UNSW SCHOOL OF PHYSICS

PHYS2111 – Quantum Mechanics

Tutorial 6 SOLUTIONS

Marko Beocanin

Question 1

Consider a two-level system with a Hamiltonian given in some basis as

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} .$$

What values of energy E allow stationary solutions of Schrödinger's equation?

A state is stationary if and only if it is an eigenstate of the Hamiltonian, in which case E must be some eigenvalue of $\hat{H}$. However, since $\hat{H}$ is already diagonal, the values of E are simply given by $\varepsilon_1, \varepsilon_2$.

Bonus comments for interested students: The "if and only if" part of this hasn't been fully elaborated in class, so I wanted to provide a small proof (basically following the answer by jacob1729 on this Quora page) that a state is stationary (ie. all expectation values are independent of time) **if and only if** it is an eigenstate of the Hamiltonian.

The (eigenstate $\implies$ stationary) direction of this proof is easy, and was outlined in class; eigenstates of the Hamiltonian can be written $|\psi(t)\rangle = |\psi_0\rangle e^{-iEt/\hbar}$, in which case it's easy to show that the exponential factors cancel out in any expectation value $\langle\psi| \hat{A} |\psi\rangle$. The other way (at least to me) is a little less obvious.

What about the (stationary $\implies$ eigenstate) direction? Well, a convenient result which you saw in Tutorial 4 is that the time-evolution of expectation values is governed by the equation

$$\frac{d}{dt}\left\langle \hat{Q} \right\rangle = \frac{i}{\hbar}\left\langle \left[\hat{H}, \hat{Q}\right] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle .$$

We've only been looking at time-independent operators, so let's assume that the final term $\frac{\partial \hat{Q}}{\partial t} = 0$; that is, that the operator has no explicit time-dependence.

Then, for a state to be stationary (ie. all expectation values are constant in time), we need the left hand side to be zero for all operators $\hat{Q}$. That means that the right hand side must also be zero, and so we have that

$$\left\langle \left[\hat{H}, \hat{Q}\right] \right\rangle = 0$$

for all operators $\hat{Q}$. Recalling our definition of expectation values, this means that for our stationary state $|\psi\rangle$, we have

$$\langle\psi| \left[\hat{H}, \hat{Q}\right] |\psi\rangle = 0.$$

Now, since $\hat{H}$ is Hermitian, we can always expand $|\psi\rangle$ in an orthonormal eigenbasis of the Hamiltonian; that is $|\psi\rangle = \sum_i c_i |\phi_i\rangle$, where $\hat{H}|\phi_i\rangle = E_i |\phi_i\rangle$. Plugging this in and representing $\hat{Q}$ as Q_{mn} in a basis of energy eigenstates, we get

$$\sum_{m,n} c_m^* c_n (E_m - E_n) Q_{mn} = 0.$$

This has to be zero for any Q_{mn}, and so each term in the sum has to vanish, and in particular each coefficient of Q_{mn} has to vanish; this gives

$$c_m^* c_n (E_m - E_n) = 0.$$

There are two possibilities: $E_m - E_n = 0$, in which case c_m, c_n may be non-zero; or $c_m^* c_n = 0$, which necessitates $c_m = c_n = 0$.

To understand this, let's think first of the case where the Hamiltonian has no degenerate energy levels. Then immediately we see that $|\psi\rangle$ has a well-defined energy E_m and is thus given by $|\psi\rangle = |\phi_m\rangle$ (up to some phase). But even in the case of a Hamiltonian with degenerate eigenvalues, we then have that $|\psi\rangle = \sum_i c_i |\phi_i\rangle$ where each $|\phi_i\rangle$ has degenerate energy $E_i = E_m$. But this is also an eigenstate of the Hamiltonian!

So, in both cases, $|\psi\rangle$ is an eigenstate of the Hamiltonian, and so we have proved our result.

Question 2

A Hamiltonian for a two-level system with interaction is given by

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & V \\ V^* & \varepsilon_2 \end{pmatrix} .$$

where V is a complex number. What are the energy eigenvalues and corresponding eigenstates?

