Practice Mid-term 2 PHYS2114.

Question 1

a) i) Laplace's equation is $\nabla^2 V = 0$. In spherical coordinates.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}.$$

given $V = \frac{A}{r} + B$ which is dependent on r but not θ or ϕ.

$\Rightarrow \frac{\partial V}{\partial \theta} = 0$ and $\frac{\partial V}{\partial \phi} = 0$.

$\Rightarrow$ second two term are zero.

$$\frac{\partial V}{\partial r} = -\frac{A}{r^2}$$

$$r^2 \frac{\partial V}{\partial r} = \frac{-r^2 A}{r^2} = -A.$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \frac{\partial}{\partial r}(A) = 0.$$

$\therefore \nabla^2 V = 0$ and this satisfies Laplace's equation.

ii) Poisson's equation tells us that $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$. As this is equal to zero $\Rightarrow$ $0 = -\frac{\rho}{\varepsilon_0} \Rightarrow \rho = 0$.

So charge density is zero inside spherical shell.

b) i) $\oint$ indicates it is an integral over a closed surface.
$\underset{\sim}{J}$ is the volume current density, $\frac{dI}{da_\perp}$, the amount of current flowing through a surface perpendicular to the current flow.
$\underset{\sim}{da}$ is an increment of surface area directed perpendicular and outwards from the ~~#~~ surface.

ii) A steady current is a continuous flow of current that has been going on forever without changing.

iii) For a steady current any charge that enters the surface must leave again. Thus the flux through the surface must be 0. If this were not true there would be a change in charge inside the surface.

$$\oint \underset{\sim}{J} \cdot \underset{\sim}{da} = \int_V (\nabla \cdot J)\, d\tau = -\int \left(\frac{\partial \rho}{\partial t}\right) d\tau = 0$$

c) i) Yes, adding free charge will induce a change in the bound charge density: $\rho_b = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f$

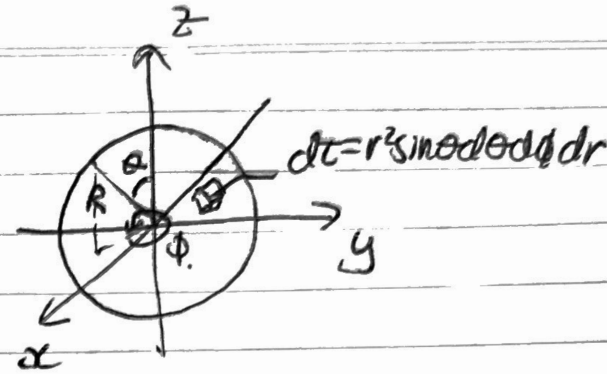
ii) Yes, this will balance the volume bound charges: bound charges in a volume is equal and opposite to those being pushed out through the surface: $\int \rho_b \, d\tau + \int \sigma_b \, dA = 0$

iii) Yes: these are the charges being added.

iv) Can't say but it seems unlikely none would be added to the surface while they are being added to the volume.

v) Described in parts above: $\rho_b = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f$; $\int \rho_b \, d\tau + \int \sigma_b \, dA = 0$
The placement of the free charges causes the bound charges to appear (separate).

Question 2



a) $Q = \int \rho \, d\tau$

Use spherical coordinates

$$Q = \int_0^{2\pi} \int_0^{\pi/2} \int_0^R k \cos\theta \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Only top half!

$$= 2\pi k \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^R \cos\theta \sin\theta \, d\theta$$

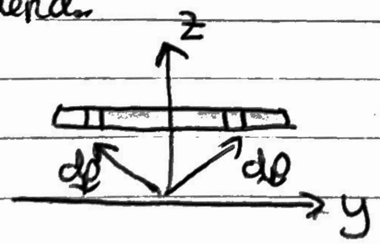
$$= \frac{2\pi k R^3}{3} \left[-\frac{1}{2}\cos^2\theta\right]_0^{\pi/2}$$

