$.

$$A(S) = \iint_D \sqrt{[(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4 - r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r\, dr}{\sqrt{4 - r^2}}\, d\theta = \int_0^{2\pi} \left[-2(4 - r^2)^{1/2} \right]_{r=0}^{r=\sqrt{3}} d\theta$$
$$= \int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

15. $z = f(x,y) = (1+x^2+y^2)^{-1}$, $f_x = -2x(1+x^2+y^2)^{-2}$, $f_y = -2y(1+x^2+y^2)^{-2}$. Then

$$A(S) = \iint_{x^2+y^2\le 1} \sqrt{[-2x(1+x^2+y^2)^{-2}]^2 + [-2y(1+x^2+y^2)^{-2}]^2 + 1}\, dA$$
$$= \iint_{x^2+y^2\le 1} \sqrt{4(x^2+y^2)(1+x^2+y^2)^{-4}+1}\, dA$$

Converting to polar coordinates we have

$$A(S) = \int_0^{2\pi}\int_0^1 \sqrt{4r^2(1+r^2)^{-4}+1}\, r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{4r^2(1+r^2)^{-4}+1}\,dr$$
$$= 2\pi\int_0^1 r\sqrt{4r^2(1+r^2)^{-4}+1}\,dr \approx 3.6258 \text{ using a calculator.}$$

16. $z = f(x,y) = \cos(x^2+y^2)$, $f_x = -2x\sin(x^2+y^2)$, $f_y = -2y\sin(x^2+y^2)$.

$$A(S) = \iint_{x^2+y^2\le 1} \sqrt{4x^2\sin^2(x^2+y^2) + 4y^2\sin^2(x^2+y^2)+1}\,dA = \iint_{x^2+y^2\le 1}\sqrt{4(x^2+y^2)\sin^2(x^2+y^2)+1}\,dA.$$

Converting to polar coordinates gives

$$A(S) = \int_0^{2\pi}\int_0^1 \sqrt{4r^2\sin^2(r^2)+1}\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^1 r\sqrt{4r^2\sin^2(r^2)+1}\,dr$$
$$= 2\pi\int_0^1 r\sqrt{4r^2\sin^2(r^2)+1}\,dr \approx 4.1073 \quad \text{using a calculator.}$$

17. (a) The midpoints of the four squares are $\left(\frac14,\frac14\right)$, $\left(\frac14,\frac34\right)$, $\left(\frac34,\frac14\right)$, and $\left(\frac34,\frac34\right)$. Here $f(x,y) = x^2+y^2$, so the Midpoint Rule gives

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2+1}\,dA = \iint_D\sqrt{(2x)^2+(2y)^2+1}\,dA$$
$$\approx \tfrac14\left(\sqrt{\left[2\left(\tfrac14\right)\right]^2+\left[2\left(\tfrac14\right)\right]^2+1} + \sqrt{\left[2\left(\tfrac14\right)\right]^2+\left[2\left(\tfrac34\right)\right]^2+1}\right.$$
$$\left. + \sqrt{\left[2\left(\tfrac34\right)\right]^2+\left[2\left(\tfrac14\right)\right]^2+1} + \sqrt{\left[2\left(\tfrac34\right)\right]^2+\left[2\left(\tfrac34\right)\right]^2+1}\right)$$
$$= \tfrac14\left(\sqrt{\tfrac32} + 2\sqrt{\tfrac72} + \sqrt{\tfrac{11}{2}}\right) \approx 1.8279$$

(b) A CAS estimates the integral to be $A(S) = \iint_D \sqrt{1+(2x)^2+(2y)^2}\,dA = \int_0^1\int_0^1\sqrt{1+4x^2+4y^2}\,dy\,dx \approx 1.8616$. This agrees with the Midpoint estimate only in the first decimal place.

18. (a) With $m = n = 2$ we have four squares with midpoints $\left(\frac12,\frac12\right)$, $\left(\frac12,\frac32\right)$, $\left(\frac32,\frac12\right)$, and $\left(\frac32,\frac32\right)$. Since $z = xy+x^2+y^2$, the Midpoint Rule gives

$$A(S) = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_D\sqrt{1+(y+2x)^2+(x+2y)^2}\,dA$$
$$\approx 1\left(\sqrt{1+\left(\tfrac32\right)^2+\left(\tfrac32\right)^2} + \sqrt{1+\left(\tfrac52\right)^2+\left(\tfrac72\right)^2} + \sqrt{1+\left(\tfrac72\right)^2+\left(\tfrac52\right)^2} + \sqrt{1+\left(\tfrac92\right)^2+\left(\tfrac92\right)^2}\right)$$
$$= \tfrac{\sqrt{22}}{2} + \tfrac{\sqrt{78}}{2} + \tfrac{\sqrt{78}}{2} + \tfrac{\sqrt{166}}{2} \approx 17.619$$

(b) Using a CAS, we have

$A(S) = \iint_D\sqrt{1+(y+2x)^2+(x+2y)^2}\,dA = \int_0^2\int_0^2\sqrt{1+(y+2x)^2+(x+2y)^2}\,dy\,dx \approx 17.7165$. This is within about 0.1 of the Midpoint Rule estimate.

19. $z = 1 + 2x + 3y + 4y^2$, so

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int_1^4 \int_0^1 \sqrt{1 + 4 + (3 + 8y)^2}\, dy\, dx = \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\, dy\, dx.$$

Using a CAS, we have $\int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\, dy\, dx = \frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left(11\sqrt{5} + 3\sqrt{14}\sqrt{5}\right) - \frac{15}{16}\ln\left(3\sqrt{5} + \sqrt{14}\sqrt{5}\right)$

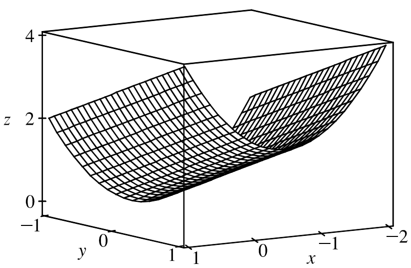
or $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\dfrac{11\sqrt{5} + 3\sqrt{70}}{3\sqrt{5} + \sqrt{70}}$.

20. $f(x, y) = 1 + x + y + x^2 \quad \Rightarrow \quad f_x = 1 + 2x,\ f_y = 1$. We use a CAS to calculate the integral

$$\begin{aligned} A(S) &= \textstyle\int_{-2}^{1} \int_{-1}^{1} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx \\ &= \textstyle\int_{-2}^{1} \int_{-1}^{1} \sqrt{(1 + 2x)^2 + 2}\, dy\, dx = 2\int_{-2}^{1} \sqrt{4x^2 + 4x + 3}\, dx \end{aligned}$$

and find that $A(S) = 3\sqrt{11} + 2\sinh^{-1}\left(\frac{3\sqrt{2}}{2}\right)$ or

$A\,(S) = 3\sqrt{11} + \ln\left(10 + 3\sqrt{11}\right)$.



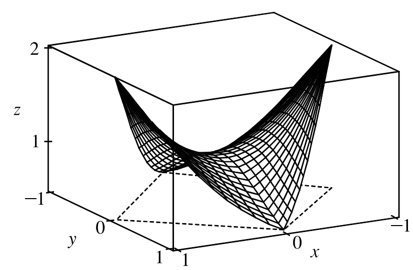
21. $f(x, y) = 1 + x^2y^2 \quad \Rightarrow \quad f_x = 2xy^2,\ f_y = 2x^2y$. We use a CAS (with precision reduced to five significant digits, to speed up the calculation) to estimate the integral

$$A(S) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2y^4 + 4x^4y^2 + 1}\, dy\, dx,$$ and find that $A(S) \approx 3.3213$.

22. Let $f(x, y) = \dfrac{1 + x^2}{1 + y^2}$. Then $f_x = \dfrac{2x}{1 + y^2}$,

$f_y = \left(1 + x^2\right)\left[-\dfrac{2y}{(1 + y^2)^2}\right] = -\dfrac{2y\left(1 + x^2\right)}{(1 + y^2)^2}$. We use a CAS

to estimate $\int_{-1}^{1} \int_{-(1-|x|)}^{1-|x|} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx \approx 2.6959$. In order to graph only the part of the surface above the square, we use $-(1 - |x|) \le y \le 1 - |x|$ as the y-range in our plot command.



23. Here $z = f(x, y) = ax + by + c$, $f_x(x, y) = a$, $f_y(x, y) = b$, so

$A(S) = \iint_D \sqrt{a^2 + b^2 + 1}\, dA = \sqrt{a^2 + b^2 + 1} \iint_D dA = \sqrt{a^2 + b^2 + 1}\, A(D)$.

24. Let S be the upper hemisphere. Then $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$, so

$$\begin{aligned} A(S) &= \textstyle\iint_D \sqrt{[-x(a^2 - x^2 - y^2)^{-1/2}]^2 + [-y(a^2 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA \\ &= \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dA = \lim_{t \to a^-} \int_0^{2\pi} \int_0^t \sqrt{\frac{r^2}{a^2 - r^2} + 1}\, r\, dr\, d\theta \\ &= \lim_{t \to a^-} \int_0^{2\pi} \int_0^t \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta = 2\pi \lim_{t \to a^-} \left[-a\sqrt{a^2 - r^2}\right]_0^t = 2\pi \lim_{t \to a^-} -a\left[\sqrt{a^2 - t^2} - a\right] \\ &= 2\pi(-a)(-a) = 2\pi a^2. \end{aligned}$$ Thus the surface area of the entire sphere is $4\pi a^2$.

25. If we project the surface onto the xz-plane, then the surface lies "above" the disk $x^2 + z^2 \le 25$ in the xz-plane.

We have $y = f(x, z) = x^2 + z^2$ and, adapting Formula 2, the area of the surface is

$$A(S) = \iint_{x^2+z^2\le 25} \sqrt{[f_x(x,z)]^2 + [f_z(x,z)]^2 + 1}\, dA = \iint_{x^2+z^2\le 25} \sqrt{4x^2 + 4z^2 + 1}\, dA$$

Converting to polar coordinates $x = r\cos\theta$, $z = r\sin\theta$ we have

$$A(S) = \int_0^{2\pi}\int_0^5 \sqrt{4r^2+1}\, r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^5 r(4r^2+1)^{1/2}\,dr = \big[\,\theta\,\big]_0^{2\pi}\left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_0^5 = \tfrac{\pi}{6}\left(101\sqrt{101} - 1\right)$$

26. First we find the area of the face of the surface that intersects the positive y-axis. As in Exercise 25, we can project the face onto the xz-plane, so the surface lies "above" the disk $x^2 + z^2 \le 1$. Then $y = f(x, z) = \sqrt{1 - z^2}$ and the area is

$$\begin{aligned} A(S) &= \iint_{x^2+z^2\le 1} \sqrt{[f_x(x,z)]^2 + [f_z(x,z)]^2 + 1}\, dA = \iint_{x^2+z^2\le 1} \sqrt{0 + \left(\frac{-z}{\sqrt{1-z^2}}\right)^2 + 1}\, dA \\ &= \iint_{x^2+z^2\le 1} \sqrt{\frac{z^2}{1-z^2} + 1}\, dA = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\,dx\,dz \\ &= 4\int_0^1\int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\,dx\,dz \qquad \text{[by the symmetry of the surface]} \end{aligned}$$

This integral is improper (when $z = 1$), so

$$A(S) = \lim_{t\to 1^-} 4\int_0^t\int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\,dx\,dz = \lim_{t\to 1^-} 4\int_0^t \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}}\,dz = \lim_{t\to 1^-} 4\int_0^t dz = \lim_{t\to 1^-} 4t = 4.$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4) = 16$.

15.6 Triple Integrals

1. $$\begin{aligned} \iiint_B xyz^2\,dV &= \int_0^1\int_0^3\int_{-1}^2 xyz^2\,dy\,dz\,dx = \int_0^1\int_0^3 \left[\tfrac12 xy^2z^2\right]_{y=-1}^{y=2} dz\,dx = \int_0^1\int_0^3 \tfrac32 xz^2\,dz\,dx \\ &= \int_0^1 \left[\tfrac12 xz^3\right]_{z=0}^{z=3} dx = \int_0^1 \tfrac{27}{2}\,x\,dx = \tfrac{27}{4}x^2\Big]_0^1 = \tfrac{27}{4} \end{aligned}$$

2. There are six different possible orders of integration.

$$\begin{aligned} \iiint_E (xy+z^2)\,dV &= \int_0^2\int_0^1\int_0^3 (xy+z^2)\,dz\,dy\,dx = \int_0^2\int_0^1 \left[xyz + \tfrac13 z^3\right]_{z=0}^{z=3} dy\,dx = \int_0^2\int_0^1 (3xy+9)\,dy\,dx \\ &= \int_0^2 \left[\tfrac32 xy^2 + 9y\right]_{y=0}^{y=1} dx = \int_0^2 \left(\tfrac32 x + 9\right) dx = \left[\tfrac34 x^2 + 9x\right]_0^2 = 21 \end{aligned}$$

$$\begin{aligned} \iiint_E (xy+z^2)\,dV &= \int_0^1\int_0^2\int_0^3 (xy+z^2)\,dz\,dx\,dy = \int_0^1\int_0^2 \left[xyz + \tfrac13 z^3\right]_{z=0}^{z=3} dx\,dy = \int_0^1\int_0^2 (3xy+9)\,dx\,dy \\ &= \int_0^1 \left[\tfrac32 x^2y + 9x\right]_{x=0}^{x=2} dy = \int_0^1 (6y+18)\,dy = \left[3y^2 + 18y\right]_0^1 = 21 \end{aligned}$$

$$\begin{aligned} \iiint_E (xy+z^2)\,dV &= \int_0^2\int_0^3\int_0^1 (xy+z^2)\,dy\,dz\,dx = \int_0^2\int_0^3 \left[\tfrac12 xy^2 + yz^2\right]_{y=0}^{y=1} dz\,dx = \int_0^2\int_0^3 \left(\tfrac12 x + z^2\right) dz\,dx \\ &= \int_0^2 \left[\tfrac12 xz + \tfrac13 z^3\right]_{z=0}^{z=3} dx = \int_0^2 \left(\tfrac32 x + 9\right) dx = \left[\tfrac34 x^2 + 9x\right]_0^2 = 21 \end{aligned}$$

[continued]

$$\iiint_E (xy+z^2)\,dV = \int_0^3\int_0^2\int_0^1 (xy+z^2)\,dy\,dx\,dz = \int_0^3\int_0^2 \left[\tfrac{1}{2}xy^2+yz^2\right]_{y=0}^{y=1} dx\,dz = \int_0^3\int_0^2 \left(\tfrac{1}{2}x+z^2\right) dx\,dz$$
$$= \int_0^3 \left[\tfrac{1}{4}x^2+xz^2\right]_{x=0}^{x=2} dz = \int_0^3 \left(1+2z^2\right) dz = \left[z+\tfrac{2}{3}z^3\right]_0^3 = 21$$

$$\iiint_E (xy+z^2)\,dV = \int_0^1\int_0^3\int_0^2 (xy+z^2)\,dx\,dz\,dy = \int_0^1\int_0^3 \left[\tfrac{1}{2}x^2y+xz^2\right]_{x=0}^{x=2} dz\,dy = \int_0^1\int_0^3 \left(2y+2z^2\right) dz\,dy$$
$$= \int_0^1 \left[2yz+\tfrac{2}{3}z^3\right]_{z=0}^{z=3} dy = \int_0^1 (6y+18)\,dy = \left[3y^2+18y\right]_0^1 = 21$$

$$\iiint_E (xy+z^2)\,dV = \int_0^3\int_0^1\int_0^2 (xy+z^2)\,dx\,dy\,dz = \int_0^3\int_0^1 \left[\tfrac{1}{2}x^2y+xz^2\right]_{x=0}^{x=2} dy\,dz = \int_0^3\int_0^1 \left(2y+2z^2\right) dy\,dz$$
$$= \int_0^3 \left[y^2+2yz^2\right]_{y=0}^{y=1} dz = \int_0^3 \left(1+2z^2\right) dz = \left[z+\tfrac{2}{3}z^3\right]_0^3 = 21$$

3. $$\int_0^2\int_0^{z^2}\int_0^{y-z}(2x-y)\,dx\,dy\,dz = \int_0^2\int_0^{z^2}\left[x^2-xy\right]_{x=0}^{x=y-z} dy\,dz = \int_0^2\int_0^{z^2}\left[(y-z)^2-(y-z)y\right] dy\,dz$$
$$= \int_0^2\int_0^{z^2}\left(z^2-yz\right) dy\,dz = \int_0^2 \left[yz^2-\tfrac{1}{2}y^2z\right]_{y=0}^{y=z^2} dz = \int_0^2 \left(z^4-\tfrac{1}{2}z^5\right) dz$$
$$= \left[\tfrac{1}{5}z^5-\tfrac{1}{12}z^6\right]_0^2 = \tfrac{32}{5}-\tfrac{64}{12} = \tfrac{16}{15}$$

4. $$\int_0^1\int_y^{2y}\int_0^{x+y} 6xy\,dz\,dx\,dy = \int_0^1\int_y^{2y}\left[6xyz\right]_{z=0}^{z=x+y} dx\,dy = \int_0^1\int_y^{2y} 6xy(x+y)\,dx\,dy = \int_0^1\int_y^{2y}(6x^2y+6xy^2)\,dx\,dy$$
$$= \int_0^1 \left[2x^3y+3x^2y^2\right]_{x=y}^{x=2y} dy = \int_0^1 23y^4\,dy = \tfrac{23}{5}y^5\Big]_0^1 = \tfrac{23}{5}$$

5. $$\int_1^2\int_0^{2z}\int_0^{\ln x} xe^{-y}\,dy\,dx\,dz = \int_1^2\int_0^{2z}\left[-xe^{-y}\right]_{y=0}^{y=\ln x} dx\,dz = \int_1^2\int_0^{2z}\left(-xe^{-\ln x}+xe^0\right) dx\,dz$$
$$= \int_1^2\int_0^{2z}(-1+x)\,dx\,dz = \int_1^2 \left[-x+\tfrac{1}{2}x^2\right]_{x=0}^{x=2z} dz$$
$$= \int_1^2 \left(-2z+2z^2\right) dz = \left[-z^2+\tfrac{2}{3}z^3\right]_1^2 = -4+\tfrac{16}{3}+1-\tfrac{2}{3} = \tfrac{5}{3}$$

6. $$\int_0^{\pi/2}\int_0^{2x}\int_0^{x+z}\cos(x-2y+z)\,dy\,dz\,dx = -\tfrac{1}{2}\int_0^{\pi/2}\int_0^{2x}\left[\sin(x-2y+z)\right]_{y=0}^{y=x+z} dz\,dx$$
$$= -\tfrac{1}{2}\int_0^{\pi/2}\int_0^{2x}\left[\sin(-x-z)-\sin(x+z)\right] dz\,dx$$
$$= -\tfrac{1}{2}\int_0^{\pi/2}\int_0^{2x}\left[-2\sin(x+z)\right] dz\,dx$$
$$= \int_0^{\pi/2}\int_0^{2x}\sin(x+z)\,dz\,dx = -\int_0^{\pi/2}\left[\cos(x+z)\right]_{z=0}^{z=2x} dx$$
$$= -\int_0^{\pi/2}(\cos 3x-\cos x)\,dx = \left[\sin x-\tfrac{1}{3}\sin 3x\right]_0^{\pi/2}$$
$$= 1-\tfrac{1}{3}(-1) = \tfrac{4}{3}$$

7. $$\int_1^3\int_{-1}^2\int_{-y}^z \frac{z}{y}\,dx\,dz\,dy = \int_1^3\int_{-1}^2 \frac{z}{y}\Big[x\Big]_{x=-y}^{x=z} dz\,dy = \int_1^3\int_{-1}^2\left(\frac{z^2}{y}+z\right) dz\,dy = \int_1^3\left[\frac{z^3}{3y}+\frac{z^2}{2}\right]_{z=-1}^{z=2} dy$$
$$= \int_1^3\left(\frac{3}{y}+\frac{3}{2}\right) dy = \left[3\ln|y|+\frac{3}{2}y\right]_1^3 = 3\ln 3+3$$

8. $$\int_0^1\int_0^1\int_0^{2-x^2-y^2} xye^z\,dz\,dy\,dx = \int_0^1\int_0^1\left[xye^z\right]_{z=0}^{z=2-x^2-y^2} dy\,dx = \int_0^1\int_0^1(xye^{2-x^2-y^2}-xy)\,dy\,dx$$
$$= \int_0^1\left[-\tfrac{1}{2}xe^{2-x^2-y^2}-\tfrac{1}{2}xy^2\right]_{y=0}^{y=1} dx = \int_0^1\left(-\tfrac{1}{2}xe^{1-x^2}-\tfrac{1}{2}x+\tfrac{1}{2}xe^{2-x^2}\right) dx$$
$$= \left[\tfrac{1}{4}e^{1-x^2}-\tfrac{1}{4}x^2-\tfrac{1}{4}e^{2-x^2}\right]_0^1 = \tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4}e-\tfrac{1}{4}e+0+\tfrac{1}{4}e^2 = \tfrac{1}{4}e^2-\tfrac{1}{2}e$$

9. (a) The solid region E can be described as $E = \{(x,y,z) \mid -1 \le x \le 1, 0 \le y \le 2 - z, 0 \le z \le 1 - x^2\}$.

Thus, $\iiint_E x\,dV = \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} x\,dy\,dz\,dx$.

(b) $$\int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} x\,dy\,dz\,dx = \int_{-1}^{1} \int_0^{1-x^2} x\Big[y\Big]_{y=0}^{y=2-z} dz\,dx = \int_{-1}^{1} \int_0^{1-x^2} (2x - xz)\,dz\,dx$$

$$= \int_{-1}^{1} \left[2xz - x\frac{z^2}{2}\right]_{z=0}^{z=1-x^2} dx = \int_{-1}^{1} \left(\frac{3}{2}x - x^3 - \frac{x^5}{2}\right) dx$$

$$= \left[\frac{3x^2}{4} - \frac{x^4}{4} - \frac{x^6}{12}\right]_{-1}^{1} = 0$$

10. (a) The solid region E can be described as $E = \{(x,y,z) \mid 0 \le x \le y, 0 \le y \le 2, 0 \le z \le 4 - y^2\}$.

Thus, $\iiint_E xy\,dV = \int_0^2 \int_0^y \int_0^{4-y^2} xy\,dz\,dx\,dy$.

(b) $$\int_0^2 \int_0^y \int_0^{4-y^2} xy\,dz\,dx\,dy = \int_0^2 \int_0^y xy\,[z]_{z=0}^{z=4-y^2}\,dx\,dy = \int_0^2 \int_0^y xy(4-y^2)\,dx\,dy = \int_0^2 \int_0^y x(4y-y^3)\,dx\,dy$$

$$= \int_0^2 (4y - y^3)\left[\tfrac{x^2}{2}\right]_{x=0}^{x=y} dy = \tfrac{1}{2}\int_0^2 (4y^3 - y^5)\,dy = \tfrac{1}{2}\left[y^4 - \tfrac{y^6}{6}\right]_0^2 = \tfrac{8}{3}$$

11. (a) The solid region E can be described as $E = \{(x,y,z) \mid 0 \le x \le 2, 0 \le y \le x^2, 0 \le z \le 2 - x\}$.

Thus, $\iiint_E (x+y)\,dV = \int_0^2 \int_0^{2-x} \int_0^{x^2} (x+y)\,dy\,dz\,dx$.

(b) $$\int_0^2 \int_0^{2-x} \int_0^{x^2} (x+y)\,dy\,dz\,dx = \int_0^2 \int_0^{2-x} \left[xy + \frac{y^2}{2}\right]_{y=0}^{y=x^2} dz\,dx = \int_0^2 \int_0^{2-x} \left(x^3 + \frac{x^4}{2}\right) dz\,dx$$

$$= \int_0^2 \left(x^3 + \frac{x^4}{2}\right)\Big[z\Big]_{z=0}^{z=2-x} dx = \int_0^2 \left(2x^3 - \frac{x^5}{2}\right) dx = \left[\frac{x^4}{2} - \frac{x^6}{12}\right]_0^2 = \frac{8}{3}$$

12. (a) The solid region E can be described as $E = \{(x,y,z) \mid z - 4 \le x \le 4 - z, -2 \le y \le 2, 0 \le z \le 4 - y^2\}$.

Thus, $\iiint_E 2\,dV = \int_{-2}^{2} \int_0^{4-y^2} \int_{z-4}^{4-z} 2\,dx\,dz\,dy$.

(b) $$\int_{-2}^{2} \int_0^{4-y^2} \int_{z-4}^{4-z} 2\,dx\,dz\,dy = \int_{-2}^{2} \int_0^{4-y^2} 2\Big[x\Big]_{x=z-4}^{x=4-z} dz\,dy = 2\int_{-2}^{2} \int_0^{4-y^2} (8 - 2z)\;dz\,dy$$

$$= 2\int_{-2}^{2} \Big[8z - z^2\Big]_{z=0}^{z=4-y^2} dy = 2\int_{-2}^{2} (16 - y^4)\,dy = 2\Big[16y - \tfrac{y^5}{5}\Big]_{-2}^{2} = \tfrac{512}{5}$$

13. $$\iiint_E y\,dV = \int_0^3 \int_0^x \int_{x-y}^{x+y} y\,dz\,dy\,dx = \int_0^3 \int_0^x \,[yz]_{z=x-y}^{z=x+y}\,dy\,dx = \int_0^3 \int_0^x 2y^2\,dy\,dx$$

$$= \int_0^3 \left[\tfrac{2}{3}y^3\right]_{y=0}^{y=x} dx = \int_0^3 \tfrac{2}{3}x^3\,dx = \tfrac{1}{6}x^4\Big]_0^3 = \tfrac{81}{6} = \tfrac{27}{2}$$

14. $$\iiint_E e^{z/y}\,dV = \int_0^1 \int_y^1 \int_0^{xy} e^{z/y}\,dz\,dx\,dy = \int_0^1 \int_y^1 \left[ye^{z/y}\right]_{z=0}^{z=xy} dx\,dy$$

$$= \int_0^1 \int_y^1 (ye^x - y)\,dx\,dy = \int_0^1 \left[ye^x - xy\right]_{x=y}^{x=1} dy = \int_0^1 \left(ey - y - ye^y + y^2\right) dy$$

$$= \left[\tfrac{1}{2}ey^2 - \tfrac{1}{2}y^2 - (y-1)e^y + \tfrac{1}{3}y^3\right]_0^1 \qquad \text{[integrate by parts]}$$

$$= \tfrac{1}{2}e - \tfrac{1}{2} + \tfrac{1}{3} - 1 = \tfrac{1}{2}e - \tfrac{7}{6}$$

15. $\iiint_E \frac{1}{x^3}\,dV = \int_0^1 \int_0^{y^2} \int_1^{z+1} \frac{1}{x^3}\,dx\,dz\,dy = \int_0^1 \int_0^{y^2} -\frac{1}{2}\left[\frac{1}{x^2}\right]_{x=1}^{x=z+1} dz\,dy$

$= -\frac{1}{2}\int_0^1 \int_0^{y^2} \left(\frac{1}{(z+1)^2} - 1\right) dz\,dy = -\frac{1}{2}\int_0^1 \left[-\frac{1}{z+1} - z\right]_{z=0}^{z=y^2} dy$

$= \frac{1}{2}\int_0^1 \left(\frac{1}{y^2+1} + y^2 - 1\right) dy = \frac{1}{2}\left[\tan^{-1} y + \frac{y^3}{3} - y\right]_0^1 = \frac{1}{2}\left[\left(\frac{\pi}{4} + \frac{1}{3} - 1\right) - 0\right]$

$= \frac{\pi}{8} - \frac{1}{3}$

16. Here $E = \{(x,y,z) \mid 0 \le x \le \pi, 0 \le y \le \pi - x, 0 \le z \le x\}$, so

$\iiint_E \sin y\,dV = \int_0^\pi \int_0^{\pi-x} \int_0^x \sin y\,dz\,dy\,dx = \int_0^\pi \int_0^{\pi-x} \left[z \sin y\right]_{z=0}^{z=x} dy\,dx = \int_0^\pi \int_0^{\pi-x} x \sin y\,dy\,dx$

$= \int_0^\pi \left[-x \cos y\right]_{y=0}^{y=\pi-x} dx = \int_0^\pi \left[-x\cos(\pi - x) + x\right] dx$

$= \left[x \sin(\pi - x) - \cos(\pi - x) + \frac{1}{2}x^2\right]_0^\pi$ [integrate by parts]

$= 0 - 1 + \frac{1}{2}\pi^2 - 0 - 1 - 0 = \frac{1}{2}\pi^2 - 2$

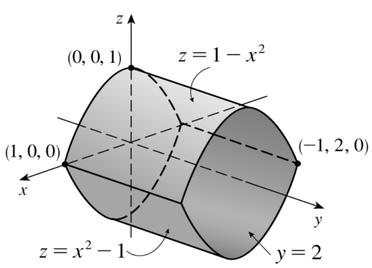
17. Here $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1 + x + y\}$, so

$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} \left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx = \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3\right]_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2})\,dx = \left[x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2}\right]_0^1 = \frac{65}{28}$

18.



Here $E = \left\{(x,y,z) \mid -1 \le x \le 1,\ 0 \le y \le 2,\ x^2 - 1 \le z \le 1 - x^2\right\}$.

Thus,

$\iiint_E (x - y)\,dV = \int_{-1}^1 \int_0^2 \int_{x^2-1}^{1-x^2} (x - y)\,dz\,dy\,dx$

$= \int_{-1}^1 \int_0^2 (x-y)(1 - x^2 - (x^2 - 1))\,dy\,dx$

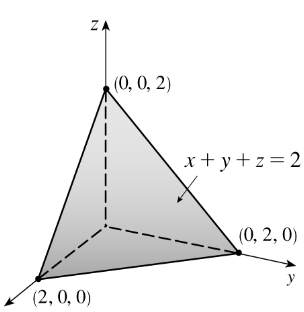
$= \int_{-1}^1 \int_0^2 (2x - 2x^3 - 2y + 2x^2y)\,dy\,dx$

$= \int_{-1}^1 \left[2xy - 2x^3y - y^2 + x^2y^2\right]_{y=0}^{y=2} dx$

$= \int_{-1}^1 (4x - 4x^3 - 4 + 4x^2)\,dx$

$= \left[2x^2 - x^4 - 4x + \frac{4}{3}x^3\right]_{-1}^1 = -\frac{5}{3} - \frac{11}{3} = -\frac{16}{3}$

19.



Here $T = \{(x,y,z) \mid 0 \le x \le 2,\ 0 \le y \le 2 - x,\ 0 \le z \le 2 - x - y\}$. Thus,

$\iiint_T y^2\,dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} y^2\,dz\,dy\,dx = \int_0^2 \int_0^{2-x} y^2(2 - x - y)\,dy\,dx$

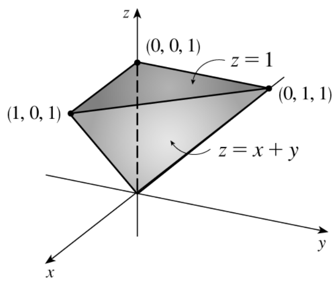
$= \int_0^2 \int_0^{2-x} \left[(2-x)y^2 - y^3\right] dy\,dx$

$= \int_0^2 \left[(2-x)\left(\frac{1}{3}y^3\right) - \frac{1}{4}y^4\right]_{y=0}^{y=2-x} dx$

$= \int_0^2 \left[\frac{1}{3}(2-x)^4 - \frac{1}{4}(2-x)^4\right] dx = \int_0^2 \frac{1}{12}(2-x)^4\,dx$

$= \left[\frac{1}{12}\left(-\frac{1}{5}\right)(2-x)^5\right]_0^2 = -\frac{1}{60}(0 - 32) = \frac{8}{15}$

20.

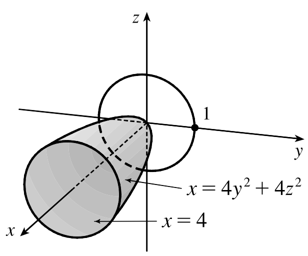


The projection of T onto the xz-plane is the triangle bounded by the lines $z = x$, $x = 0$, and $z = 1$. Then

$T = \{(x, y, z) \mid 0 \le x \le 1,\ x \le z \le 1,\ 0 \le y \le z - x\}$, and

$$\iiint_T xz\,dV = \int_0^1 \int_x^1 \int_0^{z-x} xz\,dy\,dz\,dx = \int_0^1 \int_x^1 xz(z - x)\,dz\,dx$$
$$= \int_0^1 \int_x^1 (xz^2 - x^2 z)\,dz\,dx = \int_0^1 \left[\tfrac{1}{3}xz^3 - \tfrac{1}{2}x^2 z^2\right]_{z=x}^{z=1} dx$$
$$= \int_0^1 \left(\tfrac{1}{3}x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^4 + \tfrac{1}{2}x^4\right) dx$$
$$= \left[\tfrac{1}{6}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{30}x^5\right]_0^1 = \tfrac{1}{6} - \tfrac{1}{6} + \tfrac{1}{30} = \tfrac{1}{30}$$

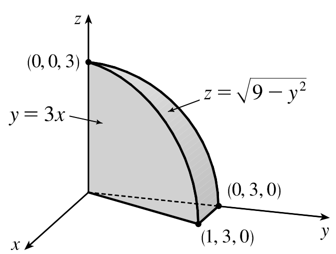
21.



The projection of E onto the yz-plane is the disk $y^2 + z^2 \le 1$. Using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we get

$$\iiint_E x\,dV = \iint_D \left[\int_{4y^2+4z^2}^{4} x\,dx\right] dA = \tfrac{1}{2} \iint_D \left[4^2 - (4y^2 + 4z^2)^2\right] dA$$
$$= 8 \int_0^{2\pi} \int_0^1 (1 - r^4)\,r\,dr\,d\theta = 8 \int_0^{2\pi} d\theta \int_0^1 (r - r^5)\,dr$$
$$= 8(2\pi)\left[\tfrac{1}{2}r^2 - \tfrac{1}{6}r^6\right]_0^1 = \tfrac{16\pi}{3}$$

22.

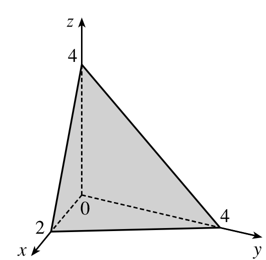


$$\int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = \int_0^1 \int_{3x}^3 \tfrac{1}{2}(9 - y^2)\,dy\,dx$$
$$= \int_0^1 \left[\tfrac{9}{2}y - \tfrac{1}{6}y^3\right]_{y=3x}^{y=3} dx$$
$$= \int_0^1 \left[9 - \tfrac{27}{2}x + \tfrac{9}{2}x^3\right] dx$$
$$= \left[9x - \tfrac{27}{4}x^2 + \tfrac{9}{8}x^4\right]_0^1 = \tfrac{27}{8}$$

23. The plane $2x + y + z = 4$ intersects the xy-plane when

$2x + y + 0 = 4 \quad\Rightarrow\quad y = 4 - 2x$, so

$E = \{(x, y, z) \mid 0 \le x \le 2, 0 \le y \le 4 - 2x, 0 \le z \le 4 - 2x - y\}$ and



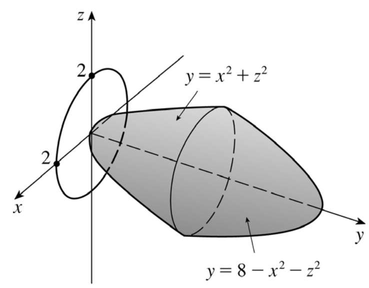
$$V = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx = \int_0^2 \int_0^{4-2x} (4 - 2x - y)\,dy\,dx$$
$$= \int_0^2 \left[4y - 2xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx$$
$$= \int_0^2 \left[4(4 - 2x) - 2x(4 - 2x) - \tfrac{1}{2}(4 - 2x)^2\right] dx$$
$$= \int_0^2 (2x^2 - 8x + 8)\,dx = \left[\tfrac{2}{3}x^3 - 4x^2 + 8x\right]_0^2 = \tfrac{16}{3}$$

24. The paraboloids intersect when $x^2 + z^2 = 8 - x^2 - z^2 \quad\Leftrightarrow\quad x^2 + z^2 = 4$, thus the intersection is the circle $x^2 + z^2 = 4$, $y = 4$. The projection of E onto the xz-plane is the disk $x^2 + z^2 \le 4$, so $E = \left\{(x, y, z) \mid x^2 + z^2 \le y \le 8 - x^2 - z^2, x^2 + z^2 \le 4\right\}$. Let $D = \left\{(x, z) \mid x^2 + z^2 \le 4\right\}$. Then using polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$, we have

$V = \iiint_E dV = \iint_D \left(\int_{x^2+z^2}^{8-x^2-z^2} dy\right) dA = \iint_D (8 - 2x^2 - 2z^2)\, dA$

$= \int_0^{2\pi} \int_0^2 (8 - 2r^2)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 (8r - 2r^3)\, dr$

$= \big[\,\theta\,\big]_0^{2\pi} \left[4r^2 - \frac{1}{2}r^4\right]_0^2 = 2\pi(16 - 8) = 16\pi$



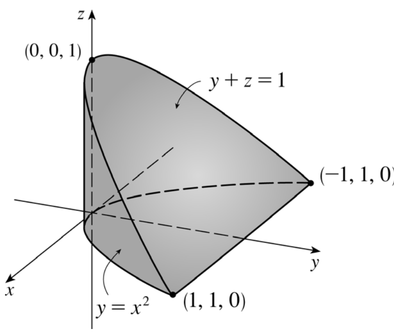
25. The plane $y + z = 1$ intersects the xy-plane in the line $y = 1$, so

$E = \left\{(x, y, z) \mid -1 \le x \le 1, x^2 \le y \le 1, 0 \le z \le 1 - y\right\}$ and

$V = \iiint_E dV = \int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\, dy\, dx = \int_{-1}^1 \int_{x^2}^1 (1 - y)\, dy\, dx$

$= \int_{-1}^1 \left[y - \frac{1}{2}y^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^1 \left(\frac{1}{2} - x^2 + \frac{1}{2}x^4\right) dx$

$= \left[\frac{1}{2}x - \frac{1}{3}x^3 + \frac{1}{10}x^5\right]_{-1}^1 = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = \frac{8}{15}$



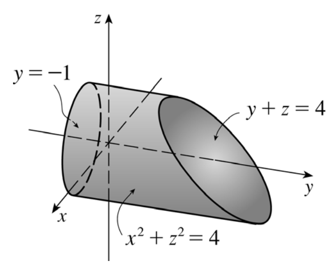
26. Here $E = \left\{(x, y, z) \mid -1 \le y \le 4 - z, x^2 + z^2 \le 4\right\}$, so

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} dy\, dz\, dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - z + 1)\, dz\, dx$$

$$= \int_{-2}^2 \left[5z - \frac{1}{2}z^2\right]_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} dx = \int_{-2}^2 10\sqrt{4 - x^2}\, dx$$

$= 10\left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^2$ [using trigonometric substitution or Formula 30 in the Table of Integrals]

$= 10\left[2\sin^{-1}(1) - 2\sin^{-1}(-1)\right] = 20\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 20\pi$



Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (5 - z)\, dz\, dx = \int_0^{2\pi} \int_0^2 (5 - r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{5}{2}r^2 - \frac{1}{3}r^3\sin\theta\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \left(10 - \frac{8}{3}\sin\theta\right) d\theta$$

$$= 10\theta + \frac{8}{3}\cos\theta\Big]_0^{2\pi} = 20\pi$$

27. (a) The wedge can be described as the region

$D = \left\{(x, y, z) \mid y^2 + z^2 \le 1, 0 \le x \le 1, 0 \le y \le x\right\}$

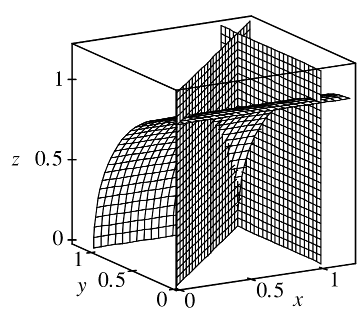
$= \left\{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le \sqrt{1 - y^2}\right\}$

So the integral expressing the volume of the wedge is

$\iiint_D dV = \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx.$

(b) A CAS gives $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx = \frac{\pi}{4} - \frac{1}{3}$.

