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A NOTE ON THE TRIPLE-ZERO LINEAR DEGENERACY: NORMAL FORMS, DYNAMICAL AND BIFURCATION BEHAVIORS OF AN UNFOLDING

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This paper is devoted to the analysis of bifurcations in a three-parameter unfolding of a linear degeneracy corresponding to a triple-zero eigenvalue. We carry out the study of codimension-two local bifurcations of equilibria (Takens–Bogdanov and Hopf-zero) and show that they are nondegenerate. This allows to put in evidence the presence of several kinds of bifurcations of periodic orbits (secondary Hopf, ...) and of global phenomena (homoclinic, heteroclinic). The results obtained are applied in the study of the Rössler equation.

Keywords: Local bifurcations; triple-zero degeneracy; normal forms; global connections.

1. Introduction

The analysis of bifurcations in codimension-one and -two linear degeneracies of equilibria of autonomous systems can be found in several textbooks on nonlinear dynamics (see [Guckenheimer & Holmes, 1983; Wiggins, 1990; Kuznetsov, 1995]). These cases correspond to a simple-zero eigenvalue (saddle-node bifurcation), a pair of pure imaginary eigenvalues (Hopf bifurcation), a double-zero eigenvalue with geometric multiplicity one (Takens–Bogdanov bifurcation), a zero and a pair of pure imaginary eigenvalues (Hopf-zero bifurcation), and two pairs of pure imaginary eigenvalues rationally independent (Hopf–Hopf interaction). Our goal here is to analyze the codimension-three case corresponding to a triple-zero eigenvalue with geometric multiplicity one. This case will be referred in this paper as triple-zero bifurcation.

On this subject, only partial results have been obtained. Namely, [Medved, 1984] showed the appearance of saddle-node, Hopf, Takens–Bogdanov and Hopf-zero linear degeneracies in the triple-zero bifurcation, but without characterizing them. Yu and Huseyin [1988] analyzed the Hopf bifurcation (their study is also useful to determine the character of the quasiperiodic motions born in the Hopf-zero bifurcation) and applied the results to the analysis of an electronic circuit.

More recently, [Dumortier & Ibáñez, 1996, 1998] have classified the triple-zero singularity up to codimension four. Ibáñez and Rodríguez [1995] have analyzed a four-parameter unfolding corresponding to a codimension-four degenerate case, that arises when a second-order normal form coefficient vanishes. In this situation, the presence of a saddle-focus homoclinic connection satisfying the Shil'nikov condition is proved.

After writing this paper, we received a work by Dumortier *et al.* [2001] where a bifurcation analysis of the unfolding of the triple-zero singularity has been addressed. Namely, they point out the presence of nondegenerate Takens–Bogdanov and Hopf-zero bifurcations in this unfolding. Nevertheless, in the quoted paper, the authors consider the normal form only up to second order, so that they cannot characterize the bifurcation loci near the Hopf-zero bifurcation (this task is done here in Theorem 5). Moreover, we carry out the local approximations of the bifurcations that arise near the Takens–Bogdanov and Hopf-zero bifurcations. Finally, the blow up (11) is also considered in [Dumortier *et al.*, 2001].

On the other hand, a number of interesting results (concerning global connections) and open problems can also be found in [Dumortier *et al.*, 2001], in particular those aspects related to the analysis of the transition map by means of the concept of traffic regulator.

Through the present paper we will assume that we have already done a reduction of the system using centre manifold theory, and then we deal with a tridimensional autonomous system. Also, we assume that we have an equilibrium point located at the origin whose linear part has been put in Jordan form, so that the linearization matrix is one Jordan block with zero eigenvalue.

The purpose of this work is to look at the complete unfolding of the triple-zero bifurcation. The analysis we perform reveals the complexity of behaviors that are involved in this important codimension-three bifurcation.

This paper is organized as follows. In Sec. 2, we start with a review of the normal form theory in the case of the triple-zero linear degeneracy. Next, we focus on obtaining the simplest normal form for the triple-zero singularity, including formulae for the coefficients of the normal form. This task is of primary importance, and has a double utility. On the one hand, we determine all the terms that are inessential in the dynamical and bifurcation behaviors. On the other hand, as we can compute the normal form coefficients, we can detect parameter values where nonlinear degeneracies take place. Close to these parameter values, even more complicated bifurcation behaviors can be expected. Numerical study without the help of this previous analysis becomes significantly difficult.

In Sec. 3 we analyze the local codimension-two bifurcations of equilibria for a three-parameter unfolding for the normal form obtained in Sec. 2. In particular, we show the presence of a nondegenerate Takens–Bogdanov bifurcation. Also, there is a nondegenerate Hopf-zero bifurcation, that falls into the case III (see [Guckenheimer & Holmes, 1983, Sec. 7.4]). Notice that this is one of the most delicate situations in the Hopf-zero bifurcation; because we can be assured of the presence of global connections and of a secondary Hopf bifurcation of periodic orbits. In a companion paper [Feire *et al.*, 2001], we study this secondary Hopf bifurcation and justify the appearance of resonance phenomena. This is done by using the Kuramoto–Sivashinsky system, which also reveals the presence of a lot of interesting bifurcations involved in the triple-zero bifurcation: period-doubling, Takens–Bogdanov bifurcations of periodic orbits and global connections corresponding to the coincidence of two one-dimensional invariant manifolds of equilibria (T-points). In a sense, from the results in [Freire *et al.*, 2001] we can say that, in the analysis of global behavior near the triple-zero linear degeneracy, the reversible systems in $\mathbb{R}^3$ (Kuramoto–Sivashinsky) play the same role as the Hamiltonian systems in $\mathbb{R}^2$, when analyzing the Takens–Bogdanov bifurcation (double-zero).

Finally, in Sec. 4 we apply the results previously achieved to the study of the Rössler equation, where a number of interesting phenomena are found. In particular, we detect the presence of cascades of period-doubling bifurcations of periodic orbits and invariant torus organized by a sequence of Takens–Bogdanov bifurcations of periodic orbits. Some chaotic attractors, born after the cascade, are presented.

2. Normal Form Analysis

The first step in the analysis of a linear degeneracy consists in the normalization of the singularity. Once one has obtained a reduction in the dimension of the problem (computing centre manifolds), one tries to simplify the system to an appropriate normal form. Sometimes, the normal form (which, in a sense, has the simplest analytical expression) is obtained by means of near-identity transformations in the state variables ($\mathcal{C}^\infty$-conjugacy), whereas in other cases a reparametrization of the time depending on the state variables is also needed ($\mathcal{C}^\infty$-equivalence).

2.1. *Preliminary results on normal forms*

Next, we review the basic ideas of computing normal forms for the triple-zero linear degeneracy. A detailed analysis, based on Lie transforms, can be found in [Gamero *et al.*, 1999].

Let us consider a three-dimensional system having an equilibrium at the origin with a zero eigenvalue with geometric multiplicity one. Choosing an adequate coordinate system, the system can be written in the form

$$\begin{aligned}\dot{x} &= y + f(x,\,y,\,z)\,,\\ \dot{y} &= z + g(x,\,y,\,z)\,,\\ \dot{z} &= h(x,\,y,\,z)\,,\end{aligned} \tag{1}$$

where f, g, h stand for the nonlinear terms.

When analyzing nonhyperbolic equilibria, it is of primary importance to put the system in a form where only the dominating nonlinear terms become apparent. In other words, it is very relevant to determine the nonlinear terms that can be removed, in order to obtain the simplest equivalent system which gives account of the original dynamics. This is done by means of the normal form theory, which establishes, based on the knowledge of the linear part, how much we can simplify the nonlinear terms degree by degree (i.e. by means of successive near-identity transformations).

In our case, it is easy to show (see [Elphick *et al.*, 1987]), that a normal form is

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= z\,,\\ \dot{z} &= \sum_{k\geq 2}\left(\sum_{j=0}^{\lfloor (k-1)/2\rfloor}(a_{k,j}x^{k-j-1}z^{j+1}\right.\\ &\qquad \left.+\,b_{k,j}x^{k-2j-1}y^{2j+1}) + \sum_{j=0}^{\lfloor k/2\rfloor} c_{k,j}x^{k-2j}y^{2j}\right),\end{aligned} \tag{2}$$

where $\lfloor\cdot\rfloor$ denotes the *floor* function.

The problem of computing normal forms for the triple-zero linear degeneracy has been addressed in [Gamero *et al.*, 1999], where an algorithm based on the Lie triangle, well suited for symbolic computations, is presented. The algorithm provides the expressions for the normal form coefficients. For instance, we present these coefficients up to third-order. Namely, denoting the third-order normal form of (1) by

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= z\,,\\ \dot{z} &= a_1xz + a_2xy + a_3x^2 + a_4y^2 + b_1x^2y\\ &\quad + b_2xz^2 + b_3x^3 + b_4x^2z + b_5y^3 + b_6xy^2\,,\end{aligned} \tag{3}$$

we have

$$\begin{aligned}a_1 &= f_{xx} + g_{xy} + h_{xz}\,,\\ a_2 &= g_{xx} + h_{xy}\,,\\ a_3 &= \frac{1}{2}h_{xx}\,,\\ a_4 &= f_{xx} + g_{xy} + \frac{1}{2}h_{yy}\,,\\ b_1 &= \frac{1}{2}g_{xxx} + \frac{1}{2}h_{xxy} + \frac{3}{2}f_{xy}g_{xx} - \frac{3}{2}g_{xy}f_{xx} + \frac{1}{6}g_{yy}g_{xx} - \frac{1}{3}g_{xz}g_{xx} - \frac{1}{2}h_{yy}f_{xx}\\ &\quad - h_{xz}f_{xx} - \frac{1}{3}h_{yz}g_{xx} - \frac{1}{3}h_{xx}h_{zz} + \frac{1}{6}h_{xy}g_{yy} + \frac{1}{2}h_{xy}f_{xy} + \frac{1}{6}h_{xy}h_{yz}\\ &\quad + \frac{7}{6}h_{xy}g_{xz} + \frac{4}{3}h_{xx}f_{xz} - \frac{1}{6}h_{xx}f_{yy} - \frac{1}{6}h_{xx}g_{yz} - h_{xz}g_{xy}\,,\\ b_2 &= -\frac{1}{4}f_{xyy} + \frac{1}{2}f_{xxz} - \frac{1}{4}g_{yyy} + \frac{1}{2}g_{xyz} - \frac{1}{4}h_{yyz} + \frac{1}{2}h_{xzz} + \frac{1}{4}f_{xx}g_{zz}\\ &\quad + \frac{1}{2}f_{yz}f_{xx} + \frac{1}{4}g_{xy}g_{zz} + \frac{1}{4}g_{yy}h_{zz} - \frac{1}{2}g_{xz}h_{zz} + \frac{1}{4}h_{yy}g_{zz} - \frac{1}{4}h_{xz}g_{zz}\\ &\quad - \frac{1}{4}f_{xz}g_{yy} - \frac{1}{2}f_{xz}f_{xy} - \frac{1}{2}f_{xz}h_{yz} - \frac{1}{2}f_{xz}g_{xz} + \frac{1}{4}f_{yy}g_{yy} + \frac{1}{4}f_{yy}f_{xy}\\ &\quad + \frac{1}{4}f_{yy}h_{yz} + \frac{1}{4}g_{yy}g_{yz} - \frac{1}{2}g_{xz}g_{yz} + \frac{1}{2}g_{xy}f_{yz} + \frac{1}{4}h_{yy}f_{yz}\,,\end{aligned} \tag{4}$$

$$b_3 = \frac{1}{6}h_{xxx} + \frac{1}{2}h_{xx}f_{xy} + \frac{1}{2}h_{xx}g_{xz} - \frac{1}{2}h_{xy}f_{xx} - \frac{1}{2}h_{xz}g_{xx}\,,$$

$$\begin{aligned}
b_4 = {} & \frac{1}{2}f_{xxx} + \frac{1}{2}g_{xxy} + \frac{1}{2}h_{xxz} - \frac{1}{2}f_{xz}g_{xx} - \frac{1}{3}g_{yy}f_{xx} + \frac{1}{6}g_{xz}f_{xx} - \frac{1}{2}g_{yz}g_{xx} \\
& - \frac{1}{2}g_{xx}h_{zz} - \frac{1}{3}h_{yz}f_{xx} - \frac{1}{2}h_{xx}g_{zz} + \frac{1}{6}h_{xz}g_{yy} + \frac{1}{2}h_{xz}f_{xy} + \frac{1}{6}h_{xz}h_{yz} \\
& + \frac{1}{6}h_{xz}g_{xz} - \frac{1}{2}h_{xx}f_{yz} + \frac{1}{6}g_{xy}g_{yy} + \frac{1}{2}g_{xy}f_{xy} + \frac{1}{6}g_{xy}h_{yz} + \frac{1}{6}g_{xy}g_{xz}\,,
\end{aligned}$$

