Finite-Difference Methods for Nonlinear Problems

For the general nonlinear boundary-value problem

$$y'' = f(x, y, y'), \qquad a \le x \le b, \qquad y(a) = \alpha, \qquad y(b) = \beta,$$

the difference method is similar to the method applied to linear problems

As in the linear case, we divide $[a, b]$ into $(N + 1)$ equal subintervals whose endpoints are at $x_i = a + ih$ for $i = 0, 1, \ldots, N + 1$. Assuming that the exact solution has a bounded fourth derivative allows us to replace $y''(x_i)$ and $y'(x_i)$ in each of the equations

$$y''(x_i) = f(x_i, y(x_i), y'(x_i))$$

by the appropriate centered-difference formula given in Eqs. (11.16) and (11.17), to obtain, for each $i = 1, 2, \ldots, N$,

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = f\left(x_i, y(x_i), \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6} y'''(\eta_i)\right) + \frac{h^2}{12} y^{(4)}(\xi_i),$$

for some ξ_i and η_i in the interval (x_{i-1}, x_{i+1}).

As in the linear case, the difference method results when the error terms are deleted and the boundary conditions employed:

$$w_0 = \alpha, \qquad w_{N+1} = \beta,$$

and

$$-\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + f\left(x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right) = 0,$$

for each $i = 1, 2, \ldots, N$.

The $N \times N$ nonlinear system obtained from this method,

$$\begin{aligned}
2w_1 - w_2 + h^2 f\left(x_1, w_1, \frac{w_2 - \alpha}{2h}\right) - \alpha &= 0,\\
-w_1 + 2w_2 - w_3 + h^2 f\left(x_2, w_2, \frac{w_3 - w_1}{2h}\right) &= 0,\\
&\ \vdots\\
-w_{N-2} + 2w_{N-1} - w_N + h^2 f\left(x_{N-1}, w_{N-1}, \frac{w_N - w_{N-2}}{2h}\right) &= 0,\\
-w_{N-1} + 2w_N + h^2 f\left(x_N, w_N, \frac{\beta - w_{N-1}}{2h}\right) - \beta &= 0,
\end{aligned} \tag{11.20}$$

To approximate the solution to this system, we use Newton's method for nonlinear systems, discussed in Section 10.2. A sequence of iterates $\{(w_1^{(k)}, w_2^{(k)}, \ldots, w_N^{(k)})^t\}$ is generated that converges to the solution of system (11.20), provided that the initial approximation $(w_1^{(0)}, w_2^{(0)}, \ldots, w_N^{(0)})^t$ is sufficiently close to the solution, $(w_1, w_2, \ldots, w_N)^t$, and that the Jacobian matrix for the system is nonsingular. For the system (11.20), the Jacobian matrix given in (11.21) is tridiagonal, and the assumptions presented at the beginning of this discussion ensure that J is a nonsingular matrix.

(11.21)

$$J(w_1, \ldots, w_N) = \begin{bmatrix} 2 + h^2 f_y\left(x_1, w_1, \frac{w_2 - \alpha}{2h}\right) & -1 + \frac{h}{2} f_{y'}\left(x_1, w_1, \frac{w_2 - \alpha}{2h}\right) & 0 & \cdots & 0 \\ -1 - \frac{h}{2} f_{y'}\left(x_2, w_2, \frac{w_3 - w_1}{2h}\right) & 2 + h^2 f_y\left(x_2, w_2, \frac{w_3 - w_1}{2h}\right) & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 + \frac{h}{2} f_{y'}\left(x_{N-1}, w_{N-1}, \frac{w_N - w_{N-2}}{2h}\right) \\ 0 & \cdots & 0 & -1 - \frac{h}{2} f_{y'}\left(x_N, w_N, \frac{\beta - w_{N-1}}{2h}\right) & 2 + h^2 f_y\left(x_N, w_N, \frac{\beta - w_{N-1}}{2h}\right) \end{bmatrix}$$

Newton's method for nonlinear systems requires that at each iteration, the $N \times N$ linear system

$$\begin{aligned} J(w_1, \ldots, w_N)(v_1, \ldots, v_n)^t = -\Bigg(& 2w_1 - w_2 - \alpha + h^2 f\left(x_1, w_1, \frac{w_2 - \alpha}{2h}\right), \\ & -w_1 + 2w_2 - w_3 + h^2 f\left(x_2, w_2, \frac{w_3 - w_1}{2h}\right), \ldots, \\ & -w_{N-2} + 2w_{N-1} - w_N \\ & + h^2 f\left(x_{N-1}, w_{N-1}, \frac{w_N - w_{N-2}}{2h}\right), \\ & -w_{N-1} + 2w_N + h^2 f\left(x_N, w_N, \frac{\beta - w_{N-1}}{2h}\right) - \beta \Bigg)^t \end{aligned}$$

be solved for $v_1, v_2, \ldots, v_N$, since

$$w_i^{(k)} = w_i^{(k-1)} + v_i, \qquad \text{for each } i = 1, 2, \ldots, N.$$

ALGORITHM 11.4

Nonlinear Finite-Difference

To approximate the solution to the nonlinear boundary-value problem

$$y'' = f(x, y, y'), \qquad a \le x \le b, \qquad y(a) = \alpha, \qquad y(b) = \beta:$$

INPUT endpoints a, b; boundary conditions α, β; integer $N \ge 1$; tolerance TOL; maximum number of iterations M.

