THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

MAY 2019

PHYS2111 Quantum Physics

Time Allowed – 2 hours

Total number of questions – 4

Use separate booklets for Questions 1 & 2 and Questions 3 & 4

Total marks: 100

This paper may be retained by the candidate.

Students must provide their own UNSW approved calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

The following information may be useful

Planck's constant h = 6.626×10^{-34} Js

Fundamental charge unit e = 1.60×10^{-19} C

Speed of light (vacuum) c = 3.0×10^{8} m/s

Electron mass = 9.1×10^{-31} kg = 0.511 MeV/c^2

Neutron mass = 1.675×10^{-27} kg = 939.6 MeV/c^2

Proton mass = 1.672×10^{-27} kg = 938.3 MeV/c^2

Boltzmann's constant k = 1.38×10^{-23} JK^{-1}

Permittivity constant ε_0 = 8.85×10^{-12} Fm^{-1}

Gravitational constant G = 6.67×10^{-11} Nm^2/kg^2

Angstrom 1 Å = 1.0×10^{-10} m

$h/m_e c$ = 2.43×10^{-12} m
1 eV = 1.60×10^{-19} J
1 J = 6.24×10^{18} eV

Time-independent Schrödinger Equation: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V\psi(x)=E\psi(x)$

Time-dependent Schrödinger Equation: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x)\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}$

Bohr-Sommerfeld equation: $\oint p\,dx = nh$

$$\int \sin^2(bx)dx = \frac{x}{2}-\frac{\sin(2bx)}{4b}$$

$$\int x\sin^2(x)dx = \frac{x^2}{4}-\frac{x\sin(2x)}{4}-\frac{\cos(2x)}{8}$$

$$\int x^2\sin^2(bx)dx = \frac{x^3}{6}-\left(\frac{x^2}{4b}-\frac{1}{8b^3}\right)\sin(2bx)-\frac{x\cos(2bx)}{4b^2}$$

$$\int_{-\infty}^{\infty} e^{-bx^2}dx = \sqrt{\frac{\pi}{b}}$$

$$\int_0^{\infty} x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$

$$\int_a^b f\frac{d^2g}{dx^2}dx = f\frac{dg}{dx}\bigg|_a^b - \int_a^b \frac{df}{dx}\frac{dg}{dx}dx$$

$$\int (a-bx)^{1/2}\,dx = -\frac{2}{3b}(a-bx)^{3/2}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx}+g\frac{df}{dx}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Pauli spin matrices: $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} dx$$

$$\int_{-\infty}^{\infty} e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{a}$$

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-(au^2+bu+c)} du = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-4ac}{4a}\right), a > 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a^3}$$

$$\int_{-\infty}^{\infty} p^2 |\Phi(p)|^2 dp = \frac{a^2\hbar^2}{2}$$

Ground state of the harmonic oscillator: $\psi_0(x) = \frac{1}{(\pi x_0^2)^{1/4}} e^{-x^2/2x_0^2}$, where $x_0^2 \equiv \frac{\hbar}{m\omega}$

Table of spin operator actions

$\sigma_x\|uu\rangle = \|du\rangle$	$\sigma_x\|ud\rangle = \|dd\rangle$	$\sigma_x\|du\rangle = \|uu\rangle$	$\sigma_x\|dd\rangle = \|ud\rangle$
$\sigma_y\|uu\rangle = i\|du\rangle$	$\sigma_y\|ud\rangle = i\|dd\rangle$	$\sigma_y\|du\rangle = -i\|uu\rangle$	$\sigma_y\|dd\rangle = -i\|ud\rangle$
$\sigma_z\|uu\rangle = \|uu\rangle$	$\sigma_z\|ud\rangle = \|ud\rangle$	$\sigma_z\|du\rangle = -\|du\rangle$	$\sigma_z\|dd\rangle = -\|dd\rangle$

Ladder operators

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}} (\mp ip + m\omega x)$$

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1}$$

$$a_-\psi_n = \sqrt{n}\psi_{n-1}$$

Question 1 (20 marks)

(i) A particle of mass m is coupled to a simple harmonic oscillator. The oscillator has frequency ω. At time $t = 0$ the particle's wave function $\Psi(x, t)$ is given by

$$\Psi(x,0) = \frac{e^{-x^2/(2\sigma^2)}}{(\pi\sigma^2)^{1/4}} \,. \tag{0.1}$$

The constant σ is unrelated to any other parameter. What is the probability that a measurement of energy at $t = 0$ finds the value of $E_0 = \hbar\omega/2$? For what value of σ is this probability equal to one?