$$\begin{aligned}
b_5 = {} & \frac{1}{6}g_{xyy} - \frac{1}{3}g_{xxz} + \frac{1}{6}h_{yyy} - \frac{1}{3}h_{xyz} - \frac{2}{9}f_{xz}f_{xx} + \frac{1}{9}f_{yy}f_{xx} + \frac{1}{18}f_{xx}h_{zz} \\
& - \frac{2}{3}f_{yz}g_{xx} + \frac{1}{9}g_{yz}f_{xx} - \frac{1}{6}g_{xx}g_{zz} + \frac{1}{18}g_{xy}h_{zz} - \frac{1}{6}h_{xx}f_{zz} - \frac{1}{2}h_{xy}g_{zz} \\
& - \frac{1}{6}h_{yy}h_{zz} + \frac{1}{18}h_{xz}h_{zz} - \frac{1}{3}f_{xy}g_{xz} - \frac{1}{18}f_{yy}g_{xy} + \frac{4}{9}f_{xz}g_{xy} - \frac{5}{18}g_{xz}g_{yy} \\
& + \frac{2}{9}g_{xz}h_{yz} + \frac{4}{9}g_{xz}^2 - \frac{2}{9}g_{xy}g_{yz} + \frac{1}{9}g_{yy}^2 - \frac{5}{18}g_{yy}h_{yz} - \frac{5}{9}h_{xz}f_{xz} + \frac{1}{9}h_{xz}f_{yy} \\
& + \frac{1}{9}h_{xz}g_{yz} - \frac{1}{3}h_{xy}f_{yz} + \frac{1}{9}h_{yz}^2 + \frac{2}{3}h_{yy}f_{xz} - \frac{1}{3}h_{yy}f_{yy} + \frac{1}{6}h_{yy}g_{yz}\,,
\end{aligned}$$

$$\begin{aligned}
b_6 = {} & f_{xxx} + g_{xxy} + \frac{1}{2}h_{xyy} - f_{xz}g_{xx} - \frac{1}{3}g_{yy}f_{xx} - \frac{1}{3}g_{xz}f_{xx} - g_{yz}g_{xx} \\
& - \frac{1}{2}g_{xx}h_{zz} - \frac{1}{3}h_{yz}f_{xx} - 2h_{xx}g_{zz} - \frac{1}{2}h_{xy}h_{zz} - \frac{1}{6}h_{xz}g_{yy} + \frac{1}{3}h_{xz}h_{yz} \\
& + \frac{1}{3}h_{xz}g_{xz} - 3h_{xx}f_{yz} + \frac{2}{3}g_{xy}g_{yy} + g_{xy}f_{xy} - \frac{1}{3}g_{xy}h_{yz} - \frac{1}{3}g_{xy}g_{xz} \\
& + 2h_{xy}f_{xz} - \frac{1}{2}h_{xy}f_{yy} + \frac{1}{6}h_{yy}g_{yy} + \frac{1}{6}h_{yy}h_{yz} + \frac{2}{3}h_{yy}g_{xz}\,,
\end{aligned}$$

where the derivatives of functions f, g, h of (1) are taken at the origin.

As pointed out in [Gamero *et al.*, 1999], the normal form (3) is not the simplest one. It is evident that the linear part of the system determines the homological operator, and consequently, the monomials appearing in the normal form. Taking into account the nonlinear part of the system, it is possible to obtain further simplifications, annihilating some terms in the normal form (2).

The key idea is that the solution of the homological equation (which provides the normal form) is nonunique. So, if we take the general solution of the homological equation, some arbitrary constants will appear in the higher-order terms of the normal form (in particular, the expressions for these coefficients are nonunique). Choosing adequately these arbitrary constants, we can annihilate some terms in the normal form (for that, we must assume that some conditions on the low-order terms hold). In fact, in [Gamero *et al.*, 1999], we define a linear operator (depending on the second-order terms) that allows to simplify the expression of the normal form of order greater than two. On the other hand, the second-order terms in the normal form can be simplified by means of a linear transformation leaving unaltered the linear part of the system (that is, taking a matrix for the linear transformation commuting with the linear part of the system).

Ushiki [1984] and Gamero *et al.* [1999] have shown that, if the nonlinear terms of system (1) satisfy $h_{xx} \neq 0$, then the monomials y^2, x^2z, y^3, xy^2 in the normal form (3) can be annihilated, obtaining the following reduced normal form or *hypernormal* form up to third order:

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= z\,, \\ \dot{z} &= a_1xz + a_2xy + a_3x^2 \\ &\quad + b_1'x^2y + b_2'xz^2 + b_3'x^3\,. \end{aligned} \tag{5}$$

2.2. *Towards the simplest normal form*

In the present paper, we will show that the third-order normal form (5) can be further simplified by considering the use of a reparametrization of the time as a function of the state variables. Namely, the following result holds:

Theorem 1. *Suppose that system (1) satisfies* $h_{xx} \neq 0$ *(that is,* $a_3 \neq 0$*). Then, this system is* $\mathcal{C}^\infty$*-equivalent to*

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= z\,, \\ \dot{z} &= A_1xz + A_2xy - \frac{1}{2}x^2 + B_1x^2y \\ &\quad + B_2xz^2 + \mathcal{O}\left(|x,\, y,\, z|^4\right)\,, \end{aligned} \tag{6}$$

where

$$A_1 = -\frac{a_1}{2a_3}\,,$$

$$A_2 = -\frac{a_2}{2a_3}\,,$$

$$B_1 = \frac{18b_1a_3 - 16b_3a_2 + 12a_4^2a_3 - 3a_2^2a_4 - 18a_4a_1a_3}{72a_3^3}\,,$$

$$B_2 = \frac{\begin{array}{c}48b_2a_3^2 + 16b_3a_1^2 + 24b_4a_4a_3 - 3a_1^2a_2a_4 \\ + 6a_1b_6a_3 - 18b_3a_1a_4 - 30a_1b_4a_3\end{array}}{192a_3^4}\,.$$

Proof. We suppose that system (1) has been reduced to the third-order normal form (3). Next, we change the time by

$$\begin{aligned} \frac{dt}{dT} &= 1 + \alpha_1x + \alpha_2y + \alpha_3z + \beta_1x^2 + \beta_2xy \\ &\quad + \beta_3xz + \beta_4y^2 + \beta_5yz + \beta_6z^2\,, \end{aligned}$$

and we make the linear change

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Note that the matrix of the above transformation commutes with the matrix A, and so, this linear transformation preserves the linear part. Let us denote the third-order normal form of the resulting system by

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= z\,, \\ \dot{z} &= A_1xz + A_2xy + A_3x^2 + A_4y^2 + B_1x^2y + B_2xz^2 \\ &\quad + B_3x^3 + B_4x^2z + B_5y^3 + B_6xy^2\,. \end{aligned}$$

Using the algorithm developed in [Gamero *et al.*, 1999], we obtain the following expressions for the above coefficients:

$$\begin{aligned} A_1 &= a_1\,, \quad A_2 = a_2\,, \quad A_3 = a_3\,, \quad A_4 = 2\delta a_3 + a_4\,, \\ B_1 &= b_1 - \frac{8}{3}a_3^2\delta^2 - \frac{8}{3}a_4a_3\delta + 2a_1a_3\delta + \frac{8}{3}\alpha_1a_2 + \frac{1}{3}a_2^2\delta\,, \\ B_2 &= b_2 + 6a_3\delta\lambda_2 + 4a_3\delta\lambda_1 - \frac{7}{2}\alpha_1a_1\delta + 3a_4\lambda_2 + 2a_4\lambda_1 - 6\alpha_3a_3\delta \\ &\quad + 6\alpha_1a_3\delta^2 - \frac{7}{2}a_1\lambda_2 - \frac{3}{2}a_1\lambda_1 - \frac{3}{2}a_1a_2\delta^2 + 3a_2\delta^3a_3 + \frac{7}{2}\alpha_3a_1 \\ &\quad - b_4\delta + 3\alpha_1a_4\delta - 3\alpha_3a_4 + \frac{3}{2}a_4\delta^2a_2\,, \\ B_3 &= b_3 + 3\alpha_1a_3\,, \\ B_4 &= b_4 - 6a_3\lambda_2 - 4a_3\lambda_1 - 6\alpha_1a_3\delta - 3a_2\delta^2a_3 + \frac{1}{3}a_1a_2\delta + \frac{5}{3}\alpha_1a_1 + 6\alpha_3a_3\,, \end{aligned}$$

$$B_5 = b_5 + a_2\lambda_2 - \frac{1}{3}a_2\lambda_1 - \frac{2}{3}\beta_1 + \frac{2}{3}\alpha_2 a_3\delta + \frac{7}{9}\alpha_1 a_2\delta + \frac{2}{3}a_3\lambda_3 + \frac{2}{3}b_1\delta$$
$$- \alpha_3 a_2 - \frac{10}{3}a_3\delta^2 a_4 + \frac{23}{9}a_3\delta^2 a_1 + \frac{20}{9}a_1 a_4\delta - 2a_3^2\delta^3$$
$$+ \frac{1}{9}\alpha_1^2 + \frac{1}{9}a_2^2\delta^2 - \frac{2}{3}\delta a_1^2 - \frac{4}{3}a_4^2\delta\,,$$
$$B_6 = b_6 - 2a_3\lambda_2 - 8a_3\lambda_1 + \frac{8}{3}\alpha_1 a_3\delta - \frac{17}{3}a_2\delta^2 a_3 + \frac{8}{3}a_1 a_2\delta + \frac{1}{3}\alpha_1 a_1$$
$$+ 2\alpha_3 a_3 + 6b_3\delta + \frac{7}{3}\alpha_1 a_4 - \frac{4}{3}a_2 a_4\delta\,,$$

where λ_1, λ_2, λ_3 are arbitrary constants that appear when the homological equation of second order is solved. We note that the above expressions depend nonlinearly upon the parameters. However, A_4 and B_3 depend linearly upon δ and α_1. Taking $\delta = -(1/2)a_4/a_3$ and $\alpha_1 = -(1/3)b_3/a_3$, we can annihilate A_4 and B_3. In fact, the third-order normal form coefficients are:

$$\begin{aligned}
&A_1 = a_1\,, \quad A_2 = a_2\,, \quad A_3 = a_3\,, \quad A_4 = 0\,,\\
&B_1 = b_1 - \frac{16b_3a_2 - 12a_4^2a_3 + 3a_2^2a_4 + 18a_4a_1a_3}{18a_3}\,,\\
&B_2 = b_2 - \frac{28\lambda_1a_1a_3^2 - 4\lambda_2a_1a_3^2 - 8b_4a_4a_3 - a_1a_2a_4^2}{16a_3^2}\,,\\
&B_3 = 0\,,\\
&B_4 = b_4 - \frac{54\lambda_1a_3^2 + 18\lambda_2a_3^2 + 10b_3a_1 + 3a_1a_2a_4}{18a_3}\,,\\
&B_5 = b_5 + \frac{\begin{array}{c}324\lambda_1a_2a_3^2 - 540\lambda_2a_2a_3^2 + 432\lambda_3a_3^3 - 216\alpha_2a_4a_3^2 - 432\beta_1a_3^2 - 24b_3a_2a_4\\ - 216b_1a_4a_3 + 54a_4^3a_3 - 306a_3a_4^2a_1 - 63a_2^2a_4^2 + 8b_3^2 + 216a_3a_4a_1^2\end{array}}{648a_3^2}\,,\\
&B_6 = b_6 - \frac{18\lambda_1a_3^2 + 126\lambda_2a_3^2 + 54b_3a_4 + 9a_2a_4^2 + 24a_1a_2a_4 + 2b_3a_1}{18a_3}\,.
\end{aligned} \tag{7}$$

Notice that the parameters λ_1, λ_2, λ_3, α_2 and β_1, appear linearly in the expressions of B_2, B_4, B_5 and B_6. The matrix of the linear system: $B_2 = 0$, $B_4 = 0$, $B_5 = 0$, $B_6 = 0$, with unknowns λ_1, λ_2, λ_3, α_2, β_1, is

$$\begin{pmatrix} 7a_1 & -a_1 & 0 & 0 & 0\\ 3a_3 & a_3 & 0 & 0 & 0\\ \frac{1}{2}a_2 & -\frac{5}{6}a_2 & \frac{2}{3}a_3 & -\frac{1}{3}a_4 & -\frac{2}{3}\\ a_3 & 7a_3 & 0 & 0 & 0 \end{pmatrix}.$$

Under the hypothesis $a_3 \neq 0$ (or equivalently $h_{xx} \neq 0$), the above matrix has rank three, and it is possible to annihilate B_4, B_5 and B_6, by selecting adequately the constants λ's. Notice that the role of parameters α_2, α_3 and β's is irrelevant, so we can change the time in the form $dT/dt = 1 + \alpha_1 x$. Also, to get the expression for B_2 in the statement of the theorem, it is necessary to obtain the values of λ_1, λ_2 by solving $B_4 = 0$, $B_6 = 0$.

