Lecture Note 5
Electromagnetic waves

We have derived Maxwell's Eq. for potentials in Lorentz gauge (Eq 36, Lecture Note 4)

Here are the Eqs.

$$\begin{cases} \left(\mu_0 \varepsilon_0 \partial_t^2 - \Delta\right) \varphi = \dfrac{\rho}{\varepsilon_0} \\ \left(\mu_0 \varepsilon_0 \partial_t^2 - \Delta\right) \vec{A} = \mu_0 \vec{J} \\ \vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \partial_t \varphi = 0 \end{cases}$$

For free space, $\rho = \vec{J} = 0$ the Eqs are

$$\begin{cases} \left(\mu_0 \varepsilon_0 \partial_t^2 - \Delta\right) \varphi = 0 \\ \left(\mu_0 \varepsilon_0 \partial_t^2 - \Delta\right) \vec{A} = 0 \\ \vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \partial_t \varphi = 0 \end{cases}$$

In this case we can combine the Coulomb and the Lorentz gauges and impose conditions

$$\vec{\nabla}\cdot\vec{A}=0, \quad \varphi=0$$

Hence we get the wave Eq.

$$\begin{cases} (\mu_0 \varepsilon_0 \partial_t^2 - \Delta)\vec{A} = 0 \\ \vec{\nabla}\cdot\vec{A} = 0, \quad \varphi = 0 \end{cases} \qquad (1)$$

applying

$$\begin{aligned} \vec{B} &= \vec{\nabla}\times\vec{A} \\ \vec{E} &= -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t} \end{aligned} \qquad (2)$$

to Eq.(1) we find that E and B obey the same wave Eqs.

$$\begin{cases} (\mu_0 \varepsilon_0 \partial_t^2 - \Delta)\vec{E} = 0 \\ (\mu_0 \varepsilon_0 \partial_t^2 - \Delta)\vec{B} = 0 \end{cases} \qquad (3)$$

Plane wave solution

Consider a wave that depends only on one coordinate X. Then Eqs.(1), (2) become

$$\left(\mu_0 \varepsilon_0 \partial_t^2 - \partial_x^2\right) f = 0 \qquad (4)$$

where $f = \vec{A}, \vec{E}, \vec{B}$.

It is easy to check that the general solution of (3) is

$$f(x,t) = f_-(x-ct) + f_+(x+ct)$$

where

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99\ 10^8\ m/s \qquad (5)$$

is speed of light in vacuum

and $f_-(\xi)$, $f_+(\eta)$ are arbitrary functions of

$$\xi = x - ct \quad , \quad \eta = x + ct$$

Indeed

$$\partial_x^2 f_- = \frac{\partial^2 f_-}{\partial \xi^2}$$

$$\partial_t^2 f_- = c^2 \frac{\partial^2 f_-}{\partial \xi^2}$$

Substitution to (3) gives

$$(\mu_0 \varepsilon_0 c^2 - 1) \frac{\partial^2 f_-}{\partial \xi^2} = 0$$

This Eq. is satisfied if (5) is valid. The proof for $f_+(\eta)$ is absolutely similar.

The solution f_- corresponds to the wave moving from left to right and the solution f_+ corresponds to the wave moving from right to left.

Consider a wave moving from left to right,
According to (1) the vector potential is $\vec{A} = \vec{A}(\xi)$

Moreover because of the second of Eqs (1)

$$\vec{\nabla} \cdot \vec{A} = \partial_x A_x = \frac{\partial A_x}{\partial \xi} = 0$$

we can conclude that $A_x = 0$ and hence the vector potential has only transverse components $A_y(\xi)$ and $A_z(\xi)$

Electric and magnetic fields are given by Eqs (2)

$$\begin{cases} \vec{E} = -\dfrac{\partial \vec{A}}{\partial t} = \left(0, c\dfrac{\partial A_y}{\partial \xi}, c\dfrac{\partial A_z}{\partial \xi}\right) \\ \vec{B} = \vec{\nabla}\times\vec{A} = \left(0, -\dfrac{\partial A_z}{\partial \xi}, \dfrac{\partial A_y}{\partial \xi}\right) \end{cases} \quad (6)$$

note that $\vec{E}\cdot\vec{B} = c\Big[-(\partial_\xi A_y)(\partial_\xi A_z)+$
$+(\partial_\xi A_z)(\partial_\xi A_y)\Big] = 0$

so, magnetic and electric fields in a wave are orthogonal to the direction of propagation and also orthogonal to each other.

