# THE UNIVERSITY OF NEW SOUTH WALES

## SCHOOL OF PHYSICS

PHYS2111 Quantum Mechanics

Final Exam

Monday 29th April 2024

2 hour written invigilated exam.

Total marks: 80 (pro-rata to 60% of course assessment)

Please note:

- Provide your answers to Q1 and Q2 in one exam book and your answers to Q3 and Q4 in a SEPARATE exam book as the two halves of the exam will be marked by different markers.
- Please write your name and student number on the first page of each exam book.
- Answer all questions concisely and legibly.
- Questions are not all worth the same marks; questions are worth the marks shown.
- You may use a university-approved calculator.

**Formula Sheet**

Planck's constant $h = 6.626 \times 10^{-34}$ Js

Fundamental charge unit $e = 1.60 \times 10^{-19}$ C

Mass of the electron = $9.11 \times 10^{-31}$ kg

Mass of the proton & neutron = $1.67 \times 10^{-27}$ kg

Boltzmann's constant = $1.380649 \times 10^{-23}$ m$^2$ kg s$^{-2}$ K$^{-1}$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Pauli spin matrices: $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[A,B] = AB - BA$

**Table of spin operator actions**

| | |
|---|---|
| $\sigma_x\lvert u\rangle = \lvert d\rangle$ | $\sigma_x\lvert d\rangle = \lvert u\rangle$ |
| $\sigma_y\lvert u\rangle = i\lvert d\rangle$ | $\sigma_y\lvert d\rangle = -i\lvert u\rangle$ |
| $\sigma_z\lvert u\rangle = \lvert u\rangle$ | $\sigma_z\lvert d\rangle = -\lvert d\rangle$ |

**Helpful Standard Integrals**

Exponential Integral: $\int_0^\infty x^n e^{-x/a} dx = n!\, a^{n+1}$

Gaussian Integrals: $\int_0^\infty x^{2n} e^{-x^2/a^2} dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$

$$\int_0^\infty x^{2n+1} e^{-x^2/a^2} dx = \frac{n!}{2}a^{2n+2}$$

**Ladder Operators**

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$$

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1}$$

$$a_-\psi_n = \sqrt{n}\psi_{n-1}$$

**Quantum Harmonic Oscillator**

$$\psi_m = \sqrt[4]{\frac{m\omega}{\pi\hbar}}(2^n n!)^{-1/2} H_m(u) e^{-u^2/2} \quad m = 0, 1, 2 \ldots \quad u = \sqrt{m\omega/\hbar}\, x$$

$$H_0 = 1, \quad H_1 = 2u, \quad H_2 = 4u^2 - 2, \quad H_3 = 8u^3 - 12u, \quad \ldots$$

**Question 1:** [20 marks total]

(a) Quantum mechanical states are defined in an $n$-dimensional vector space known as a Hilbert space.

(i) In one or two sentences, explain what a 'basis' is for a vector space. [2 marks]

(ii) What does it mean for a basis to be 'orthonormal'? [2 marks]

(iii) In one or two sentences, explain what the terms eigenvector and eigenvalue mean for some arbitrary operator $\hat{A}$. [2 marks]

(b) For a quantum spin system with spin ½, all possible spin states can be represented in a two-dimensional complex vector space with basis vectors $|u\rangle = \binom{1}{0}$ and $|d\rangle = \binom{0}{1}$.

(i) Using the Pauli matrix $\hat{\sigma}_z$ given in the equation sheet, explain briefly what the two possible outcomes for measuring the $z$-component of the spin would be. In your answer, make it clear what the two eigenvalues and their corresponding eigenvectors mean physically. [2 marks]

(ii) Suppose a quantum spin is in the state $|\chi\rangle = \frac{1}{\sqrt{6}}\binom{2}{1-i}$. If you make an observation of the $z$-component of the spin $\hat{\sigma}_z$, then what is the probability $P_u$ of obtaining spin up? What is the probability $P_d$ of obtaining spin down? [3 marks]

(iii) For the $x$-component of the spin $\hat{\sigma}_x$, show that the vector $|r\rangle = \frac{1}{\sqrt{2}}[|u\rangle + |d\rangle]$ is an eigenvector but $|u\rangle$ is not. Is this a problem? If so, explain why in one or two sentences. If not, then also explain why in one or two sentences. [3 marks]

(iv) Calculate the respective probabilities of measuring the two observable outcomes $|l\rangle$ and $|r\rangle$ for the $x$-component of the spin for $|\chi\rangle$. [2 marks]

(v) Calculate the expectation value for $\sigma_x$, which is obtained as $\langle\sigma_x\rangle = \langle\chi|\sigma_x|\chi\rangle$. Show you can get the same result using the two probabilities obtained in (iv). [4 marks]

**Question 2:** [20 marks total]

(a) Suppose Alice and Bob each have a quantum spin. Alice's spin is prepared in a state $|\Psi_A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$ and Bob's is prepared in a state $|\Psi_B\rangle = \beta_u|u\rangle + \beta_d|d\rangle$.

