THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2017

PHYS2111 Quantum Physics

Time Allowed – 2 hours

Total number of questions – 6

Use separate booklets for Questions 1 & 2, Questions 3 & 4, and Questions 5 & 6

Total marks: 120 – Questions are all of equal value (20 marks)

This paper may be retained by the candidate.

Students must provide their own UNSW approved calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

The following information may be useful

Planck's constant h = 6.626 × 10^{-34} Js

Fundamental charge unit e = 1.60 × 10^{-19} C

Speed of light (vacuum) c = 3.0 × 10^{8} m/s

Electron mass = 9.1 × 10^{-31} kg = 0.511 MeV/c^2

Neutron mass = 1.675 × 10^{-27} kg = 939.6 MeV/c^2

Proton mass = 1.672 × 10^{-27} kg = 938.3 MeV/c^2

Boltzmann's constant k = 1.38 × 10^{-23} JK^{-1}

Angstrom (Å) = 1.0 × 10^{-10} m

Permittivity constant ε_o = 8.85 × 10^{-12} Fm^{-1}

Gravitational constant G = 6.67 × 10^{-11} Nm^2/kg^2

$h/m_e c$ = 2.43 × 10^{-12} m
1 eV = 1.60 × 10^{-19} J
1 J = 6.24 × 10^{18} eV

Time-independent Schrödinger Equation: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V\psi(x)=E\psi(x)$

Time-dependent Schrödinger Equation: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x)\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}$

Bragg's law: $n\lambda = 2d\sin\theta$

Compton Shift: $\Delta\lambda = \frac{h}{mc}(1-\cos\theta)$

Bohr-Sommerfeld equation: $\oint p\,dx = nh$

$$\int \sin^2(bx)dx = \frac{x}{2}-\frac{\sin(2bx)}{4b}$$

$$\int x\sin^2(x)dx = \frac{x^2}{4}-\frac{x\sin(2x)}{4}-\frac{\cos(2x)}{8}$$

$$\int x^2\sin^2(bx)dx = \frac{x^3}{6}-\left(\frac{x^2}{4b}-\frac{1}{8b^3}\right)\sin(2bx)-\frac{x\cos(2bx)}{4b^2}$$

$$\int_{-\infty}^{\infty} e^{-bx^2}dx = \sqrt{\frac{\pi}{b}}$$

$$\int_0^{\infty} x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$

$$\int_a^b f\frac{d^2g}{dx^2}dx = f\frac{dg}{dx}\bigg|_a^b - \int_a^b \frac{df}{dx}\frac{dg}{dx}dx$$

$$\int (a-bx)^{1/2}\,dx = -\frac{2}{3b}(a-bx)^{3/2}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Pauli spin matrices: $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Visible light $\lambda \sim 400 - 700$ nm

$$E^2 = (pc)^2 + (mc^2)^2$$

$$p = mv \Big/ \sqrt{1-(v/c)^2}$$

$\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ and energies $E_n = \frac{n^2\hbar^2\pi^2}{2ma^2}$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi(x)\,dx$$

$$\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-\alpha u^2} e^{\beta u}\,du = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{-\infty} e^{-(au^2+bu+c)}\,du = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-4ac}{4a}\right), a>0$$

$$\int_{-\infty}^{\infty} x^2 e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a^3}$$

$$\int_{-\infty}^{\infty} p^2 \left|\Phi(p)\right|^2 dp = \frac{a^2\hbar^2}{2}$$

Question 1 (Marks 20)

(a) The classical wave equation is:

$$\nabla^2\psi = -k^2\psi$$

for amplitude ψ, where ∇^2 is the Laplacian operator $\partial/\partial x^2 + \partial/\partial y^2 + \partial/\partial z^2$ and $k = 2\pi/\lambda$ is the wavevector. Taking the de Broglie relation $\lambda = h/p$ and writing the conventional non-relativistic momentum in the total energy $E = K + V$, show that we can informally 'derive' the three-dimensional Schrödinger equation.

(b) Use the Fourier transform to find the wavefunction $\psi(x)$ corresponding to the momentum-space wavefunction:

$$\phi(k) = e^{-\frac{a}{b}(k-k_0)^2}$$

Question 2 (Marks 20)

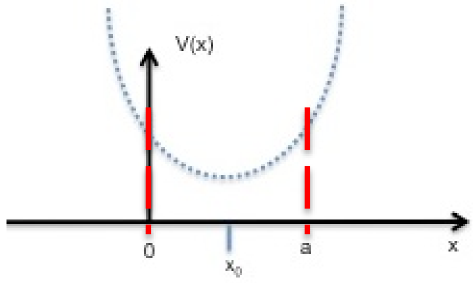
(a) A particle confined in the ground state of an infinite potential well of width *a*, that is, $0 \le x \le a$, has wavefunction

$$\psi(x) = A\sin\left(\frac{\pi x}{a}\right)$$

(i) Normalise the wavefunction

(ii) Calculate the probability that the particle will be found in the range $a/2 \le x \le 3a/4$

Question 3 (Marks 20)



Consider a particle that oscillates in a one-dimensional potential, *V(x)*, about a position x_0, between the *CLASSICAL* limits of $x = 0$ and $x = a$, as illustrated in the diagram. The minimum potential, x_0, is equidistant from the points $x = 0$ and $x = a$.

(a) Explain why you can reduce this problem to the simpler problem where the minimum energy has coordinates of $(x=0, V(x)=0)$.

(b) Draw a new potential energy v. distance diagram with your modified coordinates. Make sure you mark the *CLASSICAL* limits of oscillation of $-a/2$ and $a/2$ and the new zero position on your diagram.

