



# Banking bubbles and financial crises <sup>☆</sup>

Jianjun Miao <sup>a,b,\*</sup>, Pengfei Wang <sup>c</sup>

<sup>a</sup> Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215, USA

<sup>b</sup> CEMA, Central University of Finance and Economics, and AFR, Zhejiang University, China

<sup>c</sup> Department of Economics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong

Received 17 April 2013; final version received 4 February 2015; accepted 5 February 2015

Available online 14 February 2015

---

## Abstract

This paper develops a tractable macroeconomic model with a banking sector in which banks face endogenous borrowing constraints. There is no uncertainty about economic fundamentals. Banking bubbles can emerge through a positive feedback loop mechanism. Changes in household confidence can cause the bubbles to burst, resulting in a financial crisis. Credit policy can mitigate economic downturns. The welfare gain is larger when the government interventions are more front loaded, given that the government injects the same amount of liquidity in terms of present value. Bank capital requirements can prevent the formation of banking bubbles by limiting leverage, but if too restrictive will lead to less lending and hence lower production.

© 2015 Elsevier Inc. All rights reserved.

*JEL classification:* E2; E44; G01; G20

*Keywords:* Banking bubbles; Multiple equilibria; Financial crises; Credit policies; Capital requirements

---

---

<sup>☆</sup> We thank Zhiguo He for helpful discussions of the paper. We are especially grateful to the editor (Ricardo Lagos) and three anonymous referees for their insightful and constructive comments and suggestions. We have also benefited from comments by participants at 2013 AEA Meeting, the Macroeconomics Workshop at Shanghai University of Finance and Economics, 2013 SAIF Summer Institute of Finance, and the Conference on Financial and Macroeconomic Stability in Turkey. Jing Zhou provided excellent research assistance. First version: January 2012.

\* Corresponding author.

*E-mail addresses:* [miaoj@bu.edu](mailto:miaoj@bu.edu) (J. Miao), [pfwang@ust.hk](mailto:pfwang@ust.hk) (P. Wang).

## 1. Introduction

The recent financial crisis of 2007 resulted in the collapse of large financial institutions, the bailout of banks by national governments, and downturns in stock markets around the world. In addition, it contributed to persistently high unemployment, failures of key businesses, and declines in consumer wealth, real investment, and output. The crisis was caused by a complex interplay of valuation and liquidity problems in the U.S. banking system. Upon bursting of the U.S. housing bubble which peaked in 2007, the values of securities tied to U.S. real estate pricing began to plummet, damaging financial institutions globally. Concerns about bank solvency, declines in credit availability, and damaged investor confidence led the global stock markets to fall. Declines in bank lending reduced real investment and output, resulting in the Great Recession.

The recent financial crisis provides a challenge to macroeconomists. Traditional macroeconomic models typically assume perfect financial markets and ignore financial frictions. These models are not useful for understanding financial crises. Bernanke and Gertler [5], Carlstrom and Fuerst [11], Kiyotaki and Moore [25], and Bernanke, Gertler, and Gilchrist [6] introduce financial frictions into business cycle models. These models assume that financial frictions appear only in non-financial firms and treat financial intermediaries as a veil. They fail to capture the fact that the recent financial crisis featured a significant disruption of financial intermediation.

In this paper we develop a tractable macroeconomic model with a banking sector in which changes in household confidence can cause a financial crisis.<sup>1</sup> To focus on the impact of household confidence, we assume that there is no aggregate uncertainty about economic fundamentals. The key idea of the model is to introduce financial frictions into the banking sector in the form of endogenous borrowing constraints similar to those in Kehoe and Levine [24], Alvarez and Jermann [2], Albuquerque and Hopenhayn [1], Jermann and Quadrini [23], and Miao and Wang [31–33]. In our model households make bank deposits which become liabilities of the bank. The bank has limited commitment and may default on deposit liabilities. If the bank chooses to default, then depositors can seize a fraction of the bank capital. Instead of liquidating the seized bank capital, depositors can reorganize the bank and keep it running. The bank and depositors renegotiate deposit repayments by Nash bargaining. The threat value to depositors is the stock market value of the reorganized bank with the seized bank capital. Suppose that the bank has full bargaining power. Then deposits cannot exceed the threat value to depositors or the stock market value of the reorganized bank. This constraint is incentive compatible for both depositors and the bank. It also ensures that there is no default in an optimal contract.

We show that a banking bubble can exist given the endogenous borrowing constraint. The intuition is based on the following positive feedback loop mechanism: If both depositors and the bank hold optimistic beliefs about the bank value, then the bank wants to take more deposits and households are willing to make more deposits, in the hope that deposit repayments can be backed by a high bank value. The bank then uses the increased deposits to make more lending to non-financial firms. Consequently, the bank can make more profits, which make the bank

---

<sup>1</sup> In his testimony before the Committee on Banking, Housing, and Urban Affairs, U.S. Senate on September 23, 2008, Ben S. Bernanke argued that “investor concerns about financial institutions increased over the summer, as mortgage-related assets deteriorated further and economic activity weakened. Among the firms under the greatest pressure were Fannie Mae and Freddie Mac, Lehman Brothers, and, more recently, American International Group (AIG). As investors lost confidence in them, these companies saw their access to liquidity and capital markets increasingly impaired and their stock prices drop sharply.”

value indeed high, justifying the initial optimistic beliefs. We call this equilibrium a bubbly equilibrium.

Of course, there is another equilibrium, called the bubbleless equilibrium, in which no one believes in bubbles. In this case households place less deposits in the bank because of concerns that their deposits cannot be repaid in the future. Consequently, banks make less lending to non-financial firms, resulting in lower capital stock and lower output.

As in Blanchard and Watson [7], Weil [43], Kocherlakota [27], and Miao and Wang [31], we construct a third type of equilibrium in which households believe that banking bubbles may burst in the future with some probability. We show that, even though there is no shock to the fundamentals of the economy, changes in confidence can trigger a financial crisis. In particular, immediately following the collapse of the banking bubble, deposits shrink, lending falls, and credit spreads rise, causing both real investment and output to drop. Our model helps understand the recent Great Recession.<sup>2</sup>

As evidence, the top two panels of Fig. 1 illustrate the large fluctuations in the average market-to-book-assets ratio and the average market-to-book-equity ratio for 670 U.S. listed commercial banks.<sup>3</sup> As is well known, it is challenging to explain the stock market boom and bust based on fundamentals only. While there are other explanations, our theory suggests that the boom and bust may be due to bubbles and crashes arising from changes in confidence. The bottom two panels of Fig. 1 show that the growth rates of total deposits and total loans declined during the recent financial crisis, consistent with our model mechanism.<sup>4</sup>

During the recent financial crisis, the Federal Reserve implemented three general types of credit policies (Gertler and Kiyotaki [16]). The first is discount window lending. The Fed used the discount window to lend funds to commercial banks that in turn lent funds out to non-financial borrowers. The second is direct lending. The Fed lent directly in high-grade credit markets, funding assets that included commercial paper, agency debt, and mortgage-backed securities. The third is equity injections. The Treasury coordinated with the Fed to acquire ownership positions in commercial banks by injecting equity.

To assess the impact of credit policies, we introduce a central bank and a government in our model. We allow the central bank to act as an intermediary by borrowing funds from savers and then lending those funds to investors. Unlike private intermediaries, the central bank does not face constraints on its leverage ratio. There is no enforcement problem between the central bank and its creditors because it can commit to always honoring its debt. Following Gertler and Karadi [15] and Gertler and Kiyotaki [16], we study the aforementioned three types of credit policies, but we go one step further by analyzing the impact of different intensities of policies. We assume that the policies are applied only in the crisis period and the liquidity injections decay at an exponential rate. The government faces the choice of whether to inject a large amount of liquidity immediately which decays faster or to spread the liquidity over time which decays more slowly. We find that a more front-loaded policy leads to a larger welfare gain given that the government

<sup>2</sup> Our model is related to the view of Lucas and Stokey [28]: “What happened in September 2008 was a kind of bank run. Creditors lost confidence in the ability of investment banks to redeem short-term loans, leading to a precipitous decline in lending in the repurchase agreements (repo) market.”

<sup>3</sup> The market-to-book-assets ratio is equal to the ratio of the stock market price of the bank to the book value of assets per share. The market-to-book-equity ratio is equal to the ratio of the stock market price of the bank to the book value of equity per share. The data are quarterly and downloaded from Bloomberg and cover 670 U.S. listed commercial banks according to Global Industry Classification Standard.

<sup>4</sup> The data on deposits and loans cover all US commercial banks and are downloaded from the FRB of St Louis.

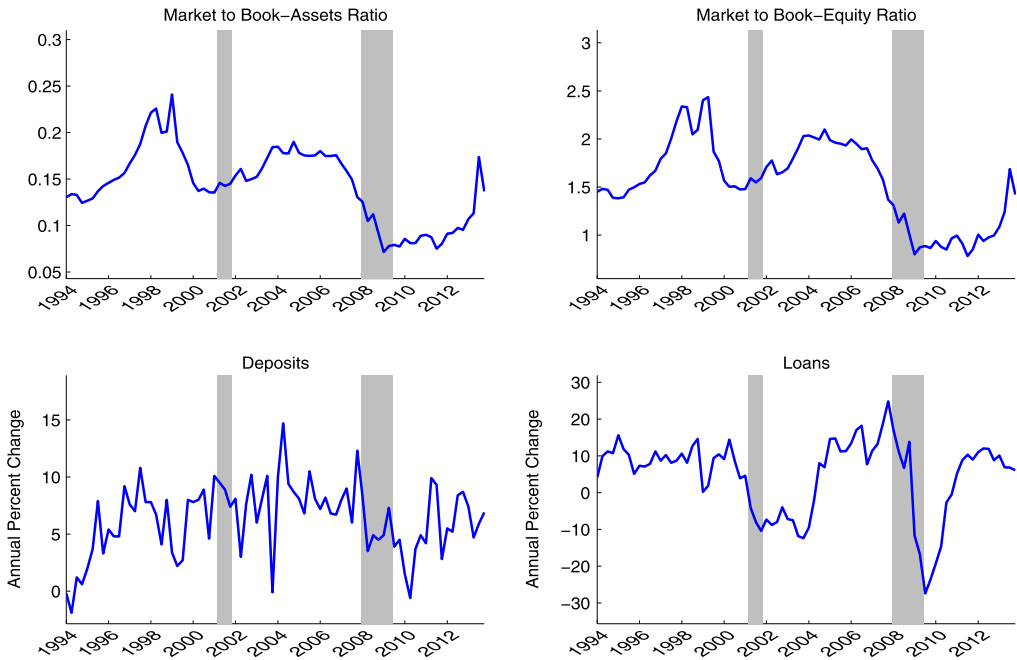


Fig. 1. Bank market-to-book ratios, total deposits, and total loans. The gray areas indicate NBER recession periods.

injects the same amount of liquidity in terms of present value.<sup>5</sup> There is a tradeoff between a large short-run stimulating effect and a small long-run recovery effect. In our numerical examples, the former effect dominates. Another difference between our study and the aforementioned studies is that we use global nonlinear solution methods instead of log-linear approximation methods. When the bubble bursts, the economy switches from one steady state to another. The log-linear approximation methods cannot be applied.

We use our model to study the role of capital requirements. Bank capital requirements ensure that banks are not making investments that increase the risk of default and that they have enough capital to sustain operating losses while still honoring deposit withdrawals. In our model there is no uncertainty about fundamentals and hence risk-taking behavior is not an issue. However, bank capital requirements can still help stabilize the banking system. We show that these requirements can help prevent the formation of a banking bubble. The intuition is that capital requirements limit leverage. When these requirements are sufficiently restrictive, banks cannot borrow excessively from households and hence the positive feedback loop discussed earlier cannot be initiated. Limiting leverage, however, comes at a cost — it will reduce lending to non-financial firms and in turn reduce investment and output.

