



Summary Statistics

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CMPSS3660 – Introduction to Data Science – Fall 2019

<https://rebrand.ly/TUDatascience>



Thanks to William Green (NYU Stern IOMS)
and Hector Corrada Bravo (UMD)
and Zico Kolter [CMU]

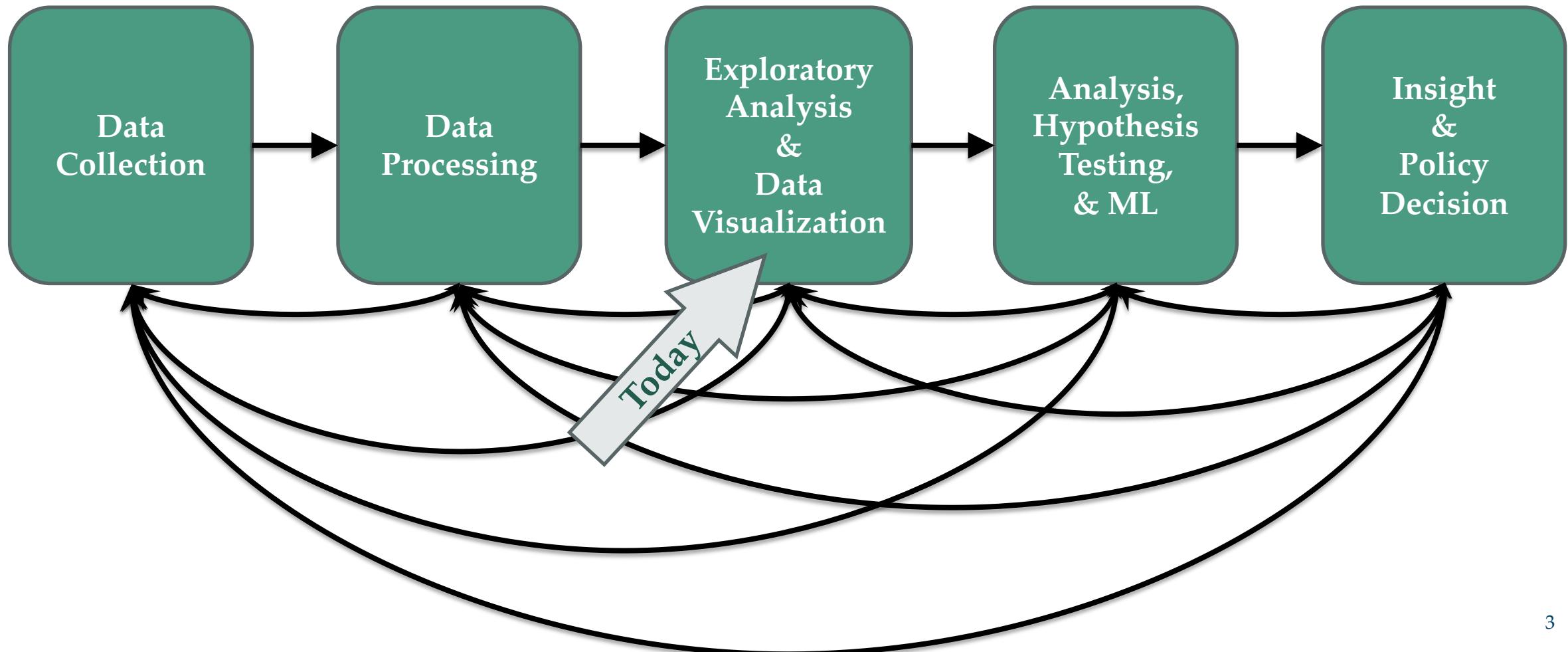
Many Thanks

Slides based off Introduction to Data Science from John P. Dickerson -
<https://cmsc320.github.io/>

Announcements

- Rest of Semester Schedule
- Lab 8 Review
- More EDA/Stats!

The Data LifeCycle



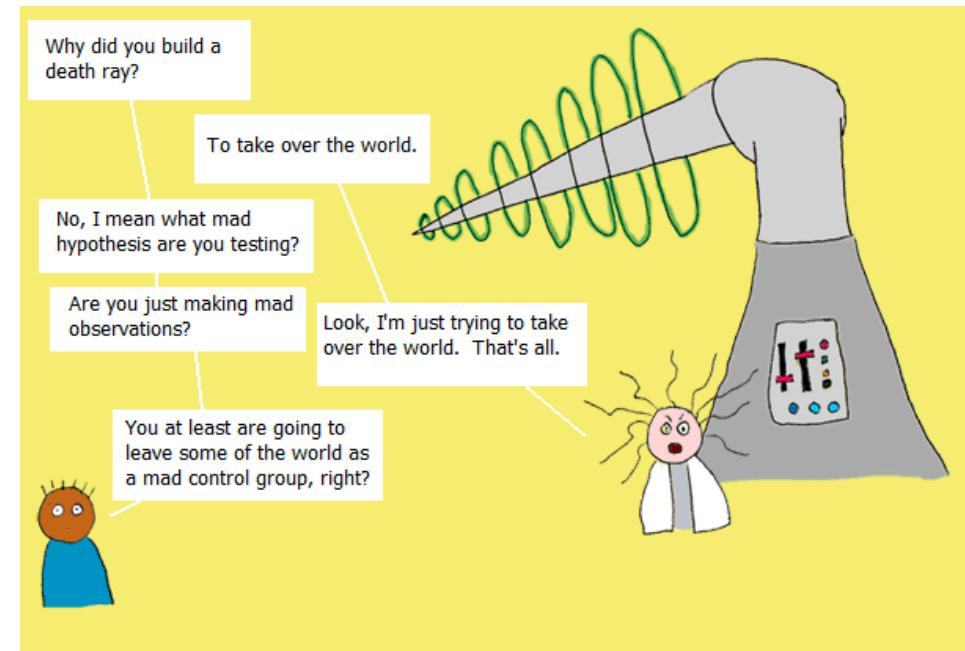
Exploratory Data Analysis

- Seen so far:
 - Manipulations that prepare datasets into tidy form.
 - Join tables and compute summaries.
 - Form relationships between different entities.
- EDA is the last step before Big Time Statistics and ML™:
 - Want to quickly “get a feel” for the data through summary statistics, visualization...
 - Spot nuances like skew, how distributed the data is, trends, how pairs of variables interact, problems.
 - Suggests which Stats/ML assumptions to make and approaches to take.



But Why?

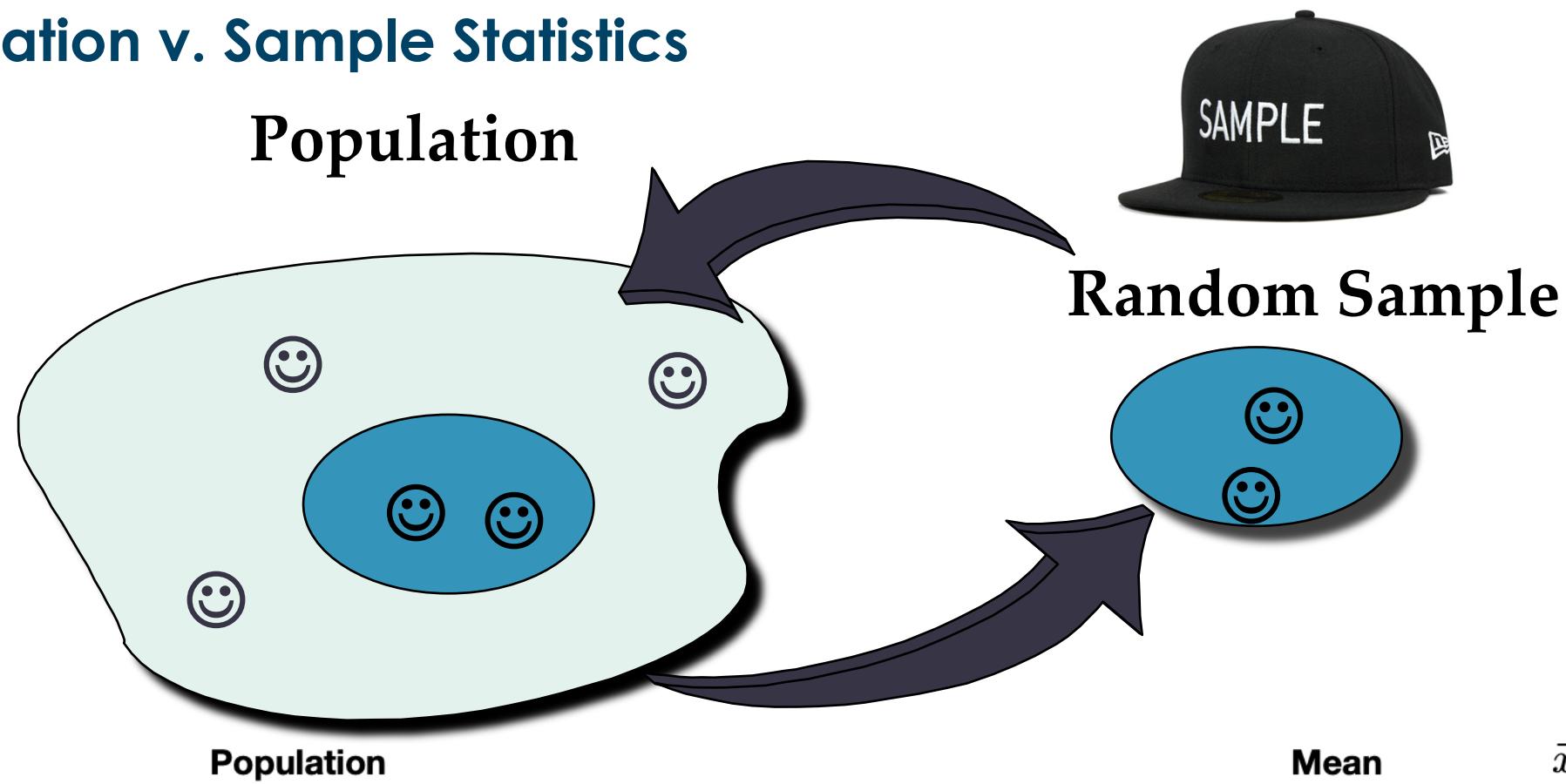
- We've only done a bit of "Science" so far...
 - Typically, "Science" is "determining some truth about the world...."
- Suppose you work for a company that is considering a redesign of their website; does their new design (design B) offer any statistical advantage to their current design (design A)?
- In a linear regression, does a certain variable impact the response?
 - Does energy consumption depend on whether or not a day is a weekday or weekend?
- Both: concerned with making actual statements about the nature of the world.



Sad truth: Most "mad scientists" are actually just mad engineers

<http://cowbirdsinlove.com/46>

Population v. Sample Statistics



Mean

$$\mu = \mathbb{E}[X]$$

Variance

$$\sigma^2 = \mathbb{E}[(X - \mu)^2]$$

Mean

$$\bar{x} = \frac{1}{m} \sum_{i=1}^m x^{(i)}$$

Variance

$$s^2 = \frac{1}{m-1} \sum_{i=1}^m (x^{(i)} - \bar{x})^2$$

[ZK]

Sample mean as random variable

The same mean is an empirical average over m independent samples from the distribution; it can also be considered as a random variable

This new random variable has the mean and variance

$$\mathbf{E}[\bar{x}] = \mathbf{E}\left[\frac{1}{m} \sum_{i=1}^m x^{(i)}\right] = \frac{1}{m} \sum_{i=1}^m \mathbf{E}[X] = \mathbf{E}[X] = \mu$$

$$\mathbf{Var}[\bar{x}] = \mathbf{Var}\left[\frac{1}{m} \sum_{i=1}^m x^{(i)}\right] = \frac{1}{m^2} \sum_{i=1}^m \mathbf{Var}[X] = \frac{\sigma^2}{m}$$

where we used the fact that for *independent* random variables X_1, X_2

$$\mathbf{Var}[X_1 + X_2] = \mathbf{Var}[X_1] + \mathbf{Var}[X_2]$$

When estimating variance of sample, we use s^2/m (the square root of this term is called the **standard error**)

Central limit theorem

Central limit theorem states further that \bar{x} (for “reasonably sized” samples, in practice $m \geq 30$) actually has a Gaussian distribution *regardless of the distribution of X*

$$\bar{x} \rightarrow \mathcal{N}\left(\mu, \frac{\sigma^2}{m}\right) \text{ (or equivalently)} \quad \frac{\bar{x} - \mu}{\sigma/m^{1/2}} \rightarrow \mathcal{N}(0,1)$$

In practice, for $m < 30$ and for estimating σ^2 using sample variance, we use a Student’s t-distribution with $m - 1$ degrees of freedom

$$\frac{\bar{x} - \mu}{s/m^{1/2}} \rightarrow T_{m-1}, \quad p(x; \nu) \propto \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

Last Week's Lesson

Having a really big sample does not assure you of an accurate result.