(Or use Formulas 30 and 87 from the Table of Integrals.)



28. Divide B into 8 cubes of size $\Delta V = 8$. With $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, the Midpoint Rule gives

$$\begin{aligned}\iiint_B \sqrt{x^2+y^2+z^2}\,dV &\approx \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f\left(\overline{x}_i,\overline{y}_j,\overline{z}_k\right)\Delta V \\ &= 8[f(1,1,1)+f(1,1,3)+f(1,3,1)+f(1,3,3)+f(3,1,1) \\ &\qquad + f(3,1,3)+f(3,3,1)+f(3,3,3)] \\ &\approx 239.64\end{aligned}$$

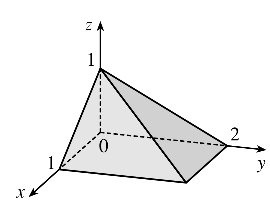
29. Here $f(x,y,z) = \cos(xyz)$ and $\Delta V = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(\overline{x}_i,\overline{y}_j,\overline{z}_k\right)\Delta V \\ &= \tfrac{1}{8}\left[f\left(\tfrac14,\tfrac14,\tfrac14\right)+f\left(\tfrac14,\tfrac14,\tfrac34\right)+f\left(\tfrac14,\tfrac34,\tfrac14\right)+f\left(\tfrac14,\tfrac34,\tfrac34\right)\right. \\ &\qquad \left.+f\left(\tfrac34,\tfrac14,\tfrac14\right)+f\left(\tfrac34,\tfrac14,\tfrac34\right)+f\left(\tfrac34,\tfrac34,\tfrac14\right)+f\left(\tfrac34,\tfrac34,\tfrac34\right)\right] \\ &= \tfrac{1}{8}\left[\cos\tfrac{1}{64}+\cos\tfrac{3}{64}+\cos\tfrac{3}{64}+\cos\tfrac{9}{64}+\cos\tfrac{3}{64}+\cos\tfrac{9}{64}+\cos\tfrac{9}{64}+\cos\tfrac{27}{64}\right] \approx 0.985\end{aligned}$$

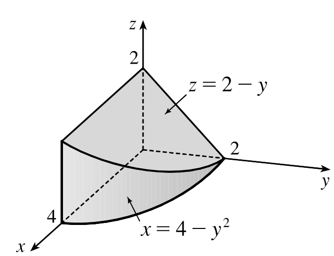
30. Here $f(x,y,z) = \sqrt{x}\,e^{xyz}$ and $\Delta V = 2\cdot\frac{1}{2}\cdot 1 = 1$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(\overline{x}_i,\overline{y}_j,\overline{z}_k\right)\Delta V \\ &= 1\left[f\left(1,\tfrac14,\tfrac12\right)+f\left(1,\tfrac14,\tfrac32\right)+f\left(1,\tfrac34,\tfrac12\right)+f\left(1,\tfrac34,\tfrac32\right)\right. \\ &\qquad \left.+f\left(3,\tfrac14,\tfrac12\right)+f\left(3,\tfrac14,\tfrac32\right)+f\left(3,\tfrac34,\tfrac12\right)+f\left(3,\tfrac34,\tfrac32\right)\right] \\ &= e^{1/8}+e^{3/8}+e^{3/8}+e^{9/8}+\sqrt{3}e^{3/8}+\sqrt{3}e^{9/8}+\sqrt{3}e^{9/8}+\sqrt{3}e^{27/8} \approx 70.932\end{aligned}$$

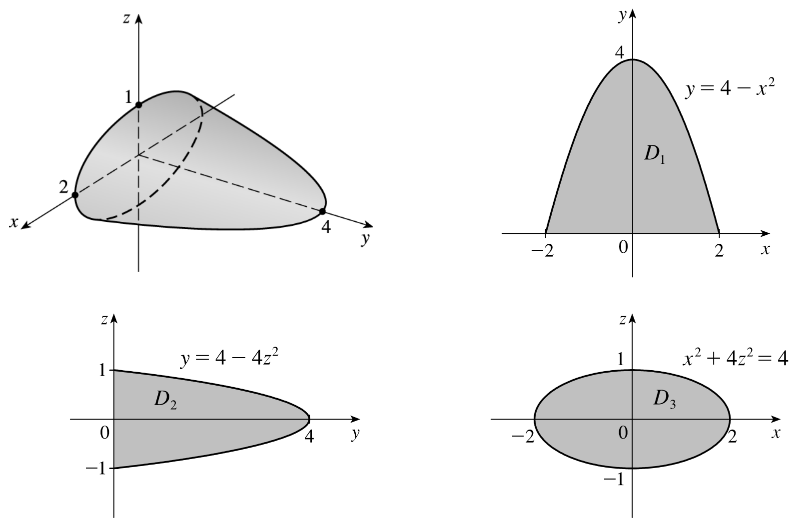
31. $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 1-x, 0 \le y \le 2-2z\}$, the solid bounded by the three coordinate planes and the planes $z = 1-x$, $y = 2-2z$.



32. $E = \left\{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le 2-y, 0 \le x \le 4-y^2\right\}$, the solid bounded by the three coordinate planes, the plane $z = 2-y$, and the cylindrical surface $x = 4-y^2$.



33.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2\} = \{(x,y) \mid 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \sqrt{4-y}\,\}$$

$$D_2 = \{(y,z) \mid 0 \le y \le 4,\ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y}\} = \{(y,z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2\}$$

$$D_3 = \{(x,z) \mid x^2 + 4z^2 \le 4\}$$

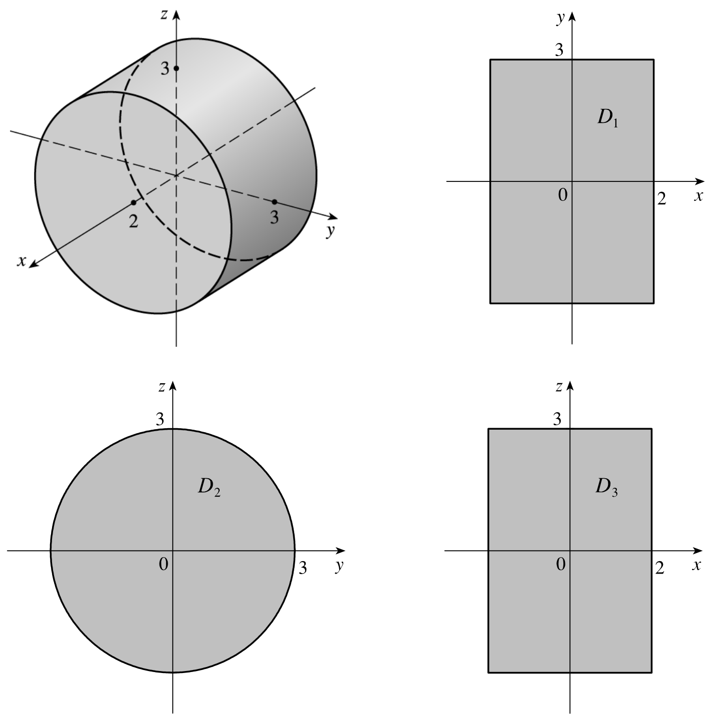
Therefore

$$\begin{aligned} E &= \left\{(x,y,z) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2,\ -\tfrac{1}{2}\sqrt{4-x^2-y} \le z \le \tfrac{1}{2}\sqrt{4-x^2-y}\,\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \sqrt{4-y},\ -\tfrac{1}{2}\sqrt{4-x^2-y} \le z \le \tfrac{1}{2}\sqrt{4-x^2-y}\,\right\} \\ &= \left\{(x,y,z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2,\ -\sqrt{4-y-4z^2} \le x \le \sqrt{4-y-4z^2}\,\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4,\ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y},\ -\sqrt{4-y-4z^2} \le x \le \sqrt{4-y-4z^2}\,\right\} \\ &= \left\{(x,y,z) \mid -2 \le x \le 2,\ -\tfrac{1}{2}\sqrt{4-x^2} \le z \le \tfrac{1}{2}\sqrt{4-x^2}, 0 \le y \le 4 - x^2 - 4z^2\right\} \\ &= \left\{(x,y,z) \mid -1 \le z \le 1,\ -\sqrt{4-4z^2} \le x \le \sqrt{4-4z^2}, 0 \le y \le 4 - x^2 - 4z^2\right\} \end{aligned}$$

Then

$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_0^{4-x^2}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dz\,dy\,dx \\ &= \int_0^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dz\,dx\,dy = \int_{-1}^{1}\int_0^{4-4z^2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dy\,dz \\ &= \int_0^4\int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dz\,dy = \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dz\,dx \\ &= \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

34.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x, y) \mid -2 \le x \le 2,\ -3 \le y \le 3\}$$

$$D_2 = \left\{(y, z) \mid y^2 + z^2 \le 9\right\}$$

$$D_3 = \{(x, z) \mid -2 \le x \le 2,\ -3 \le z \le 3\}$$

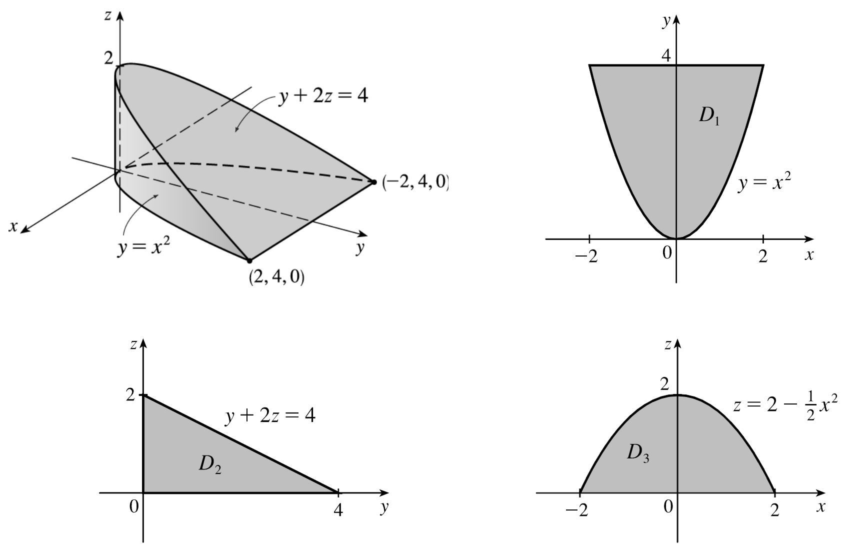
Therefore

$$\begin{aligned}
E &= \left\{(x, y, z) \mid -2 \le x \le 2,\ -3 \le y \le 3,\ -\sqrt{9 - y^2} \le z \le \sqrt{9 - y^2}\right\} \\
&= \left\{(x, y, z) \mid -3 \le y \le 3,\ -\sqrt{9 - y^2} \le z \le \sqrt{9 - y^2},\ -2 \le x \le 2\right\} \\
&= \left\{(x, y, z) \mid -3 \le z \le 3,\ -\sqrt{9 - z^2} \le y \le \sqrt{9 - z^2},\ -2 \le x \le 2\right\} \\
&= \left\{(x, y, z) \mid -2 \le x \le 2,\ -3 \le z \le 3,\ -\sqrt{9 - z^2} \le y \le \sqrt{9 - z^2}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\iiint_E f(x, y, z)\, dV &= \int_{-2}^{2} \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y, z)\, dz\, dy\, dx = \int_{-3}^{3} \int_{-2}^{2} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y, z)\, dz\, dx\, dy \\
&= \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-2}^{2} f(x, y, z)\, dx\, dz\, dy = \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{-2}^{2} f(x, y, z)\, dx\, dy\, dz \\
&= \int_{-2}^{2} \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x, y, z)\, dy\, dz\, dx = \int_{-3}^{3} \int_{-2}^{2} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x, y, z)\, dy\, dx\, dz
\end{aligned}$$

35.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \left\{(x,y) \mid -2 \le x \le 2, x^2 \le y \le 4\right\} = \left\{(x,y) \mid 0 \le y \le 4, -\sqrt{y} \le x \le \sqrt{y}\right\},$$

$$D_2 = \left\{(y,z) \mid 0 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y\right\} = \left\{(y,z) \mid 0 \le z \le 2, 0 \le y \le 4 - 2z\right\}, \text{ and}$$

$$D_3 = \left\{(x,z) \mid -2 \le x \le 2, 0 \le z \le 2 - \tfrac{1}{2}x^2\right\} = \left\{(x,z) \mid 0 \le z \le 2, -\sqrt{4-2z} \le x \le \sqrt{4-2z}\right\}$$

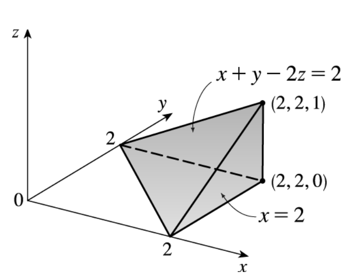
Therefore

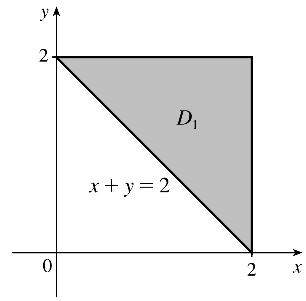
$$\begin{aligned} E &= \left\{(x,y,z) \mid -2 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4,\ -\sqrt{y} \le x \le \sqrt{y}, 0 \le z \le 2 - \tfrac{1}{2}y\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y,\ -\sqrt{y} \le x \le \sqrt{y}\right\} \\ &= \left\{(x,y,z) \mid 0 \le z \le 2, 0 \le y \le 4 - 2z,\ -\sqrt{y} \le x \le \sqrt{y}\right\} \\ &= \left\{(x,y,z) \mid -2 \le x \le 2, 0 \le z \le 2 - \tfrac{1}{2}x^2, x^2 \le y \le 4 - 2z\right\} \\ &= \left\{(x,y,z) \mid 0 \le z \le 2,\ -\sqrt{4-2z} \le x \le \sqrt{4-2z}, x^2 \le y \le 4 - 2z\right\} \end{aligned}$$

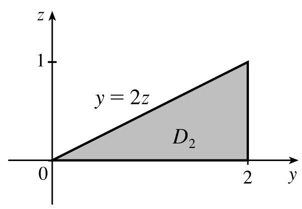
Then

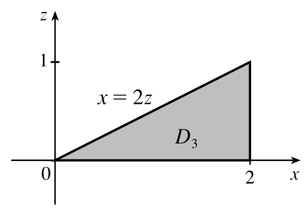
$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_{x^2}^{4}\int_0^{2-y/2} f(x,y,z)\,dz\,dy\,dx = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{2-y/2} f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^4\int_0^{2-y/2}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy = \int_0^2\int_0^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz \\ &= \int_{-2}^{2}\int_0^{2-x^2/2}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dz\,dx = \int_0^2\int_{-\sqrt{4-2z}}^{\sqrt{4-2z}}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

36.









If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid 0 \le x \le 2, 2 - x \le y \le 2\} = \{(x,y) \mid 0 \le y \le 2, 2 - y \le x \le 2\},$$

$$D_2 = \left\{(y,z) \mid 0 \le y \le 2, 0 \le z \le \tfrac{1}{2}y\right\} = \{(y,z) \mid 0 \le z \le 1, 2z \le y \le 2\}, \text{ and}$$

$$D_3 = \left\{(x,z) \mid 0 \le x \le 2, 0 \le z \le \tfrac{1}{2}x\right\} = \{(x,z) \mid 0 \le z \le 1, 2z \le x \le 2\}$$

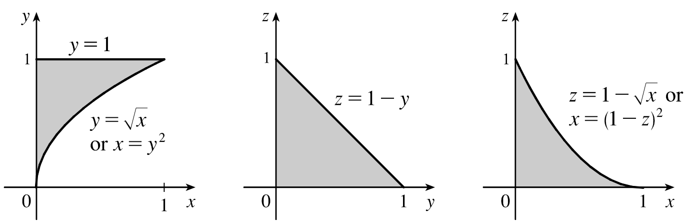
Therefore

$$\begin{aligned} E &= \left\{(x,y,z) \mid 0 \le x \le 2, 2 - x \le y \le 2, 0 \le z \le \tfrac{1}{2}(x + y - 2)\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 2, 2 - y \le x \le 2, 0 \le z \le \tfrac{1}{2}(x + y - 2)\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le \tfrac{1}{2}y, 2 - y + 2z \le x \le 2\right\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, 2z \le y \le 2, 2 - y + 2z \le x \le 2\} \\ &= \left\{(x,y,z) \mid 0 \le x \le 2, 0 \le z \le \tfrac{1}{2}x, 2 - x + 2z \le y \le 2\right\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, 2z \le x \le 2, 2 - x + 2z \le y \le 2\} \end{aligned}$$

Then

$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \int_0^2 \int_{2-x}^2 \int_0^{(x+y-2)/2} f(x,y,z)\,dz\,dy\,dx \\ &= \int_0^2 \int_{2-y}^2 \int_0^{(x+y-2)/2} f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^2 \int_0^{y/2} \int_{2-y+2z}^2 f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^1 \int_{2z}^2 \int_{2-y+2z}^2 f(x,y,z)\,dx\,dy\,dz \\ &= \int_0^2 \int_0^{x/2} \int_{2-x+2z}^2 f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1 \int_{2z}^2 \int_{2-x+2z}^2 f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

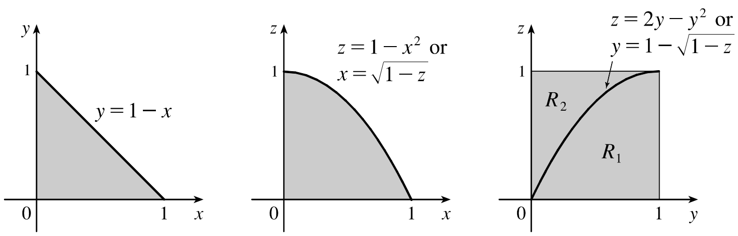
37.



The diagrams show the projections of E onto the xy-, yz-, and xz-planes. Therefore

$$\begin{aligned}\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx &= \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz \\ &= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy = \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

38.



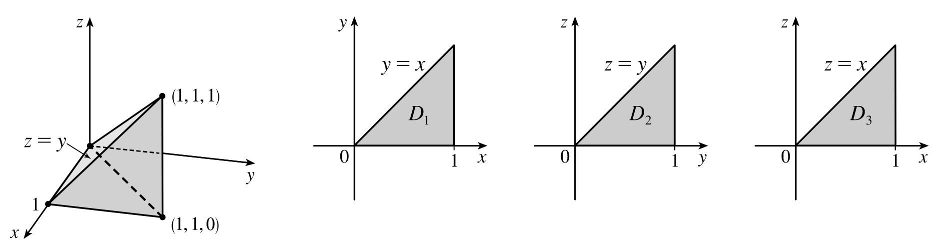
The projections of E onto the xy- and xz-planes are as in the first two diagrams and so

$$\begin{aligned}\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx &= \int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz \\ &= \int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx\end{aligned}$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1 - y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz-plane into two regions as in the third diagram. For (y, z) in R_1, $0 \le x \le 1 - y$ and for (y, z) in R_2, $0 \le x \le \sqrt{1 - z}$, and so the given integral is also equal to

$$\begin{aligned}&\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1\int_{1-\sqrt{1-z}}^1\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz \\ &\quad = \int_0^1\int_0^{2y-y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy + \int_0^1\int_{2y-y^2}^1\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy.\end{aligned}$$

39.



$\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le z \le y, y \le x \le 1, 0 \le y \le 1\}$.

If D_1, D_2, and D_3 are the projections of E onto the xy-, yz- and xz-planes then

$D_1 = \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\}$,

$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}$, and

$$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x\} = \{(x,z) \mid 0 \le z \le 1, z \le x \le 1\}.$$

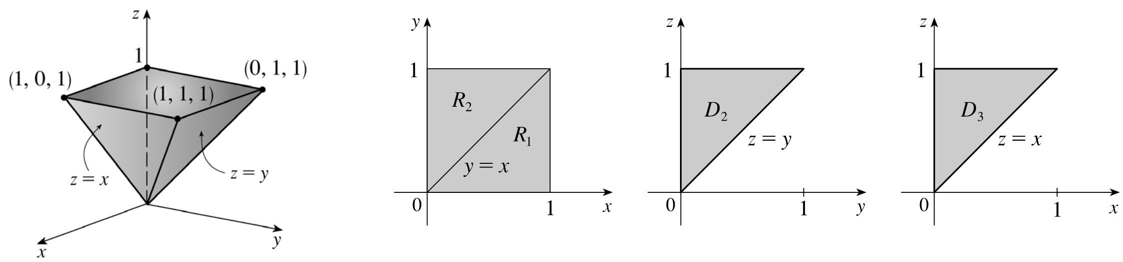
Thus we also have

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le y\} = \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, y \le x \le 1\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, y \le x \le 1\} = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le x, z \le y \le x\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le x \le 1, z \le y \le x\}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy &= \int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1 \int_z^1 \int_z^x f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

40.



$\int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dx\,dz\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le x \le z, y \le z \le 1, 0 \le y \le 1\}$.

Notice that E is bounded below by two different surfaces, so we must split the projection of E onto the xy-plane into two regions as in the second diagram. If D_1, D_2, and D_3 are the projections of E on the xy-, yz- and xz-planes then

$$\begin{aligned} D_1 = R_1 \cup R_2 &= \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\} \cup \{(x,y) \mid 0 \le x \le 1, x \le y \le 1\} \\ &= \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} \cup \{(x,y) \mid 0 \le y \le 1, 0 \le x \le y\}, \end{aligned}$$

$D_2 = \{(y,z) \mid 0 \le y \le 1, y \le z \le 1\} = \{(y,z) \mid 0 \le z \le 1, 0 \le y \le z\}$, and

$D_3 = \{(x,z) \mid 0 \le x \le 1, x \le z \le 1\} = \{(x,z) \mid 0 \le z \le 1, 0 \le x \le z\}$.

Thus we also have

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, x \le z \le 1\} \cup \{(x,y,z) \mid 0 \le x \le 1, x \le y \le 1, y \le z \le 1\} \\ &= \{(x,y,z) \mid 0 \le y \le 1, y \le x \le 1, x \le z \le 1\} \cup \{(x,y,z) \mid 0 \le y \le 1, 0 \le x \le y, y \le z \le 1\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, 0 \le y \le z, 0 \le x \le z\} = \{(x,y,z) \mid 0 \le x \le 1, x \le z \le 1, 0 \le y \le z\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, 0 \le x \le z, 0 \le y \le z\}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dx\,dz\,dy &= \int_0^1 \int_0^x \int_x^1 f(x,y,z)\,dz\,dy\,dx + \int_0^1 \int_x^1 \int_y^1 f(x,y,z)\,dz\,dy\,dx \\ &= \int_0^1 \int_y^1 \int_x^1 f(x,y,z)\,dz\,dx\,dy + \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^1 \int_0^z \int_0^z f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_x^1 \int_0^z f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1 \int_0^z \int_0^z f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

41. The region C is the solid bounded by a circular cylinder of radius 2 with axis the z-axis for $-2 \le z \le 2$. We can write $\iiint_C (4 + 5x^2yz^2)\,dV = \iiint_C 4\,dV + \iiint_C 5x^2yz^2\,dV$, but $f(x,y,z) = 5x^2yz^2$ is an odd function with respect to y. Since C is symmetrical about the xz-plane, we have $\iiint_C 5x^2yz^2\,dV = 0$. Thus $\iiint_C (4 + 5x^2yz^2)\,dV = \iiint_C 4\,dV = 4 \cdot V(E) = 4 \cdot \pi(2)^2(4) = 64\pi$.

42. We can write $\iiint_B (z^3 + \sin y + 3)\,dV = \iiint_B z^3\,dV + \iiint_B \sin y\,dV + \iiint_B 3\,dV$. But z^3 is an odd function with respect to z and the region B is symmetric about the xy-plane, so $\iiint_B z^3\,dV = 0$. Similarly, $\sin y$ is an odd function with respect to y and B is symmetric about the xz-plane, so $\iiint_B \sin y\,dV = 0$. Thus $\iiint_B (z^3 + \sin y + 3)\,dV = \iiint_B 3\,dV = 3 \cdot V(B) = 3 \cdot \frac{4}{3}\pi(1)^3 = 4\pi$.

43. The projection of E onto the xy-plane is the disk $D = \{(x,y) \mid x^2 + y^2 \le 1\}$.

$$\begin{aligned} m &= \iiint_E \rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3\,dz\right] dA = \iint_D 3(1 - x^2 - y^2)\,dA \\ &= 3\int_0^1 \int_0^{2\pi} (1 - r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} d\theta \int_0^1 (r - r^3)\,dr \\ &= 3\left[\theta\right]_0^{2\pi} \left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 = 3\,(2\pi)\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{3}{2}\pi \end{aligned}$$

$$\begin{aligned} M_{yz} &= \iiint_E x\rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3x\,dz\right] dA = \iint_D 3x(1 - x^2 - y^2)\,dA \\ &= 3\int_0^1 \int_0^{2\pi} (r\cos\theta)(1 - r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} \cos\theta\,d\theta \int_0^1 (r^2 - r^4)\,dr \\ &= 3\left[\sin\theta\right]_0^{2\pi} \left[\tfrac{1}{3}r^3 - \tfrac{1}{5}r^5\right]_0^1 = 3\,(0)\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 0 \end{aligned}$$

$$\begin{aligned} M_{xz} &= \iiint_E y\rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3y\,dz\right] dA = \iint_D 3y(1 - x^2 - y^2)\,dA \\ &= 3\int_0^1 \int_0^{2\pi} (r\sin\theta)(1 - r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} \sin\theta\,d\theta \int_0^1 (r^2 - r^4)\,dr \\ &= 3\left[-\cos\theta\right]_0^{2\pi} \left[\tfrac{1}{3}r^3 - \tfrac{1}{5}r^5\right]_0^1 = 3\,(0)\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 0 \end{aligned}$$

$$\begin{aligned} M_{xy} &= \iiint_E z\rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3z\,dz\right] dA = \iint_D \left[\tfrac{3}{2}z^2\right]_{z=0}^{z=1-x^2-y^2} dA \\ &= \tfrac{3}{2}\iint_D (1 - x^2 - y^2)^2\,dA = \tfrac{3}{2}\int_0^1 \int_0^{2\pi} (1 - r^2)^2\,r\,dr\,d\theta \\ &= \tfrac{3}{2}\int_0^{2\pi} d\theta \int_0^1 (r - 2r^3 + r^5)\,dr = \tfrac{3}{2}\left[\theta\right]_0^{2\pi} \left[\tfrac{1}{2}r^2 - \tfrac{1}{2}r^4 + \tfrac{1}{6}r^6\right]_0^1 \\ &= \tfrac{3}{2}\,(2\pi)\left(\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6}\right) = \tfrac{1}{2}\pi \end{aligned}$$

Thus the mass is $\frac{3}{2}\pi$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(0, 0, \dfrac{1}{3}\right)$.

44. $m = \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4\,dx\,dz\,dy = 4\int_{-1}^1 \int_0^{1-y^2} (1 - z)\,dz\,dy = 4\int_{-1}^1 \left[z - \frac{1}{2}z^2\right]_{z=0}^{z=1-y^2} dy = 2\int_{-1}^1 (1 - y^4)\,dy = \frac{16}{5}$,

$$\begin{aligned} M_{yz} &= \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4x\,dx\,dz\,dy = 2\int_{-1}^1 \int_0^{1-y^2} (1 - z)^2\,dz\,dy = 2\int_{-1}^1 \left[-\tfrac{1}{3}(1 - z)^3\right]_{z=0}^{z=1-y^2} dy \\ &= \tfrac{2}{3}\int_{-1}^1 (1 - y^6)\,dy = \left(\tfrac{4}{3}\right)\left(\tfrac{6}{7}\right) = \tfrac{24}{21} \end{aligned}$$

[continued]

$$M_{xz} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} 4y\,dx\,dz\,dy = \int_{-1}^{1}\int_{0}^{1-y^2} 4y(1-z)\,dz\,dy$$

$$= \int_{-1}^{1}\left[4y(1-y^2) - 2y(1-y^2)^2\right]dy = \int_{-1}^{1}(2y-2y^5)\,dy = 0 \quad \text{[the integrand is odd]}$$

$$M_{xy} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} 4z\,dx\,dz\,dy = \int_{-1}^{1}\int_{0}^{1-y^2}(4z-4z^2)\,dz\,dy = 2\int_{-1}^{1}\left[(1-y^2)^2 - \tfrac{2}{3}(1-y^2)^3\right]dy$$

$$= 2\int_{-1}^{1}\left[\tfrac{1}{3} - y^4 + \tfrac{2}{3}y^6\right]dy = \left[\tfrac{4}{3}y - \tfrac{4}{5}y^5 + \tfrac{8}{21}y^7\right]_0^1 = \tfrac{96}{105} = \tfrac{32}{35}$$

Thus, $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{5}{14}, 0, \frac{2}{7}\right)$

45. $m = \int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^a\int_0^a\left[\frac{1}{3}x^3 + xy^2 + xz^2\right]_{x=0}^{x=a} dy\,dz = \int_0^a\int_0^a\left(\frac{1}{3}a^3 + ay^2 + az^2\right)dy\,dz$

$$= \int_0^a\left[\tfrac{1}{3}a^3y + \tfrac{1}{3}ay^3 + ayz^2\right]_{y=0}^{y=a} dz = \int_0^a\left(\tfrac{2}{3}a^4 + a^2z^2\right)dz = \left[\tfrac{2}{3}a^4z + \tfrac{1}{3}a^2z^3\right]_0^a = \tfrac{2}{3}a^5 + \tfrac{1}{3}a^5 = a^5$$

$$M_{yz} = \int_0^a\int_0^a\int_0^a\left[x^3 + x(y^2+z^2)\right]dx\,dy\,dz = \int_0^a\int_0^a\left[\tfrac{1}{4}a^4 + \tfrac{1}{2}a^2(y^2+z^2)\right]dy\,dz$$

$$= \int_0^a\left(\tfrac{1}{4}a^5 + \tfrac{1}{6}a^5 + \tfrac{1}{2}a^3z^2\right)dz = \tfrac{1}{4}a^6 + \tfrac{1}{3}a^6 = \tfrac{7}{12}a^6 = M_{xz} = M_{xy} \text{ by symmetry of } E \text{ and } \rho(x,y,z)$$

Hence, $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a\right)$.

46. $m = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y - y^2\right]dy\,dx$

$$= \int_0^1\left[\tfrac{1}{2}(1-x)^3 - \tfrac{1}{3}(1-x)^3\right]dx = \tfrac{1}{6}\int_0^1(1-x)^3\,dx = \tfrac{1}{24}$$

$$M_{yz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(x-x^2)y - xy^2\right]dy\,dx$$

$$= \int_0^1\left[\tfrac{1}{2}x(1-x)^3 - \tfrac{1}{3}x(1-x)^3\right]dx = \tfrac{1}{6}\int_0^1\left(x - 3x^2 + 3x^3 - x^4\right)dx = \tfrac{1}{6}\left(\tfrac{1}{2} - 1 + \tfrac{3}{4} - \tfrac{1}{5}\right) = \tfrac{1}{120}$$

$$M_{xz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y^2 - y^3\right]dy\,dx$$

$$= \int_0^1\left[\tfrac{1}{3}(1-x)^4 - \tfrac{1}{4}(1-x)^4\right]dx = \tfrac{1}{12}\left[-\tfrac{1}{5}(1-x)^5\right]_0^1 = \tfrac{1}{60}$$

$$M_{xy} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} yz\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\tfrac{1}{2}y(1-x-y)^2\right]dy\,dx$$

$$= \tfrac{1}{2}\int_0^1\int_0^{1-x}\left[(1-x)^2y - 2(1-x)y^2 + y^3\right]dy\,dx = \tfrac{1}{2}\int_0^1\left[\tfrac{1}{2}(1-x)^4 - \tfrac{2}{3}(1-x)^4 + \tfrac{1}{4}(1-x)^4\right]dx$$

$$= \tfrac{1}{24}\int_0^1(1-x)^4\,dx = -\tfrac{1}{24}\left[\tfrac{1}{5}(1-x)^5\right]_0^1 = \tfrac{1}{120}$$

Hence, $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$.

47. $I_x = \int_0^L\int_0^L\int_0^L k(y^2+z^2)\,dz\,dy\,dx = k\int_0^L\int_0^L\left(Ly^2 + \frac{1}{3}L^3\right)dy\,dx = k\int_0^L \frac{2}{3}L^4\,dx = \frac{2}{3}kL^5$

By symmetry, $I_x = I_y = I_z = \frac{2}{3}kL^5$.

48. $I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} k(y^2+z^2)\,dx\,dy\,dz = ka\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}(y^2+z^2)\,dy\,dz$

$$= ak\int_{-c/2}^{c/2}\left[\tfrac{1}{3}y^3 + z^2y\right]_{y=-b/2}^{y=b/2} dz = ak\int_{-c/2}^{c/2}\left(\tfrac{1}{12}b^3 + bz^2\right)dz = ak\left[\tfrac{1}{12}b^3z + \tfrac{1}{3}bz^3\right]_{-c/2}^{c/2}$$

$$= ak\left(\tfrac{1}{12}b^3c + \tfrac{1}{12}bc^3\right) = \tfrac{1}{12}kabc(b^2+c^2)$$

By symmetry, $I_y = \frac{1}{12}kabc(a^2+c^2)$ and $I_z = \frac{1}{12}kabc(a^2+b^2)$.

49. $I_z = \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV = \iint\limits_{x^2+y^2\le a^2} \left[\int_0^h k(x^2+y^2)\,dz\right] dA = \iint\limits_{x^2+y^2\le a^2} k(x^2+y^2)h\,dA$

$= kh\int_0^{2\pi}\int_0^a (r^2)\,r\,dr\,d\theta = kh\int_0^{2\pi} d\theta \int_0^a r^3\,dr = kh(2\pi)\left[\frac14 r^4\right]_0^a = 2\pi kh\cdot\frac14 a^4 = \frac12\pi kha^4$

50. $I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iint\limits_{x^2+y^2\le h^2}\left[\int_{\sqrt{x^2+y^2}}^h k(x^2+y^2)\,dz\right] dA$

$= \iint\limits_{x^2+y^2\le h^2} k(x^2+y^2)\left(h-\sqrt{x^2+y^2}\right) dA = k\int_0^{2\pi}\int_0^h r^2(h-r)\,r\,dr\,d\theta$

$= k\int_0^{2\pi} d\theta\int_0^h \left(r^3h - r^4\right) dr = k(2\pi)\left[\frac14 r^4 h - \frac15 r^5\right]_0^h = 2\pi k\left(\frac14 h^5 - \frac15 h^5\right) = \frac1{10}\pi kh^5$

51. (a) $m = \int_{-1}^1\int_{x^2}^1\int_0^{1-y}\sqrt{x^2+y^2}\,dz\,dy\,dx$

(b) $(\overline{x},\overline{y},\overline{z})$ where $\overline{x} = \frac1m\int_{-1}^1\int_{x^2}^1\int_0^{1-y} x\sqrt{x^2+y^2}\,dz\,dy\,dx$, $\overline{y} = \frac1m\int_{-1}^1\int_{x^2}^1\int_0^{1-y} y\sqrt{x^2+y^2}\,dz\,dy\,dx$, and

$\overline{z} = \frac1m\int_{-1}^1\int_{x^2}^1\int_0^{1-y} z\sqrt{x^2+y^2}\,dz\,dy\,dx$.

(c) $I_z = \int_{-1}^1\int_{x^2}^1\int_0^{1-y}(x^2+y^2)\sqrt{x^2+y^2}\,dz\,dy\,dx = \int_{-1}^1\int_{x^2}^1\int_0^{1-y}(x^2+y^2)^{3/2}\,dz\,dy\,dx$

52. (a) $m = \int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

(b) $(\overline{x},\overline{y},\overline{z})$ where $\overline{x} = m^{-1}\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} x\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$,

$\overline{y} = m^{-1}\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$,

$\overline{z} = m^{-1}\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

(c) $I_z = \int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}}(x^2+y^2)(1+x+y+z)\,dz\,dx\,dy$

53. (a) $m = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y(1+x+y+z)\,dz\,dy\,dx = \frac{3\pi}{32}+\frac{11}{24}$

(b) $(\overline{x},\overline{y},\overline{z}) = \Big(m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y x(1+x+y+z)\,dz\,dy\,dx,$

$m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y y(1+x+y+z)\,dz\,dy\,dx,$

$m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y z(1+x+y+z)\,dz\,dy\,dx\Big)$

$$= \left(\frac{28}{9\pi+44}, \frac{30\pi+128}{45\pi+220}, \frac{45\pi+208}{135\pi+660}\right)$$

(c) $I_z = \displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y (x^2+y^2)(1+x+y+z)\,dz\,dy\,dx = \frac{68+15\pi}{240}$

54. (a) $m = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2 + y^2)\, dz\, dy\, dx = \frac{56}{5} = 11.2$

(b) $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} x(x^2 + y^2)\, dz\, dy\, dx \approx 0.375$,

$\overline{y} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} y(x^2 + y^2)\, dz\, dy\, dx = \frac{45\pi}{64} \approx 2.209$,

$\overline{z} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z(x^2 + y^2)\, dz\, dy\, dx = \frac{15}{16} = 0.9375$.

(c) $I_z = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2 + y^2)^2\, dz\, dy\, dx = \frac{10{,}464}{175} \approx 59.79$

55. (a) $f(x, y, z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x, y, z)\, dV = 1$. Here we have

$$\begin{aligned} \iiint_{\mathbb{R}^3} f(x, y, z)\, dV &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = \int_0^2 \int_0^2 \int_0^2 Cxyz\, dz\, dy\, dx \\ &= C \int_0^2 x\, dx \int_0^2 y\, dy \int_0^2 z\, dz = C\left[\tfrac{1}{2}x^2\right]_0^2 \left[\tfrac{1}{2}y^2\right]_0^2 \left[\tfrac{1}{2}z^2\right]_0^2 = 8C \end{aligned}$$

Then we must have $8C = 1 \quad \Rightarrow \quad C = \frac{1}{8}$.