Finally, to achieve $A_3 = -1/2$, it suffices to make a rescaling $x \to \gamma x$, $y \to \gamma y$, $z \to \gamma z$ where $\gamma = -1/2a_3$. ■

The singularities (3) are classified in [Dumortier & Ibáñez, 1996] up to codimension four. In [Ibáñez & Rodríguez, 1995], the behavior of a four-parameter unfolding is also partially studied in the codimension-four degenerate case, corresponding to the vanishing of the normal form coefficient a_3. In this situation, we have the following hypernormal form:

Theorem 2. *Suppose that system (1) satisfies $h_{xx} = 0$ (that is, $a_3 = 0$), and the coefficients a_1 and a_2 given in (4) of the normal form (3) are nonzero. Then (1) is $\mathcal{C}^\infty$-equivalent to*

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= z\,,\\ \dot{z} &= a_1xz + a_2xy + a_4y^2 + b_3x^3\\ &\quad + Bxy^2 + \mathcal{O}(|x,\,y,\,z|^4)\,,\end{aligned}$$

where all the coefficients agree with the ones given in (3), except

$$B = b_6 + \frac{\begin{aligned}&13a_4a_2^2b_4 - 9a_4a_2b_1a_1 - 21b_4a_2^2a_1\\ &\quad + 13a_1^2a_2b_1 - 48b_3a_2b_4 + 30b_3b_1a_1\end{aligned}}{a_1a_2^2}\,.$$

The proof of this theorem is similar to the above one.

3. Local Codimension-Two Bifurcations of Equilibria

In the following, we will consider the nondegenerate case $h_{xx} \neq 0$. According to Theorem 1, we consider the three-parameter unfolding of the truncated hypernormal form (6) given by

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= z\,,\\ \dot{z} &= \varepsilon_1 + \varepsilon_2 y + \varepsilon_3 z + A_1xz + A_2xy\\ &\quad - \frac{x^2}{2} + B_1x^2y + B_2xz^2\,.\end{aligned} \tag{8}$$

The above family corresponds to the normal form for the unfoldings of (1), obtained in [Dumortier & Ibáñez, 1996], truncating up to first order in the parameters and up to third order in the state variables. In fact, we have embedded the truncated normal form (6) in a three-parameter family obtained by adding a versal unfolding of the linear part.

It is easy to show that the above system has no equilibria for $\varepsilon_1 < 0$, for $\varepsilon_1 = 0$ the unique equilibrium is the origin, whereas for $\varepsilon_1 > 0$ there are two equilibria: $(x_\pm,\,0,\,0)$, where $x_\pm = \pm\sqrt{2\varepsilon_1}$.

Besides the saddle-node bifurcation of the origin at $\varepsilon_1 = 0$, there are other codimension-one bifurcations, corresponding to Hopf bifurcations of each equilibria that appear after the saddle-node. The surface where these bifurcations take place is

$$\begin{aligned}(\varepsilon_3 + A_1x_\pm)\,(\varepsilon_2 + A_2x_\pm + B_1x_\pm^2) &= x_\pm\,,\\ \varepsilon_2 + A_2x_\pm + B_1x_\pm^2 &< 0\,,\\ \varepsilon_1 &> 0\,.\end{aligned} \tag{9}$$

In this section we will analyze the linear degeneracies of codimension 2 that the origin of system (8) exhibits. There are two possibilities:

Takens–Bogdanov: $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 \neq 0$.

Hopf-zero: $\varepsilon_1 = \varepsilon_3 = 0$, $\varepsilon_2 < 0$.

Each of these bifurcations will be analyzed in next subsections.

3.1. *Takens–Bogdanov bifurcation*

It is straightforward to show that, taking $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 \neq 0$, the equilibrium at the origin of system (8) has a double-zero eigenvalue and a third eigenvalue nonzero. To describe this codimension-two linear degeneracy, we take ε_1, ε_2 as bifurcation parameters, and fix $\varepsilon_3 \neq 0$. As we will see later, there are no possibilities of nonlinear degeneracies, and then it will be sufficient to consider the system (8) truncated up to second order. In fact, we will consider $\varepsilon_3 > 0$, because in the case $\varepsilon_3 < 0$ it is enough to change the sign of t, x, z and A_2. We suspend the system considering ε_1, ε_2 as new state variables, satisfying the trivial equations $\dot{\varepsilon}_1 = 0$, $\dot{\varepsilon}_2 = 0$. The suspended system may be written as

$$\dot{\mathsf{u}} = \mathsf{A}\mathsf{u} + \varphi(\mathsf{u})\mathsf{b}\,,$$

where

$$\mathsf{u} = \begin{pmatrix} x\\ y\\ z\\ \varepsilon_1\\ \varepsilon_2\end{pmatrix},\quad \mathsf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & \varepsilon_3 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix},\quad \mathsf{b} = \begin{pmatrix} 0\\ 0\\ 1\\ 0\\ 0\end{pmatrix},$$

and $\varphi(\mathsf{u}) = \varepsilon_2 y + A_1xz + A_2xy - x^2/2$. We denote

$$\mathsf{v} = \begin{pmatrix} x_1\\ x_2\\ \varepsilon_1\\ \varepsilon_2\\ y_1\end{pmatrix},\quad \mathsf{P} = \begin{pmatrix} -\dfrac{1}{\varepsilon_3} & -\dfrac{1}{\varepsilon_3^2} & 1 - \dfrac{1}{\varepsilon_3^3} & 0 & \dfrac{1}{\varepsilon_3^3}\\[2mm] 0 & -\dfrac{1}{\varepsilon_3} & -\dfrac{1}{\varepsilon_3^2} & 0 & \dfrac{1}{\varepsilon_3^2}\\[2mm] 0 & 0 & -\dfrac{1}{\varepsilon_3} & 0 & \dfrac{1}{\varepsilon_3}\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{pmatrix}.$$

The linear change $\mathsf{u} = \mathsf{P}\mathsf{v}$ yields

$$\dot{\mathsf{v}} = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & & & & \varepsilon_3 \end{array}\right) \mathsf{v} + \varphi(\mathsf{P}\mathsf{v}) \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ \hline \varepsilon_3 \end{array}\right).$$

The center manifold for the above system can be expressed as

$$y_1 = h(x_1,\, x_2,\, \varepsilon_1,\, \varepsilon_2)\,,$$

and the corresponding reduced system on the center manifold becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \varepsilon_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \varphi(\mathsf{P}\mathsf{v}) \end{pmatrix},$$

(together with the trivial equations $\dot{\varepsilon}_1 = 0$, $\dot{\varepsilon}_2 = 0$).

Using the linear approximation of the centre manifold:

$$y_1 = h(x_1,\, x_2,\, \varepsilon_1,\, \varepsilon_2) = 0 + \mathcal{O}(2)\,,$$

we obtain the reduced system up to second order:

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= \varepsilon_1 - \frac{1 - 2\varepsilon_3 A_2 - 2\varepsilon_3^2 A_1 - 2\varepsilon_3^3 + 2\varepsilon_3^4 A_2 + 2\varepsilon_3^5 A_1 + \varepsilon_3^6}{2\varepsilon_3^6}\varepsilon_1^2 - \frac{1}{\varepsilon_3^2}\varepsilon_1\varepsilon_2 \\ &\quad - \frac{1 - \varepsilon_3 A_2 - \varepsilon_3^2 A_1 - \varepsilon_3^3}{\varepsilon_3^4} x_1\varepsilon_1 - \frac{1 - 2\varepsilon_3 A_2 - \varepsilon_3^2 A_1 - \varepsilon_3^3 + \varepsilon_3^4 A_2}{\varepsilon_3^5} x_2\varepsilon_1 \\ &\quad - \frac{1}{\varepsilon_3} x_2\varepsilon_2 - \frac{1}{2\varepsilon_3^2} x_1^2 - \frac{1 - \varepsilon_3 A_2}{\varepsilon_3^3} x_1 x_2 - \frac{1 - 2\varepsilon_3 A_2}{2\varepsilon_3^4} x_2^2\,. \end{aligned}$$

A standard normal form computation (see [Gamero *et al.*, 1991]), leads to

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= \varepsilon_1 - \frac{1 - 2\varepsilon_3 A_2 - 2\varepsilon_3^2 A_1 - 2\varepsilon_3^3 + 2\varepsilon_3^4 A_2 + 2\varepsilon_3^5 A_1 + \varepsilon_3^6}{2\varepsilon_3^6}\varepsilon_1^2 - \frac{1}{\varepsilon_3^2}\varepsilon_1\varepsilon_2 \\ &\quad - \frac{1 - \varepsilon_3 A_2 - \varepsilon_3^2 A_1 - \varepsilon_3^3}{\varepsilon_3^4} x_1\varepsilon_1 - \frac{1 - 2\varepsilon_3 A_2 - \varepsilon_3^2 A_1 - \varepsilon_3^3 + \varepsilon_3^4 A_2}{\varepsilon_3^5} x_2\varepsilon_1 - \frac{1}{\varepsilon_3} x_2\varepsilon_2 \\ &\quad - \frac{1}{2\varepsilon_3^2} x_1^2 - \frac{1 - \varepsilon_3 A_2}{\varepsilon_3^3} x_1 x_2\,. \end{aligned}$$

Note that the normal form coefficients do not vanish, and then there are no nonlinear degeneracies. Finally, we translate by

$$X = x_1 - \lambda\,, \quad Y = x_2\,,$$

and rescale

$$X \to \alpha X\,, \quad Y \to \beta Y\,, \quad t \to \frac{\alpha}{\beta} t\,,$$

where

$$\lambda = \frac{\varepsilon_1\left(\varepsilon_3^3 + \varepsilon_3^2 A_1 + \varepsilon_3 A_2 - 1\right)}{\varepsilon_3^2}\,,$$

$$\alpha = -\frac{\varepsilon_3^4}{2(\varepsilon_3 A_2 - 1)^2}\,,$$

$$\beta = \frac{\varepsilon_3^5}{4(\varepsilon_3 A_2 - 1)^3}\,.$$

Then, we get

$$\begin{aligned} \dot{X} &= Y\,, \\ \dot{Y} &= \mu_1 + \mu_2 Y + X^2 + XY\,, \end{aligned}$$

where:

$$\mu_1 = \frac{\begin{array}{r} \varepsilon_1\alpha(2\varepsilon_3^4 + \varepsilon_3^2 A_1^2\varepsilon_1 - 2\varepsilon_3^2\varepsilon_2 \\ + 2\varepsilon_3 A_1 A_2\varepsilon_1 + A_2^2\varepsilon_1) \end{array}}{2\beta^2\varepsilon_3^4}\,,$$

$$\mu_2 = \frac{\alpha(-\varepsilon_3^2\varepsilon_2 + \varepsilon_3 A_1 A_2\varepsilon_1 + A_2^2\varepsilon_1)}{\beta\varepsilon_3^3}\,.$$

As it is well known, there are several bifurcation curves in the μ_1–μ_2 parameter plane for the above system (see e.g. [Guckenheimer & Holmes, 1983]):

- Saddle-node bifurcation of the origin at $\mu_1 = 0$. Two equilibria $(\pm\sqrt{-\mu_1}, 0)$ appear for $\mu_1 < 0$.
- Hopf bifurcation of the equilibrium $(-\sqrt{-\mu_1}, 0)$ at $\mu_1 = -\mu_2^2$, $\mu_2 > 0$. This Hopf bifurcation is subcritical (recall that we are considering $\varepsilon_3 > 0$). In the case $\varepsilon_3 < 0$, the Hopf bifurcation is exhibited by the other equilibrium $(\sqrt{-\mu_1}, 0)$, and it is supercritical.
- Homoclinic connections curve of the equilibrium $(\sqrt{-\mu_1}, 0)$ at $\mu_1 = -(49/25)\mu_2^2 + \mathcal{O}(\mu_2^{5/2})$, $\mu_2 > 0$. In the X–Y plane, the homoclinic connections are repulsive. As the third eigenvalue, which determines the behavior outside the center manifold is ε_3, we conclude that the homoclinic connections are completely repulsive in the three-dimensional system. For $\varepsilon_3 < 0$, the homoclinic connections are completely attractive.

Translating the above information to the three-parameter family (8), we obtain the following result:

Theorem 3. *Consider the system* (8), *with* $\varepsilon_3 > 0$ *fixed. Then, for the critical values* $\varepsilon_1 = \varepsilon_2 = 0$, *we have a nondegenerate Takens–Bogdanov bifurcation. The singularity is of cusp type, and the following bifurcation curves emerge*:

- *Saddle-node bifurcation of the origin at* $\varepsilon_1 = 0$.
- *Subcritical Hopf bifurcation of equilibrium* $(x_-, 0, 0)$ *at*

$$\varepsilon_2 = \frac{\mathcal{O}(\varepsilon_1) + \sqrt{2\varepsilon_1\varepsilon_3^6 + \mathcal{O}(\varepsilon_1^2)}\,(\varepsilon_3 A_2 - 1)}{\varepsilon_3^4}\,,$$
$$-\varepsilon_3^2\varepsilon_2 + \varepsilon_3 A_1 A_2 \varepsilon_1 + A_2^2\varepsilon_1 > 0\,.$$

- *Repulsive homoclinic connections curve of equilibrium* $(x_+, 0, 0)$ *at*

$$\varepsilon_2 = \frac{\mathcal{O}(\varepsilon_1) + 5\sqrt{2\varepsilon_1\varepsilon_3^6 + \mathcal{O}(\varepsilon_1^2)}\,(\varepsilon_3 A_2 - 1)}{7\varepsilon_3^4}\,,$$
$$-\varepsilon_3^2\varepsilon_2 + \varepsilon_3 A_1 A_2 \varepsilon_1 + A_2^2\varepsilon_1 > 0\,.$$

Remark 1. The saddle-node bifurcation of the origin is in full concordance with the expression obtained at the end of the previous section. With respect to the Hopf bifurcation of $(x_-, 0, 0)$, we obtain an approximation (compare with (9) taking $\varepsilon_1, \varepsilon_2 \ll \varepsilon_3$).