Our paper is related to three strands of literature. First, it is related to the recent literature that incorporates a financial sector into macroeconomic models (e.g., Gertler and Kiyotaki [16], Gertler and Karadi [15], Gertler, Kiyotaki, and Queralto [18], Brunnermeier and Sannikov [10],

<sup>5</sup> Many people believed that Japan's unconventional monetary policy would have been more efficient had it been more dramatic after the bursting of the asset bubble in the early 1990s. In April of 2013, Japan's central bank announced that it would expand its asset purchase program by US\$1.4 trillion in two years.

and He and Krishnamurthy [19]). For the financial sector to play an important role, various frictions must be introduced into it to invalidate the Modigliani and Miller theorem. Frictions are typically modeled in the form of borrowing constraints. Borrowing constraints can be micro-founded by agency issues or moral hazard problems. Our paper differs from this literature in two respects. First, there is no uncertainty about economic fundamentals in our model. Unlike the studies cited earlier, which assume that financial crises are triggered by some exogenous shocks (e.g., capital quality shocks or shocks to net worth), we show that they are instead triggered by changes in agents' beliefs about the stock market value of banks. While shocks to fundamentals can cause a financial crisis, our model shows that the loss of confidence can also play an important role. Second, we introduce borrowing constraints from an optimal contracting problem with limited commitment. This contracting problem is tractable to analyze in our deterministic setup.

The second related strand of literature is the one on rational bubbles.<sup>6</sup> As is well known, generating rational bubbles in infinite-horizon models is nontrivial (Santos and Woodford [38]). Rational bubbles are often studied in the overlapping-generations framework (e.g., Tirole [41] and Martin and Ventura [29]), and can be generated in infinite-horizon models with borrowing constraints (e.g., Kocherlakota [26,27], Hirano and Yanagawa [21], and Wang and Wen [42]). One limitation of all these models is that they study bubbles on intrinsically useless assets or on assets with exogenously given payoffs. Miao and Wang [31] provide a theory of credit-driven stock price bubbles in an infinite-horizon model with production. Stock dividends are endogenously affected by bubbles. Miao and Wang [32,33], Miao, Wang, and Xu [35], and Miao, Wang, and Zhou [36] apply this theory to study other macroeconomic issues such as total factor productivity, endogenous growth, housing bubbles, and unemployment. The present paper borrows ideas from Miao and Wang [31] in that banking bubbles in this paper are created by a similar positive feedback loop mechanism.

Finally, our paper is related to the recent macroeconomics literature that explains the Great Recession as a self-fulfilling crisis (e.g., Boissay, Collard, and Smets [8], Perri and Quadrini [37], and Gertler and Kiyotaki [17]). Our paper is more closely related to the one by Gertler and Kiyotaki [17] who extend the Diamond and Dybvig [12] bank run model in a dynamic stochastic general equilibrium framework. A bank run occurs when a large number of bank customers withdraw their deposits fearing the bank might fail. To sustain a bank run equilibrium, the following positive feedback loop mechanism must be at work: As more people withdraw their deposits, the likelihood of default increases, and this encourages further withdrawals. Mass panic can cause a sound bank to fail. However, if everyone believes that the bank is sound, then no one would make large withdrawals and there would be no bank run. Both types of equilibria are self-fulfilling. The literature on bank runs typically considers an essentially static setup (or a three-period setup) without explicit dynamics. He and Xiong [20] develop a dynamic model of debt run and derive a unique equilibrium. Unlike their model, ours has multiple equilibria. Like Gertler and Kiyotaki [17], our paper differs from the bank run literature in that our model features both the financial and non-financial sectors in a dynamic macroeconomic framework. But unlike them, we focus on the question of how bubbles and crashes in banking stocks can create a financial crisis and affect the real economy.

<sup>6</sup> See Scheinkman and Xiong [39] for a model of bubbles based on heterogeneous beliefs. See Shiller [40] for a discussion of bubbles based on irrational exuberance. See Brunnermeier [9] and Miao [30] for surveys of various theories of bubbles.

The remainder of the paper proceeds as follows. Section 2 presents a baseline model. Section 3 provides equilibrium characterizations. Section 4 studies an equilibrium with stochastic bubbles and welfare implications. Section 5 analyzes the role of bank capital requirements. Section 6 examines three types of credit policies. Section 7 concludes. Technical details are relegated to Appendix A.

## 2. A baseline model

We start with a baseline model with deterministic bubbles. Consider a deterministic economy consisting of households, non-financial firms and financial intermediaries (or simply banks). We do not consider government or monetary authority for now. Time is continuous and infinite.

### 2.1. Households

Following Gertler and Karadi [15] and Gertler and Kiyotaki [16], we formulate the household sector in a way that maintains the tractability of the representative agent approach. There is a continuum of identical households of measure unity. Each household consumes, saves and supplies labor. Normalize labor supply to unity. Households save by lending funds to banks. Within each household, there are two types of members: workers and bankers. Workers supply labor and return their wages to the household. Bankers in each household manage a bank and transfers dividends back to the household. The household owns the bank and deposits funds in banks. (These banks may be owned by other households.) Within the family, there is perfect consumption insurance. All banks are identical and have a unit measure.

Households do not hold capital directly, but own all non-financial firms. They trade bank stocks in the market. The representative household derives utility from consumption  $\{C_t\}$  according to the linear utility,  $\int_0^\infty e^{-rt} C_t dt$ , where  $r$  is the subjective discount rate. Because of linear utility,  $r$  is also equal to the deposit rate. The budget constraint is given by

$$V_t d\psi_t + dD_t = \psi_t C_t^b dt + r D_t dt - C_t dt + w_t dt + \Pi_t dt, \quad (1)$$

where  $V_t$ ,  $\psi_t$ ,  $D_t$ ,  $C_t^b$ ,  $w_t$ , and  $\Pi_t$  represent the bank stock price, holdings of bank shares, deposits, bank dividends, the wage rate, and profits from non-financial firms, respectively. We normalize the supply of bank stocks to unity.

### 2.2. Banks

Banks lend funds obtained from households to non-financial firms. They act as specialists that channel funds from savers to investors. Moreover, they engage in asset transformation in that they hold long-term assets and fund these assets with short-term deposit liabilities beyond their own equity capital. Banks in our model can be interpreted as financial intermediaries that include the shadow banking system, investment banks, and commercial banks. For simplicity, we do not consider deposit insurance. Suppose that there is no liquidity risk and no interbank market. We consider a representative bank's behavior. We will make assumptions to ensure a linearity structure so that aggregation is made easy.

At each time  $t$ , let  $N_t$  denote the bank's net worth before dividends,  $D_t$  the deposits raised from households, and  $S_t$  the loans made to non-financial firms. Non-financial firms use loans to finance capital expenditures  $K_t$ . We consider financial frictions in the banking sector only. We assume that there is no financial or real friction in the non-financial firm sector, and no friction

in transferring funds between a bank and non-financial firms. Thus the price of capital is equal to unity. The bank's balance sheet satisfies

$$N_t + D_t = S_t = K_t. \quad (2)$$

The banker maximizes the stock market value of the bank  $V_t$  at any time  $t$ . Let  $V_t$  be a function of the state variable  $N_t$ , denoted by  $V_t(N_t)$ . Note that we have suppressed the aggregate state variables as arguments in the value function. This value function satisfies the dynamic programming equation

$$V_t(N_t) = \max_{C_t^b, D_t} \int_t^T e^{-r(s-t)} C_s^b ds + e^{-r(T-t)} V_T(N_T), \text{ any } T > t, \quad (3)$$

subject to some constraints to be specified next.

The first constraint is the flow-of-funds constraint given by

$$dN_t = r_{kt} N_t dt + (r_{kt} - r) D_t dt - C_t^b dt, \quad (4)$$

where  $r_{kt}$  represents the lending rate and  $r$  is the deposit rate. As long as the lending rate  $r_{kt}$  is higher than the deposit rate  $r$ , the bank prefers to keep accumulating assets until it overcomes financial frictions in the form of borrowing constraints. In the literature, there are several modeling strategies to limit bankers' ability to save to overcome borrowing constraints. For example, Gertler and Karadi [15] and Gertler and Kiyotaki [16] assume that a banker exits in the next period with a constant probability and becomes a worker. His position is then replaced by a randomly selected worker, keeping the number in each occupation constant. Brunnermeier and Sannikov [10] assume that borrowers are more impatient than lenders.

In this paper we assume that the banker must pay a fraction  $\theta \in (0, 1)$  of the bank's net worth as dividends

$$C_t^b \geq \theta N_t. \quad (5)$$

The motivation for such a constraint requires a richer micro-founded model that may involve a combination of asymmetric information and a divergence of interests between shareholders and managers.<sup>7</sup> For our purpose, one may view (5) as a modeling shortcut for forcing banks to pay out dividends, instead of keeping accumulating assets. In the Online Appendix, we show that using the approach of Gertler and Karadi [15] or Gertler and Kiyotaki [16] will not change our key insights.

The second constraint is that deposits and net worth cannot be negative, i.e.,

$$D_t \geq 0, \quad N_t \geq 0. \quad (6)$$

The third constraint is a borrowing constraint. As long as the lending rate  $r_{kt}$  is higher than the deposit rate  $r$ , the bank will expand its assets as much as possible by borrowing additional funds from households. To limit this behavior, we introduce the following borrowing constraint:

$$D_t \leq V_t(\xi N_t), \quad (7)$$

<sup>7</sup> Note that (5) rules out new equity issuance, i.e., negative dividends. In the Online Appendix, we show that, if banks can raise a sufficient amount of new equity, then they can overcome borrowing constraints and achieve the first-best.

where  $\xi \in (0, 1]$  represents the degree of financial frictions. This is the key innovation of the model and requires some explanations. The micro-foundation of this borrowing constraint is based on the optimal contract between the bank and households (depositors) with limited commitment/enforcement. It is related to the formulation of Jermann and Quadrini [23].

We consider a discrete-time approximation. Suppose that time is denoted by  $t = 0, dt, 2dt, 3dt, \dots$ . At each time  $t$ , the contract specifies a deposit  $D_t$  to the bank and a repayment  $e^{rdt} D_t$  to the households at time  $t + dt$ . The bank pays out dividends  $C_t^b dt$ . It then lends out  $N_t + D_t - C_t^b dt$  to non-financial firms and earns returns  $e^{r_k dt} (N_t + D_t - C_t^b dt)$ . Thus its flow-of-funds constraint is given by

$$N_{t+dt} = e^{r_k dt} (N_t + D_t - C_t^b dt) - e^{rdt} D_t.$$

Noting that  $e^{r_k dt} C_t^b dt \approx C_t^b dt$  up to the order of  $dt$ , we rewrite the preceding equation as

$$N_{t+dt} = e^{r_k dt} (N_t + D_t) - C_t^b dt - e^{rdt} D_t.$$

If the bank decides to repay deposits to the households, its value is given by:

$$\begin{aligned} & C_t^b dt + e^{-rdt} V_{t+dt} (N_{t+dt}) \\ &= e^{r_k dt} N_t + (e^{r_k dt} - e^{rdt}) D_t - N_{t+dt} + e^{-rdt} V_{t+dt} (N_{t+dt}), \end{aligned}$$

where we have used the preceding flow-of-funds constraint to substitute out  $C_t^b$ . However, the bank has limited commitment. It may take deposits  $D_t$  and default on the deposit liabilities by not repaying  $e^{rdt} D_t$ . If that happens, the depositors can capture a fraction  $\xi$  of the bank's end-of-period net worth (or bank capital)  $N_{t+dt}$ .<sup>8</sup> Instead of liquidating these assets, the depositors can reorganize the bank and keep it running in the next period using recovered bank capital  $\xi N_{t+dt}$ . The bank and depositors renegotiate deposit repayments by Nash bargaining. Suppose that the bank has all bargaining power. The depositors can only obtain the threat value, which is the stock market value of the reorganized bank  $e^{-rdt} V_{t+dt} (\xi N_{t+dt})$ . The bank obtains the remaining value.

Specifically, the bank and the depositors solve the following Nash bargaining problem at the end of period  $t$ ,

$$\max_{V^d} \left( e^{-rdt} V_{t+dt} (N_{t+dt}) - V^d \right)^\vartheta \left( V^d - e^{-rdt} V_{t+dt} (\xi N_{t+dt}) \right)^{1-\vartheta}$$

subject to

$$V^d \geq e^{-rdt} V_{t+dt} (\xi N_{t+dt}), \quad e^{-rdt} V_{t+dt} (N_{t+dt}) \geq V^d,$$

where  $\vartheta \in [0, 1]$  represents the bargaining power of the bank. Solving yields the value to the depositors,

$$V^d = \vartheta e^{-rdt} V_{t+dt} (\xi N_{t+dt}) + (1 - \vartheta) e^{-rdt} V_{t+dt} (N_{t+dt}).$$

When  $\vartheta = 1$ , we obtain  $V^d = e^{-rdt} V_{t+dt} (\xi N_{t+dt})$ . Thus the value to the defaulting bank is given by  $e^{-rdt} V_{t+dt} (N_{t+dt}) - e^{-rdt} V_{t+dt} (\xi N_{t+dt})$ .