This is just an eigenvalue/eigenvector problem. We have

$$\det\left(\begin{pmatrix} \varepsilon_1 - E & V \\ V^* & \varepsilon_2 - E \end{pmatrix}\right) = (\varepsilon_1 - E)(\varepsilon_2 - E) - |V|^2 = 0.$$

Expanding, this gives

$$E^2 - (\varepsilon_1 + \varepsilon_2)E + \varepsilon_1\varepsilon_2 - |V|^2 = 0$$

which can be solved using the quadratic formula and rearranged nicely to give two energy eigenvalues

$$E_\pm = \frac{\varepsilon_1 + \varepsilon_2 \pm \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + 4|V|^2)}}{2}.$$

The eigenvectors are quite ugly and unillustrative, unless you do a lot of massaging which this question doesn't ask for. So, I leave you with the un-normalized eigenvectors (to normalize them, just divide by the length):

$$\text{unnormalised } |E_\pm\rangle = \begin{pmatrix} -\frac{\varepsilon_2-\varepsilon_1\mp\sqrt{(\varepsilon_2-\varepsilon_1)^2+4|V|^2}}{2V^*} \\ 1 \end{pmatrix}$$

Question 3

A function defined on the semi-infinite domain $(0,\infty)$ is finite everywhere on the domain. It obeys the limit

$$\lim_{x\to\infty} f(x) \sim \frac{1}{x^a}.$$

Under what conditions on a is $f(x)$ square-integrable?

For a function $f(x)$ to be square integrable on this domain, we need the integral

$$\int_0^\infty |f(x)|^2\,dx < \infty.$$

Notice that we have only been given the limiting behaviour of this function as $x\to\infty$, and so we're going to assume that there's no pathological limiting behaviour near the origin (eg. $\lim_{x\to 0} f(x) \sim 1/x$, whose square integral would diverge at the origin).

With that, then, we have that

$$|f(x)|^2 \sim \frac{1}{x^{2a}}$$

as $x\to\infty$, in which case

$$\int_0^t |f(x)|^2\,dx \sim \begin{cases} \frac{1}{-2a+1}t^{-2a+1} & a\neq\frac{1}{2} \\ \ln t & a=\frac{1}{2} \end{cases}$$

as the upper bound $t\to\infty$. For this integral to converge and be finite as $t\to\infty$, then, we clearly need

$$a > \frac{1}{2}$$

which strict exclusion of $a = 1/2$ since the logarithm diverges (albeit very slowly) at infinity.

Question 4

In maths you may remember that any function can be expanded in powers of x (the Taylor expansion). We can use the set of polynomials as a basis on some finite interval

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

but the polynomials x^i are not orthonormal. Use Gram-Schmidt to orthogonalise the first few powers of x on the interval $-1\le x\le 1$. Start with the function $|e_0\rangle \sim x^0 = 1$ and normalise it. Then normalise and orthogonalise $|e_1\rangle \sim x^1$, etc.

Answer:

$$|e_n\rangle = \sqrt{n+1/2}\; P_n(x), \quad (n = 0, 1, 2, ...)$$

where $P_n(x)$ are the Legendre polynomials.

Let's do this up to $\sim x^2$ to get the point across; I leave the general result to you. Let's start with the first vector

$$|e_0\rangle = 1$$

which needs to be normalized. This gives

$$\begin{aligned}|0\rangle &= \frac{|e_0\rangle}{\sqrt{\langle e_0|e_0\rangle}}\\ &= \frac{1}{\sqrt{\int_{-1}^{1} 1dt}}\\ &= \frac{1}{\sqrt{2}}.\end{aligned}$$

where I'll do all my integrals over t so you don't confuse them with the functions of x.

Now, we consider our second vector $|e_1\rangle = x$. As usual, we define the (unnormalized vector)

$$\begin{aligned}|1'\rangle &= |e_1\rangle - |0\rangle\left(\langle 0|e_1\rangle\right)\\ &= x - \frac{1}{\sqrt{2}}\left(\int_{-1}^{1} dt \frac{t}{\sqrt{2}}\right)\\ &= x\end{aligned}$$

since t is an odd function integrated over a symmetric domain, so the inner product integral vanishes. All that's left to get our second vector $|1\rangle$ is to normalize it; this gives

$$\begin{aligned}|1\rangle &= \frac{|1'\rangle}{\sqrt{\langle 1'|1'\rangle}}\\ &= \frac{x}{\sqrt{\int_{-1}^{1} t^2 dt}}\\ &= \sqrt{\frac{3}{2}}x.\end{aligned}$$

Finally, let's do $|e_2\rangle = x^2$. We have our unnormalized vector

$$\begin{aligned}|2'\rangle &= |e_2\rangle - |1\rangle(\langle 1|e_2\rangle) - |0\rangle\left(\langle 0|e_2\rangle\right)\\ &= x^2 - \frac{3}{2}x\int_{-1}^{1} t^3 dt - \frac{1}{2}\int_{-1}^{1} t^2 dt\\ &= x^2 - \frac{1}{3}.\end{aligned}$$

where the second integral vanishes by the oddness of t^3. All that's left is to normalize it, giving

$$\begin{aligned}|2\rangle &= \frac{|2'\rangle}{\sqrt{\langle 2'|2'\rangle}} \\ &= \frac{x^2-\frac{1}{3}}{\sqrt{\int_{-1}^{1}(t^2-\frac{1}{3})^2}} \\ &= \sqrt{\frac{5}{8}}(3x^2-1).\end{aligned}$$

And so on.

