The Big Dehn Surgery Graph and the link of S^3

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Abstract

In a talk at the Cornell Topology Festival in 2005, W. Thurston discussed a graph which we call "The Big Dehn Surgery Graph", $\mathcal{B}$. Here we explore this graph, particularly the link of S^3, and prove facts about the geometry and topology of $\mathcal{B}$. We also investigate some interesting subgraphs and pose what we believe are important questions about $\mathcal{B}$.

1 Introduction

In unpublished work, W. Thurston described a graph that had a vertex v_M for each closed, orientable, 3-manifold M and an edge between two distinct vertices v_M and $v_{M'}$, if there exists a Dehn surgery between M and M'. That is, there is a knot $K \subset M$ and M' is obtained by non-trivial Dehn surgery along K in M. The edges are unoriented since M is also obtained from M' via Dehn surgery. Roughly following W. Thurston, we will call this graph the Big Dehn Surgery Graph. We denote it by $\mathcal{B}$. We will sometimes denote the vertex v_M by M. If M and M' are obtained from one another via Dehn surgery along two distinct knots, we do not make two distinct edges, although this would also make an interesting graph. We immediately record some basic properties of $\mathcal{B}$. These follow from just a small amount of the extensive work that has been done in the field of Dehn surgery.

Proposition 1.1. *The graph $\mathcal{B}$ has the following basic properties:*
(i)$\mathcal{B}$ is connected; (ii)$\mathcal{B}$ has infinite valence; and (iii) $\mathcal{B}$ has infinite diameter.

The graph $\mathcal{B}$ is connected by the beautiful work of Lickorish [21] and Wallace [36] who independently showed that all closed, orientable $3-$manifolds can be obtained by surgery along a link in S^3. That every vertex v_M in $\mathcal{B}$ has infinite valence can be seen, amongst other ways, by constructing a hyperbolic knot K in M via the work of Myers in [25]. Then by work of Thurston [34] all but finitely many fillings are hyperbolic, and the volumes of the filled manifolds approach the volume of the cusped manifold. The graph $\mathcal{B}$ has infinite diameter since the rank of $H_1(M, \mathbb{R})$ can change by at most one via drilling and filling, and there are 3-manifolds with arbitrarily high rank.

The Lickorish proof explicitly constructs a link, and therefore allows us to describe the following notion of distance. A shortest path from v_{S^3} to v_M in $\mathcal{B}$ counts the minimum number of components needed for a link in S^3 to admit M as a surgery. We will refer to the number of edges in a shortest edge path between v_{M_1} and v_{M_2} as the *Lickorish path length* and denote this function by $p_L(v_{M_1}, v_{M_2})$. For example, if P denotes the Poincaré homology sphere, then $p_L(S^3, P\#P) = 2$. See section 4 for more on p_L. Lickorish path length appears in the literature as surgery distance (see [3], [18]).

The Big Dehn Surgery graph is very big. In order to get a handle on it, we will study some useful subgraphs. We denote the subgraph of a graph generated by the vertices $\{v_i\}$ by $\langle\{v_i\}\rangle$. The *link* of a vertex v is the subgraph $lk(v) = \langle w : p_l(v, w) = 1\rangle$. If there is an automorphism of $\mathcal{B}$ taking a vertex v to a vertex w, then the links of v and w are isomorphic as graphs. We study the links of vertices and a possible characterization of the link of S^3 in §3. Associated to any knot K in a manifold M is a K_∞, the complete graph on infinitely many vertices. This is denoted by $M^K_\infty = \langle v_{M'} : M' = M(K; r)\rangle$. See §3 for notation conventions.

Interestingly, not every K_∞ arises this way. We prove this in §5 and make some further observations about these subgraphs. In §6 we study the subgraph $\mathcal{B}_H$. The vertices of the subgraph $\mathcal{B}_H$ are closed hyperbolic 3-manifolds and there is an edge between two vertices v_M and v_N if there is a cusped hyperbolic 3-manifold with two fillings homeomorphic to M and N. We also study the geometry of $\mathcal{B}$ and $\mathcal{B}_H$, showing that neither is δ-hyperbolic in §7. In §7 we also construct flats of arbitrarily large dimension in $\mathcal{B}$. An infinite family of hyperbolic 3-manifolds with weight one fundamental group which are not obtained via surgery on a knot in S^3 is given in §4. Bounded subgraphs whose vertices correspond to other geometries are detailed in §8.

2 Acknowledgements

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3 The link of S^3

Here we study the links of vertices in $\mathcal{B}$, particularly the link of S^3. As above, the link of a vertex in $\mathcal{B}$ is the subgraph $lk(v) = \langle w : p_L(v, w) = 1\rangle$. If v is associated to the manifold M, the vertices in this subgraph correspond to distinct manifolds which can be obtained via Dehn surgery on knots in M. We begin with a simple proposition.

Proposition 3.1. *The link of S^3 is connected.*

Before the proof, we set notation which we will use for the remainder of the paper. A *slope* on the boundary of a 3-manifold M is an isotopy class of unoriented, simple closed curves on ∂M. We denote the result of Dehn surgery on M along a knot $K \subset M$ with filling slope r by $M(K; r)$. We denote Dehn filling along a link $K_1 \cup K_2 ... \cup K_n \subset M$ by

$M(\{K_1, ..., K_n\}; (r_1, ..., r_n))$, with a dash denoting an unfilled component. Thus the exterior of K in M is denoted $M(K; -)$ and the complement is denoted by $M \setminus K$. We will say that $M(K; -)$ or $M \setminus K$ is hyperbolic if its interior admits a complete hyperbolic metric of finite volume.

Proof. (Proposition 3.1) Consider two manifolds, X_1 and X_2, which are in the link of v_{S^3}. Then X_1 is $S^3(K_1; r_2)$, and X_2 is $S^3(K_2; r_2)$. We will think of a crossing change as a ± 1 Dehn surgery along a small unknotted component which bounds a disk such that the disk meets the knot twice. See [32] for a good explanation of why this is the same as a traditional crossing change. In particular, a crossing change takes a knot K in S^3 to a knot K' in S^3 and $S^3(K; r)$ to $S^3(K'; r')$. See [31, §9H] for an explanation of how the surgery slopes change. In particular, a meridian of K goes to a meridian of K'. Since any two knots can be changed from one to another via crossing changes [32], and having the last surgery be surgery changing the filling slope if necessary, v_{X_1} and v_{X_2} are connected in $\mathcal{B}$ by a sequence of vertices at distance one from each other. Each of these vertices corresponds to surgery on a knot in S^3. It is possible that one of these vertices corresponds to S^3 itself. This could occur either if the slope is sent to a meridian of the knot or if the knot is transformed to the unknot by the crossing change operations. The former cannot occur because a knot complement has a unique meridian by [10] and r is never sent to the meridian by the crossing change operations. To prevent the latter, we first do some crossing changes so that the K_1 is transformed into $K_1 \# K_3$, where K_3 is trefoil knot. Then we transform to $K_2 \# K_3$, and finally to K_2. As before, the last surgery is to correct the surgery coefficients. Note that transforming to a connect sum with the trefoil requires (after isotopy) only one crossing change, so there is no danger of the unknot arising during this process. $\square$

Since there are knots with arbitrarily high crossing number, the paths constructed above in the proof of Proposition 3.1 are arbitrarily long. See [9] for more on the crossing number. However, in conversations with Luisa Paoluzzi, we noticed that the link of S^3 has bounded diameter. Indeed, any surgery on a knot in S^3, $S^3(K; r)$ is at most distance three from a lens space in the links of S^3. Let CK denote a cable of K. Then there is a surgery slope s and a lens space $L(p, q)$ such that $S^3(CK; s) = S^3(K; s') \# L(p, q)$. Thus $S^3(CK; s)$ is distance 1 from a surgery along K and distance 1 from a lens space, and all of these are contained in the link. Note that this gives an alternate proof of Proposition 3.1.

One might hope to distinguish the links of vertices combinatorially in $\mathcal{B}$.

