Math 203: Spring 2022 FINAL MAKEUP Duration: 120 minutes

I pledge my honor that I did not receive nor give any help during this exam and I am aware of the University's honor code.

Student's Name:	
ID Number:	
Signature:	

Check your section

☐ Section 1 Tue. & Thu.) ☐ Section 2 (Tue. & Thu.)

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SOME INFO THAT MAY BE USEFUL.

$$dx\ dy \qquad r\ dr\ d\theta \qquad \frac{\partial(x,y)}{\partial(u,v)}du\ dv$$

$$dx\ dy\ dy \qquad r\ dr\ d\theta\ dz \qquad R^2 \sin\phi\ dR\ d\phi\ d\theta \qquad \frac{\partial(x,y,z)}{\partial(u,v,w)}du\ dv\ dw$$

$$\sqrt{\dot{x}^2+\dot{y}^2}\ dt \qquad \sqrt{1+(y')^2}\ dx \qquad \oint_{\mathcal{C}} u\ dx + v\ dy = \iint_{\mathcal{A}} (v_x - u_y)$$

$$\sqrt{f_x^2+f_y^2+1} \qquad \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|} \qquad a^2 \sin\phi\ d\phi\ d\theta$$

$$\int \sqrt{a \pm r^2}\ r\ dr = \pm\frac{1}{2}\int u^{1/2}\ du \qquad \sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$r\ dr\ d\theta\ dz \qquad r^2 \sin\phi\ dr\ d\phi\ d\theta \qquad \int_A^B \mathbf{F}\cdot d\mathbf{r} = U\Big|_A^B \qquad \oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = 0$$

$$\sqrt{f_x^2+f_y^2+1}\ dA \qquad \frac{\sqrt{U_x^2+U_y^2+U_z^2}}{|U_z|}\ dA \qquad (-f_x,-f_y,1)\ dA \qquad \frac{\nabla U}{U_z}\ dA$$

1:	2:	3:	4:	5:	6:	/120

IF YOU NEED MORE SPACE, CONTINUE AT THE BACK OF THE SHEET OF THE QUESTION PAGE

1. (i) Let $\mathcal{P}_1 : x + y + z = 0$ $\qquad$ $\mathcal{P}_2 : x + 2y + 3z = 4$. Give the reason for the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ to intersect or not and find the intersection if they intersect in vector form.

(ii) Let $\{\mathcal{P} : x + y + z = 0\}$ and the line $\{\mathcal{L} : \mathbf{x} = (-3, -3, 0) + (1, 2, 3)t\}$. Give the reason for the line to intersect the plane or not and find the intersection if it exists.

SOLUTION (i)

The reason. Using the normal vectors $\mathbf{n}_1 = (1, 1, 1)$ $\quad \mathbf{n}_2 = (1, 2, 3)$.

It is clear that $\mathbf{n}_1$ and $\mathbf{n}_2$) are not parallel to each other. Thefore, the two planes intersect.

The intersection. It is obtained by selecting $z = t$, t as a parameter and solving the two equations for x and y:

$$x + y = -t \qquad x + 2y = 4 - 3t \quad \to \quad y = 4 - 2t \qquad x = -t - y = -4 + t$$

The Eq. for the intersection: $\mathbf{x} = (x, y, z) \to \mathbf{x} = (-4+t, 4-2t, t) = (-4, 4, 0)+(1, -2, 1)t$

SOLUTION (ii)

The reason. Using the normal vector of the plane and the direction vector of the line: $\mathbf{n} = (1, 1, 1)$ $\quad \mathbf{u} = (1, 2, 3)$.

It is clear that $\mathbf{n} \cdot \mathbf{u} = (1, 1, 1) \cdot (1, 2, 3) = 6 \neq 0$, i. e.

they are not orthogonal to each other.are not parallel to each other. Thefore, the plane and the line intersect.

The intersection. It is obtained by selecting substituting the x, y, z from the line in the equation of the plane and solvinf for t:

$$x+y+z = 0 \qquad x = -3+t \qquad y = -3+2t \qquad z = 3t \quad \to \quad (-3+t)+(-3+2t)+3t = 0 \quad \to$$

$$-6 + 6t = 0 \quad \to \quad t = 1 \quad \to \quad \text{The intersetion point: } x_{int} = (-2, -1, 3)$$

2. (i) Evalute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $A:(0,0,1)$ for $xe^y + yz + ze^x = 1$.

(ii) Find the local extrema and their nature of $f(x,y) = xy + \dfrac{1}{3}(x^3 + y^3)$

SOLUTION (i)

Let's take the partials of $xe^y + yz + ze^x = 1$ with respect to x and y.

