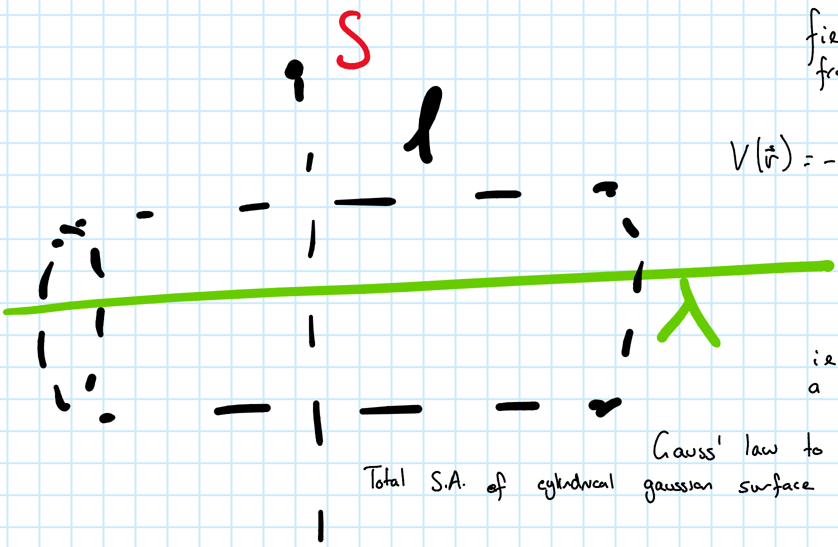
Question 1 Solution

1. In the 19th century, telegraph operators often felt shocks from long wires. Model a telegraph wire as an infinitely long line charge with uniform density λ. Find the potential a distance s from this wire. Compute the gradient of your potential, and check that it yields the correct field. These shocks inspired studies into line charges, aiding the development of safer high-voltage power lines. (Adaptation of Griffiths Problem 2.22).



field is directed radially from the charge

$$V(\vec{r}) = -\int_O^{\vec{r}} \vec{E}\cdot d\vec{l}$$

$$= -\int_a^s E \cdot dr$$

i.e. choose some point a to be our reference.

Gauss' law to find $\vec{E}$

Total S.A. of cylindrical gaussian surface is $\underbrace{2\pi r}_{\text{circumference of cylinder}} \cdot l$

Ends do not contribute as field is radial all field along z cancel via symmetry.

Gauss law $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_0^l \lambda \, dl$

$$= \frac{\lambda l}{\varepsilon_0} = E \cdot 2\pi r l$$

$d\vec{A}$ is in $\hat{r}$ direction & so is $\vec{E}$

$$\vec{E} = \frac{\lambda}{2\pi \varepsilon_0 r}\hat{r}$$

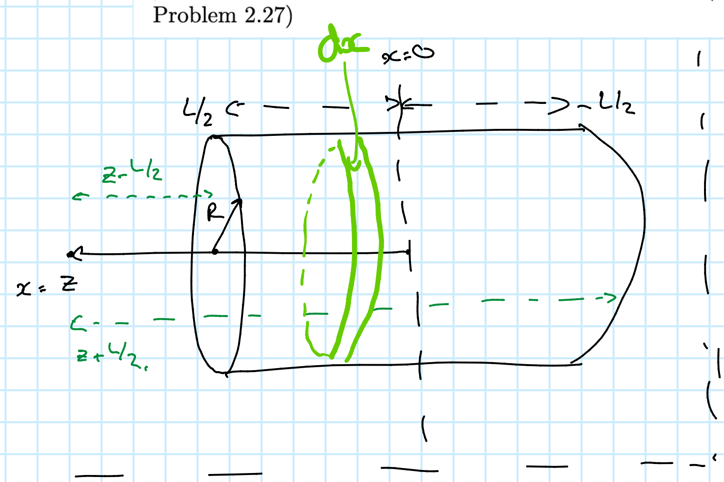
$$V(s) = -\int_a^s \frac{\lambda}{2\pi r \varepsilon_0} dr = -\left[\frac{\lambda}{2\pi\varepsilon_0}\ln|r|\right]_a^s$$

evaluate @ s

$$= \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{a}{s}\right)$$

Question 2 Solution

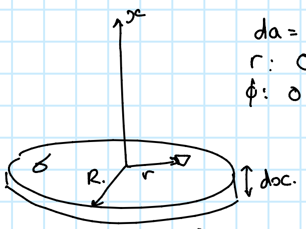
2. Find the potential on the axis of a uniformly charged solid cylinder, a distance z from the center. The length of the cylinder is L, its radius is R, and the charge density is ρ. Use your result to calculate the electric field at this point. (Assume that $z > L/2$.) (Griffiths Problem 2.27)



To solve E field using Gauss' law we need to consider directionality of the field through 3 different sides & surfaces.