Question 3.2. *Is the link of any vertex in $\mathcal{B}$ connected? of bounded diameter?*

A negative answer would lead to an obstruction to automorphisms of the graph that do not fix S^3. More generally, an answer to the following question would lead to a better understanding of how the Dehn surgery structure of a manifold relates to the homeomorphism type.

Question 3.3. *Does the graph $\mathcal{B}$ admit a non-trivial automorphism?*

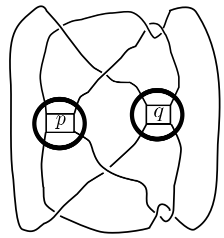


Figure 1: The Kanenobu knots $K_{p,q}$

4 Hyperbolic examples with weight one fundamental groups

A group is *weight* n if it can be normally generated by n elements. Recall that all knot groups are weight one and hence all manifolds obtained by surgery along a knot in S^3 have weight one fundamental groups. It is a folklore question if a manifold which admits a geometric structure and has a weight one fundamental group can always be realized as surgery along a knot in S^3 (see [1, Question 9.23]). The restriction to geometric manifolds is necessary since the fundamental group of $P^3 \# P^3$ is weight one, where P^3 is the Poincaré homology sphere. This cannot be surgery along a knot in S^3 since if an irreducible manifold is surgery along a non-trivial knot in S^3, one of the factors is a lens space [10] . In Theorem 4.4 we show that there are infinitely many hyperbolic 3-manifolds whose fundamental groups are normally generated by one element but which are not in the link of S^3 [Theorem 4.4]. Our technique is a generalization to the hyperbolic setting of a method of Margaret Doig, who in [6] first came up with examples that could not be obtained via surgery on a knot in S^3 using the $d-$ invariant. However, Boyer and Lines [4] exhibited a different set of small Seifert fibered spaces which are weight one but not surgery along a knot in S^3.

Before describing the hyperbolic examples, We make a few remarks regarding the weight one condition. We have the following obstruction to surgery due to James Howie:

Theorem 4.1 ([17], Corollary 4.2)**.** *Every one relator product of three cyclic groups is non-trivial.*

This implies, for example, that $M \cong L(p_1, q_1)\#L(p_2, q_2)\#L(p_3, q_3)$ is not obtained via surgery on a knot in S^3, since its fundamental group is not weight one. However, its homology is cyclic.

The following proposition extends this consequence of Howie's result to hyperbolic manifolds.

Proposition 4.2. *There are hyperbolic 3-manifolds $\{N_j\}$ with cyclic homology such that for each j, $\pi_1(N_j)$ is weight at least two.*

Proof. Just as above, $M \cong L(p_1, q_1)\#L(p_2, q_2)\#L(p_3, q_3)$ with all the p_i pairwise relatively prime. By [25, Theorem 1.1], there exists a knot $K \subset M$ such that K bounds an immersed disk in M and $M - K$ is hyperbolic. Denote $M_E = M - n(K)$, then we may assume that $\pi_1(\partial(M - n(K))) = \langle \mu, \lambda | [\mu, \lambda] = 1\rangle$, μ filling of M_E is M, and λ bounds an immersed disk in M.

This has the immediate consequence that $\Gamma_E = \pi_1(M_E)/\langle\langle\mu\rangle\rangle_{\Gamma_E} \cong \pi_1(M)$ where $\langle\langle\mu\rangle\rangle_{\Gamma_E}$ denotes the normal closure of the element μ in Γ_E.

If γ is a curve in $\partial(M - n(K))$ representing the isotopy class $\mu^r\lambda^s$, then we can denote by $M_E(\gamma)$ Dehn filling along γ. Here, $\pi_1(M_E(\gamma)) = \pi_1(M_E)/\langle\langle\mu^r\lambda^s\rangle\rangle_{\Gamma_E}$. Observe that for any r, s, $\pi_1(M_E(\gamma)) = \pi_1(M_E)/\langle\langle\mu^r\lambda^s, \mu\rangle\rangle_{\Gamma_E} = \pi_1(M_E)/\langle\langle\mu\rangle\rangle_{\Gamma_E}$ as $\lambda \in \langle\langle\mu\rangle\rangle_{\Gamma_E}$ since λ bounds an immersed disk in M. Thus, there exists a surjective homomorphism $f : \pi_1(M_E(\gamma)) \to \pi_1(M)$. In particular, $\pi_1(M_E(\gamma))$ is weight at least 2.

If we let $N_j = M_E(1/j)$ then $H_1(N_j, Z)$ is cyclic of order $p_1p_2p_3$ and by Thurston's Hyperbolic Dehn Surgery Theorem [34, Theorem 5.8.2], N_j is hyperbolic for sufficiently large j. □

Over the two papers [2, 3], Dave Auckly exhibited hyperbolic integral homology spheres that could not be surgery along a knot in S^3. However, it is unknown if these examples have weight one since his construction involves a 4-dimensional cobordism that preserves homology, but not necessarily group weight.

Margaret Doig has recently exhibited examples of manifolds admitting a Thurston geometry, but which cannot be obtained by surgery along a knot in S^3.

Theorem 4.3 ([6], Theorem 2(c))**.** *Of the infinite family of elliptic manifolds with $H_1(Y) = \mathbb{Z}_4$, only one (up to orientation preserving homeomorphism) can be realized as surgery on a knot in S^3, and that is $S^3_4(T_{2,3})$.*

Although not explicitly stated in her result, for a finite group G, the weight of G is determined by the weight G/G' (see [20]), and so the above elliptic manifolds have weight one fundamental groups.

Using similar techniques and the work of Greene and Watson in [13], we are able to exhibit hyperbolic manifolds that have weight one fundamental groups but are never surgery along a knot in S^3. Just as in Greene and Watson, our examples are the double branched covers of the knots $K_{p,q}$ (see Figure 1) where $p = -10n$, $q = 10n+3$, and $n \geq 1$. We denote these knots by K_n and their corresponding double branched covers by M_n. The techniques of the proof may require us to omit finitely many of these double branched covers from the statement of the theorem. We will use $\{X_n\}$ denote the manifolds in this (possibly) pared down set.

Theorem 4.4. *There is an infinite family of hyperbolic manifolds, $\{X_n\}$, none of which can be realized as surgery on a knot in S^3. Furthermore, these manifolds have weight one fundamental groups.*

In the following proof, we require two standard definitions from Heegaard-Floer homology (see [27], [6]). First, a rational homology sphere M is an *L-space* if the hat version of its Heegaard Floer homology is as simple as possible, namely for each Spin^c structure t of M, the hat version of $\hat{HF}(M, t)$ has a single generator and no cancelation. The *d-invariant*, $d(M, t)$ is an invariant assigned to each Spin^c structure t of M is the minimal degree of any non-torsion class of $HF^+(M, t)$ coming from $HF^\infty(M, t)$. Crudely, the d-invariant can be thought of as a way of measuring how far from S^3 a manifold is. This mentality is motivated by the argument in the proof below.

Proof. As noted above, Greene and Watson [13] study the family of knots $\{K_n\}$ and their double branched covers M_n. The manifolds M_n have the following properties:

Each M_n is an L-space ([13, Proposition 11]).

