Unit 1 Review

The Antiderivative

- Suppose you know the velocity of a projectile at every moment in time, $v(t)$. Can we find the position of the projectile at some time? $s(t)$

  we know: $\frac{d}{dt} s(t) = v(t) \Rightarrow s'(t) = v(t)$

  Since $v(t)$ is the derivative of the position function, $s(t)$, then $s(t)$ is the antiderivative of $v(t)$.

- What we are saying here is that for some function $f(x)$, there is a function $F(x)$ such that $F'(x) = f(x)$.

- There are several questions we need to ask here: Does every function have an antiderivative, how do we find the antiderivative, and is the antiderivative of a function unique?

  We will answer the first two questions later, but let's look at the last one now:

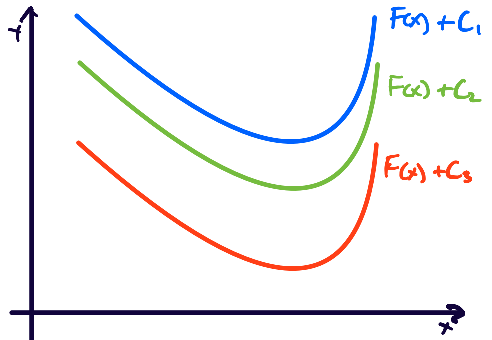
  which of the following functions are the antiderivative of $f(x) = 3x^2$?

  - $A(x) = \frac{3}{2}x^3 + 1$
  - $B(x) = x^3$
  - $C(x) = x^3 - x$
  - $D(x) = x^3 - 4$
  - $E(x) = 2x^3$
  - $F(x) = x^3 + \pi$

  From this we can see that the antiderivative is not unique, but what more can we say about it?

  If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$ is also an antiderivative of $f(x)$. Here C is an arbitrary constant called the constant of integration.

  Why should this make sense from a graphical perspective?



Remember that the derivative function just tells us the slope of the function at each point, but not where that point is located.

- Problems: find the general antiderivative of the following:

1) $f(x) = \csc^2 x - 3$

2) $f(x) = \frac{-1}{2\sqrt{x}}$

3) $f(x) = 2x^3 - 4^2$

4) $f(x) = \cos x + 1$

5) $f(x) = 2x + e^x$

- Initial Value Problems: In some cases, we can determine the value of C, given the initial information $F(x_0) = y_0$.

Ex: $f(x) = 3x^2 + 5$, $F(2) = 12$

$F(x) = x^3 + 5x + C$

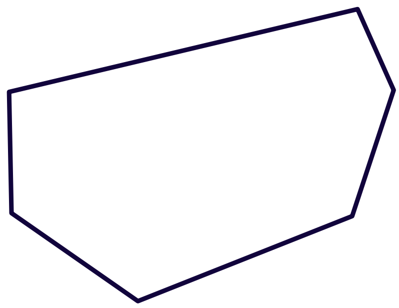
$F(2) = 2^3 + 5(2) + C = 12$

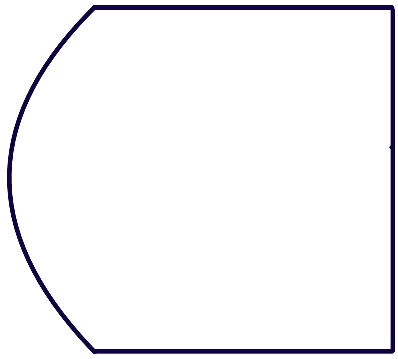
$\Rightarrow C = -6$

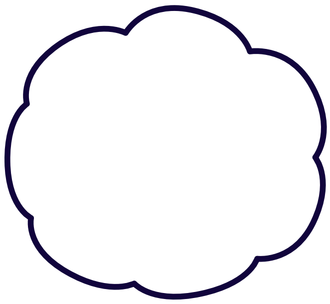
Area Under a Curve

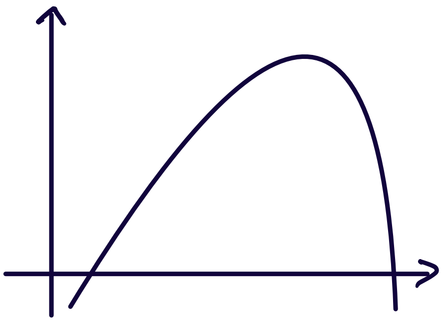
- Suppose we have an irregular shape, then how could we find its area?