Remark 2. In the case $\varepsilon_3 < 0$ fixed, we find a supercritical Hopf bifurcation of equilibrium $(x_+, 0, 0)$ at

$$\varepsilon_2 = \frac{\mathcal{O}(\varepsilon_1) - \sqrt{2\varepsilon_1\varepsilon_3^6 + \mathcal{O}(\varepsilon_1^2)}\,(\varepsilon_3 A_2 - 1)}{\varepsilon_3^4}\,,$$
$$-\varepsilon_3^2\varepsilon_2 + \varepsilon_3 A_1 A_2 \varepsilon_1 + A_2^2\varepsilon_1 > 0\,,$$

and a curve of attractive homoclinic connections of equilibrium $(x_-, 0, 0)$ at

$$\varepsilon_2 = \frac{\mathcal{O}(\varepsilon_1) - 5\sqrt{2\varepsilon_1\varepsilon_3^6 + \mathcal{O}(\varepsilon_1^2)}\,(\varepsilon_3 A_2 - 1)}{7\varepsilon_3^4}\,,$$
$$-\varepsilon_3^2\varepsilon_2 + \varepsilon_3 A_1 A_2 \varepsilon_1 + A_2^2\varepsilon_1 > 0\,.$$

3.2. *Hopf-zero bifurcation*

We now focus on the analysis of the bifurcations we find near $\varepsilon_1 = \varepsilon_3 = 0$, $\varepsilon_2 < 0$, where the equilibrium at the origin in system (8) has a pair of purely imaginary eigenvalues and a third eigenvalue zero. To analyze this codimension-two linear degeneracy, we select ε_1, ε_3 as bifurcation parameters and fix $\varepsilon_2 = -\omega_0^2 < 0$.

We suspend the system (8), adding the equations $\dot{\varepsilon}_1 = 0$, $\dot{\varepsilon}_3 = 0$. Let us denote

$$\mathsf{u} = \begin{pmatrix} x \\ y \\ z \\ \varepsilon_1 \\ \varepsilon_3 \end{pmatrix}, \quad \mathsf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\omega_0^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathsf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and $\varphi(\mathsf{u}) = \varepsilon_3 z + A_1 xz + A_2 xy - x^2/2 + B_1 x^2 y + B_2 xz^2$. The suspended system may be written as

$$\dot{\mathsf{u}} = \mathsf{A}\mathsf{u} + \varphi(\mathsf{u})\mathsf{b}\,.$$

We introduce complex variables by the linear transformation $\mathsf{u} = \mathsf{P}\mathsf{v}$, where

$$\mathsf{v} = \begin{pmatrix} \xi \\ \bar{\xi} \\ Z \\ \varepsilon_1 \\ \varepsilon_3 \end{pmatrix}, \text{ and } \mathsf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\omega_0^2} & 0 & 0 \\ \frac{i\omega_0}{2} & \frac{-i\omega_0}{2} & 0 & \frac{1}{\omega_0^2} & 0 \\ \frac{-\omega_0^2}{2} & \frac{-\omega_0^2}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The transformed system is

$$\dot{\mathsf{v}} = \mathsf{J}\mathsf{v} + \varphi(\mathsf{P}\mathsf{v})\,\mathsf{P}^{-1}\mathsf{b} \tag{10}$$

where

$$\mathsf{J} = \left(\begin{array}{cc|ccc} i\omega_0 & 0 & & & \\ 0 & -i\omega_0 & & & \\ \hline & & 0 & 1 & 0 \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \end{array}\right).$$

We notice that the second equation in the system (10) is the complex conjugate of the first one (so, we can neglect it), whereas the third one is self-conjugate.

After some computations (that are omitted for the sake of brevity), we get the following second-order normal form:

$$\dot{\xi} = i\omega_0\xi + \frac{1}{2}\varepsilon_3\xi + \frac{1 + A_1\omega_0^2 - i\omega_0 A_2}{2\omega_0^4}\xi Z,$$

$$\dot{Z} = \varepsilon_1 - \frac{1 + 2A_1\omega_0^2}{4}\xi\bar{\xi} - \frac{1}{2\omega_0^4}Z^2.$$

The above second-order normal form, in cylindrical coordinates, reads as

$$\dot{r} = \frac{\varepsilon_3}{2}r + \frac{1 + A_1\omega_0^2}{2\omega_0^4}rz,$$

$$\dot{z} = \varepsilon_1 - \frac{1 + 2\omega_0^2 A_1}{4}r^2 - \frac{1}{2\omega_0^4}z^2,$$

$$\dot{\theta} = \omega_0 - \frac{A_2}{2\omega_0^3}z.$$

The simplifications in the third-order terms are more subtle. To achieve the simplest expression we require the use of a reparametrization of time by $dt/dT = 1 + \tau_1 z$, taking τ_1 adequately (see [Algaba *et al.*, 1998]). In fact, disregarding the azimuthal component, a third-order normal form, using $\mathcal{C}^\infty$-equivalence, is

$$\dot{r} = \frac{\varepsilon_3}{2}r + \frac{1 - A_1\varepsilon_2}{2\varepsilon_2^2}rz,$$

$$\dot{z} = \varepsilon_1 - \frac{1 - 2\varepsilon_2 A_1}{4}r^2 - \frac{1}{2\varepsilon_2^2}z^2 + \frac{9A_1A_2 - 20\varepsilon_2A_1^2A_2 - 16\varepsilon_2^2B_2 + 4\varepsilon_2^3A_1B_2}{24\varepsilon_2^3(1 - 2A_1\varepsilon_2)(1 - A_1\varepsilon_2)}z^3.$$

We will take ε_2 negative so that $1 - 2A_1\varepsilon_2 > 0$. Rescaling by:

$$R = -\sqrt{\frac{1 - 2A_1\varepsilon_2}{8\varepsilon_2^2}}r, \quad Z = \frac{1}{2\varepsilon_2^2}z,$$

we get the planar family:

$$\dot{R} = \mu_1 R + aRZ,$$
$$\dot{Z} = \mu_2 - R^2 - Z^2 + cZ^3,$$

where

$$\mu_1 = \frac{\varepsilon_3}{2},$$

$$\mu_2 = \frac{\varepsilon_1}{2\varepsilon_2^2},$$

$$a = 1 - A_1\varepsilon_2 > 0, \quad \text{for } \varepsilon_2 \text{ slightly negative},$$

$$c = \frac{\varepsilon_2(9A_1A_2 - 20\varepsilon_2A_1^2A_2 - 16\varepsilon_2^2B_2 + 4\varepsilon_2^3A_1B_2}{6(1 - 2A_1\varepsilon_2)(1 - A_1\varepsilon_2)}.$$

We note that the values of A_1 and A_2 do not play any role in the type of the singularity:

Theorem 4. *The topological type of the Hopf-zero singularity for system* (*8*) *is determined up to second order and corresponds to case III, following the notation of* [*Guckenheimer & Holmes, 1983, Sec. 7.4*].

Also, we can affirm that there exist several bifurcation curves in the μ_1–μ_2 parameter plane for the above system. In this case, the coefficients A_1, A_2 have influence (really, their signs) on the character of the heteroclinic connections curve, so we assume that they do not vanish:

- Saddle-node bifurcation of the origin at $\mu_2 = 0$.
- Subcritical pitchfork bifurcation of equilibrium $(0, -\sqrt{\mu_2} - (c/2)\mu_2 + o(\mu_2))$ at

$$\mu_2 = \frac{\mu_1^2}{a^2} + c\frac{\mu_1^3}{a^3}, \quad \mu_1 > 0.$$

- Supercritical pitchfork bifurcation of equilibrium $(0, \sqrt{\mu_2} + (c/2)\mu_2 + o(\mu_2))$ at

$$\mu_2 = \frac{\mu_1^2}{a^2} + c\frac{\mu_1^3}{a^3}, \quad \mu_1 < 0\,.$$

- Hopf bifurcation of the equilibrium $(\sqrt{\mu_2 - (\mu_1^2/a^2) - c(\mu_1^3/a^3)}, -(\mu_1/a))$ at $\mu_1 = 0$, $\mu_2 > 0$. This bifurcation is subcritical for $A_1A_2 < 0$ and supercritical for $A_1A_2 > 0$.
- Heteroclinic connections curve of equilibria $(0, \pm(\sqrt{\mu_2} + (c/2)\mu_2 + o(\mu_2)))$ at

$$\mu_1 = -\frac{3a^2c}{6a+4}\mu_2 + \mathcal{O}(\mu_2^{3/2}), \quad \mu_2 > 0\,.$$

Including the azimuthal component, and recovering the original parameters, we obtain useful information about the bifurcations in the truncated Hopf-zero normal form of system (8).

Nonetheless, the correspondence of these bifurcations in the three-parameter family (8) must be done carefully. Generically, the axisymmetry of the Hopf-zero normal form will no longer be present in the original system (8). Also, the heteroclinic connection of the normal form will break giving rise to transversal heteroclinic connections in system (8). As pointed out in [Broer & Vegter, 1984; Kirk, 1991; Gaspard, 1993], we must expect that the heteroclinic connections curve of Theorem 5 is replaced by a edge shaped region bounded by two curves, corresponding to the first and last heteroclinic tangencies. Inside this region, there are heteroclinic and homoclinic orbits to each equilibrium.

In the next theorem, we summarize the bifurcations in the three-parameter unfolding (8).

Theorem 5. *Consider the system (8) and assume that $\varepsilon_2 < 0$ is fixed and the coefficients A_1, A_2 are nonzero. Then, generically, from the Hopf-zero point the following bifurcations emerge*:

- *Saddle-node bifurcation of the origin at* $\varepsilon_1 = 0$.
- *Subcritical Hopf bifurcation of equilibrium* $(-\sqrt{2\varepsilon_1}, 0, 0)$ *at* $\varepsilon_1 = (\varepsilon_2^2\varepsilon_3^2/2(1-A_1\varepsilon_2)^2) + \mathcal{O}(\varepsilon_3^3)$, $\varepsilon_3 > 0$.
- *Supercritical Hopf bifurcation of equilibrium* $(\sqrt{2\varepsilon_1}, 0, 0)$ *at* $\varepsilon_1 = (\varepsilon_2^2\varepsilon_3^2/2(1-A_1\varepsilon_2)^2) + \mathcal{O}(\varepsilon_3^3)$, $\varepsilon_3 < 0$.
- *Secondary Hopf bifurcation of periodic orbits at* $\varepsilon_3 = 0$, $\varepsilon_1 > 0$. This bifurcation gives rise to invariant tori, which are repulsive for $A_1A_2 < 0$ and attractive for $A_1A_2 > 0$.
- *Global connections (homoclinic and heteroclinic) of equilibria in an exponentially small region around the curve*

$$\varepsilon_3 = -\frac{(1-A_1\varepsilon_2)(9A_1A_2 - 20\varepsilon_2A_1^2A_2 - 16\varepsilon_2^2B_2 + 4\varepsilon_2^3A_1B_2)}{2\varepsilon_2(1-2A_1\varepsilon_2)(5-3A_1\varepsilon_2)}\varepsilon_1 + \mathcal{O}(\varepsilon_1^{3/2})\,.$$

The global connections are unstable for $A_1A_2 < 0$, and stable for $A_1A_2 > 0$.

Remark. The saddle-node bifurcation of the origin is in full concordance with the expression obtained at the end of the previous section. With respect to the Hopf bifurcations, we obtain an approximation (compare with (9) taking ε_1, $\varepsilon_3 \ll \varepsilon_2$).

3.3. *Beyond the local analysis*

Obviously, in the neighborhood of a triple-zero linear degeneracy one must expect the presence of a number of bifurcations which do not arise from the local analysis.

To make this fact evident, let us consider the following blowing-up (see [Rodríguez-Luis *et al.*, 1995; Gamero *et al.*, 1995; Dumortier *et al.*, 2001]):

$$\begin{gathered} x \to \varepsilon^3 x, \quad y \to \varepsilon^4 y, \quad z \to \varepsilon^5 z, \quad t \to \varepsilon t, \\ \varepsilon_1 \to \varepsilon^6 c, \quad \varepsilon_2 \to \varepsilon^2\lambda, \quad \varepsilon_3 \to \varepsilon^2\mu\,. \end{gathered} \tag{11}$$

System (8) becomes

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= z\,, \\ \dot{z} &= c + \lambda y - \frac{x^2}{2} + \varepsilon(\mu z + A_2xy) \\ &\quad + \varepsilon^2A_1xz + \varepsilon^4B_2x^2y + \varepsilon^7B_1xz^2\,. \end{aligned}$$

Taking $\varepsilon = 0$ and $\lambda = -1$ we obtain the Kuramoto–Sivashinsky system:

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= z\,, \\ \dot{z} &= c - y - x^2/2\,. \end{aligned} \tag{12}$$

This is a widely studied system (see e.g. [Kent & Elgin, 1991; Michelson, 1986]), whose analysis will provide valuable information about the system (8). Some features of this system (it is time-reversible and divergence free) make it a good model to study global bifurcations in the unfolding (8).

