CMIM 06 ASSIGNMENT: WEEK 10

Tuhin Choudhury

March 14, 2018

1 INTRODUCTION

This week's assignment consists of problems related to the mechanism and movement of multi-body systems. The initial part of the assignment deals with the derivation of three dimensional rotation matrix based on Euler angles. This is followed by a task of specific rotational movement where both the Euler angles and Euler parameters are determined. The next task involves deriving all elements of total derivatives $\dot{C}$ and $\ddot{C}$ for a provided set of constraint equation.
The last task involves incorporating Newton-Raphson method for a given problem where initial values are provided.

2 METHOD

To determine the Euler angles, consider a body A shown in figure 1. In order to determine the position of any particle P in the body, the following equation can be used:

$$r_{A_p} = R_A + A_A \bar{u}_{A_p} \tag{1}$$

wherer_{A_p} gives the position of the particle in terms of global coordinate system, R_A defines the location of the body reference coordinate system origin with respect to the global frame of reference and A_A is the rotational matrix which defines the rotation of the body coordinate system in a 3×3 matrix form.

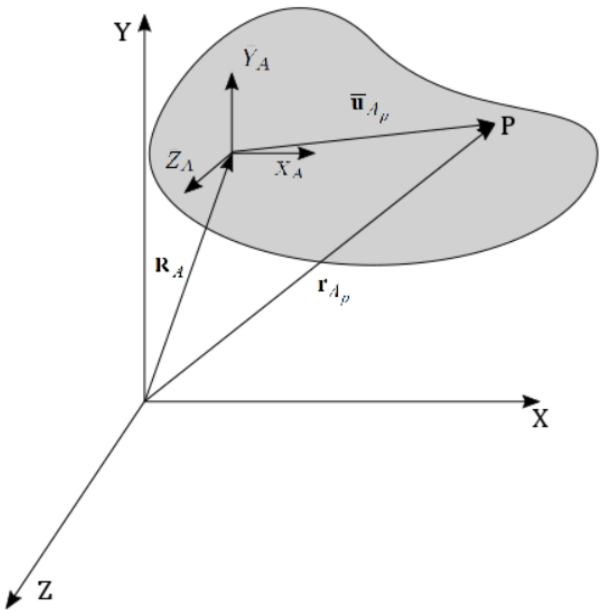

Figure 1: Spatial body A.

In three dimensional space, the rotation of the body frame of reference coordinates may occur about any of the three axes. To determine Euler angles, the rotations are required to be performed in a specific order.

By performing three successive rotations in the proper sequence a body reference coordinate system can reach any orientation. However, in this particular case, the sequence of rotation is stated as $Z - X - Z$. TThis is illustrated by Figure 2 which describes two coordinate systems XYZ and $\bar{X}\bar{Y}\bar{Z}$ which are initially coincident and then undergo subsequent rotation in the pattern of $Z - X - Z$.

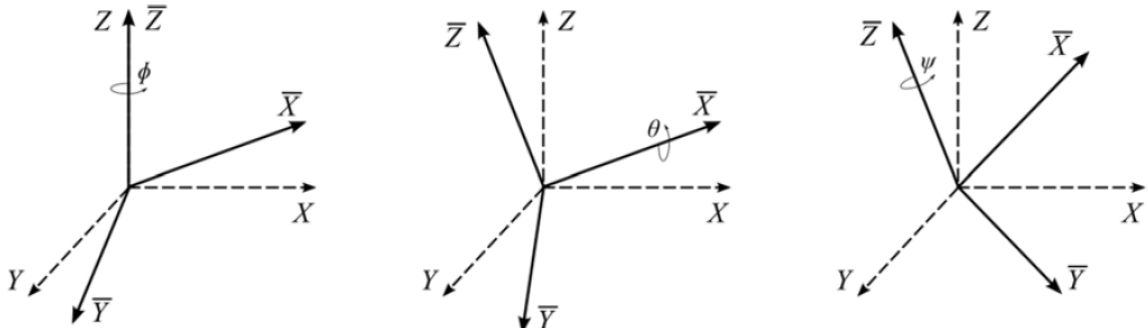

Figure 2: Consecutive rotations $Z - X - Z$.