However! Potential is scalar ⇒ do not need to worry about direction :)

We can also break up our cylinder into little disks & then superpose each disk onto z.
Lets start with potential of disk:



$da = r\,dr\,d\phi$
$r: 0 \to R$
$\phi: 0 \to 2\pi$

This is just Q1 from last week! but now we find the potential.

$$(\vec r) = \frac{1}{4\pi\varepsilon_0}\iint \frac{\sigma(\vec r')}{|\vec r|}\,da' \qquad |\vec r| = \sqrt{r^2+x^2}$$

$$V(\vec r) = \frac{1}{2\varepsilon_0}\int_0^R \frac{\sigma r}{\sqrt{r^2+x^2}}\,dr$$

$\sigma = \rho\,dx$ $[C/m^2]$ surface charge. let $u = r^2+x^2$ $u(0) = x^2$ $u(R) = R^2+x^2$
$du = 2r\,dr$

$$V_{disk} = \frac{1}{4\varepsilon_0}\int_{z^2}^{R^2+z^2} \frac{\sigma\,du}{\sqrt{u}}$$

$$= \frac{\sigma}{4\varepsilon_0}\left[2\sqrt{u}\right]_{x^2}^{R^2+x^2} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+x^2} - x\right).$$

Check V is correct by finding $\vec E$
$\vec E = -\nabla V$

$$= \left(\frac{\partial}{\partial x}\hat i + \cancel{\frac{\partial}{\partial y}}^{\,0}\hat j + \cancel{\frac{\partial}{\partial z}}^{\,0}\hat k\right)V = \frac{\sigma x}{2\varepsilon_0}\left(\frac{1}{\sqrt{R^2+x^2}} - \frac{1}{x}\right)\hat i$$

Same as Q1 last week

To find V for cylinder need to integrate over whole cylinder:

$$dV = \frac{\rho\,dx}{2\varepsilon_0}\left(\sqrt{R^2+x^2} - x\right)$$

$$V = \int_{z-L/2}^{z+L/2} \frac{\rho}{2\varepsilon_0}\left(\sqrt{R^2+x^2} - x\right)dx$$

Solve this & get $\vec E$ with $-\nabla V$

This integral can be solved with trig substitution but I will just use the result

$$= \frac{\rho}{2\varepsilon_0}\left(\int_{z-L/2}^{z+L/2}\sqrt{R^2+x^2}\,dx - \int_{z-L/2}^{z+L/2} x\,dx\right) = \frac{\rho}{2\varepsilon_0}\left(\left[\frac{x}{2}\sqrt{R^2+x^2} + \frac{R^2}{2}\ln\left|x+\sqrt{R^2+x^2}\right|\right]_{z-L/2}^{z+L/2} - \left[\frac{x^2}{2}\right]_{z-L/2}^{z+L/2}\right)$$

This looks horrible but subbing in $z+L/2$ & $z-L/2$ & using $-(z+L/2)^2 + (z-L/2)^2 = -2zL$

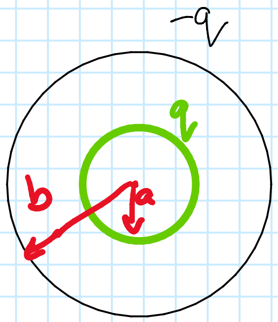
$$V = \frac{\rho}{4\varepsilon_0}\left((z+L/2)\sqrt{R^2+(z+L/2)^2} - (z-L/2)\sqrt{R^2+(z-L/2)^2} + R^2\ln\left|\frac{z+L/2+\sqrt{R^2+(z+L/2)^2}}{z-L/2+\sqrt{R^2+(z-L/2)^2}}\right| - 2zL\right)$$

$$\vec E = -\nabla V = -\hat z\frac{\partial V}{\partial z} = \frac{\rho}{2\varepsilon_0}\left[L - \sqrt{R^2+(z+L/2)^2} + \sqrt{R^2+(z-L/2)^2}\right]\hat z$$

the calculation of this is horrible! Good luck :)

