Incompressible surfaces and spunnormal form

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Abstract

Suppose M is a hyperbolic 3-manifold with torus boundary components and $\mathcal{T}$ is an ideal triangulation of M with essential edges. We show that any incompressible surface S in M that is not a virtual fiber can be isotoped into spunnormal form in $\mathcal{T}$. The proof is based directly on ideas of W. Thurston.

Keywords: spunnormal, normal, ideal triangulation, incompressible surface
MSC (2000): 57M

1 Introduction

A surface S in a closed 3-manifold M with triangulation T is called *normal* if the intersection of S with any tetrahedron in T is a finite collection of three or four sided disks. Their usefulness stems mainly from the fact, proven by Kneser, that any incompressible surface S in M can be isotoped to be in normal form in any triangulation T of M. The space of all normal surfaces in a 3-manifold with a given triangulation can be easily computed from the face-pairing equations, and this theory is well-developed. Normal surfaces and almost normal surfaces have been studied extensively and have proved very useful. See, for example, [A.T98].

Spunnormal surfaces were developed by W. Thurston as a natural extension of normal surfaces for manifolds with ideal triangulations. This form is particularly applicable for surfaces in hyperbolic manifolds with cusps. A surface S in a manifold M with an ideal triangulation $\mathcal{T}$ is called *spunnormal*, or sometimes just *normal* if the intersection of S with every (ideal) tetrahedron is a finite collection of normal quads and a possibly infinite collection of normal triangles. There is conjecturally a correspondence between some set Σ of incompressible spunnormal surfaces and incompressible surfaces that arise from splittings associated with "points at infinity" on the deformation variety of a hyperbolic 3-manifold. Such a correspondence would show that the "Culler-Shalen machine", [CS83], detects all strict boundary slopes of surfaces in Σ by Theorem 3 below. The space of spunnormal surfaces in a 3-manifold with a certain triangulation can be easily computed from the edge equations. This uses the fact that the surface must glue up around an edge. This is used for computations in a program by Culler and Dunfield, [CD]. The theory of the space of spunnormal surfaces has been developed in recent work of Kang and Rubinstein, [EK04], and Tillmann, [S.T]. Here we give an analog of Kneser's theorem for spunnormal surfaces.

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Theorem 3. *Let M be an atoroidal, acylindrical, irreducible, compact three-manifold with torus boundary components, and $\mathcal{T}$ an ideal triangulation of M with essential edges. If S is a properly embedded, two-sided, incompressible, ∂-incompressible surface in M that is not a virtual fiber then S can be isotoped into spunnormal form in $\mathcal{T}$.*

Theorem 3 is proven in section 3. In [Kan03], it was shown that the condition that S is not a fiber is necessary. In particular, the fiber of the figure-8 knot complement cannot be realized in spunnormal form in the standard ideal triangulation.

W. Thurston, [Thu], proved that three-manifolds which satisfy the hypotheses are exactly the manifolds whose interiors admit finite volume hyperbolic metrics. Finite volume hyperbolic manifolds admit ideal triangulations with geodesic, and hence essential, edges by [EP88]. By work of F. Bonahon and W. Thurston, the surface subgroups without accidental parabolics that are not virtual fiber subgroups are exactly the quasi-fuchsian subgroups. Therefore, Theorem 3 shows that quasi-fuchsian surfaces in a hyperbolic 3-manifold with cusps can be isotoped into spunnormal form. Note that since we are assuming the surface is embedded, a virtual fiber is either a fiber of a fibration of M or a semi-fiber of a semi-fibration of M. For the purposes of this paper, the most important aspect of assumption that S is not a virtual fiber is that product regions have bounded length with respect to S. This idea has been used in several other contexts, namely [CL99], and [Li02], and in initial versions of the cyclic surgery theorem, see [BCSZ]. It is discussed below in section 2 and in appendix A.

1.1 Acknowledgements

The author has benefitted from conversations about this and related work with many people, including Steve Kerckhoff, Nathan Dunfield, Stephan Tillmann, Alan Reid, Cameron Gordon, and John Luecke. Again, the main ideas of the proof are due to William Thurston.