Question 5

(Griffith's) Consider the set of all functions of the form $p(x)e^{-x^2/2}$, where $p(x)$ is a polynomial of degree $< N$ in x, on the interval $-\infty < x < \infty$. Check that they constitute an inner product space. The "natural" basis is

$$|e_0\rangle = e^{-x^2/2}, |e_1\rangle = xe^{-x^2/2}, |e_2\rangle = x^2e^{-x^2/2}, ...$$

Orthonormalize the first four of these, and comment on the result.

I leave it to you to show that this space constitutes an inner product space, with inner product given by

$$\left\langle p_1(x)e^{-x^2/2}|p_2(x)e^{-x^2/2}\right\rangle = \int_{-\infty}^{\infty} p_1(x)p_2(x)e^{-x^2}dx.$$

Now, this is basically the same sort of question as Question 4, so we'll zoom through it. I leave you to check all the integrals (and send me an email when you undoubtedly find a mistake here!). We have

$$\begin{aligned}|1\rangle &= \frac{e^{-x^2/2}}{\sqrt{\int_{-\infty}^{\infty} e^{-t^2}dt}} \\ &= \frac{1}{\pi^{1/4}}e^{-x^2/2}.\end{aligned}$$

Then we have the unnormalized

$$\begin{aligned}\left|2'\right\rangle &= xe^{-x^2/2} - \frac{1}{\sqrt{\pi}}e^{-x^2/2}\int_{-\infty}^{\infty} xe^{-x^2} \\ &= xe^{-x^2/2}\end{aligned}$$

in which case, normalizing, our second vector becomes

$$\begin{aligned}|2\rangle &= \frac{xe^{-x^2/2}}{\sqrt{\int_{-\infty}^{\infty} t^2e^{-t^2}dt}} \\ &= \frac{\sqrt{2}}{\pi^{1/4}}xe^{-x^2/2}.\end{aligned}$$

Next, we have the unnormalized

$$
\begin{aligned}
\left|3'\right\rangle &= x^2 e^{-x^2/2} - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^3 e^{-t^2} dt - \frac{1}{\sqrt{\pi}} e^{-x^2/2} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt \\
&= (x^2 - \frac{1}{2}) e^{-x^2/2}
\end{aligned}
$$

such that, after normalizing, our third vector becomes

$$
\begin{aligned}
|3\rangle &= \frac{(x^2 - \frac{1}{2}) e^{-x^2/2}}{\sqrt{\int_{-\infty}^{\infty} (t^2 - \frac{1}{2})^2 e^{-t^2/2} dt}} \\
&= \frac{\sqrt{2}}{\pi^{1/4}} (x^2 - \frac{1}{2}) e^{-x^2/2}.
\end{aligned}
$$

Finally, we have the unnormalized

$$
\begin{aligned}
\left|4'\right\rangle &= x^3 e^{-x^2/2} - \frac{2}{\sqrt{\pi}} (x^2 - \frac{1}{2}) e^{-x^2/2} \int_{-\infty}^{\infty} (t^2 - \frac{1}{2}) t^3 e^{-t^2/2} dt \\
&- \frac{2}{\sqrt{\pi}} x e^{-x^2/2} \int_{-\infty}^{\infty} t^4 e^{-t^2} dt - \frac{1}{\sqrt{\pi}} e^{-x^2} \int_{-\infty}^{\infty} t^3 e^{-t^2} dt \\
&= (x^3 - \frac{3}{2} x) e^{-x^2/2}
\end{aligned}
$$

such that, after normalizing, our fourth vector becomes

$$
\begin{aligned}
|4\rangle &= \frac{(x^3 - \frac{3}{2} x) e^{-x^2/2}}{\sqrt{\int_{-\infty}^{\infty} (t^3 - \frac{3}{2} t)^2 e^{-t^2} dt}} \\
&= \frac{2}{\sqrt{3} \pi^{1/4}} (x^3 - \frac{3}{2} x) e^{-x^2/2}.
\end{aligned}
$$

Question 6

Prove that if an operator in Hilbert space $\hat{\Omega}$ obeys the property

$$
\langle h| \hat{\Omega} |h\rangle = \langle h| \hat{\Omega} |h\rangle^* \tag{1}
$$

then it also has the property

$$
\langle f| \hat{\Omega} |g\rangle = \langle g| \hat{\Omega} |f\rangle^*
$$

where $|f\rangle$, $|g\rangle$, $|h\rangle$ are arbitrary functions. That is, both are equivalent definitions of a Hermitian operator.
Hint: Write $|h\rangle = |f\rangle + i |g\rangle$ and expand the imaginary part of Eq. (1).