<sup>8</sup> We can also assume that the depositors capture a fraction of the bank's beginning-of-period net worth  $N_t$ . In the continuous-time limit, this assumption gives the same borrowing constraint as (7). The equilibrium characterization will be unaffected.



Enforcement of the contract requires that the value of not defaulting be not smaller than the value of defaulting, i.e.,

$$\begin{aligned} & e^{r_{kt}dt} N_t + \left( e^{r_{kt}dt} - e^{rdt} \right) D_t - N_{t+dt} + e^{-rdt} V_{t+dt} (N_{t+dt}) \\ & \geq e^{r_{kt}dt} N_t + e^{r_{kt}dt} D_t - N_{t+dt} + e^{-rdt} V_{t+dt} (N_{t+dt}) - e^{-rdt} V_{t+dt} (\xi N_{t+dt}). \end{aligned}$$

This incentive constraint ensures that there is no default in an optimal contract. Simplifying this constraint yields

$$e^{rdt} D_t \leq e^{-rdt} V_{t+dt} (\xi N_{t+dt}).$$

Taking the continuous-time limits as  $dt \rightarrow 0$  yields (7).

We emphasize that the process of default, reorganization, and bargaining is off the equilibrium path. It is used as a punishment to support a no-default equilibrium studied in this paper. The incentive constraint (7) ensures that default never occurs in equilibrium. There could be many other types of punishment and incentive constraints. These incentive constraints will lead to other types of borrowing constraints. In the Online Appendix, we introduce three other types and study the conditions under which they can lead to bubbles.<sup>9</sup>

### 2.3. Non-financial firms

There is a continuum of identical non-financial firms of measure unity. Each non-financial firm produces output using a constant-returns-to-scale technology with capital and labor inputs. We may write the aggregate production function as

$$Y_t = K_t^\alpha L_t^{1-\alpha}, \quad \alpha \in (0, 1),$$

where  $Y_t$ ,  $K_t$ , and  $L_t$  represent aggregate output, capital and labor respectively. Competitive profit maximization implies

$$(1 - \alpha) K_t^\alpha L_t^{-\alpha} = w_t.$$

It follows that the gross profits per unit of capital are given by

$$\frac{Y_t - w_t L_t}{K_t} = \alpha K_t^{\alpha-1} L_t^{1-\alpha}.$$

Assume that there is no real or financial friction in the non-financial firm sector. A non-financial firm obtains funds from banks by issuing equity at the price of unity. The firm uses the funds to purchase capital  $K_t$  and returns dividends  $\alpha K_t^{\alpha-1} L_t^{1-\alpha}$  to the banks. Thus the lending rate is equal to the capital (or equity) return

$$r_{kt} = \alpha K_t^{\alpha-1} L_t^{1-\alpha} - \delta, \quad (8)$$

where  $\delta$  is the depreciation rate of capital. Eq. (8) also implies that the marginal product of capital  $\alpha K_t^{\alpha-1} L_t^{1-\alpha}$  is equal to the user cost of capital, which is equal to the sum of the lending rate  $r_{kt}$  and the depreciation rate  $\delta$ .

<sup>9</sup> In the Online Appendix, we also show that a banking bubble cannot exist in a simplified version of the models of Gertler and Karadi [15] and Gertler and Kiyotaki [16]. But a banking bubble can exist with a suitably modified dividend policy.

## 2.4. Competitive equilibrium

A competitive equilibrium consists of quantities  $\{C_t, C_t^b, K_t, N_t, D_t, L_t\}$  and prices  $\{V_t, r_{kt}, w_t\}$  such that households, bankers, and firms optimize and markets clear so that  $\psi_t = 1$ ,  $L_t = 1$ , and

$$dK_t = K_t^\alpha L_t^{1-\alpha} dt - \delta K_t dt - C_t dt. \quad (9)$$

## 3. Equilibrium characterizations

In this section we first analyze a single bank's decision problem taking prices  $\{r_{kt}, w_t\}$  as given. We then conduct aggregation and characterize equilibrium by a system of differential equations. Finally, we study bubbleless and bubbly equilibria.

### 3.1. Optimal contract

We start by solving the optimal contracting problem (3) subject to (4), (5), (6), and (7), taking  $r_{kt}$  as given. Conjecture that the stock market value of the bank takes the following form:

$$V_t(N_t) = Q_t N_t + B_t, \quad (10)$$

where  $Q_t$  and  $B_t$  are to-be-determined aggregate states that are independent of the individual bank's characteristics. We may interpret  $Q_t$  as the shadow price of net worth. Because of limited liability, we only consider the solution with  $B_t \geq 0$  for all  $t$ . We may interpret  $B_t$  as a bubble component of the stock market value of the bank. As will become clear below, one may also interpret it as a self-fulfilling component or a belief component if one wants to avoid using the term "bubble." We interpret the component  $Q_t N_t$  as the fundamental value, which is generated by the fundamental — bank capital.

Given the preceding conjectured value function, we can write the Bellman equation as

$$r(Q_t N_t + B_t) = \max_{C_t^b, D_t} C_t^b + Q_t (r_{kt} N_t + (r_{kt} - r) D_t - C_t^b) + N_t \dot{Q}_t + \dot{B}_t, \quad (11)$$

subject to (5), (6), and the borrowing constraint

$$D_t \leq \xi Q_t N_t + B_t. \quad (12)$$

We summarize the solution to the contracting problem in the following:

**Proposition 1.** Suppose that

$$0 < \xi < \frac{\theta}{r} - \frac{r}{\theta} - 1. \quad (13)$$

Then  $Q_t$ ,  $B_t$  and  $N_t$  satisfy the following differential equations in a neighborhood of the steady states characterized in Propositions 3 and 4<sup>10</sup>:

$$r Q_t = Q_t [r_{kt} + (r_{kt} - r) \xi Q_t] + \theta(1 - Q_t) + \dot{Q}_t, \quad (14)$$

$$r B_t = Q_t (r_{kt} - r) B_t + \dot{B}_t, \quad (15)$$

$$\dot{N}_t = (r_{kt} - \theta + (r_{kt} - r) \xi Q_t) N_t + (r_{kt} - r) B_t, \quad (16)$$

<sup>10</sup> We use  $\dot{X}_t$  to denote  $dX_t/dt$  for any variable  $X_t$ .

and the transversality condition

$$\lim_{T \rightarrow \infty} e^{-rT} Q_T N_T = \lim_{T \rightarrow \infty} e^{-rT} B_T = 0, \quad (17)$$

where  $N_0$  is given.

We will show in Propositions 3 and 4 that  $Q_t > 1$  and  $r_{kt} > r$  in a neighborhood of a steady state. We can then describe the intuition behind Proposition 1 as follows: If  $Q_t > 1$ , then dividend constraint (5) must bind. Paying out a dollar of dividend gives the banker a dollar of benefit. But retaining that one dollar raises the marginal value of the bank stock by  $Q_t$  dollars. If  $Q_t > 1$ , then the bank prefers to reduce dividend payment until the constraint (5) binds. If  $r_{kt} > r$ , then the bank prefers to lend as much as possible by borrowing from depositors until the borrowing constraint (12) binds.

Substituting the conjectured value function in (10) and the decision rules described above into the Bellman equation (11) and matching coefficients, we then obtain differential equations (14) and (15) for the *nonpredetermined* variables  $Q_t$  and  $B_t$ . The initial values of  $Q_t$  and  $B_t$  must be endogenously determined. Eq. (14) is an asset-pricing equation. The left-hand side of this equation gives the return on bank capital. The right-hand side consists of dividends and capital gains. Dividends consist of returns from bank lending net of deposit payments and bank payouts. Eq. (15) is also an asset-pricing equation that will be explained later. Eq. (16) follows from the flow-of-funds constraint.

There are two types of solutions to these differential equations. If both depositors and the bank believe that the bank stock has a low value in that  $B_t = 0$  for all  $t$ , then the solution to the Bellman equation is characterized by Eq. (14) only. If both depositors and the bank believe that the bank stock has a high value because it contains a bubble component  $B_t > 0$ , then the bubble relaxes the borrowing constraint (12) and allows the bank to attract more deposits  $D_t$ . This allows the bank to make more loans ( $D_t + N_t$ ) and generate more profits when  $r_{kt} > r$ , justifying the initial belief of a high value. This positive feedback loop mechanism can support a bubble.

Note that both  $B_t = 0$  and  $B_t > 0$  can be a solution to the Bellman equation because the Bellman equation does not give a contraction mapping due to the presence of the value function in the incentive constraint (7). In the partial equilibrium context with  $r_{kt}$  exogenously given,  $V_t(N_t) = Q_t N_t + B_t$  with  $B_t > 0$  gives the maximal value function and hence is the unique solution to the bank's optimization problem. However, since  $V_t(N_t)$  is the stock market value of the bank, it is prone to speculation in general equilibrium. We will show later that both  $B_t = 0$  and  $B_t > 0$  can be supported in general equilibrium under some conditions. That is, our model has multiple equilibria. In general equilibrium,  $r_{kt}$  will be endogenously determined and hence  $Q_t$  will be different in different equilibria. This reflects the usual notion of a competitive equilibrium: Given a price system, individuals optimize. If this price system also clears all markets, then it is an equilibrium system. There could be multiple equilibria with different price systems. And different price systems would generate different optimization problems with different sets of constraints.

To further understand the interpretation of  $B_t$  as a bubble, we rewrite (15) as

$$\underbrace{r B_t}_{\text{return}} = \underbrace{Q_t (r_{kt} - r) B_t}_{\text{"dividend yields"}} + \underbrace{\dot{B}_t}_{\text{capital gains}}. \quad (18)$$

This equation says that the return on the bubble is equal to the capital gains plus "dividend yields" as in the usual asset-pricing equation. The key difference is that the "dividend yields" here are

not fundamentals. In particular, if people believe  $B_t = 0$ , then the dividend yields are zero. As in Miao and Wang [31–33], the dividend yields here come from the liquidity premium provided by the bubble. The intuition is as follows. One dollar of the bubble allows the bank to relax the borrowing constraint by one dollar. This allows the bank to attract one more dollar of deposits and hence makes one more dollar of loans, thereby raising bank net worth by  $r_{kt} - r$  dollars. Thus the net benefit to the bank is  $(r_{kt} - r)$  times the shadow price  $Q_t$  of its net worth.

The equilibrium restriction on the bubble on intrinsically useless assets or on assets with exogenously given payoffs studied in the literature is different from (18). In particular, there is no dividend yield term. This means that the growth rate of the bubble is equal to the interest rate. The transversality condition in (17) can rule out such a bubble in infinite-horizon models. By contrast, because of the dividend yields term in (18), the transversality condition cannot rule out bubbles in our model.

### 3.2. Equilibrium system

We now aggregate individual decision rules and impose market-clearing conditions to derive the equilibrium system. In particular, we need to endogenize the lending rate  $r_{kt}$ .

**Proposition 2.** *Under condition (13), the three variables  $(B_t, Q_t, N_t)$  satisfy the equilibrium system consisting of (14), (15), and (16) in a neighborhood of the steady states characterized in Propositions 3 and 4, where*

$$r_{kt} = \alpha ((\xi Q_t + 1) N_t + B_t)^{\alpha-1} - \delta. \quad (19)$$

The transversality condition (17) also holds.

