THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2016

PHYS2111 Quantum Physics

Time Allowed – 3 hours

Total number of questions – 6

Use separate booklets for Questions 1 & 2, Questions 3 & 4, and Questions 5 & 6

Total marks: 180 – Questions are all of equal value (30 marks)

This paper may be retained by the candidate.

Students must provide their own UNSW approved calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

The following information may be useful

Planck's constant $h = 6.626 \times 10^{-34}$ Js

Fundamental charge unit $e = 1.60 \times 10^{-19}$ C

Speed of light (vacuum) $c = 3.0 \times 10^{8}$ m/s

Electron mass = 9.1×10^{-31} kg = 0.511 MeV/c^2

Neutron mass = 1.675×10^{-27} kg = 939.6 MeV/c^2

Proton mass = 1.672×10^{-27} kg = 938.3 MeV/c^2

Boltzmann's constant $k = 1.38 \times 10^{-23}$ JK^{-1}

Angstrom (Å) = 1.0×10^{-10} m

Permittivity constant $\varepsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1}

Gravitational constant $G = 6.67 \times 10^{-11}$ Nm^2/kg^2

$h/m_e c = 2.43 \times 10^{-12}$ m
1 eV = 1.60×10^{-19} J
1 J = 6.24×10^{18} eV

Time-independent Schrödinger Equation: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V\psi(x) = E\psi(x)$

Time-dependent Schrödinger Equation: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$

Bragg's law: $n\lambda = 2d\sin\theta$

Compton Shift: $\Delta\lambda = \frac{h}{mc}(1-\cos\theta)$

Bohr-Sommerfeld equation: $\oint p\,dx = nh$

$$\int \sin^2(bx)dx = \frac{x}{2} - \frac{\sin(2bx)}{4b}$$

$$\int x\sin^2(x)dx = \frac{x^2}{4} - \frac{x\sin(2x)}{4} - \frac{\cos(2x)}{8}$$

$$\int x^2\sin^2(bx)dx = \frac{x^3}{6} - \left(\frac{x^2}{4b} - \frac{1}{8b^3}\right)\sin(2bx) - \frac{x\cos(2bx)}{4b^2}$$

$$\int_{-\infty}^{\infty} e^{-bx^2}dx = \sqrt{\frac{\pi}{b}}$$

$$\int_0^{\infty} x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$

$$\int_a^b f\frac{d^2g}{dx^2}dx = f\frac{dg}{dx}\bigg|_a^b - \int_a^b \frac{df}{dx}\frac{dg}{dx}dx$$

$$\int (a - bx)^{1/2}\, dx = -\frac{2}{3b}(a - bx)^{3/2}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Pauli spin matrices: $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Visible light $\lambda \sim 400 - 700$ nm

$$E^2 = (pc)^2 + (mc^2)^2$$

$$p = mv \Big/ \sqrt{1 - (v/c)^2}$$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi(x)\,dx$$

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$$

$$\int_{-\infty}^{-\infty} e^{-(au^2+bu+c)} du = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2 - 4ac}{4a}\right), a > 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{2a^3}$$

$$\int_{-\infty}^{\infty} p^2 \left|\Phi(p)\right|^2 dp = \frac{a^2\hbar^2}{2}$$

Question 1 (Marks 30)

(a) A photon with energy 662 keV collides with an electron in a metal sample and is scattered causing the electron to recoil. The electron is assumed to be stationary before the collision. Write down the relativistic equation of conservation of energy and the equations for the conservation of the horizontal and vertical components of momentum for this collision *but do not solve these.*

(b) A gamma ray photon of wavelength 1.88 pm strikes a metal target and the wavelength of the scattered radiation is measured at an angle $\theta = 90°$ to the incident beam. Calculate:

- (i) the wavelength of the scattered radiation,
- (ii) the kinetic energy of the recoiling electron,
- (iii) the fraction of the energy of the incident photon lost in the collision with the electron, and
- (iv) the angle made by the trajectory of the recoiling electron with the incident beam.