We can rewrite this condition in an invariant form. If $\vec{n}$ is the unit vector in the direction of propagation then

$$\vec{B} = \frac{1}{c}\left[\vec{n} \times \vec{E}\right] \qquad (7)$$

In the above example $\vec{n} = (1, 0, 0)$ substitution of $\vec{E} = (0, c\partial_\xi A_y, c\partial_\xi A_z)$ from (6) to (7) gives $\vec{B} = (0, -\partial_\xi A_z, \partial_\xi A_y)$ which is consistent with (6)

Energy flux in electromagnetic wave

Poynting's vector, Eq. (7), page 6, Lecture Note 4

$$\vec{S} = \frac{1}{\mu_0}\left[\vec{E}\times\vec{B}\right] =$$

$$= \frac{1}{\mu_0 c}\left[\vec{E}\times\left[\vec{n}\times\vec{E}\right]\right] = \frac{1}{\mu_0 c} E^2 \vec{n} =$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E^2 \vec{n} = \frac{1}{\sqrt{\varepsilon_0 \mu_0^3}}\, B^2 \vec{n} \quad (8)$$

Relation to the energy density in the wave.

$$W = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}$$

Having in mind that $|B| = \frac{|E|}{c} = \sqrt{\mu_0\varepsilon_0}|E|$ we can rewrite W as

$$W = \frac{\varepsilon_0 E^2}{2} + \frac{\mu_0 \varepsilon_0 E^2}{2\mu_0} = \varepsilon_0 E^2$$

$$W = \frac{\varepsilon_0 B^2}{2\mu_0\varepsilon_0} + \frac{B^2}{2\mu_0} = \frac{B^2}{\mu_0}$$

Since light is moving with speed of light c

$$S = cW = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\varepsilon_0 E^2 = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E^2$$

OK, this is consistent with (8)

$$S = cW = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\frac{B^2}{\mu_0} = \frac{1}{\sqrt{\varepsilon_0\mu_0^3}}\, B^2$$

this is also consistent with (8)

Example: laser pointer

Consider a laser pointer with power $\mathcal{P} = 1\,mW$ and the beam cross section $1\,mm^2$.
Calculate amplitudes of electric and magnetic fields in the laser pointer beam.

$$S = \frac{10^{-3}}{(10^{-3})^2} = 10^3 \frac{W}{m^2} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E^2 = \sqrt{\frac{8.85\,10^{-12}}{1.256\,10^{-6}}}\, E^2$$

$$= 2.65\,10^{-3} E^2$$

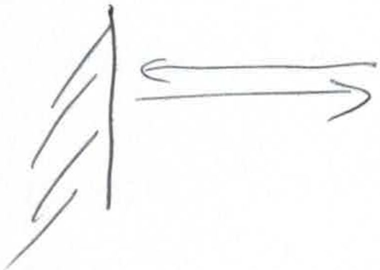
Hence $E = 0.6\,10^3\, V/m = 0.6\, kV/m$

$$S = 10^3 \frac{W}{m^2} = \frac{B^2}{\sqrt{\varepsilon_0 \mu_0^3}} = \frac{B^2}{\sqrt{8.85\,10^{-12} \cdot 1.256^3\,10^{-18}}}$$

$$= 0.24\,10^{15} B^2$$

Hence $B = \sqrt{\frac{10^{-12}}{0.24}} = 2 \cdot 10^{-6}\, T$

The laser pointer beam is reflected from a mirror.



What is the force acting on the mirror due to light pressure?

$$F = \frac{dp}{dt} = 2\frac{J}{c} = 2\frac{10^{-3}}{2.99\,10^{8}} \approx 0.7\,10^{-11} \text{ Newtons}$$

Fixed frequency solution (monochromatic)

$$E \sim e^{-i\omega t} \quad , \quad B \sim e^{-i\omega t}$$

Hence $\vec{E} = \vec{E}_0 e^{ikx - i\omega t}$ (9)

$$\vec{B} = \vec{B}_0 e^{ikx - i\omega t}$$

$\boxed{\omega = ck}$ since the argument must be $x - ct$

k is called the wave number

Electromagnetic waves frequency range

f (Hz)	λ
0	∞
$\sim 5\,10^{14}$	$6\,10^{-7}$ m
10^{30}	10^{-22} m

for comparison.