2 Product Regions

Let M be a irreducible three-manifold with boundary. Let $\mathcal{A}$ be a collection of annuli in ∂M such that 1) $\partial M - \mathcal{A}$ is incompressible in M and 2) any essential arc of $\mathcal{A}$ cannot be isotoped rel boundary into $\partial M - \mathcal{A}$. A *product disk* with respect to $\mathcal{A}$ is a map $i : I \times I \to M$ such that $i(\partial I \times I)$ is a pair of essential arcs in $\mathcal{A}$ and $i(I \times \partial I)$ is a pair of possibly immersed arcs in $\partial M - \mathcal{A}$ that cannot be isotoped rel boundary into $\mathcal{A}$.

We call $i(\partial I \times I)$ the *vertical boundary* of D and $i(I \times \partial I)$ the *horizontal boundary* of D. All proper arcs isotopic to essential arcs of $\mathcal{A}$ are called *vertical arcs*. The following is lemma 2.1 of [Li02]:

Lemma 1. *Given M and $\mathcal{A}$ as above, there is a maximal I-bundle J in M such that every product disk with respect to $\mathcal{A}$ can be isotoped into J.*

The proof is by construction; successively adding regular neighborhoods of product disks, and then filling in balls with I-bundles. The result is a I-bundle where every fiber is isotopic rel boundary to a vertical arc of a product disk. Thus fibers of J cannot be isotoped rel boundary into $M - \mathcal{A}$. Let X be an irreducible manifold with torus boundary components, and $S \subset X$ a two-sided, incompressible, ∂-incompressible surface with boundary. Then cut X along S to obtain a manifold $X - N(S)$ with boundary and a collection of annuli $\mathcal{A}$ whose union is the set of tori that contain boundary components of S. Then $X - N(S)$ and $\mathcal{A}$ satisfy 1) and 2) above. An *essential rectangle for* (X, S) *of length* n is a map $i : I \times [0, n] \to X$ such that each $i|_{[k,k+1]}, k \in [0, n-1]$ is a product disk for $X - N(S)$ with respect to $\mathcal{A}$. If S is a fiber or a semi-fiber, then there are essential rectangles for (X, S) of arbitrarily high length. This is not the case for surfaces that are not virtual fibers.

Lemma 2. *Let X be an atoroidal, acylindrical, irreducible, compact 3-manifold with torus boundary components and S a two-sided, incompressible, ∂-incompressible surface in X. Suppose that S is not a virtual fiber. Then there exists a number $P(S) \in \mathbb{N}$ such that the length of any essential rectangle for (X, S) is less than $P(S)$.*

The proof follows [Li02], although there the statement is only for manifolds with one torus boundary component. We give the proof in appendix A for the convenience of the reader.

3 Incompressible Surfaces

Theorem 3. *Let M be an atoroidal, acylindrical, irreducible, compact three-manifold with torus boundary components, and $\mathcal{T}$ an ideal triangulation of M with essential edges. If S is a properly embedded, two-sided, incompressible, ∂-incompressible surface in M that is not a virtual fiber then S can be isotoped into spunnormal form in $\mathcal{T}$.*

An *ideal tetrahedron* is a tetrahedron with its vertices removed. An *ideal triangulation* $\mathcal{T}$ of a manifold M with boundary is a expression of the interior of M as the union of ideal tetrahedra identified along their faces. A regular neighborhood of each ideal vertex is homeomorphic to $Y \times \mathbb{R}_+$, where Y is a boundary component of M. We require that $Y \times \mathbb{R}_+$ has a product foliation by normal surfaces in $\mathcal{T}$. We start with a surface S in M that satisfies the hypotheses and isotop it into spunnormal form.

Step 0: Spin S.

Since M has torus boundary components, there is a set of normal separating tori $\{t_i\}$ such that for each i one component of $int(M) - t_i$ is $t_i \times \mathbb{R}_+$. Call this side E_i, and let $E = \cup E_i$. Since our surface is properly embedded, we may choose $\{t_i\}$ such that $S \cap E$ is a union of annuli, $S^1 \times \mathbb{R}_+$. There may be several annuli in any particular E_i. Now, each E_i is foliated by normal tori $\{t_{i,j}\}$, for $j \in \mathbb{R}_+$ where $t_i = t_{i,0}$, and this gives a product structure to each E_i. S is a union of annuli between each $t_{i,j}$ and $t_{i,j+1}$. Spin S so that S is spun once between $t_{i,j}$ and $t_{i,j+1}$, as in fig 1. Let n be the number of components of S in E_i. Then the part of S between $t_{i,j}$ and $t_{i,j+1}$ is mapped n to 1 and onto $t_{i,j}$ by the projection defined by the product structure $p : (t_{i,j} \times I) \to t_{i,j}$. We define the *level* of a point $a \in t_{i,j}$ to be j, or $t_{i,j}$ if i is relevant. The level of a point not in E is -1. Let e be an edge of $\mathcal{T}$. The levels of the intersection of e and a component of $S \cap E_i$ will be j, $j + 1$, $j + 2$... as e goes out toward the cusp.