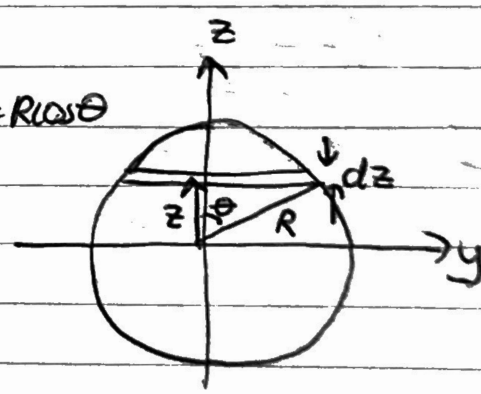
$$= \frac{\pi k R^3}{3}\left[-0+1\right]$$

$$= \frac{\pi k R^3}{3}$$

b) The negative charge in the lower half of the sphere will perfectly balance the positive charge in the top half, it is neutral. As $Q=0$ the monopole contribution dissapears.

c) $d\vec{p}$ is a vector pointing from the center of the sphere to the point with charge $dq = \rho \, d\tau$ being considered. $z = R\cos\theta$





Each point on the disk being considered has another point: $(x, y, R\cos\theta)$ and $(-x, -y, R\cos\theta)$ for which the dp_x and dp_y contributions are equal and opposite. These cancel each other out. Around the disk the dp_z contribution is always in the $+\hat{z}$ direction. These sum together.

d) $\underset{\sim}{P} = \frac{P_{tot}}{\text{volume}}$ $\Rightarrow$ need to find total dipole moment by integrating over sphere.

Have how much each disk contributes need to integrate from $z=-R$ to $z=R$. As there is a $|z|$ in the function best to split it into two sections: positive and negative z.

$$P_{tot} = \left(\int_{-R}^{0} 2\pi k z^2 (R+z) dz + \int_{0}^{R} 2\pi k z^2 (R-z) dz \right) \hat{\underset{\sim}{z}}$$

$$= 2\pi k \left(\int_{-R}^{0} (Rz^2 + z^3) dz + \int_0^R (Rz^2 - z^3) dz \right) \hat{\underset{\sim}{z}}$$

$$= 2\pi k \left(\left[\frac{Rz^3}{3} + \frac{z^4}{4} \right]_{-R}^{0} + \left[\frac{Rz^3}{3} - \frac{z^4}{4} \right]_0^R \right) \hat{\underset{\sim}{z}}$$

$$= 2\pi k \left(\frac{R^4}{3} - \frac{R^4}{4} + \frac{R^4}{3} - \frac{R^4}{4} \right) \hat{\underset{\sim}{z}}$$

$$= \cancel{2}\pi k R^4 \left(\frac{8-6}{\cancel{12}\,6} \right) \hat{\underset{\sim}{z}} = \pi k R^4 \left(\frac{2}{6} \right) \hat{\underset{\sim}{z}} = \frac{1}{3}\pi k R^4 \hat{\underset{\sim}{z}}$$

$$\Rightarrow \underset{\sim}{P} = \frac{\frac{1}{3}\cancel{\pi} k R^4 \hat{\underset{\sim}{z}}}{\frac{4}{3}\cancel{\pi} R^3} = \frac{kR}{4} \hat{\underset{\sim}{z}}$$

e) $$V_{dip} = \frac{1}{4\pi\varepsilon_0} \frac{(\frac{1}{3}\pi k R^4 \hat{\underset{\sim}{z}}) \cdot \hat{\underset{\sim}{r}}}{r^2}$$ $\hat{\underset{\sim}{z}} \cdot \hat{\underset{\sim}{r}} = \cos\theta$

$$= \frac{kR^4 \cos\theta}{12\varepsilon_0 r^2}$$

f) As the dipole contribution to the potential is not zero, it is the highest order non-zero multipole contribution and so quickly dominates the potential as r increases. Thus at large distances $V \approx V_{dip}$.

Question 3

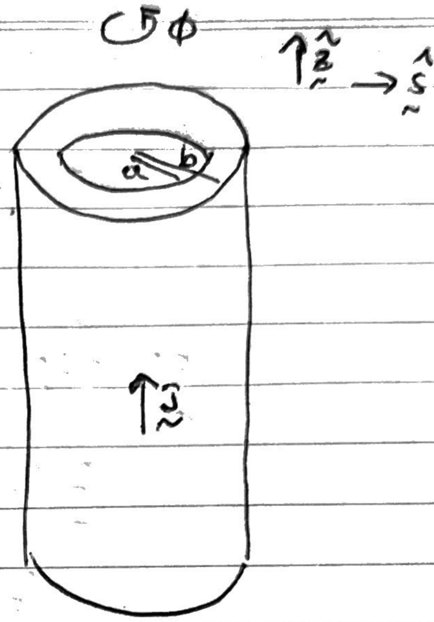
a) $\underset{\sim}{J} = \frac{dI}{da_\perp}$

