Solution:
(a) Since X and Y are independent,

$$P(X > 2|Y > 2) = P(X > 2)P(Y > 2)/P(Y > 2) = P(X > 2) = 1/8 + 1/2 = 5/8$$

(b) $E[XY] = E[X]E[Y]$ due to independence.

$$E[X] = \sum_{k=-\infty}^{\infty} kP_X(k) = 1/4 + 2/8 + 3/8 + 4/2 = 2.875$$

$$E[Y] = \sum_{k=-\infty}^{\infty} kP_Y(k) = 1/6 + 2/6 + 3/3 + 4/3 = 2.833$$

$$E[XY] = E[X]E[Y] = 2.875 \times 2.83 = 8.14$$

(c) $Var(2X + Y) = 4Var(X) + Var(Y)$ due to independence.

$$E[X^2] = \sum_{k=-\infty}^{\infty} k^2 P_X(k) = 1/4 + 4/8 + 9/8 + 16/2 = 9.875$$

$$Var(X) = E[X^2] - E[X]^2 = 9.875 - 2.875^2 = 1.609$$

$$E[Y^2] = \sum_{k=-\infty}^{\infty} k^2 P_Y(k) = 1/6 + 4/6 + 9/3 + 16/3 = 9.167$$

$$Var(Y) = E[Y^2] - E[Y]^2 = 9.167 - 2.833^2 = 1.14$$

$$Var(2X + Y) = 4 \times 1.609 + 1.14 = 7.58$$

(d) The sample space for event $A = \{11, 12, 21\}$.

$$P_{X,Y|A}(2, 2) = 0 \neq P_{X|A}(2)P_{Y|A}(2)$$

As a result, X and Y are not conditionally independent for event A.

Q 2

Solution

a) To find α:

$$\int_0^2 \alpha(2x - x^2)\,dx = 1$$

$$\Rightarrow \alpha(x^2 - \frac{1}{3}x^3)|_0^2 = \alpha(4 - \frac{8}{3}) = \frac{4}{3}\alpha = 1$$

$$\Rightarrow \alpha = \frac{3}{4}$$

b) CDF:

$$F(x) = \begin{cases} \frac{3}{4}(x^2 - \frac{1}{3}x^3) & 0 \le x \le 2 \\ 1 & x > 2 \\ 0 & x < 0 \end{cases} \tag{2}$$

c) To calculate probabilities:

$$P(X \ge 1.2) = 1 - P(X \le 1.2) = 1 - \int_0^{1.2} \frac{3}{4}(2x - x^2)\,dx = 1 - \frac{3}{4}(x^2 - \frac{1}{3}x^3)|_0^{1.2} = 1 - 0.648 = 0.352$$

$$P(0.5 < X \le 1.5) = \int_{0.5}^{1.5} \frac{3}{4}(2x - x^2)\,dx = \frac{3}{4}(x^2 - \frac{1}{3}x^3)|_{0.5}^{1.5} = 0.6875$$

$$P(X = 1) = 0$$

d) For expectation:

$$E[X] = \int_0^2 xf(x)\,dx = \frac{3}{4}\int_0^2 x(2x - x^2)\,dx = \frac{3}{4}\int_0^2 2x^2 - x^3\,dx = \frac{3}{4}(\frac{2}{3}x^3 - \frac{1}{4}x^4)|_0^2 = 1$$

e) To find variance:

$$E[X^2] = \int_0^2 x^2 f(x)\,dx = \frac{3}{4}\int_0^2 x^2(2x - x^2)\,dx = \frac{3}{4}\int_0^2 2x^3 - x^4\,dx = \frac{3}{4}(\frac{1}{2}x^4 - \frac{1}{5}x^5)|_0^2 = \frac{6}{5} = 1.2$$

$$\Rightarrow Var(X) = E[X^2] - E[X]^2 = 1.2 - 1 = 0.2$$

f)

$$P(\text{Lantern will be working more than 12 hours}) = P(\text{All 4 batteries will be working more than 12 hours})$$