Note that  $Q_t > 1$  and  $r_{kt} > r$  in a neighborhood of a steady state. Once we obtain  $B_t$ ,  $Q_t$  and  $N_t$ , we can then derive the equilibrium capital, output, consumption, and wage as follows:

$$K_t = (\xi Q_t + 1) N_t + B_t, \quad Y_t = K_t^\alpha, \quad w_t = (1 - \alpha) K_t^{\alpha-1}, \quad C_t = K_t^\alpha - \delta K_t - \dot{K}_t.$$

As discussed in the previous subsection, there may be two types of equilibria. In a bubbleless equilibrium,  $B_t = 0$  for all  $t$ . In a bubbly equilibrium,  $B_t > 0$  for all  $t$ . Note that we focus on *symmetric* bubbly equilibria in which all banks have the same size of the bubble  $B_t$ . This variable is nonpredetermined and its initial value is endogenously determined in equilibrium. It is possible to have another type of equilibria in which agents believe that a reorganized bank upon default loses a fraction of its bubble value instead of keeping all of its bubble value. In this case a bubbly equilibrium can still exist when the fraction is sufficiently close to one (see the Online Appendix). In order to deliver our insights in the simplest possible way, we will focus on symmetric equilibria characterized in Proposition 2. To analyze the existence of these equilibria, we will first study the steady state in which all aggregate variables are constant over time. We will then study local dynamics around a steady state.

### 3.3. First-best benchmark

We start with the first-best benchmark in which the lending rate is equal to the deposit rate,  $r_{kt}^{FB} = r$ . In addition, the shadow price of net worth is equal to one,  $Q_t^{FB} = 1$ . In this case there is no borrowing constraint (7). The first-best capital stock  $K^{FB}$  is equal to a constant:

$$K^{FB} = \left( \frac{r + \delta}{\alpha} \right)^{\frac{1}{\alpha-1}}.$$

Banks transfer deposits  $D^{FB} = K^{FB}$  to non-financial firms and make zero profits. The first-best consumption is equal to  $C^{FB} = (K^{FB})^\alpha - \delta K^{FB}$ .

### 3.4. Bubbleless equilibrium

Now we introduce the borrowing constraint (7) and analyze the bubbleless equilibrium in which  $B_t = 0$  for all  $t$ . We first consider the steady state. We use a variable without the subscript  $t$  to denote its steady state value and with an asterisk to denote its bubbleless steady state value.

**Proposition 3.** *If condition (13) holds, then a steady-state equilibrium  $(Q^*, N^*)$  without banking bubbles exists. In this equilibrium,*

$$r_k^* - r = \frac{r(\theta - r)}{r + \xi\theta} > 0, \quad Q^* = \frac{\theta}{r} > 1,$$

$$N^* = \frac{1}{\xi Q^* + 1} \left( \frac{r_k^* + \delta}{\alpha} \right)^{\frac{1}{\alpha-1}}.$$

Under condition (13),  $\theta > r$  so that  $Q^* > 1$  and  $r_k^* > r$ . We then apply Propositions 1 and 2 in a neighborhood of the bubbleless steady state to derive the equilibrium system. In fact, the weaker condition  $\theta > r$  is sufficient for a bubbleless equilibrium to exist.

In a steady state, Eq. (4) implies that

$$0 = r_k N + (r_k - r) D - \theta N,$$

where we have used the fact that  $C^b = \theta N$ . Thus we obtain the steady-state leverage ratio

$$\frac{K}{N} = \frac{N + D}{N} = \frac{\theta - r}{r_k - r}. \quad (20)$$

In a bubbleless steady state we substitute  $r_k^*$  given in Proposition 3 into the preceding equation and show that  $K/N > 1$ .

The bubbleless equilibrium dynamics are characterized by the system of differential equations in Proposition 2, where  $B_t = 0$  for all  $t$ . This nonlinear system has no closed-form solution. But it is straightforward to prove that the steady state is a saddle point. We can also solve the equilibrium system numerically using the finite-difference method.

### 3.5. Deterministic banking bubbles

In this subsection we study the equilibrium with banking bubbles in which  $B_t > 0$  for all  $t > 0$ . We use a variable with a superscript  $b$  to denote its bubbly steady-state value (except for  $B$ ).

**Proposition 4.** *Under condition (13), both steady-state equilibria with and without banking bubbles exist. In the bubbly steady-state equilibrium,*

$$Q^b = \frac{r + \theta}{\theta - \xi r} > 1, \quad (21)$$

$$r_k^b - r = \frac{r(\theta - r\xi)}{r + \theta} > 0, \quad (22)$$

$$\frac{B}{N^b} = \frac{-(r_k^b - \theta + (r_k^b - r)\xi Q^b)}{r_k^b - r} > 0. \quad (23)$$

Moreover,  $r_k^b < r_k^*$  and hence  $K^b > K^*$ .

Condition (13) is derived from  $B/N^b > 0$ . This condition (13) that  $Q^b > 1$  and  $r_k^b > r$ . We can then apply Propositions 1 and 2 in a neighborhood of the bubbly steady state. Moreover it implies that a bubbleless equilibrium exists by Proposition 3.

Using Propositions 3 and 4, we can easily show that  $r_k^b < r_k^*$ . Thus the capital stock in the bubbly steady state is larger than that in the bubbleless steady state. The intuition is that banking bubbles allow banks to relax borrowing constraints and attract more deposits. Thus banks make more loans to finance investment, leading to a rise in the capital stock.

Next we discuss the equilibrium dynamics. There is no closed-form solution to the bubbly equilibrium system characterized in Proposition 2. The analysis of the stability of the system is complex. Thus we present a numerical example to illustrate the solution. We need to calibrate five parameters,  $\delta$ ,  $\alpha$ ,  $r$ ,  $\theta$ , and  $\xi$ . We set  $\delta = 0.1$  to reflect 10% of the annual depreciation rate and  $\alpha = 0.33$  so that the labor share equals 2/3 of output as is commonly done in the real business cycle literature. We set  $r = 0.015$  so that the annual deposit rate is roughly equal to the average real 3-month T-Bill rate. Finally, we set  $\theta$  and  $\xi$  to match the annual credit spread of 1% and the leverage ratio of 4 as in Gertler and Kiyotaki [16].<sup>11</sup> We find that  $\theta = 0.055$  and  $\xi = 0.5556$ .

As Proposition 4 shows, the preceding parameter values imply that bubbleless and bubbly equilibria can coexist.<sup>12</sup> We find that the bubbly steady-state capital stock is equal to  $K^b = 4.35$ , which is larger than the bubbleless steady state capital stock  $K^* = 4.19$ . Moreover, the bubbly steady state  $(Q^b, B, N^b) = (1.50, 2.36, 1.09)$  is a local saddle point because there is a unique stable eigenvalue ( $-0.1097$ ) and two unstable eigenvalues ( $0.0315 \pm 0.0126i$ ) associated with the linearized system around the bubbly steady state. This implies that given an initial value in the neighborhood of the bubbly steady state for the predetermined variable  $N_t$ , there exist unique initial values for the non-predetermined variables  $Q_t$  and  $B_t$  such that  $(Q_t, B_t, N_t)$  converge to the steady state along a saddle path.

We also find that the linearized system around the bubbleless steady state  $(Q^*, 0, N^*) = (3.67, 0, 1.35)$  has two stable eigenvalues ( $-0.128, -0.033$ ) and one unstable eigenvalue ( $0.0305$ ). Thus the stable manifold for the bubbleless steady state is two dimensional. Because there is only one predetermined variable  $N_t$ , equilibria around the bubbleless steady state have indeterminacy of degree one. Formally, there is a neighborhood  $\mathcal{N} \subset \mathbb{R}_+^3$  of the bubbleless steady state  $(Q^*, 0, N^*)$  and a continuously differentiable function  $\phi^* : \mathcal{N} \rightarrow \mathbb{R}$  such that given any  $(B_0, N_0)$  there exists a unique solution  $Q_0$  to the equation  $\phi^*(Q_0, B_0, N_0) = 0$  with  $(Q_0, B_0, N_0) \in \mathcal{N}$ , and  $(Q_t, B_t, N_t)$  converges to  $(Q^*, 0, N^*)$  starting at  $(Q_0, B_0, N_0)$  as  $t$  approaches infinity. Intuitively, along the two-dimensional stable manifold, any equilibrium starting at  $N_0$  with an initial bubble  $B_0 > 0$  in the neighborhood  $\mathcal{N}$  is asymptotically bubbleless in that bubbles will burst eventually.

<sup>11</sup> It is straightforward to compute that the leverage ratio in the bubbly steady state  $K/N$  is equal to  $\frac{(\theta-r)(r+\theta)}{r(\theta-r\xi)}$ .

<sup>12</sup> The multiplicity of equilibria in our model is of a different nature than that in the indeterminacy literature surveyed in Farmer [14] or Benhabib and Farmer [4]. In that literature there are typically multiple equilibrium paths converging to a unique steady state.

#### 4. Stochastic banking bubbles

We have shown that bubbleless and bubbly equilibria can coexist under some assumptions. We now follow Blanchard and Watson [7] and Weil [43] to construct an equilibrium with stochastic bubbles. Following the continuous-time modeling of Miao and Wang [31], we assume that all agents in the economy believe that the banking bubble may burst in the future. The arrival of this event follows a Poisson process with the arrival rate  $\pi$ . After the bubble bursts, the economy enters the bubbleless equilibrium.

##### 4.1. Equilibrium

We solve for the equilibrium with stochastic bubbles by backward induction. After the bursting of the banking bubble, the economy enters the bubbleless equilibrium that was solved in Section 3. Here we write the shadow value of the net worth in the bubbleless equilibrium in a feedback form,  $Q_t^* = g(N_t)$ , where  $g$  is some function. We write the stock market value of the bank after the bubble bursts as  $V^*(N_t, Q_t^*) = Q_t^* N_t$ .

As in the case in which the bubble has not burst, the Bellman equation is given by

$$\begin{aligned} rV(N_t, Q_t, B_t) = & \max_{D_t, C_t^b} C_t^b + V_N(N_t, Q_t, B_t) (r_{kt} N_t + (r_{kt} - r) D_t - C_t^b) \\ & + V_Q(N_t, Q_t, B_t) \dot{Q}_t + V_B(N_t, Q_t, B_t) \dot{B}_t \\ & + \pi [V^*(N_t, Q_t^*) - V(N_t, Q_t, B_t)], \end{aligned} \quad (24)$$

subject to (5), (6), and (7), where  $(Q_t, B_t)$  is the aggregate state vector and  $N_t$  is the individual state. This Bellman equation describes an asset pricing equation for the bank stock. The left-hand side of Eq. (24) is the return on the bank stock. The right-hand side gives dividends  $C_t^b$  plus capital gains. Capital gains consist of four components. The first three components are due to changes in bank net worth  $N_t$ , shadow value  $Q_t$  and the bubble  $B_t$ . The last component arises from changes in beliefs. When the bubble bursts with arrival rate  $\pi$ , the bank value shifts from  $V(N_t, Q_t, B_t)$  to  $V^*(N_t, Q_t^*)$ . The following proposition characterizes the solution:

**Proposition 5.** *If  $Q_t > 1$  and  $r_{kt} > r$ , then the equilibrium  $(Q_t, B_t, N_t)$  before banking bubbles burst satisfies the system of differential equations:*

$$(r + \pi) Q_t = Q_t [r_{kt} + (r_{kt} - r) \xi Q_t] + (1 - Q_t) \theta + \pi Q_t^* + \dot{Q}_t, \quad (25)$$

$$(r + \pi) B_t = Q_t (r_{kt} - r) B_t + \dot{B}_t, \quad (26)$$

and (16), where  $r_{kt}$  satisfies Eq. (19). Here  $Q_t^* = g(N_t)$  is the equilibrium shadow price of net worth after banking bubbles burst.

As in Weil [43] and Miao and Wang [31], the possibility of bubble bursting gives a risk premium so that the return on the bubble is equal to  $r + \pi$  as revealed by Eq. (26). Since there is no closed-form solution to the system of differential equations in the proposition, we solve this system numerically by discretizing the differential equations.

