On the window (draft)

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Abstract

We show that if $M = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold with boundary, the maximal product region can be determined directly from the combinatorics of the domain of discontinuity $\Omega(\Gamma)$.

1 Introduction

Let F be a non-separating surface in a hyperbolic 3-manifold $M = \mathbf{H}^3/\Gamma$. There is a well-known dichotomy between the case when F is a fiber of a fibration of M and when F is a totally geodesic surface. Geometrically, the limit set of $\pi_1(F) \subset \Gamma$ is the entire S^2_∞ when F is a fiber, and is a round circle when F is a totally geodesic surface. Topologically, $M \setminus F$ is the product $F \times I$ in the former case, and there is no product region in $M \setminus F$ when F is totally geodesic. Here we describe what happens in the intermediate case, when $M \setminus F$ has a non-trivial product region, or window [9]. Note that as $M \setminus F$ is a hyperbolic manifold with boundary, it admits many hyperbolic structures with different limit sets. However, the results of this paper show that the combinatorics of these limit sets are determined at least in part by the size of the window. The size of the window measures how far the surface is from being a fiber and is a very useful measure of complexity. In [2], under certain algebraic hypotheses on Γ, a sequence of covers M_i of M is found containing surfaces F_i such that $M_i \setminus F_i$ has larger and larger window. This sequence terminates and the final surface is a fiber. There is also a measure of how far a surface is from being geodesic, the cusp thickness [1]. However, the relation between cusp thickness and the size of the window is not known.

We will work in a slightly more general setting. Let Γ be a finitely generated Kleinian group such that $\mathbf{H}^3/\Gamma$ is a hyperbolic manifold with incompressible boundary components. We denote the domain of discontinuity by $\Omega(\Gamma)$ and the limit set by $\Lambda(\Gamma)$. By the Ahlfors finiteness theorem [5, 8.14] $S(\Gamma) = \Omega(\Gamma)/\Gamma$ is a finite union of Riemann surfaces of finite type. By a theorem of W. Thurston [7, 7.1], if Γ is geometrically finite, then any finitely generated subgroup of Γ is also geometrically finite. Thus under these circumstances if A is a component of $\Omega(\Gamma)$, $\Lambda(Stab(A)) = \partial A$ is a K-quasicircle for some K. Here we consider the intersection of these quasi-circles and show how this determines the window. When there are no accidental parabolics, the Riemann surfaces $A/Stab(A)$ are boundary components of M. $A/Stab(A)$ has a hyperbolic structure given by the Riemann mapping $R : A \to U$, where U is the unit disk in the complex plane. We choose the Riemann maps $R_A : A \to U$ such that if $\gamma \in \Gamma$, $a \in A$, $R_A(a) = R_{\gamma(A)}(\gamma(a))$, and we denote all of these maps by R. Since the boundary of any component A of $\Omega(\Gamma)$ is a Jordan curve, R extends to a homeomorphism of the closures $\bar{R} : \bar{A} \to \bar{U}$ by [8]. We will slightly abuse notation and write "a geodesic l in A with endpoints p and q" to mean that the image of l is a geodesic in U with endpoints $\bar{R}(p)$ and $\bar{R}(q)$. Good references for Kleinian groups and hyperbolic 3-manifolds are [5] and [6] and notation largely follows these sources.

Definition 1.1. *Let Γ be a geometrically finite Kleinian group with non-empty domain of discontinuity $\Omega(\Gamma)$ such that $\mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold with incompressible boundary.*

Let A and B be two components of $\Omega(\Gamma)$, with $\bar{R} : \bar{A} \to \bar{U}$ the extended Reimann mapping. Then define

$$\mathrm{Touch}(A, B) = \partial A \cap \partial B = \Lambda(Stab(A)) \cap \Lambda(Stab(B))$$

and

$$\mathrm{Window}_A(A, B) = \bar{R}^{-1}(\mathcal{CH}(\bar{R}(\mathrm{Touch}(A, B)))$$

where $\mathcal{CH}$ denotes the convex hull in $\bar{U}$.