Ex:

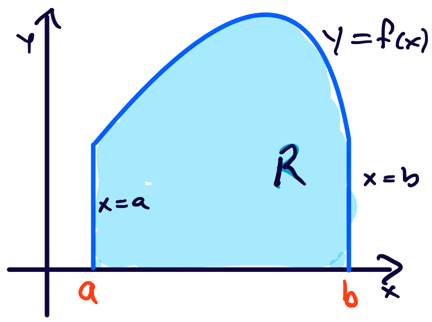




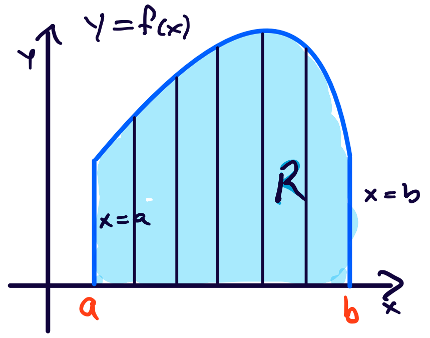




- Let's look at a situation like the last one:

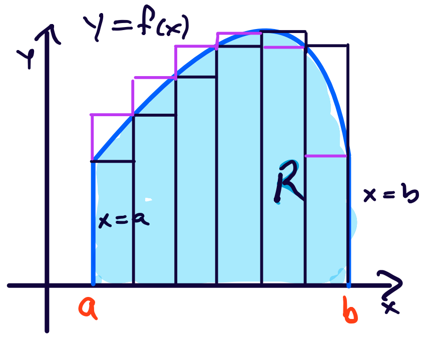


We want to find the area of R bounded by the lines $x=a$ and $x=b$ and between the x-axis and the curve $f(x)$.



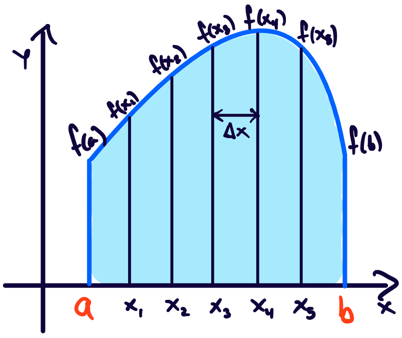
One way to approach this is to break it up into smaller pieces called subintervals. The length of each subinterval can be found by

$$\text{length} = \frac{b-a}{\#\text{ of subintervals}}$$



Now we can make rectangles out of each subinterval to approximate the area of R.

Notice that choosing the left or the right endpoint gives a different estimate.



To make this a bit more systematic, we name the length between each sub interval as Δx. This means that the area of any one rectangle will be $f(x_n) \cdot \Delta x$, since $f(x_n)$ is the height and Δx is the width.

- left sum: $A = \Delta x \cdot f(a) + \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) + \Delta x \cdot f(x_4) + \Delta x \cdot f(x_5)$

- right sum: $A = \Delta x \cdot [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(b)]$

- Mid-point sum: $A = \Delta x \cdot \left[f\left(\frac{a+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_4}{2}\right) + f\left(\frac{x_4+x_5}{2}\right) + f\left(\frac{x_5+b}{2}\right) \right]$

  - Depending on the function and the interval, these will be over or under estimating the area. Midpoint is generally more accurate, but requires more computation.

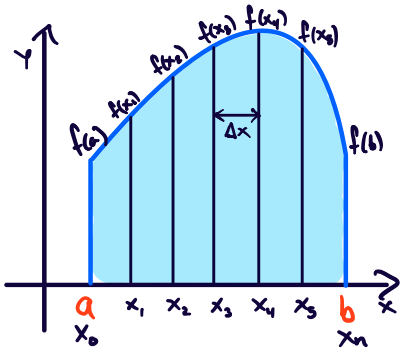
- How can we make our estimations more accurate?

- Summation Notation: as we have more subintervals, it becomes difficult to write out the sums, so we use summation notation.