The d-invariant, defined in [27] of the M_n, satisfies the following relation:

$$d(M_n, i) = 2\tau(M_n, i) - \lambda \tag{1}$$

for all $n \geq 0$ and all $i \in Spin^c(M_n)$. Here $\tau(M_n, i)$ is the Turaev torsion and $\lambda = \lambda(M_n)$ is the Casson-Walker invariant. That the Casson-Walker invariants are all identical follows from the work of Mulllins [24, Theorem 7.1] and that the knots are ribbon and have identical Jones polynomials [13, Propositions 8 and 11]. Furthermore, by [13, Proposition 14],

$$\lim_{n\to\infty} min\{\tau(M_n, i) | i \in Spin^c(M_n)\} = -\infty. \tag{2}$$

As they observe, (1) and (2) above imply:

$$\lim_{n\to\infty} min\{d(M_n, i) | i \in Spin^c(M_n)\} = -\infty. \tag{3}$$

Since the manifolds M_n are L-spaces, we may apply:

Theorem 4.5. *[27, Theorem 1.2] If a knot $K \subset S^3$ admits an L-space surgery, then the non-zero coefficients of $\Delta_K(T)$ are alternating $+1s$ and $-1s$.*

Furthermore, it is shown in [30, Theorem 1.2] that the correction terms $d(M_n, i)$ for a knot surgery $S^3_{p/q}(K)$ may be calculated as follows, for $|i| \leq p/q$, $0 < q < p$ and $c = |\lfloor i/q \rfloor|$:

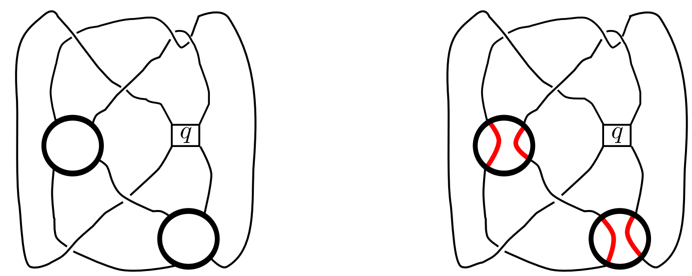
$$d(S^3_{p/q}(K), i) - d(S^3_{p/q}(U), i) = -2\sum_{j=1}^{\infty} j a_{c+j} \tag{4}$$

where the normalized Alexander polynomial of K is

$$\Delta_K(T) = a_0 + \sum_{i=1}^{n} a_i(T^i + T^{-1}).$$

We also note that Greene and Watson [13] establish that $H_1(M_n) = \mathbb{Z}/25\mathbb{Z}$. By homology considerations, if any M_n is p/q surgery on a (standardly framed) knot K in S^3, then $p = 25$. The L-space condition implies $\frac{25}{q} \geq 2g(K) - 1$ by [29, 30]. We also know that such a K is fibered by [19, 26] and that $g(K)$ is the degree of the symmetrized Alexander polynomial of K by [28], bounding the number of terms on the right hand side of Equation (4).

In addition, since M_n is an L-space, if $M_n = S^3_{p/q}(K)$, the Alexander polynomials of such a K has bounded coefficients by Theorem 4.5. Thus, the right hand side of Equation (4) is bounded and since there are only finitely many $L(p, q)$ with $p = 25$, $d(S^3_{p/q}(U), i)$ only can take on finitely many values. Therefore, $d(S^3_{p/q}(K), i)$ is bounded. However, this contradicts the limit (3), and so at most finitely many of the M_n can be surgery on any knot. Furthermore, all but at most finitely many of the M_n are hyperbolic by Lemma 4.7,



(a) We drill out two crossing regions.

(b) Then we fill as above.

Figure 2: These diagrams show K_n switching two tangle regions produces the unknot

proven below. We denote the subsequence of M_n that are hyperbolic and cannot be surgery along a knot by X_n.

Finally, we establish that $\pi_1(M_n)$ is weight one and therefore $\pi_1(X_n)$ is weight one. As noted in [13, §4.2],

$$\pi_1(M_n) = \langle a_1, a_2, a_3, a_4 | b_1, b_2, b_3, b_4\rangle \text{ with } \begin{aligned} b_1 &= (a_1^{-1}a_2)^{10n}a_4^{-1}a_1^2, \\ b_2 &= a_2^{-1}a_3(a_2^{-1}a_1)^{10n}a_2^{-2}, \\ b_3 &= (a_4^{-1}a_3)^{10n+3}a_3^{-1}a_2a_3^{-2}, \text{ and} \\ b_4 &= a_1a_4(a_3^{-1}a_4)^{10n+3}a_4^2. \end{aligned}$$

We claim $\pi_1(M_n)/\langle\langle a_1\rangle\rangle_{\pi_1(M_n)}$ is trivial. First, the relations b_1, b_2 become $a_2^{10n} = a_4$ and $a_2^{10n+2} = a_3$, respectively. Also, the relations b_3 and b_4 reduce to $a_2^{10n-1} = 1$ and $a_2^{10n-6} = 1$ respectively. The claim follows as $gcd(10n - 1, 10n - 6) = 1$. $\square$

Corollary 4.6. *For all n, $p_L(M_n, S^3) \leq 2$ and for all but at most finitely many n, $p_L(M_n, S^3) = 2$.*

Proof. Since we can produce the unknot by switching two crossing regions of the diagram for K_n as in Figure 2, the Montesinos trick shows that M_n can be obtained from surgery along a two component link in S^3. Hence, we see the upper bound $p_L(M_n, S^3) \leq 2$ and $p_L(M_n, S^3) \geq 2$ is established for all but at most finitely many n by the Theorem 4.4. $\square$

Lemma 4.7. *All but at most finitely many $\{M_n\}$ are hyperbolic.*

Proof. The Kanenobu knots K_n are all obtained by tangle filling the two boundary components of the tangle T in Figure 3, and so the manifolds $\{M_n\}$ are obtained by Dehn filling the double cover of T, which we denote by M. A triangulation for M can be obtained by inputting T labeled with cone angle $\frac{\pi}{2}$ into the computer software Orb (an orbifold version of the original Snappea) [16] to obtain an orbifold structure Q.[1] Denote by M, the double cover of Q corresponding to the unique index 2 torsion free subgroup $\pi_1^{orb}(Q)$. This computation shows that M decomposes into 8 tetrahedra. In fact, SnapPy's identify function [5] shows M is homeomorphic to 't12060' in the 8 tetrahedral census. Also, using Snappy, a set of 8 gluing equations for M are encoded by the following matrix:

[1] Detailed instructions on how to input this tangle into Orb are available on the first author's website.

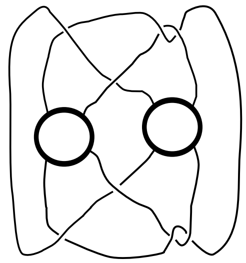


Figure 3: The tangle obtained from drilling the p and q crossing regions from $K_{p,q}$

$$\left(\begin{array}{cccccccccccccccc|c}
2 & -2 & 2 & 1 & 2 & -1 & 1 & -1 & 2 & 0 & 0 & 2 & -1 & 1 & 2 & 0 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 2 & 0 & -1 & 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & -2 & -1 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 0
\end{array}\right)$$

The coding is as follows given a row $\begin{pmatrix} a_1 & b_1 & a_2 & b_2 & \dots & a_8 & b_8 & |c \end{pmatrix}$, we produce a log equation $a_1 \log(z_1) + b_1 \log(1 - z_1) + .. + a_8 \log(z_8) + b_8 \log(1 - z_8) - c\pi \cdot i = 0$. Given such an encoding $z = (2i, \frac{1}{5} + \frac{3i}{5}, \frac{1}{5} + \frac{3i}{5}, \frac{1}{2} + \frac{i}{2}, 1 + 2i, \frac{1}{2} + i, \frac{1}{2} + \frac{i}{2}, \frac{1}{2} + i)$ is an exact solution and therefore M and 't12060' admit a complete hyperbolic structure. By Thurston's Hyperbolic Dehn Surgery Theorem [34, Theorem 5.8.2], the manifolds M_n limit to M. Thus, there are at most finitely many non-hyperbolic M_n. $\square$

5 Complete infinite subgraphs

Here we discuss an interesting property which may allow one to "see" knots in the graph $\mathcal{B}$. We also want to employ the notion of the set of neighbors of a vertex in a graph. More formally, for a graph G and a subset $\{w_i\}$ of the vertices of G, let $\langle\{w_i\}\rangle$ be the subgraph induced by these vertices. That is, the vertices of $\langle\{w_i\}\rangle$ are $\{w_i\}$, and (w_i, w_j) is an edge of $\langle\{w_i\}\rangle$ exactly when (w_i, w_j) is an edge of G. Then, as in the introduction, we define the *link* of a vertex v in G to be $\langle\{w_i\}\rangle$, for all w_i which are path length one from v.

Definition 5.1. If K is a knot in a 3-manifold M then $(M)^K_\infty = \langle\{v_{M(K;r)}\} \cup v_M\rangle$, where $\{M(K;r)\}$ is the set of 3-manifolds obtained from M via Dehn surgery on K.