To simplify the analysis, we follow Weil [43], Kocherlakota [27], and Miao and Wang [31] and focus on a particular type of stationary equilibrium with stochastic bubbles. This equilibrium has the feature that all equilibrium variables are constant before bubbles burst. After bubbles

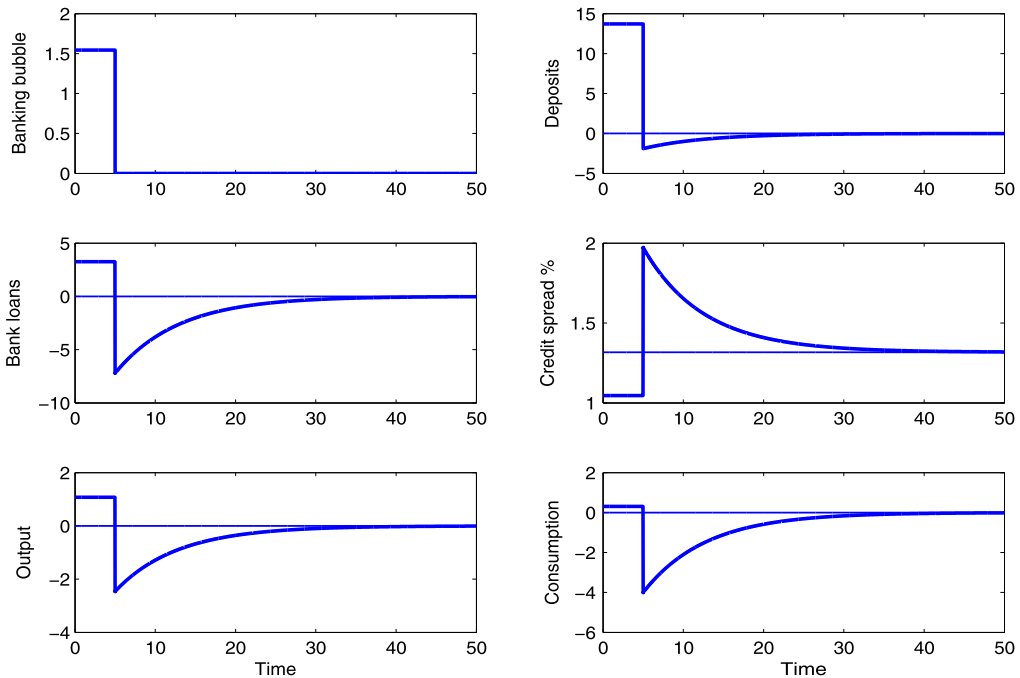


Fig. 2. The transition paths when the bubble bursts at time  $T = 5$  with the Poisson arrival rate  $\pi = 0.0125$ . Except for credit spreads, values of other variables are expressed as percentage deviation from the bubbleless steady state. The parameter values are  $r = 0.015$ ,  $\theta = 0.055$ ,  $\delta = 0.1$ ,  $\alpha = 0.33$ , and  $\xi = 0.5556$ .

burst, the economy then follows the bubbleless equilibrium path. Using Proposition 5 and setting  $B_t = B$ ,  $Q_t = Q$ ,  $N_t = N$ ,  $K_t = K$ , and  $r_{kt} = r_k$  before bubbles burst, we can derive the following system of four nonlinear equations:

$$Q = \frac{r + \pi + \theta + \pi Q^*}{\pi + \theta - \xi(r + \pi)}, \quad (27)$$

$$r + \pi = Q(r_k - r), \quad (28)$$

$$N = \frac{r_k - r}{\theta - r} K, \quad (29)$$

$$r_k = \alpha K^{\alpha-1} - \delta, \quad (30)$$

where  $Q^* = g(N)$ . Denote the solution by  $(N^s, K^s, Q^s, r_k^s)$ . To derive this solution, we need to know the function  $g$ , which can be derived by numerically solving the equilibrium after bubbles burst. We can then solve for the initial size of the bubble, denoted by  $B^s$ , using the balance sheet equation

$$K^s = \xi Q^s N^s + N^s + B^s, \quad (31)$$

and solve for the capital stock or bank loans immediately after the collapse of bubbles using the balance sheet equation

$$K^{*s} = \xi Q^* N^s + N^s.$$

Fig. 2 presents a numerical example. In this example we assume that the economy stays in the steady state with a bubble before time  $T = 5$ . The banking bubble bursts at time  $T = 5$  with the



Poisson arrival rate  $\pi = 0.0125$  because of a change in confidence. This arrival rate implies that bubble bursting is a rare event in our model and happens once in every eighty years on average. Immediately after the bursting of the bubble, the bank's borrowing constraint tightens, causing deposits to fall discontinuously at  $T = 5$ . Deposits then gradually move to the lower bubbleless steady-state level. Due to the fall in deposits, the bank's balance sheet worsens. Thus the credit spread  $(r_{kt} - r)$  rises and the bank reduces lending. This in turn causes non-financial firms to reduce investment and production. Output falls discontinuously and then gradually reaches the lower bubbleless steady-state level. This example illustrates that even though there is no shock to fundamentals of the economy, a shift in beliefs can cause a banking bubble to collapse. This in turns causes a recession.

#### 4.2. Welfare implications

What is the welfare effect of a banking bubble? The existence of a banking bubble relaxes a bank's borrowing constraints and allows the bank to make more lending, which benefits the economy. Thus the equilibrium with a deterministic banking bubble studied in Section 3.5 clearly dominates the bubbleless equilibrium studied in Section 3.4. However, changes in confidence can cause the bubble to burst as analyzed in the preceding subsection. As Fig. 2 shows, the collapse of the bubble causes the bank to cut back lending. Moreover the credit spread  $(r_{kt} - r)$  rises and output and consumption fall. Even though output and consumption are higher in the initial steady state with a bubble than in the bubbleless equilibrium, they fall below their bubbleless steady state levels immediately after the collapse of the banking bubble. Thus the welfare measured by the representative household's lifetime utility level in the stochastic bubbly equilibrium may be lower than that in the bubbleless equilibrium. We are unable to provide a theoretical characterization of this result. We thus use a numerical example to illustrate it.

We compare the steady-state welfare in the three types of equilibria. The steady-state utility levels in the equilibria with a deterministic bubble and without a bubble are, respectively, given by

$$U^b = \frac{(K^b)^\alpha - \delta K^b}{r}, \quad U^* = \frac{(K^*)^\alpha - \delta K^*}{r}.$$

We have shown in Proposition 4 that  $K^b > K^*$ . We can also show that  $K^b < K_g$  where  $\alpha K_g^{\alpha-1} = \delta$ . It follows that  $U^b > U^*$ . The steady-state utility level  $U^s$  in the equilibrium with a stochastic bubble satisfies the equation

$$rU^s = C^s + \pi (U_f(N^s) - U^s),$$

where  $C^s = (K^s)^\alpha - \delta K^s$  denotes the consumption level before the bubble bursts and  $U_f(N^s)$  denotes the utility level immediately after the bursting of the bubble. To solve for  $U_f(N^s)$ , we note that  $U_f(N_t)$  satisfies the Bellman equation in the bubbleless equilibrium

$$rU_f(N_t) = C_t + U'_f(N_t) \dot{N}_t,$$

where  $C_t = K_t^\alpha - \delta K_t - \dot{K}_t$ ,  $K_t = (\xi Q_t + 1) N_t$  and  $\dot{N}_t$  satisfies (16) with  $B_t = 0$ . We can solve the Bellman equation numerically by discretizing the differential equation.

For the parameter values used in Fig. 2, we can compute  $U^*$ ,  $U^b$ , and  $U^s$  numerically. We find that  $U^* = 79.5508$ ,  $U^b = 79.8352$ , and  $U^s = 79.5306$ . Thus the steady-state welfare in the bubbly equilibrium in which the banking bubble never bursts dominates that in the equilibrium without a bubble, which in turn dominates that in the equilibrium with a stochastic bubble.<sup>13</sup> Because the existence of a bubble depends on the beliefs and the change in beliefs can cause the collapse of a bubble, banking bubbles can make the financial system unstable and damage the macroeconomy. Such an equilibrium with a stochastic bubble is commonly observed in reality and will be used as a basis for our policy analysis. There is a role for governments to play in preventing the formation of bubbles in the first place. Furthermore, when there is a bubble that may burst with some probability in the future, there is also a role for governments to mitigate the initial undershooting of lending and investment caused by the bubble bursting. In the next two sections we study such ex ante and ex post policies.

## 5. Bank capital requirements

Bank capital requirements determine how much liquidity must be held for a certain level of assets through regulatory agencies such as the Bank for International Settlements, Federal Deposit Insurance Corporation, or Federal Reserve Board. These requirements are put place to ensure that financial institutions are not making investments that increase the risk of default and that they have enough capital to sustain operating losses while still honoring deposit withdrawals. Bank capital requirements enhance the stability of the banking system. The traditional rationale for these requirements is related to banks' risk-taking behavior. Due to limited liability and the access to secured deposits, banks have an incentive to choose risky projects which raise the probability of bank failure. Increasing the percentage of the investment funded by bank capital limits this risk-taking activity.

In this section we argue that bank capital requirements may prevent the formation of banking bubbles. These bubbles can be costly because they may burst due to a change in confidence. If the capital requirements are too stringent, then banks will lend less and charge more for loans, thereby reducing the steady-state capital stock and efficiency. We model bank capital requirements as follows:

$$N_t \geq \phi (D_t + N_t), \quad (32)$$

where  $\phi \in (0, 1)$  is the bank capital requirement ratio. We rewrite (32) as

$$D_t \leq \frac{1 - \phi}{\phi} N_t. \quad (33)$$

We incorporate this inequality to study the effects of bank capital requirements on equilibrium outcomes. Suppose that condition (13) in Proposition 4 is satisfied so that bubbleless and bubbly equilibria coexist.

<sup>13</sup> We find by numerical examples that this result holds true for a range of values of the parameter  $\pi$  that determines the probability of the bubble bursting.

**Proposition 6.** Suppose that condition (13) in Proposition 1 is satisfied. (i) If

$$\frac{1 - \phi}{\phi} > \frac{\theta - r_k^b}{r_k^b - r}, \quad (34)$$

then the bubbleless and bubbly equilibria characterized in Proposition 4 are unaffected by the bank capital requirements. (ii) If

$$0 < \frac{\theta - r_k^*}{r_k^* - r} < \frac{1 - \phi}{\phi} < \frac{\theta - r_k^b}{r_k^b - r}, \quad (35)$$

then a banking bubble cannot exist and only the bubbleless equilibrium characterized in Proposition 3 exists. (iii) If

$$0 < \frac{1 - \phi}{\phi} < \frac{\theta - r_k^*}{r_k^* - r}, \quad (36)$$

then a banking bubble cannot exist. The steady-state lending rate and the capital stock satisfy

$$r_k = \phi\theta + (1 - \phi)r > r_k^*, \quad K < K^*.$$

This proposition shows that if the bank capital requirement ratio is too small, then the requirements are ineffective in that they do not affect the equilibrium. Thus they cannot prevent the existence of banking bubbles. If the bank capital requirement ratio is too large, then the requirements can prevent the formation of banking bubbles. However, they lead to a high lending rate and low steady-state capital. If the bank capital requirement rate is in the intermediate range, then a banking bubble cannot exist and the economy is in the bubbleless equilibrium studied in Section 3.4.

## 6. Credit policies

During the recent financial crisis, many central banks around the world, including the U.S. Federal Reserve, used their powers as a lender of last resort to facilitate credit flows. As Gertler and Kiyotaki [16] point out, the Fed employed three general types of credit policies during the crisis. The first is direct lending. The Fed lent directly in high-grade credit markets, funding assets that included commercial paper, agency debt, and mortgage-backed securities. The second is discount window lending. The Fed used the discount window to lend funds to commercial banks that in turn lent funds out to non-financial borrowers. The third is equity injections. The Treasury coordinated with the Fed to acquire ownership positions in commercial banks by injecting equity.

In this section we analyze the impact of these three types of credit policies by introducing a central bank and a government in our baseline model. Before proceeding, we emphasize that, consistent with the Federal Reserve Act, these interventions are used only during crises and not during normal times. Moreover, we assume that agents have rational expectations and fully anticipate that the credit policies will be implemented whenever a bubble bursts.<sup>14</sup>

<sup>14</sup> We have also analyzed the case where credit policies are not anticipated. This analysis is simpler and available upon request.

