Unit 1 Review

The Antiderivative

- Suppose you know the velocity of a projectile at every moment in time, $v(t)$. Can we find the position of the projectile at some time? $s(t)$

  we know: $\frac{d}{dt} s(t) = v(t) \Rightarrow s'(t) = v(t)$

  Since $v(t)$ is the derivative of the position function, $s(t)$, then $s(t)$ is the antiderivative of $v(t)$.

- What we are saying here is that for some function $f(x)$, there is a function $F(x)$ such that $F'(x) = f(x)$.

- There are several questions we need to ask here: Does every function have an antiderivative, how do we find the antiderivative, and is the antiderivative of a function unique?

We will answer the first two questions later, but let's look at the last one now:

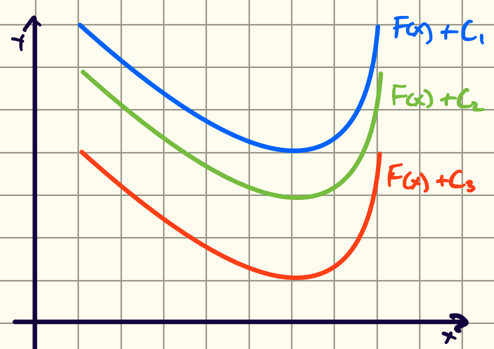
which of the following functions are the antiderivative of $f(x) = 3x^2$?

- $A(x) = \frac{3}{2}x^3 + 1$
- $B(x) = x^3$
- $C(x) = x^3 - x$
- $D(x) = x^3 - 4$
- $E(x) = 2x^3$
- $F(x) = x^3 + \pi$

From this we can see that the antiderivative is not unique, but what more can we say about it?

If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$ is also an antiderivative of $f(x)$. Here C is an arbitrary constant called the constant of integration.

Why should this make sense from a graphical perspective?



Remember that the derivative function just tells us the slope of the function at each point, but not where that point is located.

- Problems: find the general antiderivative of the following:

1) $f(x) = \csc^2 x - 3$

2) $f(x) = \frac{-1}{2\sqrt{x}}$

3) $f(x) = 2x^3 - 4^2$

4) $f(x) = \cos x + 1$

5) $f(x) = 2x + e^x$

- Initial Value Problems: In some cases, we can determine the value of C, given the initial information $F(x_0) = y_0$.

Ex: $f(x) = 3x^2 + 5$, $F(2) = 12$

$F(x) = x^3 + 5x + C$

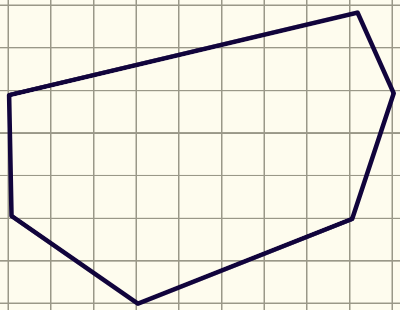
$F(2) = 2^3 + 5(2) + C = 12$

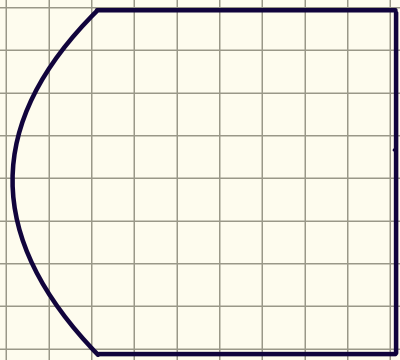
$\Rightarrow C = -6$

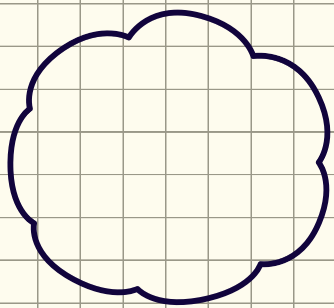
Area Under a Curve

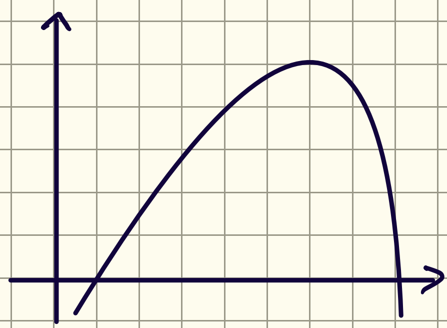
- Suppose we have an irregular shape, then how could we find its area?

