Projects of Department of Mathematics, Vol. 4 (2025), No.1, pp.xx-yy..
URL: http://www.ieu.edu.tr

NUMERICAL METHODS FOR SMOKING DYNAMICS

TUNA KUTOĞLU

ABSTRACT. This paper explores the use of Pell-Lucas polynomials to model smoking behavior dynamics across four population groups: potential smokers, light smokers, habitual smokers, and quitters. Instead of traditional methods like Runge-Kutta, a collocation-based approach using Pell-Lucas polynomials offers improved numerical stability and efficiency. Simulations demonstrate great accuracy and lower computational cost, highlighting the method's potential for broader applications in epidemiology and behavioral modeling.

1. INTRODUCTION

Smoking poses a profound challenge to public health, contributing to a substantial burden of preventable diseases, including cardiovascular conditions, lung cancer, and chronic respiratory disorders. Despite rigorous global anti-smoking campaigns, smoking prevalence remains persistently high, underscoring the necessity for innovative intervention strategies to address this ongoing crisis effectively.

A critical component of understanding smoking behavior is the phenomenon of relapse among individuals who have quit smoking. Studies in the field of behavioral science suggest that relapse is influenced by a range of factors, including psychological triggers, environmental stressors, and continued exposure to smoking peers. These findings emphasize the need for modeling approaches that account for the dynamic and recurring nature of smoking behavior.

Mathematical modeling offers a robust tool for capturing the multifaceted dynamics of smoking behavior. Recent models in the literature incorporate population segmentation and behavioral transitions to better reflect real-world smoking patterns. Building upon these approaches, this research proposes a refined framework that includes relapse, cessation, and progression dynamics within a nonlinear system of differential equations.

The model delineates the population into four key subgroups: potential smokers $P(t)$, light smokers $L(t)$, heavy smokers $S(t)$, and quitters $Q(t)$. The interactions and transitions between these subgroups are governed by a set of nonlinear differential equations, accounting for recruitment, cessation, relapse, and mortality rates. The simulation of this model employs two numerical techniques—the second-order Runge–Kutta method and a polynomial collocation method using Pell–Lucas polynomials—to explore the system's dynamics and evaluate the accuracy and efficiency of these approaches.

This paper is structured as follows: Section 3 introduces the mathematical framework and the parameters defining the model. Section 4 elaborates on the numerical methods implemented to solve the equations. Section 5 discusses the simulation results, providing insights into the implications of the model, while Section 6 concludes with a synthesis of findings and directions for future research.

Date: May 17, 2025.
2000 *Mathematics Subject Classification.* Primary 34A34, 65L05. Secondary 92D25.
Key words and phrases. Runge–Kutta method, Pell–Lucas polynomials, Smoking behavior modeling, Numerical approximation, Behavioral dynamics.
The project's advisor is Prof. Dr. Sevin Gümgüm.

2. Model Parameters

The parameters used in the smoking model are listed below:

- $\beta = 0.03$: The transmission rate of non-smokers into light smokers.
- $\mu = 0.008$: The natural death rate.
- $d = 0.004$: Smoking-related death rate.
- $\varepsilon = 0.02$: The rate at which light smokers transition into persistent smoking.
- $\delta = 0.074$: The rate at which smokers temporarily quit smoking.
- $\alpha = 0.15$: The quitting rate affected by external factors.
- $b = 0.4$: The fraction of quitters that return to the light smoker population.
- $A = 0.25$: The recruitment rate of potential smokers into the population.

3. Mathematical Model

The smoking dynamics model is described by the following system of differential equations:

$$\begin{aligned}
\frac{dP(t)}{dt} &= A - \frac{2\beta P(t)L(t)}{P(t)+L(t)} - (d+\mu)P(t),\\
\frac{dL(t)}{dt} &= \frac{2\beta P(t)L(t)}{P(t)+L(t)} - (\varepsilon+d+\mu)L(t) + b\alpha Q(t),\\
\frac{dS(t)}{dt} &= \varepsilon L(t) - \delta S(t) - (d+\mu)S(t) - (1-b)\alpha Q(t),\\
\frac{dQ(t)}{dt} &= \delta S(t) - (\mu+d)Q(t) - \alpha Q(t).
\end{aligned} \tag{3.1}$$

where P, L, S, Q represent potential smokers, light smokers, persistent smokers, and quit smokers, respectively.