It may assure you of a really solid, really bad (inaccurate) result.

- Not all randomness is created equal when it comes to random sampling of a population:
 - Ask why data are missing! MCAR, MAR, MNAR.
 - Ask how the data were collected.

Today's Lesson (and Labs): Summary Statistics

- Part of **descriptive statistics**, used to summarize data:
 - Convey lots of information with extreme simplicity
- Descriptive statistics for a variable:
 - Measures of location: mean, median, mode
 - Measure of dispersion: variance, standard deviation
- Measuring correlation of two variables:
 - Understanding correlation
 - Measuring correlation
 - Scatter plots and regression

Measures of Location

These are 30 hours of average defect data on sets of circuit boards. Roughly what is the typical value?

1.45	1.65	1.50	2.25	1.65	1.60	2.30	2.20	2.70	1.70
2.35	1.70	1.90	1.45	1.40	2.60	2.05	1.70	1.05	2.35
1.90	1.55	1.95	1.60	2.05	2.05	1.70	2.30	1.30	2.35

- Location and central tendency
 - There exists a distribution of values
 - We are interested in the “center” of the distribution
- Two measures are the **sample mean** and the **sample median**
- They look similar, and measure the same thing
- They differ systematically (and predictably) when the data are not **symmetric**.

The Mean of Aggregate Data

State	Listing	IncomePC	State	Listing	IncomePC	State	Listing	IncomePC
Hawaii	896800	24057	Rhode Island	432534	22251	Texas	266388	19857
California	713864	22493	Delaware	420845	22828	Mississippi	255774	15838
New York	668578	25999	Oregon	417551	20419	Tennessee	255064	19482
Connecticut	654859	29402	Idaho	415885	18231	Wisconsin	243006	21019
Dist.Columbia	577921	31136	Illinois	377683	23784	Michigan	241107	22333
Nevada	549187	24023	New Hampshire	361691	23434	Missouri	221773	20717
New Jersey	529201	23038	New Mexico	358369	17106	South Dakota	220708	19577
Massachusetts	521769	25616	Vermont	346469	20224	West Virginia	219275	17208
Wyoming	499674	20436	South Carolina	340066	17695	Arkansas	217659	16898
Maryland	480578	24933	North Carolina	330432	19669	Ohio	209189	20928
Utah	475060	17043	Georgia	326699	20251	Kentucky	208391	17807
Colorado	467979	22333	Alaska	324774	23788	Oklahoma	203926	17744
Arizona	448791	19001	Minnesota	306009	22453	Kansas	201389	20896
Florida	447698	21677	Maine	299796	19663	Indiana	200683	20378
Montana	446584	17865	Pennsylvania	295133	22324	Iowa	184999	20265
Virginia	443618	22594	Louisiana	280631	17651	North Dakota	173977	18546
Washington	440542	22610	Alabama	269135	18010	Nebraska	164326	20488

Average list price:

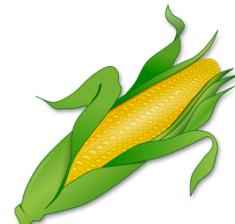
$$1/51 (\$898,800 + \$713,864 + \dots + \$164,326) = \$369,687$$

Averaging Averages?

- Hawaii's average listing = \$896,800
- Hawaii's population = 1,275,194
- Illinois' average listing = \$377,683
- Illinois' population = 12,763,371



- Illinois and Hawaii each get an equal weight of $1/51 = .019607$ when the mean is computed.
- Looks like Hawaii is getting too much influence ...



Weighted Average

$$\text{Simple average} = \overline{\text{Listing}} = \sum_{\text{States}} \text{Weight}_{\text{State}} \text{ Listing}_{\text{State}}$$

$$\text{Weight} = \frac{1}{51} = .019607$$

Illinois is 10 times as big as Hawaii. Suppose we use weights that are in proportion to the state's population. (The weights sum to 1.0.)

$\text{Weight}_{\text{State}}$ varies from .001717 for Wyoming to .121899 for California

New average is \$409,234 compared to \$369,687 without weights, an error of 11%!

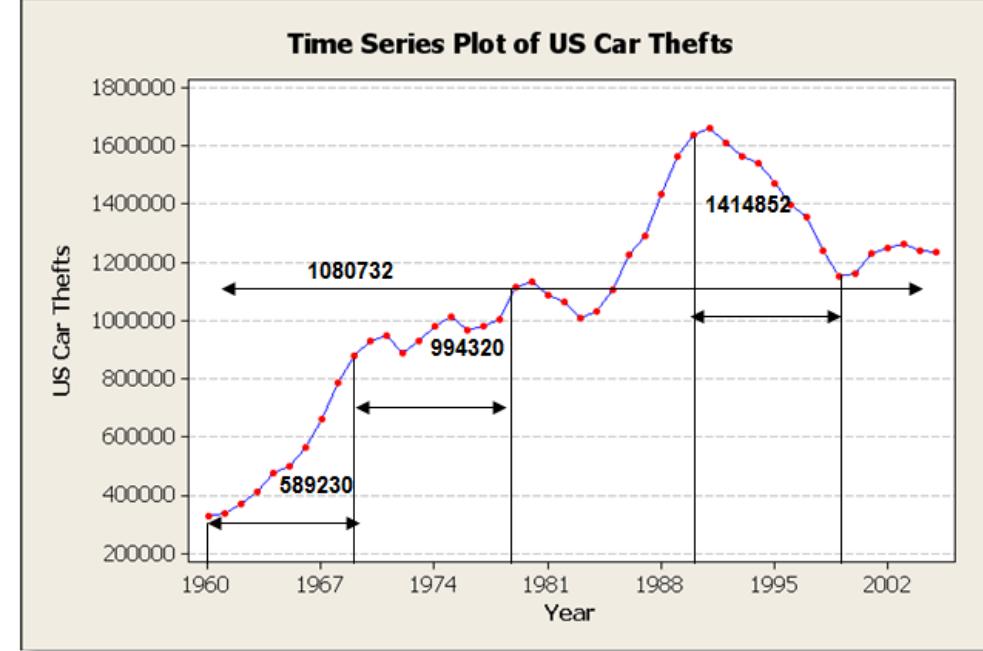
Sometimes an unequal weighting of the observations is necessary

Averages & Time Series

- Averaging **trending** time series is usually not helpful
- Mean changes completely depending on time interval
- What about **periodic** time series data ??????????

Ask yourself:

- Does the mean over the entire observation period mean anything?
- Does it estimate anything meaningful?



The Sample Median

- Median:
 - Sort the data
 - Take the middle point*
- Odd number:
 - Central observation: $\text{Med}[1,2,4,6,8,9,17]$
- Even number:
 - Midpoint between the two central observations $\text{Med}[1,2,4,6,8,9,14,17] = (6+8)/2=7$



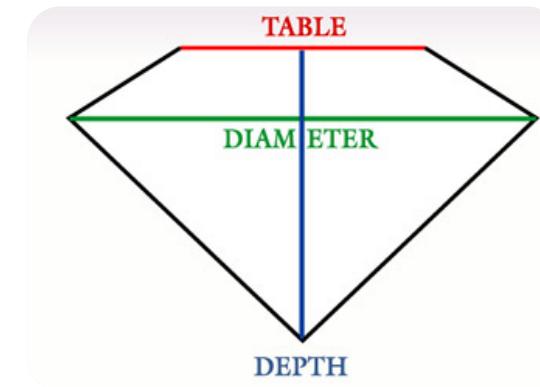
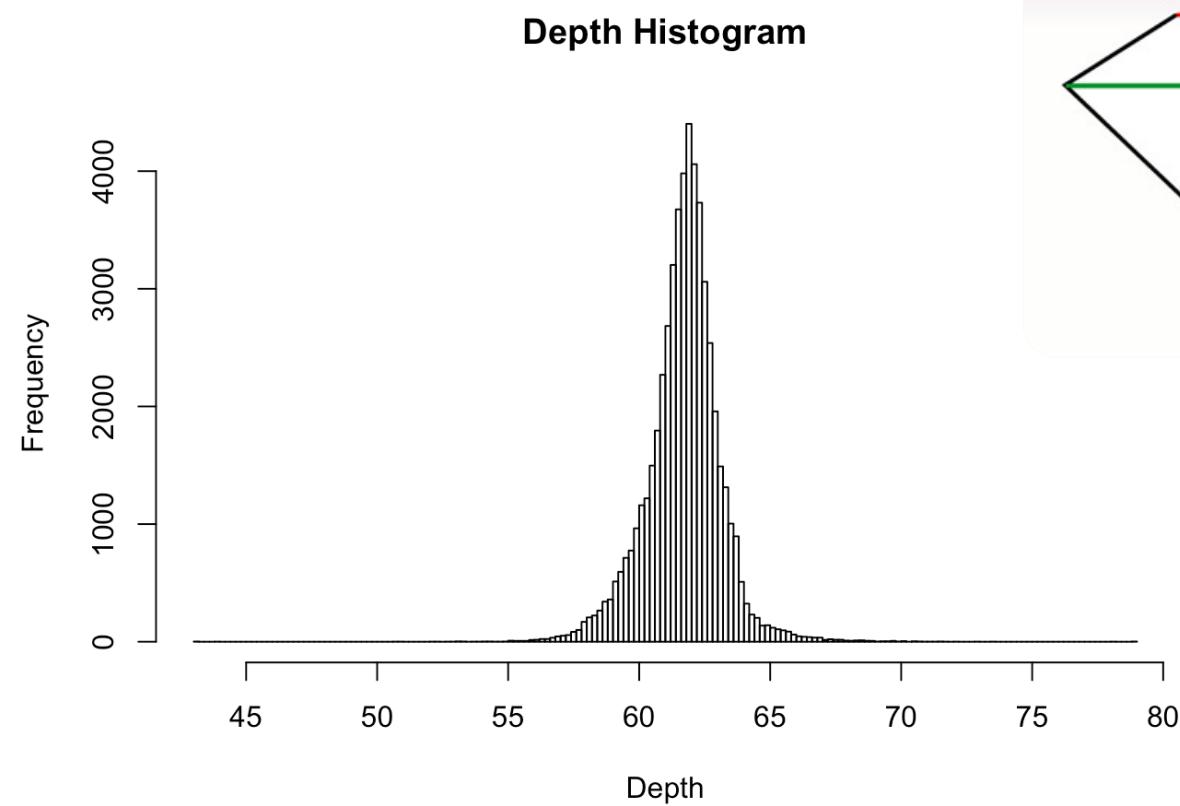
What is the center?

- The mean and median measure the **central tendency** of data
- Generally, the **center** of a dataset is a point in its range that is close to the data.
- Close? Need a **distance metric** between two points x_1 and x_2 .
- We've talked about some already!
 - Absolute deviation: $|x_1 - x_2|$
 - Squared deviation: $(x_1 - x_2)^2$
- We'll define the center based on these metrics



Dataset for this part

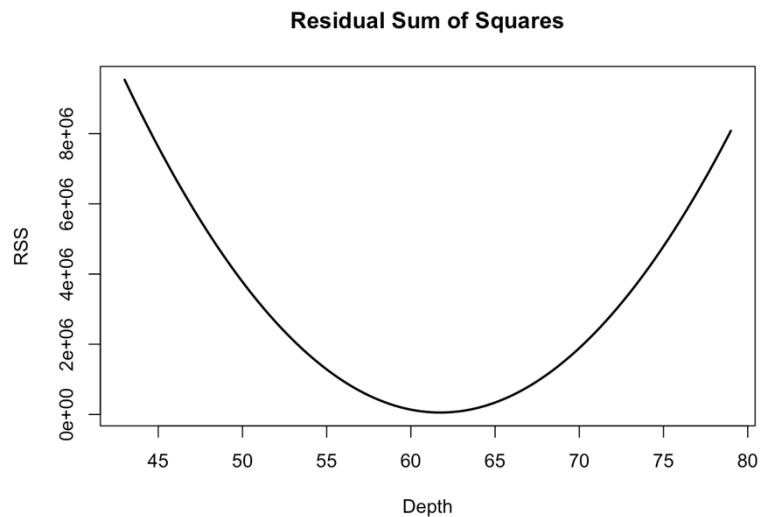
- 53,940 measurements of diamonds



The Mean Revisited

- Define a center point μ based on some function of the distance from each data point to that center point
 - Residual sum of squares (RSS) for a point μ :

$$RSS(\mu) = \frac{1}{2} \sum_{i=1}^n (x_i - \mu)^2$$



So what should our estimate of the “center” of this dataset be, based on the RSS metric?
???????????????