(b) $P(X \le 1, Y \le 1, Z \le 1) = \int_{-\infty}^1 \int_{-\infty}^1 \int_{-\infty}^1 f(x, y, z)\, dz\, dy\, dx = \int_0^1 \int_0^1 \int_0^1 \frac{1}{8}xyz\, dz\, dy\, dx$

$$= \tfrac{1}{8} \int_0^1 x\, dx \int_0^1 y\, dy \int_0^1 z\, dz = \tfrac{1}{8}\left[\tfrac{1}{2}x^2\right]_0^1 \left[\tfrac{1}{2}y^2\right]_0^1 \left[\tfrac{1}{2}z^2\right]_0^1 = \tfrac{1}{8}\left(\tfrac{1}{2}\right)^3 = \tfrac{1}{64}$$

(c) $P(X + Y + Z \le 1) = P((X, Y, Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x + y + z = 1$. The plane $x + y + z = 1$ meets the xy-plane in the line $x + y = 1$, so we have

$$\begin{aligned} P(X + Y + Z \le 1) &= \iiint_E f(x, y, z)\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \tfrac{1}{8}xyz\, dz\, dy\, dx \\ &= \tfrac{1}{8} \int_0^1 \int_0^{1-x} xy\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=1-x-y} dy\, dx = \tfrac{1}{16} \int_0^1 \int_0^{1-x} xy(1 - x - y)^2\, dy\, dx \\ &= \tfrac{1}{16} \int_0^1 \int_0^{1-x} [(x^3 - 2x^2 + x)y + (2x^2 - 2x)y^2 + xy^3]\, dy\, dx \\ &= \tfrac{1}{16} \int_0^1 \left[(x^3 - 2x^2 + x)\tfrac{1}{2}y^2 + (2x^2 - 2x)\tfrac{1}{3}y^3 + x\left(\tfrac{1}{4}y^4\right)\right]_{y=0}^{y=1-x} dx \\ &= \tfrac{1}{192} \int_0^1 (x - 4x^2 + 6x^3 - 4x^4 + x^5)\, dx = \tfrac{1}{192}\left(\tfrac{1}{30}\right) = \tfrac{1}{5760} \end{aligned}$$

56. (a) $f(x, y, z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x, y, z)\, dV = 1$. Here we have

$$\begin{aligned} \iiint_{\mathbb{R}^3} f(x, y, z)\, dV &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} Ce^{-(0.5x + 0.2y + 0.1z)}\, dz\, dy\, dx \\ &= C \int_0^{\infty} e^{-0.5x}\, dx \int_0^{\infty} e^{-0.2y}\, dy \int_0^{\infty} e^{-0.1z}\, dz \\ &= C \lim_{t \to \infty} \int_0^t e^{-0.5x}\, dx \lim_{t \to \infty} \int_0^t e^{-0.2y}\, dy \lim_{t \to \infty} \int_0^t e^{-0.1z}\, dz \\ &= C \lim_{t \to \infty} \left[-2e^{-0.5x}\right]_0^t \lim_{t \to \infty} \left[-5e^{-0.2y}\right]_0^t \lim_{t \to \infty} \left[-10e^{-0.1z}\right]_0^t \\ &= C \lim_{t \to \infty} \left[-2(e^{-0.5t} - 1)\right] \lim_{t \to \infty} \left[-5(e^{-0.2t} - 1)\right] \lim_{t \to \infty} \left[-10(e^{-0.1t} - 1)\right] \\ &= C \cdot (-2)(0 - 1) \cdot (-5)(0 - 1) \cdot (-10)(0 - 1) = 100C \end{aligned}$$

So we must have $100C = 1 \quad \Rightarrow \quad C = \frac{1}{100}$.

(b) We have no restriction on Z, so

$$\begin{aligned} P(X \le 1, Y \le 1) &= \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^\infty \tfrac{1}{100}e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\ &= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^\infty e^{-0.1z}\,dz \\ &= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1\left[-5e^{-0.2y}\right]_0^1 \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \qquad \text{[by part (a)]} \\ &= \tfrac{1}{100}(2-2e^{-0.5})(5-5e^{-0.2})(10) = (1-e^{-0.5})(1-e^{-0.2}) \approx 0.07132 \end{aligned}$$

(c) $P(X \le 1, Y \le 1, Z \le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^1 \frac{1}{100}e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx$

$$\begin{aligned} &= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^1 e^{-0.1z}\,dz \\ &= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1\left[-5e^{-0.2y}\right]_0^1\left[-10e^{-0.1z}\right]_0^1 \\ &= (1-e^{-0.5})(1-e^{-0.2})(1-e^{-0.1}) \approx 0.006787 \end{aligned}$$

57. $V(E) = L^3 \quad \Rightarrow \quad f_{\text{avg}} = \dfrac{1}{L^3}\displaystyle\int_0^L\int_0^L\int_0^L xyz\,dx\,dy\,dz = \dfrac{1}{L^3}\int_0^L x\,dx\int_0^L y\,dy\int_0^L z\,dz$

$$= \frac{1}{L^3}\left[\frac{x^2}{2}\right]_0^L\left[\frac{y^2}{2}\right]_0^L\left[\frac{z^2}{2}\right]_0^L = \frac{1}{L^3}\,\frac{L^2}{2}\,\frac{L^2}{2}\,\frac{L^2}{2} = \frac{L^3}{8}$$

58. The height of each point is given by its z-coordinate, so the average height of the points in $E = \{(x,y,z) \mid x^2+y^2+z^2 \le 1,\ z \ge 0\}$ is

$$\frac{1}{V(E)}\iiint_E z\,dV$$

Here $V(E) = \frac{1}{2}\cdot\frac{4}{3}\pi(1)^3 = \frac{2}{3}\pi$ [half the volume of a sphere], so

$$\begin{aligned} \tfrac{1}{V(E)}\iiint_E z\,dV &= \tfrac{1}{2\pi/3}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} z\,dz\,dy\,dx = \tfrac{3}{2\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=\sqrt{1-x^2-y^2}}\,dy\,dx \\ &= \tfrac{3}{2\pi}\cdot\tfrac{1}{2}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx = \tfrac{3}{4\pi}\int_0^{2\pi}\int_0^1(1-r^2)\,r\,dr\,d\theta \\ &= \tfrac{3}{4\pi}\int_0^{2\pi}d\theta\int_0^1(r-r^3)\,dr = \tfrac{3}{4\pi}(2\pi)\left[\tfrac{1}{2}r^2-\tfrac{1}{4}r^4\right]_0^1 = \tfrac{3}{2}\left(\tfrac{1}{4}\right) = \tfrac{3}{8} \end{aligned}$$

59. (a) The triple integral will attain its maximum when the integrand $1-x^2-2y^2-3z^2$ is positive in the region E and negative everywhere else. For if E contains some region F where the integrand is negative, the integral could be increased by excluding F from E, and if E fails to contain some part G of the region where the integrand is positive, the integral could be increased by including G in E. So we require that $x^2+2y^2+3z^2 \le 1$. This describes the region bounded by the ellipsoid $x^2+2y^2+3z^2=1$.

(b) The maximum value of $\iiint_E(1-x^2-2y^2-3z^2)\,dV$ occurs when E is the solid region bounded by the ellipsoid $x^2+2y^2+3z^2=1$. The projection of E on the xy-plane is the planar region bounded by the ellipse $x^2+2y^2=1$, so

$$E = \left\{(x,y,z) \mid -1 \le x \le 1, -\sqrt{\tfrac{1}{2}(1-x^2)} \le y \le \sqrt{\tfrac{1}{2}(1-x^2)}, -\sqrt{\tfrac{1}{3}(1-x^2-2y^2)} \le z \le \sqrt{\tfrac{1}{3}(1-x^2-2y^2)}\right\}$$

and

$$\iiint_E (1 - x^2 - 2y^2 - 3z^2)\,dV = \int_{-1}^{1} \int_{-\sqrt{\frac{1}{2}(1-x^2)}}^{\sqrt{\frac{1}{2}(1-x^2)}} \int_{-\sqrt{\frac{1}{3}(1-x^2-2y^2)}}^{\sqrt{\frac{1}{3}(1-x^2-2y^2)}} (1 - x^2 - 2y^2 - 3z^2)\,dz\,dy\,dx = \frac{4\sqrt{6}}{45}\,\pi$$

using a CAS.

DISCOVERY PROJECT Volumes of Hyperspheres

In this project we use V_n to denote the n-dimensional volume of an n-dimensional hypersphere.

1. The interior of the circle is the set of points $\left\{(x, y) \mid -r \le y \le r,\ -\sqrt{r^2 - y^2} \le x \le \sqrt{r^2 - y^2}\right\}$. So, substituting $y = r\sin\theta$ and then using Formula 64 from the Table of Integrals to evaluate the integral, we get

$$\begin{aligned} V_2(r) &= \int_{-r}^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx\,dy = \int_{-r}^{r} 2\sqrt{r^2 - y^2}\,dy = \int_{-\pi/2}^{\pi/2} 2r\sqrt{1 - \sin^2\theta}\ (r\cos\theta\,d\theta) \\ &= 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = 2r^2\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2r^2\left(\tfrac{\pi}{2}\right) = \pi r^2 \end{aligned}$$

2. The region of integration is

$$\left\{(x, y, z) \mid -r \le z \le r, -\sqrt{r^2 - z^2} \le y \le \sqrt{r^2 - z^2}, -\sqrt{r^2 - z^2 - y^2} \le x \le \sqrt{r^2 - z^2 - y^2}\right\}.$$

Substituting $y = \sqrt{r^2 - z^2}\sin\theta$ and using Formula 64 to integrate $\cos^2\theta$, we get

$$\begin{aligned} V_3(r) &= \int_{-r}^{r} \int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} \int_{-\sqrt{r^2-z^2-y^2}}^{\sqrt{r^2-z^2-y^2}} dx\,dy\,dz = \int_{-r}^{r} \int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} 2\sqrt{r^2 - z^2 - y^2}\,dy\,dz \\ &= \int_{-r}^{r} \int_{-\pi/2}^{\pi/2} 2\sqrt{r^2 - z^2}\sqrt{1 - \sin^2\theta}\left(\sqrt{r^2 - z^2}\cos\theta\,d\theta\right) dz \\ &= 2\left[\int_{-r}^{r} (r^2 - z^2)\,dz\right]\left[\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta\right] = 2\left(\frac{4r^3}{3}\right)\left(\frac{\pi}{2}\right) = \frac{4\pi r^3}{3} \end{aligned}$$

3. The formula for 4-dimensional hypersphere is $x^2 + y^2 + z^2 + w^2 = r^2$. Here we substitute $y = \sqrt{r^2 - w^2 - z^2}\sin\theta$ and, later, $w = r\sin\phi$. Because $\int_{-\pi/2}^{\pi/2} \cos^p\theta\,d\theta$ seems to occur frequently in these calculations, it is useful to find a general formula for that integral. From Exercises 7.1.55–56, we have

$$\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k}\,\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k+1)}$$

and from the symmetry of the sine and cosine functions, we can conclude that

$$\int_{-\pi/2}^{\pi/2} \cos^{2k} x\,dx = 2\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)\pi}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k} \tag{1}$$

$$\int_{-\pi/2}^{\pi/2} \cos^{2k+1} x\,dx = 2\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2 \cdot 2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k+1)} \tag{2}$$

[continued]

Thus
$$V_4(r)=\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}}\int_{-\sqrt{r^2-w^2-z^2-y^2}}^{\sqrt{r^2-w^2-z^2-y^2}}dx\,dy\,dz\,dw$$
$$=2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}}\sqrt{r^2-w^2-z^2-y^2}\,dy\,dz\,dw$$
$$=2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\pi/2}^{\pi/2}(r^2-w^2-z^2)\cos^2\theta\,d\theta\,dz\,dw\qquad\left[\begin{array}{l}y=\sqrt{r^2-w^2-z^2}\,\sin\theta,\\ dy=\sqrt{r^2-w^2-z^2}\,\cos\theta\,d\theta\end{array}\right]$$
$$=2\left[\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}(r^2-w^2-z^2)\,dz\,dw\right]\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right]$$
$$=2\left(\tfrac{\pi}{2}\right)\left[\int_{-r}^{r}\tfrac{4}{3}(r^2-w^2)^{3/2}\,dw\right]=\pi\left(\tfrac{4}{3}\right)\int_{-\pi/2}^{\pi/2}r^4\cos^4\phi\,d\phi\qquad\left[\begin{array}{l}w=r\sin\phi,\\ dw=r\cos\phi\,d\phi\end{array}\right]$$
$$=\frac{4\pi}{3}r^4\cdot\frac{1\cdot 3\cdot\pi}{2\cdot 4}=\frac{\pi^2r^4}{2}$$

4. By using the substitutions $x_i=\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_{i+1}^2}\cos\theta_i$ and then applying Formulas 1 and 2 from Problem 3, we can write

$$V_n(r)=\int_{-r}^{r}\int_{-\sqrt{r^2-x_n^2}}^{\sqrt{r^2-x_n^2}}\cdots\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}}dx_1\,dx_2\cdots dx_{n-1}\,dx_n$$
$$=2\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta_2\,d\theta_2\right]\left[\int_{-\pi/2}^{\pi/2}\cos^3\theta_3\,d\theta_3\right]\cdots\left[\int_{-\pi/2}^{\pi/2}\cos^{n-1}\theta_{n-1}\,d\theta_{n-1}\right]\left[\int_{-\pi/2}^{\pi/2}\cos^n\theta_n\,d\theta_n\right]r^n$$
$$=\begin{cases}\left[2\cdot\dfrac{\pi}{2}\right]\left[\dfrac{2\cdot 2}{1\cdot 3}\cdot\dfrac{1\cdot 3\pi}{2\cdot 4}\right]\left[\dfrac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\cdot\dfrac{1\cdot 3\cdot 5\pi}{2\cdot 4\cdot 6}\right]\cdots\left[\dfrac{2\cdot\cdots\cdot(n-2)}{1\cdot\cdots\cdot(n-1)}\cdot\dfrac{1\cdot\cdots\cdot(n-1)\,\pi}{2\cdot\cdots\cdot n}\right]r^n & n\text{ even}\\[2ex] 2\left[\dfrac{\pi}{2}\cdot\dfrac{2\cdot 2}{1\cdot 3}\right]\left[\dfrac{1\cdot 3\pi}{2\cdot 4}\cdot\dfrac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\right]\cdots\left[\dfrac{1\cdot\cdots\cdot(n-2)\,\pi}{2\cdot\cdots\cdot(n-1)}\cdot\dfrac{2\cdot\cdots\cdot(n-1)}{1\cdot\cdots\cdot n}\right]r^n & n\text{ odd}\end{cases}$$

By canceling within each set of brackets, we find that

$$V_n(r)=\begin{cases}\dfrac{2\pi}{2}\cdot\dfrac{2\pi}{4}\cdot\dfrac{2\pi}{6}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n=\dfrac{(2\pi)^{n/2}}{2\cdot 4\cdot 6\cdot\cdots\cdot n}r^n=\dfrac{\pi^{n/2}}{\left(\frac{1}{2}n\right)!}r^n & n\text{ even}\\[2ex] 2\cdot\dfrac{2\pi}{3}\cdot\dfrac{2\pi}{5}\cdot\dfrac{2\pi}{7}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n=\dfrac{2(2\pi)^{(n-1)/2}}{3\cdot 5\cdot 7\cdot\cdots\cdot n}r^n=\dfrac{2^n\left[\frac{1}{2}(n-1)\right]!\,\pi^{(n-1)/2}}{n!}r^n & n\text{ odd}\end{cases}$$

5. We need to show that $\lim\limits_{n\to\infty}V_n(1)=0$. We'll consider the cases of n even and n odd separately.

For $n=2k$ and $r=1$:
$$V_n(1)=\frac{\pi^{n/2}}{\left(\frac{1}{2}n\right)!}r^n=\frac{\pi^{2k/2}}{\left(\frac{1}{2}\cdot 2k\right)!}=\frac{\pi^k}{k!}$$

Then
$$0\le\pi\cdot\frac{\pi}{2}\cdot\frac{\pi}{3}\cdots\frac{\pi}{k}\le\pi\cdot\frac{\pi}{2}\cdot\frac{\pi}{3}\cdot\frac{\pi}{k}=\frac{\pi^4}{6k}\qquad[\text{for }k>3]$$

$\dfrac{\pi^4}{6k}\to 0$ as $k\to\infty\;\Rightarrow\;V_n(1)\to 0$ as $k\to\infty$ for n even by the Squeeze Theorem.

[continued]

For $n = 2k + 1$ and $r = 1$:

$$V_n(1) = \frac{2^n\left[\frac{1}{2}(n-1)\right]!\,\pi^{(n-1)/2}}{n!}\,r^n = \frac{2^{2k+1}\,k!\,\pi^k}{(2k+1)!}$$

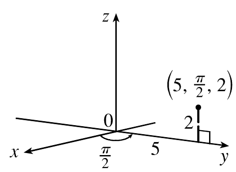
$$0 \le 2\cdot\frac{2^2\pi}{3\cdot 2}\cdot\frac{2^2(2)\pi}{5\cdot 4}\cdot\frac{2^2(3)\pi}{7\cdot 6}\cdot\dots\cdot\frac{2^2(k)\pi}{(2k+1)(2k)} \le 2\cdot\frac{2\pi}{3}\cdot\frac{2\pi}{5}\cdot\frac{2\pi}{2k+1} = \frac{2^4\pi^3}{15(2k+1)} \qquad [\text{for } k > 3]$$

$\dfrac{2^4\pi^3}{15(2k+1)} \to 0$ as $k \to \infty \quad\Rightarrow\quad V_n(1) \to 0$ as $k \to \infty$ for n odd by the Squeeze Theorem.

Thus, $\lim\limits_{n\to\infty} V_n(1) = 0$.

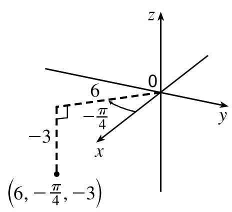
15.7 Triple Integrals in Cylindrical Coordinates

1. (a)



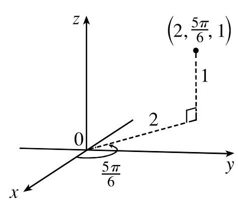
From Equations 1, $x = r\cos\theta = 5\cos\frac{\pi}{2} = 5\cdot 0 = 0$, $y = r\sin\theta = 5\sin\frac{\pi}{2} = 5\cdot 1 = 5$, and $z = 2$, so the point is $(0, 5, 2)$ in rectangular coordinates.

(b)



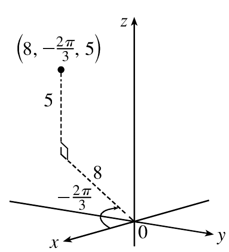
From Equations 1, $x = r\cos\theta = 6\cos\left(-\frac{\pi}{4}\right) = 6\cdot\frac{\sqrt{2}}{2} = 3\sqrt{2}$, $y = r\sin\theta = 6\sin\left(-\frac{\pi}{4}\right) = 6\left(-\frac{\sqrt{2}}{2}\right) = -3\sqrt{2}$, and $z = -3$, so the point is $\left(3\sqrt{2}, -3\sqrt{2}, -3\right)$ in rectangular coordinates.

2. (a)



From Equations 1, $x = r\cos\theta = 2\cos\frac{5\pi}{6} = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$, $y = r\sin\theta = 2\sin\frac{5\pi}{6} = 2\cdot\frac{1}{2} = 1$, and $z = 1$, so the point is $\left(-\sqrt{3}, 1, 1\right)$ in rectangular coordinates.

(b)



From Equations 1, $x = r\cos\theta = 8\cos\left(-\frac{2\pi}{3}\right) = 8\left(-\frac{1}{2}\right) = -4$, $y = r\sin\theta = 8\sin\left(-\frac{2\pi}{3}\right) = 8\left(-\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$, and $z = 5$, so the point is $\left(-4, -4\sqrt{3}, 5\right)$ in rectangular coordinates.

3. (a) $(4, 4, -3)$. From Equations 2, we have $r^2 = x^2 + y^2 = 4^2 + 4^2 = 32$, so $r = \sqrt{32}$. $\tan\theta = \frac{y}{x} = \frac{4}{4} = 1$ and the point $(4, 4)$ is in the first quadrant of the xy-plane, so $\theta = \frac{\pi}{4} + 2\pi n$. Thus, one set of cylindrical coordinates is $(r, \theta, z) = \left(4\sqrt{2}, \frac{\pi}{4}, -3\right)$.

(b) $\left(5\sqrt{3}, -5, \sqrt{3}\right)$. $r^2 = \left(5\sqrt{3}\right)^2 + (-5)^2 = 100$, so $r = 10$. $\tan\theta = \frac{-5}{5\sqrt{3}} = -\frac{1}{\sqrt{3}}$ and the point $\left(5\sqrt{3}, -5\right)$ is in the fourth quadrant of the xy-plane, so $\theta = \frac{11\pi}{6} + 2\pi n$. Thus, one set of cylindrical coordinates is $(r, \theta, z) = \left(10, -\frac{\pi}{6}, \sqrt{3}\right)$.

4. (a) $(0, -2, 9)$. $r^2 = 0^2 + (-2)^2 = 4$, so $r = 2$. $\tan\theta = 0/(-2)$ is undefined and $y < 0$, so $\theta = \frac{3\pi}{2} + 2\pi n$. Thus, one set of cylindrical coordinates is $(r, \theta, z) = \left(2, \frac{3\pi}{2}, 9\right)$.

(b) $\left(-1, \sqrt{3}, 6\right)$. $r^2 = (-1)^2 + \left(\sqrt{3}\right)^2 = 4$, so $r = 2$. $\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$ and the point $\left(-1, \sqrt{3}\right)$ is in the second quadrant of the xy-plane, so $\theta = \frac{2\pi}{3} + 2\pi n$. Thus, one set of cylindrical coordinates is $(r, \theta, z) = \left(2, \frac{2\pi}{3}, 6\right)$.

5. Since $r = 2$, the distance from any point to the z-axis is 2. Because θ and z may vary, the surface is a circular cylinder with radius 2 and axis the z-axis. (See Figure 4.)

Also, $x^2 + y^2 = r^2 = 4$, which we recognize as an equation of this cylinder.

6. Since $\theta = \frac{\pi}{6}$ but r and z may vary, the surface is a vertical plane including the z-axis and intersecting the xy-plane in the line $y = \frac{1}{\sqrt{3}}x$. (Here we are assuming that r can be negative; if we restrict $r \geq 0$, then we get a half-plane.)

7. Since $r^2 + z^2 = 4$ and $r^2 = x^2 + y^2$, we have $x^2 + y^2 + z^2 = 4$, a sphere centered at the origin with radius 2.

8. $r = 2\sin\theta \quad\Rightarrow\quad r^2 = 2r\sin\theta \quad\Rightarrow\quad x^2 + y^2 = 2y \quad\Leftrightarrow\quad x^2 + (y-1)^2 = 1$. z doesn't appear in the equation, so any horizontal trace in $z = k$ is the circle $x^2 + (y-1)^2 = 1$, $z = k$, which has center $(0, 1, k)$ and radius 1. Thus the surface is a circular cylinder with radius 1 and axis the vertical line $x = 0$, $y = 1$.

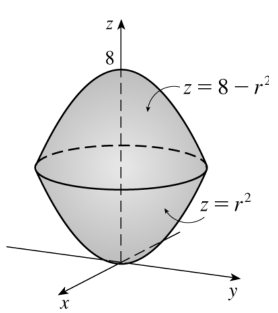
9. (a) Substituting $x^2 + y^2 = r^2$ and $x = r\cos\theta$, the equation $x^2 - x + y^2 + z^2 = 1$ becomes $r^2 - r\cos\theta + z^2 = 1$ or $z^2 = 1 + r\cos\theta - r^2$.

(b) Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation $z = x^2 - y^2$ becomes $z = (r\cos\theta)^2 - (r\sin\theta)^2 = r^2(\cos^2\theta - \sin^2\theta)$ or $z = r^2\cos 2\theta$.

10. (a) The equation $2x^2 + 2y^2 - z^2 = 4$ can be written as $2(x^2 + y^2) - z^2 = 4$ which becomes $2r^2 - z^2 = 4$ or $z^2 = 2r^2 - 4$ in cylindrical coordinates.

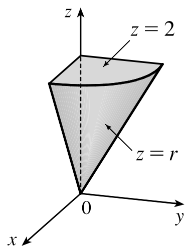
(b) Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation $2x - y + z = 1$ becomes $2r\cos\theta - r\sin\theta + z = 1$ or $z = 1 + r(\sin\theta - 2\cos\theta)$.

11.



$z = r^2 \quad\Leftrightarrow\quad z = x^2 + y^2$, a circular paraboloid opening upward with vertex the origin, and $z = 8 - r^2 \quad\Leftrightarrow\quad z = 8 - (x^2 + y^2)$, a circular paraboloid opening downward with vertex $(0, 0, 8)$. The paraboloids intersect when $r^2 = 8 - r^2 \quad\Leftrightarrow\quad r^2 = 4$. Thus $r^2 \leq z \leq 8 - r^2$ describes the solid above the paraboloid $z = x^2 + y^2$ and below the paraboloid $z = 8 - x^2 - y^2$ for $x^2 + y^2 \leq 4$.

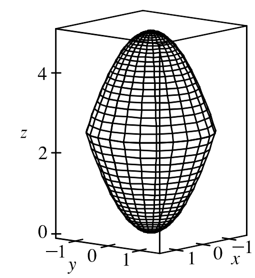
12.



$z = r = \sqrt{x^2 + y^2}$ is a cone that opens upward. Thus $r \le z \le 2$ is the region above this cone and beneath the horizontal plane $z = 2$. $0 \le \theta \le \frac{\pi}{2}$ restricts the solid to that part of this region in the first octant.

13. We can position the cylindrical shell vertically so that its axis coincides with the z-axis and its base lies in the xy-plane. If we use centimeters as the unit of measurement, then cylindrical coordinates conveniently describe the shell as $6 \le r \le 7$, $0 \le \theta \le 2\pi$, $0 \le z \le 20$.

14. In cylindrical coordinates, the equations are $z = r^2$ and $z = 5 - r^2$. The curve of intersection is $r^2 = 5 - r^2$ or $r = \sqrt{5/2}$. So we graph the surfaces in cylindrical coordinates, with $0 \le r \le \sqrt{5/2}$. In Maple, we can use the `coords=cylindrical` option in a regular `plot3d` command. In Mathematica, we can use `RevolutionPlot3D` or `ParametricPlot3D`.



15. (a) In cylindrical coordinates, the region can be described as $E = \{(r, \theta, z) \mid 0 \le r \le 1, 0 \le \theta \le \pi, 0 \le z \le 2 - r^2\}$.

Thus, $\iiint_E (x^2 + y^2)\, dV = \int_0^\pi \int_0^1 \int_0^{2-r^2} r^2 \cdot r\, dz\, dr\, d\theta$.

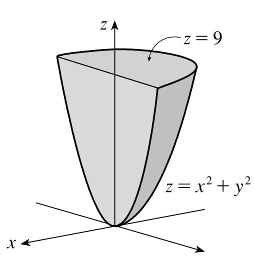
(b) $$\int_0^\pi \int_0^1 \int_0^{2-r^2} r^3\, dz\, dr\, d\theta = \int_0^\pi \int_0^1 r^3 \Big[\, z \,\Big]_{z=0}^{z=2-r^2} dr\, d\theta = \int_0^\pi \int_0^1 (2r^3 - r^5)\, dr\, d\theta$$
$$= \int_0^\pi d\theta \int_0^1 (2r^3 - r^5)\, dr = \Big[\, \theta \,\Big]_{\theta=0}^{\theta=\pi} \cdot \left[\frac{r^4}{2} - \frac{r^6}{6}\right]_{r=0}^{r=1} = \frac{\pi}{3}$$

16. (a) In cylindrical coordinates, the region E is bounded above by the paraboloid $z = 6 - r^2$ and below by the cone $z = r$. The paraboloid and cone intersect when $6 - r^2 = r \;\Rightarrow\; r^2 + r - 6 = 0 \;\Rightarrow\; r = 2$ ($r > 0$), so the region can be described as $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, r \le z \le 6 - r^2\}$. Then

$\iiint_E (xy)\, dV = \int_0^{2\pi} \int_0^2 \int_r^{6-r^2} r\cos\theta \cdot r\sin\theta \cdot r\, dz\, dr\, d\theta$.

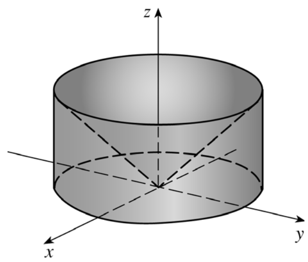
(b) $$\int_0^{2\pi} \int_0^2 \int_r^{6-r^2} r^3 \cos\theta \sin\theta\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 r^3 \cos\theta \sin\theta \Big[\, z \,\Big]_{z=r}^{z=6-r^2} dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^2 r^3 \cos\theta \sin\theta\, (6r^3 - r^4 - r^5)\, dr\, d\theta$$
$$= \int_0^{2\pi} \cos\theta \sin\theta\, d\theta \int_0^2 (6r^3 - r^4 - r^5)\, dr = 0 \cdot \int_0^2 (6r^3 - r^4 - r^5)\, dr = 0$$

17.



The region of integration represents the solid enclosed by the paraboloid $z = r^2$, $(z = x^2 + y^2)$, below the plane $z = 9$ in the second and third quadrants.

$$\int_{\pi/2}^{3\pi/2} \int_0^3 \int_{r^2}^9 r\, dz\, dr\, d\theta = \int_{\pi/2}^{3\pi/2} \int_0^3 [rz]_{z=r^2}^{z=9}\, dr\, d\theta = \int_{\pi/2}^{3\pi/2} \int_0^3 \left(9r - r^3\right) dr\, d\theta$$
$$= \int_{\pi/2}^{3\pi/2} d\theta \int_0^3 \left(9r - r^3\right) dr = \pi \left[\tfrac{9}{2} r^2 - \tfrac{r^4}{4}\right]_0^3 = \tfrac{81\pi}{4}$$

18. 

The region of integration is given in cylindrical coordinates by $E = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, 0 \le z \le r\}$. This represents the solid region enclosed by the circular cylinder $r = 2$, bounded above by the cone $z = r$, and bounded below by the xy-plane.

$$\int_0^2 \int_0^{2\pi} \int_0^r r\,dz\,d\theta\,dr = \int_0^2 \int_0^{2\pi} \left[rz\right]_{z=0}^{z=r} d\theta\,dr = \int_0^2 \int_0^{2\pi} r^2\,d\theta\,dr$$
$$= \int_0^2 r^2\,dr \int_0^{2\pi} d\theta = \left[\tfrac{1}{3}r^3\right]_0^2 \left[\theta\right]_0^{2\pi} = \tfrac{8}{3}\cdot 2\pi = \tfrac{16}{3}\pi$$

19. In cylindrical coordinates, E is given by $\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi} \int_0^4 \int_{-5}^4 \sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2\,dr \int_{-5}^4 dz$$
$$= \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{3}r^3\right]_0^4 \left[z\right]_{-5}^4 = (2\pi)\left(\tfrac{64}{3}\right)(9) = 384\pi$$

20. The paraboloid $z = x^2 + y^2 = r^2$ intersects the plane $z = 4$ in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \quad\Rightarrow\quad r = 2$, so in cylindrical coordinates, E is given by $\left\{(r,\theta,z) \,\middle|\, 0 \le \theta \le 2\pi, 0 \le r \le 2, r^2 \le z \le 4\right\}$. Thus

$$\iiint_E z\,dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 (z)\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^2 \left[\tfrac{1}{2}rz^2\right]_{z=r^2}^{z=4} dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr = 2\pi\left[4r^2 - \tfrac{1}{12}r^6\right]_0^2$$
$$= 2\pi\left(16 - \tfrac{16}{3}\right) = \tfrac{64}{3}\pi$$

21. The paraboloid $z = 4 - x^2 - y^2 = 4 - r^2$ intersects the xy-plane in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \quad\Rightarrow\quad r = 2$, so in cylindrical coordinates, E is given by $\left\{(r,\theta,z) \,\middle|\, 0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 4 - r^2\right\}$. Thus

$$\iiint_E (x+y+z)\,dV = \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\,r\,dz\,dr\,d\theta$$
$$= \int_0^{\pi/2} \int_0^2 \left[r^2(\cos\theta + \sin\theta)z + \tfrac{1}{2}rz^2\right]_{z=0}^{z=4-r^2} dr\,d\theta$$
$$= \int_0^{\pi/2} \int_0^2 \left[(4r^2 - r^4)(\cos\theta + \sin\theta) + \tfrac{1}{2}r(4-r^2)^2\right] dr\,d\theta$$
$$= \int_0^{\pi/2} \left[\left(\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5\right)(\cos\theta + \sin\theta) - \tfrac{1}{12}(4-r^2)^3\right]_{r=0}^{r=2} d\theta$$
$$= \int_0^{\pi/2} \left[\tfrac{64}{15}(\cos\theta + \sin\theta) + \tfrac{16}{3}\right] d\theta = \left[\tfrac{64}{15}(\sin\theta - \cos\theta) + \tfrac{16}{3}\theta\right]_0^{\pi/2}$$
$$= \tfrac{64}{15}(1-0) + \tfrac{16}{3}\cdot\tfrac{\pi}{2} - \tfrac{64}{15}(0-1) - 0 = \tfrac{8}{3}\pi + \tfrac{128}{15}$$

22. In cylindrical coordinates E is bounded by the planes $z = 0$, $z = r\sin\theta + 4$ and the cylinders $r = 1$ and $r = 4$, so E is given by $\{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ 1 \le r \le 4,\ 0 \le z \le r\sin\theta + 4\}$. Thus

$$\iiint_E (x-y)\,dV = \int_0^{2\pi} \int_1^4 \int_0^{r\sin\theta+4} (r\cos\theta - r\sin\theta)\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_1^4 (r^2\cos\theta - r^2\sin\theta)\left[z\right]_{z=0}^{z=r\sin\theta+4} dr\,d\theta$$
$$= \int_0^{2\pi} \int_1^4 (r^2\cos\theta - r^2\sin\theta)(r\sin\theta + 4)\,dr\,d\theta$$
$$= \int_0^{2\pi} \int_1^4 \left[r^3(\sin\theta\cos\theta - \sin^2\theta) + 4r^2(\cos\theta - \sin\theta)\right] dr\,d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{1}{4}r^4(\sin\theta\cos\theta - \sin^2\theta) + \tfrac{4}{3}r^3(\cos\theta - \sin\theta)\right]_{r=1}^{r=4} d\theta$$
$$= \int_0^{2\pi} \left[\left(64 - \tfrac{1}{4}\right)(\sin\theta\cos\theta - \sin^2\theta) + \left(\tfrac{256}{3} - \tfrac{4}{3}\right)(\cos\theta - \sin\theta)\right] d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{255}{4}(\sin\theta\cos\theta - \sin^2\theta) + 84(\cos\theta - \sin\theta)\right] d\theta$$
$$= \left[\tfrac{255}{4}\left(\tfrac{1}{2}\sin^2\theta - \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right) + 84(\sin\theta + \cos\theta)\right]_0^{2\pi} = \tfrac{255}{4}(-\pi) + 84(1) - 0 - 84(1) = -\tfrac{255}{4}\pi$$

23. In cylindrical coordinates, E is bounded by the cylinder $r = 1$, the plane $z = 0$, and the cone $z = 2r$. So

$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 0 \leq z \leq 2r\}$ and

$$\iiint_E x^2\,dV = \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left[r^3\cos^2\theta\,z\right]_{z=0}^{z=2r}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\tfrac{2}{5}r^5\cos^2\theta\right]_{r=0}^{r=1}\,d\theta = \tfrac{2}{5}\int_0^{2\pi}\cos^2\theta\,d\theta = \tfrac{2}{5}\int_0^{2\pi}\tfrac{1}{2}\left(1+\cos 2\theta\right)d\theta = \tfrac{1}{5}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = \tfrac{2\pi}{5}$$

24. In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so $E = \left\{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}\right\}$. Thus the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi}d\theta\int_0^1 2r\sqrt{4-r^2}\,dr = 2\pi\left[-\tfrac{2}{3}(4-r^2)^{3/2}\right]_0^1 = \tfrac{4}{3}\pi(8-3^{3/2})$$

25. In cylindrical coordinates, E is bounded below by the cone $z = r$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2-r^2}$. The cone and the sphere intersect when $2r^2 = 2 \quad\Rightarrow\quad r = 1$, so $E = \left\{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r \leq z \leq \sqrt{2-r^2}\right\}$ and the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 [rz]_{z=r}^{z=\sqrt{2-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2} - r^2\right)dr\,d\theta$$
$$= \int_0^{2\pi}d\theta\int_0^1\left(r\sqrt{2-r^2} - r^2\right)dr = 2\pi\left[-\tfrac{1}{3}(2-r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^1$$
$$= 2\pi\left(-\tfrac{1}{3}\right)\left(1+1-2^{3/2}\right) = -\tfrac{2}{3}\pi\left(2-2\sqrt{2}\right) = \tfrac{4}{3}\pi\left(\sqrt{2}-1\right)$$

26. In cylindrical coordinates, E is bounded below by the paraboloid $z = r^2$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2-r^2}$. The paraboloid and the sphere intersect when $r^2 + r^4 = 2 \quad\Rightarrow\quad (r^2+2)(r^2-1) = 0 \quad\Rightarrow\quad r = 1$, so $E = \left\{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r^2 \leq z \leq \sqrt{2-r^2}\right\}$ and the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 [rz]_{z=r^2}^{z=\sqrt{2-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2} - r^3\right)dr\,d\theta$$
$$= \int_0^{2\pi}d\theta\int_0^1\left(r\sqrt{2-r^2} - r^3\right)dr = 2\pi\left[-\tfrac{1}{3}(2-r^2)^{3/2} - \tfrac{1}{4}r^4\right]_0^1$$
$$= 2\pi\left(-\tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{3}\cdot 2^{3/2} - 0\right) = 2\pi\left(-\tfrac{7}{12} + \tfrac{2}{3}\sqrt{2}\right) = \left(-\tfrac{7}{6} + \tfrac{4}{3}\sqrt{2}\right)\pi$$

27. (a) In cylindrical coordinates, E is bounded above by the paraboloid $z = 24 - r^2$ and below by the cone $z = 2\sqrt{r^2}$ or $z = 2r$ $(r \geq 0)$. The surfaces intersect when

$24 - r^2 = 2r \quad\Rightarrow\quad r^2 + 2r - 24 = 0 \quad\Rightarrow\quad (r+6)(r-4) = 0 \quad\Rightarrow\quad r = 4$, so

$E = \left\{(r, \theta, z) \mid 2r \leq z \leq 24 - r^2, 0 \leq r \leq 4, 0 \leq \theta \leq 2\pi\right\}$ and the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^4\int_{2r}^{24-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r\left(24 - r^2 - 2r\right)dr\,d\theta = \int_0^{2\pi}d\theta\int_0^4\left(24r - r^3 - 2r^2\right)dr$$
$$= 2\pi\left[12r^2 - \tfrac{1}{4}r^4 - \tfrac{2}{3}r^3\right]_0^4 = 2\pi\left(192 - 64 - \tfrac{128}{3}\right) = \tfrac{512}{3}\pi$$

(b) For constant density K, $m = KV = \frac{512}{3}\pi K$ from part (a). Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

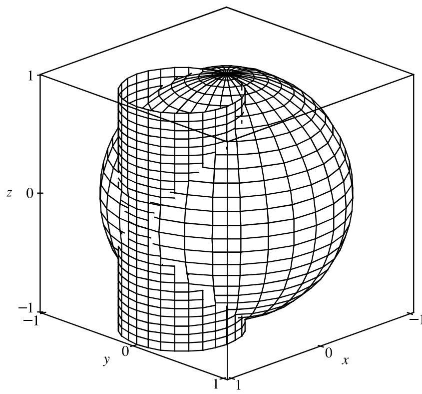
$$M_{xy} = \int_0^{2\pi}\int_0^4\int_{2r}^{24-r^2}(zK)\,r\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^4 r\left[\tfrac{1}{2}z^2\right]_{z=2r}^{z=24-r^2}dr\,d\theta$$
$$= \tfrac{K}{2}\int_0^{2\pi}\int_0^4 r[(24-r^2)^2-4r^2]\,dr\,d\theta = \tfrac{K}{2}\int_0^{2\pi}d\theta\int_0^4(576r-52r^3+r^5)\,dr$$
$$= \tfrac{K}{2}(2\pi)\left[288r^2-13r^4+\tfrac{1}{6}r^6\right]_0^4 = \pi K\left(4608-3328+\tfrac{2048}{3}\right) = \tfrac{5888}{3}\pi K$$

Thus $(\overline{x},\overline{y},\overline{z}) = \left(\dfrac{M_{yz}}{m},\dfrac{M_{xz}}{m},\dfrac{M_{xy}}{m}\right) = \left(0,0,\frac{5888\pi K/3}{512\pi K/3}\right) = \left(0,0,\frac{23}{2}\right)$.