Ex:

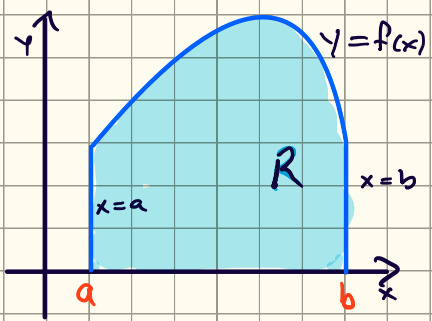




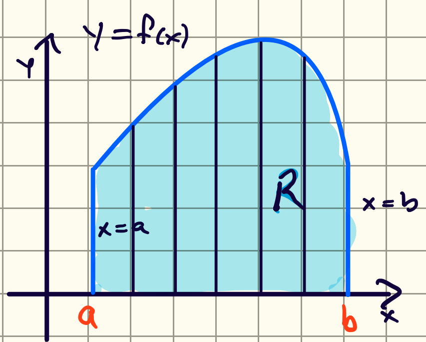




- Let's look at a situation like the last one:

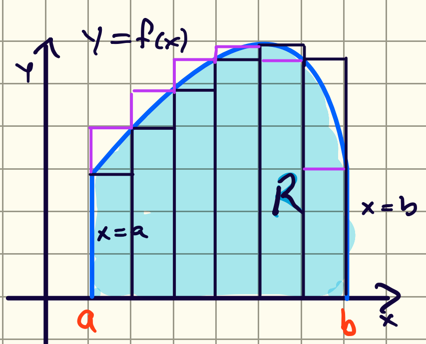


We want to find the area of R bounded by the lines $x=a$ and $x=b$ and between the x-axis and the curve $f(x)$.



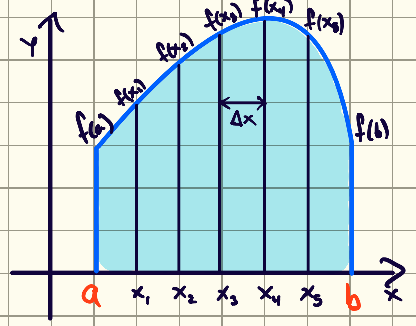
One way to approach this is to break it up into smaller pieces called subintervals. The length of each subinterval can be found by

$$\text{length} = \frac{b-a}{\#\text{ of subintervals}}$$



Now we can make rectangles out of each subinterval to approximate the area of R.

Notice that choosing the left or the right endpoint gives a different estimate.



To make this a bit more systematic, we name the length between each sub interval as Δx. This means that the area of any one rectangle will be $f(x_n) \cdot \Delta x$, since $f(x_n)$ is the height and Δx is the width.

- left sum: $A = \Delta x \cdot f(a) + \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) + \Delta x \cdot f(x_4) + \Delta x \cdot f(x_5)$

- right sum: $A = \Delta x \cdot [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(b)]$

- Mid-point sum: $A = \Delta x \cdot \left[f\left(\frac{a+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_4}{2}\right) + f\left(\frac{x_4+x_5}{2}\right) + f\left(\frac{x_5+b}{2}\right) \right]$

- Depending on the function and the interval, these will be over or under estimating the area. Midpoint is generally more accurate, but requires more computation.

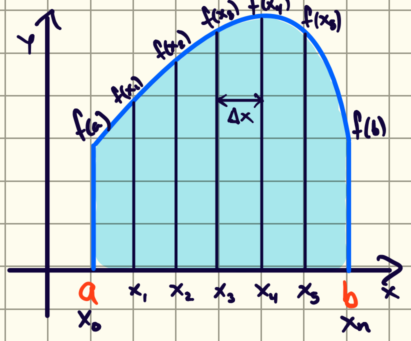
- How can we make our estimations more accurate?

- Summation Notation: as we have more subintervals, it becomes difficult to write out the sums, so we use summation notation.

In general: $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

- Important formulas:
  - $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
  - $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  - $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
  - $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
  - $\sum_{i=1}^{n} c = c \cdot n$ where c is a constant

- Riemann Sums: using summations to approximate the area



First we need to match our situation with the notation of the summations:

$$\Delta x = \frac{b-a}{n}, \quad a = x_0, \quad b = x_n$$

we also need to match our x_i values to a pattern:

$$x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ldots, \quad x_n = a + n\Delta x$$

Putting it together we get:

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x \Rightarrow \sum_{i=1}^{n} f(a + i\Delta x) \cdot \Delta x$$

Is this a left or right endpoint?

- Left endpoint: $\sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x \Rightarrow \sum_{i=1}^{n} f(a + (i-1)\Delta x) \cdot \Delta x$

- Exact area: if we take the limit as $n \to \infty$ of the summations, then the approximation becomes the actual area.

Ex: what is the area under the curve $y = x + 2$ on the interval $[1, 2]$?

$$\Delta x = \frac{1}{n} \qquad x_i = 1 + \frac{i}{n}$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right) \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n} + 3\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(\frac{1}{n} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 3\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(\frac{1}{n} \cdot \frac{n(n+1)}{2} + 3n\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(\frac{n^2+n}{2n} + 3n\right)$$

$$= \lim_{n\to\infty} \frac{1}{2} + \frac{1}{2n} + 3$$

$$= 3.5$$

- Distance: in the last section we saw that the antiderivative of velocity was distance. It turns out that the total distance traveled in some time interval is just the area under the curve of the velocity function.