size of atom $\sim 10^{-10}$ m

distance ~~of~~ between atoms in solid/liquid $\sim 10^{-10}$ m

size of proton $\sim 10^{-15}$ m

Electromagnetic waves in dielectric media

In dielectric media we have the following substitution

$\mu_0 \to \mu_0 \mu, \quad \varepsilon_0 \to \varepsilon_0 \varepsilon$

See Eq (12) p 11, Lecture Note 4

Hence the speed of light is

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \longrightarrow v = \frac{1}{\sqrt{\mu_0 \mu \varepsilon_0 \varepsilon}} = \frac{c}{n}$$

$n = \sqrt{\varepsilon \mu}$ is the index of refraction

Usually $\mu \approx 1$, hence $n \approx \sqrt{\varepsilon}$

Diamond as an example

$\varepsilon \approx 5.7 \quad , \quad n \approx 2.4$

Description in terms of n makes sense only if $\lambda \gg$ distance between atoms

Because of dissipation in the medium the index of refraction can have an imaginary part

For low dissipation the imaginary part is small.

In this case $n > 1$ and hence $v < c$.

Fixed frequency solution

$$e^{ikx - i\omega t}$$

vacuum → medium
, $\omega = ck \rightarrow \omega = vk$

$$v = \frac{c}{n} \Rightarrow k = \frac{n\omega}{c}$$

Imaginary part: $n = n' + in''$

$$e^{ikx - i\omega t} = e^{-i\omega t + i(n' + in'')\frac{\omega}{c}x} =$$

$$= e^{-i\omega t + in'\frac{\omega}{c}x} \underbrace{e^{-n''\frac{\omega}{c}x}}_{} \text{ - attenuation}$$

Polarization of monochromatic wave

$$\vec{E} = Re\left(\vec{E_0}\, e^{i\vec{k}\vec{r} - i\omega t}\right) \qquad (1)$$

$\vec{E_0} = (E_{0x}, E_{0y}, E_{0z})$ is a complex vector

$\vec{E_0}$ squared must be a complex number

$$\vec{E_0}^2 = E_{0x}^2 + E_{0y}^2 + E_{0z}^2 = |E_0^2|\, e^{-2i\alpha}$$

Let us represent $\vec{E_0}$ as

$\vec{E_0} = \vec{b}\, e^{-i\alpha}$ where $\vec{b}$ is another complex vector, but $\vec{b}^2$ is real.

$$\vec{E} = Re\left(\vec{b}\, e^{i\vec{k}\vec{r} - i\omega t - i\alpha}\right) \qquad (2)$$

$\vec{b} = \vec{b_1} + i\vec{b_2}$, $\vec{b_1}$ and $\vec{b_2}$ are real vectors.

$\vec{b}^2 = \vec{b_1}^2 - \vec{b_2}^2 + 2i\vec{b_1}\cdot\vec{b_2}$

By definition $\vec{b}^2$ must be real.

Hence $\vec{b_1}\cdot\vec{b_2} = 0$: $\vec{b_1}$ is orthogonal to $\vec{b_2}$

Let x be the direction of the wave propagation

$$\vec{K} = (K, 0, 0)$$

Let also direct y along $\vec{b_1}$

Then z is directed along $\vec{b_2}$ or opposite to $\vec{b_2}$

Hence

$$\begin{cases} E_y = b_1 \cos(\omega t - kx + \alpha) \\ E_z = \pm b_2 \sin(\omega t - kx + \alpha) \end{cases} \quad (3)$$

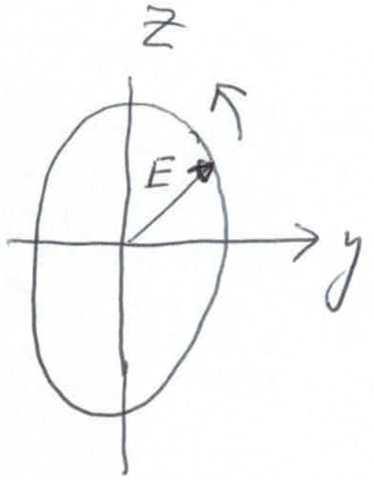
the sign (+) corresponds to $\hat{z} \parallel \vec{b_2}$
the sign (−) corresponds to $\hat{z} \parallel -\vec{b_2}$