(c) Write down an expression for the photon's fractional loss in energy K/E in the type of collision considered in (b) and hence demonstrate with a numerical estimate that the fractional energy loss for visible light is orders of magnitude less than for gamma-ray photons. (n.b., K is kinetic energy, E is total energy)

Question 2 (Marks 30)

(a) A particle is described by the one-dimensional wavefunction

$$\psi(x) = Ae^{-a^2x^2/2}$$

(i) Normalise the wavefunction

(ii) Find the momentum space wavefunction $\phi(p)$ corresponding to $\psi(x)$

(iii) Evaluate the expectation values $\langle\psi|x^2|\psi\rangle$ and $\langle\psi|p^2|\psi\rangle$

(iv) Comment on the value of the product of your answers to (iii) above

(b) The uncertainty principle can be used to *estimate* time and frequency scales governing optical transitions. Make estimates for the following:

(i) Calculate the percentage uncertainty in the frequency $\Delta\nu/\nu$ of the emitted light when a hydrogen atom makes a transition from the fourth excited state to the ground state when the lifetime of the fourth excited state is $\tau = 2.75 \times 10^{-9}$ s.

(ii) What is the lifetime of the ground state? Give the reason for your answer.

(iii) A laser pulse has a linewidth defined as the uncertainty in the output (the spectral line) wavelength $\Delta\lambda$. Find an expression relating the linewidth of the laser's spectral line to the bandwidth $\Delta\nu$, which is the uncertainty in the frequency of the spectral line.

(iv) If the duration of the laser pulse in (iii) is 1.5×10^{-9} s, estimate the linewidth of the $\lambda = 632.8$ nm beam from a HeNe laser.

Question 3 (Marks 30)

The Schrödinger equation in one dimension is described mathematically by:

$$i\hbar\frac{\partial\Psi}{dt}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V\Psi \qquad (3.1)$$

for a specific potential energy function, *V(x,t),* and for a particle of mass *m.* Ψ is the wave function, while *x* and *t* represent space and time variables respectively. Other symbols have their usual meanings.

After some mathematical manipulation, we get the time independent Schrödinger equation (TISE):

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V\psi=E\psi \qquad (3.2)$$

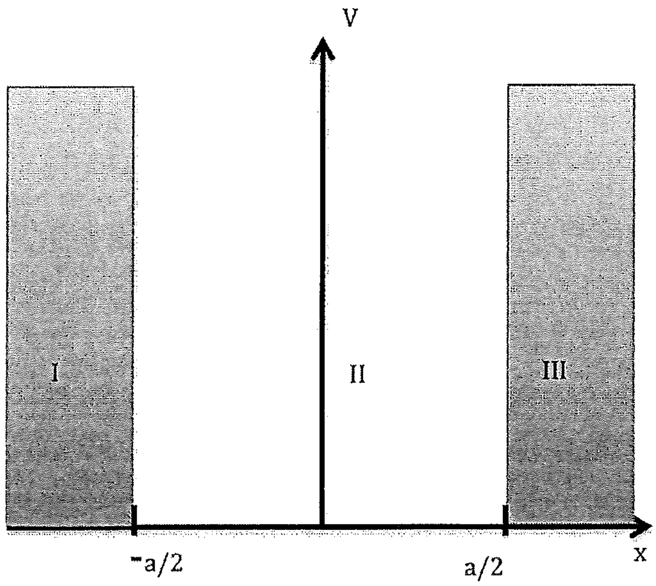
(a) While there are some mathematical manipulations required to get from Equation (3.1) to Equation (3.2), there are also some assumptions about physical quantities required.

(i) What assumptions do we make about time dependence to get from Equation (3.1) to Equation (3.2)?

(ii) Considering that Equation (3.2) is derived from Equation (3.1), how would you best describe the role of the separation constant *E* in Equation (3.2), taking into account the assumptions needed to get from Equation (3.1) to (3.2)?

(iii) Write down the Hamiltonian operator contained in Equation (3.2).

(iv) One very important physical implication for the solution to Equation (3.2) is the existence of stationary states. Define a stationary state in terms of the Hamiltonian, and explain its importance for the probability density associated with a wavefunction.