The partial with respect to x: $e^y + yz_x + z_x e^x + ze^x = 0 \quad \rightarrow \quad z_x(y + e^x) + e^y + ze^x = 0 \quad \rightarrow \quad z_x = -\dfrac{e^y + ze^x}{(y + e^x)}$

The partial with respect to y: $xe^y + yz_y + z + z_y e^x = 0 \quad \rightarrow \quad z_y(y + e^x) + xe^y + z = 0 \quad \rightarrow \quad z_y = -\dfrac{xe^y + z}{(y + e^x)}$

The values at $A:(0,0,1)$

$$z_x\Big|_A = -\frac{e^y + ze^x}{(y + e^x)}\Big|_{0,0,1)} = -\frac{1+1}{0+1} = 2$$

$$z_y\Big|_A = -\frac{xe^y + z}{(y + e^x)}\Big|_{0,0,1)} = -\frac{0+1}{0+1} = 1$$

SOLUTION (ii)

The location of the extrema:

$$f(x,y) = xy + \frac{1}{3}(x^3 + y^3) \quad \rightarrow \quad f_x = y + x^2 = 0 \qquad f_y = x + y^2 = 0 \quad \rightarrow \quad x + (-x^2)^2 = x + x^4 = x(1 + x^3) = 0$$

$$x_1 = 0 \qquad x_2 = -1 \quad \rightarrow \quad y_1 = 0 \qquad y_2 = -1 \quad \rightarrow \quad A:(0,0) \qquad B:(-1,-1)$$

Type of the extremum. Starting with the Hessian:

$$f_{xx} = 2x \qquad f_{xy} = 1 \qquad f_{yy} = 2y \quad \rightarrow \quad H = \begin{vmatrix} 2x & 1 \\ 1 & 2y \end{vmatrix} = 4xy - 1$$

At $A:(0,0)$, $\quad H = -1$. Therefore, this is a saddle point.

At $B:(-1,-1)$, $\quad H = 3$. Therefore, this is a min or max point. Becaus, at this point $f_{xx} = -2$, it is a maximum.

3. (i) Given the triangle with one corner at $O:(0,0)$ and with the two points placed symmetrically with respect to the x axis on the half circle $x^2+y^2=a^2 \quad x \geq 0$.

Find the maximum area of the triangle defined above by a method other than substitution.

(ii) Find the volume of the solid between $x^2+y^2+z^2=a^2$ and $z=\sqrt{x^2+y^2}$.

SOLUTION (i) Let the coordinates of the other points be: $A:(x,y) \quad B:(x,-y)$.

The funtion to optimize is the area of the triangle: $f(x,y)=x(2y)/2=xy$

The constraint: A and B must be on the circle: $x^2+y^2=a^2 \;\rightarrow\; g(x,y)=x^2+y^2-a^2=0$.

The three equations to solve: $\nabla f=\lambda\nabla g \qquad g=0 \quad \rightarrow \quad y=\lambda(2x) \qquad x=\lambda(2y)$

The first step is to eliminate λ. This is done by taking the ratio of the two equations:

$$\frac{y}{x}=\frac{x}{y} \quad \rightarrow \quad x=y$$

Now substituting $y=x$ in the equation of the circle: $x^2+x^2=a^2 \quad \rightarrow \quad x=y=a/\sqrt{2}$

Thus, the triangle is isoceles and the largest area is : $\mathcal{A}=a^2/2$

SOLUTION (ii)

The spherical coordinates is most appropriate. (Can also be done by cylindrical coordinates.)

OPTION 1. In spherical coordinates, , using the formula on front page for $dV=R^2\sin\phi\, dR\, d\phi\, d\theta$.

The cone axis makes an angle of 45^o. Thus, in spherical coordinates $\phi_0=\pi/4$

$$V=\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4}\int_{R=0}^{R=a} R^2\sin\phi\; dR\; d\phi\; d\theta = 2\pi\Big(-\cos\phi\Big|_0^{\pi/4}\Big)\left(\frac{a^3}{3}\right)=\frac{2\pi}{3}\;a^3\;\left(1-\frac{\sqrt{2}}{2}\right)$$

OPTION 2. In cylindrical coordinates with $dV=d\theta\; r\; dr\; dz$ and using the formula on front page for $\int\sqrt{a^2-r^2}\; r\; dr$.

$$V=\Big(\int_{\theta=0}^{\theta=2\pi} d\theta\Big)\Big(\int_{r=0}^{r=a/\sqrt{2}}\Big(\int_{z=r}^{z=\sqrt{a^2-r^2}} dz\Big)\; r\; dr\Big)=2\pi\Big(\int_{r=0}^{r=a/\sqrt{2}}(\sqrt{a^2-r^2}-r)r\; dr\Big)=\frac{2\pi}{3}\;a^3\;\left(1-\frac{\sqrt{2}}{2}\right)$$