Thus in any end E_i, an end of the edge e intersects a given component c of $S \cap E_i$ exactly once between $t_{i,j}$ and $t_{i,j+1}$, and this intersection is transverse. Since S is two-sided, a small regular neighborhood $N(S)$ of S is diffeomorphic to $S \times I$ and can be chosen so that e intersects a given component of $N(S) \cap E$ exactly once

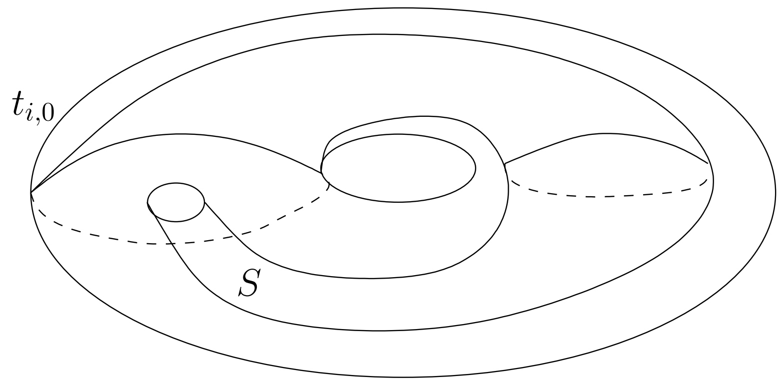

Figure 1: The surface S spun once between $t_{i,0}$ and $t_{i,1}$.

between levels $t_{i,j}$ and $t_{i,j+1}$. Pick and orientation for e and fix an identification of $N(S)$ wit $S \times I$. Label an intersection $+1$ if e enters through $S \times \{0\}$ and exits through $S \times \{1\}$, -1 if e enters through $S \times \{1\}$ and exits through $S \times \{0\}$, and 0 if e enters and exits through the same side, or is tangent to $\partial N(S)$. For a given component of $S \cap E_i$, this intersection number is the same for all j, either $+1$ or -1. If there are several components $\{c_n\}$ of $S \cap E_i$, then as e goes out further in the cusp e will intersect the c_n cyclically in some order. Different components may have different intersection numbers (either $+1$ or -1) with e in E_i.

Step 1: Remove intersections of S and the faces of $\mathcal{T}$ that are circles.

An innermost circle of the intersection of S and a face of $\mathcal{T}$ bounds a disk on S, and can be isotoped off of the face by incompressibility. There are finitely many circle intersections in any face since S is normal in the ends of M by step 0, so this process takes only finitely many steps.

Step 2: Cancel intersections of S and edges of $\mathcal{T}$.

By incompressibility and steps 0 and 1, we may assume that the intersection of S and a tetrahedron T of $\mathcal{T}$ is a union of disks that intersect the faces of T in proper arcs. Suppose that one of these disks intersects an edge e of this tetrahedron more than once. An innermost pair of intersections is connected by an arc on $S \cap T$ that bounds a disk with part of e. We can isotope across this disk to remove this pair of intersections. We need to show that only a finite number of these isotopies are required. In the case when S is closed, there are a finite number of intersections with any edge. We can define the weight of a surface to be the sum of the intersections of the surface and an edge over all edges. Pushing across a disk to remove two intersections as above will decrease this weight so the process must terminate in a finite number of steps. However, when S is not closed, there may be infinitely many intersections of S with an edge of $\mathcal{T}$. The task is now to show that when S is not a fiber or a semi-fiber this process still terminates in a finite number of steps.