Theorem 1.2. *Let Γ be a geometrically finite Kleinian group with non-empty domain of discontinuity $\Omega(\Gamma)$, $S(\Gamma) = \Omega(\Gamma)/\Gamma$ and let $M(\Gamma) = \mathbf{H}^3/\Gamma \cup S(\Gamma)$ be the associated hyperbolic manifold with boundary . Assume that all of the components of $S(\Gamma)$ are incompressible and have no accidental parabolics. Let $\pi : \mathbf{H}^3 \cup \Omega(\Gamma) \to M(\Gamma)$ and A and B vary over the components of $\Omega(\Gamma)$. Then*

$$\bigcup_{A,B} \pi(\mathrm{Window}_A(A, B))$$

with any simple closed curves thickened, is a subsurface of $S(\Gamma)$. Furthermore, it, along with regular neighborhoods of the cusps of $S(\Gamma)$, is the boundary of a maximal product region in $M(\Gamma)$.

Remark 1.3. *The product region bounded by $\bigcup_{A,B} \pi(\mathrm{Window}_A(A, B))$ and neighborhoods of the cusps is called the* window *of a hyperbolic manifold with boundary, compare [9].*

Theorem 1.2 is proven in two parts. In section 2, $\bigcup_{A,B} \pi(\mathrm{Window}_A(A, B))$ is shown to be a disjoint union of connected subsurfaces and essential simple closed curves of $S(\Gamma)$. In section 3, it is shown to be the boundary of a maximal product region of $M(\Gamma)$.

2 The window pane is a subsurface.

We state the following elementary fact as a lemma since it will be used several times.

Lemma 2.1. *Let Γ be a Kleinian group with non-empty domain of discontinuity $\Omega(\Gamma)$. If A, B, C are three simply connected components of $\Omega(\Gamma)$, then no arc connecting two points of* $\mathrm{Touch}(A, B)$ *crosses an arc connecting two points of* $\mathrm{Touch}(A, C)$ *exactly once.*

Proof. Let l and m be the arcs in A which cross and have endpoints in $\mathrm{Touch}(A, B)$ and $\mathrm{Touch}(A, C)$ respectively. Then l and an arc l' in B connecting the endpoints of l form a circle on S^2, which must be separating. There is an arc m' in C connecting the endpoints of m which does not intersect the circle $l \cup l' \subset A \cup B$. However, m' connects one side of the circle $l \cup l'$ to the other, since the endpoints of l and m are linked in ∂A, which is a contradiction. (This is sometimes explained by saying that C cannot be stretched over the limit set.) □

We will also need the following lemma of Lehto regarding K-quasicircles. Under the assumptions of Theorem 1.2, any component A of $\Omega(\Gamma)$ is simply connected and has boundary a K-quasicircle. This K varies depending on the component, but it is the same for all components in a Γ-orbit.

Lemma 2.2. *[4] Suppose C is a K-quasicircle in the complex plane passing through ∞ and z_1, z_2, and z_3 are are three points of C such that z_2 lies between z_1 and z_3. Then there is a constant depending only on K, $\lambda(K)$, such that*

$$|z_1 - z_2| + |z_2 - z_3| \leq \lambda(K)|z_1 - z_3|.$$

Lemma 2.3. *Let Γ be as in Theorem 1.2. Then the window pane $\bigcup \pi(\mathrm{Window}_A(A, B))$ where A and B range over the components of $\Omega(\Gamma)$, is a disjoint union of embedded subsurfaces of $S(\Gamma)$ and essential simple closed curves.*

Proof. We will show that each $\pi(\text{Window}_A(A, B))$ is an embedded subsurface or an essential closed curve. The components $\text{Window}_A(A, B)$ as B varies do not intersect as this would violate either convexity or Lemma 2.1. Throughout, the term boundary curve includes the case when there is one geodesic arc in $\text{Window}_A(A, B)$.

Let $\gamma_p \in Stab(A)$ be parabolic with fixed point $p \in \partial A \cap \partial B$. We claim γ_p must also be in $Stab(B)$. Conjugate Γ so that p is ∞ and $\gamma_p = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. If $B \neq \gamma_p(B)$, then B must be contained in a fundamental domain for the action of γ_p on $\mathbb{C}$, say $\{z : 0 < Re(z) \leq 1\}$, and B is either completely above or completely below the curve ∂A. ∂B is a K-quasicircle going through ∞ and we are in the situation of Lemma 2.2. However, by taking z_2 to be a point that minimizes $|Im(z)|$ on ∂B and taking two points z_1 and z_3 of ∂B with $|Im(z_1)| = |Im(z_2)| >> n$, we see that Lemma 2.2 is violated. Therefore γ_p must be in $Stab(A) \cap Stab(B)$.