(ii) Given the ladder operators and knowing their action on the stationary state of the harmonic oscillator, derive the expression of x in terms of $a_\pm$ and use it to compute $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$ (the expectation values are defined on ψ_n). Then verify that Ehrenfest's theorem, which relates expectation values of the time derivative of the momentum and the space derivative of the potential, holds.

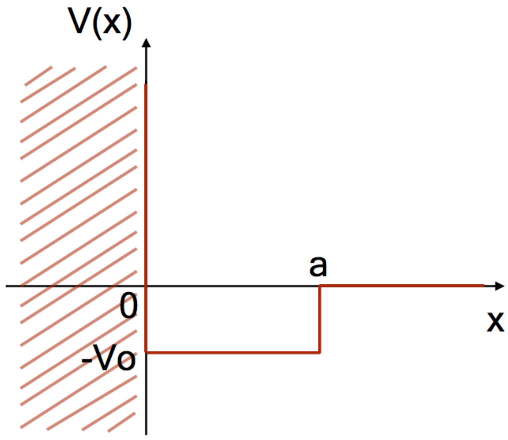
Hint: Some relevant ladder operators appear on the equation sheet.

(iii) Derive the first excited state of the harmonic oscillator using the result for the ground state.

Question 2 (30 marks)

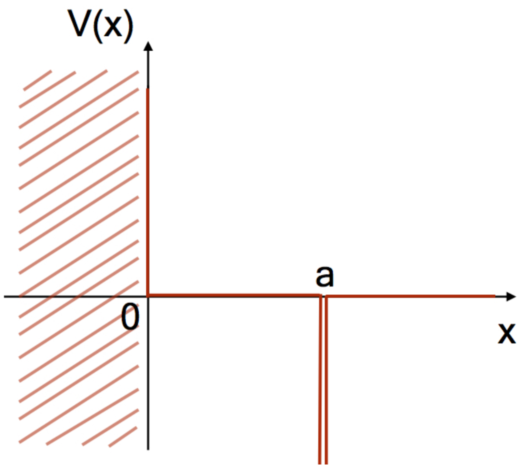
(i) A particle of mass m is confined to move in one dimension by a potential:

$$V(x) = \begin{cases} \infty & \text{if } x < 0\,, \\ -V_0 & \text{if } 0 < x < a \\ 0 & \text{if } x > a\,, \end{cases} \tag{0.2}$$



For the bound state: derive the equation relating the energy to the other physical constants (m, V_0 and a).

(ii) A particle of mass m is confined to $x > 0$ by an infinite potential at the origin. There is also a delta function potential $V(x) = -V_0\, a\, \delta(x-a)$, where $V_0 > 0$ and $a > 0$.



For the bound state: derive the equation relating the energy to the other physical constants (m, V_0 and a).

Question 3 (Marks 25)

(a) Quantum mechanical problems are often treated via formalism called matrix mechanics, where quantum mechanical states are defined in an n-dimensional vector space.

(i) In one or two sentences, explain what a 'basis' is for a vector space and what it means for a basis vector to be linearly independent.

(ii) We generally use a basis that is orthonormal, what are the two requirements for a basis to be orthonormal?

(iii) A fellow student makes the following statement "When a linear operator $\hat{L}$ operates on one of its eigenvectors $|\lambda\rangle$, the resulting vector is parallel or antiparallel to $|\lambda\rangle$." Is this statement correct? In one to two sentences, explain why.

(iv) Find the two eigenvalues λ_1 and λ_2, if the operator $\hat{L} = \begin{pmatrix} -1 & -1 \\ 2 & -4 \end{pmatrix}$

(v) Find the two corresponding eigenvectors $|\lambda_1\rangle$ and $|\lambda_2\rangle$ for $\hat{L}$.

(vi) Quickly use your answers to (iv) and (v) to demonstrate your answer to (iii). n.b., doing it for one or other of $|\lambda_1\rangle$ and $|\lambda_2\rangle$ is enough, don't waste time doing both.

(b) Observables are represented by Hermitian operators in quantum mechanics.

(i) What is the one key requirement of observables that requires operators to be strictly Hermitian in quantum mechanics?

(ii) "Probability is an observable." Is this statement true or false? Explain your reasoning in one or two sentences.

(iii) Consider the operator $\hat{L}$ again:

$$\hat{L} = \begin{pmatrix} -1 & -1 \\ 2 & -4 \end{pmatrix}$$

Is $\hat{L}$ Hermitian? If so, how do you know? If not, what is it that tells you that is isn't? (n.b. Be careful to not underthink this one).