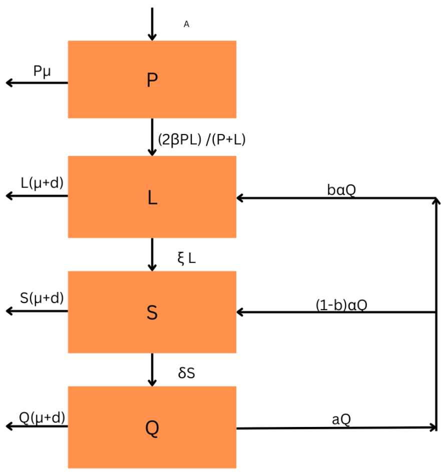

FIGURE 1. Flow chart diagram of the model.

From	To	Transition Rate
Potential Smokers (P)	Light Smokers (L)	$\frac{2\beta PL}{P+L}$
Light Smokers (L)	Smokers (S)	ξL
Smokers (S)	Quit Smokers (Q)	δS
Quit Smokers (Q)	Light Smokers (L)	baQ
Quit Smokers (Q)	Smokers (S)	$(1-b)aQ$
Potential Smokers (P)	Death	$(\mu)P$
Light Smokers (L)	Death	$(\mu+d)L$
Smokers (S)	Death	$(\mu+d)S$
Quit Smokers (Q)	Death	$(\mu+d)Q$

TABLE 1. Transitions between states in the smoking model.

4. NUMERICAL METHODS

Two numerical techniques are used to analyze the dynamics of the smoking behavior model: the second-order Runge–Kutta method and a polynomial collocation approach based on Pell–Lucas polynomials. Each method offers different advantages in terms of stability, accuracy, and computational efficiency.

4.1. **Second-Order Runge–Kutta Method.** The second-order Runge–Kutta (RK2) method represents a class of intermediate methods between the Euler method and higher-order Runge–Kutta schemes. It introduces an intermediate slope, k_2, to achieve greater precision:

$$k_1 = f(t_n, x_n), \quad (4.1) \tag{4.1}$$

$$k_2 = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_1\right), \quad (4.2) \tag{4.2}$$

$$x_{n+1} = x_n + hk_2, \quad (4.3) \tag{4.3}$$

The intermediate slope k_2 incorporates information about the system's behavior halfway through the interval h. This additional step enhances the method's ability to capture nonlinear dynamics and achieve higher accuracy, making it particularly suitable for behavioral compartment models such as the one used in this study.

4.2. **Pell–Lucas Polynomial Approximation Method.** The Pell–Lucas polynomial approximation method offers an efficient and numerically stable alternative for solving systems of nonlinear differential equations. In this approach, each function $X(t) \in \{P(t), L(t), S(t), Q(t)\}$ is approximated as a finite linear combination of Pell–Lucas polynomials:

$$X(t) \approx \sum_{n=0}^{N} c_n \cdot PL_n(\tau), \quad (4.4) \tag{4.4}$$

where c_n are the unknown coefficients, and $PL_n(\tau)$ denotes the n-th polynomial evaluated at the shifted variable τ.

Since Pell–Lucas polynomials are typically defined over the interval $(-1,1)$, a variable transformation is applied to adapt them to the model domain $t \in (0,1)$. The shift is defined as:

$$\tau = 2t - 1, \quad (4.5) \tag{4.5}$$

which allows the polynomials to be used effectively within the model's time scale.

The shifted Pell–Lucas polynomials follow the recurrence relation:

$$PL_0(\tau) = 2, \quad PL_1(\tau) = 2\tau, \quad PL_n(\tau) = 2\tau \cdot PL_{n-1}(\tau) + PL_{n-2}(\tau), \quad (4.6) \tag{4.6}$$

By substituting the polynomial expansions into the system of differential equations, the problem is reduced to solving a system of algebraic equations. This approach enhances numerical stability and enables accurate modeling of time-dependent behavior.

