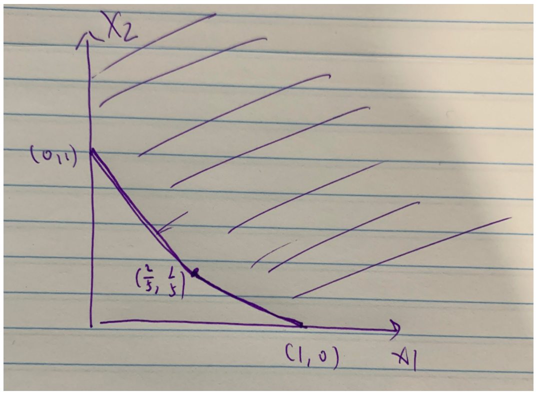
CSCI 5254 Homework 3

Tuguluke Abulitibu

October 8, 2020

Chapter 4, Basic terminology and optimality conditions

4.1



Functions	Optimal set	Optimal value
$f_0(x_1, x_2) = x_1 + x_2$	$X_{opt} = \{(2/5, 1/5)\}$	$p^* = 3/5$
$f_0(x_1, x_2) = -x_1 - x_2$	DNE	$p^* = -\infty$
$f_0(x_1, x_2) = x_1$	$X_{opt} = \{(0, x_2) \mid x_2 \geq 1\}$	$p^* = 0$
$f_0(x_1, x_2) = max\{x_1, x_2\}$	$X_{opt} = \{(1/3, 1/3)\}$	$p^* = 1/3$
$f_0(x_1, x_2) = x_1^2 + 9x_2^2$	$X_{opt} = \{(3/6, 1/6)\}$	$p^* = 1/2$

(a) $f_0(x_1, x_2) = x_1 + x_2$

By solving the system of equation, we get one of the vertexes:

$$\begin{cases} 2x_1 + x_2 = 1 \\ x_1 + 3x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 2/5 \\ x_2 = 1/5 \end{cases}$$

So $x^* = (2/5, 1/5)$, $p^* = 2/5 + 1/5 = 3/5$.

(b) $f_0(x_1, x_2) = -x_1 - x_2$

Flip part (a), we got unbound below.

(c) $f_0(x_1, x_2) = x_1$

This one is along the x_2 axis of the feasible set.

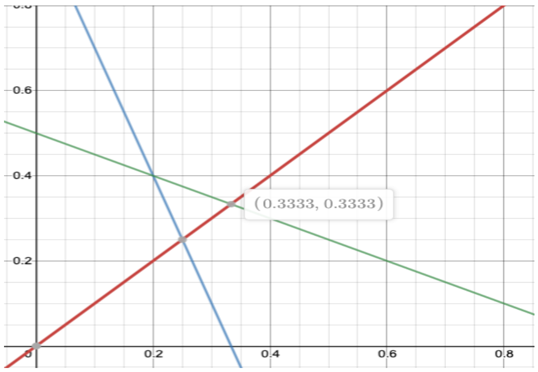


Figure 1: (d) $f_0(x_1, x_2) = max\{x_1, x_2\}$

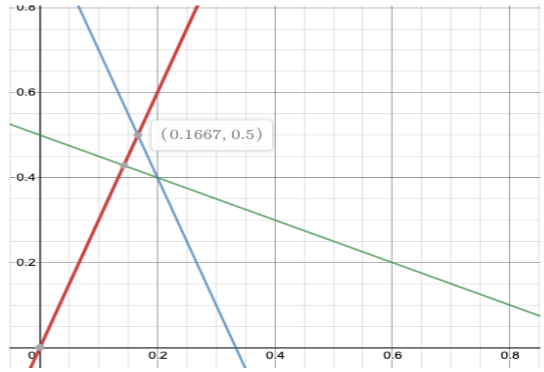


Figure 2: (e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$

(d) $f_0(x_1, x_2) = max\{x_1, x_2\}$

(d) and (e) are solved by drawing isolines.

$$\begin{cases} x_1 = x_2 \\ x_1 + 3x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 1/4 \\ x_2 = 1/4 \end{cases}$$

$$\begin{cases} 2x_1 + x_2 = 1 \\ x_1 = x_2 \end{cases} \Rightarrow \begin{cases} x_1 = 1/3 \\ x_2 = 1/3 \end{cases}$$

But $(1/4, 1/4)$ does not satisfies the condition $2x_1 + x_2 \geq 1$, so we pick $x^* = (1/3, 1/3)$, hence $p^* = 1/3$.

(e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$

$$\begin{cases} x_1^2 = 9x_2^2 \\ x_1 + 3x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 3/6 \\ x_2 = 1/6 \end{cases}$$

$$\begin{cases} 2x_1 + x_2 = 1 \\ x_1^2 = 9x_2^2 \end{cases} \Rightarrow \begin{cases} x_1 = 3/7 \\ x_2 = 1/7 \end{cases}$$

But $(3/7, 1/7)$ does not satisfies the condition $x_1 + 3x_2 \geq 1$, so we pick $x^* = (3/6, 1/6)$, hence $p^* = 3/6 + 9 * (1/6)^2 = 1/2$.