The mean Revisited

- Want the point μ that minimizes the RSS ??????????
- Find the derivative of RSS and set it to zero, solve for μ !

$$\frac{\partial}{\partial \mu} \frac{1}{2} \sum_{i=1}^n (x_i - \mu)^2 = \frac{1}{2} \sum_{i=1}^n \frac{\partial}{\partial \mu} (x_i - \mu)^2$$

$$= \frac{1}{2} \sum_{i=1}^n 2(x_i - \mu) \times (-1)$$

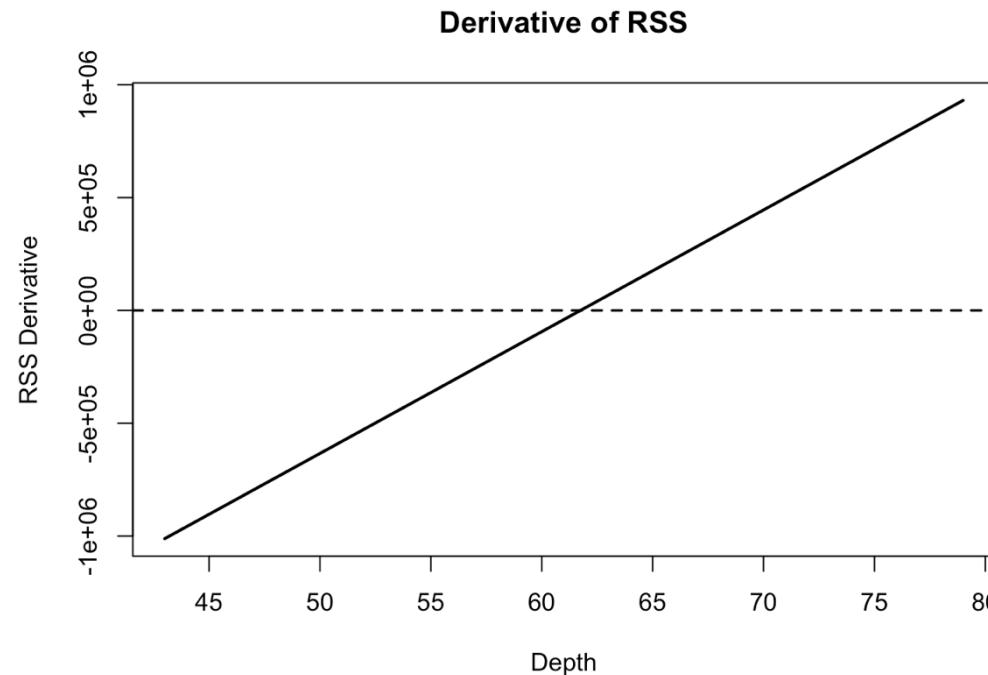
The mean revisited

$$= \frac{1}{2} \sum_{i=1}^n 2(x_i - \mu) \times (-1)$$

$$= \frac{1}{2} 2 \sum_{i=1}^n (\mu - x_i)$$

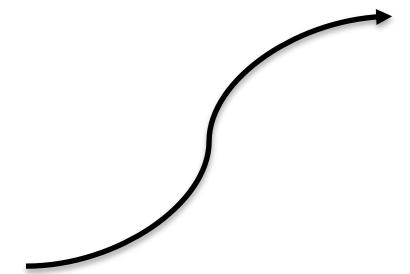
$$= \sum_{i=1}^n \mu - \sum_{i=1}^n x_i$$

$$= n\mu - \sum_{i=1}^n x_i$$



The Mean Revisited

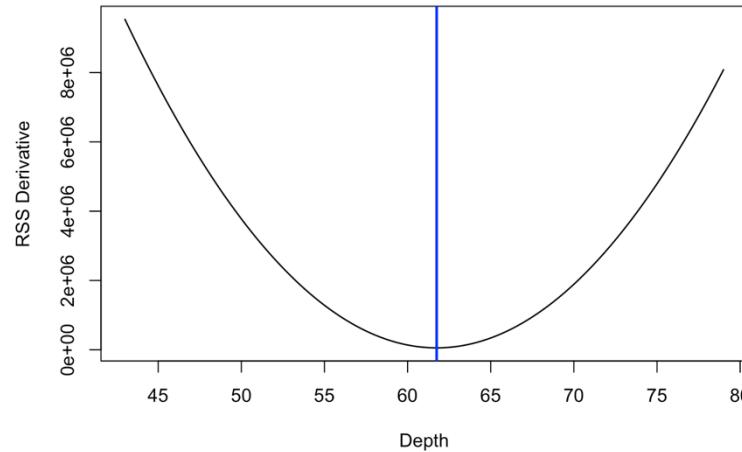
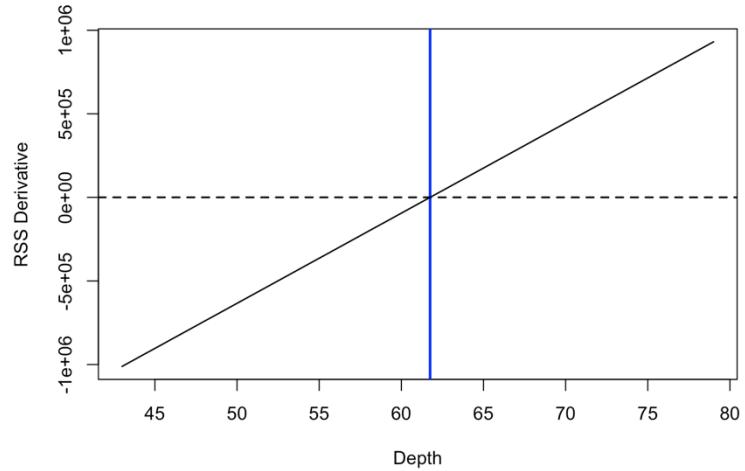
- Set the derivative to zero and solve for μ :

$$\frac{\partial}{\partial \mu} = 0 \quad n\mu = \sum_{i=1}^n x_i$$
$$n\mu - \sum_{i=1}^n x_i = 0 \quad \mu = \frac{1}{n} \sum_{i=1}^n x_i$$


The mean is the point μ that minimizes the RSS for a dataset.

The Mean Revisited

What about a
weighted average
???????

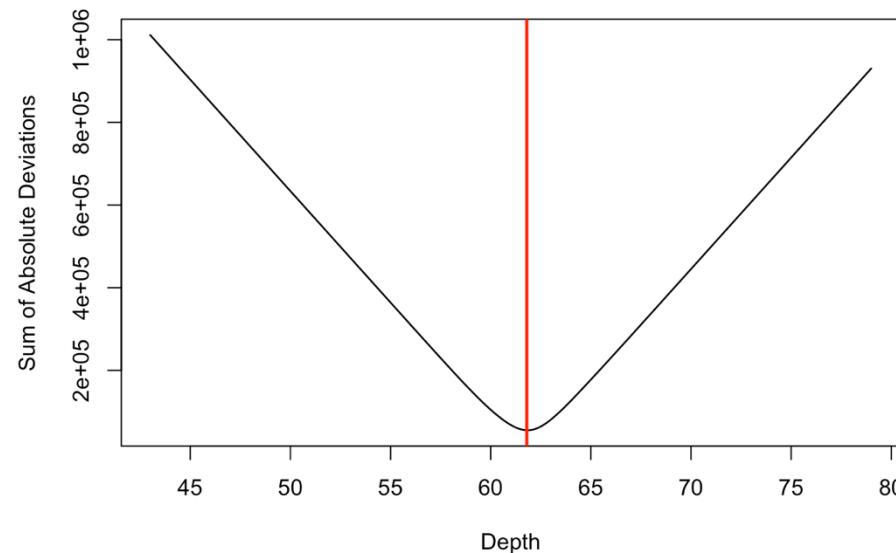


The mean is the point μ that minimizes the RSS for a dataset.

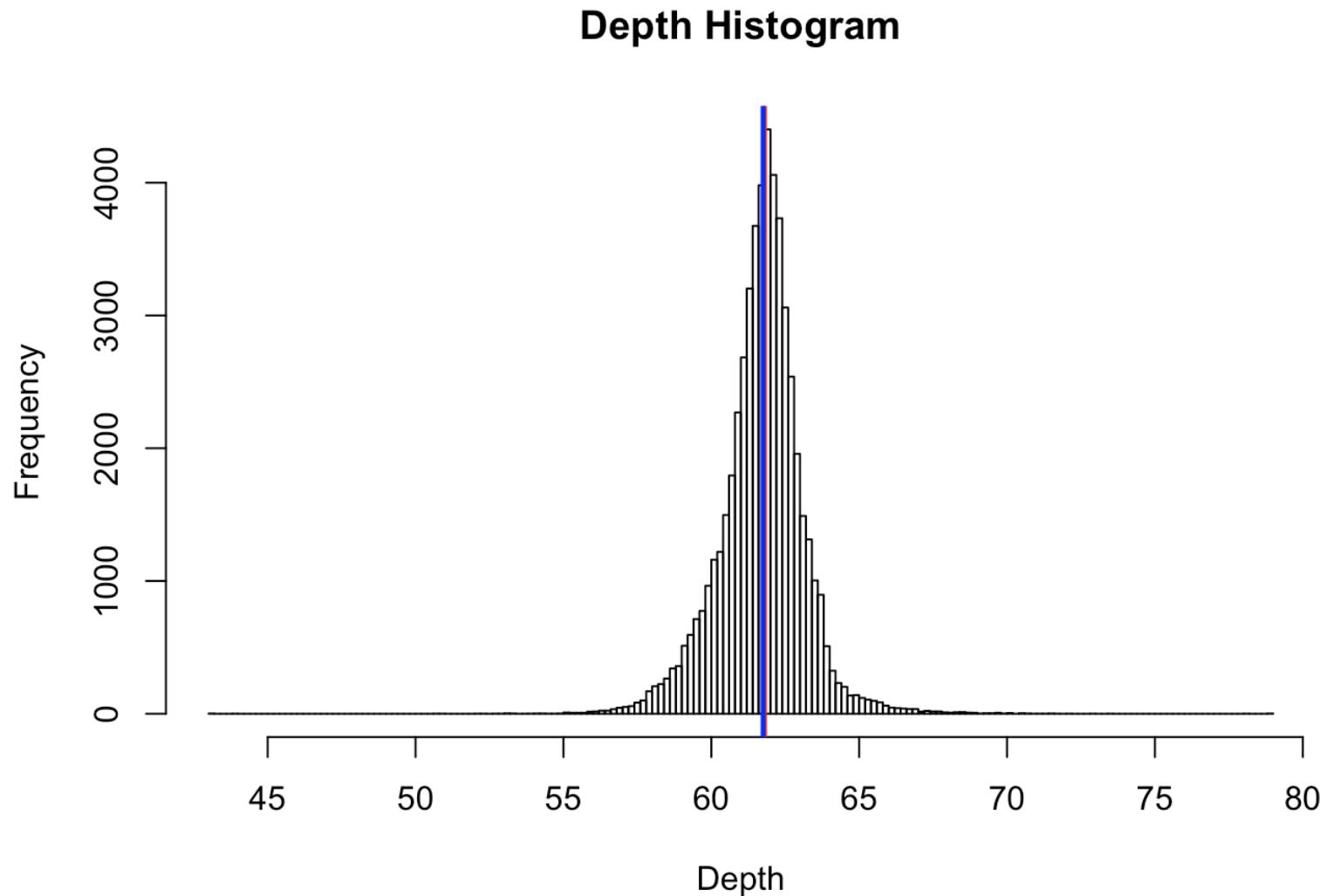
The Median Revisited

- Define a center point m based on some function of the distance from each data point to that center point.
 - The median m minimizes the sum of absolute differences:

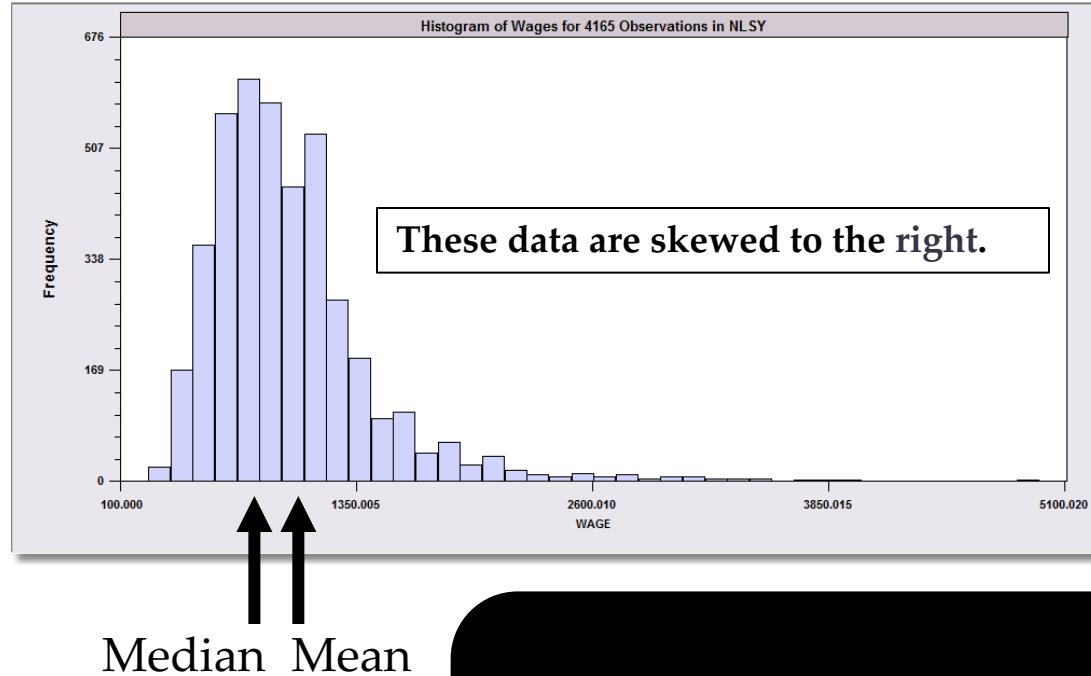
$$\sum_{i=1}^n |x_i - m|$$



Mean != Median



Skewed Data

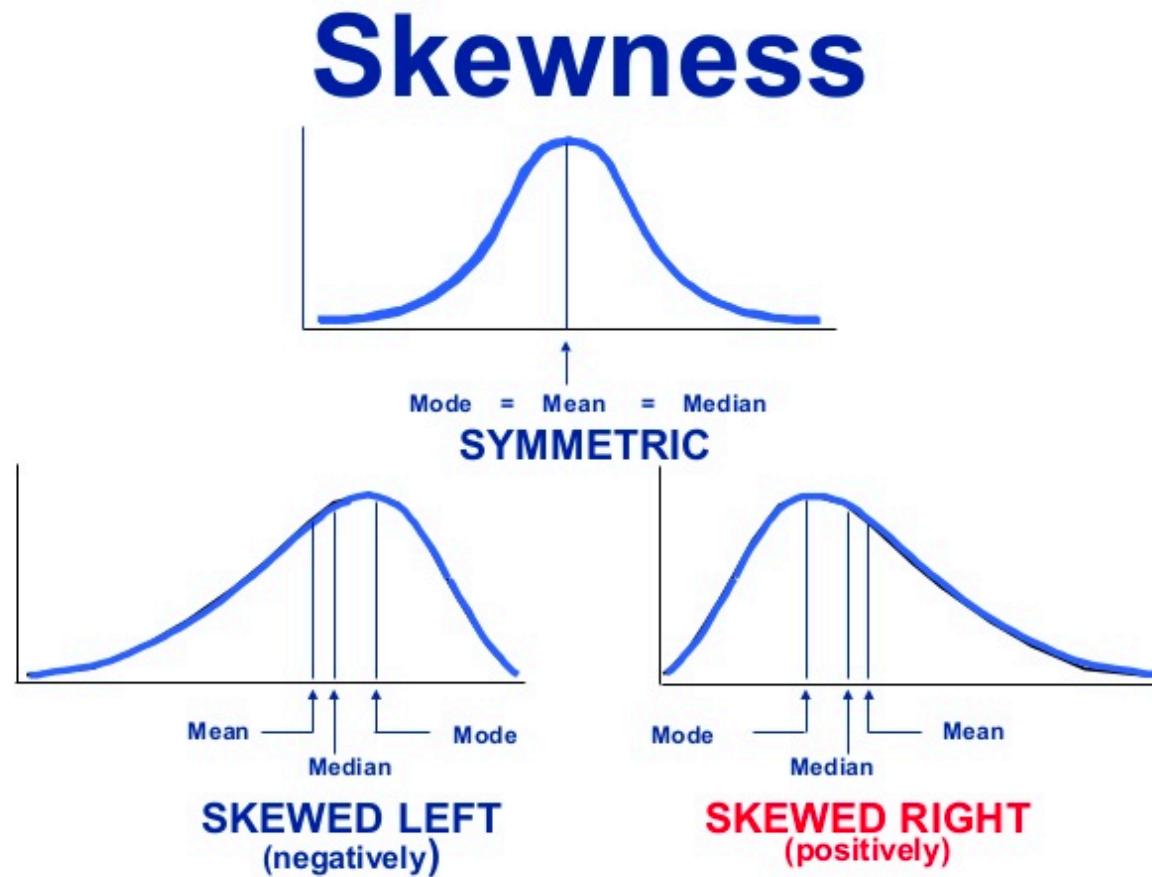


Monthly Earnings
 $N = 595$,
Median = 800
Mean = 883

The mean will exceed the median when the distribution is skewed to the right.

Skewness is in the direction of the long tail

A Visual Guide to Skew



- We skew in the direction of the long tail.
- Image:
<https://guides.douglascollege.ca/c.php?g=408742&p=2970198>

Skewness

- Extreme observations distort means but not medians.
- Outlying observations distort the mean:
 - Med $[1,2,4,6,8,9,17] = 6$
 - Mean $[1,2,4,6,8,9,17] = 6.714$
 - Med $[1,2,4,6,8,9,17000] = 6$ (still)
 - Mean $[1,2,4,6,8,9,17000] = 2432.8$ (!)
- Typically occurs when there are some outlying observations, such as in cross sections of income or wealth and/or when the sample is not very large.



DATAPoints

Income Gap Grows Wider (and Faster)

By ANNA BERNASEK

Published: August 31, 2013

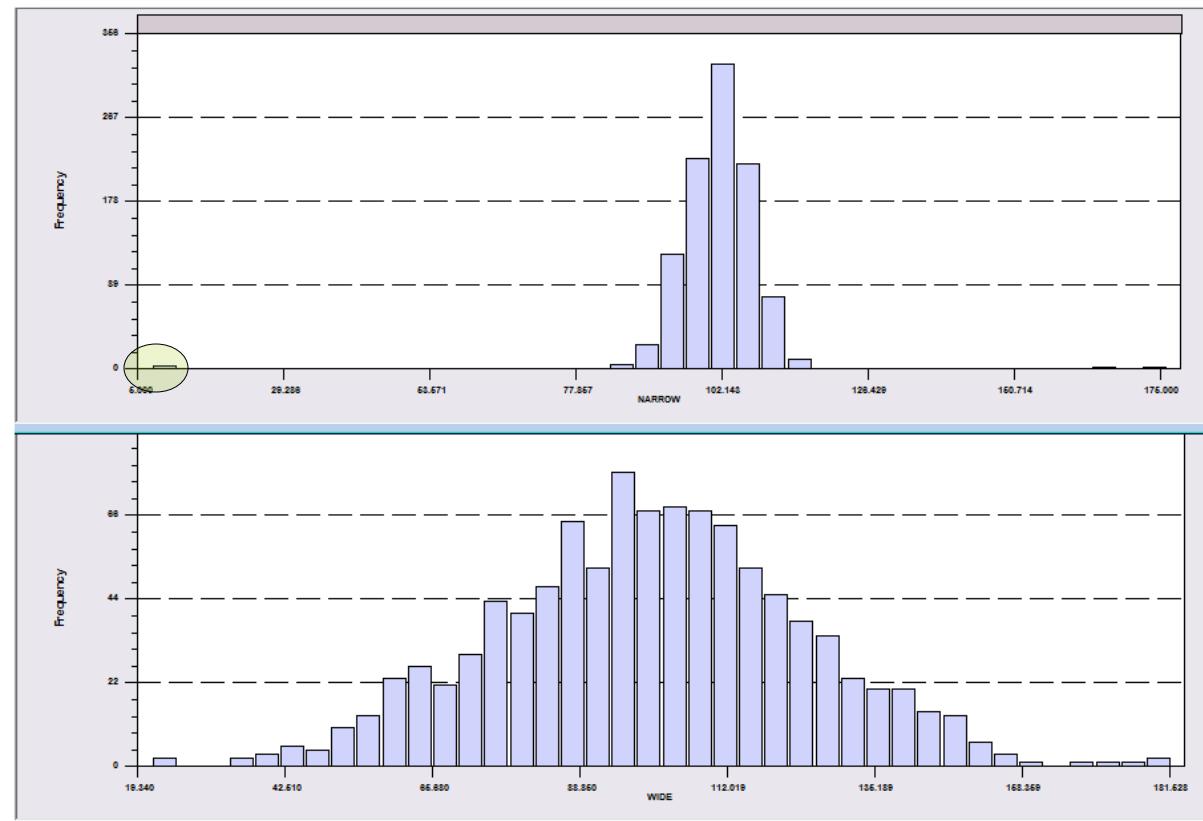
INCOME inequality in the United States has been growing for decades, but the trend appears to have accelerated during the Obama administration. One measure of this is the relationship between median and average wages.

1.7%
Increase in median annual wage
3.9%
Increase in average annual wage 2009 through 2011

The median wage is straightforward: it's the midpoint of everyone's wages. Interpreting the average, though, can be tricky. If the income of a handful of people soars while everyone else's remains the same, the entire group's average may still rise substantially. So when average wages grow faster than the median, as happened from 2009 through 2011, it means that lower earners are falling further behind those at the top.

One way to see the acceleration in inequality is to look at the ratio of average to median annual wages. From 2001 through 2008, during the George W. Bush administration, that ratio grew at 0.28 percentage point per year. From 2009 through 2011, the latest year for which the data is available, the ratio increased 1.14 percentage points annually, or roughly four times faster.

More Information Needed!

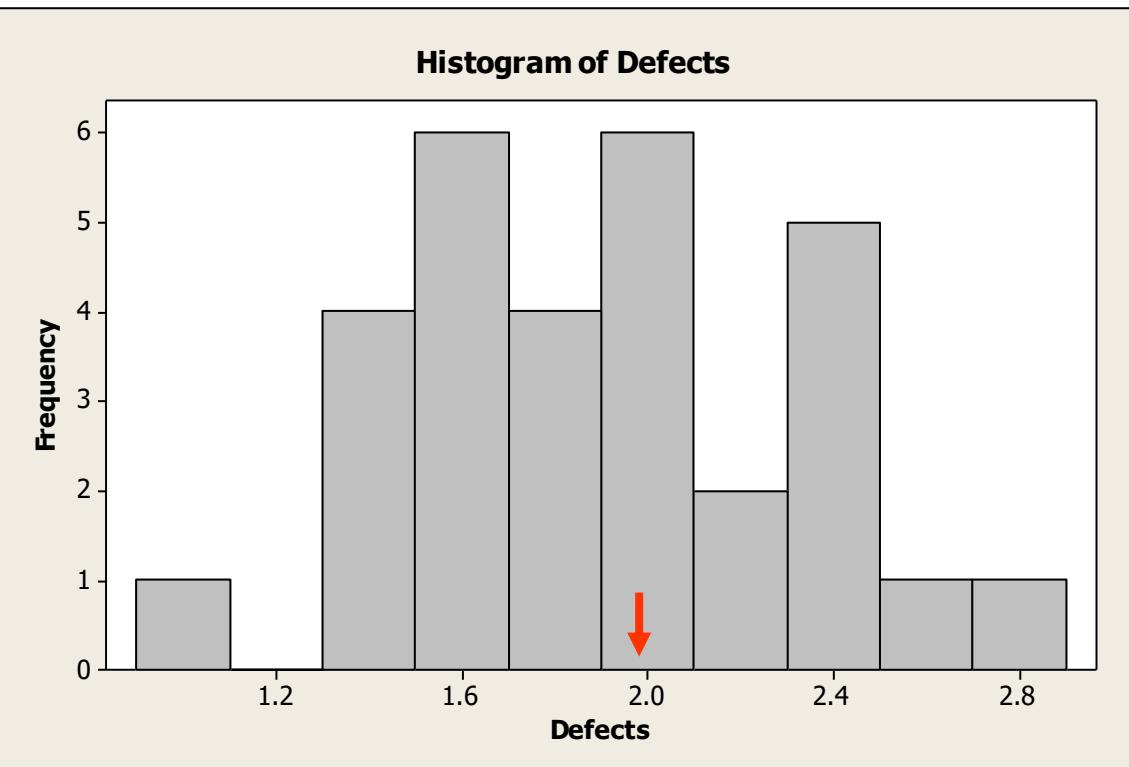


Both data sets have a mean of about 100.

Dispersion of the Observations

30 hours of average defect data on sets of circuit boards.

1.45	1.65	1.50	2.25	1.65	1.60	2.30	2.20	2.70	1.70
2.35	1.70	1.90	1.45	1.40	2.60	2.05	1.70	1.05	2.35
1.90	1.55	1.95	1.60	2.05	2.05	1.70	2.30	1.30	2.35



We quantify the variation of the values around the mean.

Note the range is from 1.05 to 2.70. This gives an idea where the data lie.

The mean plus a measure of the variation do the same job.

Range as a Measure of Dispersion

- Problems
?????????



These two data sets both have 1,000 observations that range from about 10 to about 180.

Variance & STDEV: Univariate Measures of dispersion

- Variance = $s_x^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$ or $\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$
- Standard deviation = $s_x = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}$
- The variance is commonly used statistic for spread
 - What are the units of the variance ??????????
- Standard deviation “fixes this,” can be used as an **interpretable unit of measurement.**

Variance, Aside: Why Divide by $n-1$?