28. (a) $V = \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$$= 4\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$
$$= -\tfrac{4}{3}\int_0^{\pi/2}\left[(a^2-r^2)^{3/2}\right]_{r=0}^{r=a\cos\theta}d\theta$$
$$= -\tfrac{4}{3}\int_0^{\pi/2}\left[(a^2-a^2\cos^2\theta)^{3/2}-a^3\right]d\theta$$
$$= -\tfrac{4}{3}\int_0^{\pi/2}\left[(a^2\sin^2\theta)^{3/2}-a^3\right]d\theta$$
$$= -\tfrac{4}{3}\int_0^{\pi/2}(a^3\sin^3\theta-a^3)\,d\theta$$
$$= -\frac{4a^3}{3}\int_0^{\pi/2}\left[\sin\theta\,(1-\cos^2\theta)-1\right]d\theta$$
$$= -\frac{4a^3}{3}\left[-\cos\theta+\tfrac{1}{3}\cos^3\theta-\theta\right]_0^{\pi/2} = -\frac{4a^3}{3}\left(-\tfrac{\pi}{2}+\tfrac{2}{3}\right) = \tfrac{2}{9}a^3(3\pi-4)$$

(b)



To plot the cylinder and the sphere on the same screen in Maple, we can use the sequence of commands

```
sphere:=plot3d(1,theta=0..2*Pi,phi=0..Pi,coords=spherical):
cylinder:=plot3d(cos(theta),theta=-Pi/2..Pi/2,z=-1..1,coords=cylindrical):
with(plots):
display3d({sphere,cylinder});
```

In Mathematica, we can use

```
sphere=SphericalPlot3D[1,{phi,0,Pi},{theta,0,2Pi}]
cylinder=ParametricPlot3D[{(Cos[theta])^2,Cos[theta]*Sin[theta],z},
                    {theta,-Pi/2,Pi/2},{z,-1,1}]
Show[sphere,cylinder]
```

29. The paraboloid $z = 4x^2+4y^2$ intersects the plane $z = a$ when $a = 4x^2+4y^2$ or $x^2+y^2 = \frac{1}{4}a$. So, in cylindrical coordinates, $E = \left\{(r,\theta,z)\mid 0\le r\le\frac{1}{2}\sqrt{a}, 0\le\theta\le 2\pi, 4r^2\le z\le a\right\}$. Thus

$$m = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Kr\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}(ar-4r^3)\,dr\,d\theta$$
$$= K\int_0^{2\pi}\left[\tfrac{1}{2}ar^2-r^4\right]_{r=0}^{r=\sqrt{a}/2}d\theta = K\int_0^{2\pi}\tfrac{1}{16}a^2\,d\theta = \tfrac{1}{8}a^2\pi K$$

[continued]

Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$M_{xy} = \int_0^{2\pi} \int_0^{\sqrt{a}/2} \int_{4r^2}^{a} Krz\, dz\, dr\, d\theta = K \int_0^{2\pi} \int_0^{\sqrt{a}/2} \left(\tfrac{1}{2}a^2 r - 8r^5\right) dr\, d\theta$$

$$= K \int_0^{2\pi} \left[\tfrac{1}{4}a^2 r^2 - \tfrac{4}{3}r^6\right]_{r=0}^{r=\sqrt{a}/2} d\theta = K \int_0^{2\pi} \tfrac{1}{24}a^3\, d\theta = \tfrac{1}{12}a^3 \pi K$$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{2}{3}a\right)$.

30. Since density is proportional to the distance from the z-axis, we can say $\rho(x, y, z) = K\sqrt{x^2 + y^2}$. Then

$$m = \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2 - r^2}}^{\sqrt{a^2 - r^2}} (Kr)\, r\, dz\, dr\, d\theta = 2\int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2 - r^2}} Kr^2\, dz\, dr\, d\theta = 2K \int_0^{2\pi} \int_0^a r^2 \sqrt{a^2 - r^2}\, dr\, d\theta$$

$$= 2K \int_0^{2\pi} \left[\tfrac{1}{8}r(2r^2 - a^2)\sqrt{a^2 - r^2} + \tfrac{1}{8}a^4 \sin^{-1}(r/a)\right]_{r=0}^{r=a} d\theta = 2K \int_0^{2\pi} \left[\left(\tfrac{1}{8}a^4\right)\left(\tfrac{\pi}{2}\right)\right] d\theta = \tfrac{1}{4}a^4 \pi^2 K$$

31. The region of integration is the region above the cone $z = \sqrt{x^2 + y^2}$, or $z = r$, and below the plane $z = 2$. Also, we have $-2 \le y \le 2$ with $-\sqrt{4 - y^2} \le x \le \sqrt{4 - y^2}$ which describes a circle of radius 2 in the xy-plane centered at $(0, 0)$. Thus,

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz\, dz\, dx\, dy = \int_0^{2\pi} \int_0^2 \int_r^2 (r\cos\theta)\, z\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \int_r^2 r^2 (\cos\theta)\, z\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^2 (\cos\theta) \left[\tfrac{1}{2}z^2\right]_{z=r}^{z=2} dr\, d\theta = \tfrac{1}{2}\int_0^{2\pi} \int_0^2 r^2 (\cos\theta)\left(4 - r^2\right) dr\, d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \cos\theta\, d\theta \int_0^2 \left(4r^2 - r^4\right) dr = \tfrac{1}{2}\left[\sin\theta\right]_0^{2\pi} \left[\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5\right]_0^2 = 0$$

32. The region of integration is the region above the plane $z = 0$ and below the paraboloid $z = 9 - x^2 - y^2$. Also, we have $-3 \le x \le 3$ with $0 \le y \le \sqrt{9 - x^2}$ which describes the upper half of a circle of radius 3 in the xy-plane centered at $(0, 0)$. Thus,

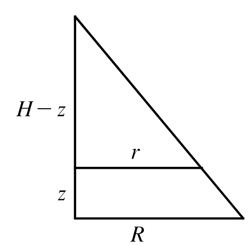
$$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2 + y^2}\, dz\, dy\, dx = \int_0^{\pi} \int_0^3 \int_0^{9-r^2} \sqrt{r^2}\, r\, dz\, dr\, d\theta = \int_0^{\pi} \int_0^3 \int_0^{9-r^2} r^2\, dz\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^3 r^2\left(9 - r^2\right) dr\, d\theta = \int_0^{\pi} d\theta \int_0^3 \left(9r^2 - r^4\right) dr$$

$$= \left[\,\theta\,\right]_0^{\pi} \left[3r^3 - \tfrac{1}{5}r^5\right]_0^3 = \pi\left(81 - \tfrac{243}{5}\right) = \tfrac{162}{5}\pi$$

33. (a) The mountain comprises a solid conical region C. The work done in lifting a small volume of material ΔV with density $g(P)$ to a height $h(P)$ above sea level is $h(P)g(P)\,\Delta V$. Summing over the whole mountain we get $W = \iiint_C h(P)g(P)\, dV$.

(b) Here C is a solid right circular cone with radius $R =$ 62,000 ft, height $H =$ 12,400 ft, and density $g(P) = 200\ \text{lb/ft}^3$ at all points P in C. We use cylindrical coordinates:



$$\frac{r}{R} = \frac{H - z}{H} = 1 - \frac{z}{H}$$

$$W = \int_0^{2\pi} \int_0^H \int_0^{R(1-z/H)} z \cdot 200r\, dr\, dz\, d\theta = 2\pi \int_0^H 200z\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R(1-z/H)} dz$$

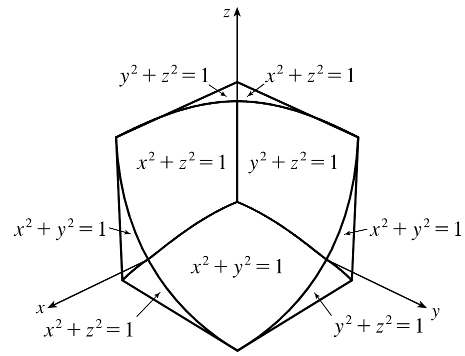
$$= 400\pi \int_0^H z\,\frac{R^2}{2}\left(1 - \frac{z}{H}\right)^2 dz = 200\pi R^2 \int_0^H \left(z - \frac{2z^2}{H} + \frac{z^3}{H^2}\right) dz$$

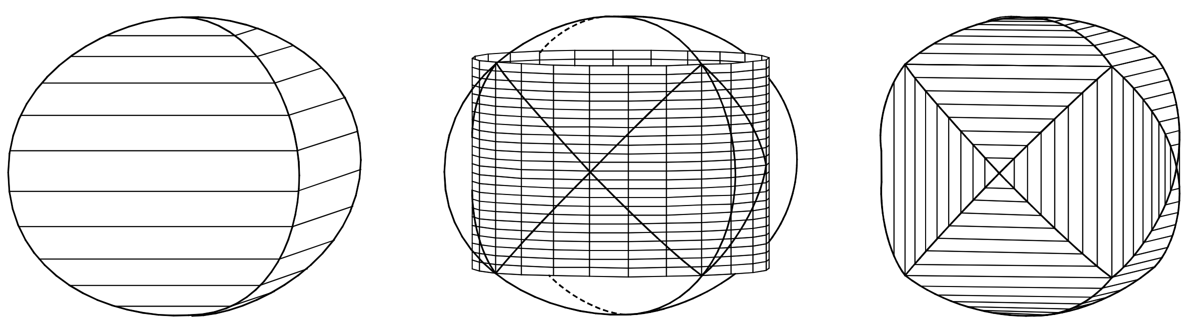
$$= 200\pi R^2 \left[\frac{z^2}{2} - \frac{2z^3}{3H} + \frac{z^4}{4H^2}\right]_0^H = 200\pi R^2 \left(\frac{H^2}{2} - \frac{2H^2}{3} + \frac{H^2}{4}\right)$$

$$= \tfrac{50}{3}\pi R^2 H^2 = \tfrac{50}{3}\pi (62{,}000)^2 (12{,}400)^2 \approx 3.1 \times 10^{19}\ \text{ft-lb}$$

DISCOVERY PROJECT The Intersection of Three Cylinders

1. The three cylinders in the illustration in the text can be visualized as representing the surfaces $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Then we sketch the solid of intersection with the coordinate axes and equations indicated. To be more precise, we start by finding the bounding curves of the solid (shown in the first graph below) enclosed by the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$: $x = \pm y = \pm\sqrt{1 - z^2}$ are the symmetric equations, and these can be expressed parametrically as $x = s$, $y = \pm s$, $z = \pm\sqrt{1 - s^2}$, $-1 \le s \le 1$. Now the cylinder $x^2 + y^2 = 1$ intersects these curves at the eight points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. The resulting solid has twelve curved faces bounded by "edges" which are arcs of circles, as shown in the third diagram. Each cylinder defines four of the twelve faces.





2. To find the volume, we split the solid into sixteen congruent pieces, one of which lies in the part of the first octant with $0 \le \theta \le \frac{\pi}{4}$. (Naturally, we use cylindrical coordinates!) This piece is described by

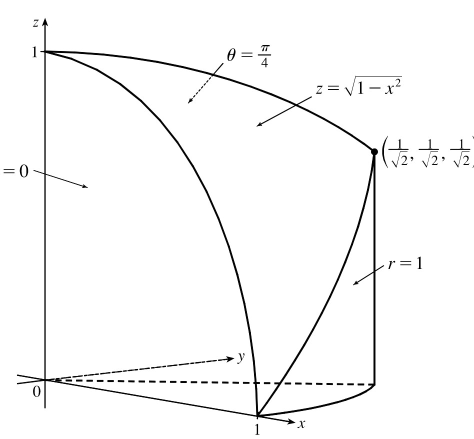
$\left\{(r, \theta, z) \mid 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{4}, 0 \le z \le \sqrt{1 - x^2}\right\}$,

and so, substituting $x = r\cos\theta$, the volume of the entire solid is

$$V = 16\int_0^{\pi/4}\int_0^1\int_0^{\sqrt{1-x^2}} r\,dz\,dr\,d\theta$$

$$= 16\int_0^{\pi/4}\int_0^1 r\sqrt{1 - r^2\cos^2\theta}\,dr\,d\theta$$

$$= 16 - 8\sqrt{2} \approx 4.6863$$

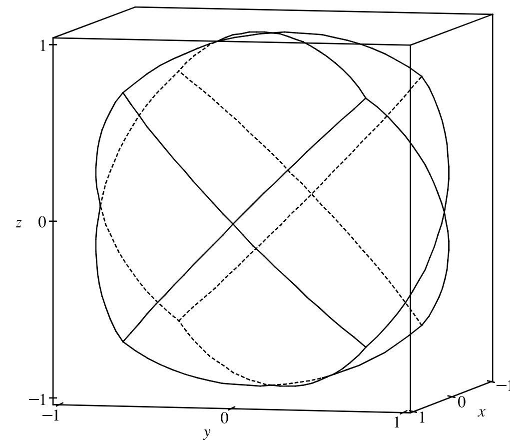


3. To graph the edges of the solid, we use parametrized curves similar to those found in Problem 1 for the intersection of two cylinders. We must restrict the parameter intervals so that each arc extends exactly to the desired vertex. One possible set of parametric equations (with all sign choices allowed) is

$x = r$, $y = \pm r$, $z = \pm\sqrt{1 - r^2}$, $-\frac{1}{\sqrt{2}} \le r \le \frac{1}{\sqrt{2}}$;

$x = \pm s$, $y = \pm\sqrt{1 - s^2}$, $z = s$, $-\frac{1}{\sqrt{2}} \le s \le \frac{1}{\sqrt{2}}$;

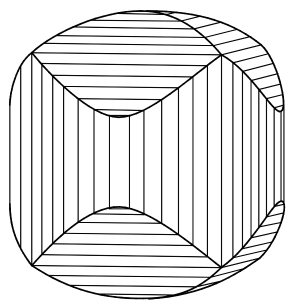
$x = \pm\sqrt{1 - t^2}$, $y = t$, $z = \pm t$, $-\frac{1}{\sqrt{2}} \le t \le \frac{1}{\sqrt{2}}$.



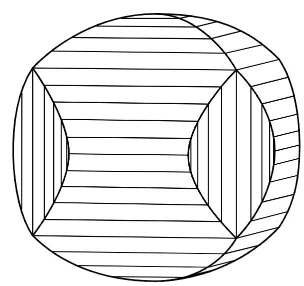
4. Let the three cylinders be $x^2 + y^2 = a^2$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$.

If $a < 1$, then the four faces defined by the cylinder $x^2 + y^2 = 1$ in Problem 1 collapse into a single face, as in the first graph. If $1 < a < \sqrt{2}$, then each pair of vertically opposed faces, defined by one of the other two cylinders, collapse into a single face, as in the second graph. If $a \ge \sqrt{2}$, then the vertical cylinder encloses the solid of intersection of the other two cylinders completely, so the solid of intersection coincides with the solid of intersection of the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, as illustrated in Problem 1.

If we were to vary b or c instead of a, we would get solids with the same shape, but differently oriented.



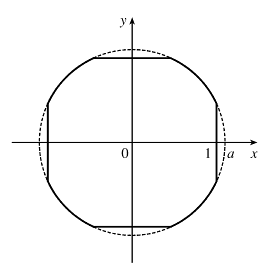
$a = 0.95, b = c = 1$

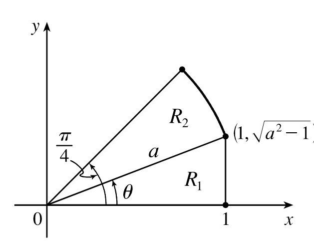


$a = 1.1, b = c = 1$

5. If $a < 1$, the solid looks similar to the first graph in Problem 4. As in Problem 2, we split the solid into sixteen congruent pieces, one of which can be described as the solid above the polar region $\left\{(r, \theta) \mid 0 \le r \le a, 0 \le \theta \le \frac{\pi}{4}\right\}$ in the xy-plane and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2\cos^2\theta}$. Thus, the total volume is $V = 16\int_0^{\pi/4}\int_0^a \sqrt{1 - r^2\cos^2\theta}\, r\, dr\, d\theta$.

If $a > 1$ and $a < \sqrt{2}$, we have a solid similar to the second graph in Problem 4. Its intersection with the xy-plane is graphed at the right. Again we split the solid into sixteen congruent pieces, one of which is the solid above the region shown in the second figure and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2\cos^2\theta}$.





[continued]

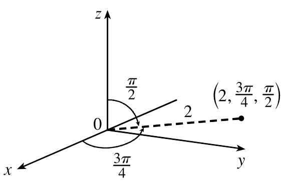
We split the region of integration where the outside boundary changes from the vertical line $x = 1$ to the circle $x^2 + y^2 = a^2$ or $r = a$. R_1 is a right triangle, so $\cos\theta = \frac{1}{a}$. Thus, the boundary between R_1 and R_2 is $\theta = \cos^{-1}\left(\frac{1}{a}\right)$ in polar coordinates, or $y = \sqrt{a^2 - 1}\, x$ in rectangular coordinates. Using rectangular coordinates for the region R_1 and polar coordinates for R_2, we find the total volume of the solid to be

$$V = 16\left[\int_0^1 \int_0^{\sqrt{a^2-1}\,x} \sqrt{1 - x^2}\, dy\, dx + \int_{\cos^{-1}(1/a)}^{\pi/4} \int_0^a \sqrt{1 - r^2\cos^2\theta}\, r\, dr\, d\theta\right]$$

If $a \geq \sqrt{2}$, the cylinder $x^2 + y^2 = 1$ completely encloses the intersection of the other two cylinders, so the solid of intersection of the three cylinders coincides with the intersection of $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as illustrated in Exercise 15.5.26. Its volume is $V = 16\int_0^1 \int_0^x \sqrt{1 - x^2}\, dy\, dx$.

15.8 Triple Integrals in Spherical Coordinates

1. (a)



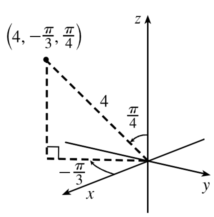
From Equations 1 with $(\rho, \theta, \phi) = \left(2, \frac{3\pi}{4}, \frac{\pi}{2}\right)$,

$x = \rho\sin\phi\cos\theta = 2\sin\frac{\pi}{2}\cos\frac{3\pi}{4} = 2(1)\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$,

$y = \rho\sin\phi\sin\theta = 2\sin\frac{\pi}{2}\sin\frac{3\pi}{4} = 2(1)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$, and

$z = \rho\cos\phi = 2\cos\frac{\pi}{2} = 2 \cdot 0 = 0$, so the point is $\left(-\sqrt{2}, \sqrt{2}, 0\right)$ in rectangular coordinates.

(b)



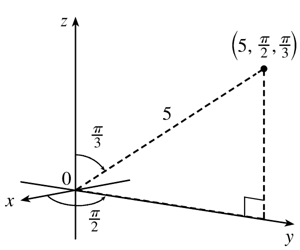
From Equations 1 with $(\rho, \theta, \phi) = \left(4, -\frac{\pi}{3}, \frac{\pi}{4}\right)$,

$x = \rho\sin\phi\cos\theta = 4\sin\frac{\pi}{4}\cos\left(-\frac{\pi}{3}\right) = 4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \sqrt{2}$,

$y = \rho\sin\phi\sin\theta = 4\sin\frac{\pi}{4}\sin\left(-\frac{\pi}{3}\right) = 4\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{6}$, and

$z = \rho\cos\phi = 4\cos\frac{\pi}{4} = 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$, so the point is $\left(\sqrt{2}, -\sqrt{6}, 2\sqrt{2}\right)$ in rectangular coordinates.

2. (a)



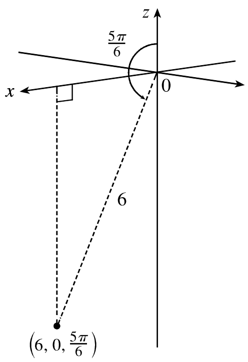
From Equations 1 with $(\rho, \theta, \phi) = \left(5, \frac{\pi}{2}, \frac{\pi}{3}\right)$,

$x = \rho\sin\phi\cos\theta = 5\sin\frac{\pi}{3}\cos\frac{\pi}{2} = 5 \cdot \frac{\sqrt{3}}{2} \cdot 0 = 0$,

$y = \rho\sin\phi\sin\theta = 5\sin\frac{\pi}{3}\sin\frac{\pi}{2} = 5 \cdot \frac{\sqrt{3}}{2} \cdot 1 = \frac{5\sqrt{3}}{2}$, and

$z = \rho\cos\phi = 5\cos\frac{\pi}{3} = 5 \cdot \frac{1}{2} = \frac{5}{2}$, so the point is $\left(0, \frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$ in rectangular coordinates.

(b)



From Equations 1 with $(\rho, \theta, \phi) = \left(6, 0, \frac{5\pi}{6}\right)$,

$x = \rho \sin\phi \cos\theta = 6 \sin\frac{5\pi}{6} \cos 0 = 6 \cdot \frac{1}{2} \cdot 1 = 3$,

$y = \rho \sin\phi \sin\theta = 6 \sin\frac{5\pi}{6} \sin 0 = 6 \cdot \frac{1}{2} \cdot 0 = 0$, and

$z = \rho \cos\phi = 6 \cos\frac{5\pi}{6} = 6\left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$, so the point is

$\left(3, 0, -3\sqrt{3}\right)$.

3. (a) From Equations 1 and 2 with $(x, y, z) = (3, 3, 0)$, $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 3^2 + 0^2} = 3\sqrt{2}$,

$\cos\phi = \dfrac{z}{\rho} = \dfrac{0}{3\sqrt{2}} = 0 \quad\Rightarrow\quad \phi = \frac{\pi}{2}$ and $\cos\theta = \dfrac{x}{\rho \sin\phi} = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}} \quad\Rightarrow\quad \theta = \dfrac{\pi}{4}$ [since $x > 0$ and $y > 0$].

Thus, spherical coordinates are $\left(3\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)$.

(b) $\rho = \sqrt{1^2 + (-\sqrt{3}\,)^2 + (2\sqrt{3}\,)^2} = 4$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \quad\Rightarrow\quad \phi = \dfrac{\pi}{6}$, and

$\cos\theta = \dfrac{x}{\rho \sin\phi} = \dfrac{1}{4 \sin\frac{\pi}{6}} = \dfrac{1}{2} \quad\Rightarrow\quad \theta = -\dfrac{\pi}{3}$ [since $x > 0$ and $y < 0$]. Thus, spherical coordinates are $\left(4, -\frac{\pi}{3}, \frac{\pi}{6}\right)$.

4. (a) $\rho = \sqrt{0^2 + 4^2 + (-4)^2} = 4\sqrt{2}$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{-4}{4\sqrt{2}} = \dfrac{-1}{\sqrt{2}} \quad\Rightarrow\quad \phi = \dfrac{3\pi}{4}$, and

$\cos\theta = \dfrac{x}{\rho \sin\phi} = \dfrac{0}{4\sqrt{2} \sin\frac{3\pi}{4}} = 0 \quad\Rightarrow\quad \theta = \dfrac{\pi}{2}$ [since $y > 0$]. Thus, spherical coordinates are $\left(4\sqrt{2}, \frac{\pi}{2}, \frac{3\pi}{4}\right)$.

(b) $\rho = \sqrt{(-2)^2 + 2^2 + (2\sqrt{6}\,)^2} = 4\sqrt{2}$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{6}}{4\sqrt{2}} = \dfrac{\sqrt{3}}{2} \quad\Rightarrow\quad \phi = \frac{\pi}{6}$, and

$\cos\theta = \dfrac{x}{\rho \sin\phi} = \dfrac{-2}{4\sqrt{2} \sin\frac{\pi}{6}} = -\dfrac{1}{\sqrt{2}} \quad\Rightarrow\quad \theta = \frac{3\pi}{4}$ [since $x < 0$ and $y > 0$]. Thus, spherical coordinates are

$\left(4\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6}\right)$.

5. Since $\phi = \frac{3\pi}{4}$, but ρ and θ can vary, the surface is the bottom half of a right circular cone with vertex at the origin and axis the negative z-axis. (See Figure 4.)

6. $\rho^2 - 3\rho + 2 = 0 \quad\Rightarrow\quad (\rho - 1)(\rho - 2) = 0 \quad\Rightarrow\quad \rho = 1$ or $\rho = 2$. Thus the equation represents two surfaces. In the case $\rho = 1$, the distance from any point to the origin is 1. Because θ and ϕ can vary, the surface is a sphere centered at the origin with radius 1. (See Figure 2.) Similarly, $\rho = 2$ is a sphere centered at the origin with radius 2.

Also, $\rho = 1 \quad\Rightarrow\quad \rho^2 = 1 \quad\Rightarrow\quad x^2 + y^2 + z^2 = 1$ which we recognize as the equation of the unit sphere, and similarly, $\rho = 2 \quad\Rightarrow\quad \rho^2 = 4 \quad\Rightarrow\quad x^2 + y^2 + z^2 = 4$.

7. From Equations 1 we have $z = \rho\cos\phi$, so $\rho\cos\phi = 1 \quad\Leftrightarrow\quad z = 1$, and the surface is the horizontal plane $z = 1$.

8. $\rho = \cos\phi \;\;\Rightarrow\;\; \rho^2 = \rho\cos\phi \;\;\Leftrightarrow\;\; x^2 + y^2 + z^2 = z \;\;\Leftrightarrow\;\; x^2 + y^2 + z^2 - z + \frac{1}{4} = \frac{1}{4} \;\;\Leftrightarrow\;\; x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$.

Therefore, the surface is a sphere of radius $\frac{1}{2}$ centered at $\left(0, 0, \frac{1}{2}\right)$.

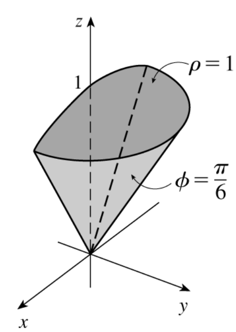
9. (a) From Equation 2 we have $\rho^2 = x^2 + y^2 + z^2$, so $x^2 + y^2 + z^2 = 9 \;\;\Leftrightarrow\;\; \rho^2 = 9 \;\;\Rightarrow\;\; \rho = 3$ (since $\rho \geq 0$).

(b) From Equations 1 we have $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $x^2 - y^2 - z^2 = 1$ becomes $(\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2 - (\rho\cos\phi)^2 = 1 \;\;\Leftrightarrow\;\; (\rho^2\sin^2\phi)(\cos^2\theta - \sin^2\theta) - \rho^2\cos^2\phi = 1 \;\;\Leftrightarrow$ $\rho^2(\sin^2\phi\cos 2\theta - \cos^2\phi) = 1$.

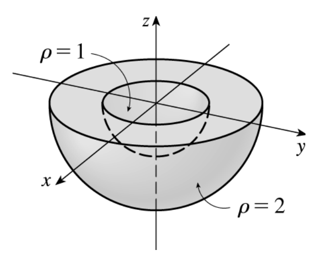
10. (a) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $z = x^2 + y^2$ becomes $\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$ or $\rho\cos\phi = \rho^2\sin^2\phi$. If $\rho \neq 0$, this becomes $\cos\phi = \rho\sin^2\phi$ or $\rho = \cos\phi\csc^2\phi$ or $\rho = \cot\phi\csc\phi$. ($\rho = 0$ corresponds to the origin which is included in the surface.)

(b) The equation $z = x^2 - y^2$ becomes $\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2$ or $\rho\cos\phi = \rho^2(\sin^2\phi)(\cos^2\theta - \sin^2\theta) \;\;\Leftrightarrow\;\; \rho\cos\phi = \rho^2\sin^2\phi\cos 2\theta$. If $\rho \neq 0$, this becomes $\cos\phi = \rho\sin^2\phi\cos 2\theta$. ($\rho = 0$ corresponds to the origin which is included in the surface.)

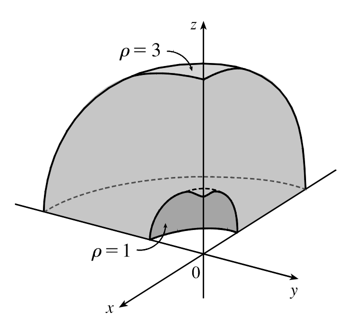
11. $\rho \leq 1$ represents the (solid) unit ball. $0 \leq \phi \leq \frac{\pi}{6}$ restricts the solid to that portion on or above the cone $\phi = \frac{\pi}{6}$, and $0 \leq \theta \leq \pi$ further restricts the solid to that portion on or to the right of the xz-plane.



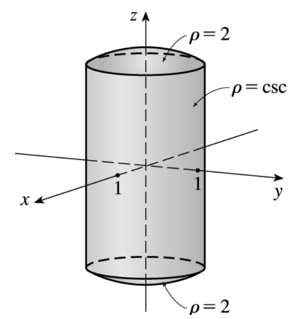
12. $1 \leq \rho \leq 2$ represents the solid region between and including the spheres of radii 1 and 2, centered at the origin. $\frac{\pi}{2} \leq \phi \leq \pi$ restricts the solid to that portion on or below the xy-plane.



13. $1 \leq \rho \leq 3$ represents the solid region between and including the spheres of radii 1 and 3, centered at the origin. $0 \leq \phi \leq \frac{\pi}{2}$ restricts the solid to that portion on or above the xy-plane. $\pi \leq \theta \leq \frac{3\pi}{2}$ further restricts the solid to the portion over the third quadrant.



14. $\rho \le 2$ represents the solid sphere of radius 2 centered at the origin. Notice that $x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Then $\rho = \csc\phi \;\Rightarrow\; \rho \sin\phi = 1 \;\Rightarrow\; \rho^2 \sin^2\phi = x^2 + y^2 = 1$, so $\rho \le \csc\phi$ restricts the solid to that portion on or inside the circular cylinder $x^2 + y^2 = 1$.

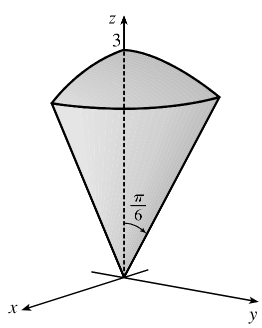


15. $x^2 + y^2 + z^2 = 4z \;\Leftrightarrow\; x^2 + y^2 + z^2 - 4z + 4 = 4 \;\Leftrightarrow\; x^2 + y^2 + (z-2)^2 = 2^2$, which is a sphere with radius 2 centered at $(0, 0, 2)$. In spherical coordinates, we have $\rho^2 = 4\rho\cos\phi \;\Leftrightarrow\; \rho^2 - 4\rho\cos\phi = 0 \;\Leftrightarrow\; \rho = 0$ or $\rho = 4\cos\phi$, so "inside the sphere" is described by $0 \le \rho \le 4\cos\phi$. The cone $z = \sqrt{x^2 + y^2}$ (see Figure 15.7.13) is described by $\phi = \frac{\pi}{4}$, so "outside the cone" is described by $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$.

16. (a) The hollow ball is a spherical shell with outer radius 15 cm and inner radius 14.5 cm. If we center the ball at the origin of the coordinate system and use centimeters as the unit of measurement, then spherical coordinates conveniently describe the hollow ball as $14.5 \le \rho \le 15$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

(b) If we position the ball as in part (a), one possibility is to take the half of the ball that is above the xy-plane. This restricts ϕ from 0 to $\pi/2$ and the hemisphere can be described by $14.5 \le \rho \le 15$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/2$.

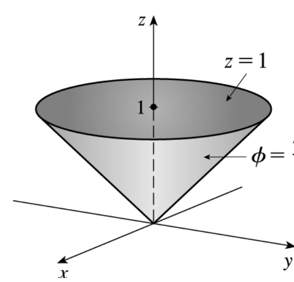
17.



The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 3,\; 0 \le \theta \le \pi/2,\; 0 \le \phi \le \pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho = 3$ and below by the cone $\phi = \pi/6$.

$$\begin{aligned} \int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/6} \sin\phi \, d\phi \int_0^{\pi/2} d\theta \int_0^3 \rho^2 \, d\rho \\ &= \big[-\cos\phi\big]_0^{\pi/6} \big[\,\theta\,\big]_0^{\pi/2} \big[\tfrac{1}{3}\rho^3\big]_0^3 \\ &= \left(1 - \frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9) = \frac{9\pi}{4}\left(2 - \sqrt{3}\right) \end{aligned}$$

18. The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le \sec\phi,\; 0 \le \theta \le 2\pi,\; 0 \le \phi \le \pi/4\}$. $\rho = \sec\phi \;\Leftrightarrow\; \rho\cos\phi = 1 \;\Leftrightarrow\; z = 1$, so E is the solid region above the cone $\phi = \pi/4$ and below the plane $z = 1$.



$$\begin{aligned} \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/4} \int_0^{2\pi} \left[\tfrac{1}{3}\rho^3 \sin\phi\right]_{\rho=0}^{\rho=\sec\phi} d\theta \, d\phi \\ &= \int_0^{\pi/4} \int_0^{2\pi} \tfrac{1}{3} \sec^3\phi \sin\phi \, d\theta \, d\phi \\ &= \tfrac{1}{3} \int_0^{\pi/4} \sec^3\phi \sin\phi \, d\phi \int_0^{2\pi} d\theta = \tfrac{1}{3} \int_0^{\pi/4} \tan\phi \sec^2\phi \, d\phi \int_0^{2\pi} d\theta \\ &= \tfrac{1}{3} \left[\tfrac{1}{2} \tan^2\phi\right]_0^{\pi/4} \big[\,\theta\,\big]_0^{2\pi} = \tfrac{1}{3}\left(\tfrac{1}{2} - 0\right)(2\pi) = \tfrac{\pi}{3} \end{aligned}$$

19. The solid E is most conveniently described if we use cylindrical coordinates:

$E = \left\{(r,\theta,z) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 3,\ 0 \le z \le 2\right\}$. Then

$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$.

20. The solid E is most conveniently described if we use spherical coordinates:

$E = \left\{(\rho,\theta,\phi) \mid 1 \le \rho \le 2,\ \frac{\pi}{2} \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2}\right\}$. Then

$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2}\int_{\pi/2}^{2\pi}\int_1^2 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$.

21. (a) The solid can be described in spherical coordinates by $E = \{(\rho,\theta,\phi) \mid 2 \le \rho \le 3, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}, \frac{\pi}{2} \le \phi \le \pi\}$.

Thus, $\iiint_E \sqrt{x^2+y^2+z^2}\,dV = \int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/2}\int_2^3 \rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$.

(b) $\int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/2}\int_2^3 \rho^3\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/2}^{\pi}\sin\phi\,d\phi\,\int_{\pi/2}^{3\pi/2}d\theta\,\int_2^3\rho^3\,d\rho$

$$= \Big[-\cos\phi\Big]_{\phi=\pi/2}^{\phi=\pi}\Big[\theta\Big]_{\theta=\pi/2}^{\theta=3\pi/2}\left[\frac{\rho^4}{4}\right]_{\rho=2}^{\rho=3} = (1)(\pi)\cdot\frac{1}{4}(81-16) = \frac{65\pi}{4}$$

22. (a) The solid can be described in spherical coordinates by $E = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 2\sqrt{2}, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{4}\}$.

Thus, $\iiint_E xy\,dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{2\sqrt{2}} \rho\sin\phi\cos\theta\cdot\rho\sin\phi\sin\theta\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$.

(b) $\int_0^{\pi/4}\int_0^{2\pi}\int_0^{2\sqrt{2}}\rho^4\sin^3\phi\cos\theta\sin\theta\,d\rho\,d\theta\,d\phi = \int_0^{\pi/4}\sin^3\phi\,d\phi\int_0^{2\pi}\cos\theta\sin\theta\,d\theta\int_0^{2\sqrt{2}}\rho^4\,d\rho$. Since

$\int_0^{2\pi}\cos\theta\sin\theta\,d\theta = \frac{1}{2}\int_0^{2\pi}\sin 2\theta\,d\theta = \frac{1}{2}\left[-\frac{1}{2}\cos 2\theta\right]_0^{2\pi} = -\frac{1}{4}(1-1) = 0$, the original iterated integral equals 0.