(b) Consider a particle in square well, as represented in the figure above:

(i) Assume the potential energy barriers ($x < -a/2$ or $x > a/2$) are infinite. Give the value for $\psi(x)$ in Regions I and III, and explain your reasoning, with reference to the TISE, Equation (3.2).

(ii) Give the value for *V* in Region II, and hence write down the TISE for Region II.

(iii) The solution to the equation you wrote down in (ii) can be written as a sum of sines and cosines:

$$\Psi(x) = A\sin(kx) + B\cos(kx) \qquad (3.3)$$

where A and B are arbitrary constants. Give the values that k will need to take in this equation.

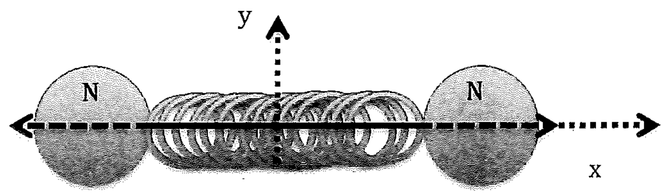
(iv) Give the boundary conditions that $\psi(x)$ must meet, for the boundaries shown in the figure. Justify your answer.

(v) Using the boundary conditions from (iv), find the range of values for the constants A and B for which Equation (3.3) will be true.

(vi) Find the energy eigenvalues for the allowed solutions, and give the first four energy levels. Sketch the wave functions for these first four energy states, marking odd and even states on your sketch.

Question 4 (Marks 30)

(a) A nitrogen molecule N_2 vibrates back and forwards along its long axis (x-axis), as shown in the figure below, resembling two particles on a spring, where the bond takes the place of the spring. For this part of the question, part (a), we will treat the molecule as a classical simple harmonic oscillator. Assume that the molecule/bond system is stretched by 0.1 nm in the x-direction, resulting in an increase in the energy of the system of 1 eV. Find the spring constant k for this bond.



(b) You are given an equation describing a Schrödinger equation with the following potential:

$$V(x)=\frac{1}{2}m\omega^2x^2-bx \tag{4.1}$$

Rather than solving the full Schrödinger equation, you can take a short cut by rewriting (4.1) as

$$V(x)=\frac{1}{2}m\omega^2(x-x_0)^2 \tag{4.2}$$

In other words, (4.2) is (4.1) with an offset in the spatial position of the potential, and a shift in the energy scale. This is useful provided you expand (4.2) so that by comparing terms with (4.1), you can find the new value for the parameter b in (4.1) in terms of the offset to the equilibrium position x_0. Once you expand (4.2) you will find an additional energy term that gives the shift in energy due to the offset in equilibrium position. Give that shift in energy in terms of the value for b you found in (4.2).

(c) For the traditional quantum harmonic oscillator with a potential of

$$V(x)=\frac{1}{2}m\omega_0^2x^2 \tag{4.3}$$

we expect a ground state wave function of the form:

$$\psi_0(x)=Ae^{-\frac{m\omega_0x^2}{2\hbar}} \tag{4.4}$$

Write down the ground state wavefunction expected for the potential in part (b), explaining your reasoning.

Question 5 (Marks 30)

(a) Observables are represented by Hermitian operators. Mathematically, this means that:

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle \qquad \text{for all } f(x) \text{ and } g(x)$$

where $\hat{Q}$ is the operator corresponding to the observable Q.

(i) In one or two sentences, explain by reference to the required properties of observables why the corresponding operator must be Hermitian.

(ii) Show that the sum of two Hermitian operators is also Hermitian

(iii) Suppose $\hat{Q}$ is Hermitian and α is a complex number. Under what condition on α will the product $\alpha\hat{Q}$ also be Hermitian?

(iv) Show that the Hamiltonian operator $\hat{H} = -(\hbar^2/2m)d^2/dx^2 + V(x)$ is Hermitian (You should assume f and g obey the normal properties of localized wavefunctions regarding what happens at $x = \pm\infty$; you may also assume $V(x)$ is real).

