Practice mid-term 1 PHYS2114

Question 1

a) i) Gauss's law is $\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\varepsilon_0}$.

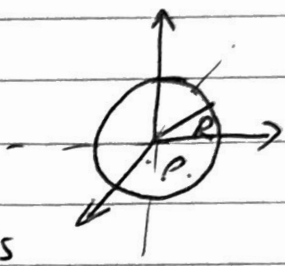
The relationship between $\underset{\sim}{E}$ and V is $\underset{\sim}{E} = -\nabla V$.
Substituting this into Gauss's law gives:

$$\nabla \cdot (-\nabla V) = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}.$$

This is Poisson's equation.

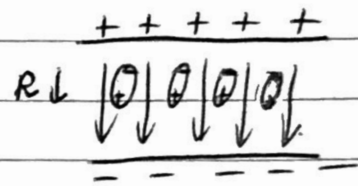
ii) Laplace's equation is:

$$\nabla^2 V = 0.$$



This holds wherever $\rho = 0$. In this situation this is in the region $r > R$.

b) Polar molecules have an electric dipole. When these are placed in an electric field they align with the field, reducing the field strength. As the dipoles are aligning the result is polarization, this happens because the dipole feels a torque and turns to align with the field.



Non-polar molecules obtain an induced dipole moment in an electric field. The positive nucleus moves in the direction of $\underset{\sim}{E}$, while the negative electron clouds move a little in the opposite direction, causing a small charge seperation. This induced dipole moment is aligned with the field, resulting in an overall polarization.

Note: When dipole moments are randomly aligned there is no polarization. Polarization is the dipole moment per unit volume.

c) A vector potential has $\nabla \cdot \underset{\sim}{A} = 0$ and $\nabla \times \underset{\sim}{A} = \underset{\sim}{B}$.

i) $\nabla \cdot \underset{\sim}{A} = \frac{\partial x^2}{\partial x} + \frac{\partial xz}{\partial y} + \frac{\partial(-2xz)}{\partial z} = 2x - 2x = 0$ ✓

$$\nabla \times \underset{\sim}{A} = \begin{vmatrix} \hat{\underset{\sim}{x}} & \hat{\underset{\sim}{y}} & \hat{\underset{\sim}{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xz & -2xz \end{vmatrix} = \hat{\underset{\sim}{x}}(0-x) + \hat{\underset{\sim}{y}}(0+2z) + \hat{\underset{\sim}{z}}(z-0)$$

Not the same as given $\underset{\sim}{B}$ so incorrect $\underset{\sim}{A}$

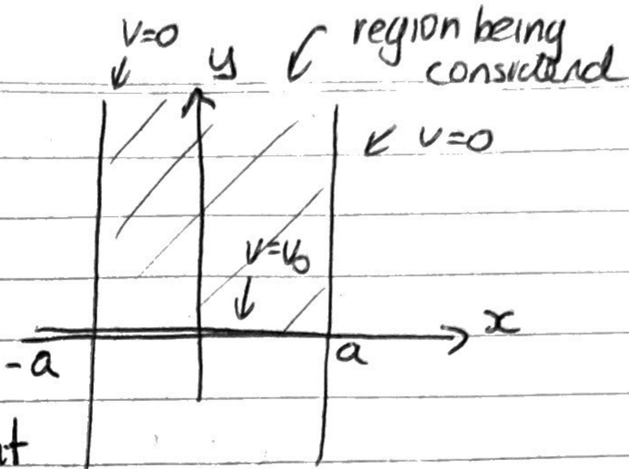
ii) $\nabla \cdot A = -2x + x + x = 0$ ✓

$$\nabla \times \underset{\sim}{A} = \begin{vmatrix} \hat{\underset{\sim}{x}} & \hat{\underset{\sim}{y}} & \hat{\underset{\sim}{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x^2 & xy & xz \end{vmatrix} = \hat{\underset{\sim}{x}}(0-0) + \hat{\underset{\sim}{y}}(0+z) + \hat{\underset{\sim}{z}}(y)$$

$$\nabla^2 \underset{\sim}{A} = -\mu_0 \underset{\sim}{J} = (-2, 0, 0) \Rightarrow \underset{\sim}{J} = \frac{2}{\mu_0} \hat{\underset{\sim}{x}}$$

Question 2

a) $V=0$ at $x=a$
$V=0$ at $x=-a$
$V=V_0$ at $y=0$
$V=0$ at $y=\infty$



b) The first uniquness theorm tells us that any solution to Laplace's equation in some volume $\mathcal{V}$ is uniquelly determined if V is known at the boundary.

c) V can not vary with z as the boundary conditions are indenendent of z, that is $V(z_1)=V(z_2)$ for all z_1, z_2. As it is indenendent of z when we differentiate it we get zero $\frac{\partial V}{\partial z}=0 \Rightarrow \frac{\partial^2 V}{\partial z^2}=0$.

d) $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$ Laplace's equation.

$\Rightarrow \frac{\partial^2 (XY)}{\partial x^2}+\frac{\partial^2 (XY)}{\partial y^2}=0$ $(*)$

$\frac{\partial (XY)}{\partial x}=X\frac{\partial Y}{\partial x}^{\to 0}+Y\frac{\partial X}{\partial x}$ $\frac{\partial Y}{\partial x}=0$ as Y is indenendent of x.

$\frac{\partial^2 (XY)}{\partial x^2}=Y\frac{\partial^2 X}{\partial x^2}$.

$\frac{\partial (XY)}{\partial y}=X\frac{\partial Y}{\partial y}+Y\frac{\partial X}{\partial y}^{\to 0}$.

$\frac{\partial^2 (XY)}{\partial y^2}=X\frac{\partial^2 Y}{\partial y^2}$.

