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Sparse Block-Encodings for Linear Combinations of Ladder Operators

(Dated: August 9, 2024)

In this work, we detail the construction of quantum circuit oracles that create block-encodings for observables described as a linear combination of products of ladder operators acting on fermionic, antifermionic, and bosonic modes. We refer to this constuction as LOBE (Ladder Operator Block-Encoding) and show how it can be used to simulate Hamiltonians involving interactions between these different types of particles. Our work builds off of similar sparse-oracle constructions in the literature, but generalizes prior works to establish a clear connection with block-encoding methods that are commonly refered to as LCU (Linear Combination of Unitaries). In addition to extnding LCU to more general observables that are given as a Linear combination of products of ladder operators, we also demonstrate how these oracles can be extended to include bosonic ladder operators. To our knowledge, this is the first block-encoding construction that allows for interactions between fermions, antifermions, and bosons, paving the way for simulation of more complicated quantum systems such as those that arise in high-energy physics.

I. INTRODUCTION

Wop, wop, wop, wop, wop, Dot, fuck 'em up
Wop, wop, wop, wop, wop, I'ma do my stuff [1]

II. THEORY

Give background of ladder operators and constructions of realistic Hamiltonians/Observables from ladder operators

A. Encoding

Here we'll discuss how we encode the physical states we are interested in terms of qubits/registers.

B. Ladder Operators

@Gus, you probably have much better language to define all of this stuff. I just needed to write something down so I could reference it in the circuit construction. Don't hesitate to scrap anything in here.

1. *Feromons and Antiferomons*

Define action of fermionic ladder operators.

Fermions (and antifermions) obey the Pauli-exclusion principle citation and therefore the occupation of a (anti)fermionic mode can only be occupied ($|1\rangle$) or unoccupied ($|0\rangle$). Fermionic (and antifermionic) ladder operators only act non-trivially on the qubits encoding the mode that the ladder operator acts on and we define their action as follows.

The fermionic creation operator is given by:

$$b_i^\dagger |n_{b_i}\rangle = \begin{cases} (-1)^{\sum_{j<i} b_j} |1\rangle & when\, |n_{b_i}\rangle\, is\, |0\rangle \\ 0 & when\, |n_{b_i}\rangle\, is\, |1\rangle \end{cases} \tag{1}$$

where b_i denotes a fermionic ladder operator on the i^{th} mode, the † indicates a creation operator, and $|n_{b_i}\rangle$ is the occupation of the i^{th} fermionic mode. An antifermionic creation operator is defined as above with the symbol d to denote that the operator acts on antifermions.

For a fermionic creation operator, if the mode being acted upon is unoccupied, then the creation operator "creates" a fermion in that mode and applies a phase determined by the parity of the occupation of the previous modes. Therefore the ordering of the modes in the encoding has an implication on the action of the operator that must be accounted for. Since fermionic modes can only be either occupied or unoccupied, then if the mode is already occupied the operator zeroes the amplitude of the quantum state, thereby "destoying" that portion of the quantum state.

The fermionic annihilation operator is given by:

$$b_i \left| n_{b_i} \right\rangle = \begin{cases} (-1)^{\sum_{j<i} b_j} \left| 0 \right\rangle & when \left| n_{b_i} \right\rangle = \left| 1 \right\rangle \\ 0 & when \left| n_{b_i} \right\rangle = \left| 0 \right\rangle \end{cases} \tag{2}$$

and the antifermionic annihilation operator is likewise defined for d instead of b.

The action of the annihilation operators is similar (and opposite) to the creation operators. If the mode is already occupied, then the annihilation operator "annihilates" the fermion at that mode by setting the occupation to zero and applies a phase based on the parity of the occupation of the preceeding modes. If the mode is unoccupied before the operator is applied, then the annihilation operator zeroes the amplitude.