Let's write $|h\rangle = |f\rangle + i |g\rangle$ where $|f\rangle, |g\rangle$ are arbitrary functions, as the hint suggests. Then our relation becomes

$$
(\langle f| - i \langle g|) \hat{\Omega} (|f + i |g\rangle\rangle) = ((\langle f| - i \langle g|) \hat{\Omega} (|f + i |g\rangle\rangle))^*
$$

or, expanding, that

$$
\begin{aligned}
&\langle f| \hat{\Omega} |f\rangle + \langle g| \hat{\Omega} |g\rangle + i[\langle f| \hat{\Omega} |g\rangle - \langle g| \hat{\Omega} |f\rangle] \\
&= \langle f| \hat{\Omega} |f\rangle^* + \langle g| \hat{\Omega} |g\rangle^* - i[\langle f| \hat{\Omega} |g\rangle - \langle g| \hat{\Omega} |f\rangle]^*.
\end{aligned}
$$

Using our identity with $h = f$ and $h = g$, we can cancel two terms on both sides of this equation, leaving us with

$$\langle f|\,\hat{\Omega}\,|g\rangle - \langle g|\,\hat{\Omega}\,|f\rangle = \langle g|\,\hat{\Omega}\,|f\rangle^* - \langle f|\,\hat{\Omega}\,|g\rangle^*$$

Now, this is tricky, and not quite alluded to in the hint from the question (I admit shamefully that I was helped by a "conversation" with ChatGPT here...) - but I can't quite see any other more straightforward way to approach it without additional assumptions on $|f\rangle, |g\rangle$. So, chugging onwards, let's do the same procedure, except this time let's take $|h\rangle = |f\rangle + |g\rangle$. Doing this and cancelling the appropriate terms should give

$$\langle f|\,\hat{\Omega}\,|g\rangle + \langle g|\,\hat{\Omega}\,|f\rangle = \langle g|\,\hat{\Omega}\,|f\rangle^* + \langle f|\,\hat{\Omega}\,|g\rangle^* .$$

We can then add these two equations together and divide both sides by two, which gives

$$\langle f|\,\hat{\Omega}\,|g\rangle = \langle g|\,\hat{\Omega}\,|f\rangle^*$$

which is our desired result.

Question 7

By directly calculating the integral,

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx. \tag{2}$$

determine the Fourier Transform pairs of the following functions,

(a)

$$f(x) = \sin(\pi x) \tag{3}$$

The easiest way to approach this is to rewrite $\sin \pi x$ in terms of complex exponentials, as

$$\sin(\pi x) = \frac{1}{2i}(e^{i\pi x} - e^{-i\pi x})$$

in which case our Fourier transform becomes

$$F(k) = \frac{1}{(\sqrt{2\pi})2i} \int (e^{i(\pi-k)x} - e^{-i(\pi+k)x})dx.$$

Now, using the definition of the δ-function in terms of complex exponentials as

$$\delta(k - k') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x}dx \tag{4}$$

along with the fact that $\delta(x) = \delta(-x)$ (which can be proven with a change of variables in the definition above), this gives

$$F(k) = \frac{\sqrt{2\pi}}{2i}\delta(k - \pi) - \delta(k + \pi).$$

This makes sense! The frequency of a sin function is fixed (up to overall sign, which can be absorbed in the coefficient out the front due to the oddness of $\sin \pi x = -\sin \pi x$), and so in the frequency domain we see two spikes at $\pm$ that frequency, with a relative sign accounting for the oddness.

(b)

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| \leq \frac{1}{2} \\ \\ 0 & \text{if} \quad |x| > \frac{1}{2} \end{cases} \tag{5}$$

This function is often called the top-hat function, for obvious reasons (hint: draw it out!). Going through the calculation, we have

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{1/2} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{-ik} [e^{-ik/2} - e^{+ik/2}] \\ &= \frac{2}{\sqrt{2\pi}k} \frac{1}{2i} [e^{ik/2} - e^{-ik/2}] \\ &= \frac{1}{\sqrt{2\pi}} \frac{\sin(k/2)}{(k/2)}. \end{aligned}$$

As a totally unrelated aside, have a chat with Peter if you want to know a wonderful link between this Fourier transform and diffraction patterns!