Consider a boundary curve of $\text{Window}_A(A, B)$. It does not limit on the fixed point of a parabolic element in $Stab(A)$. Indeed, let p and q be the endpoints of the boundary curve, with p the fixed point of a parabolic. Then q and p are both in $\text{Touch}(A, B)$ and $\gamma_p^{\pm n}$ moves q arbitrarily close to both sides of p on ∂A. Since γ_p is in $Stab(A) \cap Stab(B)$, there are points of $\text{Touch}(A, B)$ arbitrarily close to both sides of p so it cannot be the endpoint of a boundary curve of $\text{Window}_A(A, B)$. Thus the image of a boundary curve in $\pi(A)$ does not exit a cusp.

Next we show that the images of the boundary curves of $\text{Window}_A(A, B)$ are simple in $S(\Gamma)$. If not, then there is a boundary arc a of $\text{Window}_A(A, B)$ that is taken to an arc which crosses it exactly once by some element γ in $Stab(A)$. Then γ must also be in $Stab(B)$, or we would violate Lemma 2.1. Since $\gamma \in Stab(B)$, a and $\gamma(a)$, which cross, are both in $\text{Window}_A(A, B)$. Therefore a could not have been a boundary arc.

Now suppose that the image of a boundary arc a accumulates in $\pi(A)$. Then the images of a under $Stab(A)$ accumulate in A. There is a sequence of elements $\{\gamma_i\} \in Stab(A)$ such that the geodesic arcs $\gamma_i(a)$ accumulate. Thus, the endpoints of the $\gamma_i(a)$ must converge to two distinct points p and q on ∂A. Now consider the components $\gamma_i(B)$. These must be distinct components, since the boundary arcs of the single convex set $\text{Window}_A(A, B)$ do not accumulate. ∂B is a K-quasicircle, and the γ_i are conformal, so all of the $\gamma_i(\partial B)$ are also K-quasicircles. We claim that the accumulation will violate this fact. Indeed, let $z_{i,1}$ and $z_{i,3}$ be points on $\gamma_i(\partial B)$ accumulating to p and q respectively. Now consider the family of great circles on S^2_∞ going through some interior point of A. For some such great circle G (in fact infinitely many) we may pass to a subsequence such that $\gamma_i(\partial(B))$ meets l in two points $z_{i,2}$ and $z_{i,4}$. We label so that they are ordered $z_{i,2}, z_{i,4}, z_{i+1,2}, z_{i+1,4}, ...$ along G. Thus there is a sequence $\{z_{i,2}\}$ where each $z_{i,2}$ lies between $z_{i,1}$ and $z_{i,3}$ on $\gamma_i(B)$ and $\{z_{i,2}\}$ accumulates to a point r on G. The $z_{i,4}$ must also accumulate to r. By conjugating $z_{i,4}$ to infinity, we will eventually violate Lemma 2.2. Indeed, since $\{z_{i,4}\}$ converges to r, we may take r to be infinity and assume that the conjugating maps converge to the identity. Then the conjugated $\{z_{i,1}\}$ converges to p, the conjugated $\{z_{i,3}\}$ converges to q, and the distances $|z_{i,1} - z_{i,2}|$ and $|z_{i,2} - z_{i,3}|$ are increasing without bound.

We have shown that the images of the boundary arcs of $\text{Window}_A(A, B)$ in $S(\Gamma)$ are simple and that they do not accumulate or exit a cusp of $S(\Gamma)$. Therefore their images in $S(\Gamma)$ are essential simple closed curves. Since π is a covering map, the lemma follows. □

The following is essentially contained in the above proof, but we state it separately since it will be used below.

Lemma 2.4. *Let Γ be as in Lemma 2.3. If A and B are components of $\Omega(\Gamma)$, $\gamma \in Stab(A)$ and the fixed points of γ are in* $\text{Touch}(A, B)$*, then $\gamma \in Stab(B)$.*

Proof. The case when γ is parabolic is proved directly in the beginning of the proof of Lemma 2.3 above. Assume γ is hyperbolic with fixed points p and q in $\text{Touch}(A, B)$. Then if $\gamma(B) \neq B$, the sequence of components $\{\gamma^n(B)\}$ will eventually violate Lemma 2.2 as in the proof that the boundary curves do not accumulate above. □

Corollary 2.5. *Let Γ be as above and A and B two components of $\Omega(\Gamma)$. Then* $\mathrm{Touch}(A, B)$ *contains either 0, 1, 2 or infinitely many points.*

3 Maximal product regions

Definition 3.1. *A map f of an annulus or Möbius band $(F, \partial F)$ into $(M, \partial M)$ is called* essential *if i) $f_* : \pi_1(F) \to \pi_1(M)$ is monic and ii) if α is a proper arc of F such that $F \setminus \alpha$ is simply connected, then $f(\alpha)$ is not isotopic into the boundary ∂M.*

An arc as in condition (ii) is called a *spanning arc*. The main result of [3] is that if there is an essential map f of an annulus or Möbius band F such that $f(\partial F)$ is a homeomorphism, then there is an essential embedding g such that $f(\partial F) = g(\partial F)$.

