Working Title: Block Encoding the Renormalized Quartic Oscillator Hamiltonian

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Abstract

I. INTRODUCTION

A. The Quartic Oscillator Model

The quartic oscillator can be viewed as a quantum field theory as ϕ^4 constrained to one point in space time. Althernatively, it can be viewed as an extension of the quantum simple harmonic oscillator with a quartic potential term added, e.g.:

$$H = \frac{\dot{\phi}^2}{2} + \frac{m^2\phi^2}{2} + gm^3\phi^4. \tag{I.1}$$

In second quantized form, the Hamiltonian can be expressed in terms of creation and annihilation operators as

$$H = a^\dagger a + g(a + a^\dagger)^4 \tag{I.2}$$

where $m = 1$, and g is an arbitrary coupling constant. The normalized eigenstates of the free part of the Hamiltonian, H_0 are $|k\rangle = (k!)^{-1/2}a^k\,|0\rangle$. The goal is thus, given this basis, finding the ground state eigenvalue of the problem when the quartic potential is turned on ($g \neq 0$).

In the basis of states defined above, the Hamiltonian matrix, $H_{kl} \equiv \langle k|H|l\rangle$ is 5-band-diagonal. We first must pick some cutoff on the basis states to get an explicit matrix form. In general, we want this cutoff, N, to be very large, such that diagonalizing this $N\times N$ Hamiltonian gives a very good approximation to the true ground state. Note that in this model, when it is said that there is no divergence, it means that as $N \to \infty$, the ground state energy E_0 tends to the true ground state of the infinite problem. In more complicated field theories, this won't be true and will require more advanced renormalization techniques.

This $N \times N$ matrix can be diagonalized exactly, or simulated to give an approximation to E_0; however, this will be very expensive. We should be explicit in what we mean by expensive? Alternatively, we can "renormalize" the model, allowing quantum simulation to be performed. This is done with Gaussian elimination.

II. RENORMALIZATION

The quartic oscillator is a model that is not studied here within the context of quantum field theories; however, we can utilize techniques from these theories to reduce the resources needed to simulate this model. In quantum field theories, renormalization is the process of removing infinities that arise when doing calculations beyond leading order. Here, Gaussian elimination is performed to systematically remove one row and column from the "infinitely" large Hamiltonian, thus reducing the dimension of the matrix. Although there are no divergences that arise in this model, Gaussian elimination still is useful. This is done approximately, since pure Gaussian elimination isn't any cheaper than diagonalizing the infinite matrix.

Guassian elimination starts by writing down the eigenvalue equation:

$$H\psi^{(0)} = E_0\psi^{(0)} \tag{II.1}$$

where $\psi^{(0)}$ is a $N \times 1$ vector with entries $\psi_0, \dots \psi_{N-1}$ and H is a $N \times N$ matrix. Writing $H\psi = E\psi$ for the ground state in matrix form gives N coupled equations. To perform Gaussian elimination, one writes down the equation corresponding to $\psi^{(0)}_{N-1}$:

$$\sum_{l=0}^{N-1} H_{N-1,l}\psi_l = E_0\psi_{N-1} \tag{II.2}$$

From here, we rewrite this equation in terms of ψ_{N-1} and plug this into the equation for $\psi^{(0)}_{N-2}$. We say this removes a row and column from the matrix because now there is one less independent degree of freedom. The resulting set of equations is now given as:

$$\sum_{l=0}^{N-2}\left(H_{N-2,l} + \frac{H_{N-2,N}H_{N,l}}{E_0 - H_{N,N}}\right)\psi_{N-2} = E_0\psi_{N-2} \tag{II.3}$$

A problem now becomes evident: the eigenvalue equation for ψ_{N-2} now depends on E_0 in two places. This is where an approximation comes in that allows us to "approximately" Gaussian eliminate the Hamiltonian matrix, leading to a recurrsion relation that shrinks the size of the matrix. We assume that the ground state eigenvalue is small compared to the bottom-right-most matrix element (which is the large-energy corner). Thus, we can let $E_0 - H_{N,N} \approx -H_{N,N}$.

Now, a recurrsion relation can be defined on each matrix element H_{kl} where the "new" matrix after one Gaussian elimination step is one row and column smaller:

$$H^{(j)}_{kl} = H^{(j-1)}_{kl} - g\frac{H^{(j-1)}_{k,N-j}H^{(j-1)}_{N-j,l}}{H^{j-1}_{N-j,N-j}} \tag{II.4}$$

What are LOBE estimates for the "infinite" Hamiltonian matrix vs. the renormalized matrix. How do we quantify if this trade-off is useful?