Question 3 Solution

3. Consider two concentric spherical shells, of radii a and b. Suppose the inner one carries a charge q, and the outer one a charge $-q$ (both of them uniformly distributed over the surface). Calculate the energy of this configuration,
(a) using $W = \frac{\epsilon_0}{2}\int E^2 d\tau$ (all space), and
(b) using $W_1 + W_2 + \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d\tau$ and that the energy of a uniformly charged sphere with radius R and charge q is $W = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}$ (Griffiths Problem 2.36).

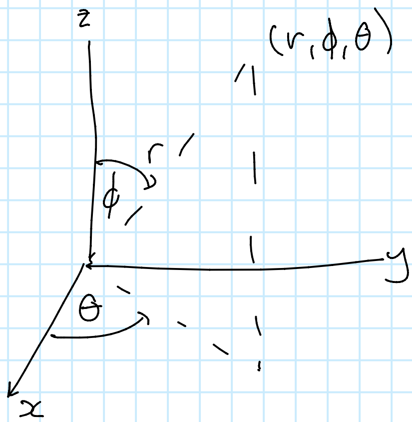


a) $W = \frac{\varepsilon_0}{2}\int E^2 d\tau$

work required to assemble charge distribution (need to do work against electric field)

$E = 0$ if $r < a$ (no q_{enc})
$E = 0$ if $r > a$ ($q_{enc} = q - q = 0$).

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ $(a < r < b)$



$$W = \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_a^b \left(\frac{1}{r^2}\right)^2 \underbrace{4\pi r^2 dr}$$

$d\tau = r^2 \sin\phi \, dr \, d\phi \, d\theta$ evaluated for ϕ & θ

ϕ is polar angle θ is azimuthal angle often interchangeable

$$W = \frac{q^2}{8\pi\varepsilon_0}\int_a^b \frac{1}{r^2} dr = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

b) $W_1 + W_2 + \varepsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau$.

$W_1 = \frac{1}{8\pi\varepsilon_0} q^2/a$, $W_2 = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{b}$

$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ $(r > a)$

$\vec{E}_2 = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ $(r > b)$.

$W_1 + W_2 = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right)$

$\varepsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau = \varepsilon_0 \int_b^\infty \frac{q}{4\pi\varepsilon_0 r^2} \cdot -\frac{q}{4\pi\varepsilon_0 r^2} 4\pi r^2 dr$

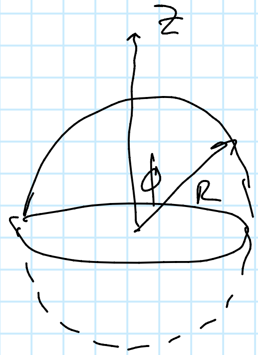
limit $b \to \infty$ as $E_2 = 0$ for $r < b$.

$$= -\int_b^\infty \frac{q^2}{4\pi\varepsilon_0 r^2} dr = -\frac{q^2}{4\pi\varepsilon_0 b}.$$

$$\Rightarrow W_{total} = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right).$$

Question 4 Solution

4. A spherical satellite in geostationary orbit has a radius of $R = 1.0\,\text{m}$. The satellite's two hemispheres are held together by 10 titanium bolts, each with a radius of $r = 1.0\,\text{cm}$. Titanium has a tensile strength of $\sigma = 300\,\text{MPa}$. Calculate the maximum charge Q_{max} the satellite can accumulate before the electrostatic repulsion between the hemispheres exceeds the bolts' mechanical strength. (Adaptation of Griffiths Problem 2.42)



Charge accumulates on surface as conductor. ⇒ if E is inside conductor flux charges would redistribute to produce a field to cancel the initial E.

$\rho = 0$ inside conductor

$E = 0 \quad \nabla \cdot E = \rho/\varepsilon_0$.

All points on a conductor are an equipotential ⇒ E ⊥ to surface just outside conductor otherwise charge will flow over the surface.