Definition 3.2. *We say that a map P from $W \times I$ into a manifold M with boundary ∂M is a* product region map *if $P(W \times \{0\})$ and $P(W \times \{1\})$ are embeddings into ∂M, $P(W \times \{t\})$ is an embedding into $int(M)$ for all $t \in (0, 1)$, and the restriction of P to each component of $\{\partial W\} \times I$ is essential.*

Note that the above definition implies that no $P(\{x\} \times I)$ is isotopic rel boundary into ∂M, a common requirement.

Lemma 3.3. *Suppose A and B are components of $\Omega(\Gamma)$ such that* $\mathrm{Touch}(A, B)$ *contains more than one point. Then there is a product region map P of $W \times I$ into $M = H^3/\Gamma$ such that $W \simeq \pi(\mathrm{Window}_A(A, B)) \simeq \pi(\mathrm{Window}_B(A, B))$.*

Proof. First we show that a boundary curve $\bar{a}$ of $\pi(\mathrm{Window}_A(A, B))$ is isotopic to a boundary curve of $\pi(\mathrm{Window}_B(A, B))$. By Lemma 2.3, $\bar{a}$ is a simple closed curve, and hence a preimage a of $\bar{a}$ in the boundary of $\mathrm{Window}_A(A, B)$ is left invariant by some γ in Γ. By Lemma 2.4 this γ must also leave the boundary curve a' of $\mathrm{Window}_B(A, B)$ with the same endpoints as a invariant. There is a copy of $\mathbb{R} \times I$ in $\mathbf{H}^3 \cup \Omega(\Gamma)$ such that $\mathbb{R} \times \{0\} = a$ and $\mathbb{R} \times \{1\} = a'$ and $\mathbb{R} \times \{t\}$ is the unique geodesic in $\mathbf{H}^3$ left invariant by γ, for some $t \in [0, 1]$. This $\mathbb{R} \times I$ maps down to an annulus in $M(\Gamma)$ which is essential since any spanning arc will lift to an arc with endpoints in different components of $\Omega(\Gamma)$. By Theorem 1 of [3], there is an embedded essential annulus with the same boundary. Therefore the image of a boundary curve of $\mathrm{Window}_A(A, B)$ is isotopic to the image of a boundary curve of $\mathrm{Window}_B(A, B)$ in $M(\Gamma)$.

Now assume $\pi(\mathrm{Window}_A(A, B))$ is not a pair of pants or a simple closed curve. Then there is a non-trivial simple closed geodesic $\bar{l}$ of $\pi(\mathrm{Window}_A(A, B))$ which is not isotopic to any of the boundary components. $\bar{l}$ is the image of a geodesic arc l in the interior of $\mathrm{Window}_A(A, B)$, and there is a $\delta \in \Gamma$ which stabilizes A and fixes the endpoints of l, which are in $\mathrm{Touch}(A, B)$. There is a geodesic arc l' in B with the same endpoints as l stabilized by δ. We claim that the image $\pi(\bar{l}')$ is also embedded. If not, then there is an element β of the stabilizer of B such that $\beta(l')$ intersects β. If $\beta \notin Stab(B)$, then we violate lemma 2.1. If $\beta \in Stab(B)$, then $\pi(\bar{l})$ would not be a simple closed curve. Therefore, $\pi(\bar{l}')$ is a simple closed curve in $\pi(B)$ and it is isotopic to $\pi(\bar{l})$ in $\pi(A)$ by the argument above for the boundary curves.