In particular, as the triple-zero unfolding (8) can be written as a small perturbation of the Kuramoto–Sivashinsky system, we are able to justify the appearance of resonance phenomena in the secondary Hopf bifurcation and the presence of several interesting bifurcations: period-doubling, Takens–Bogdanov bifurcations of periodic orbits and global connections corresponding to the coincidence of two one-dimensional invariant manifolds of equilibria (see [Freire *et al.*, 2001]).

4. Application to Rössler Equation

This last section is devoted to apply the results achieved in previous sections to the Rössler equation. This equation was proposed by [Rössler, 1979] as a simple mathematical model exhibiting chaotic behavior. Although the interest of the Rössler equation is mainly pedagogical, its simplicity has attracted the attention of researchers in last decades. The equations we will consider are

$$\begin{aligned} \dot{x} &= -y - z\,, \\ \dot{y} &= x + ay\,, \\ \dot{z} &= bx - cz + xz\,, \end{aligned} \tag{13}$$

where a, b, $c \in \mathbb{R}$ are parameters. The origin is always an equilibrium point. Also, there is another equilibrium at $(c-ab, (ab-c)/a, (c-ab)/a)$, whenever $a \neq 0$. It is a straightforward computation to show that the origin undergoes the following linear degeneracies:

- A transcritical bifurcation **T** at $ab = c$, $ac \neq b+1$.
- A Hopf bifurcation **H** at $a^2c - ac^2 - a + bc = 0$, $ac < b + 1$.
- A Takens–Bogdanov bifurcation **TB** at $b = 1/(a^2 - 1)$, $c = a/(a^2 - 1)$, $a \neq 0, \pm 1, \pm\sqrt{2}$.
- A Hopf-zero linear degeneracy **HZ** at $b = 1$, $a = c \in (-\sqrt{2}, \sqrt{2})$ and also at $a = c = 0$, $b > -1$.
- A triple-zero linear degeneracy **TZ** at $a = c = \pm\sqrt{2}$, $b = 1$ and also at $a = c = 0$, $b = -1$.

In the subsequent analysis, we will consider the triple-zero bifurcation at $a = c = \sqrt{2}$, $b = 1$. To put the Rössler equation in an appropriate form near these critical values, we perform the following transformation:

$$u = y\,, \quad v = x + ay\,, \quad w = ax + (a^2 - 1)y - z\,.$$

System (13) becomes

$$\begin{aligned} \dot{u} &= v\,, \\ \dot{v} &= w\,, \\ \dot{w} &= (ab - c)u + (ac - b - 1)v + (a - c)w \\ &\quad - au^2 + (a^2 + 1)uv - auw - av^2 + vw\,. \end{aligned}$$

Translating by:

$$U = u - \frac{ab - c}{2a}\,, \quad V = v\,, \quad W = w\,,$$

we obtain

$$\begin{aligned} \dot{U} &= V\,, \\ \dot{V} &= W\,, \\ \dot{W} &= \varepsilon_1 + \varepsilon_2 V + \varepsilon_3 W - aU^2 \\ &\quad + (a^2 + 1)UV - aUW - aV^2 + VW\,, \end{aligned} \tag{14}$$

where

$$\varepsilon_1 = \frac{(ab - c)^2}{4a}\,,$$

$$\varepsilon_2 = \frac{a^2c - ab - 2a + a^3b - c}{2a}\,,$$

$$\varepsilon_3 = \frac{2a - c - ab}{2}\,.$$

Notice that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ for the critical values $a = c = \sqrt{2}$, $b = 1$. Moreover, near these critical values, we have $\varepsilon_1 > 0$. Then, this bifurcation

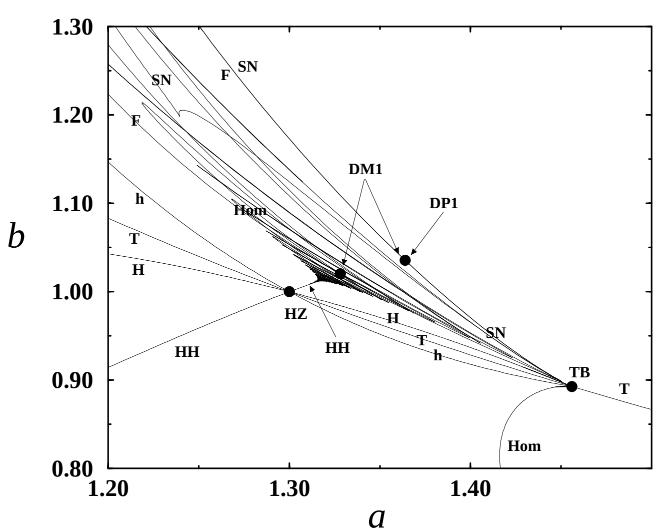


Fig. 1. Partial bifurcation set for the Rössler system (13) with $c = 1.3$.

is degenerate with respect to the parameters, because the origin exhibits a transcritical bifurcation in system (13); whereas in system (8) it exhibits a saddle-node bifurcation. This degeneracy has some minor effects in the bifurcation sets we will obtain. Namely, the bifurcation sets will be folded around the transcritical bifurcation, that plays the role of the saddle-node bifurcation.

At the critical values, we obtain the following expressions for the hypernormal form coefficients in (6):

$$A_1 = -\frac{1}{2} < 0\,, \quad A_2 = \frac{3\sqrt{2}}{4} > 0\,,$$

$$B_1 = -\frac{5}{24}\,, \quad B_2 = \frac{\sqrt{2}}{64}\,.$$

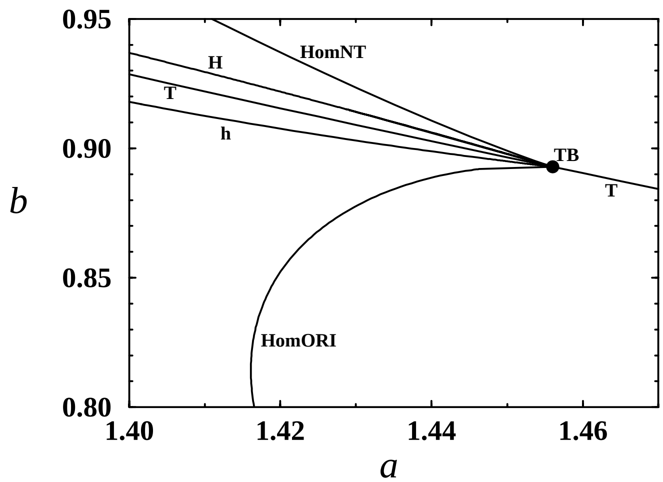


Fig. 2. Zoom of the bifurcation set of Fig. 1 near the Takens–Bogdanov point.

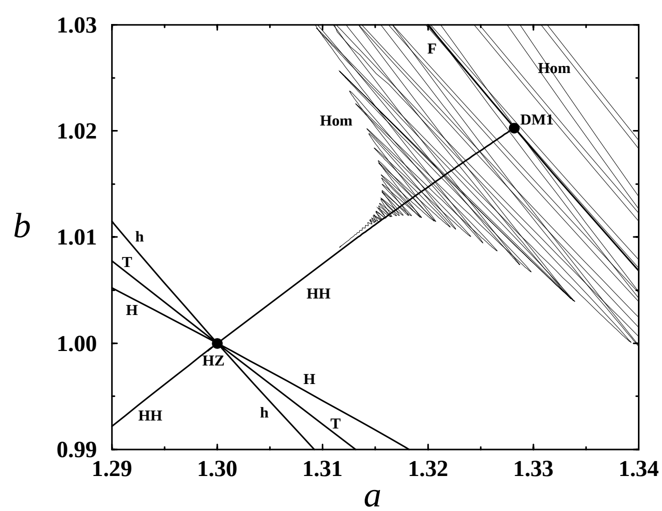
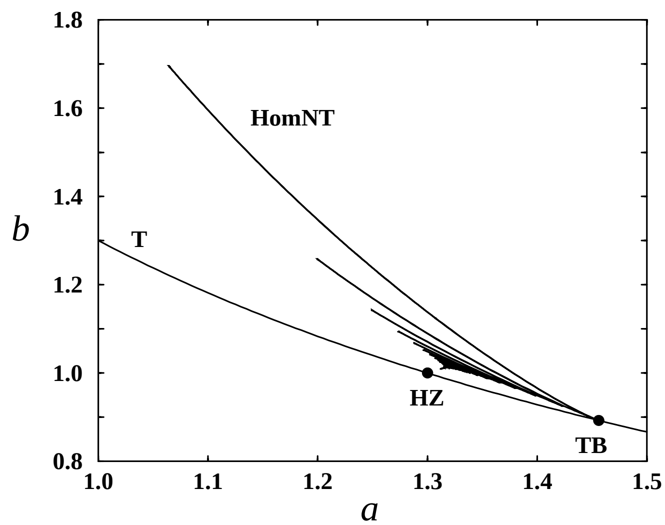


Fig. 3. Zoom of the bifurcation set of Fig. 1 near the Hopf-zero point.

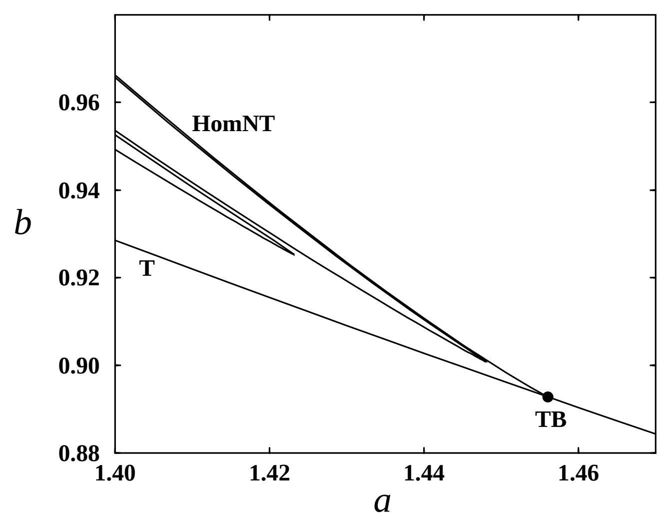
Applying Theorems 3 and 5, we can characterize the local bifurcations we expect to find in system (13) near the critical values $a = c = \sqrt{2}$, $b = 1$. In particular, we can be assured that repulsive invariant tori exist near these critical values.

In the numerical analysis, carried out with AUTO (see [Doedel *et al.*, 1995]) and Dstool (see [Back *et al.*, 1992]), we will fix $c = 1.3$, near to the triple-zero degeneracy ($c = \sqrt{2}$).

Figure 1 shows a partial bifurcation set in the (a, b)-plane, near the triple-zero value. In this figure, we can see the transcritical bifurcation curve **T**. The Takens–Bogdanov **TB** and Hopf-zero **HZ** points are located on this curve. As commented



(a)



(b)

Fig. 4. (a) Homoclinic connections of nontrivial equilibria curve joining the Takens–Bogdanov and Hopf-zero points. (b) Zoom of the neighborhood of TB point.

before, the bifurcation curves are folded around **T**. Although we will focus mainly on the bifurcations located above **T**, in Fig. 1 we have also drawn some curves below it.

In the rest of this section, we will try to understand the appearance of such a rich bifurcation behavior, partially shown in Fig. 1. Namely, the presence of codimension-one (Hopf of the origin **H**, Hopf of the nontrivial equilibrium **h**, period-doubling **F**, saddle-node of periodic orbits **SN**, homoclinic connections of equilibria **Hom**, secondary Hopf bifurcation of periodic orbits **HH**) and codimension-two (Takens–Bogdanov **TB**, Hopf-zero **HZ**, several kinds of Takens–Bogdanov bifurcations of periodic orbits: **DM1**, **DP1**) bifurcations.

In order to clarify the bifurcation set of Fig. 1, we have included some magnifications of the neighborhoods of Takens–Bogdanov and Hopf-zero points. The numerical results presented in Figs. 2 and 3 corroborate the results of Theorems 3 and 5.

Near the Takens–Bogdanov point **TB** (see Fig. 2) we can observe the transcritical bifurcation **T** (corresponding to the saddle-node), the Hopf bifurcation of the origin **H**, and the homoclinic connections curve of the nontrivial equilibrium **HomNT**. Below **T**, we can see the corresponding folded curves: the Hopf bifurcation of the nontrivial equilibrium **h**, and the homoclinic connections curve of the origin **HomORI**.

With respect to the Hopf-zero point (see Fig. 3), we show the transcritical bifurcation **T**, the two Hopf bifurcations **H** and **h**, and the secondary Hopf bifurcation of periodic orbits curve **HH**. Moreover, we can see two homoclinic connections curves that are located in the homoclinic region with edge shape (see Theorem 5). In this picture, both homoclinic curves are labeled **Hom**, because it is very complicated to distinguish them. In Figs. 4 and 5 we present them separately, and will be labeled as above: **HomNT** and **HomORI**.

