THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2018

PHYS2111 Quantum Physics

Time Allowed – 2 hours

Total number of questions – 6

Use separate booklets for Questions 1 & 2, Questions 3 & 4, and Questions 5 & 6

Total marks: 120 – Questions are all of equal value (20 marks)

This paper may be retained by the candidate.

Students must provide their own UNSW approved calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

The following information may be useful

Planck's constant $h = 6.626 \times 10^{-34}$ Js

Fundamental charge unit $e = 1.60 \times 10^{-19}$ C

Speed of light (vacuum) $c = 3.0 \times 10^{8}$ m/s

Electron mass = 9.1×10^{-31} kg = 0.511 MeV/c^2

Neutron mass = 1.675×10^{-27} kg = 939.6 MeV/c^2

Proton mass = 1.672×10^{-27} kg = 938.3 MeV/c^2

Boltzmann's constant $k = 1.38 \times 10^{-23}$ JK^{-1}

Permittivity constant $\varepsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1}

Gravitational constant $G = 6.67 \times 10^{-11}$ Nm^2/kg^2

Angstrom 1 Å = 1.0×10^{-10} m

$h/m_e c = 2.43 \times 10^{-12}$ m
1 eV = 1.60×10^{-19} J
1 J = 6.24×10^{18} eV

Time-independent Schrödinger Equation: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V\psi(x) = E\psi(x)$

Time-dependent Schrödinger Equation: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$

Bragg's law: $n\lambda = 2d\sin\theta$

Compton Shift: $\Delta\lambda = \frac{h}{mc}(1-\cos\theta)$

Bohr-Sommerfeld equation: $\oint p\,dx = nh$

$$\int \sin^2(bx)dx = \frac{x}{2} - \frac{\sin(2bx)}{4b}$$

$$\int x\sin^2(x)dx = \frac{x^2}{4} - \frac{x\sin(2x)}{4} - \frac{\cos(2x)}{8}$$

$$\int x^2\sin^2(bx)dx = \frac{x^3}{6} - \left(\frac{x^2}{4b} - \frac{1}{8b^3}\right)\sin(2bx) - \frac{x\cos(2bx)}{4b^2}$$

$$\int_{-\infty}^{\infty} e^{-bx^2}dx = \sqrt{\frac{\pi}{b}}$$

$$\int_0^{\infty} x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$

$$\int_a^b f\frac{d^2g}{dx^2}dx = f\frac{dg}{dx}\bigg|_a^b - \int_a^b \frac{df}{dx}\frac{dg}{dx}dx$$

$$\int (a - bx)^{1/2}\, dx = -\frac{2}{3b}(a - bx)^{3/2}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Pauli spin matrices: $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Visible light $\lambda \sim 400 - 700$ nm

$$E^2 = (pc)^2 + (mc^2)^2$$

$$p = mv/\sqrt{1 - (v/c)^2}$$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} dx$$

$$\int_{-\infty}^{\infty} e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{a}$$

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-(au^2+bu+c)} du = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2 - 4ac}{4a}\right), a > 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a^3}$$

$$\int_{-\infty}^{\infty} p^2 |\Phi(p)|^2 dp = \frac{a^2\hbar^2}{2}$$

A classical oscillator has frequency $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ and energy $E = \frac{1}{2}kA^2$

Question 1 (Marks 20)

(a) Derive the expression for the total energy density of a black body radiator as a function of temperature.

(b) The Compton effect describes the increase in the wavelength of a photon when it is scattered by a charged particle, usually an electron:

(i) Sketch the geometry for the Compton effect, include relevant angles and expressions for the momenta and energy.

(ii) Write down -- but do not solve -- expressions for energy and momentum conservation describing the photon-electron collision in the Compton effect

(iii) Consider a photon colliding with a stationary, free electron that is assumed to be at rest before the collision. If the electron's speed after the collision is αc, where c is the speed of light and $\alpha << 1$, show that the electron's kinetic energy is a fraction α of the incident photon energy.

Question 2 (Marks 20)

(a) Obtain expressions for the energy levels and wave functions for a particle confined inside a potential 'box' with side lengths a, b and c, as shown in Fig. 1.