OUTPUT approximations w_i to $y(x_i)$ for each $i = 0, 1, \ldots, N + 1$ or a message that the maximum number of iterations was exceeded.

Step 1 Set $h = (b - a)/(N + 1)$;
$w_0 = \alpha$;
$w_{N+1} = \beta$.

Step 2 $\boxed{\text{For } i = 1, \ldots, N \text{ set } w_i = \alpha + i\left(\dfrac{\beta - \alpha}{b - a}\right)h.}$ IMPORTANTE !

Step 3 Set $k = 1$.

Step 4 While $k \le M$ do Steps 5–18.

Step 5 Set $x = a + h$;
$t = (w_2 - \alpha)/(2h)$;
$a_1 = 2 + h^2 f_y(x, w_1, t)$; $\partial f/\partial w_1$
$b_1 = -1 + (h/2) f_{y'}(x, w_1, t)$; $\partial f/\partial w_2$
$d_1 = -(2w_1 - w_2 - \alpha + h^2 f(x, w_1, t))$.

Step 6 For $i = 2, \ldots, N - 1$
set $x = a + ih$;
$t = (w_{i+1} - w_{i-1})/(2h)$;
$a_i = 2 + h^2 f_y(x, w_i, t)$; $\partial f/\partial w_i$
$b_i = -1 + (h/2) f_{y'}(x, w_i, t)$; $\partial f/\partial w_{i+1}$
$c_i = -1 - (h/2) f_{y'}(x, w_i, t)$; $\partial f/\partial w_{i-1}$
$d_i = -(2w_i - w_{i+1} - w_{i-1} + h^2 f(x, w_i, t))$.

Step 7 Set $x = b - h$;
$t = (\beta - w_{N-1})/(2h)$;
$a_N = 2 + h^2 f_y(x, w_N, t)$; $\partial f/\partial w_N$
$c_N = -1 - (h/2) f_{y'}(x, w_N, t)$; $\partial f/\partial w_{N-1}$
$d_N = -(2w_N - w_{N-1} - \beta + h^2 f(x, w_N, t))$.

Step 8 Set $l_1 = a_1$; (*Steps* 8–14 *solve a tridiagonal linear system using Algorithm* 6.7.)
$u_1 = b_1/a_1$.

Step 9 For $i = 2, \ldots, N - 1$ set $l_i = a_i - c_i u_{i-1}$;
$u_i = b_i/l_i$.

Step 10 Set $l_N = a_N - c_N u_{N-1}$.

Step 11 Set $z_1 = d_1/l_1$.

Step 12 For $i = 2, \ldots, N$ set $z_i = (d_i - c_i z_{i-1})/l_i$.

Step 13 Set $v_N = z_N$;
$w_N = w_N + v_N$.

Step 14 For $i = N - 1, \ldots, 1$ set $v_i = z_i - u_i v_{i+1}$;
$w_i = w_i + v_i$.

Step 15 If $\|\mathbf{v}\| \leq TOL$ then do Steps 16 and 17.

Step 16 For $i = 0, \ldots, N + 1$ set $x = a + ih$;
OUTPUT (x, w_i).

Step 17 STOP. *(Procedure completed successfully.)*

Step 18 Set $k = k + 1$.

Step 19 OUTPUT ('Maximum number of iterations exceeded');
(Procedure completed unsuccessfully.)
STOP.

It can be shown (see Isaacson and Keller [78], page 433) that this Nonlinear Finite-Difference method is of order $O(h^2)$, so the stopping criteria in Step 15 could be based on the condition that $|v_j| = O(h^2)$, for each $j = 1, 2, \ldots, N$.

Since a good initial approximation is required when the satisfaction of conditions (i), (ii), and (iii) given at the beginning of this presentation cannot be verified, an upper bound for k should be specified and, if exceeded, a new initial approximation or a reduction in step size considered. The initial approximations $w_i^{(0)}$ to w_i, for each $i = 1, 2, \ldots, N$, are obtained in Step 2 by passing a straight line through (a, α) and (b, β) and evaluating at x_i.