It is important to note the behavior of nontrivial equilibria homoclinic curve **HomNT**, that starts from **TB**. It goes far from the Takens–Bogdanov point **TB** and later goes back and approaches the

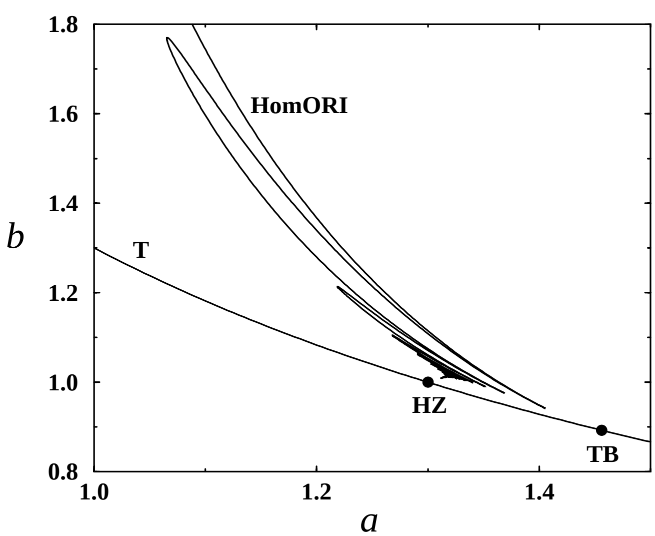
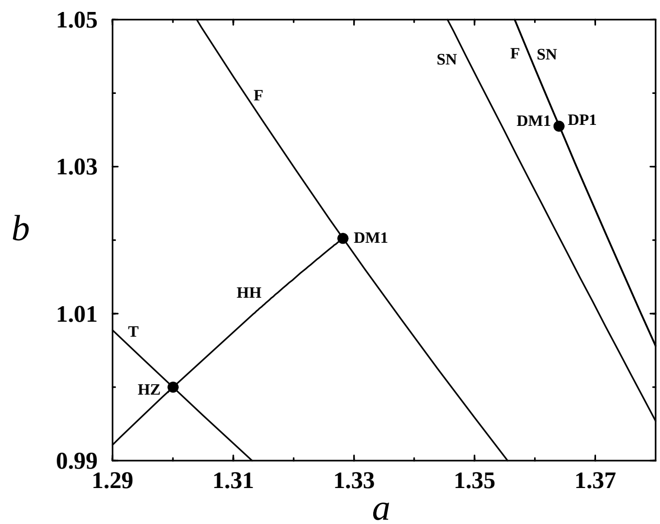
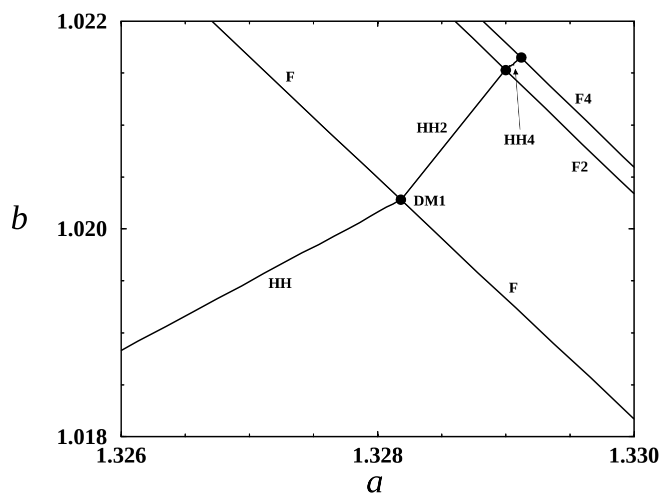


Fig. 5. Homoclinic connections of the origin curve starting from the Hopf-zero point.



(a)



(b)

Fig. 6. (a) Secondary Hopf bifurcation of periodic orbits curve composed by two branches (one of them, bounded by DM1 and DP1, is indistinguishable). (b) Cascade of period-doubling and secondary Hopf bifurcation of periodic orbits.

Hopf-zero point **HZ** wiggling (see Fig. 4). This is an interesting phenomenon because two bifurcation points: **TB** (essentially planar) and **HZ** (tridimensional) are joined by the homoclinic connections curve **HomNT**.

The remaining figures correspond to some phenomena related to the global analysis (see Sec. 3.3). In Fig. 6 we focus on the curve of secondary Hopf bifurcation of periodic orbits, **HH**, located above the transcritical bifurcation **T**. This curve has two branches.

One of them starts from **HZ** and finishes at **DM1**, where there is a degeneracy of the periodic orbit that undergoes the secondary Hopf bifurcation, corresponding to a double -1 Floquet multiplier. The point **DM1** corresponds to the intersection of **HH** with a period-doubling curve **F**. This is the starting point of an accumulating sequence of period-doubling bifurcations. In the figure we have also represented the first steps in the sequence: the curves **HH2** and **HH4**, of *double-period* and *four-period* tori, respectively, and the period-doubling curves **F2** and **F4**. In Fig. 7 we have considered the parameter values $a = 1.32858$, $b = 1.02087$, $c = 1.3$, for which we have taken a section with the plane $z = -0.02$, where we can see the scenario

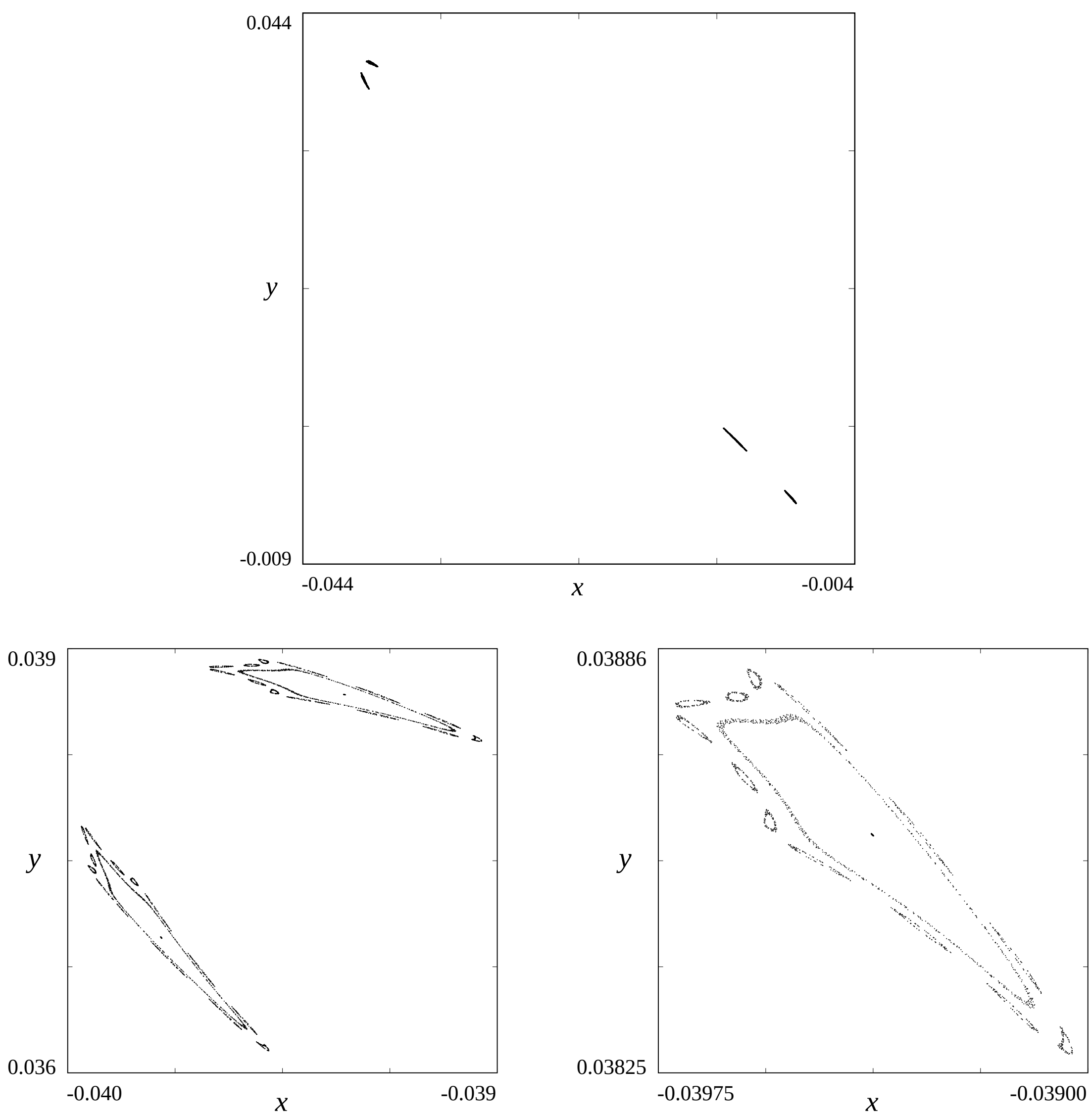


Fig. 7. Section in the plane $z = -0.02$ of a broken double-torus in the Rössler system (13) for $a = 1.32858$, $b = 1.02087$, $c = 1.3$. In the lower part, we have included two magnifications of the upper zone.

after the breakdown of a *double-period* torus due to resonances.

The second branch of **HH** (see Fig. 6) is very tiny and it is bounded by two degeneracies of the periodic orbit: **DM1** and **DP1**, corresponding to double -1 and double $+1$ Floquet multipliers, respectively.

The above phenomena can be explained because the Rössler system can be treated as a perturbation of the Kuramoto–Sivashinsky system. Namely, performing the blowing-up (11) in (14) leads to

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= z\,,\\ \dot{z} &= c + \lambda y - ax^2 + \varepsilon(\mu z + (a^2+1)xy)\\ &\quad - a\varepsilon^2(y^2 + xz) + \varepsilon^3 yz\,.\end{aligned}$$

Then, it is enough to take $\varepsilon = 0$ and rescale to arrive to the Kuramoto–Sivashinsky system (12). Moreover, the shape of the curve **HH** reflects the *noose bifurcation* that undergoes (12) (see [Kent & Elgin, 1991]), which persists when the parameter ϵ is included. For a detailed analysis, we refer to [Freire *et al.*, 2001].

In the rest of this section, we will look for chaotic attractors in the Rössler system, near the triple-zero degeneracy. Namely, the study of the triple-zero bifurcation has allowed (via the Kuramoto–Sivashinsky system (12)), to detect the double -1 degeneracy **DM1**, where the secondary Hopf bifurcation **HH** ends. From **DM1**, a period-doubling bifurcation of periodic orbits **F** emerges. It is enough to look for the cascade of period-doubling bifurcations of periodic orbits, in the

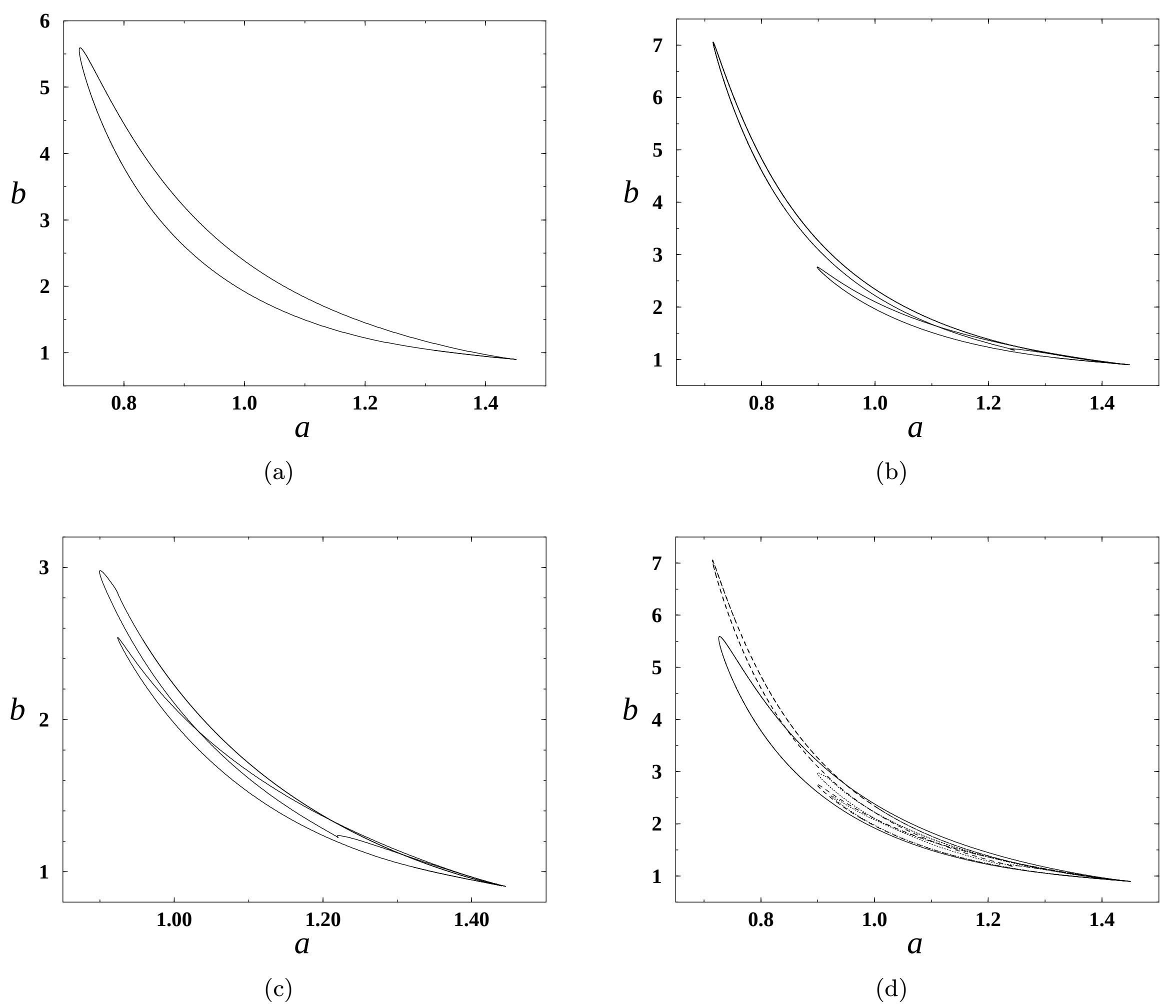


Fig. 8. Shape of the period-doubling curves (a) **F**, (b) **F2** and (c) **F4**. The three curves are represented together in (d) **F** (solid line), **F2** (dashed line) and **F4** (dotted line).

