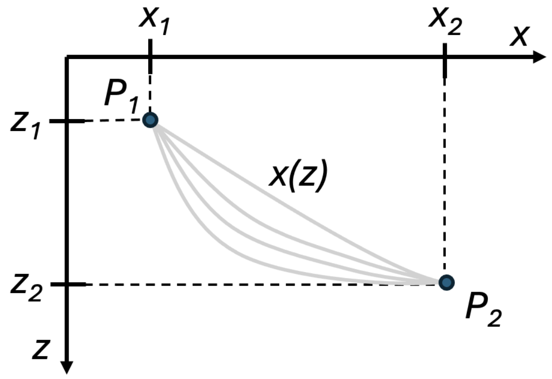
Classical Mechanics and Special Relativity (PHYS2113)

Tutorial Problem Sheet 2

1. The brachistochrone problem

This is a classic problem involving the calculus of variations: consider two points P_1 and P_2 in a uniform gravitational field pointing in the z-direction. The points are connected with a frictionless slide parameterised by a function $x(z)$. For which $x(z)$ is the travel time of an object starting at rest at P_1 and sliding to P_2 minimal?



a) Show that the travel time from P_1 to P_2 is given by

$$T_{12} = \frac{1}{\sqrt{2g}} \int_{z_1}^{z_2} \mathrm{d}z \, \frac{\sqrt{x'(z)^2 + 1}}{\sqrt{z}}. \tag{1}$$

Hint: make use of the fact that the total energy of the system is conserved.

b) Use the Euler-Lagrange equation and show that for an appropriate choice of integration constant the solution for $x(z)$ can be written as the following integral over z:

$$x(z) = \int \mathrm{d}z \, \sqrt{\frac{z}{2a - z}}, \tag{2}$$

where a is a constant.

c) Solve the integral in Equation (2) by substituting $z \to \theta$ with $z = a(1 - \cos\theta)$, and show that the solution can be written in parametric form as

$$\begin{aligned} x(\theta) &= a(\theta - \sin\theta) + x_1 &(3)\\ z(\theta) &= a(1 - \cos\theta) + z_1. &(4) \end{aligned}$$

This curve is known as a cycloid.
Hint: You may find the following trigonometric identities useful here:

$$\begin{aligned} 1 - \cos\theta &= \sin^2 \tfrac{\theta}{2} &(5)\\ 1 + \cos\theta &= \cos^2 \tfrac{\theta}{2} &(6)\\ \sin\theta &= 2 \sin\tfrac{\theta}{2} \cos\tfrac{\theta}{2}. &(7) \end{aligned}$$

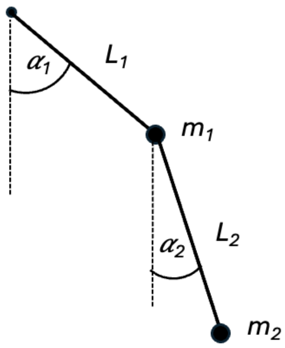
2. Equivalent Lagrangians

The Lagrange function of a system is not uniquely determined. This is obvious if we consider for instance two Lagrangians that differ by a constant, i.e., $\tilde{\mathcal{L}} = \mathcal{L} + c$. Both result in identical equations of motion and are therefore physically equivalent. Show that two Lagrangians are also physically equivalent if they differ by the *total* time derivative of an arbitrary function that depends on the coordinates q_i and time t:

$$\tilde{\mathcal{L}}(q_i, \dot{q}_i, t) = \mathcal{L}(q_i, \dot{q}_i, t) + \frac{\mathrm{d}}{\mathrm{d}t} F(q_i, t). \tag{8}$$

3. Double pendulum

Consider a double pendulum consisting of two masses m_1 and m_2 attached to massless rods of lenghts L_1 and L_2, respectively:



a) Find a Lagrangian as a function of the angles α_1 and α_2 and their time derivatives.

b) Determine the equations of motion of the system. (No need to solve them, unless you're feeling really brave.)