Proposition 5.2. *For any closed 3-manifold M and knot $K \subset M$, M^K_∞ is a K_∞.*

Proof. That every vertex in M^K_∞ is connected to every other one is a consequence of the definition. We just need to observe that there are infinitely many different manifolds in this subgraph. If $M \setminus K$ admits a hyperbolic structure, then all but finitely many fillings are hyperbolic. Furthermore, the volumes approach the volume of $M \setminus K$ and so there are infinitely many different homeomorphism types. If $M \setminus K$ is Seifert-fibered (including the unknot complement in S^3), it is Seifert-fibered over an orbifold O with boundary. The fillings r can be chosen so that the result is Seifert-fibered over an orbifold where the boundary component of O is replaced with a cone point of arbitrarily high order, so the Seifert-fibered spaces are not homeomorphic. If $M \setminus K$ admits a decomposition along incompressible tori, then, infinitely many fillings have this same decomposition [12]. Then the boundary of $M \setminus K$ is in either a hyperbolic piece or a Seifert-fibered piece, and the above arguments apply. □

Note that sometimes, the intersection of two K_∞ subgraphs arising from fillings on knot complements may intersect in a K_∞. For example, let U be the unknot and $T_{r,s}$ a torus knot. Let $(S^3)^U_\infty$ be the K_∞ associated to $S^3 \setminus U$, and $(S^3)^{T_{r,s}}_\infty$ be the K_∞ associated to $S^3 \setminus T_{r,s}$. Then $(S^3)^U_\infty \cap (S^3)^{T_{r,s}}_\infty$ is a K_∞ where each vertex is a lens space (see [23]). However, this phenomena cannot happen for hyperbolic manifolds.

Proposition 5.3. *If $M \setminus K$ and $M' \setminus K'$ are hyperbolic and not isomorphic, then the subgraphs $(M)^K_\infty$ and $(M')^{K'}_\infty$ have at most finitely many vertices in common.*

Proof. Assume that $(M)^K_\infty$ and $(M')^{K'}_\infty$ have infinitely many vertices in common. Then infinitely many of these are hyperbolic. Denote this set by $\{N_i\}_i \in \mathbb{N}$. Choose a basepoint in the thick part of each N_i. Then the geometric limit of the N_i is $M \setminus K$ and it is also $M' \setminus K'$. See [14] for background on geometric limits. □

5.1 Subgraphs which do not arise from filling

Proposition 5.4. *There is a K_∞ of small Seifert fibered spaces that does not come from surgery along a one cusped manifold.*

Proof. We will construct a family $M_{i,j}$, $i \in \{1, 2, 3, 4\}, j \in \mathbb{N}$ of Seifert fibered spaces over an orbifold with base space S^2 and negative Euler characteristic. We follow notation in [15]. In particular, we denote a closed Seifert fibered space by $SFS(F; \alpha_1/\beta_1, ..., \alpha_n/\beta_n)$ where F is the underlying space of the base orbifold. The Seifert fibered invariants of the exceptional fibers are a_i/b_i, which are allowed to take values in $\mathbb{Q}$. The cone points of the base orbifold will have multiplicities β_i. Two Seifert fiberings $SFS(F; \alpha_1/\beta_1, ..., \alpha_n/\beta_n)$ and $SFS(F'; \alpha'_1/\beta'_1, ..., \alpha'_m/\beta'_m)$ are isomorphic by a fiber preserving diffeomorphism if and only if after possibly permuting indices, $\alpha_i/\beta_i \equiv \alpha'_i/\beta'_i \mod 1$ and, if F is closed, $\sum \alpha_i/\beta_i = \sum \alpha'_i/\beta'_i$. [15, Proposition 2.1].

Let $\{a_1/b_1, a_2/b_2, a_3/b_3, a_4/b_4\}$ be four distinct rational numbers, such that $0 < a_i/b_i < 1$, $\sum 1/b_i < 1$ and a_i, b_i relatively prime. Let $M_{4,0}$ be the Seifert fibered space over S^2 with three exceptional fibers labeled by a_i/b_i $(i = 1, 2, 3)$. We can define $M_{1,0}$, $M_{2,0}$, $M_{3,0}$ similarly. The condition $\sum 1/b_i < 1$ ensures that each manifold will be Seifert fibered over a hyperbolic orbifold.

Note that each manifold $M_{i,0}$ has no common exceptional fibers with the others $\mod 1$, and manifolds with fibrations over hyperbolic base orbifolds have unique Seifert fibered structures [33, Theorem 3.8].

Observation 5.5. The set of manifolds $\{M_{i,0}\}$ form a K_4 in $\mathcal{B}$.

We will now construct a K_∞ which consists of infinitely many of these K_4. Note that if we add 1 to each Seifert invariant of each exceptional fiber above, we get another K_4. This is distinct since the sum of the Seifert invariants is not equal. Each vertex in the new K_4 is connected to each vertex of the previous K_4 as, for example $SFS(S^2; a_1/b_1+1, a_2/b_2+1, a_3/b_3+1) \equiv SFS(S^2; a_1/b_1+3, a_2/b_2, a_3/b_3) \equiv SFS(S^2; a_1/b_1, a_2/b_2+3, a_3/b_3) \equiv SFS(S^2; a_1/b_1, a_2/b_2, a_3/b_3+3)$. Dehn surgery along one of the exceptional fibers can result in any manifold which is a vertex of the original K_4. Continuing this way, we have a K_∞, parametrized by (i,j), where $i \in \{1,2,3,4\}$ and $j \in \mathbb{N}$. Specifically, $M_{i,j}$ is as follows:

$$\begin{aligned}
\{M_{1,j} &= SFS(S^2; a_2/b_2+j, a_3/b_3+j, a_4/b_4+j),\\
M_{2,j} &= SFS(S^2; a_1/b_1+j, a_3/b_3+j, a_4/b_4+j),\\
M_{3,j} &= SFS(S^2; a_1/b_1+j, a_2/b_2+j, a_4/b_4+j),\\
M_{4,j} &= SFS(S^2; a_1/b_1+j, a_2/b_2+j, a_3/b_3+j)\}.
\end{aligned}$$

Assume this K_∞ comes from filling a one cusped manifold M. First, by [10], M must be irreducible. Indeed if it were irreducible there would be a two-sphere that did not bound a ball in M. If the sphere is non-separating it will remain non-separating in any filling. If it is separating, there is at most one filling of a knot in a ball which will make it a ball.

Next we observe that each $M_{i,j}$ is a small Seifert fibered space and in particular non-Haken. We claim that M cannot contain an essential torus. If there is an essential torus T in M, it compresses in each filling. A compressible torus in an irreducible manifold bounds a solid torus. Thus all fillings of M where the torus compresses correspond to a filling of M with the subspace bounded by the torus and ∂M removed. Thus we may assume M is geometric and does not contain an essential torus. If it is not hyperbolic, it is a small Seifert sub-manifold of one of the above manifolds and so M must be Seifert fibered over the disk with at most two exceptional fibers. Each $M_{i,j}$ admits a unique Seifert fibration (see [33, Theorem 3.8]). Any choice of two elements $\{a_i/b_i\}$ to label the exceptional fibers of M will disagree with two exceptional fibers in one of the M_i, which is a contradiction to the existence of such an M. Thus M must be hyperbolic. However, there are infinitely many Seifert fibered manifolds that come from surgery on M. This contradicts Thurston's Hyperbolic Dehn Surgery Theorem [34, Theorem 5.8.2]. $\square$

We can also find finite complete graphs which do not arise from surgeries along a fixed knot in a fixed 3-manifold, as pointed out by Tao Li. For example, let M, N, and C be irreducible manifolds such that M and N are integral homology spheres, C has finite cyclic homology and v_N, v_M and v_C form a K_3 in $\mathcal{B}$. This can be obtained since, for example, S^3, the Poincaré homology sphere, and a lens space are all surgeries on the trefoil knot. Then the vertices $v_{M\#N}$, $v_{M\#C}$ and $v_{C\#N}$ also form a K_3 subgraph of $\mathcal{B}$. The claim is that there is no P and K such that all three associated manifolds are obtained by surgery on P along a knot K. Indeed, such a $P \setminus N(K)$ would have to be irreducible, since the components of the sphere decomposition of the manifolds in question are different. Then one can apply [11] to conclude that the filling slopes must all be distance one from each other where the distance between two slopes in $\partial(P \setminus N(K))$ is their algebraic intersection number. If we denote the filling slope for $M \# N$ by $(1,0)$, then the filling slope of $M \# C$ and of $N \# C$ must be $(n,1)$ and $(n+1,1)$ respectively. This is a contradiction to the fact that $M \# C$ and $N \# C$ have the same finite cyclic homology.