$\Rightarrow$ sub into $(*)$ $Y\cdot\frac{\partial^2 X}{\partial x^2}+Y\frac{\partial^2 Y}{\partial y^2}=0$.

Divide by XY $\Rightarrow$ $\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=0$.

e) This equation is true for all x and y. We can fix y and vary x. As we do that $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$ does not change (because y is fixed). $\frac{1}{X}\frac{\partial^2 X}{\partial x^2}$ must be equal to this for all x, that is it is always equal to the same thing and so must be constant.

f) $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k^2$ $\qquad$ $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2$.

$X = A\sin kx + B\cos kx$. $\qquad$ $Y = Ce^{ky} + De^{-ky}$

$\frac{\partial X}{\partial x} = Ak\cos(kx) - Bk\sin kx$. $\qquad$ $\frac{\partial Y}{\partial y} = Cke^{ky} - Dke^{-ky}$

$\frac{\partial^2 X}{\partial x^2} = -Ak^2\sin(kx) - Bk^2\cos(kx)$ $\qquad$ $\frac{\partial^2 Y}{\partial y^2} = Ck^2e^{ky} + Dk^2e^{-ky}$

$= -k^2 X$. $\qquad$ $= k^2 Y$.

ie $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k^2$ as required. $\qquad$ ie $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2$ as required.

g) In x direction: $V=0$ at $\pm a \Rightarrow$ this is an even function, need to have $X(-a) = X(+a)$. This is true for the cos function but not sin.

$\Rightarrow 0 = B\cos(ka) \Rightarrow ka = \frac{(n+1)\pi}{2}$. $\quad n = 0, 1, 2, 3$

$\Rightarrow X = B\cos\left(\frac{a(n+1)\pi}{2a}\right)$.

In y direction: as $y \to \infty$ $Y \to 0$. $\Rightarrow$ must have $C = 0$.

$Y = De^{-ky} = De^{-(n+1)\pi y/2a}$.

h) $V = XY = BD\cos\left(\frac{a(n+1)\pi}{2a}\right)e^{-(n+1)\pi y/2a}$.

$= C_n\cos\left(\frac{a(n+1)\pi}{2a}\right)e^{-(n+1)\pi y/2a}$. where C_n is a constant.

i) Have not yet matched the boundary condition $V = V_0$ at $y = 0$. Would need to find a series solution to the above expression (can use Fourier series) that meets this requirement. As V obeys law of superposition adding terms that satisfy V will result in a function that still satisfies V. This will meet the requirements.

Question 3 $\qquad \underset{\sim}{P} = kr^2 \hat{\underset{\sim}{r}}$

a) $\sigma_b = \underset{\sim}{P} \cdot \hat{\underset{\sim}{n}}$

as this is a sphere $\hat{\underset{\sim}{n}} = \hat{\underset{\sim}{r}}$

$\Rightarrow \sigma_b = kr^2 \hat{\underset{\sim}{n}} \cdot \hat{\underset{\sim}{r}} = kr^2$. all on surface so equal to kR^2.

$\rho_b = -\nabla \cdot P$

Using spherical co-ordinates

$$= -\left(\frac{1}{r^2} \cdot \frac{\partial}{\partial r}(kr^4)\right)$$

$$= -\frac{1}{r^2} 4kr^3 = -4k \cdot r$$

b) I expect these charges to be equal and opposite as no bound charge has left the sphere.

Total surface bound charge $= Q_{surface,b} = \sigma_b \cdot 4\pi R^2 = kR^2 \cdot 4\pi R^2$

$$= 4\pi k R^4.$$

Total volume bound charge $= Q_{volume,b} = \int \rho_b \, \partial\tau$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} -4kr \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi.$$

$$= 2\pi \int_0^{\pi} -4k\left[\frac{r^4}{4}\right]_0^R \sin\theta \, d\theta$$

$$= -\frac{8\pi k}{4} R^4 \left[-\cos\theta\right]_0^{\pi}$$

$$= -2\pi k R^4 [+1+1] = -4\pi k R^4.$$

These are equal and opposite as expected.

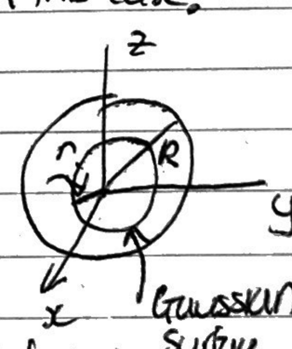
c) Use Gauss's law, the charges are just the bound charges in this case.

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{q_{enc}}{\varepsilon_0} = \frac{-4\pi k r^{4}}{\varepsilon_0} = E \cdot 4\pi r^2.$$

surface area

expression derived in part b.

$$\Rightarrow \underset{\sim}{E} = -\frac{kr^2}{\varepsilon_0} \hat{\underset{\sim}{r}}$$



d) No free charges, just bound charges here so would expect it to be 0, but can check:

$$\underset{\sim}{D} = \varepsilon_0 \underset{\sim}{E} + \underset{\sim}{P} = -kr^2 \hat{\underset{\sim}{r}} + kr^2 \hat{\underset{\sim}{r}} = 0.$$

f) $V = -\int \underset{\sim}{E} \cdot \underset{\sim}{dr}$

$$= -\int_{\infty}^{R} 0 \cdot dr - \int_{R}^{r} -\frac{kr^2}{\varepsilon_0} dr.$$

Bring from $\infty \rightarrow R$, then $R \rightarrow r$.

$$= \frac{k}{\varepsilon_0}\left[\frac{r^3}{3}\right]_R^r$$

$$= \frac{k}{3\varepsilon_0}(r^3 - R^3).$$

↑

e) $\underset{\sim}{E} = 0$ I switched the order!

There is no charge inside a Gaussian surface outside the sphere.