(c) The potential energy in simple harmonic motion under both classical and quantum regimes is given by:

$$V(x) = \frac{1}{2}kx^2 \tag{3.1}$$

where k is the spring constant. Considering that

$$\omega = \sqrt{\frac{k}{m}} \tag{3.2}$$

where ω is the angular frequency of the oscillation and m is the mass of the particle, find the potential energy of the particle as a function of ω, m and x.

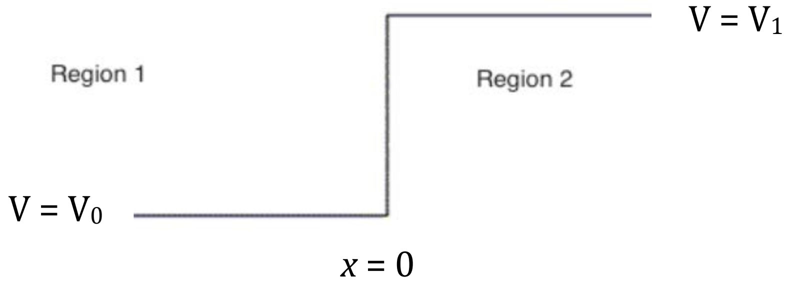
(d) Use this information to find the time-independent Schrödinger equation (TISE) for a quantum simple harmonic oscillator.

(e) Show that the equation below is a solution to your equation for the TISE for an oscillator from part (d).

$$\psi(x) = C_0 e^{-\alpha x^2} \tag{3.3}$$

Question 4 (Marks 20)

A particle of energy E travelling from left to right (Region 1 to Region 2), encounters a potential barrier that jumps suddenly from $E = V_0$ to $E = V_1$, as shown in the diagram



(a) Consider the classical case, where the particle is a macroscopic object such as a billiard ball:

(i) If $E < V_1$, what happens to the trajectory of the billiard ball when it encounters the potential barrier at $x = 0$?

(ii) If $E > V_1$, what happens to the trajectory of the macroscopic particle when it encounters the potential barrier at $x = 0$? Do not forget that V_0 and V_1 are potential energy terms. How does its velocity change?

(b) We now consider the case where the particle is small enough to be governed by the rules of quantum mechanics. The particle has total energy E in Region 1. Use the TISE to write down the equations for the particles in Regions 1 and 2.

(c) Is the scenario we are discussing here time-independent? Justify your answer.

(d) Given that:

$$E_{kin} = \frac{\hbar^2 k^2}{2m} \tag{4.1}$$

find expressions for k in regions 1 and 2 (k_1 and k_2 respectively).

(e) Are the particles we are discussing here "free" particles? Justify your answer.

(f) What particular difficulties are involved in normalising the probability functions for free particles?

Question 5 (Marks 20)

(a) One of the key mathematical concepts in quantum mechanics is that of eigenvectors and eigenvalues.

(i) In one or two sentences, explain what the terms eigenvector and eigenvalue mean for some arbitrary operator $\hat{A}$

(ii) What is an eigenfunction and an eigenstate? How are they different to an eigenvector?

(iii) Suppose that f and g are two eigenvectors of an operator $\widehat{M}$ with the same eigenvalue m. Show that any linear combination of f and g is itself an eigenvector of $\widehat{M}$ with eigenvalue m.

(b) In quantum mechanics, observables are represented by Hermitian operators.

(i) What does it mean for an operator to be Hermitian?

(ii) Why do we require observables to be represented by Hermitian operators in quantum mechanics?

(iii) Consider the following operator T:

$$T = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$$

How would you know at first sight whether T is Hermitian?

(iv) Is T Hermitian? If so, get $T^\dagger$ and use it prove $T^\dagger = T$. If not, why not?

(v) Find the two eigenvalues of T and show they are real.

(vi) Using your answer to (v), write T in its diagonalised form.

(vii) Find the two eigenstates of T? (n.b. be sure it's clear which eigenstate belongs to each of the two eigenvalues)

Question 6 (Marks 20)

(a) An operator L has two normalised eigenstates $|\lambda_1\rangle$ and $|\lambda_2\rangle$ with eigenvalues λ_1 and λ_2 respectively. Operator M has normalised eigenstates $|\mu_1\rangle$ and $|\mu_2\rangle$ with eigenvalues μ_1 and μ_2. The eigenstates are related by $|\lambda_1\rangle = (3|\mu_1\rangle + 4|\mu_2\rangle)/5$ and $|\lambda_2\rangle = (4|\mu_1\rangle - 3|\mu_2\rangle)/5$.

(i) Observable L was measured and the value λ_1 was obtained. What is the state of the system immediately after this measurement?

(ii) If M is now measured, what are the possible results, and what are their probabilities?

(iii) If we were to measure L again right after the measurement of M, what would be the probability of getting λ_1?

(iv) Would the answer to iii) be different if M is not actually measured? Why? What would it be?

(b) A commutator of two operators $\hat{A}$ and $\hat{B}$ is defined as $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

(i) Show that the commutator $[x,p^2] = 2i\hbar p$

(ii) In lectures we obtain the uncertainty principle as:

$$\sigma_A^2\sigma_B^2 = \left(\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right)^2$$

Use this to obtain the well-known position-momentum uncertainty relation $\Delta x \Delta p_x \geq \hbar/2$.

(iii) You can measure measure p_y and x simultaneously with absolute certainty because the corresponding commutator is zero. Briefly explain why this commutator is zero.

(c) An important principle in quantum mechanics is *completeness*. In particular, the eigenvectors of an observable operator are *complete*, which means any state can be written as a linear combination of the set of eigenvectors.

Show that two non-commuting operators cannot have a completely overlapping set of common eigenfunctions. (Hint: this is one of those problems where you could solve what the problem is not, rather than what it is. But there are other approaches too. 😊)