### 6.1. Direct lending

We start by analyzing the impact of direct lending. Because the collapse of a banking bubble tightens banks' borrowing constraints, households reduce deposits and banks reduce lending, causing a crisis. The central bank can help maintain credit flows by lending directly to non-financial firms. Suppose that the central bank lends  $\Psi_T$  to non-financial firms immediately after the bursting of the bubble at some random time  $T$ . We assume that the volume of direct lending at time  $t$  satisfies

$$\Psi_t = \Psi_T e^{-\lambda(t-T)} \quad \text{for } t \geq T, \quad (37)$$

where  $\lambda$  describes the decay rate. When  $t \rightarrow \infty$ ,  $\Psi_t \rightarrow 0$ , so the central bank will eventually exit from the rescue program and the economy will reach the bubbleless steady state studied in Section 3.4. Assume that the government issues government debt to households to finance lending to non-financial firms. There is no government spending and the government runs a balanced budget by raising lump-sum taxes or making transfers.

Given the direct lending policy, the firm's capital is financed by loans from commercial banks and loans from the central bank; that is,

$$K_t = N_t + D_t + \Psi_t \quad \text{for } t \geq T. \quad (38)$$

We will derive the equilibrium with a stochastic bubble and with credit policy by backward induction. We first start with the case after the bubble bursts and then go back to solve the case before the bubble bursts.

As in Section 3, we can show that if  $Q_t > 1$  and  $r_{kt} > r$  for  $t \geq T$ , then the constraints (5) and (7) bind. We can also show that the stock market value of the bank is given by  $V^*(N_t, Q_t) = Q_t N_t$  for  $t \geq T$ . It follows that  $D_t = \xi Q_t N_t$  and the equilibrium system for  $(Q_t, K_t, N_t, r_{kt})$  after the bubble bursts is given by (14) and

$$\dot{N}_t = (r_{kt} - \theta + (r_{kt} - r)\xi Q_t) N_t, \quad (39)$$

$$K_t = N_t (1 + \xi Q_t) + \Psi_t, \quad (40)$$

$$r_{kt} = \alpha K_t^{\alpha-1} - \delta, \quad (41)$$

for  $t \geq T$ . The solution must satisfy the usual transversality condition. Eq. (39) follows from (16) by setting  $B_t = 0$ . Eq. (40) follows from (38). The last equation (41) follows from (8) and the market-clearing condition  $L_t = 1$ . The solution will be in a feedback form  $Q_t^* = g(N_t, \Psi_t)$  for some function  $g$ .<sup>15</sup> The steady state is characterized by Proposition 3. In a neighborhood of this steady state, we have  $Q_t^* > 1$  and  $r_{kt} > r$ . We can solve the equilibrium system using numerical methods and derive the function  $g$ .

We now turn to the case where the bubble does not burst. As in Section 4, we assume that the economy is in the steady state for  $t < T$ . Since the credit policy is fully anticipated by rational agents, the initial steady state before the collapse of the bubble will take into account the credit policy. Immediately after the collapse of the bubble at  $T$ , the equilibrium moves along the new saddle path and  $Q_T$  jumps to  $Q_T^* = g(N_T, \Psi_T)$ . As in Section 4, we can show that the initial steady state  $(N, Q, K, r_k, B)$  satisfies equations (27)–(31), where  $Q^* = g(N, \Psi)$ . We also need  $Q > 1$  and  $r_k > r$  in the initial steady state with a bubble. This condition can be verified by numerical examples.

<sup>15</sup> Without risk of confusion, we use the same notation  $g$  as in Section 4.

## 6.2. Discount window lending

In this subsection we assume that the central bank makes discount window lending to the financial intermediaries instead of lending directly to non-financial firms. The central bank lends  $M_T$  to the financial intermediaries at the discount rate  $r$  at some random time  $T$  when the bubble bursts. Let  $M_t$  satisfy

$$M_t = M_T e^{-\lambda(t-T)} \quad \text{for } t \geq T. \quad (42)$$

Assume that the government issues debt to households to finance lending to banks. There is no government spending and the government runs a balanced budget by raising lump-sum taxes or making transfers.

Suppose that discount window loans are distributed to each bank according to its share of net worth. Specifically, bank  $j \in [0, 1]$  receives loans  $M_t^j = M_t N_t^j / N_t$ , where  $M_t = \int M_t^j dj$  and  $N_t = \int N_t^j dj$ .

Bank  $j$ 's balance sheet becomes  $N_t^j + M_t^j + D_t^j = S_t^j$  for  $t \geq T$ . Its net worth satisfies the law of motion

$$dN_t^j = r_{kt} N_t^j dt + (r_{kt} - r) D_t^j dt + (r_{kt} - r) M_t^j dt - C_t^{jb} dt.$$

Aggregate loans satisfy  $K_t = \int S_t^j dj$ . The bank also faces the borrowing constraint

$$D_t^j \leq V_t \left( \xi N_t^j \right).$$

Following the proof of [Proposition 2](#), we can show that if  $Q_t > 1$  and  $r_{kt} > r$ , the following system of differential equations governs the dynamics of the economy after the bubble bursts:

$$\begin{aligned} r Q_t &= Q_t [r_{kt} + (r_{kt} - r) (\xi Q_t + M_t / N_t)] + \theta(1 - Q_t) + \dot{Q}_t, \\ \dot{N}_t &= [(r_{kt} - \theta) + (r_{kt} - r) \xi Q_t] N_t + M_t (r_{kt} - r), \end{aligned}$$

for  $t \geq T$ . The steady state is characterized as in [Proposition 3](#). By continuity, the conditions  $Q_t > 1$  and  $r_{kt} > r$  are satisfied in the neighborhood of the steady state. We write the solution for  $Q_t$  in a feedback form  $Q_t^* = g(N_t, M_t)$  for some function  $g$ .

As in [Section 4](#) we can show that the initial steady state  $(N, Q, K, r_k, B)$  before the bubble bursts satisfies the system of nonlinear equations [\(27\)–\(31\)](#), where  $Q^* = g(N, M)$ . The policy variable  $M$  affects the initial steady state because we have assumed that rational agents anticipate the credit policy.

## 6.3. Equity injections

Under the equity injections policy, the fiscal authority coordinates with the monetary authority to acquire ownership positions in banks. Immediately after the bursting of the bubble at some random time  $T$ , the government injects equity  $N_T^g$  into banks and owns a fraction  $s \in (0, 1)$  of the bank stocks. For simplicity, assume that the government never sells its holdings of the bank stocks. The government issues debt to households to finance equity injections. There is no government spending and the government runs a balanced budget by raising lump-sum taxes or making transfers.

Suppose that  $N_T^g$  is fairly priced in that it is equal to the market value of the shares

$$N_T^g = s \int_0^{\infty} e^{-r\tau} C_{T+\tau}^b d\tau, \quad (43)$$

where  $C_{T+\tau}^b$  denotes bank dividends. This equation implies that we can use either  $s$  or  $N_T^g$  as a policy variable. Without risk of confusion, we only consider a representative bank's behavior and suppress bank index  $j$ . The bank's balance sheet becomes  $S_T + S_T^g = N_T + N_T^g + D_T$ , where  $S_T^g = N_T^g$  is the loan intermediated by the government.

We first solve for the equilibrium after the bubble bursts for  $t \geq T$ . As in Section 3, we can show that the market value of the bank to the private agents is given by

$$V^*(N_t, Q_t) = (1-s) \int_0^{\infty} e^{-r\tau} C_{t+\tau}^b d\tau = Q_t N_t,$$

where  $Q_t$  and  $N_t$  satisfy the differential equations (39) and

$$rQ_t = Q_t[r_{kt} + (r_{kt} - r)\xi Q_t] + \theta(1-s-Q_t) + \dot{Q}_t, \quad (44)$$

when  $Q_t > 1-s$  and  $r_{kt} > r$ . Let the solution in a feedback form be  $Q_t^* = g(N_t)$ . We can easily derive the bubbleless steady state with the equity injections policy and show that when  $\theta > r$ ,

$$Q^* = \frac{(1-s)\theta}{r} > 1-s, \quad r_k^* - r = \frac{r(\theta-r)}{r+\xi(1-s)\theta} > 0.$$

Thus the conditions  $Q_t > 1-s$  and  $r_{kt} > r$  are satisfied in the steady state without a bubble and hence in a neighborhood of this steady state by continuity.

Since the government injects equity  $N_T^g$  and acquires a fraction  $s$  of the total shares at time  $T$  when the bubble bursts, it follows from (43) that

$$N_T^g = \frac{s}{1-s} V^*(N_T + N_T^g, Q_T) = \frac{s}{1-s} Q_T^* (N_T + N_T^g).$$

Solving this equation yields

$$N_T^g = \frac{\frac{s}{1-s} Q_T^* N_T}{1 - \frac{s}{1-s} Q_T^*}, \quad (45)$$

where  $Q_T^* = g(N_T + N_T^g)$ .

Next consider the equilibrium before the bubble bursts. Let  $V(N_t, Q_t, B_t)$  denote the bank's value function before the bubble bursts. It satisfies the Bellman equation

$$\begin{aligned} rV(N_t, Q_t, B_t) = & \max_{D_t, C_t^b} C_t^b + V_N(N_t, Q_t, B_t) (r_{kt}N_t + (r_{kt} - r)D_t - C_t^b) \\ & + V_Q(N_t, Q_t, B_t) \dot{Q}_t + V_B(N_t, Q_t, B_t) \dot{B}_t \\ & + \pi [V^*(N_t + N_t^g, Q_t) - V(N_t, Q_t, B_t)], \end{aligned}$$

where the last line reflects the fact that when the bubble bursts at time  $t$  with Poisson arrival rate  $\pi$ , the bank value switches from  $V(N_t, Q_t, B_t)$  to  $V^*(N_t + N_t^g, Q_t)$ . Moreover,  $V^*(N_t + N_t^g, Q_t)$  indicates that private agents anticipate the equity injections policy. We conjecture that the value function takes the form  $V(N_t, Q_t, B_t) = Q_t N_t + B_t$ . Substituting this conjecture and Eq. (45) at  $T = t$  into the preceding Bellman equation, we can derive the equilibrium system with four Eqs. (26), (16), (19) and

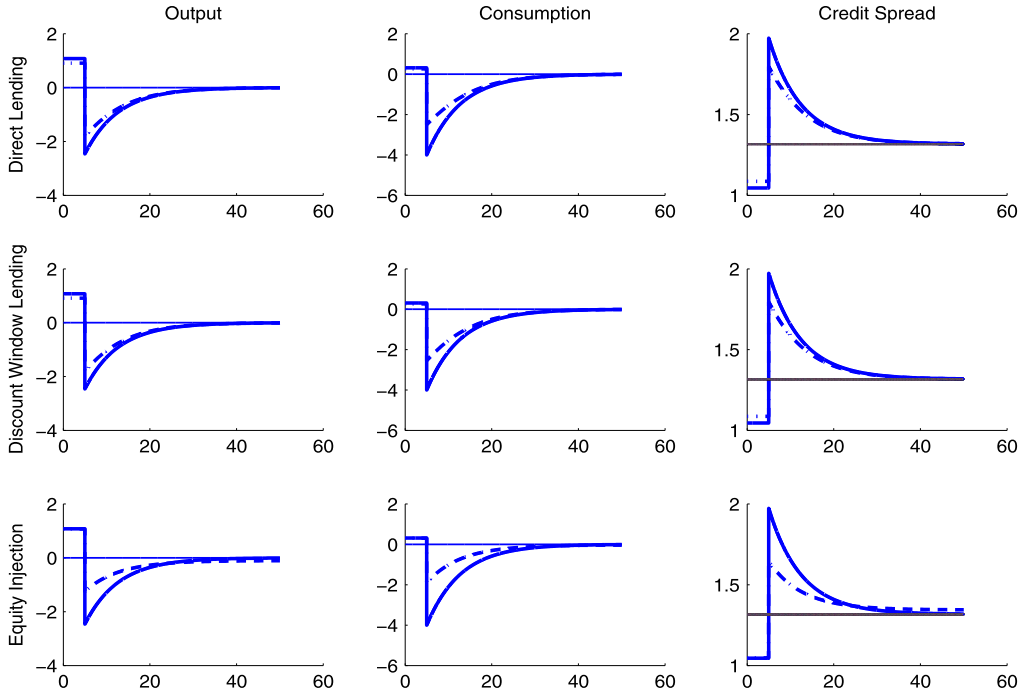


Fig. 3. Comparison of credit policies. Except for the credit spread, other variables are represented as percentage deviation from the bubbleless equilibrium. Solid lines: equilibrium with stochastic bubbles and without policy interventions. Dashed lines: equilibrium with stochastic bubbles and with policy interventions.

$$(r + \pi) Q_t = Q_t [r_{kt} + (r_{kt} - r) \xi Q_t] + (1 - Q_t) \theta + \pi Q_t^* \left[ 1 + \frac{\frac{s}{1-s} Q_t^*}{1 - \frac{s}{1-s} Q_t^*} \right] + \dot{Q}_t,$$

for four variables  $(N_t, Q_t, B_t, r_{kt})$ , where  $Q_t^* = g(N_t + N_t^g)$ . We need the condition  $Q_t > 1$  and  $r_{kt} > r$  to be satisfied in a neighborhood of the steady state.