2. *Bosos*

$$a_i^\dagger \left| n_{a_i} \right\rangle = \begin{cases} \sqrt{n_{a_i} + 1} \left| n_{a_i} + 1 \right\rangle & when \left| n_{a_i} \right\rangle \neq \left| \Omega \right\rangle \\ 0 & when \left| n_{a_i} \right\rangle = \left| \Omega \right\rangle \end{cases} \tag{3}$$

where a_i denotes a bosonic ladder operator on the i^{th} mode, the † indicates a creation operator, $\left| n_{a_i} \right\rangle$ is the occupation of the i^{th} bosonic mode, and Ω is the maximum allowable bosonic occupation.

$$a_i \left| n_{a_i} \right\rangle = \begin{cases} \sqrt{n_{a_i}} \left| n_{a_i} - 1 \right\rangle & when \left| n_{a_i} \right\rangle \neq \left| 0 \right\rangle \\ 0 & when \left| n_{a_i} \right\rangle = \left| 0 \right\rangle \end{cases} \tag{4}$$

3. *Commutation Rules*

C. Observables

1. *Products of Ladder Operators (Terms)*

We define a *term* (T) as a product of ladder operators that can act on fermionic, antifermionic, and bosonic modes:

$$T = \prod_{m=0}^{M-1} c_m \tag{5}$$

where M is the number of ladder operators in the term and $c_m \in \{b_i, b_i^\dagger, d_i, d_i^\dagger, a_i, a_i^\dagger\}$.

The ladder operators (c_m) can be reordered arbitrarily with the introduction of additional terms due to the commutation rules described in II B 3. In this work, we will have a preference for *normal ordering* of the operators to obey the following structure:

$$T = \Big(\prod_i (\delta_{b_i^\dagger} b_i^\dagger)(\delta_{b_i} b_i) \Big) \Big(\prod_i (\delta_{d_i^\dagger} d_i^\dagger)(\delta_{d_i} d_i) \Big) \Big(\prod_i (\delta_{a_i^\dagger} (a_i^\dagger)^r)(\delta_{a_i} (a_i)^s) \Big) \tag{6}$$

where δ takes the value 0 or 1 to denote if the operator is active in the term and the values r and s are positive integers $\in [0, \Omega]$ and denote the exponent of the bosonic ladder operators acting on that particular bosonic mode.

Describe number/occupation operators acting on fermions/antifermions and bosons and rewrite previous equation for T including these operators.

2. *Linear Combinations of Terms*

We can write Hamiltonians (or observables) in the form of linear combinations of terms:

$$H = \sum_{l=0}^{L-1} \alpha_l T_l \tag{7}$$

where L is the total number of terms and α_l is a real-valued coefficient associated with the term T_l.

III. LADDER OPERATOR BLOCK-ENCODING (LOBE)

In this section of the text, we'll do the following:

A. Defining "Block-Encoding"

B. Prior Works

1. *Linear Combination of Unitaries*

2. *Sparse Block-Encoding of Pairing Hamiltonians*

stuff from Liu et al Might ask you to fill this out @Gus

C. Circuit Construction

In Figure 1, we define the LOBE circuit in terms of generic oracles. Disregarding the (optional) control qubit ($|ctrl\rangle$), the LOBE circuit makes use of 5 qubit registers: $|index\rangle$, $|valid\rangle$, $|coeff\rangle$, $|\psi\rangle$, and $|0^{\otimes \alpha}\rangle$.

The register denoted $|index\rangle$ is referred to as the index register and is used to index the terms in the Hamiltonian as is done in LCU constructions. The integer representations of the computational basis states of the index register corresponds to the indices l in Eq. 7. The minimum number of qubits required for this register is given by:

$$Q_{index} = \lceil \log_2 L \rceil \tag{8}$$

The register denoted $|valid\rangle$ consists of a single qubit and is referred to as the validation qubit. It serves the same purpose as in [2] which is to denote whether or not the term at the current index (T_l) will annihilate the quantum state. If the term will annihilate the state, then the validation qubit remains in the $|1\rangle$ state such that the branch of the wavefunction stays outside the desired subspace of the block-encoding. If the term will not annihilate the state, then the validation qubit gets flipped to the $|0\rangle$ state for the term T_l.

