A GIRL HAS NO NAME
Computer Science 81
Homework 3
2/7/17

1, 2, 3, 4, EC

1. [20 points] Give natural deduction proofs of the following, using constructive rules only.

A. [4 points] $\exists x.\ (R(x) \to B(x)) \ \vdash\ (\forall x. R(x)) \to (\exists x.\, B(x))$

e.g., $R(x) =$ "x is realistic", $B(x) =$ "x is believable".

B. [4 points] $(\forall x.\, G(x)) \vee (\forall x.\, B(x)) \ \vdash\ \forall x.\, (G(x) \vee B(x))$

e.g., $G(x) =$ "x is good", $B(x) =$ "x is bad".

C. [4 points] $\forall x.\, (H(j) \to T(x)) \ \vdash\ H(j) \to (\forall x.\, T(x))$

e.g., $j =$ "I/me", $H(x) =$ "x is hungry", $T(x) =$ "x is tasty".

D. [4 points]
$\neg\exists x.\, (G(x)) \vee (\forall x. F(x)), C(j) \to \forall x.\, D(x) \ \vdash\ \forall y. \forall z. ((\neg G(z) \vee F(y)) \wedge (C(j) \to D(y)))$

e.g., $G(x) =$ "x is a ghost", $F(x) =$ "x is fictional",
$j =$ "I/me", $C(x) =$ "x is confident", $D(x) =$ "x is doable".

E. [4 points]
$\forall x.\, (\neg S(x,x)), \forall x. \forall y. \forall z. (S(x,y) \wedge S(y,z) \to S(x,z)) \ \vdash\ \forall x.\, \forall y. (S(x,y) \to \neg S(y,x))$

e.g., $S(a,b) =$ "movie a is a sequel to movie b".

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2. [16 points] Give natural deduction proofs of the following. (Use LEM or contradiction.)

A. [8 points] $\forall x. (C(x) \vee E(x)) \ \vdash \ (\forall x. C(x)) \vee (\exists x. E(x))$

e.g., $C(x) =$ "x is cheap", $E(x) =$ "x is expensive".

B. [8 points] $\exists x. \top \ \vdash \ \exists x. (D(x) \to \forall y. D(y))$

The "drinkers paradox": at any moment in any nonempty bar,
there is a person such that
if they are drinking beer, everyone is drinking beer.

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3. [8 points] It is typical mathematical practice to write so-called *bounded quantifiers* such as "$\forall x \in S.\, \Phi$" or "$\exists x \leq n.\, \Phi$", where we are quantifying not over <u>all</u> individuals, but only a <u>subset</u> (e.g., the members of S, or the individuals less-than-or-equal-to n, or just grutors). In class, we mentioned that we can express the same idea using Jape's unbounded $\forall$ and $\exists$ as follows:

$$\begin{array}{lll} \exists x \in S.\, P(x) \text{ becomes } \exists x.\, ((x \in S) \wedge P(x)) & & \\ \exists x \leq n.\, P(x) \text{ becomes } \exists x.\, ((x \leq n) \wedge P(x)) & & \text{etc.} \end{array}$$

$$\begin{array}{lll} \forall x \in S.\, P(x) \text{ becomes } \forall x.\, ((x \in S) \rightarrow P(x)) & & \\ \forall x \leq n.\, P(x) \text{ becomes } \forall x.\, ((x \leq n) \rightarrow P(x)) & & \text{etc.} \end{array}$$

We also emphasized the two quantifiers translate differently: bounded-$\exists$ becomes a conjunction, while bounded-$\forall$ becomes an implication. Your job for this problem is to confirm that the difference is necessary.

A. [4 points] Describe a model (where the set of individuals is the set $\mathbb{N}$ of natural numbers (nonnegative integers), and the relation $\leq$ is interpreted as the usual less-than-or-equal-to relation on $\mathbb{N}$) that makes

$$\forall x \leq n.\, f(x) \leq m$$

or equivalently

$$\forall x.\, (x \leq n \rightarrow f(x) \leq m)$$

true, but that makes

$$\forall x.\, (x \leq n \;\wedge\; f(x) \leq m)$$

false. (You will need to complete the model by giving interpretations of the function f and the constants n and m).

B. [4 points] Describe a model (where the set of individuals is the set $\mathbb{N}$ of natural numbers (nonnegative integers), and the relation $\leq$ is interpreted as the usual less-than-or-equal-to relation on $\mathbb{N}$) that makes

$$\exists x.\, (x \leq n \rightarrow f(x) \leq m)$$

true, but that makes

$$\exists x \leq n.\, f(x) \leq m$$

or equivalently

$$\exists x.\, (x \leq n \;\wedge\; f(x) \leq m)$$

false.

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4. [1 easy point] Please wait until youre done with the rest of the assignment to answer this quick survey:

A. How long (in hours) did you spend working on this assignment?

B. What was the most interesting thing you learned while answering these problems? (Were sure there was *something* you learned.)

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Extra Credit. [2 measly points] The formula $\neg\neg F \to F$ is a tautology according to the proof tables, but is not provable from the constructive rules; you need a "classical" rule like proof-by-contradiction or LEM (or for the shortest proof) $\neg\neg$-elimination!)

But even constructive logicians agree that $\neg\neg F \to F$ "isnt false", because $\vdash\ \neg\neg(\neg\neg F \to F)$ is provable with purely constructive rules. Provide such a proof.

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