THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

PHYS2111 Quantum Mechanics

Final Exam

3rd May 2022

2 hr Online Exam + 15 min for download/reading + 30 min for capture/upload.

Total marks: 80 (pro-rata to 60% of course assessment)

Please note:

- You are allowed to use your textbook and notes during the exam.
- Prepare two files, one with your responses to Q1 and Q2 and one with your responses to Q3 and Q4 for upload to Moodle by the end of the allotted exam time.
- When uploading, please use the following naming convention: PHYSXXXX zID Surname (e.g., PHYS2111 z5556789 Smith).
- Please write your name and student number on the first page of your submission.
- Answer all questions concisely and legibly.
- Questions are not all worth the same marks; questions are worth the marks shown.
- You may use a university-approved calculator.

Formula Sheet

Planck's constant $h = 6.626 \times 10^{-34}$ Js

Fundamental charge unit $e = 1.60 \times 10^{-19}$ C

Mass of the electron = 9.11×10^{-31} kg

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Pauli spin matrices: $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[A,B] = AB - BA$

Table of spin operator actions

$\sigma_x\lvert u\rangle = \lvert d\rangle$	$\sigma_x\lvert d\rangle = \lvert u\rangle$
$\sigma_y\lvert u\rangle = i\lvert d\rangle$	$\sigma_y\lvert d\rangle = -i\lvert u\rangle$
$\sigma_z\lvert u\rangle = \lvert u\rangle$	$\sigma_z\lvert d\rangle = -\lvert d\rangle$

Helpful Standard Integrals

Exponential Integral: $\int_0^\infty x^n e^{-x/a} dx = n!\, a^{n+1}$

Gaussian Integrals: $\int_0^\infty x^{2n} e^{-x^2/a^2} dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$

$\int_0^\infty x^{2n+1} e^{-x^2/a^2} dx = \frac{n!}{2} a^{2n+2}$

Fourier transform: $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x) e^{-ikx}$

Ladder operators

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$$

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$$

$$a_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

Question 1: [22 marks total]

A quantum spin is in the state:

$$|\chi\rangle = A\begin{pmatrix}1+2i\\2\end{pmatrix}$$

(a) Briefly explain why we care about the actual value of A and don't just set $A = 1$, i.e., just ignore A. [1 mark]

(b) Determine the constant A. What is the name of the process of obtaining A? [3 marks]

(c) If you measured σ_z for this spin, what values could you get? Briefly explain how you know this given the form of the matrix operator $\hat{\sigma}_z$ [2 marks]

(d) Briefly explain what a projection operator is in quantum mechanics. Obtain the respective projection operators for the possible measurement outcomes you found in (c). [2 marks]

(e) Calculate the probability of obtaining each of the possible values you found in (c)? What is the most likely outcome if you were to do a measurement of σ_z? [3 marks]

Note it is helpful but not necessary to use the projection operators from (d), i.e., you will not lose marks if you obtain them correctly from another approach.

(f) Briefly explain what the expectation value means as a concept in quantum mechanics. Provide a basic equation for the expectation value and use your answers to (c) and (d) to calculate the expectation value for σ_z for the state $|\chi\rangle$. [2 marks]

(g) Show that the expectation value for σ_z can also be obtained as $\langle\sigma_z\rangle = \langle\chi|\sigma_z|\chi\rangle$. Note: I'm just asking you to show that this gives the same value for $\langle\sigma_z\rangle$ as (f), not for a rigorous general proof. [3 marks]

(h) Use your answers to (f) and (g), and the concept of projection operators from (e), to show that (f) and (g) should give exactly the same answer for the expectation value of σ_z for the state $|\chi\rangle$. [3 marks]

Note that an answer that is just a variant of the proof from my notes in Lecture 4 is not sufficient because that proof avoids projection operators entirely. That said, I expect the answer to this to be algebraic, just getting the same numerical outcome as (f) and (g) is insufficient.

(i) The generalised uncertainty principle for two observables A and B is given by:

$$\Delta A \Delta B \geq \left|\frac{i}{2}\langle[A,B]\rangle\right|$$

Briefly explain what the square and angled brackets in this equation mean [1 mark].

(j) Use the generalised uncertainty principle to obtain a value for $\Delta\sigma_x\Delta\sigma_y$ for the quantum spin prior to doing a measurement of σ_z. [2 marks]

Question 2: [18 marks total]

(a) Consider a generic observable $\hat{A}$, which we allow to have a complete set of orthonormal eigenvectors $|e_n\rangle$ such that:

$$\hat{A}|e_n\rangle = a_n|e_n\rangle$$

where a_n are the corresponding eigenvalues and $n = 1, 2, 3\ldots$

(i) A generic state $|\alpha\rangle$ can be written as follows:

$$|\alpha\rangle = \sum_n c_n|e_n\rangle$$

Briefly explain why this expression makes sense and what requirements there are on the coefficients c_n, if any. In doing so, if there are any requirements for c_n with respect to a_n, comment on those also. [3 marks]

(ii) Show that $\hat{A}$ can be written as follows: [5 marks]

$$\hat{A} = \sum_n a_n|e_n\rangle\langle e_n|$$

Hint: start by considering the action of the operator $\hat{A}$ on the generic state $|\alpha\rangle$, i.e., $\hat{A}|\alpha\rangle$. You will need to get an $\langle e_n|$ to enter the equation and you might want to think about how the coefficients c_n are linked to an inner product that contains $\langle e_n|$.

(iii) Briefly demonstrate that (ii) is true by using the Pauli matrix σ_z as a simple example. [2 marks]

(b) You are at a café having a study group and a fellow student says, "I don't really get the difference between superposition and entanglement."