#### 6.4. Comparison of credit policies

In this subsection we compare the impact of the preceding three types of credit policies numerically. We set parameter values as in Fig. 2. Suppose that the bubble bursts at time  $T = 5$  with the Poisson arrival rate  $\pi = 0.0125$ . In addition, set  $\Psi_T = 0.0329$  and  $\lambda = 0.2$ . This means that the direct lending policy lasts for  $1/\lambda = 5$  years and the present value of the lending  $\Psi_T / (r + \lambda)$  accounts for 9.41% of output prior to the bursting of the bubble. We assume that the present value of the injected liquidity is the same for all policies. We can then pin down the values for  $M_T$  and  $s$ .

Fig. 3 presents the transition paths of output, consumption, and credit spreads. Table 1 presents the welfare gain in terms of percentage increases in consumption relative to the equilibrium with a stochastic bubble and without credit policy. Fig. 3 shows that the three types of credit policies have a similar impact on the economy. They can mitigate the large fall of output and consumption and the surge of credit spreads at the time when the bubble bursts. Moreover, the

Table 1  
Welfare gains in terms of percentage increases in consumption.

Deterministic steady state	0.3830
Bubbleless steady state	0.0254
Discount window lending ( $\lambda = 0.2$ )	0.0235
Discount window lending ( $\lambda = 0.5$ )	0.0498
Discount window lending ( $\lambda = 4/3$ )	0.1228
Direct lending ( $\lambda = 0.2$ )	0.0229
Direct lending ( $\lambda = 0.5$ )	0.0494
Direct lending ( $\lambda = 4/3$ )	0.1224
Equity injections	0.0873

Note: Parameter values are the same as those used in Fig. 3. Assume that the present value of the injected liquidity is the same for all policies. The welfare is measured in terms of consumption as percentage deviation from the stochastic bubbly equilibrium.

steady-state output and consumption change prior to the bursting of the bubble when a credit policy is implemented because credit policies are assumed to be anticipated by private agents.<sup>16</sup>

Three additional features stand out. First, the discount window lending policy has a slightly larger stimulating effect because of the leverage effect (not visible from Fig. 3). Injecting funds in banks raises their net worth and improves their balance sheets. This allows banks to make more lending under the discount window lending policy than under the direct lending policy. The welfare gain from the discount window lending policy relative to the equilibrium with stochastic bubble and without policy is also larger as revealed in Table 1.

Second, the bubbleless steady-state output under the equity injections policy is lower than that under the other two policies. The reason is that the government holds a fraction of the bank shares under the equity injections policy so that the market value of banks to the private agents is lower. The agency issue emphasized by Jensen and Meckling [22] causes the banks to intermediate less funds to non-financial firms in the long run.

Third, the equity injections policy has a much larger short-run stimulating effect than the other two policies. As a result, the welfare gain is also larger as shown in Table 1. The main reason is the following: under the equity injections policy, the government injects a one-time lump-sum amount of liquidity into the economy upon the bursting of a bubble, the time when the liquidity is most needed. For the discount window policy, when the same amount of liquidity is injected over a period of time until the government exits, the short-run stimulating effect is weaker. Even though the economy recovers faster in the long-run, the welfare gain is smaller.

Table 1 compares the impact of different intensities  $\lambda$  of the direct lending and discount window policies. Holding the present value of injected funds fixed, a larger  $\lambda$  means that the injections are more front loaded. We find that a larger  $\lambda$  has a larger stimulating effect. In particular, output drops less immediately after the collapse of the banking bubble. This is because more liquidity is immediately injected when the policy intervention is more front loaded. But in the medium run, output recovers slower because the government intervention lasts shorter. The overall impact on the economy balances the short-run large stimulating effect and the medium-run slow recovery effect. In terms of welfare, we find that the former effect dominates for two reasons. First, immediately after the collapse of a banking bubble, the credit spread surges and

<sup>16</sup> They would be unchanged if credit policies are unanticipated.



a credit crunch occurs. This is the time when liquidity is most needed. Injecting more liquidity immediately helps the economy better. Second, due to discounting in utility, an initial smaller drop in output and consumption accompanied by a slower recovery in the future is more desirable than an initial larger drop in output and consumption accompanied by a faster recovery in the future.

Table 1 also shows that when liquidity injections are sufficiently front loaded, e.g.,  $\lambda = 4/3$ , the discount window lending policy dominates the direct lending policy, which in turn dominates the equity injections policy when the same amount of liquidity (in terms of present value) is injected into the economy. As discussed earlier, the equity injections policy discourages long-run capital accumulation because the private ownership of banks is reduced. This negative effect makes the equity injections policy the least desirable, when the liquidity injections for the other two policies are sufficiently front loaded.

How do we compare the welfare effects of the ex ante bank capital requirements policy studied in Section 5 and the ex post stabilization policies studied in this section? We have shown that an equilibrium with a stochastic bubble can be dominated by a bubbleless equilibrium. Thus the capital requirements policy that achieves the bubbleless equilibrium characterized in Proposition 6 can improve welfare relative to the equilibrium with a stochastic bubble. Table 1 shows that this ex ante policy can raise consumption by 0.0254%. But if the ex post stabilization policies are sufficiently front loaded, then they dominate the ex ante capital requirements policy. For example, when  $\lambda = 4/3$ , the discount window lending and direct lending policies raise consumption by 0.1228% and 0.1224% respectively. In addition, the equity injections policy also dominates the bank capital requirements policy.

We should mention that we have abstracted away from several important elements in our policy analysis. First, we have not considered the moral hazard issue. Bankers or firms may take away the injected funds. They may either invest in bad projects or put those funds in their own pockets.<sup>17</sup> Second, we have not considered efficiency costs of credit policy. These costs could reflect the costs of raising funds via government debt or taxes. They could also reflect the cost of evaluating and monitoring borrowers that is above and beyond what a private intermediary (that has specific knowledge of a particular market) would pay. Some of these issues have been analyzed by Gertler and Kiyotaki [16] and Gertler and Karadi [15]. In a related model with outside equity, Gertler, Kiyotaki, and Queralto [18] find that an ex post stabilization policy can generate moral hazard by making banks take on higher leverage. This possibility might give further justification for ex ante capital requirements.

## 7. Conclusion

We have developed a tractable continuous-time macroeconomic model with a banking sector in which banks face endogenous borrowing constraints. There is no uncertainty about economic fundamentals. A positive feedback loop mechanism generates a banking bubble. Changes in people's beliefs can cause the banking bubble to collapse. Immediately after the collapse of the bubble, households withdraw deposits and the bank reduces lending to non-financial firms.

<sup>17</sup> As in Gertler and Kiyotaki [16], we have assumed that the government or the central bank can do something that commercial banks cannot. In particular, with direct lending, the government is not subject to the same agency friction as commercial banks; with discount window lending, the extra borrowing does not influence the amount of deposits commercial banks can raise; and the government can inject equity financed by debt or taxes, while commercial banks cannot.

Consequently, non-financial firms cut back investment, causing output to fall. The economy then transits from a good steady state to a bad steady state. The existence of a stochastic banking bubble reduces welfare. Credit policy can mitigate the downturn immediately after the collapse of the banking bubble. The welfare gain is larger when the government interventions are more front loaded given that the government injects the same amount of liquidity in terms of present value. We also show that bank capital requirements can prevent the formation of a banking bubble. However, they limit leverage and hence reduce lending, causing investment and output to fall.

Our model is stylized and can be generalized in many dimensions. First, it would be interesting to introduce uncertainty about fundamentals into the model. It is likely that both uncertainty and beliefs are important during financial crises. The existing literature (Gertler and Kiyotaki [16]) typically assumes that a financial crisis is triggered by a capital quality shock or a shock to the net worth. This approach ignores the boom and bust of the stock market value of banks or the large fluctuation in the market-to-book ratio. Linking the stock market value of banks to real allocation may help empirically distinguish between the existing approach and ours. Moreover the existing approach needs large exogenous shocks to capital quality or net worth to generate a crisis. Identifying different sources of shocks may also help distinguish different approaches empirically. We leave this topic for future research. Second, it is straightforward to introduce risk aversion and endogenous labor choice in the model (see Miao and Wang [33] and Miao, Wang, and Xu [34]). This makes the model more realistic, but complicates the analysis without changing our key insights. Third, it would be interesting to introduce recurrent banking bubbles (e.g., Martin and Ventura [29], Wang and Wen [42], and Miao, Wang, and Xu [34]) and study their quantitative implications.

## Appendix A. Proofs

**Proof of Proposition 1.** We write the bank's value function as  $V_t(N_t) = V(N_t, Q_t, B_t)$ . By standard dynamic programming theory, it satisfies the Bellman equation

$$\begin{aligned} rV(N_t, Q_t, B_t) = \max_{C_t^b, D_t} & C_t^b + V_N(N_t, Q_t, B_t) \left( r_{kt} N_t + (r_{kt} - r) D_t - C_t^b \right) \\ & + V_Q(N_t, Q_t, B_t) \dot{Q}_t + V_B(N_t, Q_t, B_t) \dot{B}_t, \end{aligned}$$

subject to constraints (5), (6) and (7). We conjecture that the value function takes the form in (10). Substituting this conjecture into the preceding Bellman equation yields (11).

By condition (13),  $Q_t > 1$  and  $r_{kt} > r$  in a neighborhood of the steady states characterized in Propositions 3 and 4. It follows that constraint (5) binds so that  $C_t^b = \theta N_t$ . When  $r_{kt} > r$ , the borrowing constraint (7) binds so that

$$D_t = V_t(N_t) = N_t \xi Q_t + B_t.$$

Substituting the solutions for  $C_t^b$  and  $D_t$  into the Bellman equation (11) yields

$$\begin{aligned} r(Q_t N_t + B_t) = & Q_t [r_{kt} N_t + (r_{kt} - r)(N_t \xi Q_t + B_t)] + (1 - Q_t) \theta N_t \\ & + N_t \dot{Q}_t + \dot{B}_t. \end{aligned} \quad (\text{A.1})$$

Matching coefficients for  $N_t$  and the remaining terms on the two sides of the equation yields (14) and (15). Substituting the solutions  $C_t^b = \theta N_t$  and  $D_t = N_t \xi Q_t + B_t$  into the flow-of-funds constraint (4) yields (16).

For the bank value function to be finite in an infinite-horizon optimization problem, the following transversality condition must hold:

$$\lim_{T \rightarrow \infty} e^{-rT} V_T(N_T) = 0.$$

Given linear utility, this condition is also necessary for optimality in the household's problem after setting  $\psi_t = 1$  by Araujo and Scheinkman [3] and Ekeland and Scheinkman [13]. Since  $V_T(N_T) = Q_T N_T + B_T$  and since  $Q_T$ ,  $N_T$ , and  $B_T$  are all nonnegative, we obtain (17).  $\square$

**Proof of Proposition 2.** By condition (13),  $Q_t > 1$  and  $r_{kt} > r$  in a neighborhood of the steady states characterized in Propositions 3 and 4. It follows from Proposition 1 that  $Q_t$ ,  $B_t$ , and  $N_t$  satisfy Eqs. (14), (15), and (16). Eq. (19) follows from (2), (8), and the market-clearing condition  $L_t = 1$ .  $\square$

**Proof of Proposition 3.** By Proposition 2,  $Q$  and  $N$  in the steady state satisfy

$$rQ = (r_k + (r_k - r)\xi Q)Q + \theta(1 - Q), \quad (\text{A.2})$$

$$(r_k - \theta + (r_k - r)\xi Q)N = 0, \quad (\text{A.3})$$

where

$$r_k = \alpha(\xi QN + N)^{\alpha-1} - \delta. \quad (\text{A.4})$$

It follows from (A.3) that  $(r_k - r)\xi Q = \theta - r_k$ . Substituting this expression into Eq. (A.2) yields  $Q^* = \theta/r$ . We then use Eqs. (A.3) and (A.4) to solve for  $r_k^*$  and  $N^*$ , respectively.  $\square$

**Proof of Proposition 4.** By Proposition 2, the following conditions hold in the bubbly steady-state equilibrium:

$$rQ = (r_k + (r_k - r)\xi Q)Q + \theta(1 - Q), \quad (\text{A.5})$$

$$rB = Q(r_k - r)B, \quad (\text{A.6})$$

$$0 = (r_k - \theta + (r_k - r)\xi Q)N + (r_k - r)B, \quad (\text{A.7})$$

where

$$r_k = \alpha((\xi Q + 1)N + B)^{\alpha-1} - \delta. \quad (\text{A.8})$$

By Eq. (A.6),

$$r = (r_k - r)Q. \quad (\text{A.9})$$

Substituting this equation into (A.5) yields

$$rQ = r_kQ + \xi rQ + \theta(1 - Q). \quad (\text{A.10})$$

Combining the preceding two equations yields (21). Condition (13) ensures  $\theta > \xi r$  so that  $Q^b > 1$ . Substituting (21) into (A.9) yields (22). Again condition (13) ensures that  $r_k > r$ . Substituting Eqs. (21) and (22) into (A.7), we obtain Eq. (23). We can check that  $B > 0$  if and only if condition (13) holds.

