Erratum to *Groups with S^2 Bowditch boundary*

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Abstract

The purpose of this erratum is to correct the proof of Lemma 3.1 in [TW20].

1 The result

The following statement appears in [TW20, Lemma 3.1].

Theorem 1. *Let X be a compact metric space. Assume that there exists a surjection $\pi : X \to S^2$ such that (i) there exists a countable dense subset $Z \subset S^2$ so that the restriction of π to $\pi^{-1}(S^2 \setminus Z)$ is injective, and (ii) for each $w \in Z$, the space X_w obtained from X by collapsing each $\pi^{-1}(z)$ to a point for $z \neq w$ is homeomorphic to a closed disk $\mathbb{D}^2$. Then X is homeomorphic to the Sierpinski curve.*

The proof in [TW20] is not complete, as pointed out to us by Lucas H. R. Souza, whom we kindly thank.

About the error. The proof in [TW20] attempts to show that any two spaces X, X' as in the statement are homeomorphic by expressing $X = \lim X(k)$ as an inverse limit, and similarly for X', and constructing a homeomorphism $X \to X'$ by showing that the associated inverse systems $\{X(k)\}$ and $\{X'(k)\}$ are isomorphic. This is done inductively. The base case is a theorem of Bennett [Ben72], which says that any two countable dense subsets of S^2 differ by a homeomorphism $\phi : S^2 \to S^2$. Given this, we want to obtain $\phi_k : X(k) \to X'(k)$ by a "blowup" of ϕ. However, given the non-explicit nature of Bennett's result, it is not clear that one can construct ϕ_k in this manner. In our argument, we attempt to obtain ϕ_k as an extension of a map $\phi_{k-1}\big|$ that is claimed to be uniformly continuous, but this assertion is not justified.

The fix. We provide a different approach that is closer to Whyburn's classical result [Why58, Thm. 3] that characterizes the Sierpinski curve as the unique locally-connected, 1-dimensional continuum in S^2 whose complement is a union of open disks whose boundaries are disjoint.

2 Setup for the proof

Let (X, π, Z) be as in the Theorem 1. We call X (or more precisely the tuple (X, π, Z)) an *$\mathcal{S}$-space*. We will show that any $\mathcal{S}$-space is homeomorphic to a Sierpinski carpet in Section 3. In this section we collect some basic facts about $\mathcal{S}$-spaces that we use to prove the Theorem 1 in Section 3.

Given (X, π, Z), we denote $\mathcal{C} = \{\pi^{-1}(z) : z \in Z\}$. By condition (ii) of the Theorem 1, each $C \in \mathcal{C}$ is an embedded circle in X. We call these circles *peripheral*.

Lemma 2 (Diameter of peripheral circles)**.** *Let X be a $\mathcal{S}$-space. For any $d > 0$, there are only finitely many peripheral circles with diameter $> d$.*

Proof. Suppose for a contradiction that there are infinitely many $C_1, C_2, \ldots$ of diameter $> d$. Choose $x_i, y_i \in C_i$ of distance $> d$. After passing to a subsequences, we may assume that $x_i \to x$ and $y_i \to y$ with $x \neq y$.

If x, y belong to the same peripheral circle $C = \pi^{-1}(w)$, we consider the quotient X_w (collapsing each $\pi^{-1}(z)$ to a point for $z \neq w$) and observe that x, y cannot be separated by open sets in X_w, which contradicts the assumption that $X_w \cong \mathbb{D}^2$. Similarly, if x, y do not belong to the same peripheral circle, we consider the quotient of X by collapsing each $C \in \mathcal{C}$ to a point, and observe that this space is not Hausdorff; on the other hand this quotient is S^2 by assumption, a contradiction. $\square$

Lemma 3 (Quotients of $\mathcal{S}$-spaces)**.** *Let X be an $\mathcal{S}$-space, and let $\mathcal{C}_0 \subset \mathcal{C}$ be a finite collection of k peripheral circles. The space $X(\mathcal{C}_0)$ obtained by collapsing each $C \in \mathcal{C} \setminus \mathcal{C}_0$ to a point is homeomorphic to the compact surface of genus 0 with k boundary components.*

Proof. This is explained in [TW20] in the proof of Lemma 3.1 (this argument is independent of the aforementioned error). $\square$

Lemma 4 (Subdividing an $\mathcal{S}$-space)**.** *Let X be an $\mathcal{S}$-space.*

(i) If $S \subset X$ is an embedded circle disjoint from the peripheral circles, then the closure of each component of $X \setminus S \subset X$ is an $\mathcal{S}$-space.

(ii) More generally, if G is a finite, connected graph embedded in X so that each peripheral circle is either contained in or disjoint from G, then G decomposes X into a union of $\mathcal{S}$-spaces, one for each component of $X \setminus G$.

Proof. (i) By assumption, $\pi(S) \subset S^2$ is an embedded circle. By the Jordan curve theorem, this circle separates S^2 into two closed disks D_1, D_2 with common boundary $\pi(S)$. Then $X \setminus S$ has two components with respective closures $X_1 = \pi^{-1}(D_1)$

and $X_2 = \pi^{-1}(D_2)$. Observe that the quotient map $X_i \to D_i/\partial D_i = S^2$ induces an $\mathcal{S}$-space structure on X_i.