(i) Write an expression for the product state $|\Psi_P\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$ in terms of kets of the form of $|uu\rangle$, $|ud\rangle$, etc. [3 marks]

(ii) We can also define a singlet state $|\Psi_S\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$. In lectures we showed that a singlet state cannot be a product state by treating the singlet state as a product state and showing this generates an inconsistency in the prefactors α,β.

For a product state, the spin polarization principle $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1$ holds. This provides an alternate route to proving that a singlet state is not a product state. Quickly obtain $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ for the singlet state to show that this state violates the spin polarization and is thus not a product state. [4 marks]

Hints: the table of spin operator actions in the equation sheet will be very helpful. You may ignore all $1/\sqrt{2}$'s for efficiency. Avoid writing huge repetitive strings of algebra, just do enough to get to a point where you can clearly reason to the answer.

(iii) Comment in one or two sentences on the significance of the singlet state in quantum mechanics. [2 marks]

(iv) In one or two sentences, explain what a partially entangled state is. [2 marks]

(b) In a few sentences at most, explain what the concept of completeness means within the context of quantum mechanics. [2 marks]

(c) Suppose I propose to write an expression for completeness as:

$$\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

where $\psi_n(x)$ are eigenfunctions and $c_n$ are complex coefficients. Is this expression correct? Why/Why not? Explain in one or two sentences at most. [2 marks]

(d) Suppose that operators $\hat{P}$ and $\hat{Q}$ have a common set of common eigenvectors:

$$\hat{P}\psi_n(x) = p_n\psi_n(x) \qquad \text{and} \qquad \hat{Q}\psi_n(x) = q_n\psi_n(x)$$

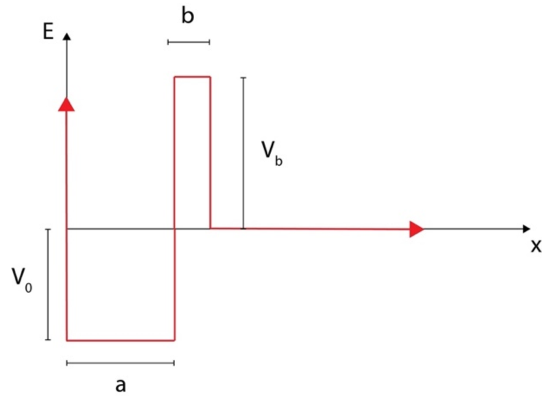
Show that the commutator $[\hat{P}, \hat{Q}] = 0$ for any function $\psi(x)$ in the Hilbert space. [5 marks]

**Question 3:** [20 marks total]

Alpha decay in nuclear physics occurs when a high energy alpha particle $He^{2+}$, consisting of two protons and two neutrons, is ejected from the nucleus along with some energy, leaving behind a new element. Let's consider the process of the alpha decay from americium, ${}^{243}_{95}Am$.

a) The alpha particle emitted during the main decay path has an energy of 5.48 MeV. Use the de Broglie relation to determine the length scale on which quantum mechanical effects are observed. How does this compare with the size of the nucleus which is approximately 5-10 fm?

Let's approximate the nuclear potential holding the alpha particle inside the nucleus in terms of one-dimensional square wells and barriers. This will appear as in the diagram below:




It consists of a finite potential well of depth $V_0 = -10$ MeV and width of $a = 5$ fm, next to a potential barrier of height $V_b$ and width $b = 1$ fm. At $x = 0$, the potential barrier is infinite, and at distances greater than the barrier the potential energy is 0.

b) Using the time-independent Schrödinger equation, write down an expression for the wave function and wavevector in each region. DO NOT SOLVE!

c) Assuming initially that the barrier is infinitely high ($V_b = \infty$), determine the lowest quantum state and corresponding energy that would be greater than 0.

d) Assume now that the barrier is large, but finite, describe *qualitatively* how the energy eigenvalues and corresponding wavefunction would be modified compared to the results in c).

In the case where the probability of transmitting through the barrier is very low, we can approximate the wavefunction inside the barrier by:

$$\psi(x) \sim \psi(a)e^{-\kappa(x-a)}$$

where $\kappa$ is the wavevector from b)

e) If the barrier height is 10 MeV and the thickness is 10 fm, what is the approximate probability that the alpha in the quantum state predicted in c) will have of tunnelling through the barrier?

**Question 4:** [20 marks total]

Consider a one-dimensional harmonic oscillator described by the potential:

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

A particle in the harmonic oscillator potential starts out in the state:

$$\Psi(x, 0) = A[6\psi_0(x) + 8\psi_1(x)]$$

where $\psi_0$ and $\psi_1$ are the lowest two stationary eigenstates.

a) Use the normalisation condition to find $A$.

b) Write down expressions for $\Psi(x, t)$ in terms of $\psi_0$ and $\psi_1$

c) If you measure the energy of the particle, what values might you get and with what probability.

d) Use the analytical form of $\psi_0$ and $\psi_1$ given in the formula sheet to calculate $\langle x \rangle$.

e) Use Ehrenfest's theorem together with the results in d) to determine $\langle p \rangle$.