Briefly explain the difference between the two concepts, and state whether the two concepts can coexist. The ideal answer is about 6-8 sentences but can be shorter or longer bearing in mind that conciseness is strongly encouraged. Quality of answer matters more than quantity. Sensible examples may be given if this is helpful. [3 marks]

(c) Assume Charlie has prepared two quantum spins in the singlet state $|\Psi_S\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$.

(i) Alice now measures σ_z and Bob measures τ_z. As you know from lectures, the expectation value of $\sigma_z\tau_z$ is -1 for the singlet state (you do not need to calculate this). Briefly explain what $\langle\sigma_z\tau_z\rangle = -1$ says about the outcomes of Alice and Bob's measurements of the z-components of their respective quantum spins. [1 mark]

(ii) Alice and Bob decide to do a different experiment with a freshly prepared singlet state: Alice will measure σ_y, Bob will measure τ_y, and they will see if any correlations hold in that instance. Calculate the relevant expectation value needed to make predictions about this experiment and use the resulting value to explain what will happen. Briefly comment on whether this outcome is expected or unexpected, and why. [4 marks]

Note: The table of spin operator actions in the equation sheet will be helpful here. Note that Alice's operator σ acts on the left letter in the ket and Bob's operator τ acts on the right letter in the ket. The table for τ is not given but should be easy to infer from the table for σ.

Question 3: [20 marks total]

A particle of mass m is coupled to a simple harmonic oscillator can be described by the Hamiltonian,

$$\widehat{H} = \hbar\omega(\hat{a}_{\pm}\hat{a}_{\mp} \pm \frac{1}{2})$$

where $\hat{a}_{\pm}$ are the raising and lowering operators, defined in the usual way.

(a) Starting with the Hamiltonian above, derive an expression for $\hat{a}_{-}\hat{a}_{+}\psi_n$ in terms of the corresponding energy eigenvalue. [4 marks]

(b) Derive an expression for the momentum operator, $\hat{p}$, in terms of the ladder operators $\hat{a}_{\pm}$ and use it to compute the kinetic energy for n^{th} stationary state. [6 marks]

A particle is in the ground state of the harmonic oscillator with corresponding wave function,

$$\Psi(x,t) = \frac{1}{\sqrt[4]{\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

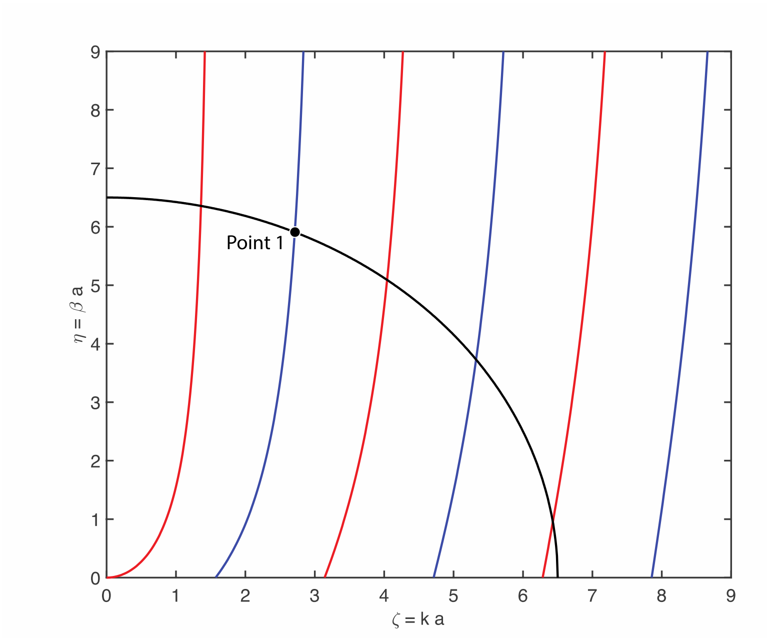
(c) The harmonic potential is suddenly removed such that the new potential felt by the electron is $V = 0$. If the wave function in (a) represents the initial conditions $\Psi(x,0)$, determine an expression for the time dependent wavefunction $\Psi(x,t)$. [4 marks]

(d) Determine the group velocity of the free electron described by (c) and give a physical justification for your answer. [3 marks]

(e) Describe qualitatively the motion of the free electron described by (c) at subsequent times and compare with what one might expect from classical behaviour. [3 marks]

Question 4: [20 marks total]

The following graph can be used to determine the boundary conditions for an electron confined to a finite one-dimensional potential well, centred about $x = 0$, with a width of $2a$. The x-axis of the plot is $\zeta = ka$ where k is the free space wave vector inside the well, and the y-axis is $\eta = \beta a$ where β is the decay constant of the wave function inside the barrier. The red lines are a plot of $\eta = \zeta \tan \zeta$ and the blue lines are a plot of $\eta = -\zeta \cot \zeta$. The black line is a plot of $\eta^2 + \zeta^2 = R^2$, where $R = 6.5$.



(a) How many bound states are supported by this quantum well? How many are odd and how many are even. In one sentence justify your answer [3 marks]

(b) If the width of the well is 2 nm, determine the height of the barrier expressed in terms of electron volts? [3 marks]

(c) The point marked on the graph (Point 1) is given by $\{\zeta, \eta\} = \{2.71,\ 5.91\}$. Use this to calculate the energy of this bound state. [4 marks]

(d) Calculate the energy of the corresponding level for the equivalent infinite square well potential and comment on the result. [3 marks]

For this state, the corresponding normalisation coefficients for the wave function in the well and barrier regions are 2.924×10^4 and 4.486×10^6, respectively.

(e) What is the probability that the electron will be found in the barrier layers for this state [4 marks]

(f) Explain ***with justification*** how the probability of being in the barrier will change for higher quantum number states. [3 marks]