(ii) Let $\mathcal{C}_0 \subset \mathcal{C}$ be the collection of peripheral circles contained in G, and consider the quotient $X(\mathcal{C}_0)$. By Lemma 3, $X(\mathcal{C}_0)$ is a genus 0 surface. The graph G embeds in $X(\mathcal{C}_0)$, is connected, and contains $\partial X(\mathcal{C}_0)$, so it subdivides $X(\mathcal{C}_0)$ into a collection of closed disks. The pre-image of each disk in X has a natural $\mathcal{S}$-space structure, similar to (i). $\square$

Given a graph $G \subset X$ as in Lemma 4, we say that G *subdivides* X into the $\mathcal{S}$-spaces provided by Lemma 4, which we call the *components* of the subdivision. We define the *mesh* of G as the maximum diameter of the components of its subdivision.

The following lemma is analogous to [Why58, Lem. 1]. This lemma may be viewed as the main tool used in the proof Theorem 1.

Lemma 5. *Let X, X' be $\mathcal{S}$-spaces with peripheral circles $\mathcal{C}, \mathcal{C}'$, respectively. Given $C_0 \in \mathcal{C}$ and $C_0' \in \mathcal{C}'$, a homeomorphism $h_0 : C_0 \to C_0'$, and $\epsilon > 0$, there exist graphs G and G' with $C_0 \subset G \subset X$ and $C_0' \subset G' \subset X$, each with mesh $< \epsilon$ and a homeomorphism $h : G \to G'$ extending h_0.*

Proof. The proof is nearly identical to the proof of [Why58, Lem. 1], even though our setup is slightly different. Take $\mathcal{C}_0 \subset \mathcal{C}$ and $\mathcal{C}_0' \subset \mathcal{C}'$ equal-sized collections of peripheral circles containing all the peripheral circles with diameter $\geq \epsilon$. We can choose $\mathcal{C}_0, \mathcal{C}_0'$ finite by Lemma 2. By Lemma 3, there is a homeomorphism $f : X(\mathcal{C}_0) \to X'(\mathcal{C}_0')$ that extends the given homeomorphism $h_0 : C_0 \to C_0'$ (here we are abusing notation slightly by identifying the $C_0 \subset X$ with its homeomorphic copy in $X(\mathcal{C}_0)$).

Let $Z_0 \subset X(\mathcal{C}_0)$ be the image of the collapsed peripheral circles under the quotient $X \to X(\mathcal{C}_0)$, and define $Z_0' \subset X'(\mathcal{C}_0')$ similarly. Then $f(Z_0) \cup Z_0' \subset X'(\mathcal{C}_0')$ is a countable collection of points, and for any $\delta > 0$, we can find a graph $\bar{G}' \subset X'(\mathcal{C}_0')$ containing $\partial X'(\mathcal{C}_0')$ of mesh $< \delta$ that is disjoint from $f(Z_0) \cup Z_0'$. The graphs $\bar{G} = f^{-1}(G')$ and $\bar{G}'$ lift homeomorphically to $G \subset X$ and $G' \subset X'$. By construction, point-preimages of $X \to X(\mathcal{C}_0)$ have diameter $< \epsilon$. Therefore, since X and $X(\mathcal{C}_0)$ are compact, if δ is sufficiently small, then $G \subset X$ will have mesh $< \epsilon$. See [Why58, Lem. 2] for a proof of this fact. The same goes for $G' \subset X'$.

Finally, observe that $f\big| : \bar{G} \to \bar{G}'$ lifts to the desired homeomorphism $h : G \to G'$. $\square$

3 The corrected proof

The Sierpinski curve is an $\mathcal{S}$-space, as explained in [TW20, Proof of Lemma 3.1]. Thus to prove the theorem, it suffices to show that any two $\mathcal{S}$-spaces are homeomorphic. This argument is almost identical to the proof of [Why58, Thm. 3]. We sketch the argument and refer to [Why58] for additional details.

Let (X, π, Z) and (X', π', Z') be two $\mathcal{S}$-spaces with peripheral circles $\mathcal{C}$ and $\mathcal{C}'$, respectively. For each $n \geq 1$, we construct graphs $G_n \subset X$ and $G'_n \subset X'$ satisfying (1) G_n and G'_n have mesh $< \frac{1}{n}$ and (2) $G_n \subset G_{n+1}$ and $G'_n \subset G'_{n+1}$. In addition, we construct homeomorphisms $h_n : G_n \to G'_n$ with h_{n+1} extending h_n.

First we explain how to construct a homeomorphism $X \to X'$ given the existence of the maps $h_n : G_n \to G'_n$. First, these homeomorphisms induce a homeomorphism h between $G := \bigcup G_n$ and $G' := \bigcup G'_n$. Since G_n, G'_n have mesh $\to 0$, $G \subset X$ and $G' \subset X'$ are dense. Since adjacent components of the subdivision of G_n go to adjacent components of the subdivision of G'_n, the map $h : G \to G'$ is uniformly continuous. See [Why58, last two paragraphs of the proof of Theorem 3] for a detailed proof. Therefore h extends to a homeomorphism $X \to X'$.

It remains to construct G_n, G'_n, and h_n. We proceed inductively. First choose arbitrarily $C_0 \in \mathcal{C}$, $C'_0 \in \mathcal{C}'$ and a homeomorphism $h_0 : C_0 \to C'_0$, and apply Lemma 5 with $\epsilon = 1$ to obtain $h_1 : G_1 \to G'_1$. Now G_1 subdivides X, and observe that each component is an $\mathcal{S}$-space with a "preferred" peripheral circle, namely the unique one intersecting G_1 nontrivially. Note also that there is a natural correspondence between the components of the subdivisions of $G_1 \subset X$ and $G'_1 \subset X'$. For the induction step, given G_n, G'_n, h_n, we apply Lemma 5 to each pair of corresponding components of the subdivisions $G_n \subset X$ and $G'_n \subset X'$, taking $\epsilon = \frac{1}{n}$ and using the preferred peripheral circles and h_n as input. □

References

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