23. In spherical coordinates, B is represented by $\{(\rho,\theta,\phi) \mid 0 \le \rho \le 5, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$. Thus

$$\begin{aligned}\iiint_B (x^2+y^2+z^2)^2\,dV &= \int_0^{\pi}\int_0^{2\pi}\int_0^5(\rho^2)^2\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\sin\phi\,d\phi\,\int_0^{2\pi}d\theta\,\int_0^5\rho^6\,d\rho\\ &= \big[-\cos\phi\big]_0^{\pi}\,\big[\theta\big]_0^{2\pi}\,\big[\tfrac{1}{7}\rho^7\big]_0^5 = (2)(2\pi)\big(\tfrac{78{,}125}{7}\big)\\ &= \tfrac{312{,}500}{7}\pi \approx 140{,}249.7\end{aligned}$$

24. In spherical coordinates, E is represented by $\left\{(\rho,\theta,\phi) \,\middle|\, 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{3}\right\}$. Thus

$$\begin{aligned}\iiint_E y^2z^2\,dV &= \int_0^{\pi/3}\int_0^{2\pi}\int_0^1(\rho\sin\phi\sin\theta)^2(\rho\cos\phi)^2\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\\ &= \int_0^{\pi/3}\sin^3\phi\cos^2\phi\,d\phi\,\int_0^{2\pi}\sin^2\theta\,d\theta\,\int_0^1\rho^6\,d\rho\\ &= \int_0^{\pi/3}(1-\cos^2\phi)\cos^2\phi\sin\phi\,d\phi\,\int_0^{2\pi}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta\,\int_0^1\rho^6\,d\rho\\ &= \left[\tfrac{1}{5}\cos^5\phi - \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/3}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{7}\rho^7\right]_0^1\\ &= \left[\tfrac{1}{5}\left(\tfrac{1}{2}\right)^5 - \tfrac{1}{3}\left(\tfrac{1}{2}\right)^3 - \tfrac{1}{5} + \tfrac{1}{3}\right](\pi - 0)\left(\tfrac{1}{7} - 0\right) = \tfrac{47}{480}\cdot\pi\cdot\tfrac{1}{7} = \tfrac{47}{3360}\pi\end{aligned}$$

25. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi) \mid 2 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$ and

$x^2+y^2 = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = \rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) = \rho^2\sin^2\phi$. Thus

[continued]

$$\iiint_E (x^2+y^2)\,dV = \int_0^\pi \int_0^{2\pi} \int_2^3 (\rho^2 \sin^2\phi)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi \sin^3\phi\,d\phi \int_0^{2\pi} d\theta \int_2^3 \rho^4\,d\rho$$

$$= \int_0^\pi (1-\cos^2\phi)\,\sin\phi\,d\phi\,\big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac15\rho^5\big]_2^3 = \big[-\cos\phi + \tfrac13\cos^3\phi\big]_0^\pi\,(2\pi)\cdot\tfrac15(243-32)$$

$$= \big(1-\tfrac13+1-\tfrac13\big)\,(2\pi)\big(\tfrac{211}{5}\big) = \tfrac{1688\pi}{15}$$

26. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi)\mid 0\le\rho\le3, 0\le\theta\le\pi, 0\le\phi\le\pi\}$. Thus

$$\iiint_E y^2\,dV = \int_0^\pi \int_0^\pi \int_0^3 (\rho\sin\phi\sin\theta)^2\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi \sin^3\phi\,d\phi \int_0^\pi \sin^2\theta\,d\theta \int_0^3 \rho^4\,d\rho$$

$$= \int_0^\pi (1-\cos^2\phi)\,\sin\phi\,d\phi \int_0^\pi \tfrac12(1-\cos2\theta)\,d\theta \int_0^3 \rho^4\,d\rho$$

$$= \big[-\cos\phi+\tfrac13\cos^3\phi\big]_0^\pi\,\big[\tfrac12\big(\theta-\tfrac12\sin2\theta\big)\big]_0^\pi\,\big[\tfrac15\rho^5\big]_0^3$$

$$= \big(\tfrac23+\tfrac23\big)\big(\tfrac12\pi\big)\big(\tfrac15(243)\big) = \big(\tfrac43\big)\big(\tfrac\pi2\big)\big(\tfrac{243}{5}\big) = \tfrac{162\pi}{5}$$

27. In spherical coordinates, E is represented by $\big\{(\rho,\theta,\phi)\,\big|\,0\le\rho\le1, 0\le\theta\le\frac\pi2, 0\le\phi\le\frac\pi2\big\}$. Thus

$$\iiint_E xe^{x^2+y^2+z^2}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta)e^{\rho^2}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\sin^2\phi\,d\phi\int_0^{\pi/2}\cos\theta\,d\theta\int_0^1\rho^3e^{\rho^2}\,d\rho$$

$$= \int_0^{\pi/2}\tfrac12(1-\cos2\phi)\,d\phi\int_0^{\pi/2}\cos\theta\,d\theta\left(\tfrac12\rho^2e^{\rho^2}\Big]_0^1-\int_0^1\rho e^{\rho^2}\,d\rho\right)$$

$$\left[\text{integrate by parts with } u=\rho^2,\, dv=\rho e^{\rho^2}d\rho\right]$$

$$= \big[\tfrac12\phi-\tfrac14\sin2\phi\big]_0^{\pi/2}\,[\sin\theta]_0^{\pi/2}\,\Big[\tfrac12\rho^2e^{\rho^2}-\tfrac12e^{\rho^2}\Big]_0^1 = \big(\tfrac\pi4-0\big)(1-0)\big(0+\tfrac12\big) = \tfrac\pi8$$

28. In spherical coordinates, the cone $z=\sqrt{x^2+y^2}$ is equivalent to $\phi=\pi/4$ (as in Example 4) and E is represented by $\{(\rho,\theta,\phi)\mid 1\le\rho\le2,\ 0\le\theta\le2\pi,\ 0\le\phi\le\pi/4\}$. Also $\sqrt{x^2+y^2+z^2}=\sqrt{\rho^2}=\rho$, so

$$\iiint_E\sqrt{x^2+y^2+z^2}\,dV = \int_0^{\pi/4}\int_0^{2\pi}\int_1^2\rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/4}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_1^2\rho^3\,d\rho$$

$$= [-\cos\phi]_0^{\pi/4}\,\big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac14\rho^4\big]_1^2 = \Big(-\tfrac{\sqrt2}{2}+1\Big)(2\pi)\cdot\tfrac14(16-1) = \tfrac{15}{2}\pi\Big(1-\tfrac{\sqrt2}{2}\Big)$$

29. The solid region is given by $E=\big\{(\rho,\theta,\phi)\mid 0\le\rho\le a, 0\le\theta\le2\pi, \frac\pi6\le\phi\le\frac\pi3\big\}$ and its volume is

$$V=\iiint_E dV = \int_{\pi/6}^{\pi/3}\int_0^{2\pi}\int_0^a\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/6}^{\pi/3}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^a\rho^2\,d\rho$$

$$= [-\cos\phi]_{\pi/6}^{\pi/3}\,[\theta]_0^{2\pi}\,\big[\tfrac13\rho^3\big]_0^a = \Big(-\tfrac12+\tfrac{\sqrt3}{2}\Big)(2\pi)\big(\tfrac13a^3\big) = \tfrac{\sqrt3-1}{3}\pi a^3$$

30. If we center the ball at the origin, then the ball is given by $B=\{(\rho,\theta,\phi)\mid 0\le\rho\le a, 0\le\theta\le2\pi, 0\le\phi\le\pi\}$ and the distance from any point (x,y,z) in the ball to the center $(0,0,0)$ is $\sqrt{x^2+y^2+z^2}=\rho$. Thus the average distance is

$$\frac{1}{V(B)}\iiint_B\rho\,dV = \frac{1}{\frac43\pi a^3}\int_0^\pi\int_0^{2\pi}\int_0^a\rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{3}{4\pi a^3}\int_0^\pi\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^a\rho^3\,d\rho$$

$$= \frac{3}{4\pi a^3}\,\big[-\cos\phi\big]_0^\pi\,\big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac14\rho^4\big]_0^a = \frac{3}{4\pi a^3}(2)(2\pi)\big(\tfrac14a^4\big) = \tfrac34a$$

31. (a) Since $\rho = 4\cos\phi$ implies $\rho^2 = 4\rho\cos\phi \quad\Leftrightarrow\quad x^2+y^2+z^2 = 4z \quad\Leftrightarrow\quad x^2+y^2+(z-2)^2 = 4$, the equation is that of a sphere of radius 2 with center at $(0,0,2)$. Thus

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left[\tfrac{1}{3}\rho^3\right]_{\rho=0}^{\rho=4\cos\phi} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left(\tfrac{64}{3}\cos^3\phi\right)\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\tfrac{16}{3}\cos^4\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} -\tfrac{16}{3}\left(\tfrac{1}{16}-1\right) d\theta = 5\theta\Big]_0^{2\pi} = 10\pi$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \cos\phi\sin\phi\left(64\cos^4\phi\right) d\phi \, d\theta$$

$$= \int_0^{2\pi} 64\left[-\tfrac{1}{6}\cos^6\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} \tfrac{21}{2} \, d\theta = 21\pi$$

Hence, $(\overline{x},\overline{y},\overline{z}) = (0,0,21\pi/(10\pi)) = (0,0,2.1)$.

32. In spherical coordinates, the sphere $x^2+y^2+z^2 = 4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2+y^2}$ is represented by $\phi = \frac{\pi}{4}$ (as in Example 4). Thus, the solid is given by $\left\{(\rho,\theta,\phi) \,\middle|\, 0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ \frac{\pi}{4} \le \phi \le \frac{\pi}{2}\right\}$ and

$$V = \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/4}^{\pi/2} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2 \, d\rho$$

$$= \left[-\cos\phi\right]_{\pi/4}^{\pi/2}\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{3}\rho^3\right]_0^2 = \left(\tfrac{\sqrt{2}}{2}\right)(2\pi)\left(\tfrac{8}{3}\right) = \tfrac{8\sqrt{2}\,\pi}{3}$$

33. (a) By the symmetry of the region, $M_{yz} = 0$ and $M_{xz} = 0$. Assuming constant density K,

$m = \iiint_E K \, dV = K\iiint_E dV = \frac{\pi}{8}K$ (from Example 4). Then

$$M_{xy} = \iiint_E z\,K \, dV = K\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} (\rho\cos\phi)\,\rho^2\sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi}\int_0^{\pi/4} \sin\phi\cos\phi\left[\tfrac{1}{4}\rho^4\right]_{\rho=0}^{\rho=\cos\phi} d\phi \, d\theta$$

$$= \tfrac{1}{4}K\int_0^{2\pi}\int_0^{\pi/4} \sin\phi\cos\phi\left(\cos^4\phi\right) d\phi \, d\theta = \tfrac{1}{4}K\int_0^{2\pi} d\theta \int_0^{\pi/4} \cos^5\phi\sin\phi \, d\phi$$

$$= \tfrac{1}{4}K\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{6}\cos^6\phi\right]_0^{\pi/4} = \tfrac{1}{4}K(2\pi)\left(-\tfrac{1}{6}\right)\left[\left(\tfrac{\sqrt{2}}{2}\right)^6 - 1\right] = -\tfrac{\pi}{12}K\left(-\tfrac{7}{8}\right) = \tfrac{7\pi}{96}K$$

Thus, the centroid is $(\overline{x},\overline{y},\overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(0, 0, \dfrac{7\pi K/96}{\pi K/8}\right) = \left(0,0,\frac{7}{12}\right)$.

(b) As in Exercise 25, $x^2+y^2 = \rho^2\sin^2\phi$ and

$$I_z = \iiint_E (x^2+y^2)\,K \, dV = K\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} (\rho^2\sin^2\phi)\,\rho^2\sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi}\int_0^{\pi/4} \sin^3\phi\left[\tfrac{1}{5}\rho^5\right]_{\rho=0}^{\rho=\cos\phi} d\phi \, d\theta$$

$$= \tfrac{1}{5}K\int_0^{2\pi}\int_0^{\pi/4} \sin^3\phi\cos^5\phi \, d\phi \, d\theta = \tfrac{1}{5}K\int_0^{2\pi} d\theta \int_0^{\pi/4} \cos^5\phi\left(1-\cos^2\phi\right)\sin\phi \, d\phi$$

$$= \tfrac{1}{5}K\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{6}\cos^6\phi + \tfrac{1}{8}\cos^8\phi\right]_0^{\pi/4}$$

$$= \tfrac{1}{5}K(2\pi)\left[-\tfrac{1}{6}\left(\tfrac{\sqrt{2}}{2}\right)^6 + \tfrac{1}{8}\left(\tfrac{\sqrt{2}}{2}\right)^8 + \tfrac{1}{6} - \tfrac{1}{8}\right] = \tfrac{2\pi}{5}K\left(\tfrac{11}{384}\right) = \tfrac{11\pi}{960}K$$

34. (a) Placing the center of the base at $(0,0,0)$, $\rho(x,y,z) = K\sqrt{x^2+y^2+z^2}$ is the density function. So

$$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3\sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi \, d\phi \int_0^a \rho^3 \, d\rho$$

$$= K\left[\,\theta\,\right]_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{4}\rho^4\right]_0^a = K(2\pi)(1)\left(\tfrac{1}{4}a^4\right) = \tfrac{1}{2}\pi K a^4$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$\begin{aligned} M_{xy} &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4 \sin\phi\cos\phi\, d\rho\, d\phi\, d\theta \\ &= K\int_0^{2\pi} d\theta\, \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi\, \int_0^a \rho^4\, d\rho \\ &= K\big[\,\theta\,\big]_0^{2\pi}\, \big[\tfrac{1}{2}\sin^2\phi\big]_0^{\pi/2}\, \big[\tfrac{1}{5}\rho^5\big]_0^a = K(2\pi)\big(\tfrac{1}{2}\big)\big(\tfrac{1}{5}a^5\big) = \tfrac{1}{5}\pi K a^5 \end{aligned}$$

Hence, $(\overline{x},\overline{y},\overline{z}) = \big(0,0,\tfrac{2}{5}a\big)$.

(c) $I_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (K\rho^3\sin\phi)(\rho^2\sin^2\phi)\, d\rho\, d\phi\, d\theta = K\int_0^{2\pi} d\theta\, \int_0^{\pi/2}\sin^3\phi\, d\phi\, \int_0^a \rho^5\, d\rho$

$= K\big[\,\theta\,\big]_0^{2\pi}\, \big[-\cos\phi + \tfrac{1}{3}\cos^3\phi\big]_0^{\pi/2}\, \big[\tfrac{1}{6}\rho^6\big]_0^a = K(2\pi)\big(\tfrac{2}{3}\big)\big(\tfrac{1}{6}a^6\big) = \tfrac{2}{9}\pi K a^6$

35. (a) The density function is $\rho(x,y,z) = K$, a constant, and by the symmetry of the problem $M_{xz} = M_{yz} = 0$. Then

$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3 \sin\phi\, \cos\phi\, d\rho\, d\phi\, d\theta = \tfrac{1}{2}\pi K a^4 \int_0^{\pi/2} \sin\phi\, \cos\phi\, d\phi = \tfrac{1}{8}\pi K a^4$. But the mass is

$K\cdot(\text{volume of the hemisphere}) = \tfrac{2}{3}\pi K a^3$, so the centroid is $\big(0,0,\tfrac{3}{8}a\big)$.

(b) Place the center of the base at $(0,0,0)$; the density function is $\rho(x,y,z) = K$. By symmetry, the moments of inertia about any two such diameters will be equal, so we just need to find I_x:

$$\begin{aligned} I_x &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (K\rho^2\sin\phi)\,\rho^2\,(\sin^2\phi\,\sin^2\theta + \cos^2\phi)\, d\rho\, d\phi\, d\theta \\ &= K\int_0^{2\pi}\int_0^{\pi/2}(\sin^3\phi\,\sin^2\theta + \sin\phi\,\cos^2\phi)\big(\tfrac{1}{5}a^5\big)\, d\phi\, d\theta \\ &= \tfrac{1}{5}Ka^5\int_0^{2\pi}\big[\sin^2\theta\,\big(-\cos\phi + \tfrac{1}{3}\cos^3\phi\big) + \big(-\tfrac{1}{3}\cos^3\phi\big)\big]_{\phi=0}^{\phi=\pi/2}\, d\theta = \tfrac{1}{5}Ka^5\int_0^{2\pi}\big[\tfrac{2}{3}\sin^2\theta + \tfrac{1}{3}\big]\, d\theta \\ &= \tfrac{1}{5}Ka^5\big[\tfrac{2}{3}\big(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\big) + \tfrac{1}{3}\theta\big]_0^{2\pi} = \tfrac{1}{5}Ka^5\big[\tfrac{2}{3}(\pi - 0) + \tfrac{1}{3}(2\pi - 0)\big] = \tfrac{4}{15}\pi K a^5 \end{aligned}$$

36. Place the center of the base at $(0,0,0)$, then the density is $\rho(x,y,z) = Kz$, K a constant. Then

$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (K\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = 2\pi K\int_0^{\pi/2}\cos\phi\sin\phi\cdot\tfrac{1}{4}a^4\, d\phi = \tfrac{1}{2}\pi K a^4\big[-\tfrac{1}{4}\cos 2\phi\big]_0^{\pi/2} = \tfrac{\pi}{4}Ka^4$.

By the symmetry of the problem $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4\cos^2\phi\sin\phi\, d\rho\, d\phi\, d\theta = \tfrac{2}{5}\pi K a^5\int_0^{\pi/2}\cos^2\phi\sin\phi\, d\phi = \tfrac{2}{5}\pi K a^5\big[-\tfrac{1}{3}\cos^3\theta\big]_0^{\pi/2} = \tfrac{2}{15}\pi K a^5$.

Hence, $(\overline{x},\overline{y},\overline{z}) = \big(0,0,\tfrac{8}{15}a\big)$.

37. In spherical coordinates $z = \sqrt{x^2+y^2}$ becomes $\phi = \frac{\pi}{4}$ (as in Example 4). Then

$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} d\theta\, \int_0^{\pi/4}\sin\phi\, d\phi\, \int_0^1 \rho^2\, d\rho = 2\pi\left(-\frac{\sqrt{2}}{2} + 1\right)\big(\tfrac{1}{3}\big) = \tfrac{1}{3}\pi\big(2 - \sqrt{2}\big)$,

$M_{xy} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^3\sin\phi\cos\phi\, d\rho\, d\phi\, d\theta = 2\pi\big[-\tfrac{1}{4}\cos 2\phi\big]_0^{\pi/4}\big(\tfrac{1}{4}\big) = \tfrac{\pi}{8}$ and by symmetry $M_{yz} = M_{xz} = 0$.

Hence, $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\dfrac{3}{8\big(2-\sqrt{2}\big)}\right)$.

38. Place the center of the sphere at $(0, 0, 0)$, let the diameter of intersection be along the z-axis, one of the planes be the xz-plane and the other be the plane whose angle with the xz-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by

$V = \int_0^{\pi/6} \int_0^{\pi} \int_0^a \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{\pi/6} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^a \rho^2 \, d\rho = \frac{\pi}{6}(2)\left(\frac{1}{3}a^3\right) = \frac{1}{9}\pi a^3.$

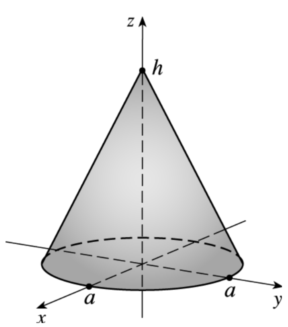
39. (a) If we orient the cylinder so that its axis is the z-axis and its base lies in the xy-plane, then the cylinder is described, in cylindrical coordinates, by $E = \{(r, \theta, z) \mid 0 \leq r \leq a, \ 0 \leq \theta \leq 2\pi, \ 0 \leq z \leq h\}$. Assuming constant density K, the moment of inertia about its axis (the z-axis) is

$$\begin{aligned} I_z &= \iiint_E (x^2 + y^2)\, \rho(x, y, z)\, dV = \int_0^{2\pi} \int_0^a \int_0^h K(r^2)\, r \, dz \, dr \, d\theta = K \int_0^{2\pi} d\theta \int_0^a r^3 \, dr \int_0^h dz \\ &= K \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^a \left[z\right]_0^h = K\,(2\pi)\left(\tfrac{1}{4}a^4\right)(h) = \tfrac{1}{2}\pi K a^4 h \end{aligned}$$

(b) By symmetry, the moments of inertia about any two diameters of the base will be equal, and one of the diameters lies on the x-axis, so we compute:

$$\begin{aligned} I_x &= \iiint_E (y^2 + z^2)\, \rho(x, y, z)\, dV = \int_0^{2\pi} \int_0^a \int_0^h K(r^2 \sin^2\theta + z^2)\, r \, dz \, dr \, d\theta \\ &= K \int_0^{2\pi} \int_0^a \int_0^h r^3 \sin^2\theta \, dz \, dr \, d\theta + K \int_0^{2\pi} \int_0^a \int_0^h r z^2 \, dz \, dr \, d\theta \\ &= K \int_0^{2\pi} \sin^2\theta \, d\theta \int_0^a r^3 \, dr \int_0^h dz + K \int_0^{2\pi} d\theta \int_0^a r \, dr \int_0^h z^2 \, dz \\ &= K \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^a \left[z\right]_0^h + K \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{2}r^2\right]_0^a \left[\tfrac{1}{3}z^3\right]_0^h \\ &= K\,(\pi)\left(\tfrac{1}{4}a^4\right)(h) + K\,(2\pi)\left(\tfrac{1}{2}a^2\right)\left(\tfrac{1}{3}h^3\right) = \tfrac{1}{12}\pi K a^2 h (3a^2 + 4h^2) \end{aligned}$$

40.



Orient the cone so that its axis is the z-axis and its base lies in the xy-plane, as shown in the figure. (Then the z-axis is the axis of the cone and the x-axis contains a diameter of the base.) A right circular cone with axis the z-axis and vertex at the origin has equation $z^2 = c^2(x^2 + y^2)$. Here we have the bottom frustum, shifted upward h units, and with $c^2 = h^2/a^2$ so that the cone includes the point $(a, 0, 0)$. Thus an equation of the cone in rectangular coordinates is $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$, $0 \leq z \leq h$. In cylindrical coordinates, the cone is described by

$$E = \left\{(r, \theta, z) \mid 0 \leq r \leq a, \ 0 \leq \theta \leq 2\pi, \ 0 \leq z \leq h\left(1 - \tfrac{1}{a}r\right)\right\}$$

(a) Assuming constant density K, the moment of inertia about its axis (the z-axis) is

$$\begin{aligned} I_z &= \iiint_E (x^2 + y^2)\, \rho(x, y, z)\, dV = \int_0^{2\pi} \int_0^a \int_0^{h(1 - r/a)} K(r^2)\, r \, dz \, dr \, d\theta \\ &= K \int_0^{2\pi} \int_0^a \left[r^3 z\right]_{z=0}^{z=h(1-r/a)} dr \, d\theta = K \int_0^{2\pi} \int_0^a r^3 h \left(1 - \tfrac{1}{a}r\right) dr \, d\theta \\ &= Kh \int_0^{2\pi} d\theta \int_0^a \left(r^3 - \tfrac{1}{a}r^4\right) dr = Kh \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{4}r^4 - \tfrac{1}{5a}r^5\right]_0^a \\ &= Kh\,(2\pi)\left(\tfrac{1}{4}a^4 - \tfrac{1}{5}a^4\right) = \tfrac{1}{10}\pi K a^4 h \end{aligned}$$

(b) By symmetry, the moments of intertia about any two diameters of the base will be equal, and one of the diameters lies on the x-axis, so we compute:

$$\begin{aligned}
I_x &= \iiint_E (y^2+z^2)\,\rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^a\int_0^{h(1-r/a)} K(r^2\sin^2\theta + z^2)\,r\,dz\,dr\,d\theta \\
&= K\int_0^{2\pi}\int_0^a \left[(r^3\sin^2\theta)z + \tfrac{1}{3}rz^3\right]_{z=0}^{z=h(1-r/a)} dr\,d\theta \\
&= K\int_0^{2\pi}\int_0^a \left[(r^3\sin^2\theta)\left(h\left(1-\tfrac{1}{a}r\right)\right) + \tfrac{1}{3}r\left(h\left(1-\tfrac{1}{a}r\right)\right)^3\right] dr\,d\theta \\
&= Kh\int_0^{2\pi}\int_0^a (r^3\sin^2\theta)\left(1-\tfrac{1}{a}r\right)\,dr\,d\theta + Kh^3\int_0^{2\pi}\int_0^a \tfrac{1}{3}r\left(1-\tfrac{1}{a}r\right)^3 dr\,d\theta \\
&= Kh\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^a\left(r^3-\tfrac{1}{a}r^4\right)dr + \tfrac{1}{3}Kh^3\int_0^{2\pi}d\theta\int_0^a\left(r-\tfrac{3}{a}r^2+\tfrac{3}{a^2}r^3-\tfrac{1}{a^3}r^4\right)dr \\
&= Kh\left[\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4-\tfrac{1}{5a}r^5\right]_0^a + \tfrac{1}{3}Kh^3\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{2}r^2-\tfrac{1}{a}r^3+\tfrac{3}{4a^2}r^4-\tfrac{1}{5a^3}r^5\right]_0^a \\
&= Kh\,(\pi)\left(\tfrac{1}{4}a^4-\tfrac{1}{5}a^4\right) + \tfrac{1}{3}Kh^3\,(2\pi)\left(\tfrac{1}{2}a^2-a^2+\tfrac{3}{4}a^2-\tfrac{1}{5}a^2\right) \\
&= \pi Kh\left(\tfrac{1}{20}a^4\right) + \tfrac{2}{3}\pi Kh^3\left(\tfrac{1}{20}a^2\right) = \pi Ka^2h\left(\tfrac{1}{20}a^2+\tfrac{1}{30}h^2\right)
\end{aligned}$$

41. In cylindrical coordinates the paraboloid is given by $z=r^2$ and the plane by $z=2r\sin\theta$ and the projection of the intersection onto the xy-plane is the circle $r=2\sin\theta$. Then $\iiint_E z\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} rz\,dz\,dr\,d\theta = \frac{5\pi}{6}$ [using a CAS].

42. (a) The region enclosed by the torus is $\{(\rho,\theta,\phi)\mid 0\le\theta\le 2\pi,\ 0\le\phi\le\pi,\ 0\le\rho\le\sin\phi\}$, so its volume is

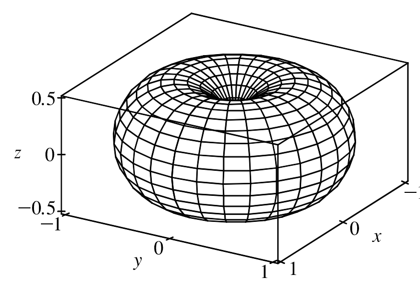
$$V=\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\pi}\tfrac{1}{3}\sin^4\phi\,d\phi = \tfrac{2}{3}\pi\left[\tfrac{3}{8}\phi-\tfrac{1}{4}\sin 2\phi+\tfrac{1}{16}\sin 4\phi\right]_0^{\pi} = \tfrac{1}{4}\pi^2.$$

(b) In Maple, we can plot the torus using the command

```
plot3d(sin(phi),theta=0..2*Pi,
       phi=0..Pi,coords=spherical);.
```

In Mathematica, use

```
SphericalPlot3D[Sin[phi],{phi,0,Pi},{theta,0,2Pi}].
```



43. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$. The region E of integration is the region above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=2$ in the first octant. Because E is in the first octant we have $0\le\theta\le\frac{\pi}{2}$. The cone has equation $\phi=\frac{\pi}{4}$ (as in Example 4), so $0\le\phi\le\frac{\pi}{4}$, and $0\le\rho\le\sqrt{2}$. Then the integral becomes

$$\begin{aligned}
&\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}}(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&\quad= \int_0^{\pi/4}\sin^3\phi\,d\phi\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^{\sqrt{2}}\rho^4\,d\rho = \left(\int_0^{\pi/4}\left(1-\cos^2\phi\right)\sin\phi\,d\phi\right)\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^{\sqrt{2}} \\
&\quad= \left[\tfrac{1}{3}\cos^3\phi-\cos\phi\right]_0^{\pi/4}\cdot\tfrac{1}{2}\cdot\tfrac{1}{5}\left(\sqrt{2}\right)^5 = \left[\tfrac{\sqrt{2}}{12}-\tfrac{\sqrt{2}}{2}-\left(\tfrac{1}{3}-1\right)\right]\cdot\tfrac{2\sqrt{2}}{5} = \tfrac{4\sqrt{2}-5}{15}
\end{aligned}$$

44. $\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}(x^2z+y^2z+z^3)\,dz\,dx\,dy$. The region of integration is the solid sphere $x^2+y^2+z^2\le a^2$, so $0\le\theta\le 2\pi$, $0\le\phi\le\pi$, and $0\le\rho\le a$. Also $x^2z+y^2z+z^3=(x^2+y^2+z^2)z=\rho^2z=\rho^3\cos\phi$, so the integral becomes

$$\int_0^{\pi}\int_0^{2\pi}\int_0^a\left(\rho^3\cos\phi\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_0^{\pi}\sin\phi\cos\phi\,d\phi\int_0^{2\pi}d\theta\int_0^a\rho^5\,d\rho=\left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi}\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{6}\rho^6\right]_0^a=0$$

45. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}}(x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx$. The region of integration is the solid sphere $x^2+y^2+(z-2)^2\le 4$, or equivalently, $\rho^2\sin^2\phi+(\rho\cos\phi-2)^2=\rho^2-4\rho\cos\phi+4\le 4 \quad\Rightarrow\quad \rho\le 4\cos\phi$, so $0\le\theta\le 2\pi$, $0\le\phi\le\frac{\pi}{2}$, and $0\le\rho\le 4\cos\phi$. Also $(x^2+y^2+z^2)^{3/2}=(\rho^2)^{3/2}=\rho^3$, so the integral becomes

$$\begin{aligned}\int_0^{\pi/2}\int_0^{2\pi}\int_0^{4\cos\phi}\left(\rho^3\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi&=\int_0^{\pi/2}\int_0^{2\pi}\sin\phi\left[\tfrac{1}{6}\rho^6\right]_{\rho=0}^{\rho=4\cos\phi}d\theta\,d\phi\\&=\tfrac{1}{6}\int_0^{\pi/2}\int_0^{2\pi}\sin\phi\left(4096\cos^6\phi\right)d\theta\,d\phi\\&=\tfrac{1}{6}(4096)\int_0^{\pi/2}\cos^6\phi\sin\phi\,d\phi\int_0^{2\pi}d\theta=\tfrac{2048}{3}\left[-\tfrac{1}{7}\cos^7\phi\right]_0^{\pi/2}\left[\theta\right]_0^{2\pi}\\&=\tfrac{2048}{3}\left(\tfrac{1}{7}\right)(2\pi)=\tfrac{4096\pi}{21}\end{aligned}$$

46. The solid region between the ground and an altitude of 5 km (5000 m) is given by

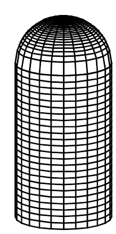
$$E=\left\{(\rho,\theta,\phi)\mid 6.370\times10^6\le\rho\le 6.375\times10^6, 0\le\theta\le 2\pi, 0\le\phi\le\pi\right\}.$$

Then the mass of the atmosphere in this region is

$$\begin{aligned}m=\iiint_E\delta\,dV&=\int_0^{2\pi}\int_0^{\pi}\int_{6.370\times10^6}^{6.375\times10^6}(619.09-0.000097\rho)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\&=\int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi\,d\phi\int_{6.370\times10^6}^{6.375\times10^6}\left(619.09\rho^2-0.000097\rho^3\right)d\rho\\&=\left[\theta\right]_0^{2\pi}\left[-\cos\phi\right]_0^{\pi}\left[\tfrac{619.09}{3}\rho^3-\tfrac{0.000097}{4}\rho^4\right]_{6.370\times10^6}^{6.375\times10^6}\\&=(2\pi)(2)\left[\tfrac{619.09}{3}\left((6.375\times10^6)^3-(6.370\times10^6)^3\right)-\tfrac{0.000097}{4}\left((6.375\times10^6)^4-(6.370\times10^6)^4\right)\right]\\&\approx 4\pi\left(1.944\times10^{17}\right)\approx 2.44\times10^{18}\text{ kg}\end{aligned}$$

47. In cylindrical coordinates, the equation of the cylinder is $r=3$, $0\le z\le 10$.

The hemisphere is the upper part of the sphere radius 3, center $(0,0,10)$, equation $r^2+(z-10)^2=3^2$, $z\ge 10$. In Maple, we can use the `coords=cylindrical` option in a regular `plot3d` command. In Mathematica, we can use `ParametricPlot3D`.



48. We begin by finding the positions of Los Angeles and Montréal in spherical coordinates, using the method described in the exercise:

Montréal	Los Angeles
$\rho=3960$ mi	$\rho=3960$ mi
$\theta=360^\circ-\ 73.60^\circ=286.40^\circ$	$\theta=360^\circ-\ 118.25^\circ=241.75^\circ$
$\phi=90^\circ-\ 45.50^\circ=44.50^\circ$	$\phi=90^\circ-\ 34.06^\circ=55.94^\circ$

[continued]

Now we change the above to Cartesian coordinates using $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$ to get two position vectors of length 3960 mi (since both cities must lie on the surface of the earth). In particular:

$$\text{Montréal: } \langle 783.67, -2662.67, 2824.47\rangle \qquad \text{Los Angeles: } \langle -1552.80, -2889.91, 2217.84\rangle$$

To find the angle γ between these two vectors we use the dot product:

$\langle 783.67, -2662.67, 2824.47\rangle \cdot \langle -1552.80, -2889.91, 2217.84\rangle = (3960)^2\cos\gamma \quad\Rightarrow\quad \cos\gamma \approx 0.8126 \quad\Rightarrow$

$\gamma \approx 0.6223$ rad. The great circle distance between the cities is $s = \rho\gamma \approx 3960(0.6223) \approx 2464$ mi.

49. If E is the solid enclosed by the surface $\rho = 1 + \frac{1}{5}\sin 6\theta\,\sin 5\phi$, it can be described in spherical coordinates as $E = \left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 1 + \frac{1}{5}\sin 6\theta\sin 5\phi, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\right\}$. Its volume is given by $V(E) = \iiint_E dV = \int_0^\pi \int_0^{2\pi} \int_0^{1+(\sin 6\theta\sin 5\phi)/5} \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{136\pi}{99}$ [using a CAS].

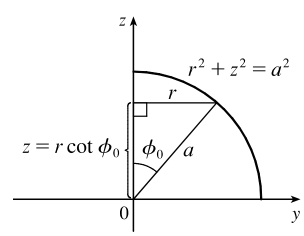
50.
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = \lim_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho e^{-\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \lim_{R\to\infty}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi\,d\phi\int_0^{R}\rho^3 e^{-\rho^2}\,d\rho$$

Now use integration by parts with $u = \rho^2$, $dv = \rho e^{-\rho^2}\,d\rho$ to get

$$\lim_{R\to\infty}2\pi(2)\left(\rho^2\left(-\tfrac{1}{2}\right)e^{-\rho^2}\Big]_0^R - \int_0^R 2\rho\left(-\tfrac{1}{2}\right)e^{-\rho^2}\,d\rho\right) = \lim_{R\to\infty}4\pi\left(-\tfrac{1}{2}R^2e^{-R^2} + \left[-\tfrac{1}{2}e^{-\rho^2}\right]_0^R\right)$$
$$= 4\pi\lim_{R\to\infty}\left[-\tfrac{1}{2}R^2e^{-R^2} - \tfrac{1}{2}e^{-R^2} + \tfrac{1}{2}\right] = 4\pi\left(\tfrac{1}{2}\right) = 2\pi$$

(Note that $R^2e^{-R^2} \to 0$ as $R \to \infty$ by l'Hospital's Rule.)

51. (a) From the diagram, $z = r\cot\phi_0$ to $z = \sqrt{a^2 - r^2}$, $r = 0$ to $r = a\sin\phi_0$ (or use $a^2 - r^2 = r^2\cot^2\phi_0$). Thus

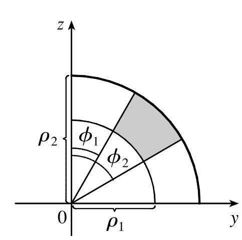


$$V = \int_0^{2\pi}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$
$$= 2\pi\int_0^{a\sin\phi_0}\left(r\sqrt{a^2-r^2} - r^2\cot\phi_0\right)dr$$
$$= \tfrac{2\pi}{3}\left[-(a^2-r^2)^{3/2} - r^3\cot\phi_0\right]_0^{a\sin\phi_0}$$
$$= \tfrac{2\pi}{3}\left[-\left(a^2 - a^2\sin^2\phi_0\right)^{3/2} - a^3\sin^3\phi_0\cot\phi_0 + a^3\right]$$
$$= \tfrac{2}{3}\pi a^3\left[1 - \left(\cos^3\phi_0 + \sin^2\phi_0\cos\phi_0\right)\right] = \tfrac{2}{3}\pi a^3(1 - \cos\phi_0)$$

(b) The wedge in question is the shaded area rotated from $\theta = \theta_1$ to $\theta = \theta_2$.

Letting

V_{ij} = volume of the region bounded by the sphere of radius ρ_i and the cone with angle ϕ_j ($\theta = \theta_1$ to θ_2)



and letting V be the volume of the wedge, we have

[continued]

$$\begin{aligned}V &= (V_{22} - V_{21}) - (V_{12} - V_{11})\\ &= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\rho_2^3(1 - \cos\phi_2) - \rho_2^3(1 - \cos\phi_1) - \rho_1^3(1 - \cos\phi_2) + \rho_1^3(1 - \cos\phi_1)\right]\\ &= \tfrac{1}{3}(\theta_2 - \theta_1)\left[(\rho_2^3 - \rho_1^3)(1 - \cos\phi_2) - (\rho_2^3 - \rho_1^3)(1 - \cos\phi_1)\right] = \tfrac{1}{3}(\theta_2 - \theta_1)\left[(\rho_2^3 - \rho_1^3)(\cos\phi_1 - \cos\phi_2)\right]\end{aligned}$$

Or: Show that $V = \displaystyle\int_{\theta_1}^{\theta_2}\int_{\rho_1\sin\phi_1}^{\rho_2\sin\phi_2}\int_{r\cot\phi_2}^{r\cot\phi_1} r\,dz\,dr\,d\theta$.

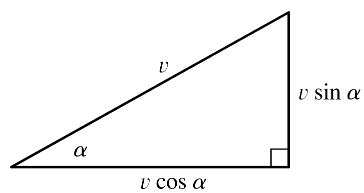
(c) By the Mean Value Theorem with $f(\rho) = \rho^3$ there exists some $\tilde{\rho}$ with $\rho_1 \le \tilde{\rho} \le \rho_2$ such that $f(\rho_2) - f(\rho_1) = f'(\tilde{\rho})(\rho_2 - \rho_1)$ or $\rho_2^3 - \rho_1^3 = 3\tilde{\rho}^2\Delta\rho$. Similarly, with $f(\phi) = \cos\phi$ there exists some $\tilde{\phi}$ with $\phi_1 \le \tilde{\phi} \le \phi_2$ such that $\cos\phi_2 - \cos\phi_1 = \left(-\sin\tilde{\phi}\right)\Delta\phi$. Substituting into the result from (b) gives $\Delta V = \frac{1}{3}(\theta_2 - \theta_1)(3\tilde{\rho}^2\,\Delta\rho)(\sin\tilde{\phi})\;\Delta\phi = \tilde{\rho}^2\sin\tilde{\phi}\,\Delta\rho\,\Delta\theta\,\Delta\phi$.

APPLIED PROJECT Roller Derby

1. $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad\Rightarrow\quad 2gh = v^2 + \dfrac{I}{m}\left(\dfrac{v}{r}\right)^2 \quad\Rightarrow\quad 2gh = v^2\left(1 + \dfrac{I}{mr^2}\right) \quad\Rightarrow\quad v^2 = \dfrac{2gh}{1 + I^*}$.

2. The vertical component of the speed is $dy/dt = v\sin\alpha$, so by the same reasoning as used in Problem 1,

$$\frac{dy}{dt} = \sqrt{\frac{2gy}{1 + I^*}}\sin\alpha = \sqrt{\frac{2g}{1 + I^*}}\sin\alpha\,\sqrt{y}.$$



3. Solving the separable differential equation, we get $\dfrac{dy}{\sqrt{y}} = \sqrt{\dfrac{2g}{1 + I^*}}\sin\alpha\,dt \quad\Rightarrow\quad 2\sqrt{y} = \sqrt{\dfrac{2g}{1 + I^*}}(\sin\alpha)t + C$.

But $y = 0$ when $t = 0$, so $C = 0$ and we have $2\sqrt{y} = \sqrt{\dfrac{2g}{1 + I^*}}(\sin\alpha)t$. Solving for t when $y = h$ gives

$$T = \frac{2\sqrt{h}}{\sin\alpha}\sqrt{\frac{1 + I^*}{2g}} = \sqrt{\frac{2h(1 + I^*)}{g\sin^2\alpha}}.$$

4. Assume that the length of each cylinder is ℓ. Then the density of the solid cylinder is $\dfrac{m}{\pi r^2\ell}$, and from Formulas 15.6.16, its moment of inertia (using cylindrical coordinates) is

$$I_z = \iiint (x^2 + y^2)\frac{m}{\pi r^2\ell}\,dV = \frac{m}{\pi r^2\ell}\int_0^{\ell}\int_0^{2\pi}\int_0^r R^2\,R\,dR\,d\theta\,dz = \frac{m}{\pi r^2\ell}\cdot\ell\cdot 2\pi\cdot\left[\tfrac{1}{4}R^4\right]_0^r = \frac{mr^2}{2}$$

and so $I^* = \dfrac{I_z}{mr^2} = \dfrac{1}{2}$.

For the hollow cylinder, we consider its entire mass to lie a distance r from the axis of rotation, so $x^2 + y^2 = r^2$ is a constant. We express the density in terms of mass per unit area as $\rho = \dfrac{m}{2\pi r\ell}$, and then the moment of inertia is calculated as a double integral: $I_z = \displaystyle\iint (x^2 + y^2)\frac{m}{2\pi r\ell}\,dA = \frac{mr^2}{2\pi r\ell}\iint dA = mr^2$, so $I^* = \dfrac{I_z}{mr^2} = 1$.