The examples above of pathological behavior involve manifolds which are not hyperbolic. This leads us to consider the Hyperbolic Big Dehn Surgery Graph, $\mathcal{B}_H$.

6 The subgraph for hyperbolic manifolds

Definition 6.1. Let $\mathcal{B}_H$ be the subgraph of $\mathcal{B}$ where the vertices correspond to closed hyperbolic 3-manifolds, and there is an edge between two vertices v_M and v_N exactly when there is a one-cusped hyperbolic 3-manifold P with two fillings homeomorphic to M and N.

Note that there is not necessary a hyperbolic Dehn surgery between M and N in our definition.

As mentioned above in Section 5, this part of the graph has the nice property that if two different K_∞ graphs that arise as $M(K)$ and $M(K')$ intersect, they must do so in finitely many vertices. We conjecture that the combinatorics of this subgraph may reveal more of geometry and topology than in the full graph. For the same reasons as $\mathcal{B}$, $\mathcal{B}_H$ is infinite valence and infinite diameter. We show here that it is connected, using the work of Myers. Let Y be a compact orientable 3-manifold, possibly with boundary. Following Myers, we say that Y is *excellent* if it is irreducible and boundary irreducible, not a 3-ball, every properly embedded incompressible surface of zero Euler characteristic is isotopic into the boundary, and it contains a two-sided properly embedded incompressible surface. These manifolds are known by Thurston [35, Thm 1.2] to admit hyperbolic structures. By slight abuse of notation, if a properly embedded 1-manifold $K \subset Y$ has excellent exterior, we will call K excellent.

Theorem 6.2. *(Myers) Let M be a compact connected 3-manifold whose boundary does not contain 2-spheres or projective planes. Let J be a compact properly embedded 1-manifold in M. Then J is homotopic rel ∂J to a excellent 1-manifold K.*

Lemma 6.3. *([25, Lemma 2.1]) If each component of Y is excellent, $F_1 \cup F_2$ and $cl(\partial Y \setminus (F_1 \cup F_2))$ are incompressible in Y, and each component of $F_1 \cup F_2$ has negative Euler characteristic, then X is excellent.*

Theorem 6.4. *Suppose M_0 and M_n are closed hyperbolic 3-manifolds such that the associated vertices v_{M_0} and v_{M_n} are connected via a path of length n in $\mathcal{B}$. Then v_{M_0} and v_{M_n} are connected via a path of length $n+2$ in $\mathcal{B}_H$.*

Proof. Observe that under this hypothesis, there is an n-component link, $\{a_1, ..., a_n\}$ in M_0 and closed manifolds $M_1, ..., M_n$ such that

$$M_i = M_0(\{a_1, ..., a_n\}; (\beta_1, \beta_2, ..., \beta_i, \alpha_{i+1}, ..., \alpha_n)), i \in \{0, .., n\}$$

We will find a knot k in $M_0 \setminus \{a_1, ..., a_n\}$ and a slope r such that the closed manifolds

$$N_i = M_0(\{a_1, ..., a_n, k\}; (\beta_1, ...\beta_i, \alpha_{i+1}, ..., \alpha_n, r)), i \in \{0, .., n\}$$

are hyperbolic. Each N_i is obtained from M_i via Dehn surgery on k with slope r. The knot k and slope r will also have the property that the 1-cusped manifolds

$$P_i = M_0(\{a_1, ..., a_n, k\}; (\beta_1, ..., \beta_{i-1}, -, \alpha_{i+1} ..., \alpha_n, r)),$$

$$Q_0 = M_0(\{a_1, ..., a_n, k\}; (\alpha_1, ..., \alpha_n, -)), \text{and } Q_n = M_0(\{a_1, ..., a_n, k\}; (\beta_1, ..., \beta_n, -))$$

are hyperbolic. We will use Myers' Theorem 6.2 and Lemma 6.3, stated above. We will also use the fact, proven in Lemma 6.5, that, given a $T^2 \times I$ and two slopes x and y on $T^2 \times \{0\}$, there is an arc A in $T^2 \times I$ with endpoints on $T^2 \times \{1\}$ such that the exterior H_A of A in $T^2 \times I$ is excellent. Furthermore, the results of Dehn filling H_A along the slopes x and y are excellent.

Now we prove the existence of a knot k in the exterior of the link $\{a_1, ..., a_n\}$ in M_0 with the desired properties. First fix a homeomorphism h_i of a neighborhood $N(\partial N(a_i))$ of each $\partial N(a_i)$ with $T^2 \times I$. For each component a_i, we construct an arc A_i in $T^2 \times I$ such that: (i) $\partial A_i \subset T^2 \times \{1\}$; (ii) the exterior of A_i in $T^2 \times I$ is excellent; and (iii) the results of filling the exterior of A_i along the slopes $h_i(\alpha_i)$ and $h_i(\beta_i)$ on $T^2 \times \{0\}$ are excellent. This is done in Lemma 6.5 below. Now by Myers' Theorem, stated as 6.2 above, there is an excellent collection of arcs $\{B_i\}$ in $M_0 \setminus \{N(N(a_i))\}$ such that B_i connects an endpoint of A_i to one of $A_{i+1} \mod n$. Then we claim the following:

1. $k = \cup_n (A_i \cup B_i)$ is an excellent knot in $M_0 \setminus \{N(a_i)\}$.
2. The result of filling along any choice of α_i or β_i for any subset of the a_i is excellent.

The fact that the union of arcs in (1) above is a knot follows from the recipe. The fact that the exterior in (1) is excellent follows Myers' Lemma 6.3 above and the fact that each $T^2 \times I \setminus N(A_i)$ is excellent and that the exterior of the union of the B_i is excellent. Similarly, since each $T^2 \times I \setminus N(A_i)$ filled along α_i or β_i is excellent, Myers' gluing Lemma 6.3 yields that filling any subset of the a_i along α_i or β_i is excellent. Thus, in particular, Q_0 and Q_n above are hyperbolic.

Let k be a knot in $M_0 \setminus \{N(a_i)\}$ having property (1) above. Choose a slope r on $\partial N(k)$ such that r lies outside of the finite set of slopes that makes any one of the closed manifolds N_i or the cusped manifolds P_i not hyperbolic.

Then the path $M_0, Q_0, N_0, P_1, N_1, P_2, ..., N_n, Q_n, M_n$ is a path in $\mathcal{B}_H$ connecting v_{M_0} and v_{M_n}. Here the M_i and N_i are closed hyperbolic manifolds (represented by vertices in $\mathcal{B}_H$) and the P_i and Q_i are cusped hyperbolic manifolds (represented by edges in $\mathcal{B}_H$). $\square$

Lemma 6.5. *Given $T^2 \times [0,1]$ and two isotopy classes of curves x and y on $T^2 \times \{0\}$, there is an arc A with endpoints on $T^2 \times \{1\}$ such that:*

1. *$T^2 \times I \setminus N(A)$ is excellent.*
2. *The results of filling $T^2 \times I \setminus N(A)$ along the slopes x and y are excellent.*

Proof. By Myers theorem 6.2 there exists an arc E in $T^2 \times I$ with endpoints on $T^2 \times \{1\}$ such that the exterior $T^2 \times I \setminus N(E)$ is excellent. The arc we will use is E, wrapped around enough to make filling along 2 specified slopes x and y hyperbolic. We detail this wrapping around below.