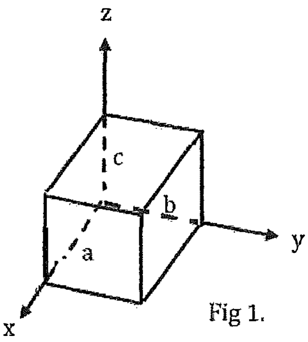


Fig 1.

(b) A particle in the ground state confined to the region $0 \le x \le a$ and is described by the wave function:

$$\psi(x) = A\sin\left(\frac{\pi x}{a}\right)$$

where A is a normalisation constant.

(i) The normalisation condition gives:

$$\int_0^a |\psi|2dx = \frac{A^2}{2}\int_0^a dx - \frac{A^2}{2}\int_0^a \cos\left(\frac{2\pi x}{a}\right)dx$$

where the second term does not contribute since the integral:

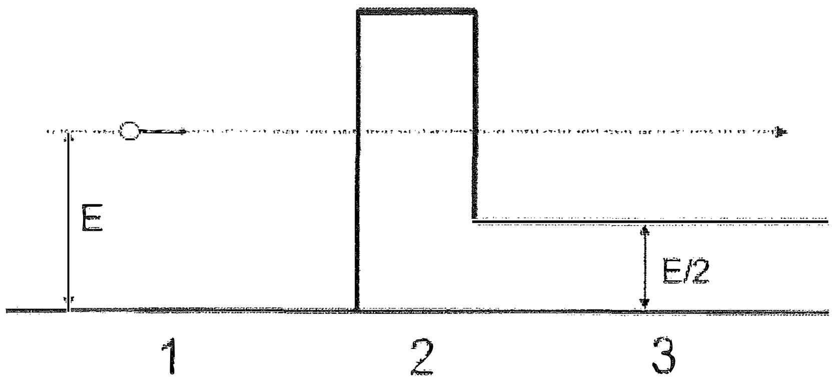
$$\int_0^a \cos\left(\frac{2\pi x}{a}\right)dx = 0$$

Hence write the full expression for the normalized wavefunction.

(ii) With the wavefunction described in Part (i) above, calculate the probability that the particle will be found in the interval $a/2 \le x \le 3a/4$.

Question 3 (Marks 20)

A particle with an energy E is tunneling through a potential barrier (see figure below). Note that such a tunneling process takes place for example at a metal-insulator-semiconductor thin film structure used in semiconducting industry.



(a) Sketch the form of the wavefunction expected in each of the 3 regions and give the functional forms of these wavefunctions. It is not required to give the precise mathematical expression of the final wavefunctions but the type of the waves. Explain the choice of the wavefunction for each region in a few words.

(b) What are the mathematical boundary conditions for the waves when the particle enters and leaves the barrier?

(c) Derive the equations that must be solved to obtain the constants appearing in the wavefunctions (do not actually solve for them).

(d) If the potential energy in Region 3 is exactly $E/2$, calculate the ratio of the wavelength in Regions 1 and 3.

Question 4 (Marks 20)

The Schrödinger equation of a harmonic oscillator is given by:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+\frac{1}{2}m\omega^2x^2\psi(x) \quad = \quad E\psi(x)$$

and the first eigenfunctions are given by:

$$\psi_0(u) \quad = \quad A_0\ e^{-u^2/2}$$
$$\psi_1(u) \quad = \quad A_1\ u\ e^{-u^2/2}$$

where $u = (m\omega/\hbar)^{1/2}x$

(a) Calculate the normalization constants A_0 and A_1.

(b) For $\psi_0(u)$ and $\psi_1(u)$: Calculate the expectation values of $\bar{x}$ and $\bar{p}$.

(c) Calculate the energy of the wavefunction ψ_0.

(d) An atom of the mass $m = 4.85\times10^{-23}$ g is vibrating. The energy of a vibrational level is $E = 7.2$ meV. What is the amplitude (displacement) of the atom in the classical limit?

Question 5 (Marks 20)

(a) Quantum mechanical states are defined in an n-dimensional vector space known as a Hilbert space.

(i) In one or two sentences, explain what a 'basis' is for a vector space.

(ii) What does it mean for a basis to be 'orthonormal'?

(iii) In one or two sentences, explain what the terms eigenfunction and eigenvalue mean for some arbitrary operator $\hat{A}$.