By this process we may cut up the surfaces $\pi(\mathrm{Window}_A(A, B))$ and $\pi(\mathrm{Window}_B(A, B))$ into pairs of pants such that the boundaries of the pants are isotopic to each other in pairs. We claim that these pairs of pants themselves are isotopic through $M(\Gamma)$. Consider some pair of pants P in $\pi(\mathrm{Window}_A(A, B))$. Let x y and z be the boundary curves. These are isotopic to boundary curves x' y' and z', respectively, of a pair of pants P' in $\pi(\mathrm{Window}_B(A, B))$. Denote the embedded annuli that realize these isotopies by X Y and Z, respectively. There are shortest arcs, $\bar{d}$, $\bar{e}$, $\bar{f}$, connecting the boundary components of P on P and these arcs correspond to arcs $\bar{d}'$ $\bar{e}'$ and $\bar{f}'$ on a pair of pants in $\pi(\mathrm{Window}_B(A, B))$. If d connects the boundary components x and y, then the endpoints of $\{\bar{d}, \bar{d}'\}$ are connected by arcs two arcs D and $'$ on X and Y. Then $\bar{d} \cup D \cup \bar{d}' \cup D'$ forms a loop which is covered by a loop in $\mathbf{H}^3$ which bounds a disk in $\mathbf{H}^3$. Thus, by Dehn's lemma $\bar{d} \cup D \cup \bar{d}' \cup D'$ bounds an embedded disk, and the arcs $\bar{d}$ and $\bar{d}'$ are isotopic though $M(\Gamma)$. The same holds for the pairs $\{\bar{e}, \bar{e}'\}$ and $\{\bar{f}, \bar{f}'\}$. Consider a sphere in $M(\Gamma)$ consisting of one component of P cut along $\bar{d}, \bar{e}\bar{f}$, the 3 disks realizing the isotopies between $\{\bar{d}, \bar{d}'\}$, $\{\bar{e}, \bar{e}'\}$ and $\{\bar{f}, \bar{f}'\}$, one component of P' cut along $\bar{d}', \bar{e}', \bar{f}'$ and and half of each of the annuli X, Y and Z. This sphere must bound a ball in $M(\Gamma)$ by irreducibility, so the halves of pairs of pants are isotopic. By the exact same argument, the other halves of P and P' are isotopic by an isotopy that only intersects the previous one in disks so P and P' are isotopic. This shows that there is a map ϕ of $W \times I$ into $M(\Gamma)$ where $W \simeq \pi(\mathrm{Window}_A(A, B)) \simeq \pi(\mathrm{Window}_B(A, B))$ such that $\phi(W \times \{0\}) \subset \pi(A)$ and $\phi(W \times \{1\}) \subset \pi(B)$ where each $\phi(W \times \{t\})$ is an embedding. The restriction of ϕ to $\partial W \times I$ is essential by the construction at the beginning of the proof.

$\square$

Definition 3.4. *The* window *of a hyperbolic manifold with boundary is the union of the product regions constructed in Lemma 3.3 above, adding regular neighborhoods of any 2-manifold components of these product regions, and adding regular neighborhoods of any rank 1 cusps.*

The window is maximal in the following sense.

Lemma 3.5. *Let M be as in Theorem 1.2. If $f : (F, \partial F) \to (M, \partial M)$ is an essential map of an annulus or Möbius band into $(M, \partial M)$, then $f(F, \partial F)$ is isotopic rel boundary into the window.*

Proof. If any component of $f(\partial F)$ is peripheral, then $f(F)$ is an annulus and both components of $f(\partial F)$ are peripheral, since there are no accidental parabolics. In this case we are done by the product structure of a cusp. Otherwise the components of $f(\partial F)$ are essential non-peripheral curves. Let $\hat{f}(S^1 \times I)$ be a map of an annulus into $(M, \partial M)$ with image $f(F)$. Consider the lift $\tilde{f}(\mathbb{R} \times I)$ to the universal cover $\mathbf{H}^3 \cap \Omega(\Gamma)$. Then $\tilde{f}(\mathbb{R} \times \{0\})$ and $\tilde{f}(\mathbb{R} \times \{1\})$ are in different components A and B, respectively of $\Omega(\Gamma)$ since $f(F)$ is essential. Furthermore, each $\tilde{f}(\mathbb{R} \times \{t\})$ limits on the same two points of $\Lambda(\Gamma)$ since the $\hat{f}(S^1 \times \{t\})$ are all freely homotopic. Then $\tilde{f}(\mathbb{R} \times \{0\})$ and $\tilde{f}(\mathbb{R} \times \{1\})$ respectively are isotopic in $\Omega(\Gamma)$ to geodesics with these same endpoints, and these geodesics are in $\mathrm{Window}_A(A, B)$ and $\mathrm{Window}_B(A, B)$, respectively. The $\hat{f}(S^1 \times \{0\})$ and $\hat{f}(S^1 \times \{1\})$ are isotopic to the images of these geodesics on ∂M. The rest of the annulus is isotopic into the window by the irreducibility of M. $\square$

Theorem 1.2 follows immediately from Lemmas 2.3, 3.3 and 3.5.

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