We will assume that there is an infinite sequence of moves that simplify the intersection of S and $\mathcal{T}$ as in steps 1 and 2, and reach a contradiction when S is a virtual fiber. Whenever we push S across an edge of $\mathcal{T}$ as in step 2, we cancel two intersections of S and the edge. If this happens at any point in our infinite sequence of moves, we call these two intersections a *canceling pair* of intersections. There must be some edge,

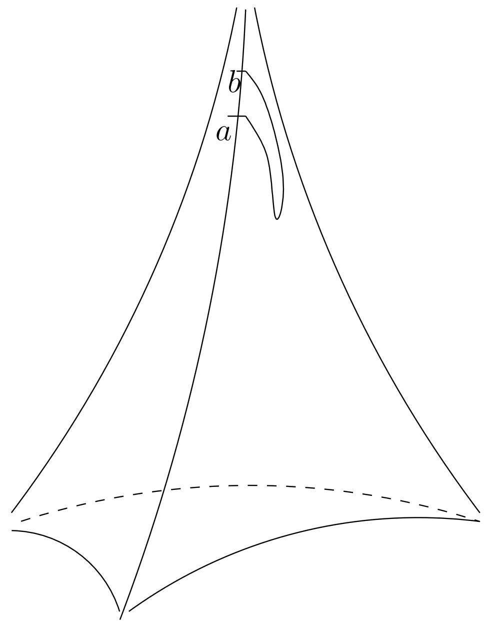

Figure 2: Part of the intersection of S with a tetrahedron on $\mathcal{T}$. Here the canceling pair $\{a, b\}$ gives us a boundary compression in M.

e, that has an infinite number of canceling pairs. There are finitely many canceling pairs in the compact part, $M - E$. Either the pairs of canceling intersections approach one end of e or not.

Suppose that the pairs approach one end of e. There is a pair in some E_i, where the surface has been spun. This pair $\{a, b\}$ is connected by an arc on S that bounds a disk D in $int(M)$ with part of e. Pushing this arc on S across this disk by the sequence of isotopies removes these two intersections. The two sides of $N(S)$ are well-defined, so one of these intersections with $N(S)$ must enter through $S \times \{0\}$ and exit through $S \times \{1\}$, while the other must do the opposite. Therefore a and b must be on different components of $S \cap E_i$. Since a and b are in E_i which is a product, we can extend the disk D all the way out to the ideal boundary of $int(M)$, the boundary of M. Then pushing across this disk is a boundary compression for S in M, which was ruled out in the assumptions. See figure 2.

This shows that if there is a canceling pair (a, b) in E, then a and b must be in different components of $e \cap E$, as in figure 3.

Proceed through the infinite sequence of moves until all the canceling pairs are in E, and on edges that have infinitely many canceling pairs. Then each cancellation is across an arc that connects the two components of the intersection of an edge and E, as in figure 3. There is at least one edge, e, with infinitely many canceling pairs, and thus e has canceling pairs past an arbitrarily high level t. Each canceling pair (a, b) is connected by a long arc on S. The sequence of isotopies (cancellations) pushing sub-arcs of this arc across edges of $\mathcal{T}$ will eventually cancel the pair (a, b). At a given point, there are only finitely many canceling pairs that are connected by an arc lying completely in a face or a tetrahedron of $\mathcal{T}$. Therefore, we can choose t large enough so that an arc connecting a canceling pair on e past level t intersects r edges and must be isotoped across r edges before canceling the intersection, for r arbitrarily large.

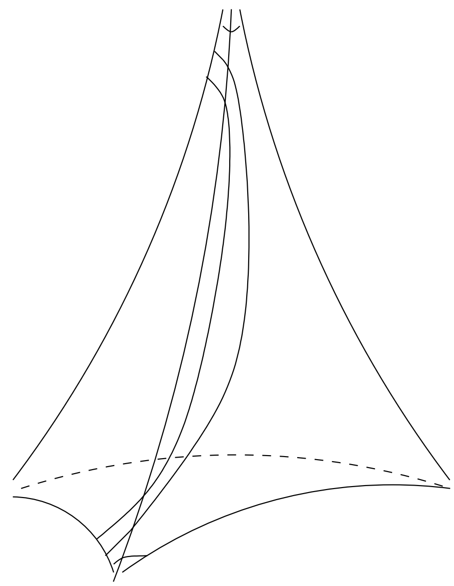

Figure 3: Some canceling pairs along an edge. The outermost pair cancels at some point during the infinite sequence of moves.