5. The volume of such a ball is $\frac{4}{3}\pi(r^3 - a^3) = \frac{4}{3}\pi[r^3 - (br)^3] = \frac{4}{3}\pi r^3(1 - b^3)$, and so its density is $\dfrac{m}{\frac{4}{3}\pi r^3(1-b^3)}$. Now

$$\begin{aligned} I_z &= \iiint (x^2 + y^2)\,\frac{m}{\frac{4}{3}\pi r^3(1-b^3)}\,dV \\ &= \frac{m}{\frac{4}{3}\pi r^3(1-b^3)} \int_a^r \int_0^{2\pi} \int_0^{\pi} (\rho^2 \sin^2\phi)(\rho^2 \sin\phi)\,d\phi\,d\theta\,d\rho \qquad \text{[from Formula 15.8.3 and Exercise 15.8.25]} \\ &= \frac{m}{\frac{4}{3}\pi r^3(1-b^3)} \left[\frac{\rho^5}{5}\right]_a^r \cdot \Big[\theta\Big]_0^{2\pi} \cdot \left[-\frac{(2+\sin^2\phi)\cos\phi}{3}\right]_0^{\pi} \qquad \text{[from Formula 67 in the Table of Integrals]} \\ &= \frac{m}{\frac{4}{3}\pi r^3(1-b^3)} \cdot \frac{r^5 - a^5}{5} \cdot 2\pi \cdot \frac{4}{3} = \frac{2mr^5(1-b^5)}{5r^3(1-b^3)} = \frac{2(1-b^5)mr^2}{5(1-b^3)} \end{aligned}$$

Therefore, $I^* = \dfrac{I_z}{mr^2} = \dfrac{2(1-b^5)}{5(1-b^3)}$. Since a represents the inner radius, $a \to 0$ corresponds to a solid ball, and $a \to r$ corresponds to a hollow ball.

6. For a solid ball, $a \to 0 \;\Rightarrow\; b \to 0$, so $I^* = \lim\limits_{b\to 0} \dfrac{2(1-b^5)}{5(1-b^3)} = \dfrac{2}{5}$. For a hollow ball, $a \to r \;\Rightarrow\; b \to 1$, so

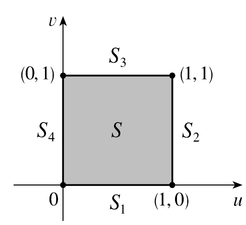
$$I^* = \lim_{b\to 1} \frac{2(1-b^5)}{5(1-b^3)} = \frac{2}{5} \lim_{b\to 1} \frac{-5b^4}{-3b^2} = \frac{2}{5}\left(\frac{5}{3}\right) = \frac{2}{3} \qquad \text{[by l'Hospital's Rule]}$$

Note: We could instead have calculated $I^* = \lim\limits_{b\to 1} \dfrac{2(1-b)(1+b+b^2+b^3+b^4)}{5(1-b)(1+b+b^2)} = \dfrac{2\cdot 5}{5\cdot 3} = \dfrac{2}{3}$.

Thus the objects finish in the following order: solid ball $\left(I^* = \frac{2}{5}\right)$, solid cylinder $\left(I^* = \frac{1}{2}\right)$, hollow ball $\left(I^* = \frac{2}{3}\right)$, hollow cylinder $(I^* = 1)$.

15.9 Change of Variables in Multiple Integrals

1. For Exercise 1(a)–(f), we refer to the figure. Each transformation maps the boundary of S to the boundary of one of the images (I–VI).



(a) Along S_1, $v = 0$, $0 \le u \le 1$, so $x = u + v = u$ and $y = u - v = u \;\Rightarrow\; y = x$, $0 \le x \le 1$. [Note that only images V and VI have $y = x$ as a boundary.] Along S_2, $u = 1$, $0 \le v \le 1$, so $x = 1 + v$ and $y = 1 - v$. Eliminating v gives $x + y = 2$, $1 \le x \le 2$. Along S_3, $v = 1$, $0 \le u \le 1 \;\Rightarrow\; (x)x = u + 1$ and $y = u - 1$. Eliminating u gives $y - x = -2$, $1 \le x \le 2$. Finally, along S_4, $u = 0$, $0 \le v \le 1 \;\Rightarrow\; x = v$ and $y = -v \;\Rightarrow\; y = -x$, $0 \le x \le 1$. Thus, VI is the image of the transformation.

(b) Along S_1, $v = 0$, $0 \le u \le 1 \;\Rightarrow\; y = uv = 0$, $0 \le x \le 1$. Along S_2, $u = 1$, $0 \le v \le 1 \;\Rightarrow\; x = u - v = 1 - v$, $y = v \;\Rightarrow\; y = 1 - x$, $0 \le x \le 1$. Along S_3, $v = 1$, $0 \le u \le 1 \;\Rightarrow\; x = u - 1$, $y = u \;\Rightarrow\; y = x + 1$, $-1 \le x \le 0$. Finally, along S_4, $u = 0$, $0 \le v \le 1 \;\Rightarrow\; y = 0$, $-1 \le x \le 0$. Thus, I is the image of the transformation.

(c) Along S_1, $v = 0$, $0 \le u \le 1 \;\Rightarrow\; y = u \sin v = 0$ and $x = u \cos v = u \cos 0 = u \;\Rightarrow\; 0 \le x \le 1$. Along S_2, $u = 1$, $0 \le v \le 1 \;\Rightarrow\; x = \cos v$, $y = \sin v$, $0 \le v \le 1$, which are parametric equations for a circle of radius 1,

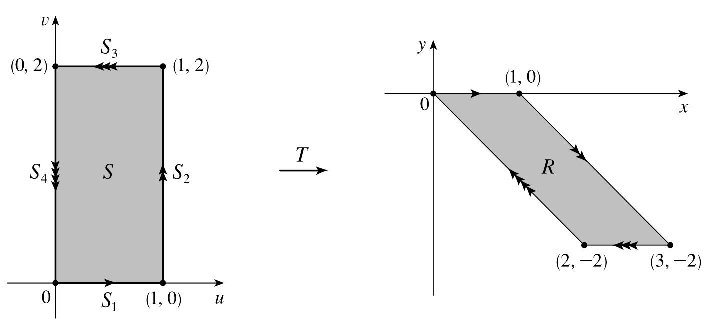
$1 \geq x \geq \cos 1$. Along S_3, $v = 1$, $0 \leq u \leq 1 \Rightarrow x = u\cos 1$, $y = u \sin 1 \Rightarrow y = \tan(1)x$, $0 \leq x \leq \cos 1$. Along S_4, $u = 0 \Rightarrow x = y = 0$. Thus, IV is the image of the transformation.

(d) Along S_1, $v = 0$, $0 \leq u \leq 1 \Rightarrow x = u - v = u$, $y = u + v^2 = u \Rightarrow y = x$, $0 \leq x \leq 1$. Along S_2, $u = 1$, $0 \leq v \leq 1 \Rightarrow x = 1 - v$, $y = 1 + v^2 \Rightarrow y = 1 + (1-x)^2$, $1 \geq x \geq 0$. Along S_3, $v = 1$, $0 \leq u \leq 1 \Rightarrow x = u - 1$, $y = u + 1 \Rightarrow y = x + 2$, $-1 \leq x \leq 0$. Finally, along S_4, $u = 0$, $0 \leq v \leq 1 \Rightarrow x = -v$, $y = v^2 \Rightarrow y = x^2$, $-1 \leq x \leq 0$. Thus, V is the image of the transformation.

(e) Along S_1, $v = 0$, $0 \leq u \leq 1 \Rightarrow y = 2v = 0$, and since $x = u + v = u$, $0 \leq x \leq 1$. Along S_2, $u = 1$, $0 \leq v \leq 1 \Rightarrow x = 1 + v$, $y = 2v \Rightarrow y = 2x - 2$, $1 \leq x \leq 2$. Along S_3, $v = 1$, $0 \leq u \leq 1 \Rightarrow y = 2v = 2$, $1 \leq x \leq 2$. Finally, along S_4, $u = 0$, $0 \leq v \leq 1 \Rightarrow x = v$, $y = 2v \Rightarrow y = 2x$, $0 \leq x \leq 1$. Thus, III is the image of the transformation.

(f) Along S_1, $v = 0$, $0 \leq u \leq 1 \Rightarrow x = uv = 0$, and since $y = u^3 - v^3 = u^3$, $0 \leq y \leq 1$. Along S_2, $u = 1$, $0 \leq u \leq 1 \Rightarrow x = v$, $y = 1 - v^3 \Rightarrow y = 1 - x^3$, $1 \geq x \geq 0$. Along S_3, $v = 1$, $0 \leq u \leq 1 \Rightarrow x = u$, $y = u^3 - 1 \Rightarrow y = x^3 - 1$, $0 \leq x \leq 1$. Finally, along S_4, $u = 0$, $0 \leq v \leq 1 \Rightarrow x = 0$, $-1 \leq y \leq 0$. Thus, II is the image of the transformation.

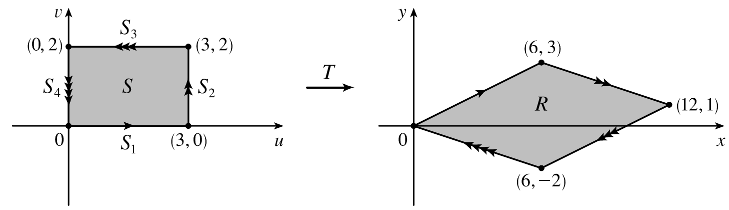
2. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0$, $0 \leq u \leq 1$, so $x = u + v = u$ and $y = -v = 0$. Therefore, the image is the line segment $y = 0$, $0 \leq x \leq 1$. S_2 is the line segment $u = 1$, $0 \leq v \leq 2$, so $x = 1 + v$ and $y = -v$. Eliminating v, we have $y = 1 - x$, $1 \leq x \leq 3$. S_3 is the line segment $v = 2$, $0 \leq u \leq 1$, so $x = u + 2$ and $y = -2 \Rightarrow$ the image is the line segment $y = -2$, $2 \leq x \leq 3$. Finally, S_4 is the line segment $u = 0$, $0 \leq v \leq 2$, so $x = v$ and $y = -v \Rightarrow$ the image is the line segment $y = -x$, $0 \leq x \leq 2$. The image of the set S is the region R shown in the xy-plane, a parallelogram bounded by these four line segments.



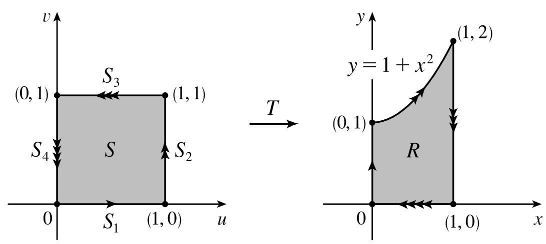
3. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0$, $0 \leq u \leq 3$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u, we have $x = 2y$, $0 \leq x \leq 6$. S_2 is the line segment $u = 3$, $0 \leq v \leq 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \Rightarrow x = 6 + 3(3 - y) = 15 - 3y$, $6 \leq x \leq 12$. S_3 is the line segment $v = 2$, $0 \leq u \leq 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \Rightarrow x = 2y + 10$, $6 \leq x \leq 12$. Finally, S_4 is the segment $u = 0$, $0 \leq v \leq 2$, so $x = 3v$ and $y = -v \Rightarrow x = -3y$, $0 \leq x \leq 6$.

[continued]

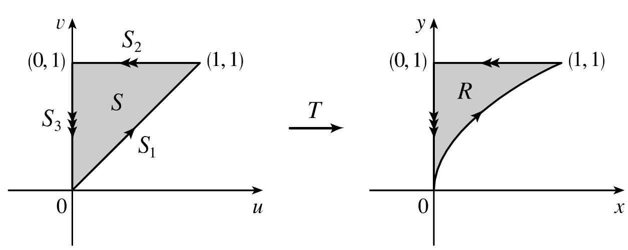
The image of set S is the region R shown in the xy-plane, a parallelogram bounded by these four segments.



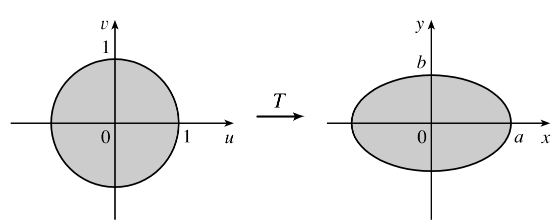
4. S_1 is the line segment $v = 0$, $0 \leq u \leq 1$, so $x = v = 0$ and $y = u(1 + v^2) = u$. Since $0 \leq u \leq 1$, the image is the line segment $x = 0$, $0 \leq y \leq 1$. S_2 is the segment $u = 1$, $0 \leq v \leq 1$, so $x = v$ and $y = u(1 + v^2) = 1 + x^2$. Thus the image is the portion of the parabola $y = 1 + x^2$ for $0 \leq x \leq 1$. S_3 is the segment $v = 1$, $0 \leq u \leq 1$, so $x = 1$ and $y = 2u$. The image is the segment $x = 1$, $0 \leq y \leq 2$. S_4 is described by $u = 0$, $0 \leq v \leq 1$, so $0 \leq x = v \leq 1$ and $y = u(1 + v^2) = 0$. The image is the line segment $y = 0$, $0 \leq x \leq 1$. Thus, the image of S is the region R bounded by the parabola $y = 1 + x^2$, the x-axis, and the lines $x = 0$, $x = 1$.



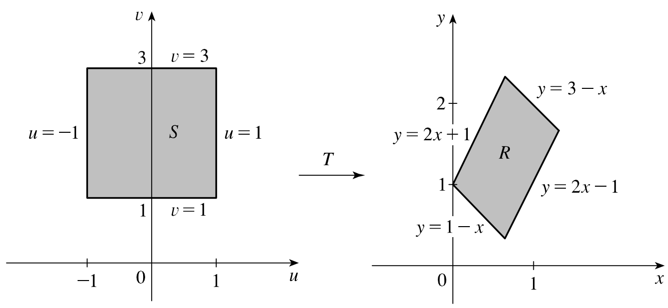
5. S_1 is the line segment $u = v$, $0 \leq u \leq 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \leq u \leq 1$, the image is the portion of the parabola $x = y^2$, $0 \leq y \leq 1$. S_2 is the segment $v = 1$, $0 \leq u \leq 1$, thus $y = v = 1$ and $x = u^2$, so $0 \leq x \leq 1$. The image is the line segment $y = 1$, $0 \leq x \leq 1$. S_3 is the segment $u = 0$, $0 \leq v \leq 1$, so $x = u^2 = 0$ and $y = v \Rightarrow 0 \leq y \leq 1$. The image is the segment $x = 0$, $0 \leq y \leq 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x = y^2$, the y-axis, and the line $y = 1$.



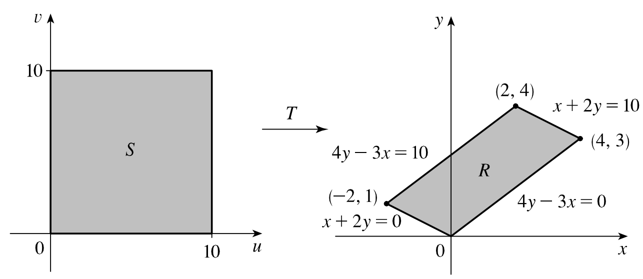
6. Substituting $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$ into $u^2 + v^2 \leq 1$ gives $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$, so the image of $u^2 + v^2 \leq 1$ is the elliptical region $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$.



7. R is a parallelogram enclosed by the parallel lines $y = 2x - 1$, $y = 2x + 1$ and the parallel lines $y = 1 - x$, $y = 3 - x$. The first pair of equations can be written as $y - 2x = -1$, $y - 2x = 1$. If we let $u = y - 2x$ then these lines are mapped to the vertical lines $u = -1$, $u = 1$ in the uv-plane. Similarly, the second pair of equations can be written as $x + y = 1$, $x + y = 3$, and setting $v = x + y$ maps these lines to the horizontal lines $v = 1$, $v = 3$ in the uv-plane. Boundary curves are mapped to boundary curves under a transformation, so here the equations $u = y - 2x$, $v = x + y$ define a transformation T^{-1} that maps R in the xy-plane to the square S enclosed by the lines $u = -1$, $u = 1$, $v = 1$, $v = 3$ in the uv-plane. To find the transformation T that maps S to R we solve $u = y - 2x$, $v = x + y$ for x, y: Subtracting the first equation from the second gives $v - u = 3x \quad \Rightarrow \quad x = \frac{1}{3}(v - u)$ and adding twice the second equation to the first gives $u + 2v = 3y \quad \Rightarrow$ $y = \frac{1}{3}(u + 2v)$. Thus one possible transformation T (there are many) is given by $x = \frac{1}{3}(v - u)$, $y = \frac{1}{3}(u + 2v)$.

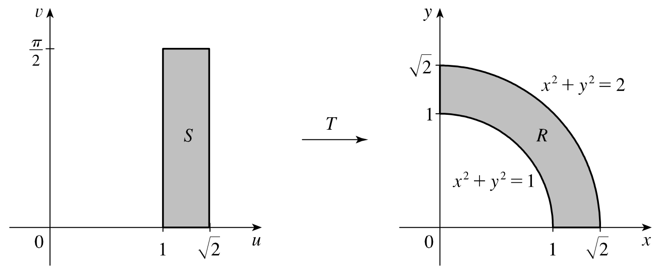


8. The boundaries of the parallelogram R are the lines $y = \frac{3}{4}x$ or $4y - 3x = 0$, $y = \frac{3}{4}x + \frac{5}{2}$ or $4y - 3x = 10$, $y = -\frac{1}{2}x$ or $x + 2y = 0$, $y = -\frac{1}{2}x + 5$ or $x + 2y = 10$. Setting $u = 4y - 3x$ and $v = x + 2y$ defines a transformation T^{-1} that maps R in the xy-plane to the square S enclosed by the lines $u = 0$, $u = 10$, $v = 0$, $v = 10$ in the uv-plane. Solving $u = 4y - 3x$, $v = x + 2y$ for x and y gives $2v - u = 5x \quad \Rightarrow \quad x = \frac{1}{5}(2v - u)$, $u + 3v = 10y \quad \Rightarrow \quad y = \frac{1}{10}(u + 3v)$. Thus one possible transformation T is given by $x = \frac{1}{5}(2v - u)$, $y = \frac{1}{10}(u + 3v)$.

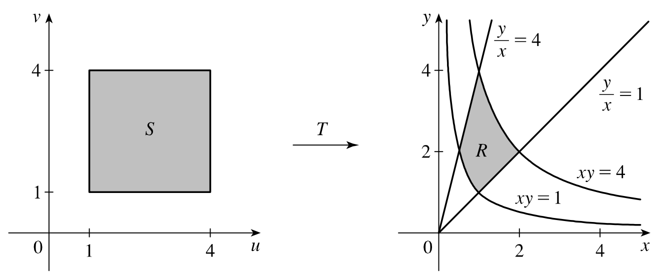


9. R is a portion of an annular region (see the figure) that is easily described in polar coordinates as $R = \left\{(r, \theta) \mid 1 \leq r \leq \sqrt{2},\ 0 \leq \theta \leq \pi/2\right\}$. If we converted a double integral over R to polar coordinates the resulting region of integration is a rectangle (in the $r\theta$-plane), so we can create a transformation T here by letting u play the role of r and v the role of θ. Thus T is defined by $x = u \cos v$, $y = u \sin v$ and T maps the rectangle

$S = \left\{(u,v) \mid 1 \le u \le \sqrt{2},\ 0 \le v \le \pi/2\right\}$ in the uv-plane to R in the xy-plane.



10. The boundaries of the region R are the curves $y = 1/x$ or $xy = 1$, $y = 4/x$ or $xy = 4$, $y = x$ or $y/x = 1$, $y = 4x$ or $y/x = 4$. Setting $u = xy$ and $v = y/x$ defines a transformation T^{-1} that maps R in the xy-plane to the square S enclosed by the lines $u = 1$, $u = 4$, $v = 1$, $v = 4$ in the uv-plane. Solving $u = xy$, $v = y/x$ for x and y gives $x^2 = u/v \quad \Rightarrow$ $x = \sqrt{u/v}$ [since x, y, u, v are all positive], $y^2 = uv \quad \Rightarrow \quad y = \sqrt{uv}$. Thus one possible transformation T is given by $x = \sqrt{u/v}$, $y = \sqrt{uv}$.



11. $x = 2u + v,\ y = 4u - v$.

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = (2)(-1) - (1)(4) = -6$.

12. $x = u^2 + uv,\ y = uv^2$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2u+v & u \\ v^2 & 2uv \end{vmatrix} = (2u+v)(2uv) - u(v^2) = 4u^2v + 2uv^2 - uv^2 = 4u^2v + uv^2$$

13. $x = s\cos t,\ y = s\sin t$.

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix} = \begin{vmatrix} \cos t & -s\sin t \\ \sin t & s\cos t \end{vmatrix} = s\cos^2 t - (-s\sin^2 t) = s(\cos^2 t + \sin^2 t) = s$$

14. $x = pe^q,\ y = qe^p$.

$$\frac{\partial(x,y)}{\partial(p,q)} = \begin{vmatrix} \partial x/\partial p & \partial x/\partial q \\ \partial y/\partial p & \partial y/\partial q \end{vmatrix} = \begin{vmatrix} e^q & pe^q \\ qe^p & e^p \end{vmatrix} = e^q e^p - pe^q \cdot qe^p = e^{p+q} - pqe^{p+q} = (1-pq)e^{p+q}$$

15. $x = uv,\ y = vw,\ z = wu.$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} v & u & 0 \\ 0 & w & v \\ w & 0 & u \end{vmatrix} = v\begin{vmatrix} w & v \\ 0 & u \end{vmatrix} - u\begin{vmatrix} 0 & v \\ w & u \end{vmatrix} + 0\begin{vmatrix} 0 & w \\ w & 0 \end{vmatrix}$$

$$= v(uw - 0) - u(0 - vw) + 0 = uvw + uvw = 2uvw$$

16. $x = u + vw,\ y = v + wu,\ z = w + uv.$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} 1 & w & v \\ w & 1 & u \\ v & u & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & u \\ u & 1 \end{vmatrix} - w\begin{vmatrix} w & u \\ v & 1 \end{vmatrix} + v\begin{vmatrix} w & 1 \\ v & u \end{vmatrix}$$

$$= 1(1 - u^2) - w\,(w - uv) + v\,(uw - v)$$

$$= 1 - u^2 - w^2 + uvw + uvw - v^2 = 1 + 2uvw - u^2 - v^2 - w^2$$

17. $x = 2u + v, y = u + 2v \quad\Rightarrow\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$. The integrand $x - 3y = (2u + v) - 3(u + 2v) = -u - 5v$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0,0)$ and $(2,1)$ is $y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u + v) \quad\Rightarrow\quad v = 0$; the line through $(2,1)$ and $(1,2)$ is $x + y = 3$ which is the image of $(2u + v) + (u + 2v) = 3 \quad\Rightarrow\quad u + v = 1$; the line through $(0,0)$ and $(1,2)$ is $y = 2x$ which is the image of $u + 2v = 2(2u + v) \quad\Rightarrow\quad u = 0$. Thus S is the triangle $0 \le v \le 1 - u$, $0 \le u \le 1$ in the uv-plane and

$$\iint_R (x - 3y)\,dA = \int_0^1 \int_0^{1-u} (-u - 5v)\,|3|\;dv\,du = -3\int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_{v=0}^{v=1-u}\;du$$

$$= -3\int_0^1 \left(u - u^2 + \tfrac{5}{2}(1-u)^2\right)\,du = -3\left[\tfrac{1}{2}u^2 - \tfrac{1}{3}u^3 - \tfrac{5}{6}(1-u)^3\right]_0^1 = -3\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{5}{6}\right) = -3$$

18. $x = \frac{1}{4}(u + v), y = \frac{1}{4}(u - 3u) \quad\Rightarrow\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = \dfrac{1}{4}$. The integrand $4x + 8y = 4 \cdot \frac{1}{4}(u + v) + 8 \cdot \frac{1}{4}(v - 3u) = 3v - 5u$. R is a parallelogram bounded by the lines $x - y = -4$, $x - y = 4$, $3x + y = 0$, $3x + y = 8$. Since $u = x - y$ and $v = 3x + y$, R is the image of the rectangle enclosed by the lines $u = -4$, $u = 4$, $v = 0$, and $v = 8$. Thus

$$\iint_R (4x + 8y)\,dA = \int_{-4}^4 \int_0^8 (3v - 5u)\left|\tfrac{1}{4}\right|\,dv\,du = \tfrac{1}{4}\int_{-4}^4 \left[\tfrac{3}{2}v^2 - 5uv\right]_{v=0}^{v=8}\,du$$

$$= \tfrac{1}{4}\int_{-4}^4 (96 - 40u)\,du = \tfrac{1}{4}\left[96u - 20u^2\right]_{-4}^4 = 192$$

19. $x = 2u, y = 3v \quad\Rightarrow\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$. The integrand $x^2 = 4u^2$. The planar ellipse $9x^2 + 4y^2 \le 36$ is the image of the disk $u^2 + v^2 \le 1$. Thus,

[continued]

$$\iint_R x^2\,dA = \iint\limits_{u^2+v^2\le 1} (4u^2)(6)\,du\,dv = \int_0^{2\pi}\int_0^1 (24r^2\cos^2\theta)\,r\,dr\,d\theta = 24\int_0^{2\pi}\cos^2\theta\,d\theta\int_0^1 r^3\,dr$$
$$= 24\left[\tfrac12 x + \tfrac14\sin 2x\right]_0^{2\pi}\left[\tfrac14 r^4\right]_0^1 = 24(\pi)\left(\tfrac14\right) = 6\pi$$

20. $x = \sqrt{2}\,u - \sqrt{2/3}\,v$, $y = \sqrt{2}\,u + \sqrt{2/3}\,v \quad\Rightarrow\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \dfrac{4}{\sqrt{3}}$. The integrand

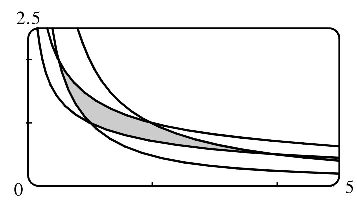
$x^2 - xy + y^2 = 2u^2 + 2v^2$. The planar ellipse $x^2 - xy + y^2 \le 2$ is the image of the disk $u^2 + v^2 \le 1$. Thus,

$$\iint_R (x^2 - xy + y^2)\,dA = \iint\limits_{u^2+v^2\le 1} (2u^2 + 2v^2)\left(\tfrac{4}{\sqrt{3}}\,du\,dv\right) = \int_0^{2\pi}\int_0^1 \tfrac{8}{\sqrt{3}}r^3\,dr\,d\theta = \tfrac{4\pi}{\sqrt{3}}$$

21. $x = u/v$, $y = v \quad\Rightarrow\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$. The integrand $xy = u$. The line $y = x$ $[v = u/v]$ is the image of the parabola $v^2 = u$ and the line $y = 3x$ $[v = 3u/v]$ is the image of the parabola $v^2 = 3u$. The hyperbolas $xy = 1$ and $xy = 3$ are the images of the lines $u = 1$ and $u = 3$, respectively. Thus,

$$\iint_R xy\,dA = \int_1^3\int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right)dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right)du = \int_1^3 u\ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3.$$

22. $u = xy$, $v = xy^2$. To solve for x and y in terms of u and v, try dividing.



$\dfrac{xy^2}{xy} = \dfrac{v}{u} \quad\Rightarrow\quad y = \dfrac{v}{u}$. Also, $\dfrac{(xy)^2}{xy^2} = \dfrac{u^2}{v} \quad\Rightarrow\quad x = \dfrac{u^2}{v}$. Then

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{vmatrix} = \dfrac{1}{v}$. The integrand $y^2 = v^2/u^2$. R is the image of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$. Thus,

$$\iint_R y^2\,dA = \int_1^2\int_1^2 \frac{v^2}{u^2}\left(\frac{1}{v}\right)du\,dv = \int_1^2 \frac{v}{2}\,dv = \frac{3}{4}$$

23. (a) $x = au$, $y = bv$, $z = cw \quad\Rightarrow\quad \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$. Since $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$, $w = \dfrac{z}{c}$, the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \le 1$. Thus,

$$\iiint_E dV = \iiint\limits_{u^2+v^2+w^2\,\le\,1} abc\,du\,dv\,dw = (abc)(\text{volume of the ball}) = \tfrac43\pi abc$$

(b) If we approximate the surface of the earth by the ellipsoid $\dfrac{x^2}{6378^2} + \dfrac{y^2}{6378^2} + \dfrac{z^2}{6356^2} = 1$, then we can estimate the volume of the earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV = \frac43\pi(6378)(6378)(6356) \approx 1.083 \times 10^{12}\ \text{km}^3$.

(c) The moment of intertia about the z-axis is $I_z = \iiint_E \left(x^2 + y^2\right)\rho(x,y,z)\,dV$, where E is the solid enclosed by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. As in part (a), we use the transformation $x = au$, $y = bv$, $z = cw$, so $\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| = abc$ and

$$I_z = \iiint_E \left(x^2+y^2\right) k\,dV = \iiint\limits_{u^2+v^2+w^2 \le 1} k(a^2u^2+b^2v^2)(abc)\,du\,dv\,dw$$

$$= abck \int_0^{\pi}\int_0^{2\pi}\int_0^1 (a^2\rho^2\sin^2\phi\cos^2\theta + b^2\rho^2\sin^2\phi\sin^2\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= abck\left[a^2\int_0^{\pi}\int_0^{2\pi}\int_0^1 (\rho^2\sin^2\phi\cos^2\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi + b^2\int_0^{\pi}\int_0^{2\pi}\int_0^1 (\rho^2\sin^2\phi\sin^2\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\right]$$

$$= a^3bck\int_0^{\pi}\sin^3\phi\,d\phi\int_0^{2\pi}\cos^2\theta\,d\theta\int_0^1\rho^4\,d\rho + ab^3ck\int_0^{\pi}\sin^3\phi\,d\phi\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^1\rho^4\,d\rho$$

$$= a^3bck\left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi}\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{5}\rho^5\right]_0^1 + ab^3ck\left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{5}\rho^5\right]_0^1$$

$$= a^3bck\left(\tfrac{4}{3}\right)(\pi)\left(\tfrac{1}{5}\right) + ab^3ck\left(\tfrac{4}{3}\right)(\pi)\left(\tfrac{1}{5}\right) = \tfrac{4}{15}\pi(a^2+b^2)abck$$

24. R is the region enclosed by the curves $xy=a$, $xy=b$, $xy^{1.4}=c$, and $xy^{1.4}=d$, so if we let $u=xy$ and $v=xy^{1.4}$ then R is the image of the rectangle enclosed by the lines $u=a$, $u=b$ $(a<b)$ and $v=c$, $v=d$ $(c<d)$. Now $x=u/y \quad\Rightarrow\quad v=(u/y)y^{1.4}=uy^{0.4} \quad\Rightarrow\quad y^{0.4}=u^{-1}v \quad\Rightarrow\quad y=(u^{-1}v)^{1/0.4}=u^{-2.5}v^{2.5}$ and $x=uy^{-1}=u(u^{-2.5}v^{2.5})^{-1}=u^{3.5}v^{-2.5}$, so

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3.5u^{2.5}v^{-2.5} & -2.5u^{3.5}v^{-3.5} \\ -2.5u^{-3.5}v^{2.5} & 2.5u^{-2.5}v^{1.5} \end{vmatrix} = 8.75v^{-1} - 6.25v^{-1} = 2.5v^{-1}$$

Thus the area of R, and the work done by the engine, is

$$\iint_R dA = \int_a^b\int_c^d \left|2.5v^{-1}\right|\,dv\,du = 2.5\int_a^b du\int_c^d (1/v)\,dv = 2.5\big[u\big]_a^b\,\big[\ln|v|\,\big]_c^d = 2.5(b-a)(\ln d - \ln c) = 2.5(b-a)\ln\frac{d}{c}.$$

25. Letting $u=x-2y$ and $v=3x-y$, we have $x=\frac{1}{5}(2v-u)$ and $y=\frac{1}{5}(v-3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \dfrac{1}{5}$.

R is the image of the rectangle enclosed by the lines $u=0$, $u=4$, $v=1$, and $v=8$. Thus,

$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_0^4\int_1^8 \frac{u}{v}\left|\frac{1}{5}\right| dv\,du = \frac{1}{5}\int_0^4 u\,du\int_1^8 \frac{1}{v}\,dv = \tfrac{1}{5}\left[\tfrac{1}{2}u^2\right]_0^4\big[\ln|v|\,\big]_1^8 = \tfrac{8}{5}\ln 8$$

26. Letting $u=x+y$ and $v=x-y$, we have $x=\frac{1}{2}(u+v)$ and $y=\frac{1}{2}(u-v)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\dfrac{1}{2}$.

R is the image of the rectangle enclosed by the lines $u=0$, $u=3$, $v=0$, and $v=2$. Thus,

$$\iint_R (x+y)\,e^{x^2-y^2}\,dA = \int_0^3\int_0^2 ue^{uv}\left|-\tfrac{1}{2}\right| dv\,du = \tfrac{1}{2}\int_0^3\left[e^{uv}\right]_{v=0}^{v=2} du = \tfrac{1}{2}\int_0^3 (e^{2u}-1)\,du$$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}e^{2u} - u\right]_0^3 = \tfrac{1}{2}\left(\tfrac{1}{2}e^6 - 3 - \tfrac{1}{2}\right) = \tfrac{1}{4}(e^6-7)$$

27. Letting $u=y-x$ and $v=y+x$, we have $x=\frac{1}{2}(v-u)$ and $y=\frac{1}{2}(u+v)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\dfrac{1}{2}$.

R is the image of the trapezoidal region with vertices $(-1,1)$, $(-2,2)$, $(2,2)$, and $(1,1)$. Thus,

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \int_1^2\int_{-v}^{v} \cos\frac{u}{v}\left|-\frac{1}{2}\right| du\,dv = \frac{1}{2}\int_1^2\left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v} dv$$

$$= \tfrac{1}{2}\int_1^2 [v\sin(1) - v\sin(-1)]\,dv = \tfrac{1}{2}\int_1^2 2v\sin 1\,dv$$

$$= \sin 1\left[\tfrac{1}{2}v^2\right]_1^2 = \tfrac{3}{2}\sin 1$$

28. Letting $u = 3x$ and $v = 2y$, we have $9x^2 + 4y^2 = u^2 + v^2$, $x = \frac{1}{3}u$, and $y = \frac{1}{2}v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/3 & 0 \\ 0 & 1/2 \end{vmatrix} = \dfrac{1}{6}$.

R is the image of the quarter-disk D given by $u^2 + v^2 \le 1$, $u \ge 0$, $v \ge 0$. Thus,

$$\iint_R \sin(9x^2 + 4y^2)\,dA = \iint_D \tfrac{1}{6}\sin(u^2 + v^2)\,du\,dv = \int_0^{\pi/2}\int_0^1 \tfrac{1}{6}\sin(r^2)\,r\,dr\,d\theta = \tfrac{\pi}{12}\left[-\tfrac{1}{2}\cos r^2\right]_0^1 = \tfrac{\pi}{24}(1 - \cos 1)$$

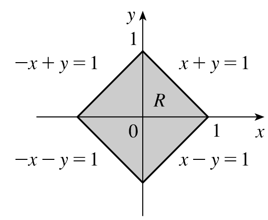
29. See the figure. Leting $u = x + y$ and $v = -x + y$, we have $x = \frac{1}{2}(u - v)$ and $y = \frac{1}{2}(u + v)$.

Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$. Now $|u| = |x + y| \le |x| + |y| \le 1 \;\Rightarrow$

$-1 \le u \le 1$, and $|v| = |-x + y| \le |x| + |y| \le 1 \;\Rightarrow\; -1 \le v \le 1$.

R is the image of the square region with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$.

Thus, $\iint_R e^{x+y}\,dA = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u\,du\,dv = \frac{1}{2}\left[e^u\right]_{-1}^{1}\left[v\right]_{-1}^{1} = e - e^{-1}$.



30. Letting $u = x + y$ and $v = y/x$, we have $x = \dfrac{u}{1+v}$ and $y = \dfrac{uv}{1+v}$. Then

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/(1+v) & -u/(1+v)^2 \\ v/(1+v) & u/(1+v)^2 \end{vmatrix} = \frac{u + uv}{(1+v)^3} = \frac{u}{(1+v)^2}$$

R is the image of the rectangle enclosed by the lines $u = 1$, $u = 3$, $v = 2$, and $v = 1/2$. Thus,

$$\begin{aligned} \iint_R \frac{y}{x}\,dA &= \int_{1/2}^{2}\int_1^3 v\left(\frac{u}{(1+v)^2}\right)du\,dv = \int_{1/2}^{2}\frac{v}{(1+v)^2}\,dv\int_1^3 u\,du \\ &= \left[\ln|1+v| + \frac{1}{1+v}\right]_{1/2}^{2}\left[\frac{u^2}{2}\right]_1^3 \qquad \left[\text{Use the substitution } w = 1+v \text{ for the first integral}\right] \\ &= \left(\ln 3 + \tfrac{1}{3} - \ln\tfrac{3}{2} - \tfrac{2}{3}\right)\cdot 4 = 4\ln 2 - \tfrac{4}{3} \end{aligned}$$

31. Letting $u = x + y$ and $v = y$, we have $x = u - v$ and $y = v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$.

R is the image under T of the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Thus,

$$\iint_R f(x+y)\,dA = \int_0^1\int_0^u (1)\,f(u)\,dv\,du = \int_0^1 f(u)\left[\,v\,\right]_{v=0}^{v=u}du = \int_0^1 u f(u)\,du, \text{ as desired.}$$

15 Review

TRUE-FALSE QUIZ

1. This is true by Fubini's Theorem.

2. False. $\int_0^1\int_0^x \sqrt{x + y^2}\,dy\,dx$ describes the region of integration as a Type I region. To reverse the order of integration, we must consider the region as a Type II region: $\int_0^1\int_y^1 \sqrt{x + y^2}\,dx\,dy$.

3. True by Equation 15.1.11.

4. $\int_{-1}^{1}\int_{0}^{1} e^{x^2+y^2}\sin y\,dx\,dy = \left(\int_0^1 e^{x^2}\,dx\right)\left(\int_{-1}^{1} e^{y^2}\sin y\,dy\right) = \left(\int_0^1 e^{x^2}\,dx\right)(0) = 0$, since $e^{y^2}\sin y$ is an odd function.

Therefore the statement is true.

5. True. By Equation 15.1.11 we can write $\int_0^1\int_0^1 f(x)\,f(y)\,dy\,dx = \int_0^1 f(x)\,dx\,\int_0^1 f(y)\,dy$. But $\int_0^1 f(y)\,dy = \int_0^1 f(x)\,dx$ so this becomes $\int_0^1 f(x)\,dx\,\int_0^1 f(x)\,dx = \left[\int_0^1 f(x)\,dx\right]^2$.

6. This statement is true because in the given region, $\left(x^2+\sqrt{y}\right)\sin(x^2y^2) \le (1+2)(1) = 3$, so

$\int_1^4\int_0^1 \left(x^2+\sqrt{y}\right)\sin(x^2y^2)\,dx\,dy \le \int_1^4\int_0^1 3\,dA = 3A(D) = 3(3) = 9$.

7. True. $\iint_D \sqrt{4-x^2-y^2}\,dA =$ the volume under the surface $x^2+y^2+z^2=4$ and above the xy-plane

$= \frac{1}{2}\left(\text{the volume of the sphere } x^2+y^2+z^2=4\right) = \frac{1}{2}\cdot\frac{4}{3}\pi(2)^3 = \frac{16}{3}\pi$

8. True. The moment of inertia about the z-axis of a solid E with constant density k is

$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iiint_E (kr^2)\,r\,dz\,dr\,d\theta = \iiint_E kr^3\,dz\,dr\,d\theta$.

9. The volume enclosed by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=2$ is, in cylindrical coordinates,

$V = \int_0^{2\pi}\int_0^2\int_r^2 r\,dz\,dr\,d\theta \ne \int_0^{2\pi}\int_0^2\int_r^2 dz\,dr\,d\theta$, so the assertion is false.