(iv) A fellow student says "An eigenfunction, eigenvector and an eigenstate are really just different ways of writing the same physical thing in quantum mechanics." Is this correct?

(b) A beam of spin-½ electrons goes through a series of Stern-Gerlach type measurements (i.e., like the 'Q-meter' we discussed in lectures) as follows:

1. The first measurement accepts $s_z = +\hbar/2$ (spin up) and rejects $s_z = -\hbar/2$ (spin down).
2. The second measurement accepts $s_n = +\hbar/2$ and rejects $s_n = -\hbar/2$, where s_n is the eigenvalue of the operator $\mathbf{S}.\hat{\boldsymbol{n}}$ with $\hat{\boldsymbol{n}}$ making an angle β with respect to the z-axis in the xz-plane.
3. The third measurement accepts $S_z = -\hbar/2$ and rejects $S_z = +\hbar/2$.

The incident beam is not spin-polarized in any way (i.e., spins point randomly).

(i) Write an expression for the basis state $|s_n+\rangle$ for the second measurement in terms of $|u\rangle$ and $|d\rangle$ and the angle β. (n.b. this state corresponds to 'spin up' along n, and there is a corresponding state $|s_n-\rangle$ for 'spin down'. You may ignore $|s_n-\rangle$ here.)

(ii) What is the intensity of the final $S_z = -\hbar/2$ beam if the intensity of the incoming beam is normalized to 1? This should be an expression in terms of β.

(iii) What angle β would you use to maximise the intensity of the final $S_z = -\hbar/2$ beam? What would the maximum intensity be?

(c) Assume a generic spin state $|\chi\rangle = a|u\rangle + b|d\rangle$

(i) Find the eigenvalues and eigenvectors $|\chi_+\rangle$ and $|\chi_-\rangle$ for $S_y = (\hbar/2)\sigma_y$. Show all working; simply stating the correct answer will not get any marks.

(ii) If you measured S_y on a particle in the general state $|\chi\rangle$, what values might you get and what is the probability of each? Show that the probabilities add up to 1.

(iii) If you measured S_y^2, what values might you get and with what probability?

Question 6 (Marks 20)

(a) A commutator of two operators $\hat{A}$ and $\hat{B}$ is defined as $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

(i) What does it mean mathematically for two operators to 'not commute'? In one or two sentences, explain what the physical implications are for non-commuting observables.

(ii) Show that $[\sigma_x, \sigma_y] = 2i\sigma_z$.

(iii) Show that $(\hat{A}\hat{B})^{\dagger} = \hat{B}^{\dagger}\hat{A}^{\dagger}$ (n.b. the matrix relation $(AB)^T = B^T A^T$ may be useful)

(iv) Show also that $[\hat{A}, \hat{B}]^{\dagger} = -[\hat{A}, \hat{B}]$ providing both $\hat{A}$ and $\hat{B}$ are Hermitian (n.b., the answer to (iii) will be helpful here)

(b) For two observables L and M we can obtain the generalized uncertainty principle as:

$$\Delta L \Delta M = \left| \frac{i}{2} \langle [\hat{L}, \hat{M}] \rangle \right|$$

(i) Assume a single spin prepared in the state $|\chi\rangle = |u\rangle$. Use the generalized uncertainty principle to show that $\Delta S_x \Delta S_y \geq \hbar^2/4$. Note, you will need to convert the result given in Question 6(a)(ii) above to $[S_x, S_y]$ to do this; be sure to show working. You will also need $\langle S_z \rangle$, briefly explain the value that you use for this.

(ii) Suppose instead that the single spin is in the arbitrary state $|\chi\rangle = a|u\rangle + b|d\rangle$. What will the uncertainty relation for $\Delta S_x \Delta S_y$ become? Briefly explain the physical meaning of this result.

(c) An important principle in quantum mechanics is *completeness*. In particular, the eigenvectors of an observable operator are *complete*, which means any state can be written as a linear combination of the set of eigenvectors.

For observables A and B, there exists a set of simultaneous eigenvectors of A and B $\{|a,b\rangle\}$. Is it possible from this to always conclude that $[A,B] = 0$. If your answer is yes, prove the assertion. If the answer is no, give a counterexample.