- Remember: we are typically calculating the mean / median / variance / etc of a **sample** of a population.
 - Want that {mean, median, variance, ...} to be an “unbiased” estimate of the true population’s {mean, median, variance, ...}
- **Unbiased? Consider variance ...**
 1. Look at every possible sample of the population
 2. Compute sample variance of each population
 3. Is the average of those variances equal to the population variance? If so, then this is an “unbiased” estimator.
- Extra Reading: https://en.wikipedia.org/wiki/Bias_of_an_estimator

Variance, Aside: Why Divide by $n-1$?

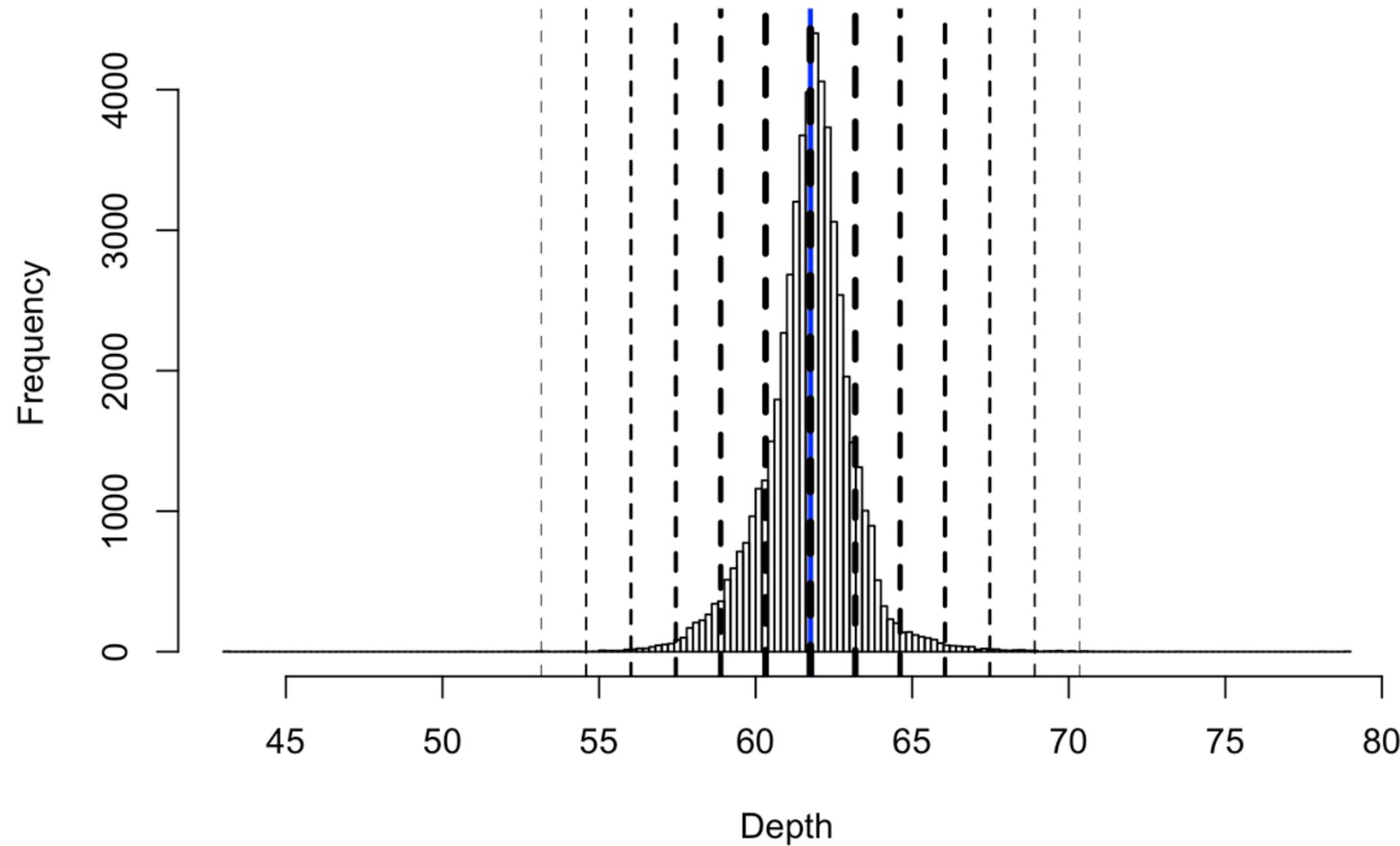
- Dividing by $n-1$ in the sample variance computation leads to an unbiased estimate of the population variance

- Intuition. Fix a sample ...
 - Variance measures distribution around a mean
 - Sampled values are, on average, closer to sample mean than to true population mean
 - So, we will underestimate the true variance slightly
 - Using $n-1$ instead of n makes our variance calculation bigger

- This “embiggening” impacts smaller n more than larger n
 - Larger samples are better estimates of population
 - If sample is the population, just divide by n ...

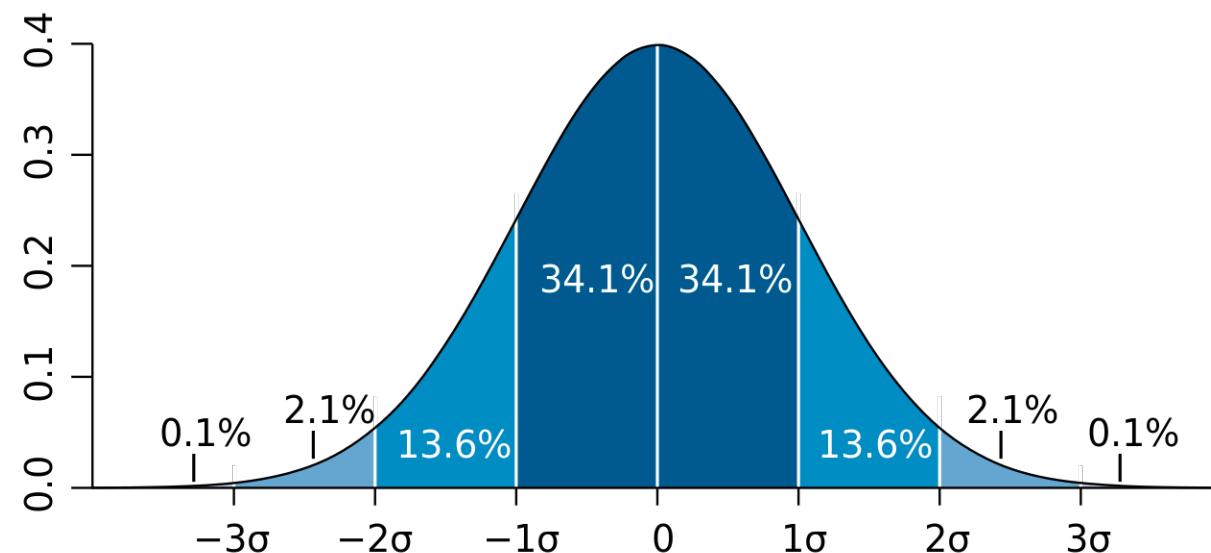


Depth Histogram

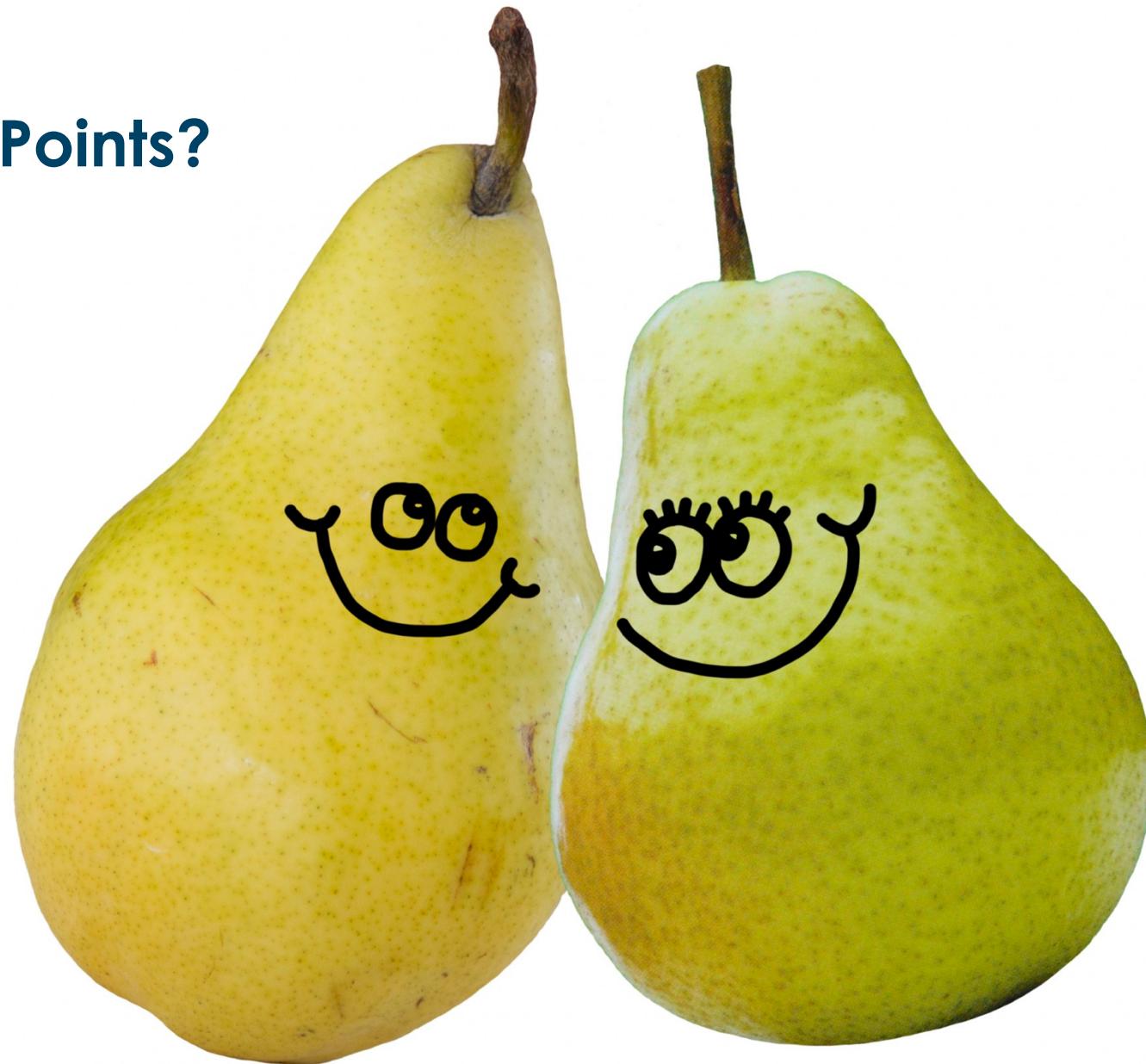


Using “standard deviations from the mean” as a unit

SDs	Proportion	Interpretation
1	0.68	68% of the data is within ± 1 sds
2	0.95	95% of the data is within ± 2 sds
3	0.9973	99.73% of the data is within ± 3 sds
4	0.999937	99.9937% of the data is within ± 4 sds
5	0.9999994	99.999943% of the data is within ± 5 sds
6	1	99.999998% of the data is within ± 6 sds



Pairs of data Points?