Fix $T^2 \times I$ up to isotopy. Let m be an oriented slope on $T^2 \times \{0\}$. Let A_m be an essential annulus bounded by m and a curve m' on $T^2 \times \{1\}$. Let l be a slope that has intersection number 1 with m. There are homeomorphisms $f_m, f_l : T^2 \times I \to T^2 \times I$ obtained by cutting along A_m and A_l, twisting once, and then gluing back by the identity on this annulus. We twist so that an oriented $f_m(pm + ql) = pm + (q+1)l$ and $f_l(pm + ql) = (p+1)m + ql$, in the original isotopy class of $T^2 \times I$. Furthermore, given an $n \in \mathbb{N}$ and an oriented slope t, there is an f, which is a composition of f_m and f_l such that the oriented intersection of t and m and t and l is larger than n.

Now let H_E be the exterior of E in $T^2 \times I$. There is a subsurface $D = T^2 \times \{1\} \setminus N(\partial E)$ of the boundary such that it and its complement are incompressible. Thus we may apply Myers gluing to the double along D, DH_E and conclude that it is excellent, hence hyperbolic. The manifold DH_E is the exterior of a knot in $T^2 \times [0,2]$. We say that filling along the components $T^2 \times \{0\}$ and $T^2 \times \{2\}$ such that the filling is the double along D of a filling of H_E is a *symmetric* filling. Then, by Thurston's Hyperbolic Dehn Surgery Theorem [34, Theorem 5.8.2], all but finitely many symmetric fillings of DH_E are hyperbolic. (Note that the filling curves have the same length in the complete structure on DH_E) The maps f_m and f_l extend naturally to DH_E (by restriction to H_R and doubling) and take symmetric slopes to symmetric slopes. Thus there is a map f, which can be taken to be of the form $f_m^n f_l^p$, such that filling DH_E symmetrically along $f^{-1}(x)$ and $f^{-1}(y)$ is hyperbolic. Then the arc $A = f(E)$ in $T^2 \times I$ has the property that filling along x and y is hyperbolic. Indeed doubling the exterior of A results in $f(DH_E)$ which is hyperbolic when symmetrically Dehn filled along x and y. $\square$

7 Obstructions to $\delta-$hyperbolicity

We begin with a lemma which will help us to compute the exact distance in some simple examples.

Lemma 7.1. *Let M_1 and M_2 be closed orientable 3-manifolds and let $0 \le m \le n$ and p a prime. If $\pi_1(M_1) \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^n$ and $\pi_1(M_2) \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^m$ but $\pi_1(M_2) \not\twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^{m+1}$, then*

$$p_L(M_1, M_2) \ge n - m.$$

Proof. Let K be a knot in a closed manifold M, and let w be a word in $\pi_1(M(K;-))$. We claim that if $\phi : \pi_1(M(K;-)) \to (\mathbb{Z}/p\mathbb{Z})^n$ is a surjection, then ϕ induces a surjection ϕ'

from $\pi_1(M(K;-)/\langle\langle w\rangle\rangle$ to $(\mathbb{Z}/p\mathbb{Z})^n$ or $(\mathbb{Z}/p\mathbb{Z})^{n-1}$. Indeed, the image of w under ϕ is either trivial or non-trivial. If it is trivial, then ϕ induces a surjection $\phi' : \pi_1(M(K;-))/\langle\langle w\rangle\rangle$ to $(\mathbb{Z}/p\mathbb{Z})^n = (\mathbb{Z}/p\mathbb{Z})^n / \langle\langle \phi(w)\rangle\rangle$. If $\phi(w)$ is non-trivial, then it is order p in $(\mathbb{Z}/p\mathbb{Z})^n$, since every element is order p. Then there is a minimal generating set of $(\mathbb{Z}/p\mathbb{Z})^n$ where $\phi(w)$ is a basis element. Then $\phi' : \pi_1(M(K;-))/\langle\langle w\rangle\rangle \to (\mathbb{Z}/p\mathbb{Z})^{(n-1)} = (\mathbb{Z}/p\mathbb{Z})^n / \langle\langle \phi(w)\rangle\rangle$ is a surjection. This proves the claim.

We note that if $\pi_1(M(K;-))/\langle\langle w\rangle\rangle$ surjects $(\mathbb{Z}/p\mathbb{Z})^n$, then $\pi_1(M(K;-)))$ does as well, since there is a presentation of the two groups which differs only by the addition of a relation. Then the claim implies that the maximum n such that $\pi_1(N)$ surjects $(\mathbb{Z}/p\mathbb{Z})^n$ can change by at most 1 under the operation of Dehn surgery along a knot in N, and the lemma follows. ☐

Theorem 7.2. *$\mathcal{B}$ is not δ-hyperbolic.*

Proof. For each n, let U_n be the unlink in S^3 with n components with the natural homological framing. Then we will consider the manifolds:

$$\begin{tikzcd}
A_4 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[d, dash] \\
A_3 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & C_{1,3} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[d, dash] \\
A_2 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & C_{2,2} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[d, dash] \\
A_1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & C_{3,1} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \circ \arrow[d, dash] \\
S^3 \arrow[r, dash] & B_1 \arrow[r, dash] & B_2 \arrow[r, dash] & B_3 \arrow[r, dash] & B_4
\end{tikzcd}$$

$$A_j = S^3(\{U_{2n}\}; (\tfrac{p_1}{1}, \tfrac{p_1}{1}, ...\tfrac{1}{0}, \tfrac{1}{0})),$$
$$B_k = S^3(\{U_{2n}\}; (\tfrac{1}{0}, \tfrac{1}{0}, ..., \tfrac{p_1p_2}{1}, \tfrac{p_1p_2}{1})), \text{ and}$$
$$C_{j,k} = S^3(\{U_{2n}\}; (\tfrac{p_1}{1}, ...\tfrac{1}{0}, \tfrac{1}{0}, \tfrac{p_1p_2}{1}, \tfrac{p_1p_2}{1}, ...\tfrac{1}{0}, \tfrac{1}{0})).$$

In other words, the surgeries on the first n components are either $\frac{p_1}{1}$ or $\frac{1}{0}$, with surgery on the first j components being $\frac{p_1}{1}$. Let p_1 and p_2 be distinct primes. The surgeries on the second n components are either $\frac{p_1p_2}{1}$ or $\frac{1}{0}$, with the first k being $\frac{p_1p_2}{1}$.

Then:

$$H_1(A_j, \mathbb{Z}) = (\mathbb{Z}/p_1\mathbb{Z})^j,$$
$$H_1(B_k, \mathbb{Z}) = (\mathbb{Z}/\ p_1p_2\mathbb{Z})^k, \text{ and}$$
$$H_1(C_{j,k}, \mathbb{Z}) = (\mathbb{Z}/p_1\mathbb{Z})^j \oplus (\mathbb{Z}/p_1p_2\mathbb{Z})^k.$$

Then, by repeated use of Lemma 7.1, since every map to an abelian group factors through the homology, the distances between these manifolds are as in the diagram. $\square$

Corollary 7.3. *$\mathcal{B}$ has a quasi-flat based at S^3. Furthermore, $\mathcal{B}$ has a quasi-flat based at each vertex v_M.*

Proof. By choosing four distinct primes p_1, p_2, p_3 and p_4, the graph $\mathcal{B}$ can be seen to exhibit a quasi-flat based at S^3, by the above argument. The vertices of such a quasi-flat are:

$$E_j = S^3(\{U_{2n}\}; (\tfrac{p_1}{1}, \tfrac{p_1}{1}, \ldots \tfrac{1}{0}, \tfrac{1}{0})),$$
$$N_j = (S^3(\{U_{2n}\}; (\tfrac{1}{0}, \tfrac{1}{0}, \ldots., \tfrac{p_1p_2}{1}, \tfrac{p_1p_2}{1})),$$
$$W_j = S^3(\{U_{2n}\}; (\tfrac{p_1p_2p_3}{1}, \tfrac{p_1p_2p_3}{1}, \ldots \tfrac{1}{0}, \tfrac{1}{0})), \text{ and}$$
$$S_j = S^3(\{U_{2n}\}; (\tfrac{1}{0}, \tfrac{1}{0}, \ldots., \tfrac{p_1p_2p_3p_4}{1}, \tfrac{p_1p_2p_3p_4}{1})).$$

In fact, if the manifolds E_j, N_j, W_j and S_j are connect summed with a given closed orientable manifold M, then by the same homology argument as above, there is a large quasi-flat based at M. $\square$

Remark 7.4. This construction can be adapted to construct quasi-flats quasi-isometric to $\mathbb{Z}^m$ for arbitrarily large m.