In general: $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$

- Important formulas:
  - $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
  - $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  - $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
  - $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
  - $\sum_{i=1}^{n} c = c \cdot n$ where c is a constant

- Riemann Sums: using summations to approximate the area



First we need to match our situation with the notation of the summations:

$$\Delta x = \frac{b-a}{n}, \quad a = x_0, \quad b = x_n$$

we also need to match our x_i values to a pattern:

$$x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ldots, \quad x_n = a + n\Delta x$$

Putting it together we get:

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x \Rightarrow \sum_{i=1}^{n} f(a + i\Delta x) \cdot \Delta x$$

Is this a left or right endpoint?

- Left endpoint: $\sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x \Rightarrow \sum_{i=1}^{n} f(a + (i-1)\Delta x) \cdot \Delta x$

- Exact area: if we take the limit as $n \to \infty$ of the summations, then the approximation becomes the actual area.

Ex: what is the area under the curve $y = x+2$ on the interval $[1, 2]$?

$$\Delta x = \frac{1}{n} \qquad x_i = 1 + \frac{i}{n}$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right) \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n} + 3\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(\frac{1}{n} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 3\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(\frac{1}{n} \cdot \frac{n(n+1)}{2} + 3n\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(\frac{n^2+n}{2n} + 3n\right)$$

$$= \lim_{n\to\infty} \frac{1}{2} + \frac{1}{2n} + 3$$

$$= 3.5$$

- Distance: in the last section we saw that the antiderivative of velocity was distance. It turns out that the total distance traveled in some time interval is just the area under the curve of the velocity function.

The Definite Integral

Definition: $A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

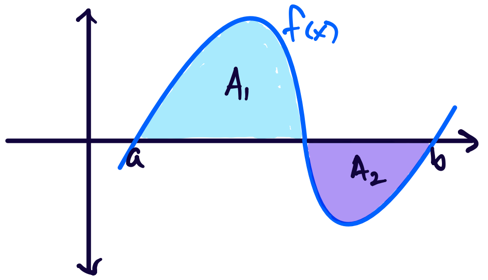
this says that the area under the curve $f(x)$ on the interval $[a,b]$ is the same as the limit of the Riemann sum on that interval as n goes to infinity.

Notation: $\int$ represent an elongated S to mean sum,

a and b are the interval, and dx is the infintesmal and also shows the variable of integration. Here f(x) is called the integrand, which is the part being integrated.

Note: the definite integral finds the net area, meaning the result could be positive or negative.

Ex:



$$\int_a^b f(x)\,dx = A_1 - A_2$$

in other words, area below the x-axis is considered negative area.

Theorem: If f(x) is continuous on an interval $[a,b]$, then f(x) is integrable on $[a,b]$.

So as long as a function is continuous on a closed interval, then we can integrate it.

The rest of the section is confusing notation to represent an integral and show the limit of the Riemann Sum as $n \to \infty$. Then they have you calculate integrals using different types of sums, like we did last section.

Properties of the Definite Integral

- Upper and lower limits: the order in which you choose the limits changes the answer:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

also, if $a=b$ then $\int_a^b f(x)\,dx = 0$

- Other properties:

$$\int_a^b c\,dx = c(b-a) \quad \text{where } c \text{ is a constant}$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$$

Problems: Evaluate the following definite integrals

1) $\int_0^3 f(x)\,dx$ if $\int_3^0 f(x)\,dx = -16$

2) $\int_{-1}^2 (x^2+3)\,dx$ if $\int_{-1}^2 x^2\,dx = 3$

3) $\int_0^1 [f(x)+g(x)-h(x)]\,dx$ if $\int_0^1 f(x)\,dx = 7$, $\int_0^1 g(x)\,dx = -5$, and $\int_0^1 h(x)\,dx = 12$

4) $\int_\pi^\pi \cos x\,dx$

5) $\int_{-2}^0 (4x^2-3x)\,dx$ if $\int_{-2}^0 x^2\,dx = \frac{8}{3}$ and $\int_{-2}^0 x\,dx = -2$