Since there are finitely many edges in $\mathcal{T}$, so we can also choose t large enough so that such an arc must be isotoped across some particular edge r times before it cancels the intersection. Call this arc α. By relabeling, we will assume that α connects a canceling pair on the edge e, there are at least r cancellations of α across e, and that the first cancellation of α across e occurs at levels 0 and r_b. Since S has been spun, the next cancellation of α along e occurs at least at levels 1 and $r_b + 1$, and there are cancellations up to levels r and $r + r_b$. We will use this to get a contradiction to lemma 2. The general idea is that this isotopy sweeps out a product disk of length at least r. Since r is arbitrarily large, this is a contradiction to the assumption that S is not a virtual fiber.

Case 1: S is non-separating: We have an arc α with endpoints the canceling pair (a, b) on e with a and b at levels r and $r + r_b$ respectively. Cut M along the tori $t_{i,r+}$ at level $r+$, with $r+ >> r + r_b$. Then we have a manifold X with several torus boundary components. We will continue to call the intersections of S, E, and e with X: S, E, and e. Now consider $\hat{S}$ in X, where is $\hat{S}$ is identical to S outside E but $\hat{S}$ goes straight out from $t_{i,0}$ to $t_{i,r+}$ in each of the E_i. Take the infinite cyclic cover $\tilde{X}$ of X with respect to $\hat{S}$. Consider a lift $\tilde{S}$ of S (the "spun S"). The lift of $S \cap (X - E)$ is contained in a lift $\hat{S}_0$ of $\hat{S}$, since $\hat{S}$ and S coincide in the complement of E. If S has n boundary components on $t_{i,r+}$, then a component of $S \cap E_i$ will pass though nr components of $\hat{S} \cap E_i$ as it goes out r levels. This is because S will hit all the components of $\hat{S} \cap E_i$ each time it is spun around E_i. Let $\tilde{\alpha}$ be the lift of the arc α in $\tilde{S}$. $\tilde{\alpha}$ is contained in a lift $\hat{S}_0$ of $\hat{S}$ except for the lift of the part of α that is in E, see figure 4. The isotopy of α arising from the cancellations will lift to an isotopy of $\tilde{\alpha}$ fixing $\partial\tilde{\alpha}$ that pushes $\tilde{\alpha}$ across r lifts of e and removes (cancels) these intersections. These cancellations occur at higher and higher levels and the associated isotopies are the identity outside the level of the cancellations. Therefore, we are pulling $\tilde{\alpha}$ past intersections of $\tilde{S}$ and lifts of $\hat{S}$. After the last cancellation, we will have pulled $\tilde{\alpha}$ above all lifts of $\hat{S}$ that intersect $\tilde{\alpha}$ at levels r or lower.

Now there is also an arc $\hat{\alpha}$ on $\hat{S}$ that is identical to α except that $\hat{\alpha}$ goes straight out in the product structure from $t_{i,0}$ to $t_{i,r+}$. $\hat{\alpha}$ is isotopic to α rel ∂X by following the spinning. This isotopy lifts to an isotopy of the lift of $\hat{\alpha}$ in $\hat{S}_0$ to $\tilde{\alpha}$. $\hat{\alpha}$ intersects at most two of the torus boundary components of X. Each of these torus boundary components contains some number of boundary components of $\hat{S}$. Let n be the higher number of boundary components. Then this isotopy takes an arc that lies on one lift of $\hat{S}$ to an arc that intersects at least nr lifts of $\hat{S}$. This isotopy is the identity outside of E.

Consider the isotopy $\hat{\alpha}$ to $\tilde{\alpha}$ followed by the isotopy of $\tilde{\alpha}$ up past the lift of $\hat{S}$ at level r. This takes the lift of $\hat{\alpha}$ from level 0 past level r.

We claim that the trace of these two isotopies maps down to a product disk for $(X, \hat{S})$ of length at least nr. Call the image of the trace of the two isotopies D. The edge e is essential, and $\hat{\alpha}$ is isotopic to e, so the horizontal boundary components of D in $X - N(\hat{S})$ are not isotopic into the vertical boundary. Also, successive horizontal boundary components of D come from different lifts of $\hat{S}$. Therefore the vertical boundary components of D are essential. Since there are at least nr intersections of D with $\hat{S}$, D is a product disk of length nr as claimed.