(b) Quantum mechanical states are defined in an n-dimensional vector space known as a Hilbert space.

(i) In one or two sentences, explain what a 'basis' is for a vector space.

(ii) What does it mean for a basis to be 'orthonormal'?

(iii) In one or two sentences, explain what the terms eigenfunction and eigenvalue mean for some arbitrary operator $\hat{A}$

(c) A classic 'two level' system in quantum mechanics is the spin-1/2 particle. Here all possible spin states can be represented in a two-dimensional vector space with basis vectors $|\chi_\uparrow\rangle = \binom{1}{0}$ and $|\chi_\downarrow\rangle = \binom{0}{1}$. (n.b., the relevant Pauli spin matrices are on the equation sheet).

(i) Suppose a spin-1/2 particle is in the state $|\chi\rangle = \frac{1}{\sqrt{6}}\binom{1+i}{2}$. If you make an observation of the z-component of the spin $\hat{S}_z = -\frac{\hbar}{2}\hat{\sigma}_z$, what are the respective probabilities of measuring $S_z = +\hbar/2$ and measuring $S_z = -\hbar/2$? Show these add to one as expected.

(ii) For the x-component of the spin $\hat{S}_x = -\frac{\hbar}{2}\hat{\sigma}_x$, find the eigenvectors $|\chi_\rightarrow\rangle$ and $|\chi_\leftarrow\rangle$ that correspond to the eigenvalues $+\hbar/2$ and $-\hbar/2$ respectively. It may help to attack this using a generic spin state $|\chi\rangle = \binom{a}{b}$. Be sure your final eigenvectors are properly normalised.

(iii) Calculate the respective probabilities of measuring $S_x = +\hbar/2$ and measuring $S_x = -\hbar/2$ if you make an observation of the x-component of the spin S_x for the spin state $|\chi\rangle$ in (i). Show these also add to one.

(iv) Calculate the expectatation value for S_x, which is obtained as $\langle S_x\rangle = \langle\chi|S_x|\chi\rangle$. Show you can get the same result using the two probabilities obtained in (iii).

Question 6 (Marks 30)

(a) Consider the quantum mechanical analog to the classical problem of a ball bouncing elastically on a horizontal plane situated at $z = 0$. Here we will assume this ball is a neutron (as this is an experiment that has actually been done). You may use eV as your energy units if you find this more convenient than J.

(i) Using your knowledge of gravity from first year physics, come up with a compelling potential energy function $V(z)$. Be careful to define this function properly for all z accounting for the plane at $z = 0$. Sketch a graph of $V(z)$ versus z.

(ii) Use the Bohr-Sommerfeld quantization rule to estimate the energy levels E_n for this system. (n.b., do not attempt an exact solution using the Schrodinger equation).

(b) A commutator of two operators $\hat{A}$ and $\hat{B}$ is defined as $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

(i) What does it mean mathematically for two operators to 'not commute'? In one or two sentences, explain what the physical implications are for non-commuting observables.

(ii) Prove that the following identity is true: $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$.

(iii) Show that $[f(x), \hat{p}] = i\hbar\frac{df}{dx}$ for any function $f(x)$. Hint: you may want to strategically introduce another function $g(x)$ to help achieve this.

(c) In the lectures we derived an important relation:

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H}, \hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle \qquad (1)$$

(i) Using the momentum operator, show that this relation can be used to give you Ehrenfest's theorem $\frac{d\langle p\rangle}{dt} = -\langle\frac{\partial V}{\partial x}\rangle$

(ii) Briefly explain what Ehrenfest's theorem means physically.

(iii) Think back to the scenario in (a) above. What intuition about the bouncing neutron can you obtain from a consideration of Ehrenfest's theorem? If this makes sense to you, explain why? If the result is instead counter-intuitive, explain why?

(iv) Suppose that $\hat{Q} = \hat{H}$ for the Equation 1 above, what principle of physics emerges?

(v) Suppose instead that $\hat{Q} = \hat{1}$, what principle of physics emerges this time? Hint: think carefully about what an expectation value is.