4. (i) Calculate $\quad I = \int_{x=-\infty}^{\infty} e^{-x^2/2} dx$

(ii) Given $\mathbf{u} = (e^x + 3x^2y,\ e^{-y} + x^3 - 4y^3)$ and the circle C with radius r = 2 centered at the origin of the coordinate system.

$$\text{Evaluate} \quad \mathcal{I} = \int_A^B \mathbf{u} \cdot d\mathbf{x} \quad \text{with the points } A : (1,\ 0),\ B : (1,\ 0).$$

SOLUTION (i) (See Lecture note for Week , page)

This integral can not be done directly. Let's integrate its square:

$$I^2 = \left(\int_{x=-\infty}^{\infty} e^{-x^2/2} dx \right) \left(\int_{y=-\infty}^{\infty} e^{-y^2/2} dy \right) \quad \rightarrow$$

Let's rewrite this integral as a double integral and in polar coordinates:

$$I^2 = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-x^2/2}\ e^{-y^2/2}\ dx\ dy = \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) \left(\int_{r=0}^{r=\infty} e^{-r^2/2}\ r\ dr \right) \quad \rightarrow$$

For the integral in r, let's change variables as $r^2/2 = u \quad r\ dr = du \quad \rightarrow$

$$I^2 = 2\pi \int_{r=0}^{r=\infty} e^{-u}\ du = -e^{-u}\Big|_{r=0}^{r=\infty} = -e^{-r^2}\Big|_{r=0}^{r=\infty} = -(0-1) \quad \rightarrow$$

$$I^2 = 2\pi \quad \rightarrow \quad I = \sqrt{2\pi}$$

SOLUTION (ii) (See take home quiz 8)

This integral is too difficult to do directly. So, let's try to see if $\mathbf{u} = (u, v)$ is conservative.

$u_y = 3x^2 \qquad v_x = 3x^2 \quad \rightarrow \quad \mathbf{u}$ is conservative. Therefore, a potential f exists.

Finding the potential.

$$F_x = u = e^x + 3x^2y \quad \rightarrow \quad F = e^x + x^3y + g(y)$$

$$v = e^{-y} + x^3 - 4y^3 = F_y \quad \rightarrow \quad x^3 + g'(y) = e^{-y} + x^3 - 4y^3 \quad \rightarrow \quad g'(y) = e^{-y} - 4y^3 \quad \rightarrow$$

$$g(y) = -e^{-y} - y^4 \quad \rightarrow \quad F = e^x + x^3y - e^{-y} - y^4$$

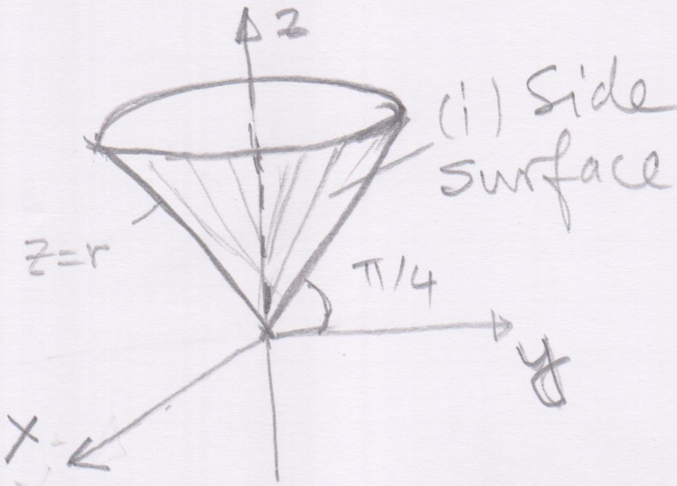
Now, the integral by using the theorem for conservative fields:

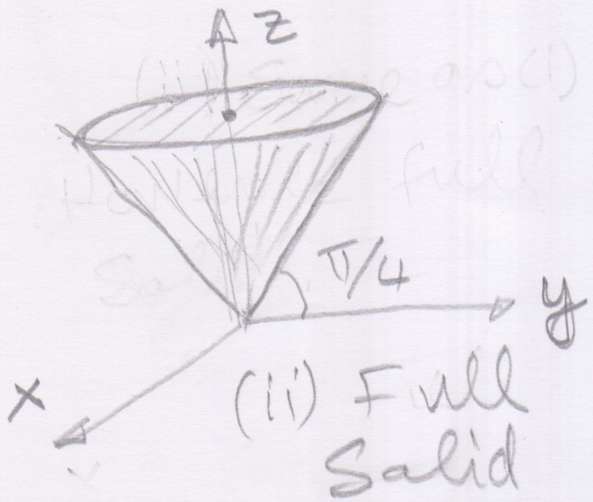
$$I = \int_A^B \mathbf{u}.d\mathbf{r} = F(B)\text{–}F(A) \quad \rightarrow \quad I = (e^x + x^3y - e^{-y} - y4)\Big|_{(-1,0)}^{(1,0)} \quad \rightarrow$$

$$I = (e - -1 + 0\text{–}1\text{–}0)\text{–}(e + 0 - 10) = e - -1 - e \quad \rightarrow \quad I = -2\sinh(1)$$

5. (i) Sketch the surface $z = \sqrt{x^2 + y^2}$, $0 \le z \le a$ and calculate its area using multiple integrals.

