# UNSW Sydney

# TERM 1 2020 EXAMINATIONS

# PHYS2111: Quantum Physics

1. TIME ALLOWED – 2 hours

2. READING TIME – 10 minutes

3. THIS EXAMINATION PAPER HAS 7 PAGES

4. TOTAL NUMBER OF QUESTIONS – 4

5. TOTAL MARKS AVAILABLE – 100

6. MARKS AVAILABLE FOR EACH QUESTION ARE SHOWN IN THE EXAMINATION PAPER

7. ALL ANSWERS MUST BE WRITTEN IN INK. EXCEPT WHERE THEY ARE EXPRESSLY REQUIRED, PENCILS MAY BE USED ONLY FOR DRAWING, SKETCHING OR GRAPHICAL WORK

8. THIS PAPER MAY BE RETAINED BY CANDIDATE

9. CANDIDATES MAY BRING TO THE EXAMINATION: UNSW APPROVED CALCULATOR

10. THE FOLLOWING MATERIALS WILL BE PROVIDED: Exam booklets

**The following information may be useful**

Planck's constant $h = 6.626 \times 10^{-34}$ Js

Fundamental charge unit $e = 1.60 \times 10^{-19}$ C

Speed of light (vacuum) $c = 3.0 \times 10^{8}$ m/s

Electron mass = $9.1 \times 10^{-31}$ kg = 0.511 MeV/$c^2$

Neutron mass = $1.675 \times 10^{-27}$ kg = 939.6 MeV/$c^2$

Proton mass = $1.672 \times 10^{-27}$ kg = 938.3 MeV/$c^2$

Boltzmann's constant $k = 1.38 \times 10^{-23}$ JK$^{-1}$

Permittivity constant $\varepsilon_o = 8.85 \times 10^{-12}$ Fm$^{-1}$

Gravitational constant $G = 6.67 \times 10^{-11}$ Nm$^2$/kg$^2$

Angstrom 1 Å = $1.0 \times 10^{-10}$ m

$h/m_e c = 2.43 \times 10^{-12}$ m
1 eV = $1.60 \times 10^{-19}$ J
1 J = $6.24 \times 10^{18}$ eV

Time-independent Schrödinger Equation: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V\psi(x) = E\psi(x)$

Time-dependent Schrödinger Equation: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$

Bohr-Sommerfeld equation: $\oint p\,dx = nh$

$$\int \sin^2(bx)dx = \frac{x}{2} - \frac{\sin(2bx)}{4b}$$

$$\int x\sin^2(x)dx = \frac{x^2}{4} - \frac{x\sin(2x)}{4} - \frac{\cos(2x)}{8}$$

$$\int x^2\sin^2(bx)dx = \frac{x^3}{6} - \left(\frac{x^2}{4b} - \frac{1}{8b^3}\right)\sin(2bx) - \frac{x\cos(2bx)}{4b^2}$$

$$\int_{-\infty}^{\infty} e^{-bx^2}dx = \sqrt{\frac{\pi}{b}}$$

$$\int_0^{\infty} x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$

$$\int_a^b f\frac{d^2g}{dx^2}dx = f\frac{dg}{dx}\bigg|_a^b - \int_a^b \frac{df}{dx}\frac{dg}{dx}dx$$

$$\int (a-bx)^{1/2}\,dx = -\frac{2}{3b}(a-bx)^{3/2}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Pauli spin matrices: $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)e^{-ipx/\hbar}dx$$

$$\int_{-\infty}^{\infty} e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{a}$$

$$\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-(au^2+bu+c)}du = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-4ac}{4a}\right), a > 0$$

$$\int_{-\infty}^{\infty} x^2e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a^3}$$

$$\int_{-\infty}^{\infty} p^2|\Phi(p)|^2dp = \frac{a^2\hbar^2}{2}$$

Ground state of the harmonic oscillator: $\psi_0(x) = \frac{1}{(\pi x_0^2)^{1/4}}e^{-x^2/2x_0^2}$, where $x_0^2 \equiv \frac{\hbar}{m\omega}$

$[A,B] = AB - BA$

$[A,B]^\dagger = [B^\dagger,A^\dagger]$

**Table of spin operator actions**

| | | | |
|---|---|---|---|
| $\sigma_x\|uu\rangle = \|du\rangle$ | $\sigma_x\|ud\rangle = \|dd\rangle$ | $\sigma_x\|du\rangle = \|uu\rangle$ | $\sigma_x\|dd\rangle = \|ud\rangle$ |
| $\sigma_y\|uu\rangle = i\|du\rangle$ | $\sigma_y\|ud\rangle = i\|dd\rangle$ | $\sigma_y\|du\rangle = -i\|uu\rangle$ | $\sigma_y\|dd\rangle = -i\|ud\rangle$ |
| $\sigma_z\|uu\rangle = \|uu\rangle$ | $\sigma_z\|ud\rangle = \|ud\rangle$ | $\sigma_z\|du\rangle = -\|du\rangle$ | $\sigma_z\|dd\rangle = -\|dd\rangle$ |

Note that $\sigma$ acts on the left letter in the ket. The table for $\tau$ is similar but $\tau$ acts on the right letter in the ket instead.