Using equation

$$\alpha K^{\alpha-1} = r_k^b + \delta,$$

we can solve for the bubbly steady-state aggregate capital stock  $K^b$ . Note that Eq. (20) also holds for the bubbly equilibrium. Substituting  $r_k^b$  in (22) and  $K^b$  into this equation, we can solve for the bubbly steady-state net worth  $N^b$ . Finally, we use Eq. (23) to solve for the steady-state bubble  $B$ .  $\square$

**Proof of Proposition 5.** After the bubble bursts, the economy enters the bubbleless equilibrium and the bank value function is given by  $V_t^*(N_t) = Q_t^* N_t$ . Conjecture that the value function before the bubble bursts takes the following form:

$$V(N_t; Q_t, B_t) = Q_t N_t + B_t.$$

Thus the borrowing constraint becomes

$$D_t \leq \xi Q_t N_t + B_t.$$

Substituting the preceding conjecture into the Bellman equation (24) yields

$$\begin{aligned} r(Q_t N_t + B_t) = \max_{C_t^b, D_t} & C_t^b + Q_t \left( r_{kt} N_t + (r_{kt} - r) D_t - C_t^b \right) \\ & + \pi \left[ Q_t^* N_t - (Q_t N_t + B_t) \right] + N_t \dot{Q}_t + \dot{B}_t. \end{aligned}$$

By this equation, if  $Q_t > 1$ , then the constraint (5) binds so that  $C_t^b = \theta N_t$ . If  $r_{kt} > r$ , then the borrowing constraint binds so that

$$D_t = \xi Q_t N_t + B_t.$$

Substituting this binding constraint into the Bellman equation yields

$$\begin{aligned} r(Q_t N_t + B_t) = (1 - Q_t) \theta N_t + Q_t r_{kt} N_t + Q_t (r_{kt} - r) (\xi Q_t N_t + B_t) \\ + \pi \left[ Q_t^* N_t - (Q_t N_t + B_t) \right] + N_t \dot{Q}_t + \dot{B}_t. \end{aligned}$$

Matching coefficients of  $N_t$  and terms not related to  $N_t$  on the two sides of this equation delivers the equations given in Proposition 5.  $\square$

**Proof of Proposition 6.** Ignore capital requirements given in (33) for now. Then by assumption, the bubbly and bubbleless equilibria coexist. The borrowing constraint for the bubbly equilibrium is given by

$$D_t \leq \xi Q_t N_t + B_t.$$

In the bubbly steady state,

$$\frac{D^b}{N^b} \leq \xi Q^b + \frac{B}{N^b} = \frac{\theta - r_k^b}{r_k^b - r},$$

where the equality follows from Proposition 4. In the bubbleless steady state, Proposition 3 implies that

$$\frac{D^*}{N^*} \leq \xi Q^* = \frac{\theta - r_k^*}{r_k^* - r}.$$

By Proposition 4,

$$\frac{\theta - r_k^b}{r_k^b - r} > \frac{\theta - r_k^*}{r_k^* - r}.$$

Now we introduce capital requirements given in (33). If condition (34) is satisfied, then the capital requirements are ineffective in the neighborhood of the bubbly steady state. If condition (35) holds, then the capital requirement constraint (33) binds in the neighborhood of the bubbly steady state, which prevents the existence of the bubbly equilibrium. But it does not bind around the bubbleless steady state characterized in Proposition 3. Thus a bubbleless equilibrium around that steady state still exists. Finally if condition (36) holds, then the capital requirement constraint (33) binds. The new steady state is given by

$$\frac{D}{N} = \frac{1 - \phi}{\phi}. \quad (\text{A.11})$$

In the steady state, Eq. (4) becomes

$$r_k N + (r_k - r) D - C^b = 0.$$

Since  $C^b = \theta N$ ,

$$\frac{D}{N} = \frac{\theta - r_k}{r_k - r}. \quad (\text{A.12})$$

Using (A.11) and (A.12) yields

$$r_k = \phi\theta + (1 - \phi)r.$$

Condition (36) implies that  $r_k > r^*$  and hence  $K < K^*$ .  $\square$

## Appendix B. Supplementary material

Supplementary material related to this article can be found online at <http://dx.doi.org/10.1016/j.jet.2015.02.004>.

## References

- [1] R. Albuquerque, H.A. Hopenhayn, Optimal lending contracts and firm dynamics, *Rev. Econ. Stud.* 71 (2004) 285–315.
- [2] F. Alvarez, U.J. Jermann, Efficiency, equilibrium, and asset pricing with risk of default, *Econometrica* 68 (2000) 775–798.
- [3] A. Araujo, J.A. Scheinkman, Maximum principle and transversality condition for concave infinite horizon economic models, *J. Econ. Theory* 30 (1983) 1–16.
- [4] J. Benhabib, R. Farmer, Indeterminacy and sunspots in macroeconomics, in: J.B. Taylor, M. Woodford (Eds.), *Handbook of Macroeconomics*, vol. 1A, North-Holland, Amsterdam, 1999, pp. 387–448.
- [5] B. Bernanke, M. Gertler, Agency costs, net worth, and business fluctuations, *Amer. Econ. Rev.* 79 (1989) 14–31.
- [6] B. Bernanke, M. Gertler, S. Gilchrist, Financial accelerator in a quantitative business cycle framework, in: J.B. Taylor, M. Woodford (Eds.), *Handbook of Macroeconomics*, vol. 1A, North-Holland, Amsterdam, 1999, pp. 1341–1393.
- [7] O. Blanchard, M. Watson, Bubbles, rational expectations and financial markets, Harvard Institute of Economic Research Working Paper No. 945, 1982.
- [8] F. Boissay, F. Collard, F. Smets, Booms and systemic banking crises, working paper, European Central Bank, 2013.
- [9] M.K. Brunnermeier, Bubbles, in: S. Durlauf, L. Blume (Eds.), *The New Palgrave Dictionary of Economics*, Palgrave Macmillan, New York, 2008.
- [10] M.K. Brunnermeier, Y. Sannikov, A macroeconomic model with a financial sector, *Amer. Econ. Rev.* 104 (2014) 379–421.

- [11] C.T. Carlstrom, T.S. Fuerst, Agency costs, net worth, and business fluctuations: a computable general equilibrium analysis, *Amer. Econ. Rev.* 87 (1997) 893–910.
- [12] D.W. Diamond, P.H. Dybvig, Bank runs, deposit insurance, and liquidity, *J. Polit. Economy* 91 (1983) 401–419.
- [13] I. Ekeland, J.A. Scheinkman, Transversality conditions for some infinite horizon discrete time optimization problems, *Math. Operations Res.* 11 (1986) 216–229.
- [14] R.E.A. Farmer, *The Macroeconomics of Self-Fulfilling Prophecies*, second ed., MIT Press, Cambridge, 1999.
- [15] M. Gertler, P. Karadi, A model of unconventional monetary policy, *J. Monet. Econ.* 58 (2011) 17–34.
- [16] M. Gertler, N. Kiyotaki, Financial intermediation and credit policy in business cycle analysis, in: B. Friedman, M. Woodford (Eds.), *Handbook of Monetary Economics*, vol. 3A, 2010, pp. 547–599.
- [17] M. Gertler, N. Kiyotaki, Banking, liquidity and bank runs in an infinite horizon economy, working paper, New York University and Princeton University, 2013.
- [18] M. Gertler, N. Kiyotaki, A. Queralto, Financial crises, bank risk exposure and government financial policy, *J. Monet. Econ.* 59 (2012) 517–534.
- [19] Z. He, A. Krishnamurthy, A model of capital and crises, *Rev. Econ. Stud.* 79 (2012) 735–777.
- [20] Z. He, W. Xiong, Dynamic debt runs, *Rev. Finan. Stud.* 25 (2012) 1799–1843.
- [21] T. Hirano, N. Yanagawa, Asset bubbles, endogenous growth and financial frictions, working paper, University of Tokyo, 2011.
- [22] M.C. Jensen, W.H. Meckling, Theory of the firm: managerial behavior, agency costs and ownership structure, *J. Finan. Econ.* 3 (1976) 305–360.
- [23] U. Jermann, V. Quadrini, Macroeconomic effects of financial shocks, *Amer. Econ. Rev.* 102 (2012) 238–271.
- [24] T.J. Kehoe, D.K. Levine, Debt-constrained asset markets, *Rev. Econ. Stud.* 60 (1993) 865–888.
- [25] N. Kiyotaki, J. Moore, Credit cycles, *J. Polit. Economy* 105 (1997) 211–248.
- [26] N. Kocherlakota, Bubbles and constraints on debt accumulation, *J. Econ. Theory* 57 (1992) 245–256.
- [27] N. Kocherlakota, Bursting bubbles: consequences and cures, working paper, University of Minnesota, 2009.
- [28] R.E. Lucas, N. Stokey, Liquidity crisis, Economic Policy Paper 11-3, Federal Reserve Bank of Minneapolis, 2011.
- [29] A. Martin, J. Ventura, Economic growth with bubbles, *Amer. Econ. Rev.* 102 (2012) 3033–3058.
- [30] J. Miao, Introduction to economic theory of bubbles, *J. Math. Econ.* 53 (2014) 130–136.
- [31] J. Miao, P. Wang, Bubbles and credit constraints, working paper, Boston University, 2011.
- [32] J. Miao, P. Wang, Bubbles and total factor productivity, *Amer. Econ. Rev.* 102 (2012) 82–87.
- [33] J. Miao, P. Wang, Sectoral bubbles, misallocation, and endogenous growth, *J. Math. Econ.* 53 (2014) 153–163.
- [34] J. Miao, P. Wang, Z. Xu, A Bayesian DSGE model of stock market bubbles and business cycles, working paper, Boston University, 2013.
- [35] J. Miao, P. Wang, L. Xu, Stock market bubbles and unemployment, working paper, Boston University, 2013.
- [36] J. Miao, P. Wang, J. Zhou, Housing bubbles and policy analysis, working paper, Boston University, 2014.
- [37] F. Perri, V. Quadrini, International recessions, working paper, University of Southern California, 2013.
- [38] M.S. Santos, M. Woodford, Rational asset pricing bubbles, *Econometrica* 65 (1997) 19–58.
- [39] J.A. Scheinkman, W. Xiong, Overconfidence and speculative bubbles, *J. Polit. Economy* 111 (2003) 1183–1219.
- [40] R.J. Shiller, *Irrational Exuberance*, second ed., Princeton University Press, 2005.
- [41] J. Tirole, Asset bubbles and overlapping generations, *Econometrica* 53 (1985) 1499–1528.
- [42] P. Wang, Y. Wen, Speculative bubbles and financial crisis, *Am. Econ. J. Macroecon.* 4 (2012) 184–221.
- [43] P. Weil, Confidence and the real value of money in an overlapping generations economy, *Quart. J. Econ.* 102 (1987) 1–22.