CSCI 5254 Homework 1

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Chapter 2, Definition of convexity

2.11

2-D

Hyperbolic set S = $\{x \in \mathbf{R}^2_+ \mid x_1x_2 \geq 1\}$

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be in set S,
for $0 \leq \theta \leq 1$,

$$(\theta x + (1-\theta)y) = (\theta x_1 + (1-\theta)y_1)\,(\theta x_2 + (1-\theta)y_2.)$$

$$= \theta^2 x_1x_2 + \theta x_1(1-\theta)y_2 + (1-\theta)\theta y_1x_2 + (1-\theta)^2 y_1y_2$$

$$\geq \theta^2 + 2\theta(1-\theta)^2 \,(\text{by definition of S})$$

$$= \cancel{\theta^2} + \cancel{2\theta} - \cancel{2\theta^2} + 1 - \cancel{2\theta} + \cancel{\theta^2} = 1$$

We showed that $\forall x = (x_1, x_2)$ and $y = (y_1, y_2) \in S$, and any θ with $0 \leq \theta \leq 1$
$(\theta x + (1-\theta)y)$ was also in set S, hence convex. □

n-D

Hyperbolic set S = $\{x \in \mathbf{R}^n \mid \Pi_{i=1}^n x_i \geq 1\}$

Proof. Let $(x_1, x_2, \ldots, x_n)$ and$(y_1, y_2, \ldots, y_n)$ be in set S,
for $0 \leq \theta \leq 1$,

$$\Pi_{i=1}^n[\theta x_i + (1-\theta)y_i]$$

$$\geq \Pi_{i=1}^n[x_i^\theta y_i^{(1-\theta)}] \,(\text{by hint } a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b)$$

$$= \Pi_{i=1}^n[x_i^\theta]\Pi_{i=1}^n[y_i^{(1-\theta)}]$$

$$\geq 1 = 1 \,(\text{by definition of S})$$

We showed that $\forall x_i$ and $y_i \in S$, and any θ with $0 \leq \theta \leq 1$
$\theta x_i + (1-\theta)y_i$ was also in set S, hence convex. □

2.12(c, e, f, g)

c

Wedge set S $= \{x \in \mathbf{R}^n \mid a_1^T x \le b_1, a_2^T x \le b_2\}$

Proof. Let x_1, x_2 be in set S,
for $0 \le \theta \le 1$,

$$a_1^T[\theta x_1 + (1-\theta)x_2] = a_1^T[\theta x_1] + a_1^T[(1-\theta)x_2]$$

$$= \theta a_1^T x_1 + a_1^T - \theta a_1^T x_2$$

$$\le \cancel{\theta b_1} + b_1 - \cancel{\theta b_1} \text{ (by definition of S)}$$

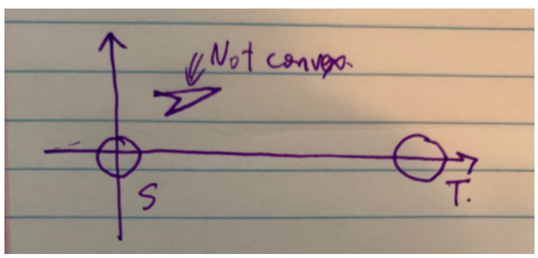
the same can be apply to

$$a_1^T[\theta x_1 + (1-\theta)x_2] \le b_2.$$

We showed that $\forall x_1$ and $x_2 \in S$, and any θ with $0 \le \theta \le 1$
$\theta x_1 + (1-\theta)x_2$ was also in set S, hence convex. □

e

This not convex. Counter example:



f

Proof. Let $y \in S_2$, since S_1 is convex, then $S_1 \setminus y$ is convex $\forall y \in S_2$

$$\{x \mid x + S_2 \subseteq S_1\} = \cap_{y \in S_2}\{S_1 \setminus y\}$$

Which is convex, since convexity is preserved by intersection. □

g

Proof. Aproach 1:
Let x_1 and x_2 be in set $S = \{x \mid \|x-a\|_2 \le \theta \|x-b\|_2\}$,
for $0 \le \alpha \le 1$,

$$\|\alpha x_1 + (1-\alpha)x_2 - a\|_2 = \|\alpha x_1 + x_2 - \alpha x_2 - a - \alpha a + \alpha a\|_2$$

$$= \|\alpha x_1 - \alpha a + x_2 - a - \alpha x_2 + \alpha a\|_2 \le \|\alpha x_1 - \alpha a\|_2 + \|x_2 - a\|_2 - \|\alpha x_2 - \alpha a\|_2$$

$$\le \alpha\theta \|x_1 - b\|_2 + \|x_2 - b\|_2 - \alpha\theta \|x_2 - b\|_2 \le {}^{1}\theta[\alpha \|x_1 - b\|_2 + (1-\alpha)\|x_1 - b\|_2]$$

[1]With some trig inequality plus perturbation, we should be getting this step, but I failed to do that, so we will have to use Office Hour's suggestion, which is Approach 2

We showed that $\forall x_1$ and $x_2 \in S$, and any α with $0 \leq \alpha \leq 1$
$\theta x_1 + (1-\alpha)x_2$ was also in set S, hence convex.