**Ladder operators**

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$$

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1}$$

$$a_-\psi_n = \sqrt{n}\psi_{n-1}$$

**Question 1 (25 Marks)**

(a) One of the key mathematical concepts in quantum mechanics is that of eigenvectors and eigenvalues.

(i) In one or two sentences, explain what the terms ‘eigenvector’ and ‘eigenvalue’ mean for some arbitrary operator $\hat{A}$.

(ii) A fellow student makes the statement “Eigenvector, eigenfunction, eigenstate, they’re really just different ways of representing the same essential thing.” Would you agree or disagree? Briefly explain your reasoning (1-2 sentences).

(b) In quantum mechanics, observables are represented by Hermitian operators.

(i) What does it mean for an operator to be Hermitian? In a Hermitian matrix, what do we automatically know about terms on the diagonal and pairs of off-diagonal terms symmetric about the diagonal?

(ii) Why do we require observables to be represented by Hermitian operators in quantum mechanics?

(c) In the course we have often considered a ‘quantum spin’, the quantum equivalent of a particle with angular momentum. You learn about classical angular momentum in first year. Briefly discuss how the outcome of measuring the *z*-component of the angular momentum differs between the classical and quantum cases. Marks will be allocated based on the conciseness and effectiveness of your answer rather than length alone. Stick to what is important in answering this question.

(d) For a quantum spin system with spin ½, all possible spin states can be represented in a two-dimensional complex vector space with basis vectors $|u\rangle = \binom{1}{0}$ and $|d\rangle = \binom{0}{1}$.

(i) Using the Pauli matrix $\hat{\sigma}_z$ given in the equation sheet, explain briefly what the two possible outcomes for measuring the *z*-component of the spin would be. In your answer, make it clear what the two eigenvalues and their corresponding eigenvectors mean physically.

(ii) Suppose a quantum spin is in the state $|\chi\rangle = \frac{1}{\sqrt{6}}\binom{1+i}{2}$. If you make an observation of the *z*-component of the spin $\hat{\sigma}_z$, then what are the respective probabilities the two possible outcomes? Show that these add to 1 as expected.

(iii) For the *x*-component of the spin $\hat{\sigma}_x$, find the eigenvectors $|r\rangle$ and $|l\rangle$ that correspond to the two possible measurement outcomes. Be sure your final eigenvectors are properly normalized. (n.b., there is more than one way to do this; one is faster than the other).

(iv) The vectors you obtain in (iii) are not $|u\rangle$ and $|d\rangle$. Is this a problem? Explain your answer in no more than 2-3 sentences.

(v) Calculate the respective probabilities of measuring the two observable outcomes for the *x*-component of the spin for $|\chi\rangle$. Show these add to 1.

(vi) Calculate the expectation value for $\sigma_x$, which is obtained as $\langle\sigma_x\rangle = \langle\chi|\sigma_x|\chi\rangle$. Show you can get the same result using the two probabilities obtained in (iv).

**Question 2 (25 Marks)**

(a) A commutator of two operators $\hat{A}$ and $\hat{B}$ is defined as $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

(i) What does it mean mathematically for two operators to 'not commute'? In 1-2 sentences, explain the physical implications for non-commuting observables.

(ii) Show that the commutator $i[\hat{A},\hat{B}]$ is Hermitian if $\hat{A}$ and $\hat{B}$ are both Hermitian.

(iii) Show that $[\sigma_x,\sigma_y] = 2i\sigma_z$.

(b) For two observables *L* and *M* we can obtain the generalized uncertainty principle as:

$$\Delta L \Delta M \geq \left|\frac{i}{2}\langle[\hat{L},\hat{M}]\rangle\right|$$

(i) Assume a single spin prepared in the state $|u\rangle$. Use the generalized uncertainty principle to show that $\Delta\sigma_x\Delta\sigma_y \geq 1$.