Case 2: S is separating: We will proceed as above but without passing to a cover. Again, cut M along the tori $t_{i,r+}$ to get a manifold X with torus boundary components. $\hat{S}$ is identical to S except that it goes straight out in the product structure from $t_{i,0}$ to $t_{i,r+}$. From the spinning in step 0, the intersection of an edge e and S will rotate among the components of $E_i \cap S$ as we progress out towards the cusp. Since S is separating, there are at least two components of $E_i \cap S$, for each i. Therefore, as e goes out towards the cusp, consecutive intersections $e \cap S$ are on different components of $E_i \cap S$.

As in case 1, there is an arc α on S that connects the canceling pair at level r and $r + r_b$, and an arc $\hat{\alpha}$ on $\hat{S}$ that is isotopic to α. This isotopy follows the spinning and is the identity outside of E. α intersects e in at least r pairs in E, so α has been spun at least r times. Each time α is spun around a component E_i of E, it intersects all the components of $\hat{S} \cap E_i$. Therefore, the isotopy from $\hat{\alpha}$ to α takes α to an arc that intersects $\hat{S}$ at least nr times, where n is the maximum number of components of $\hat{S}$ in the components of E that α intersects. Then α is isotopic to e out to levels r and $r + r_b$ by our sequence of cancellations. This isotopy will go across e every time a cancellation occurs. Since we are canceling intersections with e, this is will remove all intersections of α and e out to levels r and $r + r_b$. The isotopy between given by our sequence of cancellations fixes the endpoints, and takes α to an arc that does not intersect $\hat{S}$ out to levels r and $r + r_b$, so the trace of the two isotopies is a square D that intersects $\hat{S}$ nr times. We need to show that each component of D in $X - \hat{S}$ is product disk. $\hat{S}$ is connected to itself with a union of annuli $\mathcal{A}$ in $\partial(X - \hat{S})$. Consider a component of D in $X - \hat{S}$. There are two horizontal arcs of ∂D on $\hat{S} \subset \partial(X - \hat{S})$. These horizontal arcs are not isotopic into $\mathcal{A}$ because they are isotopic to e, which is an essential edge. The vertical arcs connect the horizontal arcs. As we spin $\hat{\alpha}$ around to meet α, we are tracing out the vertical arcs on ∂M. These vertical arcs will meet all the boundary components of $\hat{S}$ cyclically. Since $\hat{S}$ is separating, there are at least two such boundary components on each boundary torus. Therefore, vertical arcs of D connect different boundary components of $\hat{S}$ and must be essential. Therefore D is a product disk as claimed. □

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A Proof of lemma 2

Lemma 2 *Let X be an atoroidal, acylindrical, irreducible, compact 3-manifold with torus boundary components and S a two-sided, incompressible, ∂-incompressible surface in X. Suppose that S is not a virtual fiber. Then there exists a number $P(S) \in \mathbb{N}$ such that the length of any essential rectangle for (X, S) is less than $P(S)$.*

The proof follows [Li02]. By taking two parallel copies of S if necessary, S is separating. Without loss of generality, $X - S$ has exactly two components, M_1 and M_2.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the annuli in M_1 and M_2, respectively, that are the result of cutting ∂X along the curves ∂S. Let J_1 be the maximal product disk I-bundle for M_1