Inside $\bar{E} = 0$, outside $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$

$\vec{f} = \sigma E_{avg}$, $\quad \sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$ ← unit direction normal

Just outside. $E_{out} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\hat{r}$

$$E_{avg} = \frac{E_{inside} + E_{out}}{2} = \frac{1}{8\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

$f_z = \sigma (E_{avg})_z$ ⇒ want force in north → south direction.

force per unit area.

σ = uniform surface charge $= \frac{Q}{4\pi R^2}$ $\qquad (E_{avg})_z = E_{avg}\cos\phi$

$$F_z = \iint f_z \, da = \int_0^{2\pi}\int_0^{\pi/2} \frac{Q}{4\pi R^2}\,\frac{1}{8\pi\varepsilon_0}\,\frac{Q}{R^2}\cos\phi \, R^2 \sin\phi \, d\phi \, d\theta$$

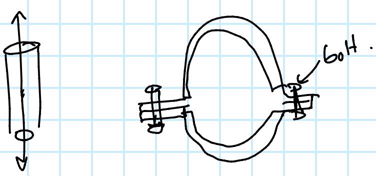
sub $u \to \sin\phi$ or $\cos\phi$

$$\Rightarrow F_z = \frac{Q^2}{32\pi R^2 \varepsilon_0}$$

Now max force bolt can hold is 300MPa tensile strength (pressure/stress)

Assume uniform force on each bolt & no pretension force.

$$A_{total\ bolt} = 10 \times \pi r^2 = 10 \times \pi \times (1e\text{-}2)^2$$



$$F_{max} = 300\text{MPa} \times A_{total} = F_z$$

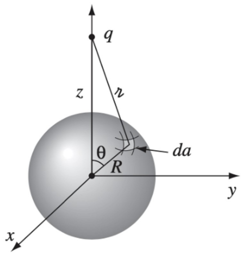
$$\Rightarrow Q^2 = 32\pi \overset{1m}{R^2}\varepsilon_0 \times 300e{+}6 \times 10 \times \pi \times (1e\text{-}2)^2$$

$$Q = \sqrt{0.000838} = 0.029\ C = 2.9\,mC.$$

Question 5 Solution

5. Find the average potential over a spherical surface of radius R due to a point charge q located inside (same in image below, only with $z < R$). (In this case, of course, Laplace's equation does not hold within the sphere.) Show that, in general, $V_{avg} = V_{center} + \frac{Q_{enc}}{4\pi\epsilon_0 R}$, where V_{center} is the potential at the center due to all the external charges, and Q_{enc} is the total enclosed charge. (Griffiths Problem 3.1).

here θ is polar angle.



$$V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\varepsilon_0 R}$$

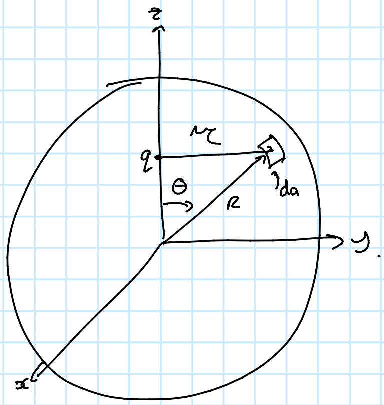
↑ potential if charge was @ centre of sphere (arises due to external charge)

Laplaces equation:

We want to find E using coulombs law.
→ but hard for all but the simplest geometries.

Can use V but integral still hard!

Change our approach to differential form.



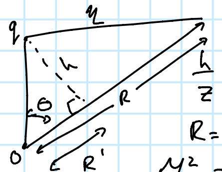
$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho \quad \text{with adequate B.C vs equivalent}$$

to

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\eta}\rho(\vec{r}\,')\,d\tau'$$

Consider regions with no charge then $\rho = 0$

$\nabla^2 V = 0$ ⇒ in region of interest with no charge.