From the initial configuration, coordinate system $\bar{X}\bar{Y}\bar{Z}$ is rotated about axis $\bar{Z}$ by an angle ϕ. Rotation matrix resulting from this rotation can be written as:

$$\mathbf{A_1} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

The coordinate system $\bar{X}\bar{Y}\bar{Z}$ is successively rotated about current $\bar{X}$ axis by an angle θ. Rotation matrix resulting from this second rotation can be written as:

$$\mathbf{A_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{3}$$

As a final rotation, $\bar{X}\bar{Y}\bar{Z}$ is successively rotated about current $\bar{Z}$, axis by an angle ψ . In this case, rotation matrix can be written as:

$$\mathbf{A_3} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

By combining successive rotations,a rotation matrix can be formed between the original coordinate system XYZ and the trice rotated final coordinate system $\bar{X}\bar{Y}\bar{Z}$ as follows:

$$\mathbf{A} = \mathbf{A_1}\mathbf{A_2}\mathbf{A_3} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

By conducting the matrix multiplications, matrix $\mathbf{A}$ can be expressed as :

$$\mathbf{A} = \begin{bmatrix} \cos(\phi)\cos(\psi) - \sin(\phi)\cos(\theta)\sin(\psi) & -\cos(\phi)\sin(\psi) - \sin(\phi)\cos(\theta)\cos(\psi) & \sin(\phi)\sin(\theta) \\ \sin(\phi)\cos(\psi) + \cos(\phi)\cos(\theta)\sin(\psi) & -\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\theta)\cos(\psi) & -\cos(\phi)\sin(\theta) \\ \sin(\theta)\sin(\psi) & \sin(\theta)\cos(\psi) & \cos(\theta) \end{bmatrix} \tag{6}$$

The three angles ϕ ,θ and ψ are called Euler's angles and $\mathbf{A}$ is a spatial rotational matrix based on the usage of Euler angles.Accordingly, in the method of Euler angles, a body orientation is defined by three rotational parameters that can be expressed in vector form as $\theta = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^T$

The Euler parameters are given by $\theta = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^{\mathbf{T}}$. Each individual Euler parameter can be simply obtained from the equations:

$$\theta_0 = \cos(\frac{\theta}{2}) \tag{7}$$

$$\theta_1 = v_1 \sin(\frac{\theta}{2}) \tag{8}$$

$$\theta_2 = v_2 \sin(\frac{\theta}{2}) \tag{9}$$

$$\theta_3 = v_3 \cos(\frac{\theta}{2}) \tag{10}$$

Here v_1,v_2 and v_3 are components of unit vector $\mathbf{v}$ about which the rotation takes place. Therefore vector v can be expressed as $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{\mathbf{T}}$

For the next part of the assignment, a system is provided which have the constraint equations (C) and generalized coordinates ($\mathbf{q}$) as

$$C = \begin{bmatrix} x^2 + y + \sqrt{z} + \sin\phi_1 \\ xy + xz + y\sin\phi_3 + t^3 \\ \sin\phi_2 + x^{\frac{3}{2}} + t \end{bmatrix} \tag{11}$$

$$\mathbf{q} = \begin{bmatrix} x & y & z & \phi_1 & \phi_2 & \phi_3 \end{bmatrix}^T \tag{12}$$

From the given parameters, the total derivatives $\dot{C}$ and $\ddot{C}$ are given as:

$$\dot{C} = C_q\dot{\mathbf{q}} + C_t \tag{13}$$

$$\ddot{C} = C_q\ddot{\mathbf{q}} + (C_q\dot{\mathbf{q}})_q\dot{\mathbf{q}} + 2C_{qt}\dot{\mathbf{q}} + C_{tt} \tag{14}$$

For the last task, a system is provided as shown in the figure (3)

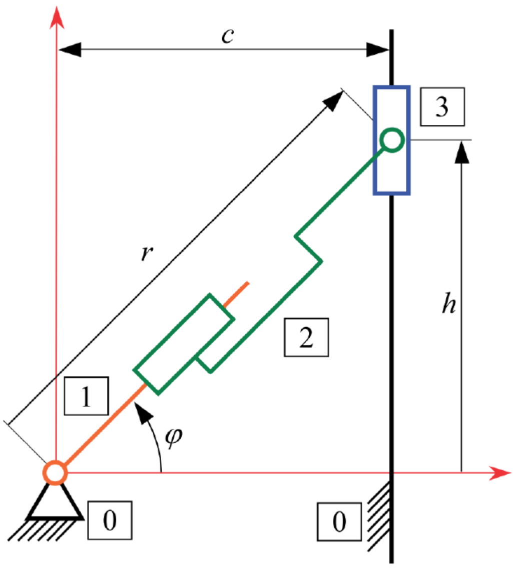

Figure 3: Multibody system.

In this system the two generalized coordinates are r and ϕ. From the figure, the two constraint equations for this are:

$$C_1 = c^2 + h^2 - r^2 = 0 \tag{15}$$

$$C2 = c\tan\phi - h = 0 \tag{16}$$

3 RESULTS AND DISCUSSIONS

In this section, the numerical calculations are carried out and the final results are obtained for respective tasks.