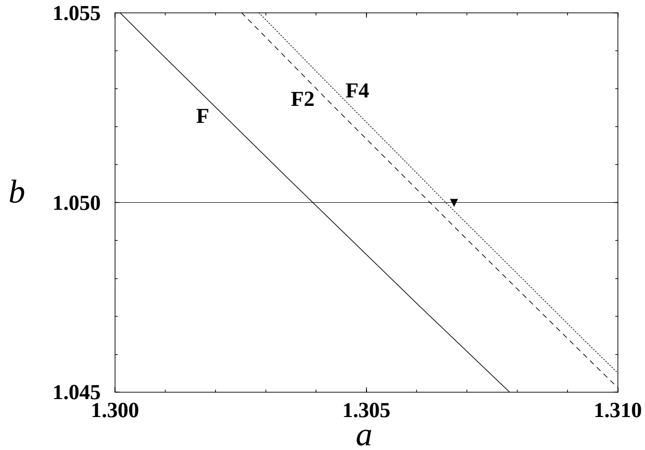


Fig. 9. Period-doubling curves **F**, **F2**, **F4** near $b = 1.05$. Triangle denotes the parameter values where the chaotic attractor has been detected.

zone where they are stable, to detect the chaotic attractors.

We remark that the existence of the cascade of period-doubling bifurcations of periodic orbits has been detected taking advantage of the analysis of the triple-zero bifurcation.

As above, we consider $c = 1.3$ fixed. In Fig. 8, we show the shape of the period-doubling curves **F**, **F2** and **F4**. They are all closed curves, although **F2** and **F4** have several loops.

Figure 9 shows a zoom of Fig. 8(d), around the value $b = 1.05$. Notice that this window is located over **DM1**. The reason is that the periodic orbit that undergoes the first period-doubling bifurcation **F** changes its stability at **DM1** (where the secondary Hopf bifurcation of periodic orbits takes

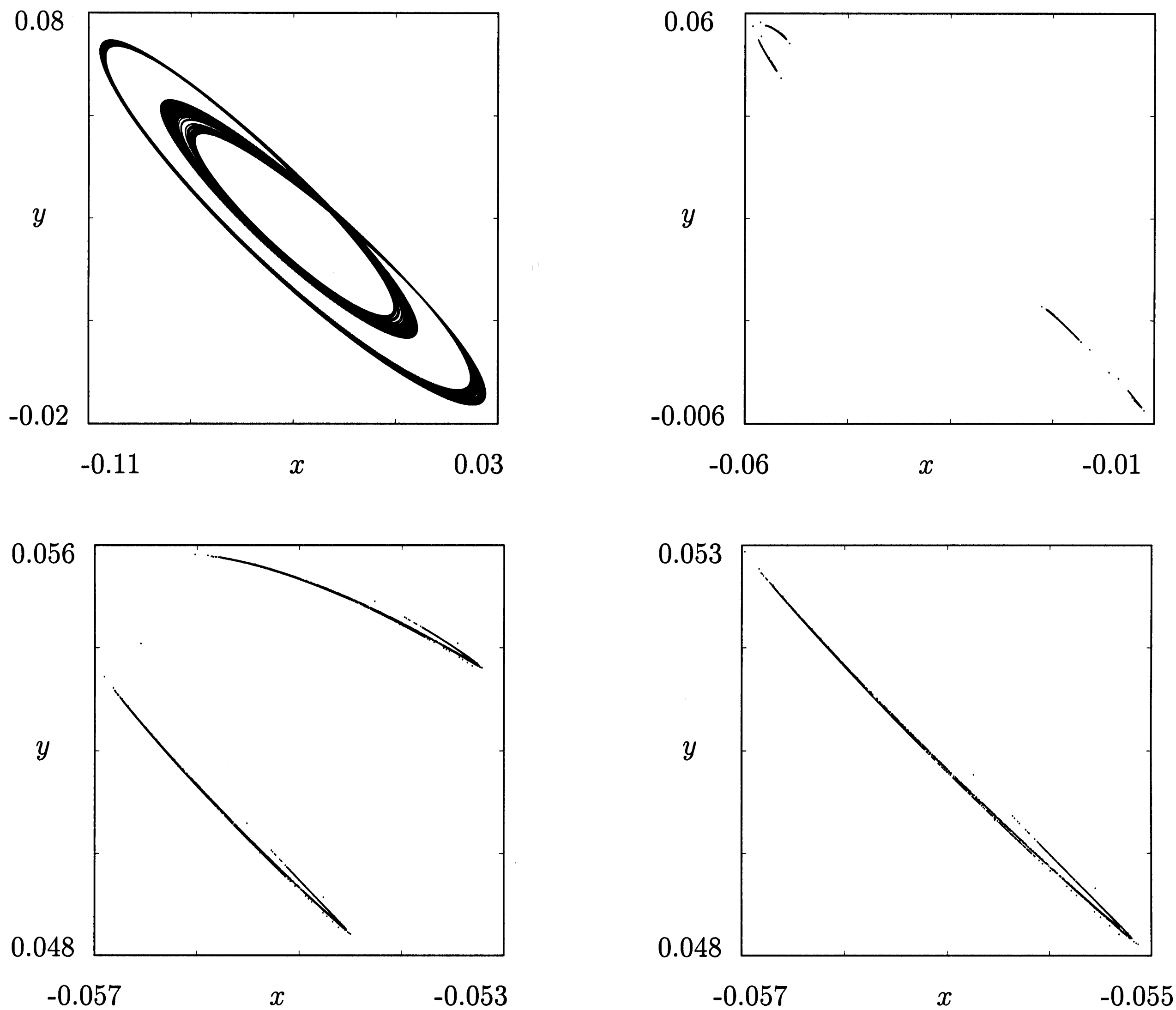


Fig. 10. Chaotic attractors for $a = 1.3066735$, $b = 1.05$, $c = 1.3$. On the left we present a projection onto the xy-plane. On the right, we draw a Poincaré section in the same plane. In the lower part, we can see two zooms of the Poincaré section.

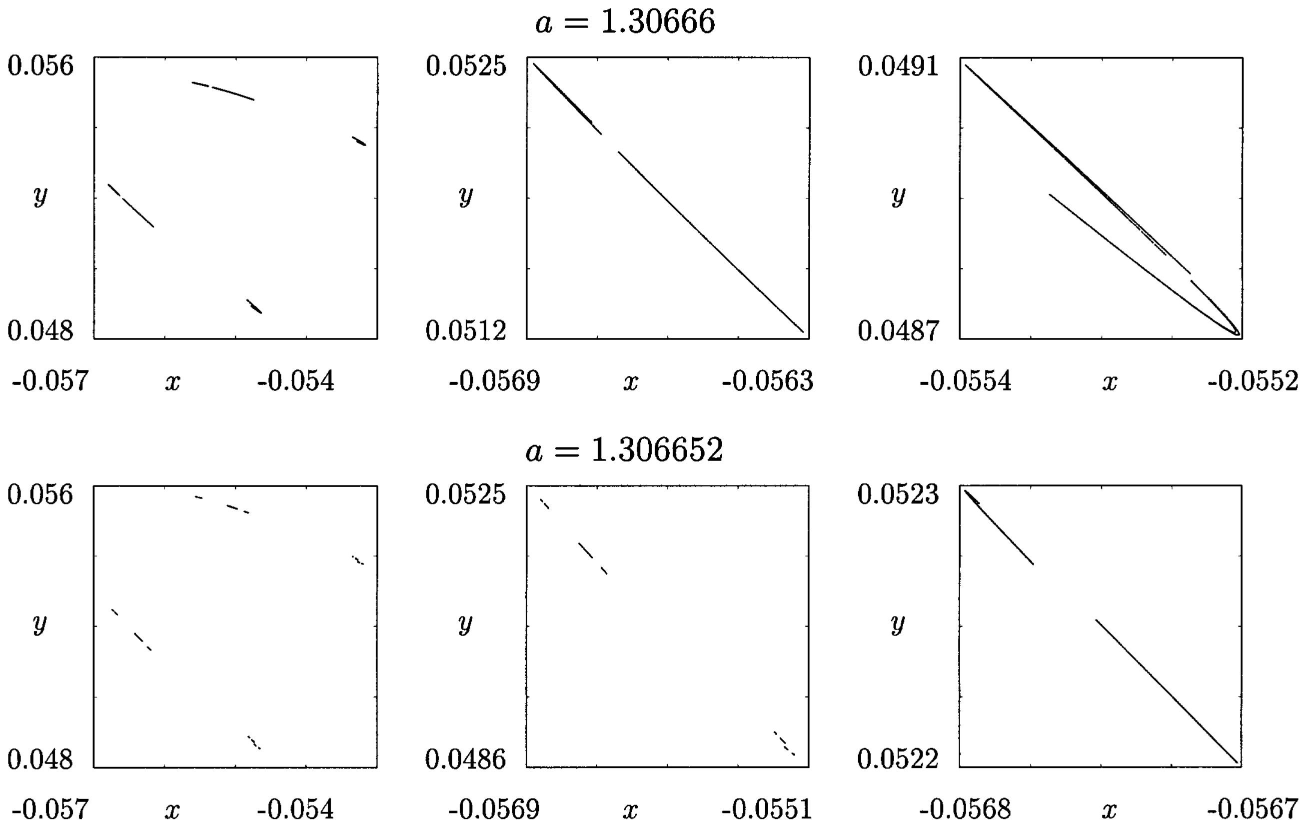


Fig. 11. Evolution of the chaotic attractors moving a ($a = 1.30666$ and 1.306652).

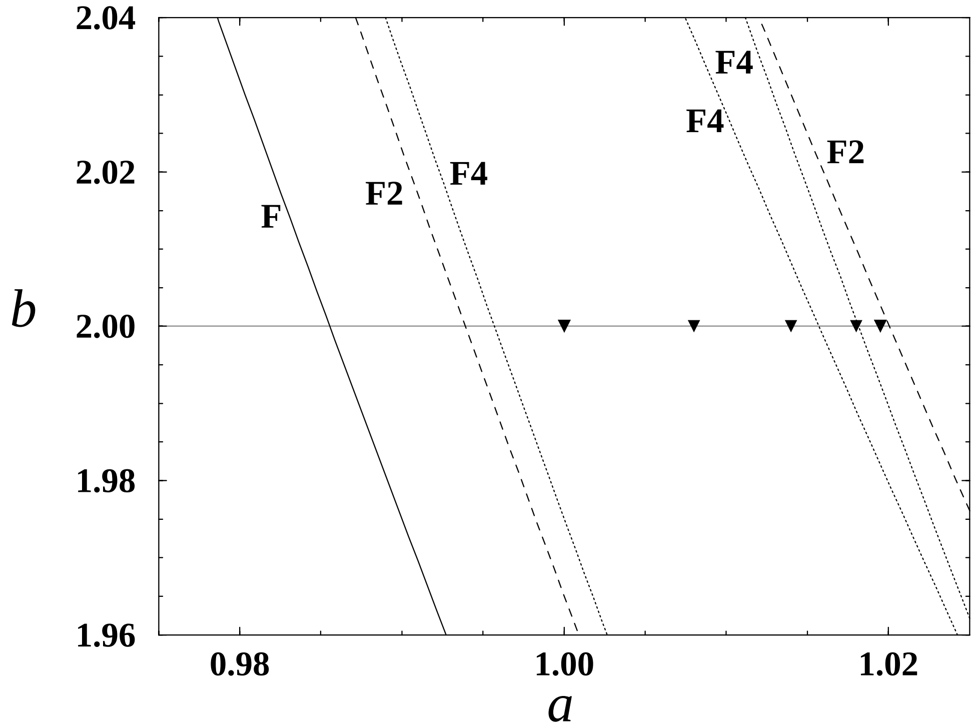


Fig. 12. Period-doubling curves **F**, **F2**, **F4** near $b = 2$. Triangles denote the parameter values selected for the chaotic attractors of Figs. 14 and 15.