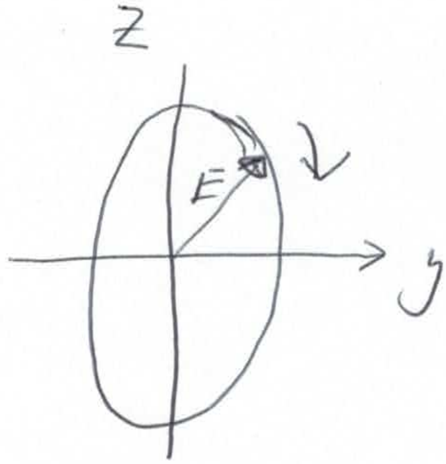
Thus

$$\boxed{\frac{E_y^2}{b_1^2} + \frac{E_z^2}{b_2^2} = 1} \quad (4)$$

This is equation of an ellipse.

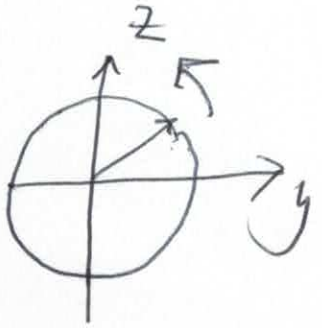
Hence, the most general is elliptic polarization.

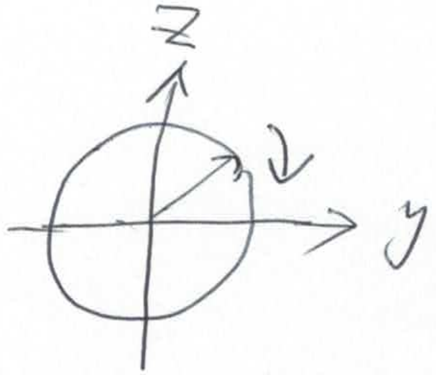




If $|b_1| = |b_2|$: ellipse → circle.

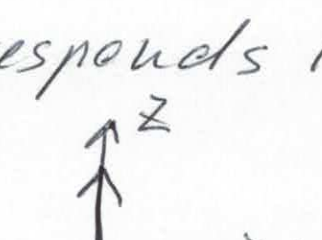
This is the circular polarization



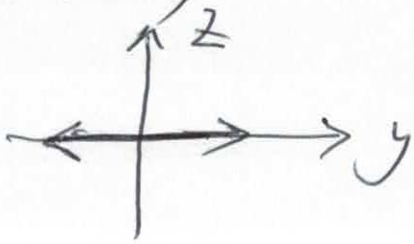


Linear polarization corresponds to

$b_1 = 0$ (z-polarization)



or to $b_2 = 0$ (y-polarization)



Partially polarized wave

$$\vec{E} = \vec{E}_0(t)\, e^{-i\omega t}$$

where $\vec{E}_0(t)$ very slowly depends on time. Slow compared to ω.

Properties of a partially polarized wave are determined by time average

$$\overline{E_\alpha E_\beta^*} = \overline{E_{0\alpha}(t) e^{-i\omega t} E_{0\beta}^*(t) e^{i\omega t}} =$$

$$= \overline{E_{0\alpha}(t)\, E_{0\beta}^*(t)}$$

Here $\alpha, \beta = y, z$

Intensity of the wave is proportional to

$$I = \sum_{\alpha} \overline{E_{0\alpha}(t) E^*_{0\alpha}(t)} = |E_{0y}|^2 + |E_{0z}|^2$$

The light polarization is characterised by 2×2 matrix

$$\rho_{\alpha\beta} = \frac{\overline{E_{0\alpha}(t)\, E^*_{0\beta}(t)}}{I}$$

In modern terminology this is called polarization density matrix.