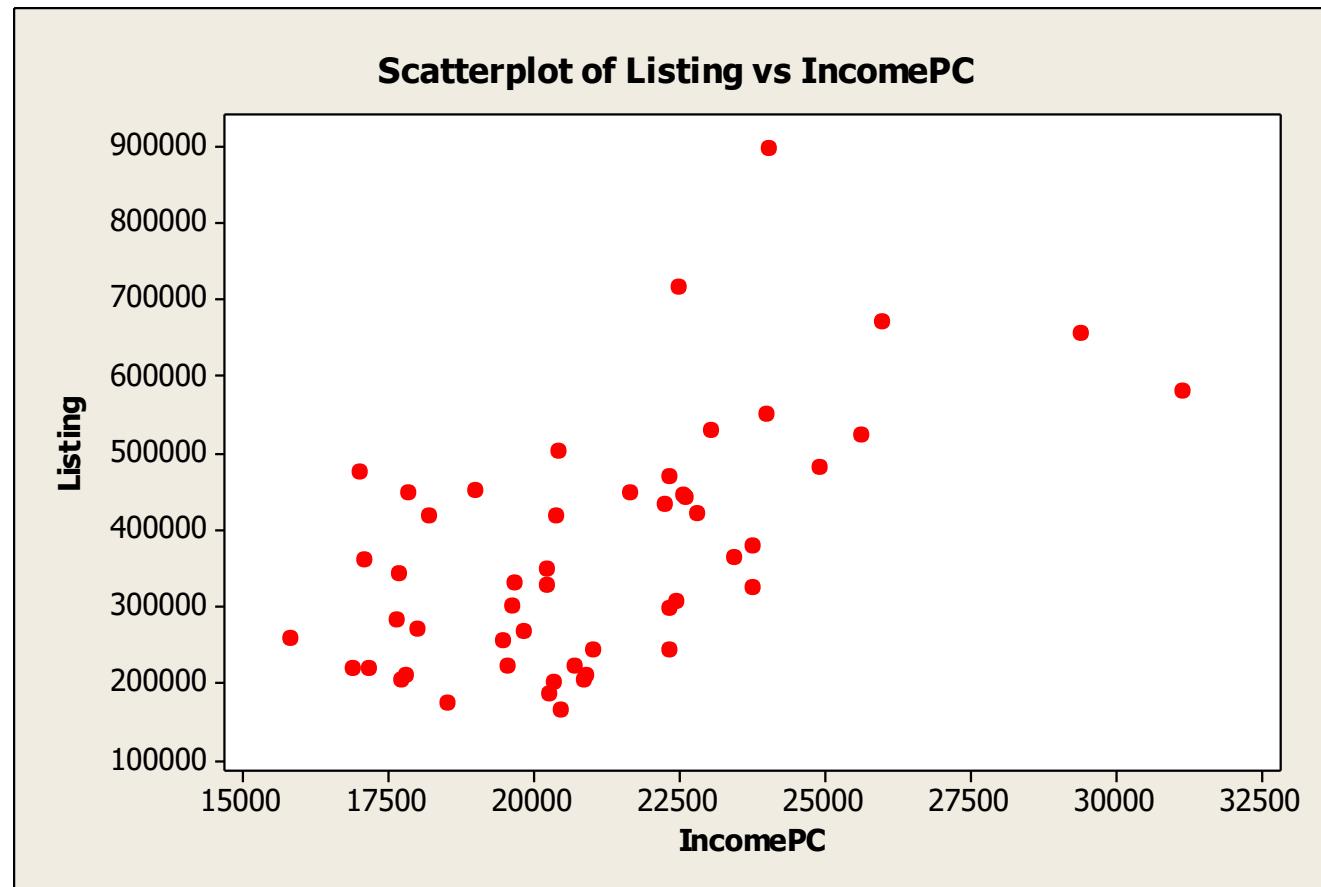
Correlation

- Variables Y and X vary together
- Causality vs. Correlation: Does movement in X “cause” movement in Y in some metaphysical sense?
- Correlation
 - Simultaneous movement through a statistical relationship
 - Simultaneous variation “induced” by the variation of a common third effect

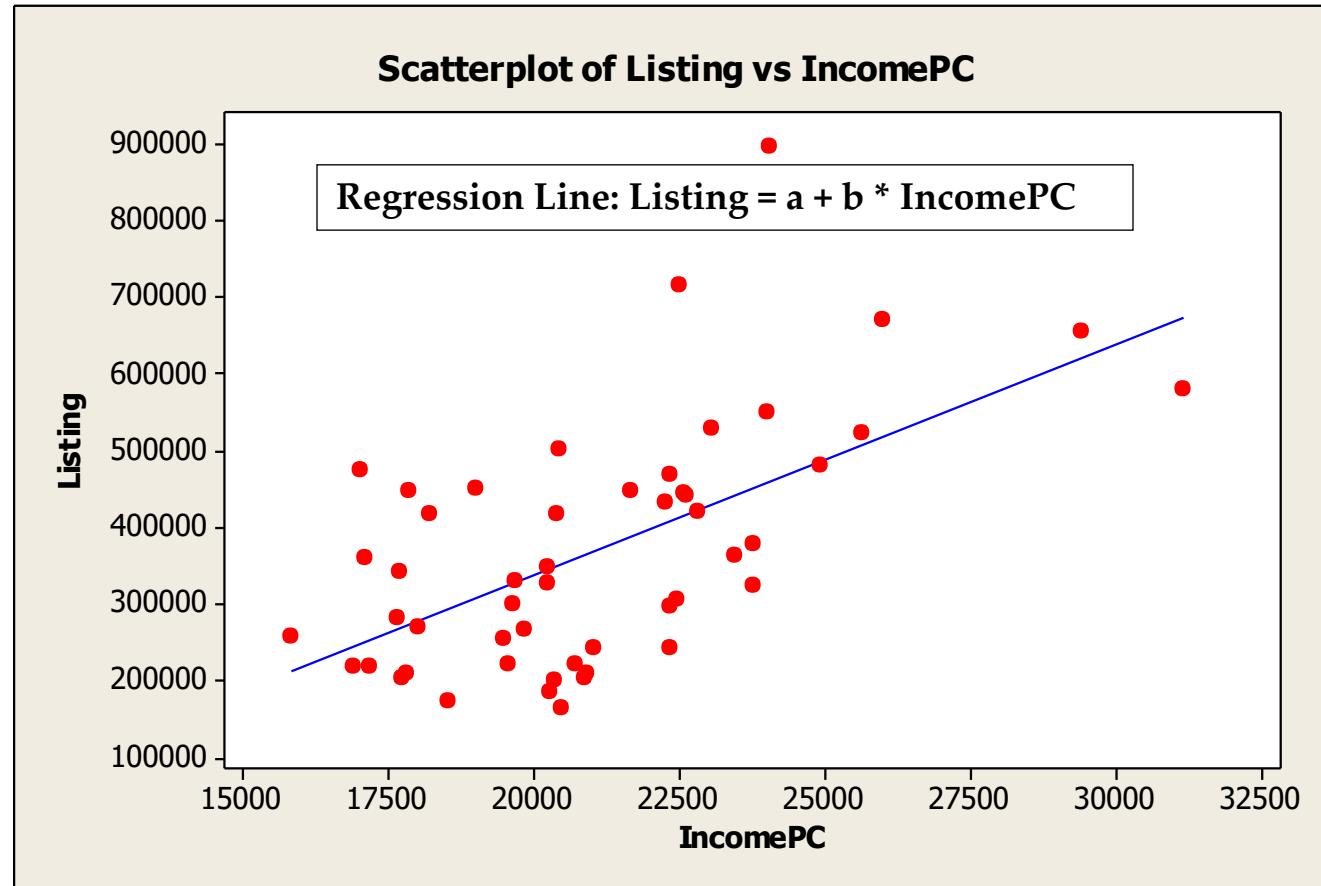
House Prices & Per Capita Income

State	Listing	IncomePC	State	Listing	IncomePC	State	Listing	IncomePC
Hawaii	896800	24057	Rhode Island	432534	22251	Texas	266388	19857
California	713864	22493	Delaware	420845	22828	Mississippi	255774	15838
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Washington	440542	22610	Alabama	269135	18010	Nebraska	164326	20488

Scatter Plot Suggests Positive Correlation

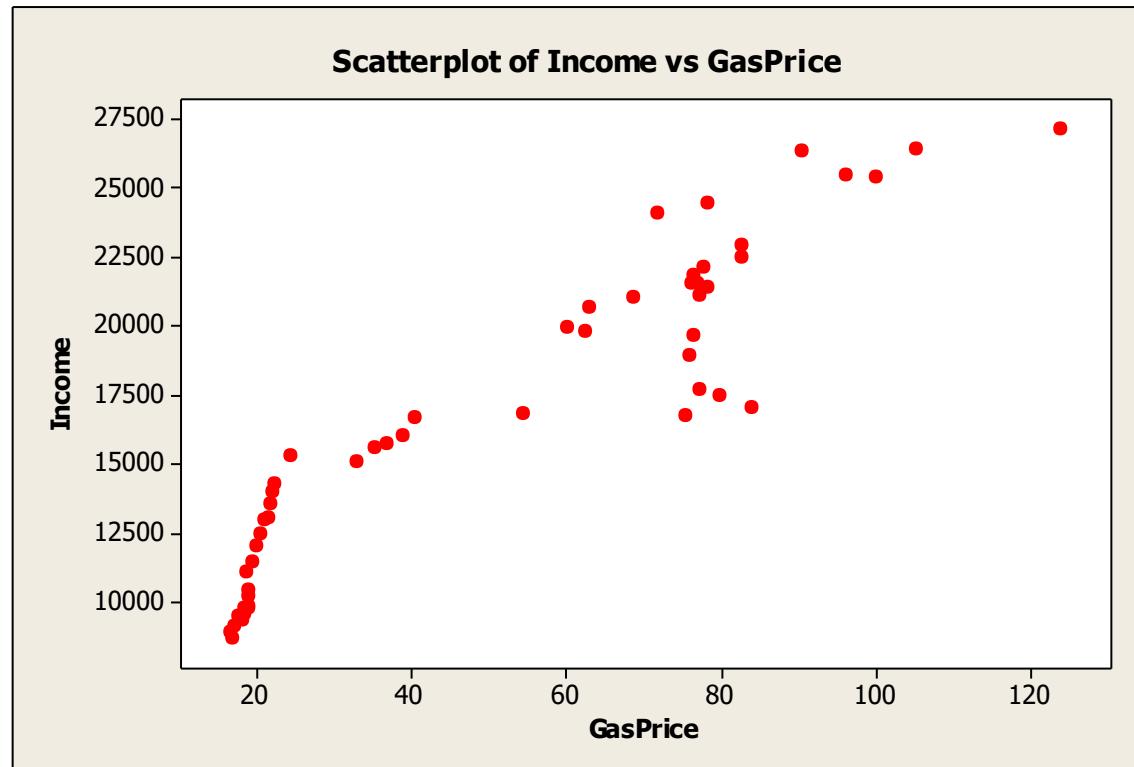


Linear Regression Measures Correlation



Correlation Is Not Causation

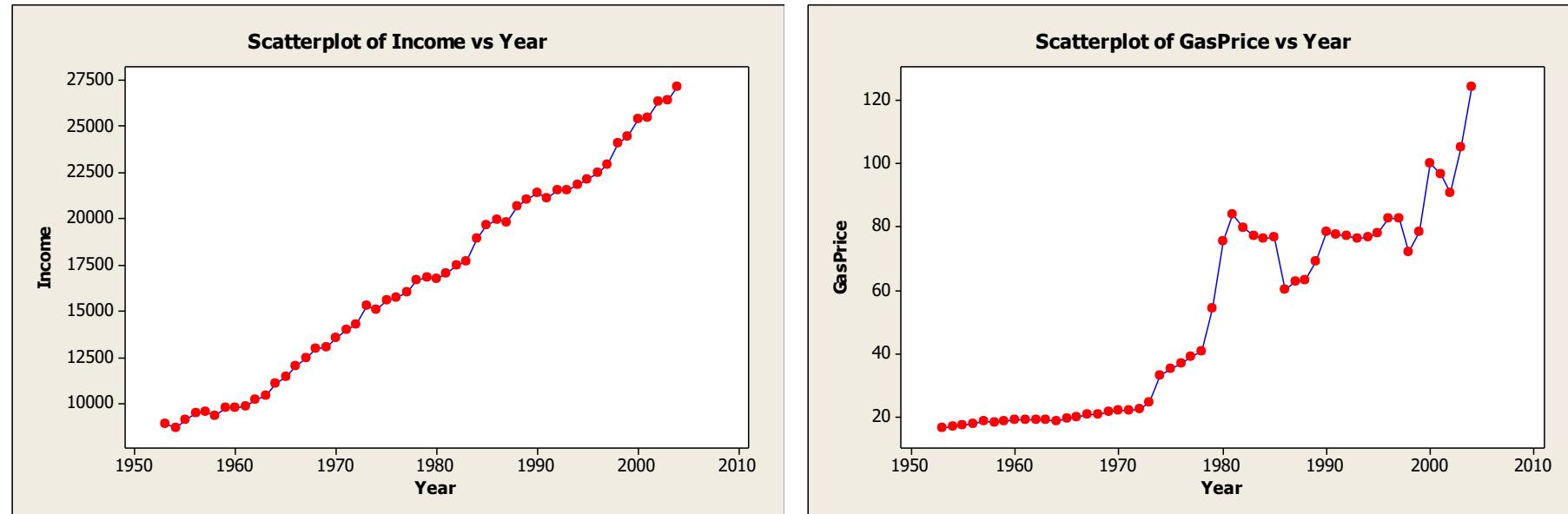
- Price and income seem to be **positively correlated**.



Does a rise in
income cause a
rise in gas prices
???????????????

A Hidden Relationship

Not positively “related” to each other;
both positively related to “time.”



“Related” ...?