The method is particularly suitable for long-term behavioral simulations where smoothness and computational efficiency are essential.

5. Results and Discussion

Numerical simulations are conducted to evaluate the performance of each method. Separate figures illustrate the dynamics of the four subpopulations (P,L,S,Q).

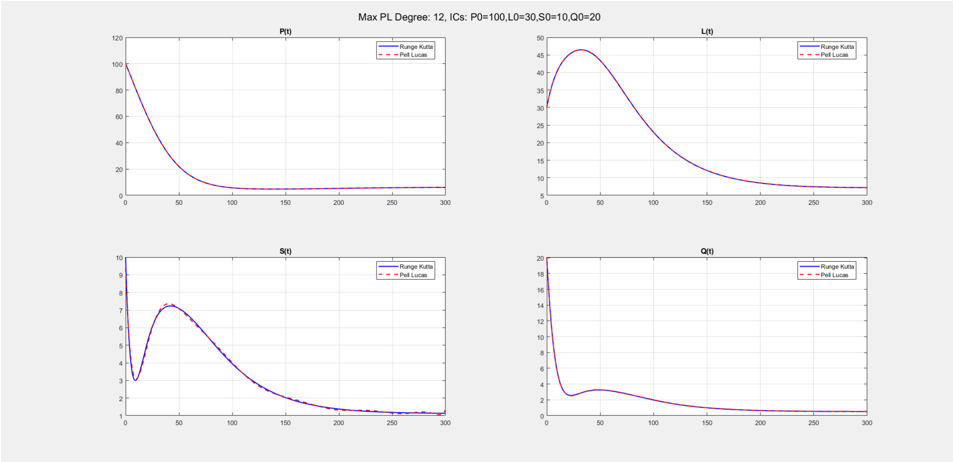

Figure 2. *Dynamics of Potential Smokers $P(t)$, Light Smokers $L(t)$, Smokers $S(t)$, and Quit Smokers $Q(t)$ using Runge–Kutta and Pell–Lucas polynomial methods. Initial conditions: $P_0 = 100$, $L_0 = 30$, $S_0 = 10$, $Q_0 = 20$.*

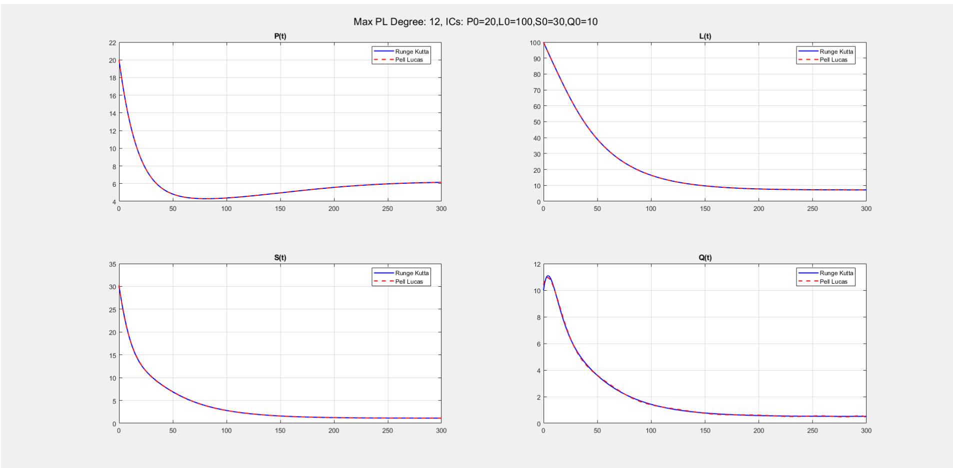

Figure 3. *Dynamics of Potential Smokers $P(t)$, Light Smokers $L(t)$, Smokers $S(t)$, and Quit Smokers $Q(t)$ using Runge–Kutta and Pell–Lucas polynomial methods. Initial conditions: $P_0 = 20$, $L_0 = 100$, $S_0 = 30$, $Q_0 = 10$.*

5.1. **Model Behavior with Different Initial Conditions.** These figures illustrate the model's predictions for Potential Smokers $P(t)$, Light Smokers $L(t)$, Habitual Smokers $S(t)$, and Quit Smokers $Q(t)$ under varying initial population sizes. The simulation results are obtained using the Pell–Lucas polynomial approximation method.