Note that by definition trace of the matrix is equal to 1

$$\mathrm{tr}\,\rho = \rho_{\alpha\alpha} = \rho_{yy} + \rho_{zz} = 1$$

↑ Einstein notation

By definition the density matrix

$$\rho_{\alpha\beta} = \frac{1}{I} \overline{E_{0\alpha}(t) E^{*}_{0\beta}(t)}$$

is hermitian

$$\rho_{\alpha\beta} = \rho^{*}_{\beta\alpha}$$

For a fully polarized wave (elliptic, circular or linear polarization) the vector $\vec{E_0}$ is time independent. Hence

$$\det \rho_{\alpha\beta} = |\rho_{\alpha\beta}| = 0$$

Indeed

$$\rho_{\alpha\beta} \sim E_{0\alpha} E_{0\beta}^{*} = \begin{pmatrix} E_{0y} E_{0y}^{*} & E_{0y} E_{0z}^{*} \\ E_{0z} E_{0y}^{*} & E_{0z} E_{0z}^{*} \end{pmatrix}$$

$$\det \rho_{\alpha\beta} = E_{0y} E_{0y}^{*} E_{0z} E_{0z}^{*} - E_{0z} E_{0y} E_{0y} E_{0z}^{*} = 0$$

For a fully unpolarized wave all directions are equivalent and hence

$$\rho_{\alpha\beta} = \frac{1}{2} \delta_{\alpha\beta}$$

Here $\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq 0 \end{cases}$

is Kronecker symbol.

The coefficient $\frac{1}{2}$ in

$\rho_{\alpha\beta} = \frac{1}{2}\delta_{\alpha\beta}$ comes from the

requirement that trace is equal to 1

$$tr\,\rho = \rho_{\alpha\alpha} = \frac{1}{2} + \frac{1}{2} = 1$$

For the fully unpolarized wave the determinant of the density matrix is

$$\det \rho = \begin{vmatrix} 1/2 & 0 \\ 0 & 1/2 \end{vmatrix} = \frac{1}{4}$$

Thus for fully polarized wave $|\rho_{\alpha\beta}| = 0$
and for fully unpolarized wave $|\rho_{\alpha\beta}| = 1/4$

For a partially polarized wave

$$|\rho_{\alpha\beta}| = \frac{1}{4}(1-P^2) \quad , 0 \le P \le 1$$

P is called the degree of polarization.

In the most general case a 2x2 hermitian matrix with trace equal to 1 can be represented as

$$\rho_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 1+\xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 1-\xi_3 \end{pmatrix}$$

where ξ_1, ξ_2, ξ_3 are real numbers. Determinant of the matrix is

$$|\rho_{\alpha\beta}| = \frac{1}{4}(1-\xi_1^2-\xi_2^2-\xi_3^2)$$

Hence the corresponding degree of polarization is

$$P = \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}$$

Since $0 \leq p \leq 1$ we conclude that

$$-1 \leq \xi_1 \leq +1$$

$$-1 \leq \xi_2 \leq +1$$

$$-1 \leq \xi_3 \leq +1$$

They also must satisfy the inequality $\xi_1^2 + \xi_2^2 + \xi_3^2 \leq 1$

The parameters ξ_1, ξ_2, ξ_3 are called the Stokes parameters. They describe the most general partial or full polarization of an electromagnetic wave

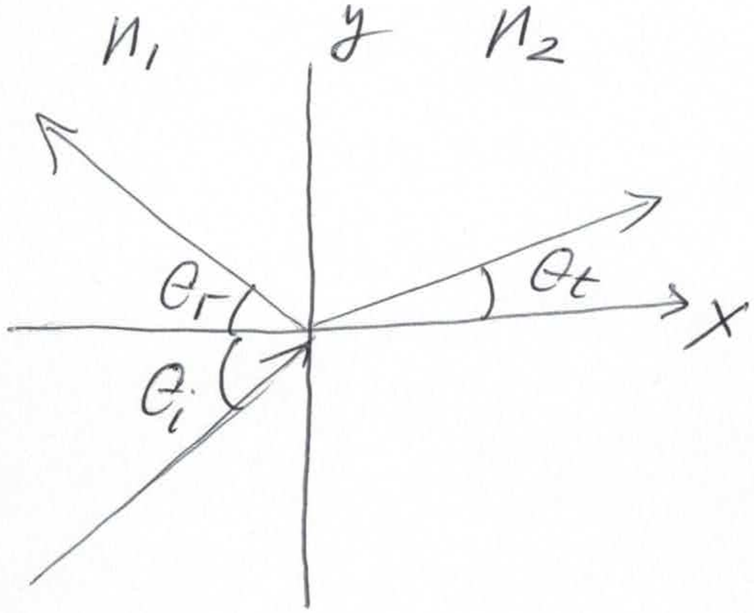
Examples:

$$\begin{cases} \text{linear polarization along } y: \\ \xi_3 = 1, \ \xi_1 = \xi_2 = 0 \end{cases}$$

$$\begin{cases} \text{linear polarization along } z: \\ \xi_3 = -1, \ \xi_1 = \xi_2 = 0 \end{cases}$$

$$\begin{cases} \text{unpolarized wave:} \\ \xi_1 = \xi_2 = \xi_3 = 0 \end{cases}$$