4.3

Proof. From textbook example we know $\nabla f_0(x) = Px + q$, and:

$$\begin{bmatrix} 13 & 12 & -2 \\ 23 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Hence, optimality condition is

$$\nabla f_0(x^*)^T(y - x) = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \\ y_3 - x_3 \end{bmatrix} = -1(y_1 - 1) + 0(y_2 - \frac{1}{2}) + 2(y_3 + 1) \geq 0$$

which can only be true when

$$\begin{cases} y_1 - 1 \leq 0 \\ y_3 + 1 \geq 0 \end{cases} \Rightarrow \forall -1 \leq y_i \leq 1$$

we showed that $x^* = (1, 1/2, -1)$ is optimal. $\square$

4.7

(a)

- Domain of the objective function $\{x \in \mathbf{dom} f_0 \mid c^T x + d > 0\}$ is convex since f_0 is convex.
- Sublevel set $S_\alpha = \{x \in \mathbf{dom} f_0 \mid f_0(x)/(c^T x + d) \leq \alpha\}$ is convex since $c^T x + d > 0$ for $f_0(x) \leq \alpha(c^T x + d)$.

hence this is a quasiconvex optimization.

(b)

From hint that g_i is perspective of f_i, we have $g_i(y,t) = tf_i(y/t)$, we can transform the problem into

$$\begin{array}{ll} \text{minimize} & tg_0(\frac{y}{t}) \\ \text{subject to} & tg_i(\frac{y}{t}) \leq 0 \\ & ay = bt \\ & c^T y + dt = 1 \end{array} \tag{1}$$

Let

$$\begin{cases} t = \dfrac{1}{c^T x + d} \\ y = xt = \dfrac{x}{c^T x + d} \end{cases}$$

Hence with algebra:

$$\begin{array}{ll} \text{minimize} & f_0(x)/(c^T x + d) \\ \text{subject to} & f_i(x) \leq 0, \; i = 1, \ldots, m \\ & Ax = b \end{array} \tag{2}$$

is equivalent to our problem (1), since

- $Ax = b \Longleftrightarrow Ay = bt$
- $f_0(x)/(c^T x + d) = \dfrac{f_0(y/t)}{1/t} \Longleftrightarrow tg_0(\frac{y}{t})$
- $f_i(x) \leq 0 \Longleftrightarrow tg_i(\frac{y}{t}) \leq$ for $t > 0$
- $C^T y + dt = C^T \dfrac{x}{c^T x + d)} + \dfrac{d}{c^T x + d)} = 1$

Chapter 4, Linear optimization problems

4.8

(a)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \end{array}$$

1. If it is infeasible, then $p^* = +\infty$
2. If it is feasible, by finding the spatial solution. Let $x = \tilde{x} + A^T z = \tilde{x} + y$, then

$$\begin{array}{ll} \text{minimize} & c^T x = c^T \tilde{x} + c^T y \\ \text{subject to} & y \in Null(A) \end{array}$$

   if $c \perp Null(A) \Rightarrow c^T y = 0$, then we have $c^T x = c^T \tilde{x} + c^T y = c^T \tilde{x}$, where $A\tilde{x} = b$, and $c = A^T \lambda$, then we have

$$p^* = c^T \tilde{x} = (A^T \lambda)^T (A^{-1} b) = \lambda^T b$$

3. If it is feasible and if $c \not\perp Null(A)$, then it is unbounded from both direction, hence $p^* = -\infty$

(c)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & l \preceq x \preceq u \end{array}$$

It is a box constrain:

$$\begin{array}{ll} \text{minimize} & \sum_i c_i x_i \\ \text{subject to} & l_i \leq x_i \leq u_i \, \forall i \end{array}$$

therefore

$$\begin{cases} \text{if } c_i > 0 & \text{lowerbound , hence } x^* = l_i, p^* = c * l_i \\ \text{if } c_i = 0 & \text{between lowerbound and upperbound, hence } x^* \in [l_i, u_i], p^* = c * x^* \\ \text{if } c_i < 0 & \text{upperbound , hence } x^* = u_i, p^* = c * u_i \end{cases}$$

4.11

(b)

$$\text{minimize} \quad \|Ax - b\|_1$$

The term can be rewritten as: $\|Ax - b\|_1 = \sum_{i=1}^n |Ax_i - b_i|$, then setting $|Ax_i - b_i| \leq t_i$, we have

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & |Ax_i - b_i| \leq t_i \end{array}$$

that is:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & Ax_i - b_i \geq -t_i \, \forall i \\ & Ax_i - b_i \leq t_i \, \forall i \end{array}$$

Written in LP:

$$\boxed{\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & Ax - b \succeq -t \\ & Ax - b \preceq t \end{array}}$$

Since $-t_i \leq a_i^T x - b_i \leq t_i \, \forall i \Leftrightarrow \|a_i^T x - b_i\| \leq t_i \, \forall i$, it is easy to see the we can get the optimal solution at $\|a_i^T x - b_i\| = t_i$, hence we say the optimal solution of the norm and it's LP problem are the same.

