Physics 84
Homework 5
3/9/17

1, 2, 3, 4

1) To Code or not to Code?
We know the three-qubit bit-flip code can correct any single-qubit bit-flip error. Let us consider an error channel in which each qubit experiences a bit flip with probability p, and no bit flip with probability $(1-p)$.

(a) If we use no error correction, so a one-qubit state $|\psi\rangle$ is represented by the state of a single physical qubit, what is the probability $P(error)$ that the state will undergo an uncorrected bit-flip error?

(b) If we use the 3-qubit bit-flip code to represent the one-qubit state $|\psi\rangle$ in the states of three physical qubits, what is the probability $P(coded-error)$ that the state will undergo an uncorrected error? Remember that the bit-flip code is capable of diagnosing and correcting a bit flip on any one of the three physical qubits.

(c) Show that $P(coded-error) < P(error)$, meaning that coding improves the reliability of information transmission, as long as $p < \frac{1}{2}$. Why is this restriction on p not surprising? If we know we have a channel with $p > \frac{1}{2}$, can you think of a strategy for recovering from bit-flip errors?

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2) Smallest Code for the Job

Suppose we need to encode a general single-qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the states of n physical qubits. Without focusing on a particular error process or a particular code, we can make some general statements about the nature of the encoding situation. Some n-qubit physical state $|\mathbf{0}_L\rangle$ will represent $|0\rangle$, and some orthogonal n-qubit physical state $|\mathbf{1}_L\rangle$ will represent $|1\rangle$. Thus the initial encoded states $\alpha|\mathbf{0}_L\rangle + \beta|\mathbf{1}_L\rangle$ can be found in a two-dimensional subspace – the "code space" – of the overall 2^n-dimensional state space for n qubits. Each particular error process maps the code space to some other two-dimensional subspace.

(a) A bit-flip code diagnoses and corrects a bit-flip error in any one of the n qubits. It also diagnoses the "no-error" state and corrects *it* by leaving it alone. To permit this kind of reliable diagnosis of $(n+1)$ different conditions, requiring $(n+1)$ different correction protocols, it is necessary that the code space be mapped to $(n+1)$ mutually orthogonal subspaces by the $(n+1)$ different error or no-error processes. Show that this reasoning leads to the requirement

$$2^n \geq 2(n+1)$$

and thus

$$n \geq 3.$$

Thus the 3-qubit bit-flip code is the smallest possible code that can correct bit-flip errors.

(b) Suppose we require an n-qubit code that can correct a bit flip, a phase flip, or a combined bit-and-phase flip on any one of the n qubits. Develop a condition for the allowed values of n, and use it to show that $n \geq 5$ for such a code.

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3) Extensions of the 3-Qubit Bit-Flip Code

We saw in class how to correct single bit-flip errors with the 3-qubit bit-flip code. In that version of the error-correction protocol, we measured our two ancilla qubits in the $\{|0\rangle, |1\rangle\}$ basis, giving four possible measurement results. Each measurement result instructed us to perform a particular error-correcting operation on the three code qubits.

(a) Draw an error-correction circuit that performs the same 3-qubit bit-flip correction without explicit measurement of the ancilla qubits. (Instead of measurement of the ancilla qubits and conditional operations on the code qubits, try controlled-U and/or controlled-controlled-U gates with the ancilla qubits as controls and the code qubits as targets.) Your circuit can use any one-, two-, or three-qubit gates we have learned about in class, including CNOT and Toffoli gates.

(b) At an arbitrary stage of a long quantum computation, the three code qubits may be entangled with an external set of qubits, in a state such as

$$\alpha |000\rangle |\psi\rangle_{ext} + \beta |111\rangle |\psi_{\perp}\rangle_{ext}$$

Convince yourself (and me!) that our error correction procedure protects even an entangled state like this against single bit-flip errors on the three code qubits.

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4) 9-Qubit Shor Encoding Circuit

Recall that the 9-qubit Shor code encodes $|0\rangle$ and $|1\rangle$ as

$$|\mathbf{0}_L\rangle = \frac{\big(|000\rangle + |111\rangle\big)\big(|000\rangle + |111\rangle\big)\big(|000\rangle + |111\rangle\big)}{2\sqrt{2}},$$
$$|\mathbf{1}_L\rangle = \frac{\big(|000\rangle - |111\rangle\big)\big(|000\rangle - |111\rangle\big)\big(|000\rangle - |111\rangle\big)}{2\sqrt{2}}.$$

Design a circuit that begins with a single qubit in state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and eight qubits in state $|0\rangle$, and produces all nine qubits in the encoded state $\alpha\,|\mathbf{0}_L\rangle + \beta\,|\mathbf{1}_L\rangle$. Explain or demonstrate how your circuit produces the desired output. [Hint: Remember that the Shor code is a concatenation, or nesting, of the three-qubit phase flip code and the three-qubit bit flip code. Draw inspiration from the encoding circuits for each of two codes individually.]

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