6) if $\int_2^4 f(x)\,dx = 7$, $\int_2^4 g(x)\,dx = -2$, and $\int_2^3 h(x)\,dx = 4$, find $\left[\int_2^4 f(x) - g(x)\,dx\right]^2 + \int_3^2 h(x)\,dx$

7) $\int_{-14}^{17} 8\,dx$

Comparison Properties of the Definite Integral

Properties:

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Problems:

1) Give bounds for $\int_0^2 x^3-1\,dx$

2) If $5-2x \geq f(x)$, estimate $\int_0^1 f(x)\,dx$

3) if $a \leq \int_0^{\pi/2} \cos x\,dx \leq b$, find a and b

4) For $f(x)$ positive on $[a,b]$, $\int_a^b f(x)\,dx \geq (2k-7)$ find k

The Fundamental Theorem of Calculus

Theorem:

Let f be continuous on $[a,b]$. If F is an antiderivative of f on $[a,b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Notation: we often write $\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{x=b}$, where the vertical bar means to evaluate $F(x)$ as the difference between $x=b$ and $x=a$.

Ex: $\int_1^3 2x\,dx = x^2\Big|_1^3 = 3^2 - 1^2 = 8$

We can reframe the theorem as follows:

If f is continuous on the interval $[a,b]$, then g can be defined by $g(x) = \int_a^x f(t)\,dt$, where $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

This last part also means we can say $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Ex 1: Find $\frac{dy}{dx}$ if $y = \int_0^x \sec t\,dt$

From the definition we just found:

$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ $\quad a=0$ & $f(t) = \sec t$

$\Rightarrow \frac{dy}{dx} = \sec x$

Ex 2: Find $\frac{dy}{dx}$ if $y = \int_0^{2x} 2t^3\,dt$

using the first representation: $F(x) = \frac{x^4}{2}$

$\Rightarrow y = \frac{x^4}{2}\Big|_0^{2x} = 8x^4 - 0$

$\Rightarrow \frac{dy}{dx} = 32x^3$ $\quad$ is this result surprising?

Problems: Evaluate the following

1) $\int_{-4}^{0} x^3 + 4x^2 \, dx$

2) $\int_{-1}^{2} 2x^4 - 3x^2 \, dx$

3) $\int_{0}^{\pi} \sin x \, dx$

4) $\int_{0}^{\pi/4} \sec x \cdot \tan x \, dx$

5) $\int_{0}^{3} \frac{1}{3\sqrt[3]{x^2}} \, dx$

6) $\int_{0}^{5} 5x^2 + 3x \, dx$

7) $\int_{0}^{\pi} e^{-x} + \cos x \, dx$

Problems:

1) Find $f'(x)$ if $f(x) = \int_{0}^{x} \sqrt{1+t^2} \, dt$

2) Find $f'(x)$ if $f(x) = \int_{x}^{2x} \cos t \, dt$

3) Find $f'(x)$ if $f(x) = \int_{0}^{3x} \sec t \cdot \tan t \, dt$

4) Find $f'(x)$ if $f(x) = \int_{x}^{2} 3t^2 - 2t + 1 \, dt$

Problems:

1) For $\frac{dy}{dx} = -\cos x$, find the initial value when $y=1$ and $x=0$

2) For $\frac{dy}{dx} = \sec^2 x$, find the initial value when $y=\pi$ and $x=\pi/4$

3) For $f'(x) = e^x + x$, find the initial value when $f(0)=5$

4) For $f'(x) = 6x^2 - 14x + 8$, find the initial value when $f(2)=17$

Indefinite Integrals

- Recall the fundamental theorem of calculus:

$$g(x) = \int_a^x f(t)\,dt \quad \text{or} \quad \int_a^b f(t)\,dt = F(b) - F(a)$$

in either case, $g(x)$ and $F(x)$ are antiderivatives of $f(x)$.

This leads to our definition of an indefinite integral:

$$\int f(x)\,dx = F(x) + C$$

where $F'(x) = f(x)$ and C is a constant.

Ex: $\int 4x^3\,dx = x^4 + C$

- Definite vs. Indefinite Integrals

definite integrals are a number (area under a curve over a defined interval), whereas an indefinite integral is a family of functions.