$$= (P(X \ge 1.2))^4 = (0.352)^4 \quad = 0.015$$

January 11, 2023

1 Question 3

a)

$$p(Y_1 > 1) = 1 - p(Y_1 < 1) = 0.25$$
$$p(Y_1 < 1) = 0.75$$

Let's write the Z value as

$$p(Z < \frac{1-X}{1.5}) = 0.75$$

Using the normal table

$$\frac{1-X}{1.5} = 0.68 \Rightarrow 1 - X = 1.02 \Rightarrow X = -0.02$$

b) Using the Z value

$$p(Z < \frac{1-X}{1.5}) = p(Z < \frac{2-X}{2})$$
$$\frac{1-X}{1.5} = \frac{2-X}{2}$$
$$\frac{2}{3} - \frac{2}{3}X = 1 - \frac{X}{2} \Rightarrow X = -2$$

c) Canceled

d) For the first receiver:

$$p(0.5 < Y_1 < 2) = p(Y_1 < 2) - p(Y_1 < 0.5)$$
$$= p(Z < \frac{2-1}{1.5} = \frac{2}{3}) - p(Z < \frac{0.5-1}{1.5} = \frac{-1}{3})$$
$$= 0.7457 - 0.3707 = 0.3747$$

For the second receiver:

$$p(0.5 < Y_2 < 2) = p(Y_2 < 2) - p(Y_2 < 0.5)$$
$$= p(Z < \frac{2-1}{2} = \frac{1}{2}) - p(Z < \frac{0.5-1}{2} = -0.25)$$
$$= 0.6915 - 0.4013 = 0.2902$$

p(only one of the Y values is in that range)=0.3747(1-0.2902)+0.2902(1-0.3747)=0.2659+0.1814=0.4473

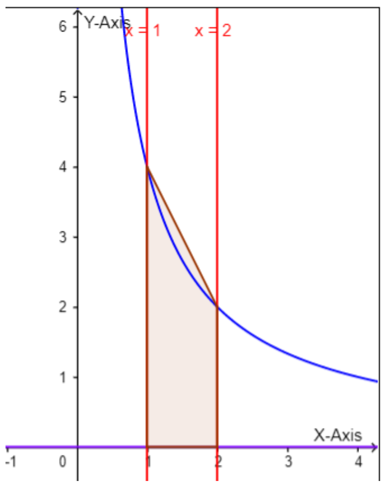
Question 4 (25 points)

Let X and Y be jointly distributed random variables whose joint probability density function is given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{x}{4}, & 1 < x < 2, 0 < xy < 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (3 points) Draw the region where $f_{X,Y}(x,y) > 0$

Answer:



(b) (7 points) Find $P(Y > X)$. *Hint: Drawing will help.*

Answer:

$$P(Y > X) = 1 - P(Y < X)=1-\left(\int_1^2 \int_0^x \frac{x}{4}\, dx\, dy\right)$$

$$P(Y > X) = 1 - \frac{7}{12} = \frac{5}{12}$$

(c) (8 points) Find the marginal probability density function of Y, i.e., $f_Y(y)$. *Hint: You need to differentiate two cases.*

Answer:

Two cases need to be evaluated:

Case 1: $0 \leq y \leq 2$

$$\int_1^2 \frac{x}{4}\, dx = 3/8$$

Case 2: $2 \leq y \leq 4$

$$\int_1^{4/y} \frac{x}{4}\, dx = \frac{16}{8\, y^2} - \frac{1}{8}$$

(d) (7 points) Find $E[Y \mid Y < 2]$.

Answer:

First, we have to find the conditional density $f_{Y|(y<2)} = \dfrac{f_Y(y<2)}{P(y<2)}$

From part c, we know that $f_{Y|Y}(y < 2) = \dfrac{3}{8}$. Also, $P(y < 2) = \displaystyle\int_0^2 \frac{3}{8}\, dy = \frac{3}{4}$

Therefore, $E[f_{Y|(y<2)}] = \displaystyle\int_0^2 \frac{1}{2}\, dy = 1$