As shown in the graphs, the model consistently converges to stable states over time, regardless of the starting values assigned to each subgroup. This behavior indicates the system's robustness and dynamic equilibrium characteristics.

For instance, Fig. 1 shows the decline of the potential smoker population $P(t)$, which stabilizes near zero across all scenarios. Fig. 2 demonstrates the transient nature of light smokers $L(t)$, where an initial rise is followed by a steady decay. In Fig. 3, the smoker population $S(t)$ rises in the short term before gradually decreasing, while Fig. 4 illustrates the growth and eventual stabilization of quitters $Q(t)$ as more individuals exit smoking behavior.

This analysis highlights the model's capacity to capture realistic behavioral transitions and its numerical stability when simulated using the Pell–Lucas method. Even with highly different initial states, the overall trends reflect expected population dynamics, reinforcing the model's utility in predictive epidemiological applications.

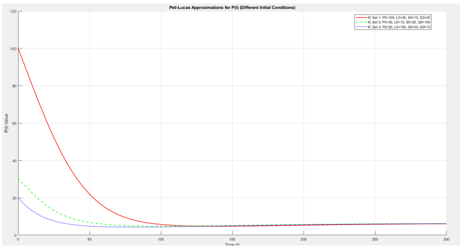

Figure 4a. $P(t)$

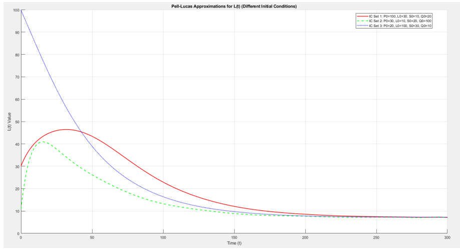

Figure 4b. $L(t)$

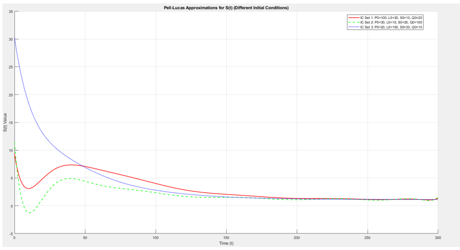

Figure 4c. $S(t)$

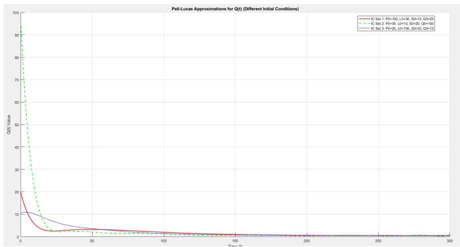

Figure 4d. $Q(t)$

Figure 4. *Model dynamics under varying initial conditions using the Pell–Lucas polynomial approximation method.*

5.2. **Discussion.** The simulation results highlight the strengths of the second-order Runge–Kutta and Pell–Lucas polynomial approximation methods in accurately modeling the dynamics of smoking behavior. The Runge–Kutta method demonstrates strong stability and precision, particularly in capturing nonlinear interactions and transitional behaviors over time. Its use of intermediate slope evaluations allows it to closely follow the system's true trajectory, making it highly suitable for modeling behavioral compartments.

On the other hand, the Pell–Lucas polynomial approximation method offers notable advantages in terms of smoothness and long-term accuracy. By transforming the domain and representing each function as a combination of orthogonal polynomials, it effectively reduces numerical instability and provides reliable approximations even under varying initial conditions. This method is especially advantageous when simulating systems where analytical smoothness and structural integrity are essential.

Overall, both methods prove effective for different purposes: Runge–Kutta excels in step-based numerical integration for short- to medium-term predictions, while the Pell–Lucas method is particularly well-suited for generating smooth, stable solutions over extended time horizons.