Fresnel Eqs. for reflection/transmission of electromagnetic wave from a boundary between two dielectric media



This is the incidence plane. The boundary is $x=0$.

n_1, n_2 - indexes of refraction

$x \le 0$ $\begin{cases} i: \text{incident wave} & \vec{E}_i\, e^{i\vec{k}_i\vec{r}-i\omega t} \\ r: \text{reflected wave} & \vec{E}_r\, e^{i\vec{k}_r\vec{r}-i\omega t} \end{cases}$

$x \ge 0$, t: transmitted wave $\vec{E}_t\, e^{i\vec{k}_t\vec{r}-i\omega t}$

$\vec{k}_i, \vec{k}_r, \vec{k}_t$ have only x and y components because xy is the incidence plane. All z-components are equal to zero.

We have to satisfy a boundary condition at the interface:
$x = 0$, y is arbitrary.

The boundary condition looks like

$$(\ldots)e^{ik_{iy}y - i\omega t} + (\ldots)e^{ik_{ry}y - i\omega t} = (\ldots)e^{ik_{ty}y - i\omega t}$$

We will specify $(\ldots)$ later.

The condition can be satisfied only if

$$\boxed{k_{iy} = k_{ry} = k_{ty}}$$

$$\begin{cases} k_{iy} = k_i \sin\theta_i = \dfrac{n_1\omega}{c}\sin\theta_i \\ k_{ry} = k_r \sin\theta_r = \dfrac{n_1\omega}{c}\sin\theta_r \\ k_{ty} = k_t \sin\theta_t = \dfrac{n_2\omega}{c}\sin\theta_t \end{cases}$$

Hence

1) $\boxed{\theta_i = \theta_r}$: the reflection angle is equal to the incidence angle.

2) $\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t}$ Snell's law

Below I denote $\boxed{\begin{array}{l} \theta_i = \theta_r = \theta_1 \\ \theta_t = \theta_2 \end{array}}$

In this notation Snell's low reads

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

There are no free (external) charges in the reflection problem.

Hence Maxwell's Eq read (Eq 12, page 11, Lecture Note 4)

$$\begin{cases} \vec{\nabla}\vec{D}=0 \\ \vec{\nabla}\vec{B}=0 \\ \vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{H}=\dfrac{\partial\vec{D}}{\partial t} \\ \vec{D}=\varepsilon_0\mu\vec{E} \\ \vec{B}=\mu_0\mu\vec{H} \end{cases}$$

Hence, at an interface we have the following conditions

$$\begin{cases} D_{1\perp}=D_{2\perp} \\ E_{1\parallel}=E_{2\parallel} \\ B_{1\perp}=B_{2\perp} \\ H_{1\parallel}=H_{2\parallel} \end{cases} \Longrightarrow \begin{cases} \varepsilon_1 E_{1\perp}=\varepsilon_2 E_{2\perp} \\ E_{1\parallel}=E_{2\parallel} \\ B_{1\perp}=B_{2\perp} \\ \dfrac{1}{\mu_1}B_{1\parallel}=\dfrac{1}{\mu_2}B_{2\parallel} \end{cases} \quad (1)$$

There are two different polarizations

TM: Electric field lies in the incidence plane. Hence magnetic field is perpendicular to the plane.

This is why it is called

TM, "transverse magnetic"

TE: Magnetic field lies in the incidence plane. Electric field is perpendicular to the plane. This is TE, "transverse electric"

Let us consider nonmagnetic materials, $\mu_1 = \mu_2 = 1$. In this case $n_1 = \sqrt{\varepsilon_1}$, $n_2 = \sqrt{\varepsilon_2}$.