The register denoted $|coeff\rangle$ is referred to as the coefficient register and is used to apply the coefficients associate with the term T_l. These coefficients include both the coefficients of the terms in the linear combination (α_l) as well as the coefficients associated with the bosonic ladder operators. One qubit is used to apply the α_l coefficient while a separate qubit will be required for each bosonic operator (defined as a creation, annihilation, or occupation operator acting with a positive integer-valued exponent) in the term. If we let K denote the maximum number of bosonic operators within a single term, then the number of qubits in the coefficient register is given by:

$$Q_{coeff} = K + 1 \tag{9}$$

In addition to the system register - of which the encoding was described in Section II A - the LOBE cicuit makes use of 5 additional qubit registers. The top register is a

D. Hamiltonian Rescaling

E. Analytical Cost Analysis

Detail analytic cost of different variations

F. Example

Step-by-step example (intention is to move this to an appendix)

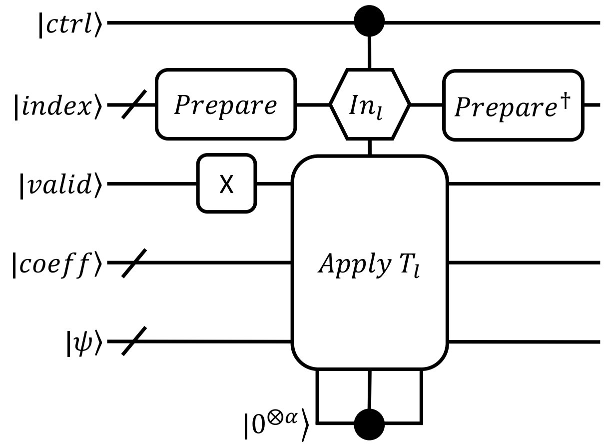

FIG. 1. **Ladder Operator Block-Encoding.**

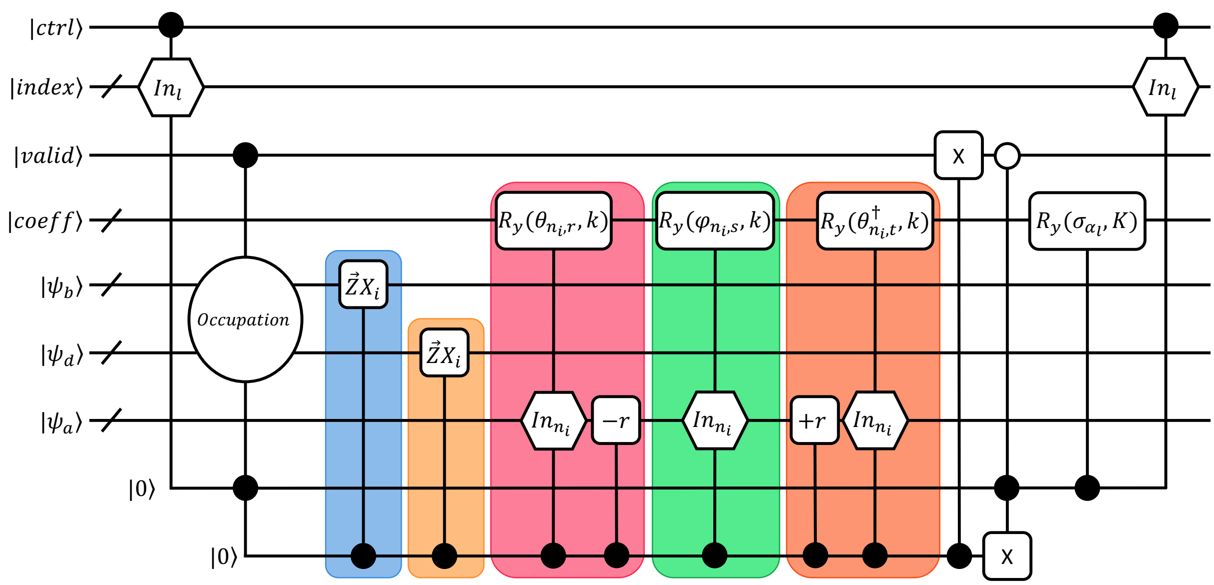

FIG. 2. **Ladder Operator Term Oracle.**

IV. RESULTS / NUMERICAL BENCHMARKING

here we'll "benchmark" (aka numerically compute the cost - number of qubits and gates - of creating a lobe block-encoding) for some systems

ideas of systems to benchmark on:

- Fermi-Hubbard
- Something with just bosons
- Something with fermions, antifermions, and bosons

V. CONCLUSIONS

[1] K. Lamar, *Not Like Us*, 1st ed. (Interscope Records, 2024).

[2] D. Liu, W. Du, L. Lin, J. P. Vary, and C. Yang, An efficient quantum circuit for block encoding a pairing hamiltonian, arXiv preprint arXiv:2402.11205 (2024).