(ii) Suppose instead that the single spin is in the state $|\chi\rangle = \frac{|u\rangle}{\sqrt{2}} + \frac{|d\rangle}{\sqrt{2}}$. What will the uncertainty relation $\Delta\sigma_x\Delta\sigma_y$ become? Briefly explain the physical meaning.

**Note:** For each of (i) and (ii) above you will need $\langle\sigma_z\rangle$, it is acceptable to briefly explain the value you use for this rather than calculate it explicitly.

(c) Assume Charlie has prepared two spins in the singlet state $|\Psi_S\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$.

(i) Alice now measures $\sigma_z$ and Bob measures $\tau_z$. What is the expectation value of $\sigma_z\tau_z$? The table of spin operator actions in the equation sheet will be helpful here. Note that Alice's operator $\sigma$ acts on the left letter in the ket and Bob's operator $\tau$ acts on the right letter in the ket. The table for $\tau$ is not given but should be easy to infer.

(ii) What does your result in (i) say about the outcomes of Alice and Bob's measurements of their spins if they measure their respective *z*-components?

(iii) There isn't enough time to make you calculate this, but $\langle\sigma_x\tau_y\rangle = 0$ and not $-1$ (you can confirm it for yourself after the exam if you don't believe me). What does this mean when you consider this means Alice measuring the *x*-component of her spin and Bob measuring the *y*-component of his spin.

(d) There is an interesting postulate in quantum mechanics that says any $2 \times 2$ Hermitian matrix *L* can we written as the sum of the three Pauli matrices and the identity matrix:

$$\hat{L} = a\sigma_x + b\sigma_y + c\sigma_z + dI$$

where *a*, *b*, *c* and *d* are all real numbers. Verify this postulate.

**Question 3 (25 marks)**

(i) Consider the Hamiltonian for the harmonic oscillator

$$H = \frac{1}{2m}\left[p^2 + (m\omega x)^2\right] \ , \tag{0.1}$$

and the ladder operators

$$a_\pm = \frac{1}{\sqrt{2\,\hbar\,m\,\omega}}\,(\mp i\,p + m\,\omega\,x) \ , \tag{0.2}$$

where $p \equiv -i\hbar\,\frac{\partial}{\partial x}$ is the momentum operator. Verify the following equations:

(a) $[a_-, a_+] = 1$.

(b) $H = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)$.

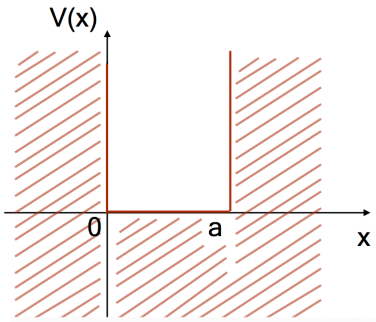
(ii) Knowing that $a_-\psi_0(x) = 0$ for the ground state, derive $\psi_0(x)$ (don't forget to normalize it).

**Question 4 (25 marks)**

(i) Consider the stationary states $\psi_n(x)$, with $n = 1, 2, 3, ...$, for the infinite square well potential and the corresponding energy levels $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

- Knowing that the $\psi_n(x)$ are orthonormal, find $\Psi(x,t)$ for the following initial conditions:
$$\Psi(x,0) = A\,(4\,\psi_1 + \psi_2)\ , \tag{0.3}$$
where $\psi_1$ is the ground state, $\psi_2$ is the first excited state and A is a constant. *For this question, you DO NOT need to specify what the explicit form of the $\psi_n$ functions is but you DO NEED to determine A.*

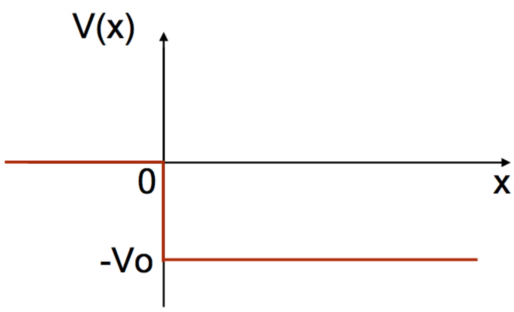
- What is the probability that a measurement of the energy returns the value $E = \frac{9\pi^2\hbar^2}{2ma^2}$? *Explain your answer.*




(ii) Consider the following potential

$$V(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ -V_0 & \text{if } x > 0 \end{cases} \tag{0.4}$$

where $V_0 > 0$, and a particle of mass $m$ and energy $E > 0$ approaching it from the left (i.e. from $x < 0$). What is the value of the reflection coefficient for $E = 2V_0$?




**END OF EXAM**