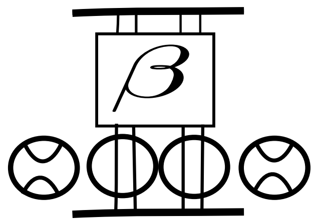
The behavior of homology under Dehn filling is a key property of the $2n$ component split link in the argument above. The pairwise linking number of the components of that link is 0. The rest of this section will be devoted to finding an $2n+1$ component link that has similar behavior. First, we construct hyperbolic link where each one of the pairwise linking numbers is 0. This is accomplished as follows:

Lemma 7.5. *There is a knot K in the complement of the m component split link L_m such that $S^3(\{L_m \cup K\}; -)$ is hyperbolic and all components have pairwise linking number 0.*

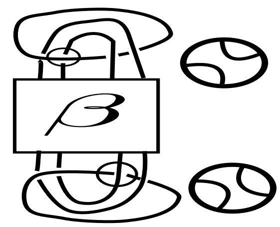
Proof. Consider the link exterior $M = S^3(\{L_m\}; -)$ and label each component by C_i. By [25], we can drill out a set of arcs a_i $1 \leq i < m$ such that a_i connects a neighborhood of the ith component with the $(i+1)$st component and a_m connects the last component to the first. Furthermore, orient each a_i such that a_i is based on a neighborhood of the ith component. For convenience denote the last arc by a_0 and a_n. If D_i is the disk in S^3 with C_i as a boundary, collectively the set $\{a_i\}$ will have an oriented intersection number λ_i with D_i.

Next, consider H_i as the neighborhood of C_i in S^3 and let $A_i = H_i \cap D_i$. With a slight abuse of notation we consider H_i as homeomorphic to $T^2 \times I$ with the marked annulus A_i embedded in it. Again by [25], we can drill out an arc in any homotopy class, and so if the oriented arc connects the end point of a_{i-1} to the a_i can intersect A_i, we may prescribe the oriented intersection number. Here, we choose $-\lambda_i$ to be this intersection number.

Let $K = \cup a_i$ be an oriented knot in $S^3(\{L_m\}; -)$. For each component C_i of L_m, the disk D_i is also a Seifert surface for C_i. Thus, the pairwise linking number of C_i and K is the oriented intersection number of K with D_i, $0 = \lambda_i - \lambda_i$. $\square$



(a) Replacement of these rational tangles gives a null-homotopic two bridge link in $S^2 \times S^1$.



(b) Replacement of these rational tangles gives a null-homotopic two bridge link in S^3.

Figure 4: Rational Tangle replacements on quotients of solv manifolds.

In the above proof, there is a special component of the link represented by K such that drilling out K from $S^3(\{L_m\}; -)$ is hyperbolic. We call this component of the link $L_m \cup K$ the *Myers component*. Although for a general n-component link an $(n-1)$-component must be specified to determine the Myers component, in the context below we hope it is clear which component we mean.

Theorem 7.6. *$\mathcal{B}_H$ is not δ-hyperbolic.*

Proof. Using the link $L' = L_n \cup K$ as in Lemma 7.5, we have that $S^3(\{L'\}; -)$ is hyperbolic and each pair of components has linking number 0. This condition implies that K, an embedded curve, is null homologous in

$$S^3(\{L'\}; (\tfrac{r_1}{s_1}, ..., \tfrac{r_n}{s_n}, \tfrac{1}{0})) \cong L(r_1, s_1)\#...\#L(r_n, s_n),$$

since the homology class of K is determined by the sum of the oriented mod r_i intersection number with the Seifert surface of the ith component of L_n. One can be observe this directly by consideration of K as curve in $S^3(\{L'\}; (\frac{1}{0}, ..., \frac{1}{0}, \frac{r_i}{s_i}, \frac{1}{0}, ..., \frac{1}{0}, \frac{1}{0})) \cong L(r_i, s_i)$.

Thus,

$$H_1(S^3(\{L'\}; (\tfrac{r_1}{s_1}, ..., \tfrac{r_n}{s_n}, \tfrac{1}{q'})), \mathbb{Z}) = \mathbb{Z}/r_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/r_n\mathbb{Z},$$

and so we can choose surgery coefficients such that the homology of the fillings behaves analogously to the manifolds A_j,B_k, and $C_{j,k}$ as in the proof of Theorem 7.2.

Finally, we remark that choosing sufficiently large choices of primes p_1 and p_2 and a large choice of q', the manifolds obtained by filling the first n components of $S^3(\{L'\}; (-, ..., -, \frac{1}{q'}))$ by either $\frac{p_1}{1}$ or $\frac{p_1 p_2}{1}$ is hyperbolic by Thurston's Hyperbolic Dehn Surgery Theorem [34, Theorem 5.8.2]. $\square$

8 Global Structure, Lickorish path length, and geometries.

Here we delve into the more global structure of $\mathcal{B}$. Specifically, we ask: where do certain types of manifolds lie in the graph? The general idea is that "simpler" manifolds lie close to S^3.

Theorem 8.1. *If M is a closed orientable 3-manifold which admits a Solv, Nil, $\mathbb{E}^3$, $S^2 \times \mathbb{R}$ or S^3 geometry, then $p_L(M, S^3) \leq 5$.*

The remainder of this section is dedicated to case analysis that establishes this theorem. Although many of these arguments are well known, they are compiled here for the sake of completeness. Also, where possible, we try to compute $p_L(M, S^3)$ exactly. Finally, these arguments extensively use the so called Montesinos trick, i.e. a surgery along a strongly invertible link in S^3 can also be realized a double branched cover of a link obtained from rational tangle of an unknot diagram.

As noted in the theorem above, this section focuses on five of the eight Thurston geometries. Three of these geometries, Solv, Nil, $\mathbb{E}^3$, only contain manifolds that are torus bundles or finitely covered by torus bundles. Thus, we denote by $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$, the mapping torus $T^2 \times I/((x, 0) \sim (f(x), 1))$ where $f \colon T^2 \to T^2$ is a homeomorphism and the induced action of f on $\pi_1(T^2)$ is equivalent to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$.

We begin with the following proposition computing an upper bound on $p_L(S^3, M)$ if M is Seifert fibered with base orbifold S^2.

Proposition 8.2. *If M is a Seifert fibered space over S^2 with n exceptional fibers, then $p_L(S^3, M) \leq n - 1$.*

Proof. Remove Seifert fibered torus neighborhoods of $n-2$ critical fibers. The resulting manifold $M \setminus \cup_i T_i$, is Seifert fibered over an $n-2$ punctured disk with two singular fibers. Then Dehn fill so that the filling slopes intersect once with the induced Seifert fibered structure on the boundary components of $M \setminus \cup_i T_i$. The resulting manifold will be Seifert fibered over a sphere with two cone points, and thus is a lens space. Since lens spaces are path length one from S^3, the theorem follows. □

Remark 8.3. Martelli and Petronio [22, pp. 1001] realize all Seifert fibered manifolds of the form $RP^2((2, 1), (t+u, u))$ as $((-3, 1), (-3, 1), (t, u))$ filling of the Magic Manifold, also known as the minimally twisted three component link complement in S^3.

Elliptic geometry. Any manifold M covered by S^3 is a Seifert fibered space over S^2 with at most 3 exceptional fibers or a Seifert fibered space over RP^2 with at most 1 exceptional fiber. In the first case, Proposition 8.2 shows $p_L(M, S^3) \leq 2$. In the second case, we can use Remark 8.3 with $(t, u) = (1, 0)$ to see that there is a Seifert fibered space over $RP^2(2)$ that is path length two from S^3. Performing (p, q) surgery along the exceptional fiber will yield any Seifert fibered space M over $RP^2(n)$ and so $p_L(M, S^3) \leq 3$.