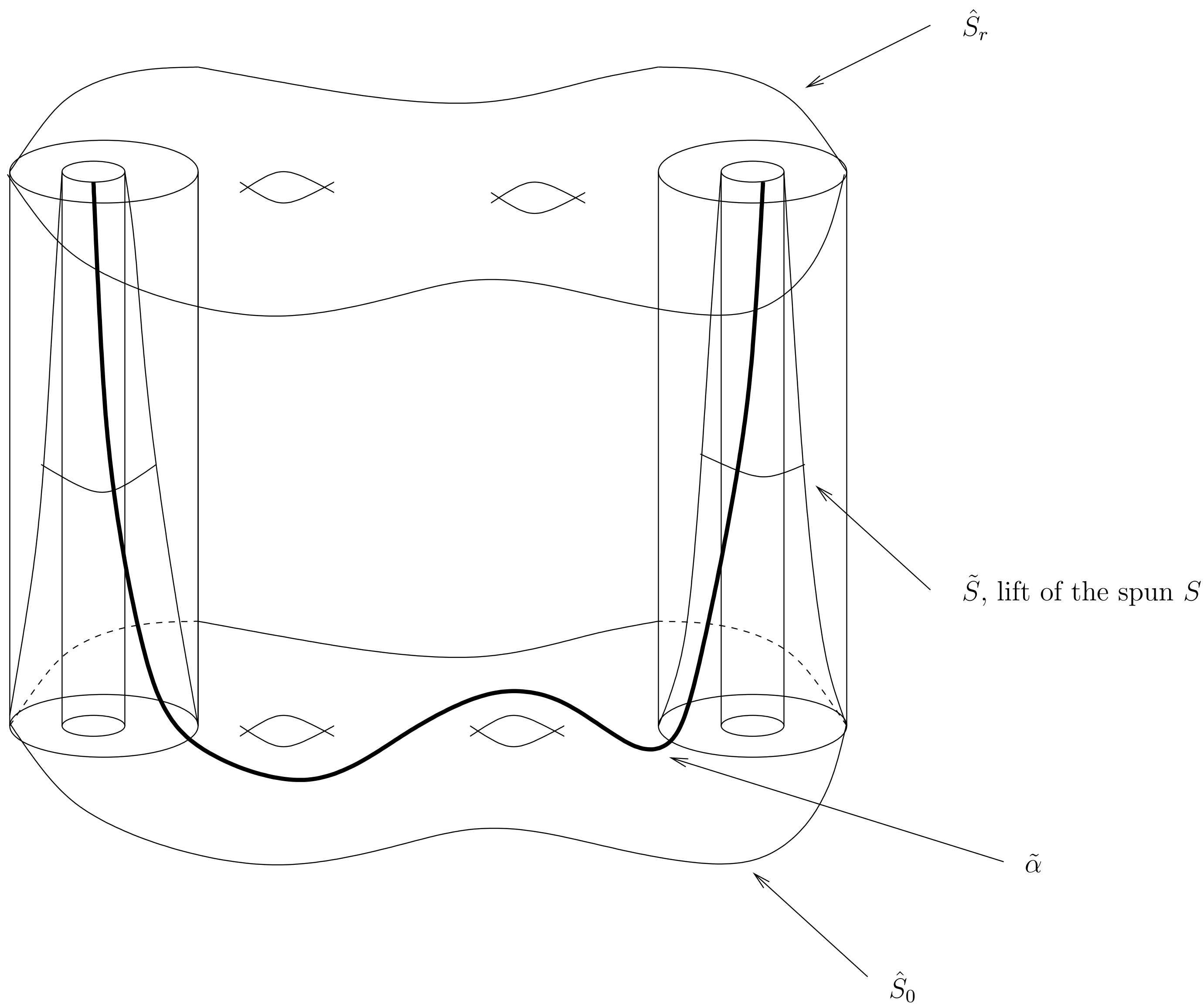

Figure 4: A lift $\tilde{\alpha}$ of the arc α. $\hat{S}_r$ is the lift of $\hat{S}$ that intersects a lift of S at level r.

and $\mathcal{A}_1$ and J_2 the maximal product disk I-bundle for M_2 and $\mathcal{A}_2$. Let $S_i = \partial M_i - \mathcal{A}_i$ be the copy of S on each component M_i, and $C_i = \partial J_i \cap S_i$. Since X is not a fiber bundle or a semi-bundle, at least one of the C_i, say C_1, is not equal to all of S_1. There is an involution $\tau_i, i \in \{1,2\}$, that takes C_i to itself, by switching the endpoints of each fiber in the fiber bundles J_i. $\phi : S_1 \to S_2$ is the gluing map.

Let R_1 and R_2 be subsurfaces of a surface S. We say that R_1 and R_2 are equivalent ($R_1 \sim R_2$) if R_1 is isotopic to R_2 after adding disk components of the complements $S - R_i$ and then removing any disk components from the resulting surfaces. By [Li02][Prop. 2.2], if R_1 and R_2 are subsurfaces of S with $\partial S \subset R_1 \cap R_2$, then there are subsurfaces $R_1' \sim R_1$ and $R_2' \sim R_2$ such that if a non-trivial curve can be homotoped into R_1 and R_2, it can be homotoped into $R_1' \cap R_2'$. We denote the set of surfaces equivalent to R by $[R]$, and do not distinguish between $[R]$ and a properly chosen element of $[R]$, so that $[R_1] \cap [R_2]$ denotes $[R_1' \cap R_2']$.