(ii) Sketch the solid $z = \sqrt{x^2 + y^2}$, $z \le a$ and calculate its volume using multiple integrals.





Sketches

SOLUTION (i) Let's identify $F(x, y, z) = z^2 - (x^2 + y^2) = 0$. Thus,

$$F_x = -2x \quad F_y = -2y \quad F_z = 2z \quad \rightarrow \quad dS = \frac{\sqrt{F_x^2 + F_y^2 + F_y^2}}{F_z}\, dA = \frac{4x^2 + 4y^2 + 4z^2}{2z} \quad \rightarrow .$$

$$dS = \frac{8z^2}{2z}\, dA = \sqrt{2}\, dA \quad \rightarrow \quad S = \sqrt{2} \int\int dA = \sqrt{2}A = \sqrt{2}\pi a^2$$

SOLUTION (ii) Cylindrical coordinates are appropriate. Thus:

$$V = \int\int\int r\, dr\, d\theta\, dz = Big(\int_{\theta=0}^{\theta=2\pi} d\theta\Big)\Big(\int_{r=0}^{r=a} \Big(\int_{z=r}^{z=a} dz\Big) r\, dr\Big) = 2\pi \Big(\int_{r=0}^{r=a} (a-r) r\, dr\Big) \quad \rightarrow$$

$$V = 2\pi \Big(a\frac{r^2}{2} - \frac{r^3}{3}\Big)\Big|_{r=0}^{r=a} = 2\pi a^3 \Big(\frac{1}{2} - \frac{1}{3}\Big) = 2\pi a^3 \frac{1}{6} \quad \rightarrow$$

$$V = \frac{1}{3}\pi a^3$$

6. (i) Calculate $\mathcal{I} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS$ with $\mathbf{F} = (3x,\ 4z,\ 6x) \equiv 3x\mathbf{i} + 4z\mathbf{j} + 6x\mathbf{k}$

S is the part of the paraboloid $z = 9 - x^2 - y^2$ above the xy-plane.

(ii) Calculate $\quad \mathcal{I} = \displaystyle\iint_{\mathcal{S}} \mathbf{u}\cdot\mathbf{n}\, dS \quad$ where $\quad \mathbf{u} = (0,\ yz,\ 0)$

S is the surface of the half sphere with $z \geq 0$ centered at the origin of the coordinate system and with radius a.

SOLUTION (i)

Doing this calculation directly is quite difficult. It is easier to use the Stokes theorem and calculate a line integral. Thus, with $z = 0$ and $dz = 0$ on $\mathcal{C}$:

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{x} \quad \to \quad \mathcal{I} = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{x} = \oint_{\mathcal{C}} (3x\ dx + 0 + 0) \quad \to$$

$$\mathcal{I} = 3\int_{\theta=0}^{\theta=2\pi} 3\cos\theta \sin\theta\ d\theta = \frac{9}{2}\sin^2\theta\Big|_0^{2\pi} = 0 - 0 = 0$$

Note. It is also clear that the vector $\mathbf{u} = (3x, 0, 0)$ is conservative. Thus, by the theorem and also the definition on conservative fields, $\oint_{\mathcal{C}} 3x\ dx = 0$.

SOLUTION (ii) It is easier to use the Divergence (Gauss') theorem and calculate a volume integral. Thus, with $z = 0$ and $dz = 0$ on $\mathcal{C}$:

$$\mathcal{I} = \int\int\int_{\mathcal{V}} \nabla \cdot \mathbf{u}\ dV \qquad \nabla \cdot \mathbf{u} = \nabla \cdot (0,\ yz,\ 0) = 0 + z + 0 = z \quad \to \quad \mathcal{I} = \int\int\int_{\mathcal{V}} z\ dV$$

The spherical coordinates are most appropriate. With $z = R\cos\phi,\ dV = R^2 \sin\phi\ dR\ d\phi\ d\theta$ and the separability of the three integrals:

$$I = \left(\int_{\theta=0}^{\theta=2\pi} d\theta\right)\left(\int_{\phi=0}^{\phi=\pi/2} \cos\phi\sin\phi\ d\phi\right)\left(\int_{R=0}^{R=a} R^3\ dR\right) = 2\pi\left(\frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{\phi=\pi/2}\right)\frac{a^4}{4} \quad \to$$

$$I = 2\pi\left(\frac{1}{2}\right)\frac{a^4}{4} \quad \to \quad \mathcal{I} = \frac{\pi}{4}a^4$$