3.1 Task 1

The first task, the given Euler angles ($Z - X - Z$ sequence) are given as $\alpha = 45°$, $\beta = 45°$ and $\gamma = 45°$. Therefore the spatial rotation matrix can be derived from (6) as

$$\mathbf{A} = \begin{bmatrix} 0.1464 & -0.8536 & 0.5 \\ 0.8536 & -0.1464 & -0.5 \\ 0.5 & 0.5 & 0.7071 \end{bmatrix}$$

3.2 Task 2

For the second task, the given rotation is $\phi = \frac{\pi}{6}$ about axis y_0. In this case, Euler angles with $Z - X - Z$ sequence can be acquired by rotating frame 1 about the Z-axis of frame 0 by $\frac{\pi}{2}$. That way, the current X-axis of frame 1 will be coincident with the Y-axis of frame 0. After this, the rotation of X-axis of frame 1 by $\phi = \frac{\pi}{6}$ can be carried out about y_0. Finally, the Z-axis is rotated counter wise by $\frac{\pi}{2}$ in order to compensate effect the rotation . Therefore, the final Euler angles are,

$$\theta = \begin{bmatrix} \pi/2 & \pi/6 & -\pi/2 \end{bmatrix}^T$$

For the Euler parameters, the rotations of $\theta = \frac{\pi}{6}$ occurs about the y_0 axis and hence the vector v has a unit magnitude component in the $Y - axis$.

$$\mathbf{v} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathbf{T}} \tag{17}$$

Therefore Euler parameters can be calculated from equations (7)(8)(9) and (10). The resulting Euler parameters are :

$$\theta = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} 0.965 & 0 & 0.25 & 0 \end{bmatrix}^{\mathbf{T}} \tag{18}$$

3.3 Task 3

The third task is to derive all the elements used in the total derivatives $\dot{C}$ and $\ddot{C}$ given in equation (13) and (14) of chapter 2. Using the constraint equations of the system provided in equation (11), we have

First total derivative

$$\dot{C} = \begin{bmatrix} \dot{y} + 2\dot{x}x + \dot{\phi}_1 \cos\phi_1 + \frac{\dot{z}}{2\sqrt{z}} \\ 3t^2 + \dot{x}(y+z) + \dot{z}x + \dot{y}(x + \sin\phi_3) + \dot{\phi}_3 y \cos\phi_3 \\ \dot{\phi}_2 \cos\phi 2 + 3\dot{x}\frac{\sqrt{x}}{2} + 1 \end{bmatrix}$$

and second total derivative can be given by

$$\ddot{C} = \begin{bmatrix} \ddot{y} - \dot{\phi}_1^2 \sin\phi_1 + 2\ddot{x}x + \ddot{\phi}_1 \cos\phi_1 + \frac{\ddot{z}}{2\sqrt{z}} + 2\dot{x}^2 - \frac{(\dot{z}^2)}{4z^{\frac{3}{2}}} \\ 6t + \ddot{x}(y+z) + \dot{y}(\dot{x} + \dot{\phi}_3 \cos\phi_3) + \dot{x}\dot{z} + \ddot{z}x + \ddot{y}(x + \sin\phi_3) + \dot{\phi}_3(\dot{y}\cos\phi_3) - \dot{\phi}_3 y \sin\phi_3) + \dot{x}(\dot{y} + \dot{z}) + \ddot{\phi}_3 y \cos(\phi_3) \\ \ddot{\phi}_2 \cos(\phi 2) - \dot{\phi}_2^2 \sin\phi_2 + \frac{3\ddot{x}\sqrt{x}}{2} + \frac{3\dot{x}^2}{4\sqrt{x}} \end{bmatrix}$$

3.4 Task 4

For the final task, the system shown in figure (3), chapter 2, is considered along with its constraint equations (15) and (16). The initial values were given as $r = 4m$ and $\phi = \frac{\pi}{4}$. These along with the dimensions $h = 4m$ and $c = 3m$ were used in the constraint equation for implementing the Newton Raphson method in matlab. The solutions obtained by this process are:

$$\phi = 0.9273rad$$
$$r = 5m$$

4 CONCLUSION

In conclusion, this assignment provided a way of revising through the kinematics of multibody system. The Euler angles and euler parameters are essential basics and the derivations in refreshing the knowledge. Similarly, the total derivations of constraint equations helped in practicing both simple matlab algebra and writing equations in latex, which will be essential for the upcoming assignments. Finally, the task based on newton raphsons method was simple yet helpful in understanding how the solution is derived.