$\frac{h}{z} = \sin\theta$, $\cos\theta = \frac{R'}{z}$

$R = R' + R''$

$\eta^2 = R''^2 + h^2$

$\eta^2 = R''^2 + z^2\sin^2\theta$

$\eta^2 = (R - z\cos\theta)^2 + z^2\sin\theta$
$= R^2 + z^2 - 2rz\cos\theta$

$V(\vec{r}) = \frac{1}{4\pi R^2}\oint V\,da$. potential due to charge q is $\frac{1}{4\pi\varepsilon_0}\frac{q}{\eta}$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\varepsilon_0}\int_{sphere}\left[z^2 + R^2 - 2zR\cos\theta\right]^{-1/2} R^2\sin\theta\,d\theta\,d\phi$$

sub $u = \cos\theta$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{2zR}\sqrt{z^2+R^2-2zR\cos\theta}\Big|_0^{\pi}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{2zR}\left(\sqrt{z^2+R^2+2zR} - \sqrt{z^2+R^2-2zR}\right)$$

→ $(z^2+R^2-2zR) = (z-R)^2 = (R-z)^2$ however $\sqrt{\ } > 0$ if $z < R$

then $(z^2+R^2-2zR)^{1/2} = R - z$

$$V_{ave} = \frac{q}{4\pi\varepsilon_0}\frac{1}{2zR}\left[z + R - (R - z)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \quad \text{avg due to interior charges}$$

In general we need to sum potential from outside & inside

$$V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\varepsilon_0 R}$$ → V_{center} from another example.

Question 6 Solution

Conceptual exam problem:

6. i) Give an example of a case where the electric field is not continuous.
ii) For the case you have given is the potential continuous? Demonstrate this.

Electric field is not continuous when you cross a boundary with surface charge. I.e. $\hat{n}$ changes sign.

Solution from Griffiths.

Example 2.5. An infinite plane carries a uniform surface charge σ. Find its electric field.

Solution

Draw a "Gaussian pillbox," extending equal distances above and below the plane (Fig. 2.22). Apply Gauss's law to this surface:

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

In this case, $Q_{\text{enc}} = \sigma A$, where A is the area of the lid of the pillbox. By symmetry, $\mathbf{E}$ points away from the plane (upward for points above, downward for points below). So the top and bottom surfaces yield

$$\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}|,$$

whereas the sides contribute nothing. Thus

$$2A\,|\mathbf{E}| = \frac{1}{\epsilon_0} \sigma A,$$

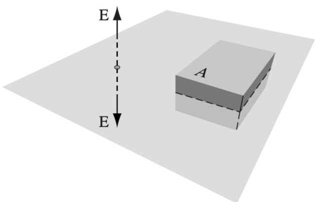


FIGURE 2.22

or

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}, \tag{2.17}$$

where $\hat{\mathbf{n}}$ is a unit vector pointing away from the surface. In Prob. 2.6, you obtained this same result by a much more laborious method.

The potential, meanwhile, is continuous across *any* boundary (Fig. 2.38), since

$$V_{\text{above}} - V_{\text{below}} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l};$$

as the path length shrinks to zero, so too does the integral:

$$V_{\text{above}} = V_{\text{below}}. \tag{2.34}$$

However, the *gradient* of V inherits the discontinuity in $\mathbf{E}$; since $\mathbf{E} = -\nabla V$, Eq. 2.33 implies that

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0} \sigma \hat{\mathbf{n}}, \tag{2.35}$$

or, more conveniently,

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0} \sigma, \tag{2.36}$$

where

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}} \tag{2.37}$$

denotes the **normal derivative** of V (that is, the rate of change in the direction perpendicular to the surface).

Please note that these boundary conditions relate the fields and potentials *just* above and *just* below the surface. For example, the derivatives in Eq. 2.36 are the *limiting* values as we approach the surface from either side.

Question 7 Solution

Calculated exam problem:

7. The electric potential of some configuration is given by the expression

$$V(\mathbf{r}) = A\frac{e^{-\lambda r}}{r},$$

$\rightsquigarrow$ negative cloud } kind of.
$\leftarrow$ point charge

where A and λ are constants with appropriate units.
You may make use of the following relationships involving delta functions:

$$f(x)\delta(x) = f(0)\delta(x),$$

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\mathbf{r}),$$

$$\int \delta^3(r)d\tau = 1.$$

And the following standard integral:

$$\int_0^\infty re^{-ar}dr = \frac{1}{a^2}.$$

i) Find an expression for the electric filed $\mathbf{E}(\mathbf{r})$.
ii) Find an expression for the charge density $\rho(r)$.
iii) What is the total charge (through all space)?
Note: An example of a potential with this form is the Yukawa Potential which describes the potential between nucleons (protons and neutrons) in an atomic nucleus. This is a short-range interation dropping off rapidly with distance, it can be thought of as a screened