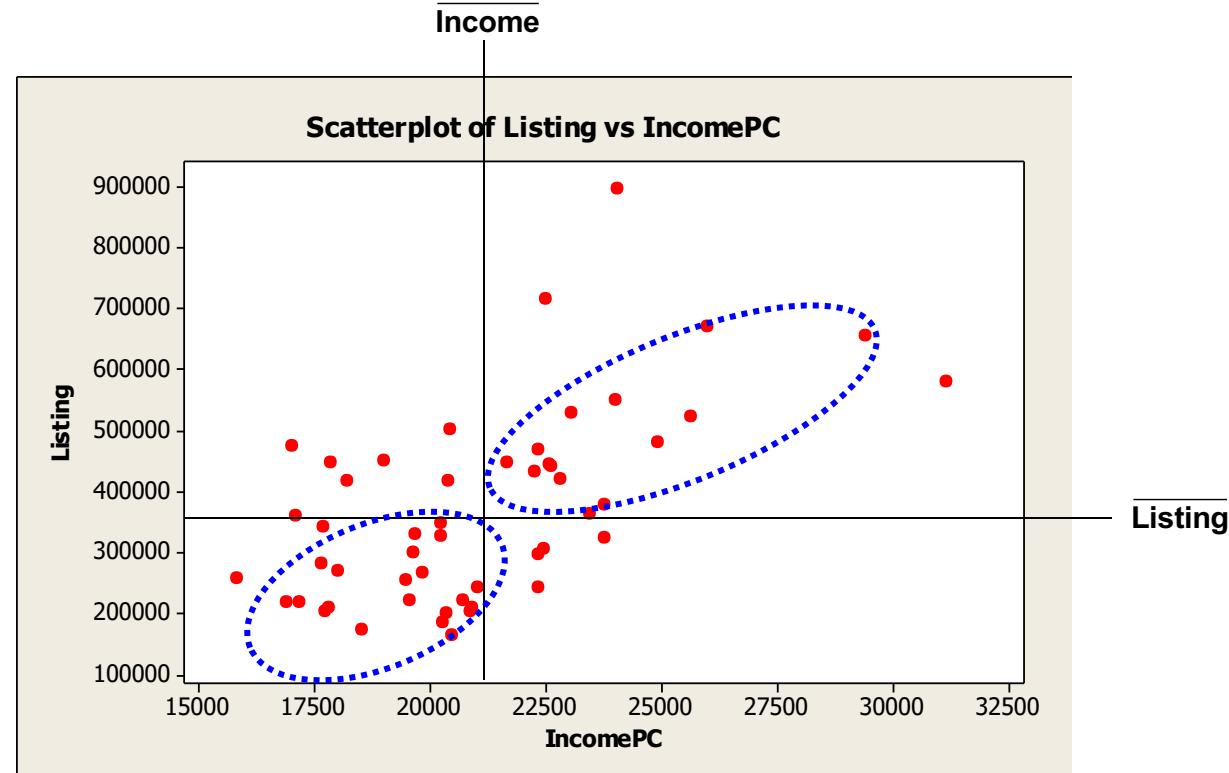
- Want to capture: some variable X varies in the same direction and at the same scale as some other variable Y

$$cov(x, y) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

- What happens if:
 - X varies in the opposite direction as Y ????????
 - X varies in the same direction as Y ????????
- What are the units of the covariance ????????
- Pearson’s correlation coefficient is **unitless** in [-1,+1]:

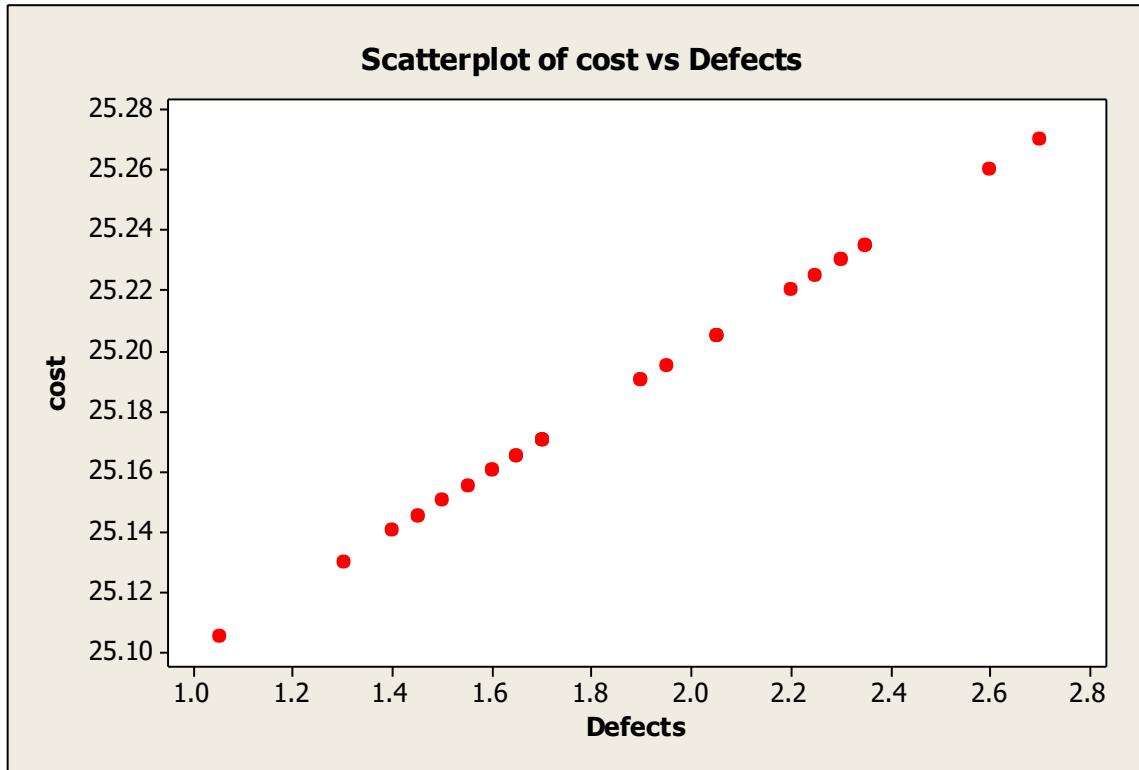
$$cor(x, y) = \frac{cov(x, y)}{sd(x)sd(y)}$$

Correlation

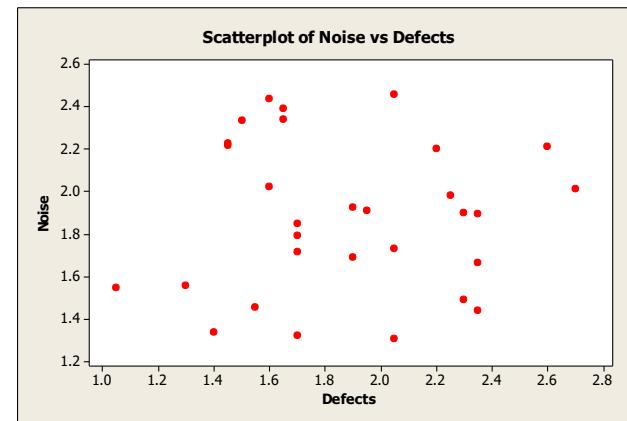


$$r_{\text{Income}, \text{Listing}} = +0.591$$

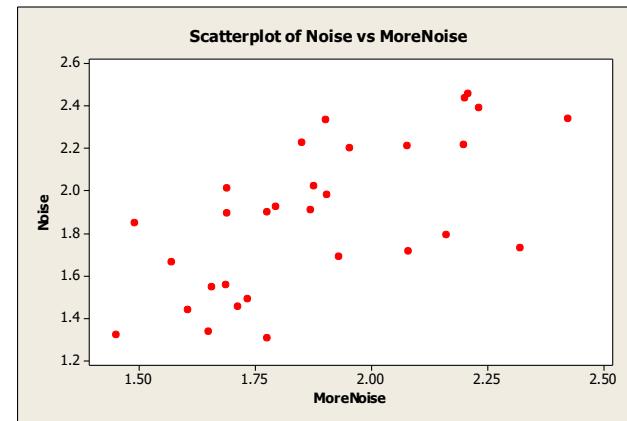
Correlations



$r = +1.0$

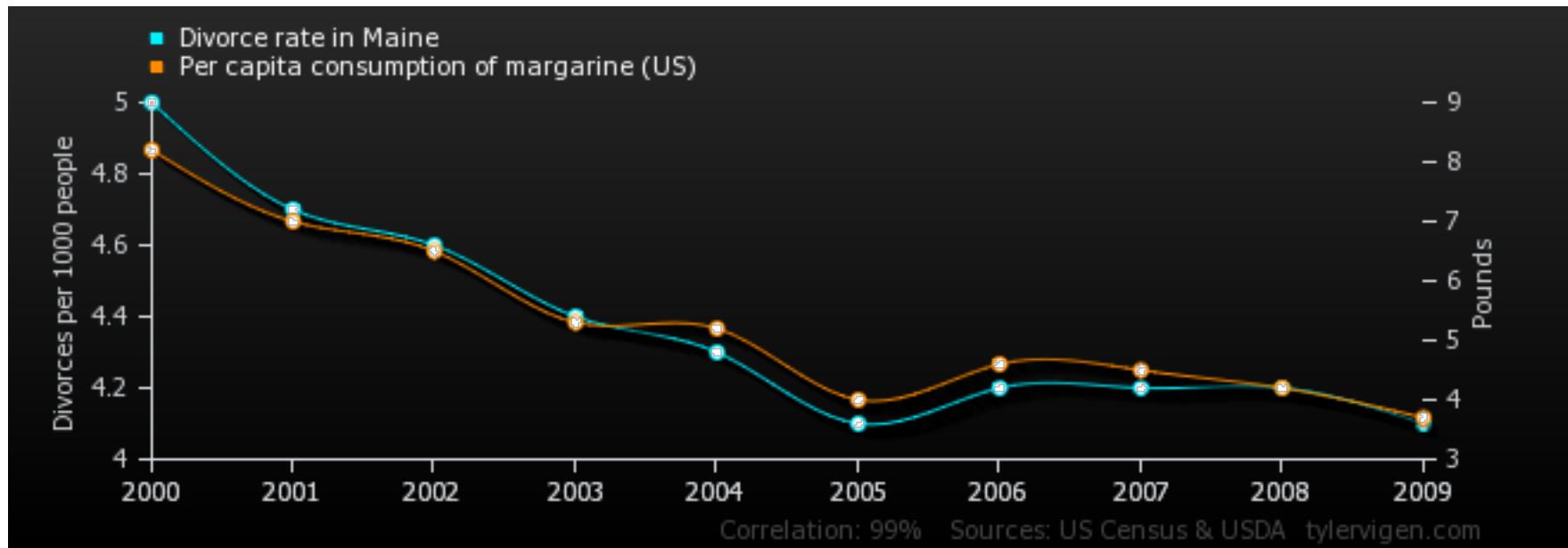


$r = 0.0$



$r = +0.5$

Correlation is not causation!!!

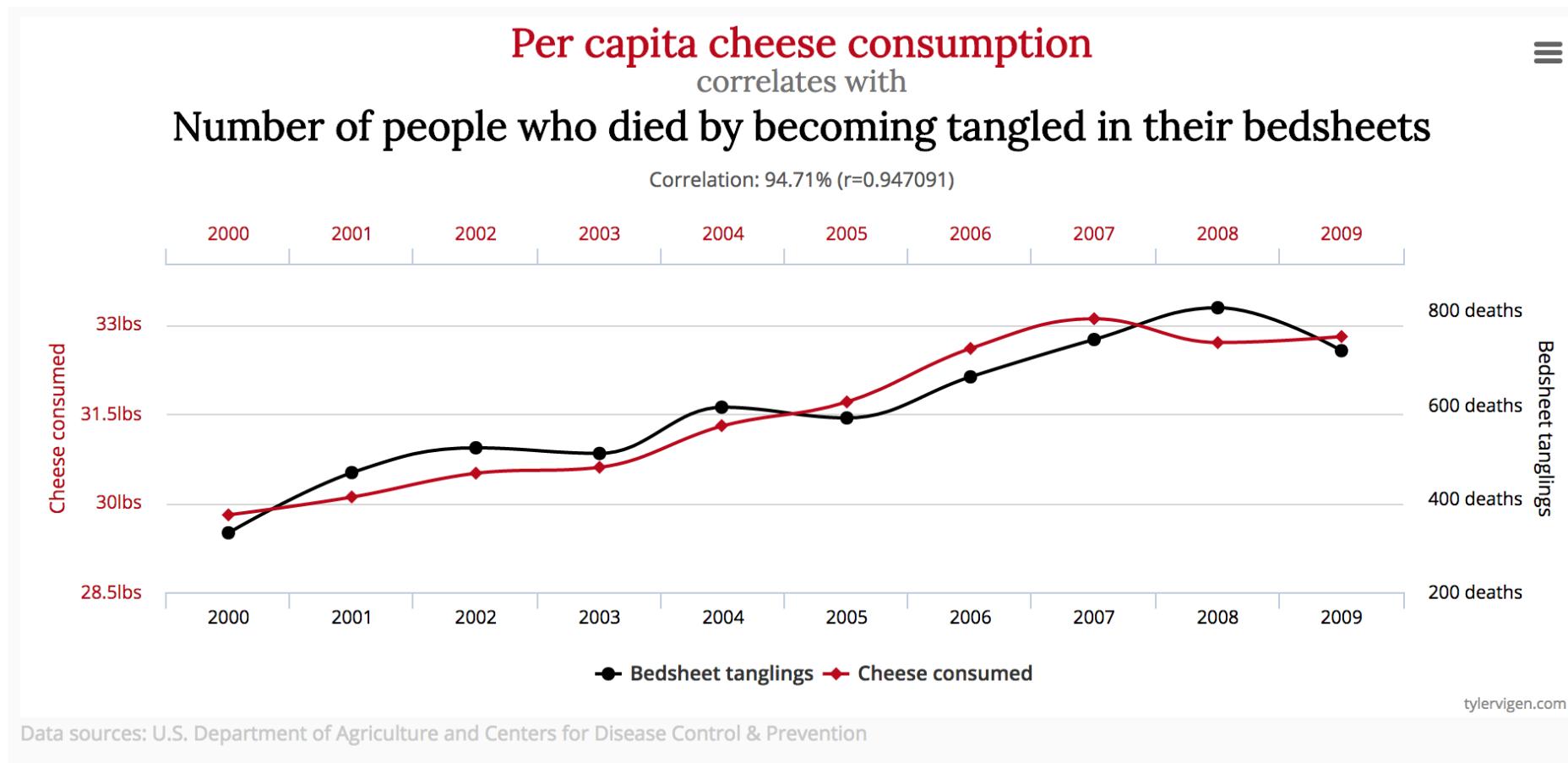


	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Divorce rate in Maine										
Divorces per 1000 people (US Census)	5	4.7	4.6	4.4	4.3	4.1	4.2	4.2	4.2	4.1
Per capita consumption of margarine (US)	8.2	7	6.5	5.3	5.2	4	4.6	4.5	4.2	3.7
Pounds (USDA)										

$r=0.993$

???????????