(c) Consider a three-dimensional vector space spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. Two possible quantum states are given by the kets $|\alpha\rangle$ and $|\beta\rangle$ as:

$$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle \text{ and } |\beta\rangle = i|1\rangle + 2|3\rangle$$

(i) Construct $\langle\alpha|$ and $\langle\beta|$ in terms of the dual basis $\langle 1|$, $\langle 2|$ and $\langle 3|$. (n.b., it should be as simple as writing these down).

(ii) Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$ and confirm that $\langle\beta|\alpha\rangle^* = \langle\alpha|\beta\rangle$

(iii) Find the 3×3 matrix for the operator $\hat{O} = |\alpha\rangle\langle\beta|$ in this basis.

(iv) Is $|\alpha\rangle$ above properly normalized? If not, obtain the normalized form for $|\alpha\rangle$.

(v) Suppose we prepare the quantum state $|\alpha\rangle$, go have lunch, come back and perform a measurement. What will be the probability that we obtain the outcome $|2\rangle$?

(vi) Suppose that, without otherwise perturbing the system, we now go have dinner, come back and remeasure. What is the probability that we get $|1\rangle$? Briefly comment on why you get the answer that you do.

Question 4 (Marks 25)

(a) For any two operators $\hat{A}$ and $\hat{B}$ we can define a mathematical object called the commutator $[\hat{A}, \hat{B}]$, which plays an important role in quantum mechanics.

(i) Write an expression for the commutator $[\hat{A}, \hat{B}]$ in terms of the operators $\hat{A}$ and $\hat{B}$.

(ii) In one or two sentences explain what it means mathematically for two operators to 'not commute' and what the physical implications are for non-commuting observables.

(iii) Show that the commutator $[x, p_x^2] = 2i\hbar p_x$. (n.b., it's ok to use p instead of p_x here to save writing time, I'm just using it here to be unambiguous for later parts.)

(iv) In lectures we obtain the generalized uncertainty principle as: $\Delta A \Delta B \geq \left| \frac{i}{2} \langle [\hat{A}, \hat{B}] \rangle \right|$

Use this to obtain the well-known position-momentum uncertainty relation $\Delta x \Delta p_x \geq \hbar/2$. (n.b., you may reuse a result you get in (iii) for this)

(v) You can measure p_y and x simultaneously with absolute certainty because the corresponding commutator is zero. Briefly explain why this commutator is zero. (n.b. don't repeat algebra if you've already done it, e.g., in (iii), just refer back and explain the reason).

(vi) Suppose the product of two Hermitian operators $\hat{A}$ and $\hat{B}$ is also Hermitian. What requirement does it place on the corresponding commutator? Make a comprehensive algebraic argument to support your case, i.e., just stating the answer is not quite enough.

Hint: It may be very useful to approach this from the perspective of the expectation value of the product operator $\hat{C} = \widehat{A}\hat{B}$ and consider what happens when transferring operators from the bra side to the ket side.

(vii) I now want you to take (v) one step further and consider it in terms of your answer for (vi). Comment briefly from this perspective on why it makes sense that p_y and x are simultaneous observables but p_x and x or p_y and y are not.

(b) Suppose Alice and Bob each have a quantum spin. Alice's spin is prepared in a state $|\Psi_A\rangle = \alpha_u |u\rangle + \alpha_d |d\rangle$ and Bob's is prepared in a state $|\Psi_B\rangle = \beta_u |u\rangle + \beta_d |d\rangle$.

(i) Write an expression for the product state $|\Psi_P\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$ in terms of kets of the form of $|uu\rangle$, $|ud\rangle$, etc.

(ii) We can also define a singlet state $|\Psi_S\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$. In lectures we showed that a singlet state cannot be a product state by treating the singlet state as a product state and showing this generates an inconsistency in the prefactors α, β.

For a product state, the spin polarization principle $\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 = 1$ holds. This provides an alternate route to proving that a singlet state is not a product state. Quickly obtain $\langle \sigma_x \rangle$, $\langle \sigma_y \rangle$ and $\langle \sigma_z \rangle$ for the singlet state to show that this state violates the spin polarization and is thus not a product state.

Hints: the table of spin operator actions in the equation sheet will be very helpful. You may ignore all $1/\sqrt{2}$'s for efficiency. Avoid writing huge repetitive strings of algebra, just do enough to get to a point where you can reason to the answer.

(iii) Comment in one or two sentences on the significance of the singlet state in quantum mechanics.