EXERCISES

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x,y)\,dA$ by a Riemann sum with $m=n=3$ and the sample points the upper right corners of each square, so

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i,y_j)\,\Delta A\\ &= \Delta A\left[f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2)+f(2,3)+f(3,1)+f(3,2)+f(3,3)\right]\end{aligned}$$

Using the contour lines to estimate the function values, we have

$$\iint_R f(x,y)\,dA \approx 1[2.7+4.7+8.0+4.7+6.7+10.0+6.7+8.6+11.9] \approx 64.0$$

2. As in Exercise 1, we have $m=n=3$ and $\Delta A = 1$. Using the contour map to estimate the value of f at the center of each subsquare, we have

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A\\ &= \Delta A\left[f(0.5,0.5)+(0.5,1.5)+(0.5,2.5)+(1.5,0.5)+f(1.5,1.5)\right.\\ &\qquad \left.+f(1.5,2.5)+(2.5,0.5)+f(2.5,1.5)+f(2.5,2.5)\right]\\ &\approx 1[1.2+2.5+5.0+3.2+4.5+7.1+5.2+6.5+9.0] = 44.2\end{aligned}$$

3. $\int_1^2\int_0^2 (y+2xe^y)\,dx\,dy = \int_1^2 \left[xy+x^2e^y\right]_{x=0}^{x=2} dy = \int_1^2 (2y+4e^y)\,dy = \left[y^2+4e^y\right]_1^2$

$= 4+4e^2-1-4e = 4e^2-4e+3$

4. $\int_0^1 \int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1} dy = \int_0^1 (e^y - 1)\,dy = \left[e^y - y\right]_0^1 = e - 2$

5. $\int_0^1 \int_0^x \cos(x^2)\,dy\,dx = \int_0^1 \left[\cos(x^2)y\right]_{y=0}^{y=x} dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big]_0^1 = \frac{1}{2}\sin 1$

6. $\int_0^1 \int_x^{e^x} 3xy^2\,dy\,dx = \int_0^1 \left[xy^3\right]_{y=x}^{y=e^x} dx = \int_0^1 (xe^{3x} - x^4)\,dx = \frac{1}{3}xe^{3x}\Big]_0^1 - \int_0^1 \frac{1}{3}e^{3x}\,dx - \left[\frac{1}{5}x^5\right]_0^1$ $\left[\begin{array}{c}\text{integrate by parts}\\ \text{in the first term}\end{array}\right]$

$= \frac{1}{3}e^3 - \left[\frac{1}{9}e^{3x}\right]_0^1 - \frac{1}{5} = \frac{2}{9}e^3 - \frac{4}{45}$

7. $\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx = \int_0^\pi \int_0^1 \left[(y\sin x)z\right]_{z=0}^{z=\sqrt{1-y^2}} dy\,dx = \int_0^\pi \int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$

$= \int_0^\pi \left[-\frac{1}{3}(1-y^2)^{3/2}\sin x\right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\Big]_0^\pi = \frac{2}{3}$

8. $\int_0^1 \int_0^y \int_x^1 6xyz\,dz\,dx\,dy = \int_0^1 \int_0^y \left[3xyz^2\right]_{z=x}^{z=1} dx\,dy = \int_0^1 \int_0^y (3xy - 3x^3y)\,dx\,dy$

$= \int_0^1 \left[\frac{3}{2}x^2y - \frac{3}{4}x^4y\right]_{x=0}^{x=y} dy = \int_0^1 \left(\frac{3}{2}y^3 - \frac{3}{4}y^5\right) dy = \left[\frac{3}{8}y^4 - \frac{1}{8}y^6\right]_0^1 = \frac{1}{4}$

9. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 4, 0 \le \theta \le \pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^\pi \int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

10. The region R is a type II region that can be described as the region enclosed by the lines $y = 4 - x$, $y = 4 + x$, and the x-axis. So using rectangular coordinates, we can say $R = \{(x,y) \mid 0 \le y \le 4, y - 4 \le x \le 4 - y\}$ and $\iint_R f(x,y)\,dA = \int_0^4 \int_{y-4}^{4-y} f(x,y)\,dx\,dy$.

11. $x = r\cos\theta = 2\sqrt{3}\cos\frac{\pi}{3} = 2\sqrt{3}\cdot\frac{1}{2} = \sqrt{3}$, $y = r\sin\theta = 2\sqrt{3}\sin\frac{\pi}{3} = 2\sqrt{3}\cdot\frac{\sqrt{3}}{2} = 3$, $z = 2$, so in rectangular coordinates the point is $(\sqrt{3}, 3, 2)$. $\rho = \sqrt{r^2 + z^2} = \sqrt{12 + 4} = 4$, $\theta = \frac{\pi}{3}$, and $\cos\phi = z/\rho = \frac{1}{2}$, so $\phi = \frac{\pi}{3}$ and spherical coordinates are $\left(4, \frac{\pi}{3}, \frac{\pi}{3}\right)$.

12. $r = \sqrt{4+4} = 2\sqrt{2}$; $z = -1$; $\tan\theta = \frac{2}{2} = 1$ and the point $(2,2)$ is in the first quadrant of the xy-plane, so $\theta = \frac{\pi}{4}$. Thus in cylindrical coordinates the point is $\left(2\sqrt{2}, \frac{\pi}{4}, -1\right)$. $\rho = \sqrt{4+4+1} = 3$, $\cos\phi = z/\rho = -\frac{1}{3}$, so the spherical coordinates are $\left(3, \frac{\pi}{4}, \cos^{-1}\left(-\frac{1}{3}\right)\right)$.

13. $x = \rho\sin\phi\cos\theta = 8\sin\frac{\pi}{6}\cos\frac{\pi}{4} = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2} = 2\sqrt{2}$, $y = \rho\sin\phi\sin\theta = 8\sin\frac{\pi}{6}\sin\frac{\pi}{4} = 2\sqrt{2}$, and $z = \rho\cos\phi = 8\cos\frac{\pi}{6} = 8\cdot\frac{\sqrt{3}}{2} = 4\sqrt{3}$. Thus rectangular coordinates for the point are $\left(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3}\right)$.

$r^2 = x^2 + y^2 = 8 + 8 = 16 \quad\Rightarrow\quad r = 4$, $\theta = \frac{\pi}{4}$, and $z = 4\sqrt{3}$, so cylindrical coordinates are $\left(4, \frac{\pi}{4}, 4\sqrt{3}\right)$.

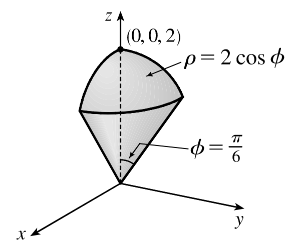
14. (a) $\theta = \frac{\pi}{4}$. In cylindrical coordinates (assuming that r can be negative), this is a vertical plane that includes the z-axis and intersects the xy-plane in the line $y = x$. In spherical coordinates, because $\rho \ge 0$ and $0 \le \phi \le \pi$, we get a vertical half-plane that includes the z-axis and intersects the xy-plane in the half-line $y = x$, $x \ge 0$.

(b) $\phi = \frac{\pi}{4}$. In spherical coordinates, this is one frustum of a circular cone with vertex the origin and axis the positive z-axis.

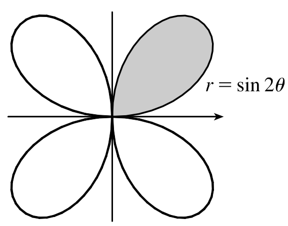
15. (a) $x^2+y^2+z^2=4$. In cylindrical coordinates, this becomes $r^2+z^2=4$. In spherical coordinates, it becomes $\rho^2=4$ or $\rho=2$.

(b) $x^2+y^2=4$. In cylindrical coordinates: $r^2=4$ or $r=2$. In spherical coordinates: $\rho^2-z^2=4$ or $\rho^2-\rho^2\cos^2\phi=4$ or $\rho^2\sin^2\phi=4$ or $\rho\sin\phi=2$.

16. $\rho=2\cos\phi \quad\Rightarrow\quad \rho^2=2\rho\cos\phi \quad\Rightarrow\quad x^2+y^2+z^2=2z \quad\Rightarrow$ $x^2+y^2+(z-1)^2=1$. This is the equation of a sphere with radius 1, centered at $(0,0,1)$. Therefore, $0\le\rho\le 2\cos\phi$ is the solid ball whose boundary is this sphere. $0\le\theta\le\frac{\pi}{2}$ and $0\le\phi\le\frac{\pi}{6}$ restrict the solid to the section of this ball that lies above the cone $\phi=\frac{\pi}{6}$ and is in the first octant.



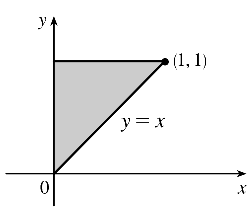
17.



The region whose area is given by $\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta$ is $\left\{(r,\theta)\mid 0\le\theta\le\frac{\pi}{2}, 0\le r\le\sin 2\theta\right\}$, which is the region inside the loop of the four-leaved rose $r=\sin 2\theta$ in the first quadrant.

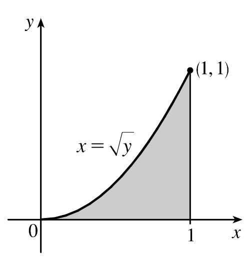
18. The solid is $\left\{(\rho,\theta,\phi)\mid 1\le\rho\le 2,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\phi\le\frac{\pi}{2}\right\}$, which is the region in the first octant on or between the two spheres $\rho=1$ and $\rho=2$.

19.



$$\int_0^1\int_x^1 \cos(y^2)\,dy\,dx=\int_0^1\int_0^y \cos(y^2)\,dx\,dy$$
$$=\int_0^1 \cos(y^2)\big[\,x\,\big]_{x=0}^{x=y}\,dy=\int_0^1 y\cos(y^2)\,dy$$
$$=\left[\tfrac{1}{2}\sin(y^2)\right]_0^1=\tfrac{1}{2}\sin 1$$

20.



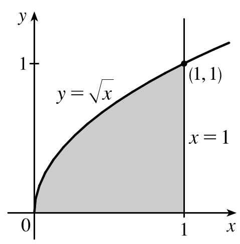
$$\int_0^1\int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy=\int_0^1\int_0^{x^2}\frac{ye^{x^2}}{x^3}\,dy\,dx=\int_0^1\frac{e^{x^2}}{x^3}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=x^2}dx$$
$$=\int_0^1 \tfrac{1}{2}xe^{x^2}\,dx=\tfrac{1}{4}e^{x^2}\Big]_0^1=\tfrac{1}{4}(e-1)$$

21. $\iint_R ye^{xy}\,dA=\int_0^3\int_0^2 ye^{xy}\,dx\,dy=\int_0^3\left[e^{xy}\right]_{x=0}^{x=2}dy=\int_0^3(e^{2y}-1)\,dy=\left[\frac{1}{2}e^{2y}-y\right]_0^3=\frac{1}{2}e^6-3-\frac{1}{2}=\frac{1}{2}e^6-\frac{7}{2}$

22. $\iint_D xy\,dA=\int_0^1\int_{y^2}^{y+2}xy\,dx\,dy=\int_0^1 y\left[\frac{1}{2}x^2\right]_{x=y^2}^{x=y+2}dy=\frac{1}{2}\int_0^1 y\left[(y+2)^2-y^4\right]dy$

$$=\tfrac{1}{2}\int_0^1(y^3+4y^2+4y-y^5)\,dy=\tfrac{1}{2}\left[\tfrac{1}{4}y^4+\tfrac{4}{3}y^3+2y^2-\tfrac{1}{6}y^6\right]_0^1=\tfrac{41}{24}$$

23.



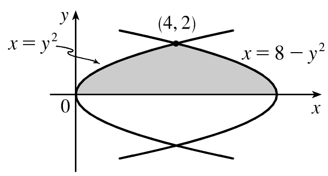
$$\iint_D \frac{y}{1+x^2}\,dA = \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}}\,dx$$

$$= \tfrac{1}{2}\int_0^1 \frac{x}{1+x^2}\,dx = \left[\tfrac{1}{4}\ln(1+x^2)\right]_0^1 = \tfrac{1}{4}\ln 2$$

24. $$\iint_D \frac{1}{1+x^2}\,dA = \int_0^1\int_x^1 \frac{1}{1+x^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\left[\,y\,\right]_{y=x}^{y=1}\,dx = \int_0^1 \frac{1-x}{1+x^2}\,dx = \int_0^1\left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right)dx$$

$$= \left[\tan^{-1}x - \tfrac{1}{2}\ln(1+x^2)\right]_0^1 = \tan^{-1}1 - \tfrac{1}{2}\ln 2 - \left(\tan^{-1}0 - \tfrac{1}{2}\ln 1\right) = \tfrac{\pi}{4} - \tfrac{1}{2}\ln 2$$

25.

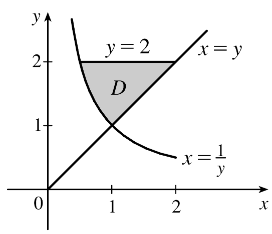


$$\iint_D y\,dA = \int_0^2\int_{y^2}^{8-y^2} y\,dx\,dy$$

$$= \int_0^2 y\left[x\right]_{x=y^2}^{x=8-y^2}dy = \int_0^2 y(8-y^2-y^2)\,dy$$

$$= \int_0^2 (8y-2y^3)\,dy = \left[4y^2 - \tfrac{1}{2}y^4\right]_0^2 = 8$$

26.

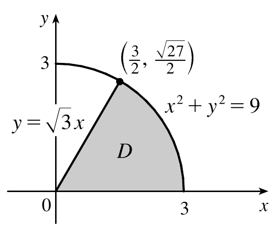


$$\iint_D y\,dA = \int_1^2\int_{1/y}^{y} y\,dx\,dy = \int_1^2 y\left(y-\frac{1}{y}\right)dy$$

$$= \int_1^2\left(y^2-1\right)dy = \left[\tfrac{1}{3}y^3 - y\right]_1^2$$

$$= \left(\tfrac{8}{3}-2\right) - \left(\tfrac{1}{3}-1\right) = \tfrac{4}{3}$$

27.



$$\iint_D \left(x^2+y^2\right)^{3/2}dA = \int_0^{\pi/3}\int_0^3 (r^2)^{3/2} r\,dr\,d\theta$$

$$= \int_0^{\pi/3}d\theta \int_0^3 r^4\,dr = \left[\,\theta\,\right]_0^{\pi/3}\left[\tfrac{1}{5}r^5\right]_0^3$$

$$= \frac{\pi}{3}\frac{3^5}{5} = \frac{81\pi}{5}$$

28. $$\iint_D x\,dA = \int_0^{\pi/2}\int_1^{\sqrt{2}}(r\cos\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\,d\theta\int_1^{\sqrt{2}} r^2\,dr = \left[\sin\theta\right]_0^{\pi/2}\left[\tfrac{1}{3}r^3\right]_1^{\sqrt{2}}$$

$$= 1\cdot\tfrac{1}{3}(2^{3/2}-1) = \tfrac{1}{3}(2^{3/2}-1)$$

29. $$\iiint_E xy\,dV = \int_0^3\int_0^x\int_0^{x+y} xy\,dz\,dy\,dx = \int_0^3\int_0^x xy\left[\,z\,\right]_{z=0}^{z=x+y}dy\,dx = \int_0^3\int_0^x xy(x+y)\,dy\,dx$$

$$= \int_0^3\int_0^x(x^2y+xy^2)\,dy\,dx = \int_0^3\left[\tfrac{1}{2}x^2y^2 + \tfrac{1}{3}xy^3\right]_{y=0}^{y=x}dx = \int_0^3\left(\tfrac{1}{2}x^4 + \tfrac{1}{3}x^4\right)dx$$

$$= \tfrac{5}{6}\int_0^3 x^4\,dx = \left[\tfrac{1}{6}x^5\right]_0^3 = \tfrac{81}{2} = 40.5$$

30. $\iiint_T xy\,dV = \int_0^{1/3}\int_0^{1-3x}\int_0^{1-3x-y} xy\,dz\,dy\,dx = \int_0^{1/3}\int_0^{1-3x} xy(1-3x-y)\,dy\,dx$

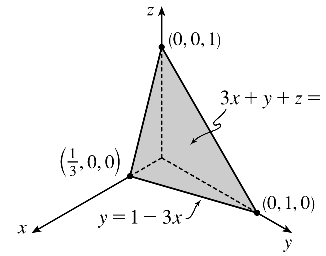
$= \int_0^{1/3}\int_0^{1-3x}(xy-3x^2y-xy^2)\,dy\,dx$

$= \int_0^{1/3}\left[\frac{1}{2}xy^2-\frac{3}{2}x^2y^2-\frac{1}{3}xy^3\right]_{y=0}^{y=1-3x}dx$

$= \int_0^{1/3}\left[\frac{1}{2}x(1-3x)^2-\frac{3}{2}x^2(1-3x)^2-\frac{1}{3}x(1-3x)^3\right]dx$

$= \int_0^{1/3}\left(\frac{1}{6}x-\frac{3}{2}x^2+\frac{9}{2}x^3-\frac{9}{2}x^4\right)dx$

$= \frac{1}{12}x^2-\frac{1}{2}x^3+\frac{9}{8}x^4-\frac{9}{10}x^5\Big]_0^{1/3} = \frac{1}{1080}$



31. $\iiint_E y^2z^2\,dV = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{1-y^2-z^2} y^2z^2\,dx\,dz\,dy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2z^2(1-y^2-z^2)\,dz\,dy$

$= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)(r^2\sin^2\theta)(1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{2}\sin 2\theta\right)^2 d\theta\,\int_0^1(r^5-r^7)\,dr$

$= \int_0^{2\pi}\frac{1}{4}\left[\frac{1}{2}(1-\cos 4\theta)\right]d\theta\,\int_0^1(r^5-r^7)\,dr = \frac{1}{8}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{2\pi}\left[\frac{1}{6}r^6-\frac{1}{8}r^8\right]_0^1$

$= \frac{1}{8}(2\pi)\left(\frac{1}{6}-\frac{1}{8}\right) = \frac{\pi}{4}\cdot\frac{1}{24} = \frac{\pi}{96}$

32. $\iiint_E z\,dV = \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{2-y} z\,dx\,dz\,dy = \int_0^1\int_0^{\sqrt{1-y^2}}(2-y)z\,dz\,dy = \int_0^1 \frac{1}{2}(2-y)(1-y^2)\,dy$

$= \int_0^1 \frac{1}{2}(2-y-2y^2+y^3)\,dy = \frac{13}{24}$

33. $\iiint_E yz\,dV = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_0^y yz\,dz\,dy\,dx = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\left[\frac{1}{2}yz^2\right]_{z=0}^{z=y}dy\,dx = \frac{1}{2}\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} y^3dy\,dx$

$= \frac{1}{2}\int_0^{\pi}\int_0^2(r\sin\theta)^3\,r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi}\sin^3\theta\,d\theta\,\int_0^2 r^4\,dr = \frac{1}{2}\int_0^{\pi}(1-\cos^2\theta)\sin\theta\,d\theta\,\int_0^2 r^4\,dr$

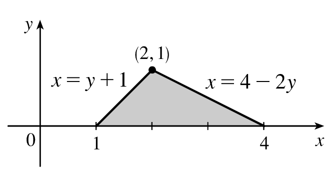
$= \frac{1}{2}\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_0^{\pi}\left[\frac{1}{5}r^5\right]_0^2 = \frac{1}{2}\left(\frac{2}{3}+\frac{2}{3}\right)\left(\frac{32}{5}\right) = \frac{64}{15}$

34. $\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1(\rho^3\cos^3\phi)\rho(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$= \int_0^{2\pi}d\theta\,\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi\,\int_0^1\rho^6\,d\rho = 2\pi\left[-\frac{1}{4}\cos^4\phi\right]_0^{\pi/2}\left(\frac{1}{7}\right) = \frac{\pi}{14}$

35. $V = \int_0^2\int_1^4(x^2+4y^2)\,dy\,dx = \int_0^2\left[x^2y+\frac{4}{3}y^3\right]_{y=1}^{y=4}dx = \int_0^2(3x^2+84)\,dx = x^3+84x\Big]_0^2 = 176$

36.

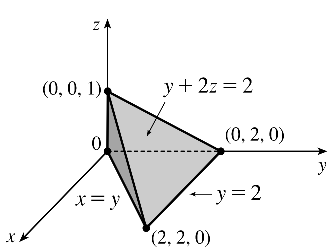


$V = \int_0^1\int_{y+1}^{4-2y}\int_0^{x^2y} dz\,dx\,dy = \int_0^1\int_{y+1}^{4-2y} x^2y\,dx\,dy$

$= \int_0^1\frac{1}{3}\left[(4-2y)^3y-(y+1)^3y\right]dy$

$= \int_0^1 3(-y^4+5y^3-11y^2+7y)\,dy = 3\left(-\frac{1}{5}+\frac{5}{4}-\frac{11}{3}+\frac{7}{2}\right) = \frac{53}{20}$

37.



$V = \int_0^2\int_0^y\int_0^{(2-y)/2} dz\,dx\,dy = \int_0^2\int_0^y\left(1-\frac{1}{2}y\right)dx\,dy$

$= \int_0^2\left(y-\frac{1}{2}y^2\right)dy = \frac{1}{2}y^2-\frac{1}{6}y^3\Big]_0^2 = \frac{2}{3}$

38. $V = \int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \Big[rz\Big]_0^{3-r\sin\theta} dr\,d\theta = \int_0^{2\pi}\int_0^2 (3r - r^2\sin\theta)\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{3}{2}r^2 - \frac{1}{3}r^3\sin\theta\right]_0^2 d\theta = \int_0^{2\pi}\left[6 - \frac{8}{3}\sin\theta\right]d\theta = \left[6\theta + \frac{8}{3}\cos\theta\right]_0^{2\pi} = 12\pi$

39. Using the wedge above the plane $z = 0$ and below the plane $z = mx$ and noting that we have the same volume for $m < 0$ as for $m > 0$ (so use $m > 0$), we have

$V = 2\int_0^{a/3}\int_0^{\sqrt{a^2-9y^2}} mx\,dx\,dy = 2\int_0^{a/3}\frac{1}{2}m(a^2 - 9y^2)\,dy = m\big[a^2y - 3y^3\big]_0^{a/3} = m\big(\frac{1}{3}a^3 - \frac{1}{9}a^3\big) = \frac{2}{9}ma^3.$

40. The paraboloid and the half-cone intersect when $x^2 + y^2 = \sqrt{x^2+y^2}$, that is when $x^2 + y^2 = 1$ or 0. So

$V = \iint_{x^2+y^2\le 1}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\,dA = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^2 - r^3)\,dr\,d\theta = \int_0^{2\pi}\big(\frac{1}{3} - \frac{1}{4}\big)\,d\theta = \frac{1}{12}(2\pi) = \frac{\pi}{6}.$

41. (a) $m = \int_0^1\int_0^{1-y^2} y\,dx\,dy = \int_0^1 (y - y^3)\,dy = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(b) $M_y = \int_0^1\int_0^{1-y^2} xy\,dx\,dy = \int_0^1 \frac{1}{2}y(1-y^2)^2\,dy = -\frac{1}{12}(1-y^2)^3\big]_0^1 = \frac{1}{12}$,

$M_x = \int_0^1\int_0^{1-y^2} y^2\,dx\,dy = \int_0^1 (y^2 - y^4)\,dy = \frac{2}{15}$. Hence $(\overline{x}, \overline{y}) = \big(\frac{1}{3}, \frac{8}{15}\big)$.

(c) $I_x = \int_0^1\int_0^{1-y^2} y^3\,dx\,dy = \int_0^1 (y^3 - y^5)\,dy = \frac{1}{12}$,

$I_y = \int_0^1\int_0^{1-y^2} yx^2\,dx\,dy = \int_0^1 \frac{1}{3}y(1-y^2)^3\,dy = -\frac{1}{24}(1-y^2)^4\big]_0^1 = \frac{1}{24}$,

$\overline{\overline{y}}^2 = I_x/m = \frac{1/12}{1/4} = \frac{1}{3} \quad\Rightarrow\quad \overline{\overline{y}} = \frac{1}{\sqrt{3}}$, and $\overline{\overline{x}}^2 = I_y/m = \frac{1/24}{1/4} = \frac{1}{6} \quad\Rightarrow\quad \overline{\overline{x}} = \frac{1}{\sqrt{6}}$.

42. (a) In polar coordinates, the lamina occupies the region $D = \{(r,\theta) \mid 0 \le r \le a,\, 0 \le \theta \le \pi/2\}$. Assuming constant density K, then $m = K\,A(D) = K \cdot \frac{1}{4}\pi a^2 = \frac{1}{4}\pi K a^2$,

$M_y = \iint_D Kx\,dA = K\int_0^{\pi/2}\int_0^a (r\cos\theta)\,r\,dr\,d\theta = K\int_0^{\pi/2}\cos\theta\,d\theta\int_0^a r^2\,dr = K\,[\sin\theta]_0^{\pi/2}\,\big[\frac{1}{3}r^3\big]_0^a = \frac{1}{3}Ka^3$, and

$M_x = \iint_D Ky\,dA = K\int_0^{\pi/2}\sin\theta\,d\theta\int_0^a r^2\,dr = K\,[-\cos\theta]_0^{\pi/2}\,\big[\frac{1}{3}r^3\big]_0^a = \frac{1}{3}Ka^3$ [by symmetry $M_y = M_x$].

Thus the centroid is $(\overline{x}, \overline{y}) = (M_y/m, M_x/m) = \big(\frac{4}{3\pi}a, \frac{4}{3\pi}a\big)$.

(b) $m = \iint_D \rho(x,y)\,dA = \iint_D xy^2\,dA = \int_0^{\pi/2}\int_0^a (r\cos\theta)(r\sin\theta)^2\,r\,dr\,d\theta$

$= \int_0^{\pi/2}\sin^2\theta\,\cos\theta\,d\theta\int_0^a r^4\,dr = \big[\frac{1}{3}\sin^3\theta\big]_0^{\pi/2}\,\big[\frac{1}{5}r^5\big]_0^a = \frac{1}{15}a^5$,

$M_y = \int_0^{\pi/2}\int_0^a r^5\cos^2\theta\sin^2\theta\,dr\,d\theta = \frac{1}{8}\big[\theta - \frac{1}{4}\sin 4\theta\big]_0^{\pi/2}\,\big[\frac{1}{6}r^6\big]_0^a = \frac{1}{96}\pi a^6$, and

$M_x = \int_0^{\pi/2}\int_0^a r^5\cos\theta\sin^3\theta\,dr\,d\theta = \big[\frac{1}{4}\sin^4\theta\big]_0^{\pi/2}\,\big[\frac{1}{6}r^6\big]_0^a = \frac{1}{24}a^6$. Hence $(\overline{x}, \overline{y}) = \big(\frac{5}{32}\pi a, \frac{5}{8}a\big)$.

43. (a) A right circular cone with axis the z-axis and vertex at the origin has equation $z^2 = c^2(x^2 + y^2)$. Here we have the bottom frustum, shifted upward h units, and with $c^2 = h^2/a^2$ so that the cone includes the point $(a, 0, 0)$. Thus an equation of the cone in rectangular coordinates is $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$, $0 \le z \le h$. In cylindrical coordinates, the cone is described by $E = \big\{(r,\theta,z) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi,\ 0 \le z \le h\,\big(1 - \frac{1}{a}r\big)\big\}$, and its volume is $V = \frac{1}{3}\pi a^2 h$. By symmetry

$M_{yz} = M_{xz} = 0$, and

$$\begin{aligned}
M_{xy} &= \int_0^{2\pi}\int_0^a\int_0^{h(1-r/a)} z\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a\left[\frac{1}{2}rz^2\right]_{z=0}^{z=h(1-r/a)} dr\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^a rh^2\left(1-\frac{r}{a}\right)^2 dr\,d\theta = \frac{1}{2}h^2\int_0^{2\pi}d\theta\int_0^a\left(r-\frac{2}{a}r^2+\frac{1}{a^2}r^3\right)dr \\
&= \tfrac{1}{2}h^2\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{2}r^2-\tfrac{2}{3a}r^3+\tfrac{1}{4a^2}r^4\right]_0^a = \tfrac{1}{2}h^2\,(2\pi)\left(\tfrac{1}{2}a^2-\tfrac{2}{3}a^2+\tfrac{1}{4}a^2\right) \\
&= \pi h^2\left(\tfrac{1}{12}a^2\right) = \tfrac{1}{12}\pi a^2h^2
\end{aligned}$$

Hence the centroid is $(\overline{x},\overline{y},\overline{z}) = \left(0,0,[\pi a^2h^2/12]/[\pi a^2h/3]\right) = \left(0,0,\tfrac{1}{4}h\right)$.

(b) The density function is $\rho = \sqrt{x^2+y^2} = \sqrt{r^2} = r$, so the moment of inertia about the cone's axis (the z-axis) is

$$\begin{aligned}
I_z &= \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^a\int_0^{h(1-r/a)}(r^2)(r)\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^a\left[r^4z\right]_{z=0}^{z=h(1-r/a)} dr\,d\theta = \int_0^{2\pi}\int_0^a r^4h\left(1-\tfrac{1}{a}r\right)dr\,d\theta \\
&= h\int_0^{2\pi}d\theta\int_0^a\left(r^4-\tfrac{1}{a}r^5\right)dr = h\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{5}r^5-\tfrac{1}{6a}r^6\right]_0^a \\
&= h\,(2\pi)\left(\tfrac{1}{5}a^5-\tfrac{1}{6}a^5\right) = \tfrac{1}{15}\pi a^5h
\end{aligned}$$

44. $1 \le z^2 \le 4 \quad\Rightarrow\quad 1/a^2 \le x^2+y^2 \le 4/a^2$. Let $D = \left\{(x,y) \mid 1/a^2 \le x^2+y^2 \le 4/a^2\right\}$. $z = f(x,y) = a\sqrt{x^2+y^2}$, so $f_x(x,y) = ax(x^2+y^2)^{-1/2}$, $f_y(x,y) = ay(x^2+y^2)^{-1/2}$, and

$$\begin{aligned}
A(S) &= \iint_D\sqrt{\frac{a^2x^2+a^2y^2}{x^2+y^2}+1}\,dA = \iint_D\sqrt{a^2+1}\,dA = \sqrt{a^2+1}\,A(D) \\
&= \sqrt{a^2+1}\left[\pi\left(\frac{2}{a}\right)^2-\pi\left(\frac{1}{a}\right)^2\right] = \frac{3\pi}{a^2}\sqrt{a^2+1}
\end{aligned}$$

45. Let D represent the given triangle; then D can be described as the area enclosed by the x- and y-axes and the line $y = 2-2x$, or equivalently $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2-2x\}$. We want to find the surface area of the part of the graph of $z = x^2+y$ that lies over D, so using Formula 15.5.3 we have

$$\begin{aligned}
A(S) &= \iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_D\sqrt{1+(2x)^2+(1)^2}\,dA = \int_0^1\int_0^{2-2x}\sqrt{2+4x^2}\,dy\,dx \\
&= \int_0^1\sqrt{2+4x^2}\,\left[\,y\,\right]_{y=0}^{y=2-2x}\,dx = \int_0^1(2-2x)\sqrt{2+4x^2}\,dx = \int_0^1 2\sqrt{2+4x^2}\,dx - \int_0^1 2x\sqrt{2+4x^2}\,dx
\end{aligned}$$

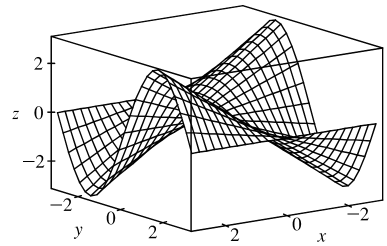
Using Formula 21 in the Table of Integrals with $a = \sqrt{2}$, $u = 2x$, and $du = 2\,dx$, we have

$\int 2\sqrt{2+4x^2}\,dx = x\sqrt{2+4x^2}+\ln\left(2x+\sqrt{2+4x^2}\right)$. If we substitute $u = 2+4x^2$ in the second integral, then $du = 8x\,dx$ and $\int 2x\sqrt{2+4x^2}\,dx = \frac{1}{4}\int\sqrt{u}\,du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2} = \frac{1}{6}(2+4x^2)^{3/2}$. Thus

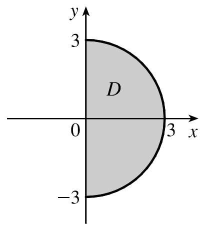
$$\begin{aligned}
A(S) &= \left[x\sqrt{2+4x^2}+\ln\left(2x+\sqrt{2+4x^2}\right)-\tfrac{1}{6}(2+4x^2)^{3/2}\right]_0^1 \\
&= \sqrt{6}+\ln\left(2+\sqrt{6}\right)-\tfrac{1}{6}(6)^{3/2}-\ln\sqrt{2}+\tfrac{\sqrt{2}}{3} = \ln\tfrac{2+\sqrt{6}}{\sqrt{2}}+\tfrac{\sqrt{2}}{3} \\
&= \ln\left(\sqrt{2}+\sqrt{3}\right)+\tfrac{\sqrt{2}}{3} \approx 1.6176
\end{aligned}$$

46. Using Formula 15.5.3 with $\partial z/\partial x = \sin y$, $\partial z/\partial y = x\cos y$, we get

$S = \int_{-\pi}^{\pi}\int_{-3}^{3}\sqrt{1+\sin^2 y + x^2\cos^2 y}\,dx\,dy \approx 62.9714.$



47.



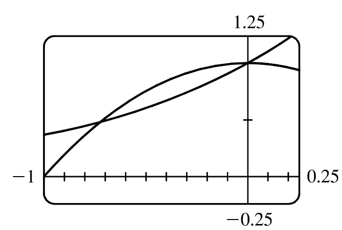
$$\begin{aligned}\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(x^3+xy^2)\,dy\,dx &= \int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x(x^2+y^2)\,dy\,dx\\ &= \int_{-\pi/2}^{\pi/2}\int_0^3 (r\cos\theta)(r^2)\,r\,dr\,d\theta\\ &= \int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta\int_0^3 r^4\,dr\\ &= \big[\sin\theta\big]_{-\pi/2}^{\pi/2}\left[\tfrac{1}{5}r^5\right]_0^3 = 2\cdot\tfrac{1}{5}(243) = \tfrac{486}{5} = 97.2\end{aligned}$$

48. The region of integration is the solid hemisphere $x^2+y^2+z^2 \le 4$, $x \ge 0$.

$$\begin{aligned}&\int_{-2}^{2}\int_0^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy\\ &\quad= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^2(\rho\sin\phi\sin\theta)^2\left(\sqrt{\rho^2}\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi\int_0^2\rho^5\,d\rho\\ &\quad= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2}\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi}\left[\tfrac{1}{6}\rho^6\right]_0^2 = \left(\tfrac{\pi}{2}\right)\left(\tfrac{2}{3}+\tfrac{2}{3}\right)\left(\tfrac{32}{3}\right) = \tfrac{64}{9}\pi\end{aligned}$$

49. From the graph, it appears that $1-x^2 = e^x$ at $x \approx -0.71$ and at $x = 0$, with $1-x^2 > e^x$ on $(-0.71, 0)$. So the desired integral is

$$\begin{aligned}\iint_D y^2\,dA &\approx \int_{-0.71}^{0}\int_{e^x}^{1-x^2} y^2\,dy\,dx\\ &= \tfrac{1}{3}\int_{-0.71}^{0}\left[(1-x^2)^3 - e^{3x}\right]dx\\ &= \tfrac{1}{3}\left[x - x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7 - \tfrac{1}{3}e^{3x}\right]_{-0.71}^{0} \approx 0.0512\end{aligned}$$



50. Let the tetrahedron be called T. The front face of T is given by the plane $x + \frac{1}{2}y + \frac{1}{3}z = 1$, or $z = 3 - 3x - \frac{3}{2}y$, which intersects the xy-plane in the line $y = 2 - 2x$. So the total mass is

$m = \iiint_T \rho(x,y,z)\,dV = \int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2}(x^2+y^2+z^2)\,dz\,dy\,dx = \frac{7}{5}$. The center of mass is

$(\overline{x},\overline{y},\overline{z}) = \left(m^{-1}\iiint_T x\rho(x,y,z)\,dV, m^{-1}\iiint_T y\rho(x,y,z)\,dV, m^{-1}\iiint_T z\rho(x,y,z)\,dV\right) = \left(\frac{4}{21}, \frac{11}{21}, \frac{8}{7}\right).$

51. (a) $f(x,y)$ is a joint density function, so we know that $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,3]\times[0,2]$, we can say

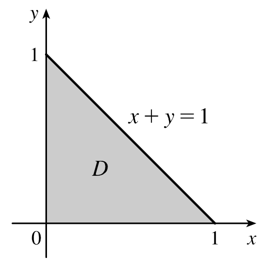
$$\begin{aligned}\iint_{\mathbb{R}^2} f(x,y)\,dA &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^3\int_0^2 C(x+y)\,dy\,dx\\ &= C\int_0^3\left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=2}dx = C\int_0^3(2x+2)\,dx = C\big[x^2+2x\big]_0^3 = 15C\end{aligned}$$

Then $15C = 1 \quad\Rightarrow\quad C = \frac{1}{15}$.

(b) $P(X \le 2, Y \ge 1) = \int_{-\infty}^{2} \int_{1}^{\infty} f(x,y)\,dy\,dx = \int_0^2 \int_1^2 \frac{1}{15}(x,y)\,dy\,dx = \frac{1}{15}\int_0^2 \left[xy + \frac{1}{2}y^2\right]_{y=1}^{y=2} dx$

$= \frac{1}{15}\int_0^2 \left(x + \frac{3}{2}\right) dx = \frac{1}{15}\left[\frac{1}{2}x^2 + \frac{3}{2}x\right]_0^2 = \frac{1}{3}$

(c) $P(X + Y \le 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$\begin{aligned} P(X+Y \le 1) &= \iint_D f(x,y)\,dA = \int_0^1 \int_0^{1-x} \tfrac{1}{15}(x+y)\,dy\,dx \\ &= \tfrac{1}{15}\int_0^1 \left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx \\ &= \tfrac{1}{15}\int_0^1 \left[x(1-x) + \tfrac{1}{2}(1-x)^2\right] dx \\ &= \tfrac{1}{30}\int_0^1 (1-x^2)\,dx = \tfrac{1}{30}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{45} \end{aligned}$$

52. Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{800}e^{-t/800} & \text{if } t \ge 0 \end{cases}$$

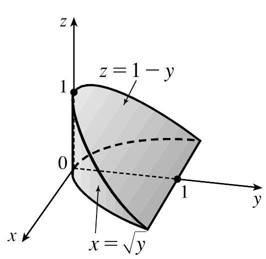
If X, Y, and Z are the lifetimes of the individual bulbs, then X, Y, and Z are independent, so the joint density function is the product of the individual density functions:

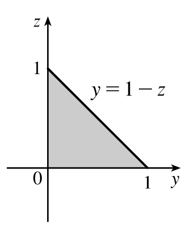
$$f(x,y,z) = \begin{cases} \frac{1}{800^3}e^{-(x+y+z)/800} & \text{if } x \ge 0, y \ge 0, z \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that all three bulbs fail within a total of 1000 hours is $P(X + Y + Z \le 1000)$, or equivalently $P((X,Y,Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x + y + z = 1000$. The plane $x + y + z = 1000$ meets the xy-plane in the line $x + y = 1000$, so we have

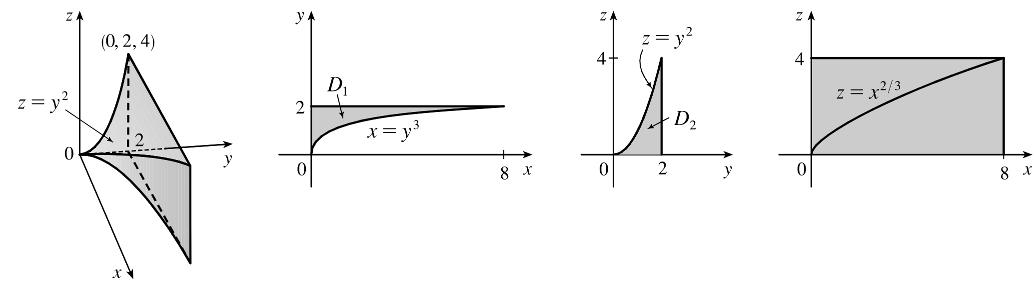
$$\begin{aligned} P(X+Y+Z \le 1000) &= \iiint_E f(x,y,z)\,dV = \int_0^{1000}\int_0^{1000-x}\int_0^{1000-x-y} \tfrac{1}{800^3}e^{-(x+y+z)/800}\,dz\,dy\,dx \\ &= \tfrac{1}{800^3}\int_0^{1000}\int_0^{1000-x} -800\left[e^{-(x+y+z)/800}\right]_{z=0}^{z=1000-x-y} dy\,dx \\ &= \tfrac{-1}{800^2}\int_0^{1000}\int_0^{1000-x}\left[e^{-5/4} - e^{-(x+y)/800}\right] dy\,dx \\ &= \tfrac{-1}{800^2}\int_0^{1000}\left[e^{-5/4}y + 800e^{-(x+y)/800}\right]_{y=0}^{y=1000-x} dx \\ &= \tfrac{-1}{800^2}\int_0^{1000}\left[e^{-5/4}(1800-x) - 800e^{-x/800}\right] dx \\ &= \tfrac{-1}{800^2}\left[-\tfrac{1}{2}e^{-5/4}(1800-x)^2 + 800^2e^{-x/800}\right]_0^{1000} \\ &= \tfrac{-1}{800^2}\left[-\tfrac{1}{2}e^{-5/4}(800)^2 + 800^2e^{-5/4} + \tfrac{1}{2}e^{-5/4}(1800)^2 - 800^2\right] \\ &= 1 - \tfrac{97}{32}e^{-5/4} \approx 0.1315 \end{aligned}$$

53.