(c)

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_1 \\ \text{subject to} & \|x\|_\infty \leq 1 \end{array}$$

From part (b), we edit it a little, we can get

$$\boxed{\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & Ax - b \succeq -t \\ & Ax - b \preceq t \\ & -\mathbf{1} \leq x \leq \mathbf{1} \end{array}}$$

The same explanation as part (b) that the optimal solution of the norm and it's LP problem are the same, with one more constrain $\|x\| \leq 1$

4.12

From this online lecture link: Slide 7.
Also problem stated that at each node we have conservation of flow: **the total flow into node i along links and the external supply, minus the total flow out along the links, equals zero.** Hence [1]

$$\boxed{\begin{array}{ll} \text{minimize} & C = \sum_{i,j=1}^{n} c_{ij}x_{ij} \\ \text{subject to} & b_i + \sum_{\{l \mid (l,i)\in E\}} x_{li} - \sum_{\{j \mid (i,j)\in E\}} x_{ij} = 0 \\ & l_{ij} \le x_{ij} \le u_{ij} \end{array}}$$

4.15

(a)

Since $\{x_i \mid x_i \in \{0,1\},\, i = 1,2,\dots,n\} \subseteq \{x_i \mid 0 \le x_i \le 1,\, i = 1,2,\dots,n\}$, it is obvious that the optimal value of the LP relaxation is a lower bound on the optimal value of the Boolean LP.
If the LP relaxation is infeasible, **then Boolean LP itself is infeasible**.

(b)

If LP relaxation has the same solution with Boolean LP, **then the optimal value of the LP relaxation is the optimal value of the Boolean LP**.

Chapter 4, Quadratic optimization problems

4.23

Adding auxiliary variable z_i for $i = 1,3,\dots,m$. Then with a bit of algebra, we got QCQP:

$$\boxed{\begin{array}{ll} \text{minimize} & \sum_i^m z_i^2 \\ \text{subject to} & y_i^2 \le z_i,\ i = 1,3,\dots,m \\ & y_i = a_i^T x - b_i,\ i = 1,3,\dots,m \end{array}}$$

Chapter 4, Semidefinite programming and conic form problems

4.40

(c)

This one is similar to Matrix norm minimization example on page 169, and Example 3.5 on page 76[2], that

$$\mathbf{epi} f = \{(x,Y,t) \mid Y \succ 0,\ x^T Y^{-1} x \le t\} = \{(x,Y,t) \mid \begin{bmatrix} Y & x \\ x^T & t \end{bmatrix} \succeq 0, Y \succ 0\}$$

Since we assume there exists at least one x with $F(x) \succ 0$ therefore, the SPD form is

$$\boxed{\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & t \end{bmatrix} \succeq 0 \end{array}}$$

In the variable x and t, where $x, t \in \mathbf{R}$.

[1] The set E is the set of directed links (i,j)

[2] Schur complement condition for positive semi-definiteness of a block matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \ge \mathbf{0}$, $C \ge \mathbf{0} \Rightarrow A - BC^{-1}B^T \ge \mathbf{0}$

4.43

(a)

$$\min \lambda_1(A(x))$$

From Linear Algebra, we know that : λ is an eigenvalue of $A(x)$ if and only if $\lambda - t$ is an eigenvalue of $A(x) - t\mathbf{I}$, hence[3]

$$\lambda_1(A(x)) \leq t \iff A(x) - t\mathbf{I} \leq 0$$

Written as SDP:

$$\boxed{\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq t\mathbf{I} \end{array}}$$

(b)

$$\min \lambda_1 - \lambda_m(A(x))$$

On top of $\min \lambda_1(A(x))$ from part (a), we need $\max \lambda_t(A(x))$, which add one more condition

$$\lambda_m(A(x)) \geq t \iff A(x) - t\mathbf{I} \geq 0$$

Hence

$$\boxed{\begin{array}{ll} \text{minimize} & t_1 - t_m \\ \text{subject to} & A(x) \preceq t_1\mathbf{I} \\ & A(x) \succeq t_m\mathbf{I} \end{array}}$$

[3]Eigenvalue decomposition for SPD matrix: $A \leq tI \iff Z^T AZ \leq Z^T tIZ \iff \sum_i \lambda_i a_i^2 \leq t \sum_i a_i^2 \iff \lambda_{max} \leq t$