Approach 2: Square both side of set $\{x \mid \|x-a\|_2 \leq \theta \|x-b\|_2\}$, we have $\{x \mid \|x-a\|_2^2 \leq \theta \|x-b\|_2^2\}$
If $\theta = 1$, then it is a halfspace, hence convex.
If $0 \leq \theta < 1$, then it is equivalent to[2],

$$\{x \mid (1-\theta^2)x^T x - 2(a-\theta^2 b)^T x + \theta^T \theta - \theta^2 b^T b \leq 0\}$$

by the look it, it actually is a ball, hence convex.

□

2.14

a

Proof. Let x_1, x_2 be in set S_a,
for $0 \leq \theta \leq 1$,

$$\inf_{y \in S} \|\theta x_1 + (1-\theta)x_2 - y\|$$

$$= \inf_{y \in S} \|\theta x_1 + x_2 - \theta x_2 - y\| = \inf_{y \in S} \|\theta x_1 + x_2 - \theta x_2 - y + \theta y - \theta y\| = \inf_{y \in S} \|\theta x_1 + \theta y + x_2 - y - \theta x_2 - \theta y\|$$

$$\leq \inf_{y \in S} \|\theta x_1 + \theta y\| + \inf_{y \in S} \|x_2 - y\| - \inf_{y \in S} \|\theta x_2 + \theta y\| \text{ (by Triangle inequality)}$$

$$\leq \cancel{\theta a} + a - \cancel{\theta a} = a \,\text{(by definition of } S_a\text{)}$$

We showed that $\forall x_1$ and $x_2 \in S_a$, and any θ with $0 \leq \theta \leq 1$
$\theta x_1 + (1-\theta)x_2$ was also in set S_a, hence convex. □

b

Proof. Let x_1, x_2 be in set S_{-a}, and $y \in S$
for $0 \leq \theta \leq 1$, and y $y \in S$

$$\|y - (\theta x_1 + (1-\theta)x_2)\| = \|y - \theta x_1 - x_2 + \theta x_2 + \theta y - \theta y\|$$

$$\leq \|y - x_2\| + \theta \|y - x_1\| - \theta \|y - x_1\| \text{ (by Triangle inequality)}$$

$$\leq a + \cancel{\theta a} - \cancel{\theta a} = a \,\text{(by definition of } S_{-a}\text{)}$$

We showed that $\forall x_1$ and $x_2 \in S_{-a}$, and any θ with $0 \leq \theta \leq 1$
$\theta x_1 + (1-\theta)x_2$ was also in set S_{-a}, hence convex. □

[2] Textbook page 97

2.15 (a, b, f, g)

We use $S = \cap\{\mathcal{H} \mid \mathcal{H} \text{ halfspace } , S \subseteq \mathcal{H}\}$[3]

a

Proof. Since

- p_i is halfspace (hence convex)
- $f(a_i) : \mathbf{R} \to \mathbf{R}$ (hence constant real).
- $\alpha \leq \sum_{i=1}^n p_i f(a_i) \leq \beta$, which is closed and bounded(hence converge).

Thus, linear combinations of halfspace, we showed that it is convex. □

b

Proof. Similar to part (a) with $p_i \leq \beta$.
Thus, a halfspace with convergnece, we showed that it is convex. □

f

$$\mathbf{var}(x) = \mathbf{E}(x - \mathbf{E}x)^2 = \mathbf{E}x^2 - (\mathbf{E}x)^2 = \sum_{i=1}^n p_i x^2 - (\sum_{i=1}^n p_i x)^2 = \sum_{i=1}^n p_i a_i^2 - (\sum_{i=1}^n p_i a_i)^2$$

here we consider p_i as coefficient, we have $-(\sum_{i=1}^n p_i a_i)^2 + \sum_{i=1}^n p_i a_i^2 \leq \alpha$, that is

$$\frac{(\sum_{i=1}^n p_i a_i)^2}{\alpha} - \frac{\sum_{i=1}^n p_i a_i^2}{\alpha} \geq 1$$

this is the complement of an ellipsoid function (which is convex), hence not convex.
Counter example: $a = (-1, 1), p = (0, 1)$ and $(1, 0)$ vs mid point $p = (1/2, 1/2)$.