- List of common integrals:

- $\int a f(x)\,dx =$
- $\int k\,dx =$
- $\int x^n\,dx =$
- $\int \frac{1}{x}\,dx =$
- $\int e^x\,dx =$
- $\int a^x\,dx =$
- $\int \sin x\,dx =$
- $\int \cos x\,dx =$
- $\int \tan x\,dx =$
- $\int \sec^2 x\,dx =$
- $\int \sec x \tan x\,dx =$
- $\int \csc^2 x\,dx =$

Problems:

1) $\int e^2\,dx$
2) $\int \sin x + 2\,dx$
3) $\int \frac{5}{x}\,dx$
4) $\int 2^x + 1\,dx$
5) $\int_0^{\pi/4} \sec^2 x\,dx$
6) $\int_0^{\pi/3} \sec x \tan x + x\,dx$
7) $\int_{-1}^{3} (3x^3 + 4x - 5)\,dx$

Applications for the Definite Integral

Net change theorem:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

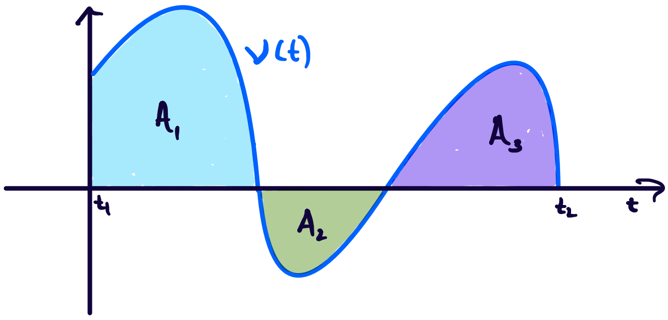
Ex: Given a velocity function $v(t) = 2t - 7$, what is the net displacement from $t=0$ to $t=6$?

$$\Rightarrow \int_0^6 2t-7\,dt = t^2 - 7t\Big|_0^6 = 36 - 42 - 0 = -6$$

What is the difference between displacement and distance?

Calculating displacement: $\int_{t_1}^{t_2} v(t)\,dt$

Calculating distance: $\int_{t_1}^{t_2} |v(t)|\,dt$



displacement: $\int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$

distance: $\int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$

From this, what issues do you see in calculating distance?

Ex: Given the velocity function $v(t) = 2t^2 - 4$, find the displacement and distance from $t=0$ to $t=3$.

displacement: $\int_0^3 (2t^2-4)\,dt = \frac{2}{3}t^3 - 4t\Big|_0^3 = 18 - 12 - 0 = 6$

distance: $2t^2 - 4 = 0 \Rightarrow t = \pm\sqrt{2}$

$$\Rightarrow \int_0^{\sqrt{2}} -(2t^2-4)\,dt + \int_{\sqrt{2}}^3 (2t^2-4)\,dt$$

$$= 4t - \frac{2}{3}t^3\Big|_0^{\sqrt{2}} + \frac{2}{3}t^3 - 4t\Big|_{\sqrt{2}}^3$$

$$= 4\sqrt{2} - \frac{4}{3}\sqrt{2} + 18 - 12 - \frac{4}{3}\sqrt{2} + 4\sqrt{2}$$

$$= \frac{16}{3}\sqrt{2} + 6 \approx 13.54\ldots$$

Problems: find the displacement and distance of the following

1) $\int_0^{\pi} \cos t \, dt$

2) $\int_{-2}^{1} 5t-3 \, dt$

3) $\int_{-1}^{2} e^t -1 \, dt$

4) $\int_1^4 \frac{1}{t} - \frac{1}{2} \, dt$

5) $\int_0^4 t^3-4t^2+3t \, dt$

6) $\int_0^{\pi/3} \sec^2 t -2 \, dt$

7) $\int_{-\pi/3}^{\pi/3} \tan t -1 \, dt$

The Substitution Rule

Consider the indefinite integral $\int 2x \cdot e^{x^2} dx$

can we use any of the basic integral formulas we have learned to solve this problem?

while none of the methods we have discussed will work, let's look at this from another angle, antiderivatives.

Want $f(x)$ such that $f'(x) = 2x \cdot e^{x^2}$. what would $f(x)$ be?