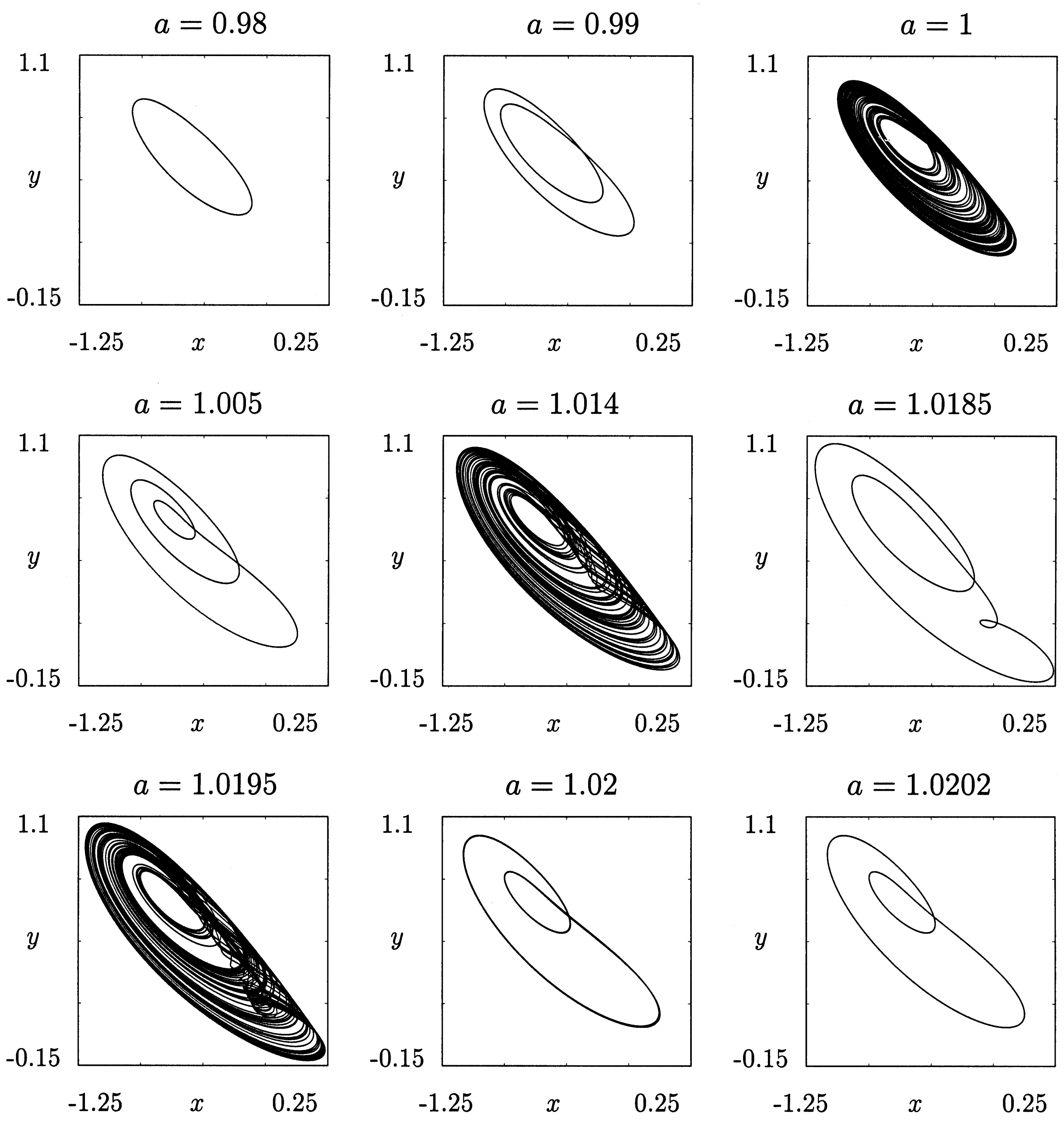


Fig. 13. Sequence of stable periodic orbits and chaotic attractors for $b = 2$, moving a ($a = 0.98$, 0.99, 1, 1.005, 1.014, 1.0185, 1.0195, 1.02, 1.0202).

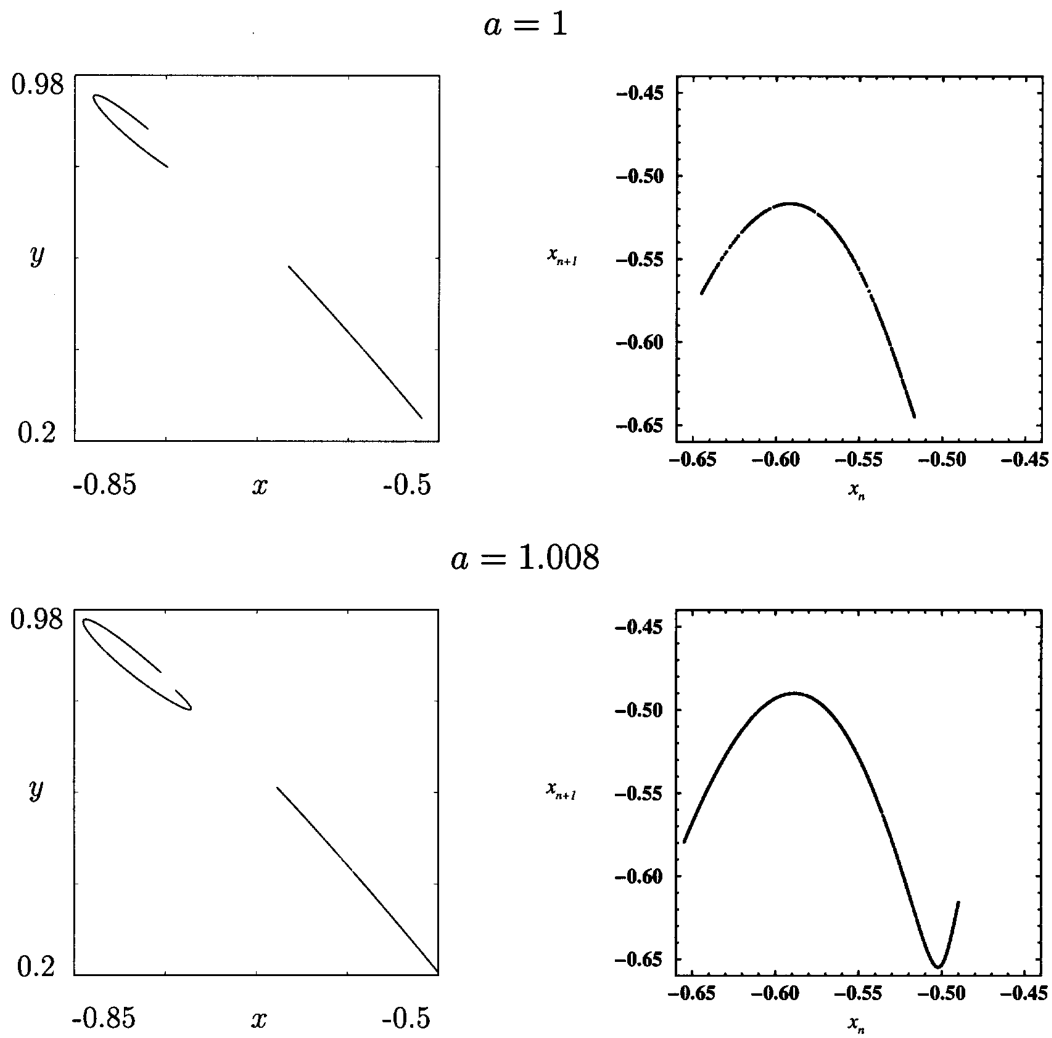


Fig. 14. Poincaré sections and next return maps of screw chaotic attractor ($a = 1$) and spiral chaotic attractor ($a = 1.008$) for $b = 2$.

place), and one can check that taking the above parameter values, the periodic orbit is stable [see Fig. 6(b)]. In fact, we are located over the cascade of secondary Hopf bifurcation of periodic orbits.

Once we have fixed this value of parameter b, we have taken some values of parameter a (marked with triangles in the figure, almost indistinguishable). For the first value $a = 1.3066735$, we have obtained the chaotic attractor of Fig. 10. We have also presented a Poincaré section in the xy-plane, and two zooms of the Poincaré section.

For the other values $a = 1.30666$ and $a = 1.306652$, we only present the Poincaré sections, where we can see how this section is broken into several pieces.

Nevertheless, this evolution of the Poincaré section takes place on a very tiny zone. So, we increase the value of parameter b. In Fig. 12 we have blown-up Fig. 8(d), around the value $b = 2$. We have also drawn on the line $b = 2$ some triangles for different values of a in order to see how the attractor evolves. In Fig. 13 we show this evolution, in a sequence combining chaotic attractors and stable periodic orbits.

Figure 14 shows the transition from spiral chaotic attractor ($a = 1$) to screw chaotic attractor ($a = 1.008$). Beyond the Poincaré sections, we have also drawn the next return maps, passing from unimodal to bimodal.

Finally, we also show in Fig. 15 how the shape of the chaotic attractors change when increasing a bit more of the parameter a. Observe that now, we have three-modal, four-modal, five-modal next return maps.

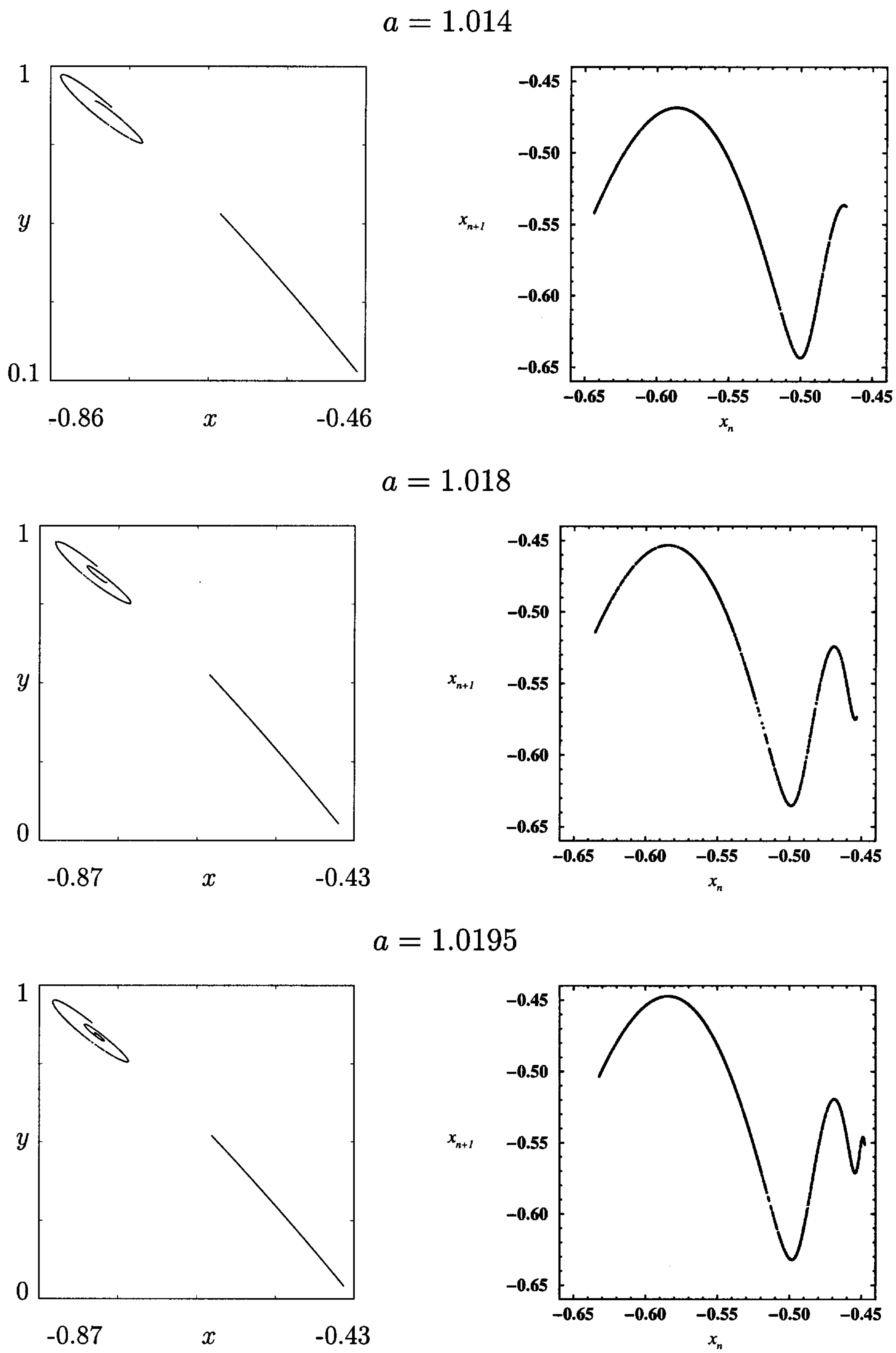


Fig. 15. Evolution of the chaotic attractors increasing a for $b = 2$.

Acknowledgments

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