Hence Eqs (1) on page (30) read

$$\left\{\begin{array}{ll} \varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp} & (i) \\ E_{1\parallel} = E_{2\parallel} & (ii) \\ B_{1\perp} = B_{2\perp} & (iii) \\ B_{1\parallel} = B_{2\parallel} & (iv) \end{array}\right. \qquad (2)$$

The relation betwen B and E in a wave follows from Faraday's law

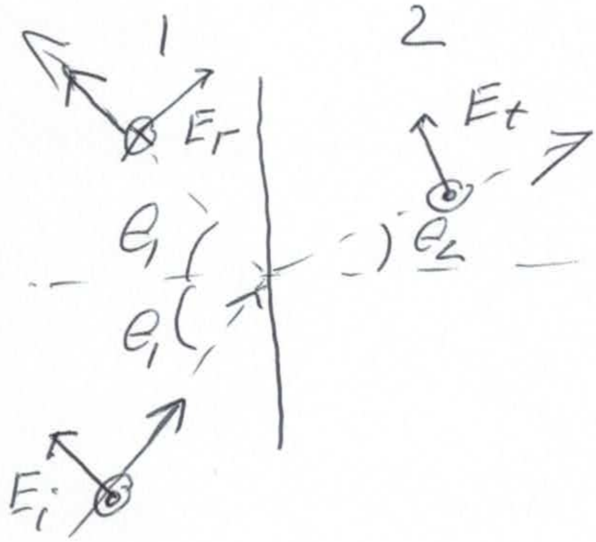
$$\vec{B} e^{i\vec{k}\vec{r} - i\omega t}, \quad \vec{E} e^{i\vec{k}\vec{r} - i\omega t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t} \Rightarrow i\vec{k} \times \vec{E} = i\omega \vec{B}$$

$$k = \frac{n\omega}{c} \longrightarrow \boxed{B = \frac{n}{c} E}$$

The relation between amplitudes.

Consider TM polarization



$$\boxed{B = \frac{n}{c} E}$$

Eq (iv): $B_{1\parallel} = B_{2\parallel} \Rightarrow (E_i - E_r)\, n_1 = E_t\, n_2$

Eq (ii): $E_{1\parallel} = E_{2\parallel} \Rightarrow E_i \cos\theta_1 + E_r \cos\theta_1 = E_t \cos\theta_2$

Eq (i): $\varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp} \Rightarrow -\varepsilon_1 E_i \sin\theta_1 + \varepsilon_1 E_r \sin\theta_1 =$

$= -\varepsilon_2 E_t \sin\theta_2$

So we got 3 Eqs

$\varepsilon_1 = n_1^2$

$\varepsilon_2 = n_2^2$

$$\begin{cases} (1) & (E_i - E_r)\, n_1 = E_t\, n_2 \\ (2) & (E_i + E_r)\cos\theta_1 = E_t \cos\theta_2 \\ (3) & (E_i - E_r)\, n_1^2 \sin\theta_1 = E_t\, n_2^2 \sin\theta_2 \end{cases}$$

devide (3)/(1) $\Rightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$

This is Snell's law that we already know. So, only Eq (1), (2) are new.

$$\begin{cases} (1)\ (E_i - E_r)\, n_1 = E_t\, n_2 \\ (2)\ (E_i + E_r)\cos\theta_1 = E_t \cos\theta_2 \end{cases}$$

(A) multiply (1) by $\cos\theta_2$, (2) by n_2 and take difference

$$E_r\,(n_2\cos\theta_1 + n_1\cos\theta_2) + E_i\cdot(n_2\cos\theta_1 - n_1\cos\theta_2) = 0$$

Hence

$$\boxed{\frac{E_r}{E_i} = \frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}}$$ Fresnel equation.

This gives the relation between the amplitude and the phase of the incident and reflected waves.

B) multiply (1) by $\cos\theta_1$ and (2) by n_1. Then take the sum

$$2E_i \cdot n_1 \cos\theta_1 = E_t(n_1\cos\theta_2 + n_2\cos\theta_1)$$

$$\boxed{\frac{E_t}{E_i} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}}$$ Fresnel equation

This gives the relation between the amplitude and the phase of the incident and reflected wave.

Tutorial problem:

Derive Fresnel equations ~~for~~

$$\frac{E_r}{E_i} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

$$\frac{E_t}{E_i} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

for TE-polarization.