Version of the coulomb potential

i) Find expression for $\vec{E}(\vec{r})$. $r>0$

$$E = -\nabla V = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)\hat{r}$$

$$\nabla = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\varphi}$$

polar angle

azimuthal angle.

$$E = -A\left(\frac{-\lambda e^{-\lambda r}}{r} - \frac{1}{r^2}e^{-\lambda r}\right)\hat{r}$$

$$= A e^{-\lambda r}\left(\frac{1+\lambda r}{r^2}\right)\hat{r}$$

ii) find charge density $\rho(r)$.

$$\nabla \cdot E = \frac{\rho(r)}{\varepsilon_0} \rightsquigarrow \text{Gauss law differential form.}$$

$$\rho(r) = \varepsilon_0 \nabla \cdot E$$

$$= \varepsilon_0 A\left\{\overbrace{e^{-\lambda r}(1+\lambda r)\vec{\nabla}\cdot}^{1}\left(\frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2}\cdot\vec{\nabla}\left(e^{-\lambda r}(1+\lambda r)\right)\right\}$$

$\rightsquigarrow$ product rule.

we have $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$

$$\left.\begin{aligned}&\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})\\ &e^{-\lambda r}(1+\lambda r)\delta^3(\vec{r}) = \delta^3(\vec{r})\end{aligned}\right\}1$$

$$\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = \hat{r}\frac{\partial}{\partial r}\left(e^{-\lambda r}(1+\lambda r)\right)$$

$$= \hat{r}\left\{-\lambda e^{-\lambda r}(1+\lambda r) + \lambda e^{-\lambda r}\right\}$$

$$= \hat{r}\left(-\lambda^2 r e^{-\lambda r}\right)$$

$$\Rightarrow \frac{\hat{r}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = -\frac{\lambda}{r}e^{-\lambda r}$$

$$\rho = \varepsilon_0 A\left[4\pi\delta^3(\vec{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right]$$

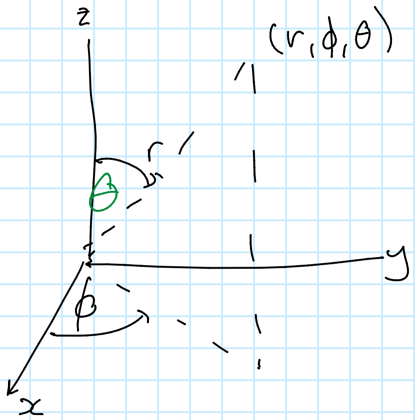
iii) Total charge is $\int \rho\, d\tau$

$$= \varepsilon_0 A\left\{4\pi\int\delta^3(\vec{r})\,d\tau - \lambda^2\int\frac{e^{-\lambda r}}{r}\underbrace{4\pi r^2 dr}_{\text{integrate over angles already}}\right\}$$

$$= \varepsilon_0 A\left(4\pi - \lambda^2 4\pi\int_0^\infty re^{-\lambda r}dr\right)$$

$$\int_0^\infty re^{-\lambda r}dr = \frac{1}{\lambda^2} \Rightarrow Q = 4\pi\varepsilon_0 A\left(1 - \frac{\lambda^2}{\lambda^2}\right) = 0$$

Derivation of spherical coord



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$$

$$\hat{r} = \frac{\partial\vec{r}}{\partial r} = \frac{\partial}{\partial r}\left(r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}\right)$$

$$= \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$\rightsquigarrow$ already normalised

$$\frac{\partial\vec{r}}{\partial\phi} = -r\sin\theta\,\hat{x} + r\sin\theta\cos\phi\,\hat{y}$$

normalise for $\hat{\phi} = \frac{1}{r\sin\theta}\frac{\partial\vec{r}}{\partial\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$$\frac{\partial\vec{r}}{\partial\theta} = r\cos\theta\cos\phi\,\hat{x} + r\cos\theta\sin\phi\,\hat{y} - r\sin\theta\,\hat{z}$$

normalise for $\hat{\theta} = \frac{1}{r}\frac{\partial\vec{r}}{\partial r} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$

To get ∇f need chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial f}{\partial\phi}\frac{\partial\phi}{\partial x}$$

Same for y & z

Compute all chain rule derivatives ie. $\frac{\partial r}{\partial x}$
need horrible Jacobians.