Notice that when we take $\frac{d}{dx} f(x)$, we must use the chain rule to get $2x \cdot e^{x^2}$, so let's review it.

Recall $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

$\Rightarrow f(g(x)) = \int f'(g(x)) \cdot g'(x)\, dx$

This is the basis of the substitution rule.

Substitution Rule: for some $u = g(x)$, $du = g'(x)\, dx$

$$\int f(g(x)) \cdot g'(x) dx = \int f(u)\, du$$

let's apply this to our previous situation: $\int 2x e^{x^2} dx$

$u = x^2$ so $du = 2x\, dx$

$\Rightarrow \int 2x e^{x^2} dx = \int e^u du$

$\Rightarrow e^u + c \Rightarrow e^{x^2} + c$

Ex: Prove $\int \tan x\, dx = \ln|\sec x| + c$ using substitution rule.

$\tan x = \frac{\sin x}{\cos x}$ so $u = \cos x$ and $du = -\sin x\, dx$

$\Rightarrow \int \tan x\, dx = -\int \frac{1}{u} du$

$\Rightarrow -\ln|u| + c \Rightarrow \ln\left|\frac{1}{u}\right| + c$

$\Rightarrow \ln\left|\frac{1}{\cos x}\right| + c \Rightarrow \ln|\sec x| + c$

Problems:

1) $\int 2x \cos 3x^2 \, dx$

2) $\int 6x^2 \sqrt{1+x^3} \, dx$

3) $\int \frac{dx}{x^2+6x+9}$

4) $\int \frac{\sqrt{x}+3}{\sqrt{x}} \, dx$

5) $\int x\sqrt{x+2} \, dx$

6) $\int (4x+9)^2 \, dx$

7) $\int 4 \sec^2 x \cdot \tan x \, dx$

Substitution Rule and Definite Integrals

$$\int_a^b f(g(x)) \cdot g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

There are two approaches to definite integrals and u-substitution

1) Do the whole process like with indefinite integrals, and once you have your $F(g(x))$ evaluate from a to b.

2) Change the bounds of integration and the evaluate using $F(u)$. Since $u=g(x)$ we use $g(a)$ and $g(b)$ for our bounds in u

Ex: $\int_0^1 2x(9-x^2)^3 \, dx$ $\quad u = 9-x^2, \; du = -2x \, dx$

1) $-\int u^3 \, du = -\frac{u^4}{4} \Rightarrow -\frac{1}{4}(9-x^2)^4 \Big|_0^1 = -1024 + 1640.25 = 616.25$

2) $u(1) = 8 \quad u(0) = 9 \Rightarrow -\int_9^8 u^3 \, du = -\frac{1}{4}u^4 \Big|_9^8 = -1024 + 1640.25$
$= 616.25$

Symmetric Functions and Integration

Recall:

- Even functions: $f(-x) = f(x)$ i.e. reflected across y-axis

  integration property: $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

  Proof: $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

  but $\int_{-a}^{0} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx$ say $u = -x$, $du = -dx$

  $\Rightarrow -\int_{0}^{a} f(-u)(-du) = \int_{0}^{a} f(u)\,du$

  $\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx$

  but u and x are arbitrary, so $\int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx$

  $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

- Odd functions: $f(-x) = -f(x)$ i.e. reflected across origin

  integration property: $\int_{-a}^{a} f(x)\,dx = 0$

  Proof: $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

  but $\int_{-a}^{0} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx$ say $u = -x$, $du = -dx$

  $\Rightarrow -\int_{0}^{a} f(-u)(-du) = -\int_{0}^{a} f(u)\,du$

  $\Rightarrow \int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx$

  but u and x are arbitrary, so $\int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx$

  $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

Problems: Are the following functions even, odd, or neither?

1) $f(x) = x \sin x \cdot \cos x$
2) $f(x) = \cos(3x)$
3) $f(x) = \sec x \cdot \tan x$
4) $f(x) = 3x^4 + 2^3$
5) $f(x) = x^2 \sin x$
6) $f(x) = 2x^5 - 3x^3 + 8x$
7) $f(x) = x^3 + 7$