Brewster's (polarizing) angle

Let us return to the TM polarization

The reflection coefficient

$$\frac{E_r}{E_i} = \frac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}$$

that we derived is zero if

$$\boxed{n_1 \cos\theta_2 = n_2 \cos\theta_1} \quad (1)$$

Ther is also Snell's law

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2} \quad (2)$$

Combining (1) and (2) we get

$$\boxed{\tan\theta_1 = \frac{n_2}{n_1}} \quad (3)$$

Mathematical check of validity of Eq (3).

From (3) we find

$$\begin{cases} \sin\theta_1 = \dfrac{n_2}{\sqrt{n_1^2+n_2^2}} \\ \cos\theta_1 = \dfrac{n_1}{\sqrt{n_1^2+n_2^2}} \end{cases} \qquad (4)$$

Substituting (4) to (1) gives

$$\cos\theta_2 = \frac{n_2}{\sqrt{n_1^2+n_2^2}} \qquad (5)$$

Substituting (4) to (2) gives

$$\sin\theta_2 = \frac{n_1}{\sqrt{n_1^2+n_2^2}} \qquad (6)$$

(5) and (6) are consistent, $\cos^2\theta_2 + \sin^2\theta_2 = 1$, and this confirms validity of (3)

The angle given by (3)

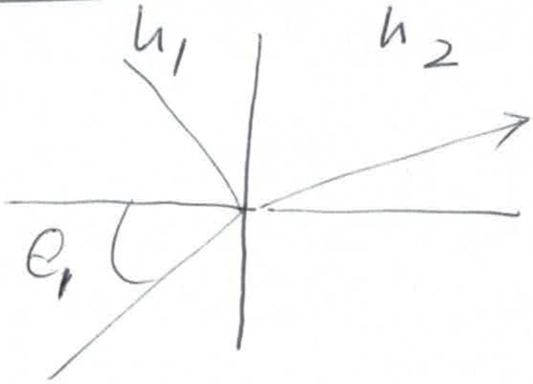
$$\tan \theta_B = \frac{n_2}{n_1}$$

is called Brewster's or polarizing angle.

The point is that if the incident wave is unpolarized it consists of 50/50 combination of TM and TE polarization.

However, if TM is not reflected, the reflected wave is 100% TE polarized.

Total internal reflection and evanescent wave

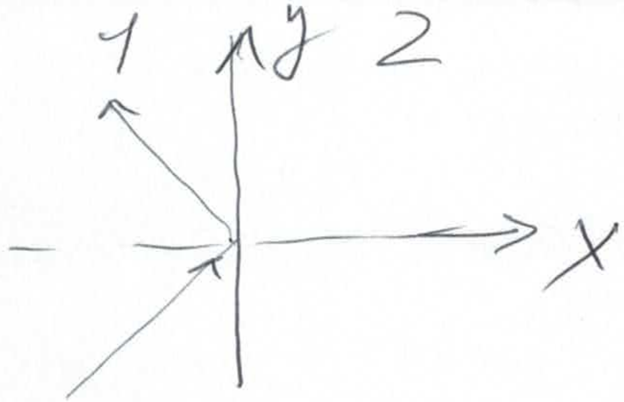


Consider $n_1 > n_2$

According to Snell's law $n_1 \sin\theta_1 =$
$= n_2 \sin\theta_2$

If the incidence angle $\theta > \theta_c$,
where $\theta_c = \arcsin \frac{n_2}{n_1}$ Snell's law cannot be satisfied.

So, there is no transmitted wave.
This is called total internal reflection,



Absence of a transmitted wave does not imply that there is no field at $x>0$.

There is a field, but it decays exponentially with x. You can see this from momentum conservation = the matching condition $k_{iy} = k_{ry} = k_{ty}$

$$k_t^2 = \frac{n_2^2\omega^2}{c^2} = k_{tx}^2 + k_{ty}^2 = k_{tx}^2 + k_i^2 \sin^2\theta_1$$

$$\Rightarrow k_{tx} = \sqrt{\frac{n_2^2\omega^2}{c^2} - \frac{n_1^2\omega^2}{c^2}\sin^2\theta_1} =$$

$$= \frac{n_1\omega}{c}\sqrt{\frac{n_2^2}{n_1^2} - \sin^2\theta_1}$$

When $\theta_1 > \theta_c$ the x-component of k_t wave vector becomes imaginary. $k_{tx} = i|k_{tx}|$

Hence the field

$$E \sim e^{ik_{tx}x} = e^{-|k_{tx}|x}$$ decays exponentially.

This is called "evanescent wave".