6. Conclusion

The results of the smoking model provide valuable insights into the behaviors of smokers and how these behaviors evolve over time. For individuals who smoke consistently, the model indicates that their behavior generally remains constant, making it difficult for them to quit smoking and leading to a longer duration of smoking. This suggests that individuals with long-term smoking habits have strong dependencies, requiring stronger and long-term interventions to encourage them to quit smoking. Long-term smoking has serious health implications, and the process of quitting is more complex for these individuals.

Individuals who smoke occasionally, on the other hand, show more variable behavior in the model, transitioning between smoking and non-smoking. This group is more sensitive to external factors such as social environment, stress, or environmental influences, which can cause fluctuations in their decision to continue smoking or quit. Targeted preventive measures for this group are critical in reducing the initiation rate of smoking and preventing continued smoking.

For potential smokers who have not yet started smoking, the model shows that the risk of smoking initiation increases due to factors like environmental influences or peer pressure. This highlights the importance of early interventions and preventive health policies for individuals who have not smoked. Such measures play a significant role in preventing the initiation of smoking.

For individuals who decide to quit smoking, the model shows that this decision can be sustained over time and their behavior can stabilize. This emphasizes the importance of time and ongoing programs in the smoking cessation process. Long-term cessation requires constant and careful monitoring, making it essential for smoking cessation programs to be effective not only in the short term but also in the long term.

The methods used in this study provide two distinct perspectives on the system dynamics. The second-order Runge–Kutta method offers improved accuracy and numerical stability by incorporating intermediate slope estimates, allowing it to track nonlinear transitions more effectively. In contrast, the Pell–Lucas polynomial approximation method enables a smooth and efficient representation of behavioral trends over time by expressing state variables as combinations of orthogonal basis functions. This polynomial approach demonstrates strong potential in long-term behavior modeling due to its ability to approximate global dynamics with reduced numerical noise.

In conclusion, the findings of this model provide valuable insights into smoking behaviors and contribute to the development of more effective health policies for different groups. As smoking behavior varies from person to person, tailored intervention strategies are required for each group. Specifically, strategies to support individuals who are new to smoking or those who smoke occasionally in quitting smoking are of critical importance in public health policies.

Future work could include incorporating delayed terms into the model to reflect time lags in behavioral responses. Such an extension would improve the model's accuracy and better capture the real-world dynamics of smoking behaviors, where changes in habits often occur with some delay.

REFERENCES

[1] Rahman GU, Agarwal RP, Din Q. Mathematical analysis of giving up smoking model via harmonic mean type incidence rate. Appl Math Comput 2019;354: 128–48.

[2] Atangana A, Shafiq A. Differential and integral operators with constant fractional order and variable fractional dimension. Chaos, Solitons Fractals 2019;127: 226–43.

[3] Huo H-F, Zhu C-C. Influence of relapse in a giving up smoking model. AbstrAppl Anal 2013:525461, 12 pages. http://dx.doi.org/10.1155/2013/525461.

[4] Alzaid, S. S., & Alkahtani, B. S. T. (2021). Asymptotic analysis of a giving up smoking model with relapse and harmonic mean type incidence rate. Results in Physics, 28, 104437.

[5] Zhang Z, Zou J, Upadhyay RK. Stability and Hopf bifurcation of a delayed giving up smoking model with harmonic mean type incidence rate and relapse. Results Phys 2020;19:103619. https://doi.org/10.1016/j.rinp.2020.103619.

[6] Guo Y, Liu Z, Tan Y, Liu Y. Modeling and analysis of a stochastic giving-up-smoking model with quit smoking duration. Math Biosci Eng 2023;20(12):20576–20598. https://doi.org/10.3934/mbe.2023910.

Tuna Kutoğlu is a senior student in the Department of Mathematics, Izmir University of Economics. His research interests include Artificial Intelligence, speech processing, and machine learning.

IZMIR UNIVERSITY OF ECONOMICS, DEPARTMENT OF MATHEMATICS, IZMIR, TURKEY

Email address: tuna.kutoglu@std.ieu.edu.tr