$S^2 \times \mathbb{R}$ **geometry.** There are two compact oriented manifolds covered by $S^2 \times \mathbb{R}$, $S^2 \times S^1$ and $RP^3 \# RP^3$. In these cases, $p_L(S^2 \times S^1, S^3) = 1$ ($\frac{0}{1}$ surgery on the unknot) and $p_L(RP^3 \# RP^3, S^3) = 2$, which has non-cyclic homology and can be realized as surgery on the two component unlink.

Euclidean geometry. There are six closed orientable manifolds admitting an $\mathbb{E}^3$ geometry, the three torus T^3, as well as Seifert fibered spaces over the base orbifolds $S^2(2, 2, 2, 2)$, $S^2(2, 4, 4)$, $S^2(3, 3, 3)$, $S^2(2, 3, 6)$ and $RP^2(2, 2)$. (As noted in [33], the Seifert fibered space over $S^2(2,2,2,2)$ is homeomorphic to a Seifert fibered space over the Klein bottle.) T^3 can be realized as $(\frac{0}{1}, \frac{0}{1}, \frac{0}{1})$ surgery on the Borromean rings, a 3 component link complement. If M has base orbifold $S^2(2,2,2,2)$ then $p_L(M, S^3) \geq 3$ by integral homology however $p_L(M, S^3) \leq 3$ by Proposition 8.2. By integral homology computations, the manifolds with base orbifold either $S^2(2,4,4)$ or $S^2(3,3,3)$ are both path length at least two. The Euclidean Seifert fibered space with base orbifold $S^2(2,3,6)$ can be realized as surgery on the trefoil (see [23] for example).

Finally, we have that orientable Euclidean manifold HW over $RP^2(2,2)$, sometimes called the Hantzsche Wendt manifold. By Remark 8.3, HW can be realized as $(\frac{-3}{1}, \frac{-3}{1}, \frac{-t}{u})$ surgery on the Magic manifold if $t = u = 1$. This surgery together with homology considerations shows $p_L(HW, S^3) = 2$.

Nil geometry. For our purposes, there are four types of (orientable) Nil manifolds: Seifert fibered spaces over the torus, Seifert fibered spaces over the Klein bottle, Seifert fibered spaces over S^2 with 3 or 4 exceptional fibers, and Seifert fibered spaces over the $RP^2(2,2)$. We note by Proposition 8.2, Nil manifolds that are Seifert fibered spaces over S^2 with 3 or 4 exceptional fibers are path length at most three from S^3 and any Seifert fibered space over $RP^2(2,2)$ can be realized as $(\frac{-3}{1}, \frac{-3}{1}, \frac{-t}{u})$ surgery if $t + u = \pm 2$ by Remark 8.3.

We now turn our attention to the Nil manifolds that are Seifert fibered spaces over the torus. The Heisenberg manifold H can be realized as $(-\frac{1}{1}, \frac{0}{1}, \frac{0}{1})$ surgery from the Borromean rings. However, since all of the components of the Borromean rings are unknotted, $-\frac{1}{1}$ on one component produces surgery on a two component link in S^3 and so by lower bound obtained from integral homology, $p_L(H, S^3) = 2$.

We can realize all other Nil Seifert fibered spaces over the torus as surgeries along a 3 component link. First do the two surgeries as above and consider H as $T^2_{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$. By drilling out a curve in a torus fiber we can add a Dehn twist with $\frac{1}{n-1}$ surgery ($n \neq 0$). The result being $M = T^2_{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}}$. All Nil manifolds that are Seifert fibered over the torus can be expressed this way. Also for $n = \pm 1$, we obtain H. Otherwise, $H_1(M, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, and so $p_L(M, S^3) \geq 3$. In this case, we can see these results are sharp.

All orientable Nil manifolds over the Klein bottle can be obtained by $(1, b)$ surgery on along a regular fiber in the orientable Euclidean Klein bottle bundle E_K. E_K is shown above to be path length three. Hence, all orientable Nil Klein bottle bundles are path length four or less.

Finally, just as in the Euclidean case, if M is surgery along a knot in S^3, the base orbifold of M must be $S^2(2,3,6)$. We point out that all such manifolds can be obtained by surgery along the trefoil knot (again see [23] for example).

Solv geometry. Solv manifolds are either torus bundles over S^1 or the union of twisted I bundles over the Klein bottle (see [33, Thm 4.17]). Work of Dunbar provides an orbifold analog to this statement, namely if Q is an orientable Solv orbifold, Q is either a manifold as above or an orbifold with fiber $S^2(2,2,2,2)$ over S^1 or the union of twisted I bundles with fiber $S^2(2,2,2,2)$ (see [8, Prop 1.1]). Using these two results, we can obtain the following proposition.

Proposition 8.4. *(i)If M is a torus bundle admitting a solv geometric structure, then*

$p_L(M, S^3) \leq 5$.

(ii) If M is the union of twisted I bundles over the Klein bottle admitting an orientable solv structure, then $p_L(M, S^3) \leq 3$.

Proof. (i) By [8], M admits a 2-fold quotient Q such that the base space of Q is $S^1 \times S^2$ and the singular locus is a four strand braid B. Although that paper is careful to classify such braids, the details will not be relevant to this argument. Using the Montesinos trick, we have a sequence of tangle replacements to get from Q to the trivial two strand braid in $S^1 \times S^2$. The first two replacements of this sequence are shown in Figure 4(b). The resulting link is two-bridge and therefore a single rational tangle replacement yields the unknot. The trivial two strand braid can be obtained from a single rational tangle replacement on the unknot. Hence, $p_L(M, S^1 \times S^2) \leq 4$ and $p_L(M, S^3) \leq 5$.

(ii) Let M is the union of twisted I bundles over the Klein bottle admitting an orientable solv structure. Then M is the 2-fold quotient of $\tilde{M}$ a solv torus bundle. Moreover, $\pi_1(\tilde{M})$ is the index 2 subgroup of $\pi_1(M)$ elements that preserve the orientation of every fiber of M and we may consider $\pi_1(M) = \{\pi_1(\tilde{M}), \rho\pi_1(\tilde{M})\}$ where ρ is the composition of a translation t in a fiber and a symmetry of Solv taking the form $\langle x, y, z\rangle \rightarrow \langle y, x, -z\rangle$ or $\langle -y, -x, -z\rangle$.

Denote by $Q \cong M/\langle t\rangle$ the 2-fold quotient of M by t. The base space of Q is S^3 and a singular set isotopic to the link picture in Figure 4(a). The rational tangle replacements in that figure yield a two-bridge link and so the double branched cover of the resulting link is a lens space. A lens space is path length one from S^3, completing the proof. $\square$

The fundamental group of a solv torus bundle or the union of twisted I bundles over the Klein bottle admitting an orientable solv structure has rank at most three, and so it has weight at most three. We make no claim that the path length bounds in this case are sharp. In fact, for homological reasons, the weight of the fundamental groups of solv torus bundles is at least two, except in the case that $M = T^2_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ with $a + d = 3$. Up to conjugation in $SL(2, \mathbb{Z})$, there is one such matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $T^2_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}$ is well known to be $\frac{0}{1}$ surgery on the figure 8 knot complement.

The following remark summarizes many of the manifolds discussed in this section that are known to be surgery along a knot in S^3. The reader is referred to Margaret Doig's work [6, 7] for a more comprehensive treatment of which manifolds admitting an elliptic geometric structure can be obtained from surgery along a knot in S^3.

Remark 8.5. If M admits a $S^1 \times S^2$, Nil, Euclidean or solv geometric structure and has cyclic homology, M can be obtained from surgery along a knot in S^3. In particular, $S^1 \times S^2$ can be obtained from surgery along the unknot, Euclidean and Nil Seifert fibered spaces over $S^2(2, 3, 6)$ can be obtained from surgery along the trefoil, and the solv torus bundle $T^2_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}$ can be obtained by surgery along the figure 8 knot. Though not explicitly stated above, we point out that all other solv torus bundles and all solv rational homology spheres have non-cyclic homology.

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