If $C_i' \subset [C_i]$, then we can isotope J_i so $J_i \cap S_i = C_i'$, and define τ_i coherently. Then $\tau_1([\phi^{-1}(\tau_2([\phi(C_1)] \cap [C_2]))] \cap [C_1])$ is the part of $[C_1]$ that contains both horizontal boundaries of essential rectangles of length 2 that start in M_2. Let $F_1 = C_1$, $E_k = \tau_1([\phi^{-1}(\tau_2([\phi(F_k)] \cap [C_2]))] \cap [C_1])$ and $F_{k+1} = F_k \cap E_k$. Note that a regular neighborhood of $\partial\mathcal{A}_1 \subset S_1$ is contained in every F_k. The main point of the proof is that unless $[F_k] = [\partial\mathcal{A}_1]$, $[F_{k+1}] \subsetneq [F_k]$.

Suppose $[F_{k+1}] = [F_k]$. Then 1) $[\phi(F_k)] = [\phi(F_k)] \cap [C_2]$, 2)$[\phi^{-1}(\tau_2([\phi(F_k)] \cap [C_2]))] = [\phi^{-1}(\tau_2([\phi(F_k)] \cap [C_2]))] \cap [C_1]$, and 3) $[E_k] = [F_k]$.

Since we are assuming $C_1 \neq S_1$, there is a non-trivial boundary curve of $[F_k]$ unless $[F_k] = [\partial\mathcal{A}_1]$. Call it γ. $\phi(\gamma)$ is also a boundary component of $[\phi(F_k)] \cap [C_2]$, by 1), and $\tau_1(\phi^{-1}(\tau_2(\phi(\gamma))))$ is a boundary component of $[E_k] = [F_k]$ by 2) and 3). Call this new curve γ_1. Then, by the same reasoning, $\tau_1(\phi^{-1}(\tau_2(\phi(\gamma_1))))$ is also a boundary curve of F_k, and we define γ_i this way for all i. Each γ_i bounds an annulus or mobius strip with γ_{i+1} for any i because γ_i is glued to $\phi(\gamma_i)$, which is isotopic to $\tau_2(\phi(\gamma_i))$ through J_2. This is glued to $\phi^{-1}(\tau_2(\phi(\gamma)))$, which is isotopic through J_1 to $\tau_1(\phi^{-1}(\tau_2(\phi(\gamma_i)))) = \gamma_{i+1}$. Since $[F_k]$ only has finitely many boundary curves, eventually γ_i is the same boundary curve as γ. We will get an immersed torus or Klein bottle, T.

We claim that T is incompressible and not boundary parallel in X. Suppose that there is a compressing disk $f : D \to X$ with $f(\partial D)$ a non-trivial curve on T. If $f(\partial D)$ is isotopic to some γ_i, then this contradicts the incompressibility of S, since γ_i is a non-trivial curve of $F_k \subset S$. Therefore, $f(\partial D)$ must intersect each γ_i and $f(\partial D) \cap M_1$ is a collection of sub-arcs of $f(\partial D)$ that can be isotoped rel boundary to vertical arcs on $T \cap J_1$. The pullback of $f(\partial D) \cap S$ to D is a collection of arcs in D that connect the pullbacks of these sub-arcs in ∂D. An outermost such arc in D cuts off a subdisk D' of D. $f(D')$ is bounded by an arc of S union a vertical arc of $T \cap J_1$. This is a contradiction, since the vertical arcs are fibers of J_1, which cannot be isotoped rel boundary into S. Therefore, T is incompressible in X. Suppose that T is a boundary parallel torus. Then there is a map $\phi : T^2 \times I \to X$ such that $\phi(T^2 \times 0) = T$ and $\phi(T^2 \times 1)$ is a component Y of ∂X. If $Y \cap S = \emptyset$, then there is a vertical arc of J_1 or J_2 that is isotopic into S rel boundary, which is a contradiction. Suppose Y contains some components of ∂S. Then the pullback of $\phi(T^2 \times I) \cap S$ is a union of annuli a_i with the property that one boundary curve of each a_i maps to a component of $Y \cap S$ and the other boundary curve maps to a component of $T \cap S$. Consider some such annulus a where one boundary curve maps to γ. Then $\phi|_a$ defines a homotopy of γ into ∂S in S, and hence γ is isotopic into ∂S by [Eps66]. This is a contradiction since

γ was not boundary parallel. Thus T is an incompressible torus that is not boundary parallel, contradicting the fact that X is atoroidal.