$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$

54. 

$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le y \le 2, 0 \le x \le y^3, 0 \le z \le y^2\}$.

If D_1, D_2, and D_3 are the projections of E onto the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid 0 \le y \le 2, 0 \le x \le y^3\} = \{(x,y) \mid 0 \le x \le 8, \sqrt[3]{x} \le y \le 2\},$$

$$D_2 = \{(y,z) \mid 0 \le z \le 4, \sqrt{z} \le y \le 2\} = \{(y,z) \mid 0 \le y \le 2, 0 \le z \le y^2\},$$

$$D_3 = \{(x,z) \mid 0 \le x \le 8, 0 \le z \le 4\}.$$

Therefore we have

$$\begin{aligned}
\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x,y,z)\,dz\,dx\,dy &= \int_0^8 \int_{\sqrt[3]{x}}^2 \int_0^{y^2} f(x,y,z)\,dz\,dy\,dx \\
&= \int_0^4 \int_{\sqrt{z}}^2 \int_0^{y^3} f(x,y,z)\,dx\,dy\,dz \\
&= \int_0^2 \int_0^{y^2} \int_0^{y^3} f(x,y,z)\,dx\,dz\,dy \\
&= \int_0^8 \int_0^{x^{2/3}} \int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dz\,dx + \int_0^8 \int_{x^{2/3}}^4 \int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dz\,dx \\
&= \int_0^4 \int_0^{z^{3/2}} \int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dx\,dz + \int_0^4 \int_{z^{3/2}}^8 \int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dx\,dz
\end{aligned}$$

55. Since $u = x - y$ and $v = x + y$, $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$.

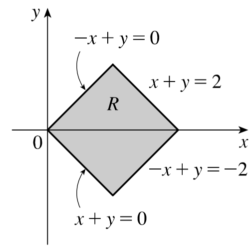
R is the image under this transformation of the square with vertices $(u,v) = (-2,2)$, $(0,2)$, $(0,4)$, and $(-2,4)$. Thus,

$$\begin{aligned}
\iint_R \frac{x-y}{x+y}\,dA &= \int_2^4 \int_{-2}^0 \frac{u}{v}\left(\frac{1}{2}\right) du\,dv = \frac{1}{2}\int_2^4 \left[\frac{u^2}{2v}\right]_{u=-2}^{u=0} dv = \frac{1}{2}\int_2^4 \left(-\frac{2}{v}\right) dv \\
&= \Big[-\ln v\Big]_2^4 = -\ln 4 + \ln 2 = -2\ln 2 + \ln 2 = -\ln 2
\end{aligned}$$

56. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw$, so

$$\begin{aligned}
V = \iiint_E dV &= \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw\,dw\,dv\,du = \int_0^1 \int_0^{1-u} 4uv(1-u-v)^2\,du \\
&= \int_0^1 \int_0^{1-u} \left[4u(1-u)^2 v - 8u(1-u)v^2 + 4uv^3\right] dv\,du \\
&= \int_0^1 \left[2u(1-u)^4 - \tfrac{8}{3}u(1-u)^4 + u(1-u)^4\right] du = \int_0^1 \tfrac{1}{3}u(1-u)^4 du \\
&= \int_0^1 \tfrac{1}{3}\left[(1-u)^4 - (1-u)^5\right] du = \tfrac{1}{3}\left[-\tfrac{1}{5}(1-u)^5 + \tfrac{1}{6}(1-u)^6\right]_0^1 = \tfrac{1}{3}\left(-\tfrac{1}{6} + \tfrac{1}{5}\right) = \tfrac{1}{90}
\end{aligned}$$

57. See the figure. Letting $u = -x + y$ and $v = x + y$, we have $x = \frac{1}{2}(v - u)$ and $y = \frac{1}{2}(v + u)$. Then



$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| = \left|-\tfrac{1}{2}\left(\tfrac{1}{2}\right) - \tfrac{1}{2}\left(\tfrac{1}{2}\right)\right| = \left|-\tfrac{1}{2}\right| = \tfrac{1}{2}.$$

R is the image under this transformation of the square region with vertices $(u,v) = (0,0)$, $(-2,0)$, $(0,2)$, and $(-2,2)$. Thus,

$$\iint_R xy\,dA = \int_0^2\int_{-2}^0 \frac{v^2 - u^2}{4}\left(\frac{1}{2}\right)du\,dv = \frac{1}{8}\int_0^2 \left[v^2u - \frac{1}{3}u^3\right]_{u=-2}^{u=0} dv$$

$$= \frac{1}{8}\int_0^2 \left(2v^2 - \frac{8}{3}\right)dv = \frac{1}{8}\left[\frac{2}{3}v^3 - \frac{8}{3}v\right]_0^2 = 0$$

This result could have been anticipated by symmetry, since the integrand is an odd function of y and R is symmetric about the x-axis.

58. (a) $$\iint_D \frac{1}{(x^2+y^2)^{n/2}}\,dA = \int_0^{2\pi}\int_r^R \frac{1}{(t^2)^{n/2}}\,t\,dt\,d\theta = 2\pi\int_r^R t^{1-n}\,dt$$

$$= \begin{cases} \left.\dfrac{2\pi}{2-n}t^{2-n}\right]_r^R = \dfrac{2\pi}{2-n}\left(R^{2-n} - r^{2-n}\right) & \text{if } n \neq 2 \\ 2\pi\ln(R/r) & \text{if } n = 2 \end{cases}$$

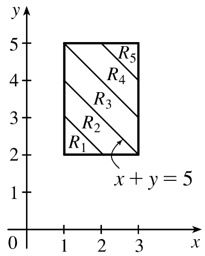
(b) The integral in part (a) has a limit as $r \to 0^+$ for all values of n such that $2 - n > 0 \quad\Leftrightarrow\quad n < 2$.

(c) $$\iiint_E \frac{1}{(x^2+y^2+z^2)^{n/2}}\,dV = \int_r^R\int_0^\pi\int_0^{2\pi} \frac{1}{(\rho^2)^{n/2}}\rho^2\sin\phi\,d\theta\,d\phi\,d\rho = 2\pi\textstyle\int_r^R\int_0^\pi \rho^{2-n}\sin\phi\,d\phi\,d\rho$$

$$= \begin{cases} \left.\dfrac{4\pi}{3-n}\,\rho^{3-n}\right]_r^R = \dfrac{4\pi}{3-n}\left(R^{3-n} - r^{3-n}\right) & \text{if } n \neq 3 \\ 4\pi\ln(R/r) & \text{if } n = 3 \end{cases}$$

(d) As $r \to 0^+$, the above integral has a limit, provided that $3 - n > 0 \quad\Leftrightarrow\quad n < 3$.

PROBLEMS PLUS

1. 

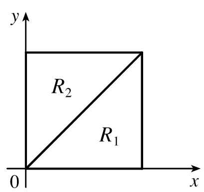
Let $R = \bigcup_{i=1}^{5} R_i$, where

$R_i = \{(x, y) \mid x + y \geq i + 2, x + y < i + 3, 1 \leq x \leq 3, 2 \leq y \leq 5\}$.

$\iint_R [\![x + y]\!]\, dA = \sum\limits_{i=1}^{5} \iint_{R_i} [\![x + y]\!]\, dA = \sum\limits_{i=1}^{5} [\![x + y]\!] \iint_{R_i} dA$, since

$[\![x + y]\!] = \text{constant} = i + 2$ for $(x, y) \in R_i$. Therefore

$$\begin{aligned} \iint_R [\![x + y]\!]\, dA &= \textstyle\sum_{i=1}^{5} (i + 2)\,[A(R_i)] \\ &= 3A(R_1) + 4A(R_2) + 5A(R_3) + 6A(R_4) + 7A(R_5) \\ &= 3\left(\tfrac{1}{2}\right) + 4\left(\tfrac{3}{2}\right) + 5(2) + 6\left(\tfrac{3}{2}\right) + 7\left(\tfrac{1}{2}\right) = 30 \end{aligned}$$

2. 

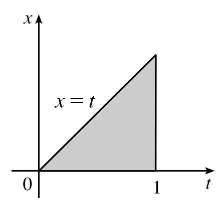
Let $R = \{(x, y) \mid 0 \leq x, y \leq 1\}$. For $x, y \in R$, $\max\left\{x^2, y^2\right\} = x^2$ if $x \geq y$, and $\max\left\{x^2, y^2\right\} = y^2$ if $x \leq y$. Therefore we divide R into two regions: $R = R_1 \cup R_2$, where $R_1 = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$ and $R_2 = \{(x, y) \mid 0 \leq y \leq 1, 0 \leq x \leq y\}$. Now $\max\left\{x^2, y^2\right\} = x^2$ for $(x, y) \in R_1$, and $\max\left\{x^2, y^2\right\} = y^2$ for $(x, y) \in R_2 \quad \Rightarrow$

$$\begin{aligned} \int_0^1 \int_0^1 e^{\max\{x^2, y^2\}}\, dy\, dx &= \iint_R e^{\max\{x^2, y^2\}}\, dA = \iint_{R_1} e^{\max\{x^2, y^2\}}\, dA + \iint_{R_2} e^{\max\{x^2, y^2\}}\, dA \\ &= \int_0^1 \int_0^x e^{x^2}\, dy\, dx + \int_0^1 \int_0^y e^{y^2}\, dx\, dy = \int_0^1 x e^{x^2}\, dx + \int_0^1 y e^{y^2}\, dy = \left. e^{x^2} \right]_0^1 = e - 1 \end{aligned}$$

3. $f_{\text{avg}} = \dfrac{1}{b - a} \displaystyle\int_a^b f(x)\, dx = \dfrac{1}{1 - 0} \displaystyle\int_0^1 \left[\int_x^1 \cos(t^2)\, dt \right] dx$

$= \int_0^1 \int_x^1 \cos(t^2)\, dt\, dx = \int_0^1 \int_0^t \cos(t^2)\, dx\, dt \quad$ [changing the order of integration]

$= \int_0^1 t \cos(t^2)\, dt = \left. \tfrac{1}{2} \sin(t^2) \right]_0^1 = \tfrac{1}{2} \sin 1$



4. To show that $\int_0^2 \int_0^x 2e^{x^2 - y^2}\, dy\, dx = \int_0^2 \int_0^x 2e^{(x+y)(x-y)}\, dy\, dx$ is equal to $\int_0^2 \int_y^{4-y} e^{xy}\, dx\, dy$, we use a change of variables on the left-hand side. Leting $u = x + y$ and $v = x - y$, we have $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Then

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

R is bounded by $\;y = x,\; x = 2,\;$ and $\;y = 0$.

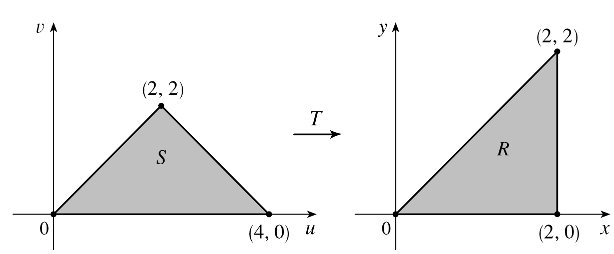
$y = x \quad \Rightarrow \quad \frac{1}{2}(u - v) = \frac{1}{2}(u + v) \quad \Rightarrow \quad v = 0;$

$x = 2 \quad \Rightarrow \quad \frac{1}{2}(u + v) = 2 \quad \Rightarrow$

$\qquad v = 4 - u$, or $u = 4 - v$;

$y = 0 \quad \Rightarrow \quad \frac{1}{2}(u - v) = 0 \quad \Rightarrow \quad u = v.$

So R is the image of S. Thus,



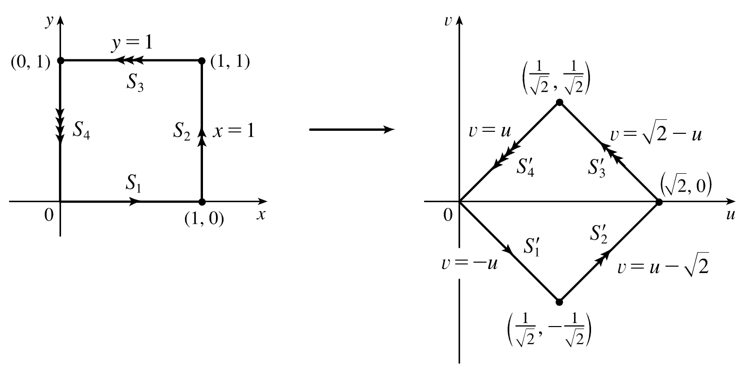
$$\int_0^2 \int_0^x 2e^{x^2-y^2}\,dy\,dx = \int_0^2 \int_0^x 2e^{(x+y)(x-y)}\,dy\,dx = \int_0^2 \int_v^{4-v} 2e^{uv}\left|-\frac{1}{2}\right|\,du\,dv$$
$$= \int_0^2 \int_v^{4-v} e^{uv}\,du\,dv,$$

which is the same as $\int_0^2 \int_y^{4-y} e^{xy}\,dx\,dy$.

5. Since $|xy| < 1$, except at $(1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xy} = \sum\limits_{n=0}^{\infty} (xy)^n$, so

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xy)^n\,dx\,dy = \sum_{n=0}^{\infty} \int_0^1 \int_0^1 (xy)^n\,dx\,dy$$
$$= \sum_{n=0}^{\infty} \left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right] = \sum_{n=0}^{\infty} \frac{1}{n+1}\cdot\frac{1}{n+1}$$
$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. Let $x = \dfrac{u-v}{\sqrt{2}}$ and $y = \dfrac{u+v}{\sqrt{2}}$. We know the region of integration in the xy-plane, so to find its image in the uv-plane we get u and v in terms of x and y, and then use the methods of Section 15.9. $x + y = \dfrac{u-v}{\sqrt{2}} + \dfrac{u+v}{\sqrt{2}} = \sqrt{2}\,u$, so $u = \dfrac{x+y}{\sqrt{2}}$, and similarly $v = \dfrac{y-x}{\sqrt{2}}$. S_1 is given by $y = 0$, $0 \le x \le 1$, so from the equations derived above, the image of S_1 is S_1': $u = \frac{1}{\sqrt{2}}x$, $v = -\frac{1}{\sqrt{2}}x$, $0 \le x \le 1$, that is, $v = -u$, $0 \le u \le \frac{1}{\sqrt{2}}$. Similarly, the image of S_2 is S_2': $v = u - \sqrt{2}$, $\frac{1}{\sqrt{2}} \le u \le \sqrt{2}$, the image of S_3 is S_3': $v = \sqrt{2} - u$, $\frac{1}{\sqrt{2}} \le u \le \sqrt{2}$, and the image of S_4 is S_4': $v - u$, $0 \le u \le \frac{1}{\sqrt{2}}$.



The Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$$

From the diagram, we see that we must evaluate two integrals: one over the region $\left\{(u,v) \mid 0 \le u \le \frac{1}{\sqrt{2}},\ -u \le v \le u\right\}$ and the other over $\left\{(u,v) \mid \frac{1}{\sqrt{2}} \le u \le \sqrt{2},\ -\sqrt{2} + u \le v \le \sqrt{2} - u\right\}$. So

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = \int_0^{\sqrt{2}/2}\int_{-u}^{u} \frac{dv\,du}{1-\left[\frac{1}{\sqrt{2}}(u+v)\right]\left[\frac{1}{\sqrt{2}}(u-v)\right]} + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{dv\,du}{1-\left[\frac{1}{\sqrt{2}}(u+v)\right]\left[\frac{1}{\sqrt{2}}(u-v)\right]}$$

$$= \int_0^{\sqrt{2}/2}\int_{-u}^{u} \frac{2\,dv\,du}{2-u^2+v^2} + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{2\,dv\,du}{2-u^2+v^2}$$

$$= 2\left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}}\left[\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-u}^{u} du + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\left[\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-\sqrt{2}+u}^{\sqrt{2}-u} du\right]$$

$$= 4\left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}}\arctan\frac{u}{\sqrt{2-u^2}}\,du + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\arctan\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\,du\right]$$

Now let $u = \sqrt{2}\sin\theta$, so $du = \sqrt{2}\cos\theta\,d\theta$ and the limits change to 0 and $\frac{\pi}{6}$ (in the first integral) and $\frac{\pi}{6}$ and $\frac{\pi}{2}$ (in the second integral). Continuing:

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = 4\left[\int_0^{\pi/6} \frac{1}{\sqrt{2-2\sin^2\theta}}\arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\right)\left(\sqrt{2}\cos\theta\,d\theta\right) + \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2-2\sin^2\theta}}\arctan\left(\frac{\sqrt{2}-\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\right)\left(\sqrt{2}\cos\theta\,d\theta\right)\right]$$

$$= 4\left[\int_0^{\pi/6} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right)d\theta + \int_{\pi/6}^{\pi/2} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2}\,(1-\sin\theta)}{\sqrt{2}\cos\theta}\right)d\theta\right]$$

$$= 4\left[\int_0^{\pi/6} \arctan(\tan\theta)\,d\theta + \int_{\pi/6}^{\pi/2} \arctan\left(\frac{1-\sin\theta}{\cos\theta}\right)d\theta\right]$$

But (following the hint)

$$\frac{1-\sin\theta}{\cos\theta} = \frac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)} = \frac{1-\left[1-2\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right]}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)} \quad \text{[half-angle formulas]}$$

$$= \frac{2\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)} = \tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)$$

Continuing:

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = 4\left[\int_0^{\pi/6} \arctan(\tan\theta)\,d\theta + \int_{\pi/6}^{\pi/2} \arctan\left(\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right)d\theta\right]$$

$$= 4\left[\int_0^{\pi/6} \theta\,d\theta + \int_{\pi/6}^{\pi/2} \left[\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right]d\theta\right] = 4\left(\left[\frac{\theta^2}{2}\right]_0^{\pi/6} + \left[\frac{\pi\theta}{4}-\frac{\theta^2}{4}\right]_{\pi/6}^{\pi/2}\right) = 4\left(\frac{3\pi^2}{72}\right) = \frac{\pi^2}{6}$$

7. (a) Since $|xyz| < 1$ except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xyz} = \sum\limits_{n=0}^{\infty}(xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right] = \sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)^3} = \frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

(b) Since $|-xyz| < 1$, except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1+xyz} = \sum\limits_{n=0}^{\infty}(-xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1+xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(-xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (-xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right] = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} = \frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n^3}$$

To evaluate this sum, we first write out a few terms: $s = 1 - \dfrac{1}{2^3} + \dfrac{1}{3^3} - \dfrac{1}{4^3} + \dfrac{1}{5^3} - \dfrac{1}{6^3} \approx 0.8998$. Notice that $a_7 = \dfrac{1}{7^3} < 0.003$. By the Alternating Series Estimation Theorem from Section 11.5, we have $|s - s_6| \le a_7 < 0.003$. This error of 0.003 will not affect the second decimal place, so we have $s \approx 0.90$.

8. $$\int_0^\infty \frac{\arctan \pi x - \arctan x}{x}\,dx = \int_0^\infty \left[\frac{\arctan yx}{x}\right]_{y=1}^{y=\pi} dx = \int_0^\infty\int_1^\pi \frac{1}{1+y^2x^2}\,dy\,dx$$

$$= \int_1^\pi\int_0^\infty \frac{1}{1+y^2x^2}\,dx\,dy = \int_1^\pi \lim_{t\to\infty}\left[\frac{\arctan yx}{y}\right]_{x=0}^{x=t} dy$$

$$= \int_1^\pi \frac{\pi}{2y}\,dy = \frac{\pi}{2}\Big[\ln y\Big]_1^\pi = \frac{\pi}{2}\ln\pi$$

9. (a) $x = r\cos\theta,\ y = r\sin\theta,\ z = z.$ Then $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ and

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial r}\right] + \sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial r}\right]$$

$$= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\,\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\,\sin\theta$$

Similarly $\dfrac{\partial u}{\partial\theta} = -\dfrac{\partial u}{\partial x}r\sin\theta + \dfrac{\partial u}{\partial y}r\cos\theta$ and

$$\frac{\partial^2 u}{\partial\theta^2} = \frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta + \frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta - 2\,\frac{\partial^2 u}{\partial y\,\partial x}r^2\sin\theta\,\cos\theta - \frac{\partial u}{\partial x}r\cos\theta - \frac{\partial u}{\partial y}r\sin\theta.$$ So

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2} &= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\,\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\,\sin\theta + \frac{\partial u}{\partial x}\frac{\cos\theta}{r} + \frac{\partial u}{\partial y}\frac{\sin\theta}{r}\\ &\quad + \frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\,\frac{\partial^2 u}{\partial y\,\partial x}\sin\theta\,\cos\theta\\ &\quad - \frac{\partial u}{\partial x}\frac{\cos\theta}{r} - \frac{\partial u}{\partial y}\frac{\sin\theta}{r} + \frac{\partial^2 u}{\partial z^2}\\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\end{aligned}$$

(b) $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi.$ Then

$$\frac{\partial u}{\partial\rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\rho} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial\rho} = \frac{\partial u}{\partial x}\sin\phi\,\cos\theta + \frac{\partial u}{\partial y}\sin\phi\,\sin\theta + \frac{\partial u}{\partial z}\cos\phi,$$ and

$$\begin{aligned}\frac{\partial^2 u}{\partial \rho^2} &= \sin\phi\cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial \rho}\right] \\ &\quad + \sin\phi\sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial \rho}\right] \\ &\quad + \cos\phi\left[\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial \rho} + \frac{\partial^2 u}{\partial x\,\partial z}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\,\partial z}\frac{\partial y}{\partial \rho}\right] \\ &= 2\,\frac{\partial^2 u}{\partial y\,\partial x}\sin^2\phi\,\sin\theta\,\cos\theta + 2\,\frac{\partial^2 u}{\partial z\,\partial x}\sin\phi\,\cos\phi\,\cos\theta + 2\,\frac{\partial^2 u}{\partial y\,\partial z}\sin\phi\,\cos\phi\,\sin\theta \\ &\quad + \frac{\partial^2 u}{\partial x^2}\sin^2\phi\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\phi\sin^2\theta + \frac{\partial^2 u}{\partial z^2}\cos^2\phi\end{aligned}$$

Similarly $\dfrac{\partial u}{\partial \phi} = \dfrac{\partial u}{\partial x}\,\rho\cos\phi\cos\theta + \dfrac{\partial u}{\partial y}\,\rho\cos\phi\sin\theta - \dfrac{\partial u}{\partial z}\,\rho\sin\phi$, and

$$\begin{aligned}\frac{\partial^2 u}{\partial \phi^2} &= 2\,\frac{\partial^2 u}{\partial y\,\partial x}\,\rho^2\cos^2\phi\,\sin\theta\,\cos\theta - 2\,\frac{\partial^2 u}{\partial x\,\partial z}\,\rho^2\sin\phi\,\cos\phi\,\cos\theta \\ &\quad - 2\,\frac{\partial^2 u}{\partial y\,\partial z}\,\rho^2\sin\phi\,\cos\phi\,\sin\theta + \frac{\partial^2 u}{\partial x^2}\,\rho^2\cos^2\phi\,\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\,\rho^2\cos^2\phi\,\sin^2\theta \\ &\quad + \frac{\partial^2 u}{\partial z^2}\,\rho^2\sin^2\phi - \frac{\partial u}{\partial x}\,\rho\sin\phi\,\cos\theta - \frac{\partial u}{\partial y}\,\rho\sin\phi\,\sin\theta - \frac{\partial u}{\partial z}\,\rho\cos\phi\end{aligned}$$

And $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}\,\rho\sin\phi\sin\theta + \dfrac{\partial u}{\partial y}\,\rho\sin\phi\cos\theta$, while

$$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2} &= -2\,\frac{\partial^2 u}{\partial y\,\partial x}\,\rho^2\sin^2\phi\,\cos\theta\,\sin\theta + \frac{\partial^2 u}{\partial x^2}\,\rho^2\sin^2\phi\,\sin^2\theta \\ &\quad + \frac{\partial^2 u}{\partial y^2}\,\rho^2\sin^2\phi\cos^2\theta - \frac{\partial u}{\partial x}\,\rho\sin\phi\,\cos\theta - \frac{\partial u}{\partial y}\,\rho\sin\phi\,\sin\theta\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} \\ &\quad = \frac{\partial^2 u}{\partial x^2}\left[(\sin^2\phi\,\cos^2\theta) + (\cos^2\phi\,\cos^2\theta) + \sin^2\theta\right] \\ &\qquad + \frac{\partial^2 u}{\partial y^2}\left[(\sin^2\phi\,\sin^2\theta) + (\cos^2\phi\,\sin^2\theta) + \cos^2\theta\right] + \frac{\partial^2 u}{\partial z^2}\left[\cos^2\phi + \sin^2\phi\right] \\ &\qquad + \frac{\partial u}{\partial x}\left[\frac{2\sin^2\phi\,\cos\theta + \cos^2\phi\,\cos\theta - \sin^2\phi\,\cos\theta - \cos\theta}{\rho\sin\phi}\right] \\ &\qquad + \frac{\partial u}{\partial y}\left[\frac{2\sin^2\phi\,\sin\theta + \cos^2\phi\,\sin\theta - \sin^2\phi\,\sin\theta - \sin\theta}{\rho\sin\phi}\right]\end{aligned}$$

But

$$2\sin^2\phi\cos\theta + \cos^2\phi\cos\theta - \sin^2\phi\cos\theta - \cos\theta = (\sin^2\phi + \cos^2\phi - 1)\,\cos\theta = 0$$

and similarly the coefficient of $\partial u/\partial y$ is 0. Also

$$\sin^2\phi\cos^2\theta + \cos^2\phi\cos^2\theta + \sin^2\theta = \cos^2\theta\,(\sin^2\phi + \cos^2\phi) + \sin^2\theta = 1$$

and similarly the coefficient of $\partial^2 u/\partial y^2$ is 1. So Laplace's Equation in spherical coordinates is as stated.

10. (a) Consider a polar division of the disk, similar to that in Figure 15.3.4, where $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi$, $0 = r_1 < r_2 < \cdots < r_m = R$, and where the polar subrectangle R_{ij}, as well as r_i^*, θ_j^*, Δr and $\Delta\theta$ are the same as in that figure. Thus $\Delta A_i = r_i^*\,\Delta r\,\Delta\theta$. The mass of R_{ij} is $\rho\,\Delta A_i$, and its distance from m is $s_{ij} \approx \sqrt{(r_i^*)^2 + d^2}$. According to Newton's Law of Gravitation, the force of attraction experienced by m due to this polar subrectangle is in the direction from m towards R_{ij} and has magnitude $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}$. The symmetry of the lamina with respect to the x- and y-axes and the position of m are such that all horizontal components of the gravitational force cancel, so that the total force is simply in the z-direction. Thus, we need only be concerned with the components of this vertical force; that is, $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}\sin\alpha$, where α is the angle between the origin, r_i^* and the mass m. Thus $\sin\alpha = \dfrac{d}{s_{ij}}$ and the previous result becomes $\dfrac{Gm\rho d\,\Delta A_i}{s_{ij}^3}$. The total attractive force is just the Riemann sum $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{Gm\rho d\,\Delta A_i}{s_{ij}^3} = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{Gm\rho d(r_i^*)\,\Delta r\,\Delta\theta}{\left[(r_i^*)^2 + d^2\right]^{3/2}}$

which becomes $\displaystyle\int_0^R\int_0^{2\pi}\frac{Gm\rho d}{(r^2+d^2)^{3/2}}\,r\,d\theta\,dr$ as $m\to\infty$ and $n\to\infty$. Therefore,

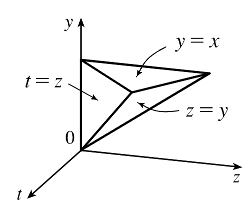
$$F = 2\pi Gm\rho d\int_0^R\frac{r}{(r^2+d^2)^{3/2}}\,dr = 2\pi Gm\rho d\left[-\frac{1}{\sqrt{r^2+d^2}}\right]_0^R = 2\pi Gm\rho d\left(\frac{1}{d} - \frac{1}{\sqrt{R^2+d^2}}\right)$$

(b) This is just the result of part (a) in the limit as $R\to\infty$. In this case $\dfrac{1}{\sqrt{R^2+d^2}}\to 0$, and we are left with

$$F = 2\pi Gm\rho d\left(\frac{1}{d} - 0\right) = 2\pi Gm\rho.$$

11. $\int_0^x\int_0^y\int_0^z f(t)\,dt\,dz\,dy = \iiint_E f(t)\,dV$, where $E = \{(t,z,y) \mid 0\le t\le z,\ 0\le z\le y,\ 0\le y\le x\}$.

If we let D be the projection of E on the yt-plane then $D = \{(y,t) \mid 0\le t\le x,\ t\le y\le x\}$. And we see from the diagram that $E = \{(t,z,y) \mid t\le z\le y,\ t\le y\le x,\ 0\le t\le x\}$. So



$$\int_0^x\int_0^y\int_0^z f(t)\,dt\,dz\,dy = \int_0^x\int_t^x\int_t^y f(t)\,dz\,dy\,dt = \int_0^x\left[\int_t^x (y-t)\,f(t)\,dy\right]dt$$

$$= \int_0^x \left[\left(\tfrac{1}{2}y^2 - ty\right)f(t)\right]_{y=t}^{y=x}\,dt = \int_0^x \left[\tfrac{1}{2}x^2 - tx - \tfrac{1}{2}t^2 + t^2\right]f(t)\,dt$$

$$= \int_0^x \left[\tfrac{1}{2}x^2 - tx + \tfrac{1}{2}t^2\right]f(t)\,dt = \int_0^x \left(\tfrac{1}{2}x^2 - 2tx + t^2\right)f(t)\,dt$$

$$= \tfrac{1}{2}\int_0^x (x-t)^2\,f(t)\,dt$$

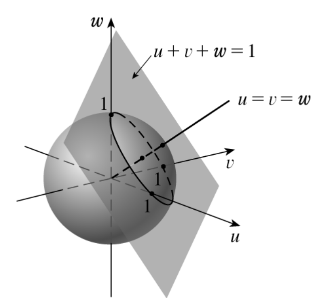
12. $n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\dfrac{1}{\sqrt{n^2+ni+j}}=\sum_{i=1}^{n}\sum_{j=1}^{n^2}\dfrac{1}{\frac{1}{n}\sqrt{n^2+ni+j}}\cdot\dfrac{1}{n^3}=\sum_{i=1}^{n}\sum_{j=1}^{n^2}\dfrac{1}{\sqrt{1+\frac{i}{n}+\frac{j}{n^2}}}\cdot\dfrac{1}{n^3}$ can be considered a double Riemann sum of the function $f(x,y)=\dfrac{1}{\sqrt{1+x+y}}$ where the square region $R=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$ is divided into subrectangles by dividing the interval $[0,1]$ on the x-axis into n subintervals, each of width $\frac{1}{n}$, and $[0,1]$ on the y-axis is divided into n^2 subintervals, each of width $\frac{1}{n^2}$. Then the area of each subrectangle is $\Delta A=\frac{1}{n^3}$, and if we take the upper right corners of the subrectangles as sample points, we have $(x_{ij}^*,y_{ij}^*)=\left(\frac{i}{n},\frac{j}{n^2}\right)$. Finally, note that $n^2\to\infty$ as $n\to\infty$, so

$$\lim_{n\to\infty}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\frac{1}{\sqrt{n^2+ni+j}}=\lim_{n,n^2\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\frac{1}{\sqrt{1+\frac{i}{n}+\frac{j}{n^2}}}\cdot\frac{1}{n^3}=\lim_{n,n^2\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n^2}f(x_{ij}^*,y_{ij}^*)\,\Delta A$$

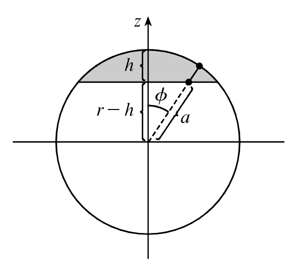
But by Definition 15.1.5 this is equal to $\iint_R f(x,y)\,dA$, so

$$\begin{aligned}\lim_{n\to\infty}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\frac{1}{\sqrt{n^2+ni+j}}&=\iint_R f(x,y)\,dA=\int_0^1\int_0^1\frac{1}{\sqrt{1+x+y}}\,dy\,dx\\&=\textstyle\int_0^1\left[2(1+x+y)^{1/2}\right]_{y=0}^{y=1}dx=2\int_0^1\left(\sqrt{2+x}-\sqrt{1+x}\right)dx\\&=2\left[\tfrac{2}{3}(2+x)^{3/2}-\tfrac{2}{3}(1+x)^{3/2}\right]_0^1=\tfrac{4}{3}(3^{3/2}-2^{3/2}-2^{3/2}+1)\\&=\tfrac{4}{3}(3\sqrt{3}-4\sqrt{2}+1)=4\sqrt{3}-\tfrac{16}{3}\sqrt{2}+\tfrac{4}{3}\end{aligned}$$

13. The volume is $V=\iiint_R dV$ where R is the solid region given. From Exercise 15.9.23(a), the transformation $x=au$, $y=bv$, $z=cw$ maps the unit ball $u^2+v^2+w^2\le 1$ to the solid ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\le 1$ with $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=abc$. The same transformation maps the plane $u+v+w=1$ to $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$. Thus the region R in xyz-space corresponds to the region S in uvw-space consisting of the smaller piece of the unit ball cut off by the plane $u+v+w=1$, a "cap of a sphere" (see the figure).



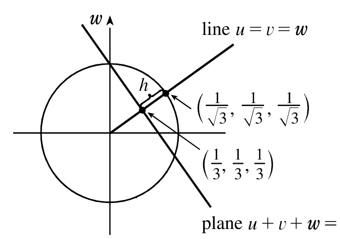
We will need to compute the volume of S, but first consider the general case where a horizontal plane slices the upper portion of a sphere of radius r to produce a cap of height h. We use spherical coordinates. From the figure, a line through the origin at angle ϕ from the z-axis intersects the plane when $\cos\phi=(r-h)/a \quad\Rightarrow\quad a=(r-h)/\cos\phi$, and the line passes through the outer rim of the cap when $a=r \quad\Rightarrow\quad \cos\phi=(r-h)/r \quad\Rightarrow\quad \phi=\cos^{-1}((r-h)/r)$. Thus the cap



is described by $\left\{(\rho, \theta, \phi) \mid (r-h)/\cos\phi \le \rho \le r, 0 \le \theta \le 2\pi, 0 \le \phi \le \cos^{-1}\left((r-h)/r\right)\right\}$ and its volume is

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{\cos^{-1}((r-h)/r)} \int_{(r-h)/\cos\phi}^{r} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\cos^{-1}((r-h)/r)} \left[\tfrac{1}{3}\rho^3 \sin\phi\right]_{\rho=(r-h)/\cos\phi}^{\rho=r} d\phi \, d\theta \\
&= \frac{1}{3} \int_0^{2\pi} \int_0^{\cos^{-1}((r-h)/r)} \left[r^3 \sin\phi - \frac{(r-h)^3}{\cos^3\phi} \sin\phi\right] d\phi \, d\theta \\
&= \tfrac{1}{3} \int_0^{2\pi} \left[-r^3 \cos\phi - \tfrac{1}{2}(r-h)^3 \cos^{-2}\phi\right]_{\phi=0}^{\phi=\cos^{-1}((r-h)/r)} d\theta \\
&= \frac{1}{3} \int_0^{2\pi} \left[-r^3 \left(\frac{r-h}{r}\right) - \frac{1}{2}(r-h)^3 \left(\frac{r-h}{r}\right)^{-2} + r^3 + \frac{1}{2}(r-h)^3\right] d\theta \\
&= \tfrac{1}{3} \int_0^{2\pi} \left(\tfrac{3}{2} r h^2 - \tfrac{1}{2} h^3\right) d\theta = \tfrac{1}{3}\left(\tfrac{3}{2} r h^2 - \tfrac{1}{2} h^3\right)(2\pi) = \pi h^2 \left(r - \tfrac{1}{3} h\right)
\end{aligned}$$

(This volume can also be computed by treating the cap as a solid of revolution and using the single variable disk method; see Exercise 6.2.61.)



To determine the height h of the cap cut from the unit ball by the plane $u + v + w = 1$, note that the line $u = v = w$ passes through the origin with direction vector $\langle 1, 1, 1 \rangle$ which is perpendicular to the plane. Therefore this line coincides with a radius of the sphere that passes through the center of the cap and h is measured along this line. The line intersects the plane at $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and the sphere at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. (See the figure.)

The distance between these points is $h = \sqrt{3\left(\frac{1}{\sqrt{3}} - \frac{1}{3}\right)^2} = \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{1}{3}\right) = 1 - \frac{1}{\sqrt{3}}$. Thus the volume of R is

$$\begin{aligned}
V &= \iiint_R dV = \iiint_S \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV = abc \iiint_S dV = abc\, V(S) \\
&= abc \cdot \pi h^2 \left(r - \tfrac{1}{3} h\right) = abc \cdot \pi \left(1 - \tfrac{1}{\sqrt{3}}\right)^2 \left[1 - \tfrac{1}{3}\left(1 - \tfrac{1}{\sqrt{3}}\right)\right] \\
&= abc\pi \left(\tfrac{4}{3} - \tfrac{2}{\sqrt{3}}\right)\left(\tfrac{2}{3} + \tfrac{1}{3\sqrt{3}}\right) = abc\pi \left(\tfrac{2}{3} - \tfrac{8}{9\sqrt{3}}\right) \approx 0.482abc
\end{aligned}$$