Just to drive the point home ...





Transformations

Transformations

- So, you've figured out that your data are:
 - Skewed
 - Have vastly different ranges across datasets and/or different units
- What do you do?

Transform the variables to:

- ease the validity and interpretation of data analyses
- change or ease the type of Stat/ML models you can use

Standardization

- Transforming the variable to a comparable metric
 - known unit
 - known mean
 - known standard deviation
 - known range
- Three ways of standardizing:
 - P-standardization (percentile scores)
 - Z-standardization (z-scores)
 - D-standardization (dichotomize a variable)

When you should always standardize

- When averaging multiple variables, e.g. when creating a socioeconomic status variable out of income and education.
- When comparing the effects of variables with unequal units, e.g. does age or education have a larger effect on income?



P-Standardization

- Every observation is assigned a number between 0 and 100, indicating the percentage of observation **beneath** it.
- Can be read from the cumulative distribution
- In case of knots: assign midpoints
- The median, quartiles, quintiles, and deciles are special cases of P-scores.

	rent	cum %	percentile
room 1	175	5,3%	5,3%
room 2	180	10,5%	10,5%
room 3	185	15,8%	15,8%
room 4	190	21,1%	21,1%
room 5	200	26,3%	26,3%
room 6	210	31,6%	36,8%
room 7	210	36,8%	36,8%
room 8	210	42,1%	36,8%
room 9	230	47,4%	47,4%
room 10	240	52,6%	55,3%
room 11	240	57,9%	55,3%
room 12	250	63,2%	65,8%
room 13	250	68,4%	65,8%
room 14	280	73,7%	73,7%
room 15	300	78,9%	81,6%
room 16	300	84,2%	81,6%
room 17	310	89,5%	89,5%
room 18	325	94,7%	94,7%
room 19	620	100,0%	100,0%

P-standardization

- Turns the variable into a ranking, i.e. it turns the variable into a ordinal variable.
- It is a **non-linear** transformation: relative distances change
- Results in a fixed mean, range, and standard deviation; $M=50$, $SD=28.6$.
 - This can change slightly due to knots
- A histogram of a P-standardized variable approximates a uniform distribution.

	rent	cum %	percentile
room 1	175	5,3%	5,3%
room 2	180	10,5%	10,5%
room 3	185	15,8%	15,8%
room 4	190	21,1%	21,1%
room 5	200	26,3%	26,3%
room 6	210	31,6%	36,8%
room 7	210	36,8%	36,8%
room 8	210	42,1%	36,8%
room 9	230	47,4%	47,4%
room 10	240	52,6%	55,3%
room 11	240	57,9%	55,3%
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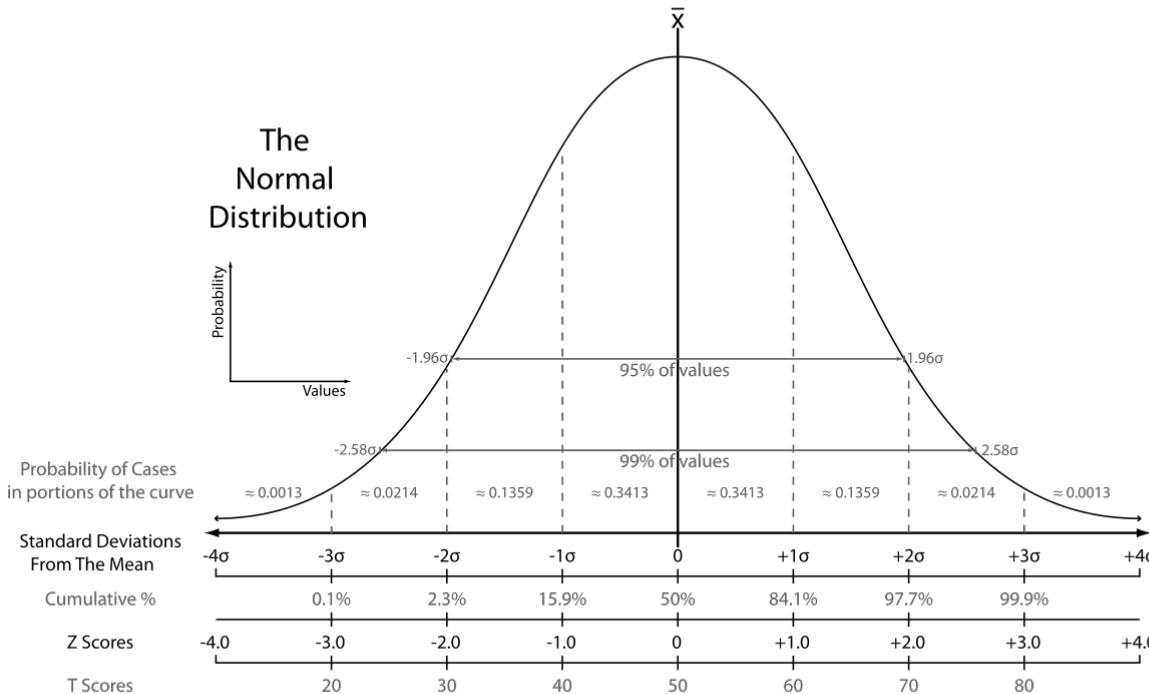
Centering And Scaling – Z-Standardization.

- Transform your data into a **unitless scale**
 - Put data into “standard deviations from the mean” units
 - This is called standardizing a variable, into standard units

- Given data points $x = x_1, x_2, \dots, x_n$:

$$z_i = \frac{(x_i - \bar{x})}{\text{sd}(x)}$$

- Translates x into a scaled and centered variable z
- What is the mean of z ??????????
- What is the standard deviation of z ??????????



Centering or Scaling

- Maybe you just want to center the data:

$$z_i = (x_i - \bar{x})$$

- What is the mean of z ??????????
- What is the standard deviation of z ??????????
- Maybe you just want to scale the data:

$$z_i = \frac{x_i}{\text{sd}(x_i)}$$

- What is the mean of z ??????????
- What is the standard deviation of z ??????????

Discrete to Continuous Variables

- Some models only work on continuous numeric data
- Convert a binary variable to a number ??????????
 - `health_insurance = {"yes", "no"} → {1, 0}`
- Why not {-1, +1} or {-10, +14}?
 - 0/1 encoding lets us say things like “if a person has healthcare then their income increases by \$X.”
 - Might need {-1,+1} for certain ML algorithms (e.g., SVM)

Discrete to Continuous Variables

- What about non-binary variables?
- My main transportation is a {BMW, Bicycle, Hovercraft}
- One option: { BMW → 1, Bicycle → 2, Hovercraft → 3 }
 - Problems ???????????
- **One-Hot Encoding:** convert a categorical variable with N values into a N-bit vector:
 - BMW → [1, 0, 0]; Bicycle → [0, 1, 0]; Hovercraft → [0, 0, 1]

```
# Converts dtype=category to one-hot-encoded cols
cols = ['my_transportation']
df = df.get_dummies( columns = cols )
```

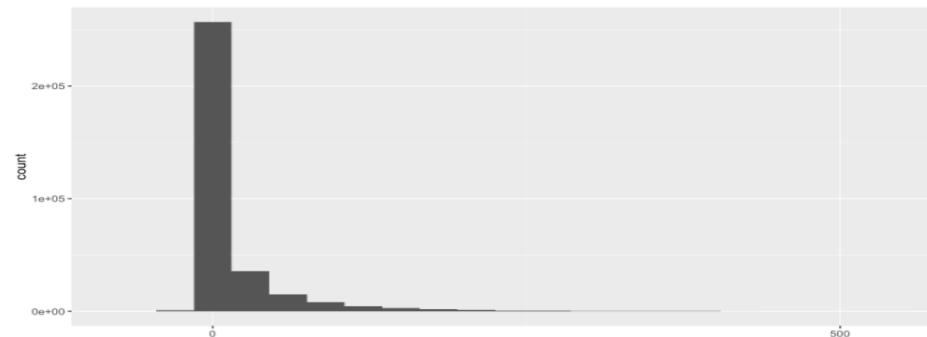
Continuous to discrete variables

- Do doctors prescribe a certain medication to older kids more often? Is there a difference in wage based on age?
- Pick a discrete set of bins, then put values into the bins
- Equal-length bins:
 - Bins have an equal-length range and skewed membership
 - Good/Bad ????????
- Equal-sized bins:
 - Bins have variable-length ranges but equal membership
 - Good/Bad ????????

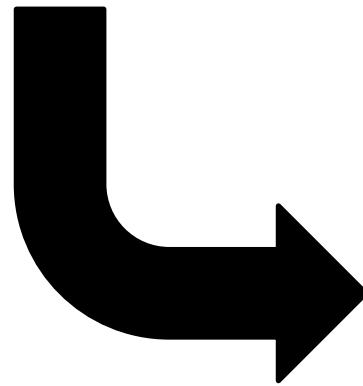
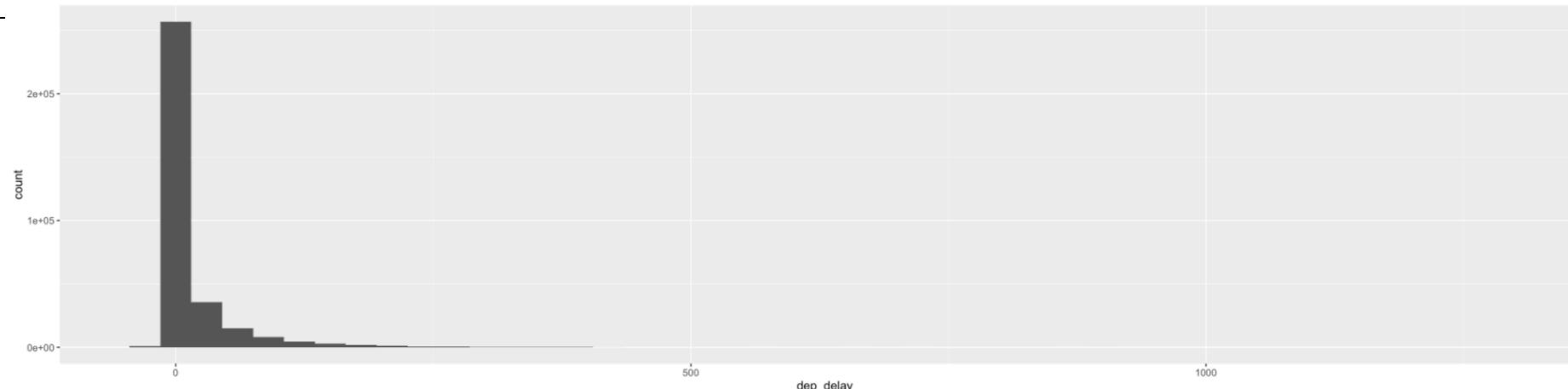


Skewed Data

- Skewed data often arises in multiplicative processes:
 - Some points float around 1, but one unlucky draw → 0
- Logarithmic transforms reduce skew:
 - If values are all positive, apply \log_2 transform
 - If some values are negative:
 - Shift all values so they are positive, apply \log_2
 - Signed log: $\text{sign}(x) * \log_2(|x| + 1)$



Skewed Data



log₂ transform
on airline
takeoff delays