$\Rightarrow \underset{\sim}{I} = \int \underset{\sim}{J} \cdot d\underset{\sim}{a}\ \hat{z}$ Use cylindrical coordinates

$= \int_0^{2\pi}\int_a^b \frac{k}{s} s\, d\phi\, ds\ \hat{z}$

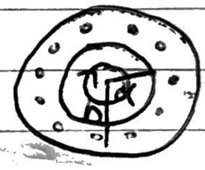
$= 2\pi k\, [s]_a^b\ \hat{z}$

$= 2\pi k (b-a)\ \hat{z}$



b) The magnetic field is in the circumferential direction. This is because of the Biot-Savart law: $\underbrace{d\underset{\sim}{I}}_{\hat{z}\text{ direction}} \times \hat{r}$ ← can choose in zs plane. $d\hat{\underset{\sim}{B}} = d\hat{\underset{\sim}{I}} \times \hat{r}$ ⇒ must be in $\hat{\phi}$ direction

From on top

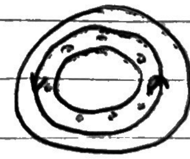


$s<a$ (inside wire)

Take Amperian loop in diagram ⟶

$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = B \cdot 2\pi s = \mu_0 I_{enc} = 0.$

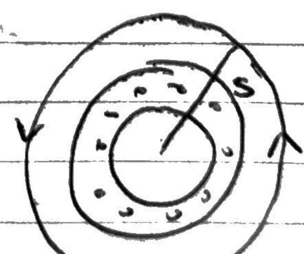
$\Rightarrow B = 0.$



$a<s<b$

$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = B \cdot 2\pi s = \mu_0 I_{enc} = \mu_0 \overbrace{2\pi k (s-a)}^{\text{Integral in part(s) with limits } a\to s.}$

$\Rightarrow \underset{\sim}{B} = \frac{\mu_0 k(s-a)}{s}\ \hat{\underset{\sim}{\phi}}$



$s>b$

$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = B\, 2\pi s = \mu_0 I_{enc} = \mu_0 2\pi k (b-a)$

$\Rightarrow \underset{\sim}{B} = \frac{\mu_0 k (b-a)}{s}\ \hat{\underset{\sim}{\phi}}$

c) $\nabla \times \underset{\sim}{A} = \underset{\sim}{B}$ and $\nabla \cdot \underset{\sim}{A} = 0$

$$\underset{\sim}{A}(s) = \mu_0 k(-s + a + a\ln(\tfrac{a}{s}))\,\hat{\underset{\sim}{z}}$$

In cylindrical coordinates.

$$\nabla \cdot \underset{\sim}{A} = \frac{\partial(\mu_0 k(-s+a+a\ln(\frac{a}{s})))}{\partial z} = 0 \quad \text{as is required.}$$

$$\nabla \times \underset{\sim}{A} = \frac{1}{s}\frac{\partial(\mu_0 k(-s+a+a\ln(\frac{a}{s})))}{\partial \phi}\hat{\underset{\sim}{s}} - \frac{\partial(\mu_0 k(-s+a+a\ln(\frac{a}{s})))}{\partial s}\hat{\underset{\sim}{\phi}}$$

$$= \mu_0 k\left(1 + \frac{\partial}{\partial s}(a\ln a - a\ln s)\right)\hat{\underset{\sim}{\phi}}$$

$$= \left(+\mu_0 k - \mu_0 k a \frac{1}{s}\right)\hat{\underset{\sim}{\phi}}$$

$$= \left(\frac{\mu_0 k s}{s} - \frac{\mu_0 k a}{s}\right)\hat{\underset{\sim}{\phi}}$$

$$= \frac{\mu_0 k(s-a)}{s}\hat{\underset{\sim}{\phi}}$$

This agrees with $\underset{\sim}{B}$ found in part (b) showing this is the correct expression for the vector potential.

d) $$\frac{\underset{\sim}{F}_{mag}}{l} = \frac{I \times \int d\underset{\sim}{l} \times \underset{\sim}{B}}{l} = \frac{2\pi k(b-a).l.B_0}{l}\hat{\underset{\sim}{y}}$$

$$= 2\pi k B_0 (b-a)\,\hat{\underset{\sim}{y}}$$