g

Proof. Similar to part (f), we have

$$\frac{(\sum_{i=1}^n p_i a_i)^2}{\alpha} - \frac{\sum_{i=1}^n p_i a_i^2}{\alpha} \leq 1$$

this can be tranformed into standard Ellipsoid format

$$\frac{(\sum_{i=1}^n [p_i - X\hat{(a_i)}])^2}{\hat{\beta}} \leq 1$$

it is in a closed ellipsoid, hence convex. □

[3] Text book page 36

Chapter 2, Operations that preserve convexity

2.19 (a, b)

a

half space set C = $\{y \mid g^T y \leq h\}(g \neq 0)$
Since $f^{-1}(C) = \{x \in \mathbf{dom} f \mid f(x) \in C\}$, and $f(x) = \frac{Ax+b}{c^T x+d}(c^T x + d > 0)$, plug them back into C, we have

$$f^{-1}(C) = \{x \in \mathbf{dom} f \mid g^T f(x) \leq h\}$$

$$\Rightarrow f^{-1}(C) = \{x \in \mathbf{dom} f \mid g^T f(x) \leq h\} \Rightarrow f^{-1}(C) = \{x \in \mathbf{dom} f \mid g^T \frac{Ax+b}{c^T x + d} \leq h\}(\text{where } c^T x + d > 0)$$

$$\Rightarrow f^{-1}(C) = \{x \in \mathbf{dom} f \mid g^T(Ax+b) \leq h(c^T x + d)\} = \{x \in \mathbf{dom} f \mid g^T Ax - hc^T x \leq hd - g^T b\}$$

$$\Rightarrow f^{-1}(C) = \{x \in \mathbf{dom} f \mid (A^T g - ch^T)^T x \leq (hd - g^T b)\}$$

Which shows that $f^{-1}(C)$ is another **halfspace**.

b

Similar to part (a), we plug known conditions into definition of polyhedron,

$$f^{-1}(C) = \{x \in \mathbf{dom} f \mid f(x) \in C\} \Rightarrow f^{-1}(C) = \{x \in \mathbf{dom} f \mid G\frac{Ax+b}{c^T x + d} \preceq h\}(\text{where } c^T x + d > 0)$$

$$\Rightarrow f^{-1}(C) = \{x \in \mathbf{dom} f \mid GAx + Gb \preceq h(c^T x + d) \Rightarrow f^{-1}(C) = \{x \in \mathbf{dom} f \mid GAx - hc^T x \preceq hd - Gb)\}$$

$$\Rightarrow f^{-1}(C) = \{x \in \mathbf{dom} f \mid (GA - hc^T)x \preceq (hd - Gb)\}$$

Which shows that $f^{-1}(C)$ is another **polyhedron**.

Chapter 2, Convex cones and generalized inequalities

2.33 (a)

Proof. 1. Convex:
Let $(x_1, x_2, \ldots, x_n)$ and$(y_1, y_2, \ldots, y_n)$ be in set K_{m+},
for $0 \leq \theta \leq 1$,

$$\theta x_1 \geq \theta x_2 \geq \ldots \geq \theta x_n \geq 0$$

$$\theta(1-\theta)y_1 \geq (1-\theta)y_2 \geq \ldots \geq (1-\theta)y_n \geq 0.$$

thus, by combination of the above

$$\theta x_1 + \theta(1-\theta)y_1 \geq \theta x_2 + (1-\theta)y_2 \geq \ldots \geq \theta x_n(1-\theta)y_n \geq 0.$$

We showed that $\forall x_i$ and $y_i \in K_{m+}$, and for any $0 \leq \theta \leq 1$
$\theta x_i + (1-\theta)y_i$ was also in set K_{m+}, hence convex.
2. Closed:
Since $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$, it can only be $x_1 = x_2 = \ldots = x_n = 0$ when $\lambda = 0$, and it includes 0, hence closed
3. Solid:
Obviously there is only nonempty interior since when $\lambda > 0$, there is only > 0, hence solid.
4. Pointed (no line):
From 2 and 3 we know that λ is nonnegative, hence pointed.
Summing all 4 properties, we showed that K_{m+} is a proper cone. □