UNSW SCHOOL OF PHYSICS

PHYS2111 – Quantum Mechanics

Tutorial 6

Question 1

Consider a two-level system with a Hamiltonian given in some basis as

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} .$$

What values of energy E allow stationary solutions of Schrödinger's equation?

Question 2

A Hamiltonian for a two-level system with interaction is given by

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & V \\ V^* & \varepsilon_2 \end{pmatrix} .$$

where V is a complex number. What are the energy eigenvalues and corresponding eigenstates?

Question 3

A function defined on the semi-infinite domain $(0, \infty)$ is finite everywhere on the domain. It obeys the limit

$$\lim_{x\to\infty} f(x) \sim \frac{1}{x^a}.$$

Under what conditions on a is $f(x)$ square-integrable?

Question 4

In maths you may remember that any function can be expanded in powers of x (the Taylor expansion). We can use the set of polynomials as a basis on some finite interval

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

but the polynomials x^i are not orthonormal. Use Gram-Schmidt to orthogonalise the first few powers of x on the interval $-1 \leq x \leq 1$. Start with the function $|e_0\rangle \sim x^0 = 1$ and normalise it. Then normalise and orthogonalise $|e_1\rangle \sim x^1$, etc.

Answer:

$$|e_n\rangle = \sqrt{n + 1/2}\; P_n(x), \quad (n = 0, 1, 2, ...)$$

where $P_n(x)$ are the Legendre polynomials.

Question 5

(Griffith's) Consider the set of all functions of the form $p(x)e^{-x^2/2}$, where $p(x)$ is a polynomial of degree $< N$ in x, on the interval $-\infty < x < \infty$. Check that they constitute an inner product space. The "natural" basis is

$$|e_0\rangle = e^{-x^2/2}, |e_1\rangle = xe^{-x^2/2}, |e_2\rangle = x^2e^{-x^2/2}, ...$$

Orthonormalize the first four of these, and comment on the result.

Question 6

Prove that if an operator in Hilbert space $\hat{\Omega}$ obeys the property

$$\langle h|\,\hat{\Omega}\,|h\rangle = \langle h|\,\hat{\Omega}\,|h\rangle^* \tag{1}$$

then it also has the property

$$\langle f|\,\hat{\Omega}\,|g\rangle = \langle g|\,\hat{\Omega}\,|f\rangle^*$$

where $|f\rangle$, $|g\rangle$, $|h\rangle$ are arbitrary functions. That is, both are equivalent definitions of a Hermitian operator.

Hint: Write $|h\rangle = |f\rangle + i\,|g\rangle$ and expand the imaginary part of Eq. (1).

Question 7

By directly calculating the integral,

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx. \tag{2}$$

determine the Fourier Transform pairs of the following functions,

(a)

$$f(x) = \sin(\pi x) \tag{3}$$

(b)

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| \leq \frac{1}{2} \\ 0 & \text{if} \quad |x| > \frac{1}{2} \end{cases} \tag{4}$$

(c)

$$e^{-\frac{1}{2}x^2} \tag{5}$$

(d)

$$e^{-|x|} \tag{6}$$

Question 8

Prove the similarity theorem.

$$f(ax) \quad \leftrightarrow \quad \frac{1}{|a|} F\left(\frac{k}{a}\right) \tag{7}$$

Question 9

Prove the shift theorem.

$$f(x-a) \quad \leftrightarrow \quad e^{-ika} F\left(k\right) \tag{8}$$

Question 10

(Bohm, Chapter 10) Using Fourier theory, calculate $\phi(k)$ and show that the following wave packet satisfies the uncertainty principle,

$$\psi(x) = \alpha_1 \exp\left[-\frac{\alpha x^2}{2}\right] \tag{9}$$

Here α_1 is used to normalise the wavefunction.

Question 11

Use the Gaussian wave function to check the equivalence of the x-space and k-space momentum operators in determining expectation value of the momentum.