The Definite Integral

Definition: $A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

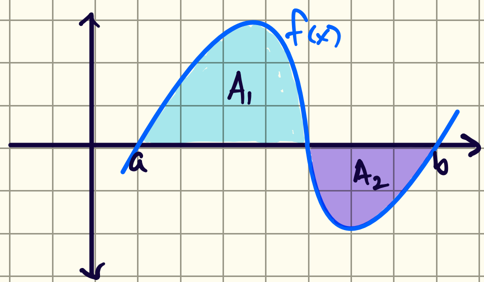
this says that the area under the curve $f(x)$ on the interval $[a, b]$ is the same as the limit of the Riemann sum on that interval as n goes to infinity.

Notation: $\int$ represent an elongated S to mean sum,

a and b are the interval, and dx is the infintesmal and also shows the variable of integration. Here f(x) is called the integrand, which is the part being integrated.

Note: the definite integral finds the net area, meaning the result could be positive or negative.

Ex:



$$\int_a^b f(x)\,dx = A_1 - A_2$$

in other words, area below the x-axis is considered negative area.

Theorem: If f(x) is continuous on an interval $[a, b]$, then f(x) is integrable on $[a, b]$.

So as long as a function is continuous on a closed interval, then we can integrate it.

The rest of the section is confusing notation to represent an integral and show the limit of the Riemann Sum as $n \to \infty$. Then they have you calculate integrals using different types of sums, like we did last section.

Properties of the Definite Integral

- Upper and lower limits: the order in which you choose the limits changes the answer:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

also, if $a=b$ then $\int_a^b f(x)\,dx = 0$

- Other properties:

$$\int_a^b c\,dx = c(b-a) \quad \text{where } c \text{ is a constant}$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$$

Problems: Evaluate the following definite integrals

1) $\int_0^3 f(x)\,dx$ if $\int_3^0 f(x)\,dx = -16$

2) $\int_{-1}^2 (x^2+3)\,dx$ if $\int_{-1}^2 x^2\,dx = 3$

3) $\int_0^1 [f(x)+g(x)-h(x)]\,dx$ if $\int_0^1 f(x)\,dx = 7$, $\int_0^1 g(x)\,dx = -5$, and $\int_0^1 h(x)\,dx = 12$

4) $\int_{\pi}^{\pi} \cos x\,dx$

5) $\int_{-2}^0 (4x^2-3x)\,dx$ if $\int_{-2}^0 x^2\,dx = \frac{8}{3}$ and $\int_{-2}^0 x\,dx = -2$

6) if $\int_2^4 f(x)\,dx = 7$, $\int_2^4 g(x)\,dx = -2$, and $\int_2^3 h(x)\,dx = 4$, find $\left[\int_2^4 f(x)-g(x)\,dx\right]^2 + \int_3^2 h(x)\,dx$

7) $\int_{-14}^{17} 8\,dx$

Comparison Properties of the Definite Integral

Properties:

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Problems:

1) Give bounds for $\int_0^2 x^3 - 1\,dx$

2) If $5-2x \geq f(x)$, estimate $\int_0^1 f(x)\,dx$

3) if $a \leq \int_0^{\pi/2} \cos x\,dx \leq b$, find a and b

4) For $f(x)$ positive on $[a,b]$, $\int_a^b f(x)\,dx \geq (2k-7)$ find k

The Fundamental Theorem of Calculus

Theorem:

Let f be continuous on $[a,b]$. If F is an antiderivative of f on $[a,b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Notation: we often write $\int_a^b f(x)dx = F(x)\Big|_{x=a}^{x=b}$, where the vertical bar means to evaluate $F(x)$ as the difference between $x=b$ and $x=a$.

Ex: $\int_1^3 2x\,dx = x^2\Big|_1^3 = 3^2 - 1^2 = 8$

We can reframe the theorem as follows:

If f is continuous on the interval $[a,b]$, then g can be defined by $g(x) = \int_a^x f(t)\,dt$, where $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

This last part also means we can say $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Ex1: Find $\frac{dy}{dx}$ if $y = \int_0^x \sec t\,dt$

From the definition we just found:

$\frac{d}{dx}\int_a^x f(t)dt = f(x)$ $\quad a=0$ & $f(t) = \sec t$

$\Rightarrow \frac{dy}{dx} = \sec x$

Ex2: Find $\frac{dy}{dx}$ if $y = \int_0^{2x} 2t^3\,dt$

using the first representation: $F(x) = \frac{x^4}{2}$

$\Rightarrow y = \frac{x^4}{2}\Big|_0^{2x} = 8x^4 - 0$

$\Rightarrow \frac{dy}{dx} = 32x^3$ $\qquad$ is this result surprising?

Problems: Evaluate the following

1) $\int_{-4}^{0} x^3 + 4x^2 \, dx$

2) $\int_{-1}^{2} 2x^4 - 3x^2 \, dx$

3) $\int_{0}^{\pi} \sin x \, dx$

4) $\int_{0}^{\pi/4} \sec x \cdot \tan x \, dx$

5) $\int_{0}^{3} \frac{1}{3\sqrt[3]{x^2}} \, dx$

6) $\int_{0}^{5} 5x^2 + 3x \, dx$

7) $\int_{0}^{\pi} e^{-x} + \cos x \, dx$

Problems:

1) Find $f'(x)$ if $f(x) = \int_{0}^{x} \sqrt{1+t^2} \, dt$

2) Find $f'(x)$ if $f(x) = \int_{x}^{2x} \cos t \, dt$

3) Find $f'(x)$ if $f(x) = \int_{0}^{3x} \sec t \cdot \tan t \, dt$

4) Find $f'(x)$ if $f(x) = \int_{x}^{2} 3t^2 - 2t + 1 \, dt$

Problems:

1) For $\frac{dy}{dx} = -\cos x$, find the initial value when $y=1$ and $x=0$

2) For $\frac{dy}{dx} = \sec^2 x$, find the initial value when $y=\pi$ and $x=\pi/4$

3) For $f'(x) = e^x + x$, find the initial value when $f(0)=5$

4) For $f'(x) = 6x^2 - 14x + 8$, find the initial value when $f(2) = 17$