(c)

$$e^{-\frac{1}{2}x^2} \tag{6}$$

Once again going through, we have

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 - ikx} dx.$$

The trick here is to complete the square inside the exponential. (We have that

$$\begin{aligned} -\frac{1}{2}x^2 - ikx &= -\frac{1}{2}(x^2 + 2ikx) \\ &= -\frac{1}{2}[(x+ik)^2 - i^2k^2] \\ &= -\frac{1}{2}(x+ik)^2 - \frac{1}{2}k^2 \end{aligned}$$

so that our integral becomes

$$F(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-ik)^2} dx$$

where I've pulled the k-dependent exponential out the front. Now, we can do a change of variables $x \to x' = x - ik$ and use the standard Gaussian integral to evaluate this. **Remark:** technically, you need some complex analysis/contour integration to justify this substitution, since the substitution should naively shift the integral bounds into the complex plane as well, but I'm going to ignore these details. This gives

$$F(k) = e^{-\frac{1}{2}k^2}$$

which, miraculously, is the same as the original function, just with $x \leftrightarrow k$!

(d)

$$e^{-|x|} \tag{7}$$

Splitting up our integral into the $x \geq 0$ and $x < 0$ contributions, we have

$$F(k) = \frac{1}{\sqrt{2\pi}}[\int_{-\infty}^{0} e^{x(1-ik)}dx + \int_{0}^{\infty} e^{-x(1+ik)}dx].$$

But the exponential is just a standard integral which can be evaluated

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}}(-\frac{1}{1-ik}[1-0] + \frac{1}{1+ik}[0-1]) \\ &= -\frac{1}{\sqrt{2\pi}}(\frac{1}{1+ik} + \frac{1}{1-ik}) \\ &= -\frac{1}{\sqrt{2\pi}}\frac{2}{1+k^2}. \end{aligned}$$

Question 8

Prove the similarity theorem.

$$f(ax) \quad \leftrightarrow \quad \frac{1}{|a|}F\left(\frac{k}{a}\right) \tag{8}$$

Perhaps the best way to approach this is just through direct substitution. Let's consider the Fourier transform of $f(ax)$, denoted $\mathcal{F}\{f(ax)\}$; this is

$$\mathcal{F}\{f(ax)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{-ikx}dx.$$

Next, we do a substitution $x \to x' = ax$, in which case $dx = dx'/a$. As for the integral bounds, the only thing that matters here is the sign of a, since our bounds are infinite (and of course, we take $a \neq 0$). If a is positive, the bounds remain unchanged and we can proceed without much further thought. However, if a is negative, the bounds should also flip. In this case, however, we can absorb the minus sign of $a = -|a|$ back into the integral to swap the bounds back to their original order (from $-\infty$ to ∞). So, in both cases, we get

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')e^{-i\frac{k}{a}x'}dx'.$$

Since it doesn't matter what variable we use to integrate (x' or x) as it's a dummy variable, one immediately sees that the remaining integral (and $1/\sqrt{2\pi}$) factor out the front is the usual Fourier transform of $f(x)$, only with k divided by a factor of a. So, we have that

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|}F(\frac{k}{a})$$

as required.

Question 9

Prove the shift theorem.

$$f(x-a) \quad \leftrightarrow \quad e^{-ika}F\left(k\right) \tag{9}$$

This runs very similarly to the previous question. Plugging in directly to our formula, we have

$$\mathcal{F}\{f(x-a)\} = \frac{1}{\sqrt{2\pi}}\int f(x-a)e^{-ikx}dx$$

wherein, as before, we can substitute $x \to x' = x - a$, which leaves the integral bounds unchanged. This gives

$$\begin{aligned}\mathcal{F}\{f(x-a)\} &= \frac{1}{\sqrt{2\pi}}\int f(x')e^{-ik(x'+a)}dx' \\ &= e^{-ika}\frac{1}{\sqrt{2\pi}}\int f(x')e^{-ikx'}dx' \\ &= e^{-ika}F(k)\end{aligned}$$

as required. Note that we could pull out that e^{-ika} factor from the integral, since we were integrating over x', not k.

Question 10

(Bohm, Chapter 10) Using Fourier theory, calculate $\phi(k)$ and show that the following wave packet satisfies the uncertainty principle,

$$\psi(x) = \alpha_1 \exp\left[-\frac{\alpha x^2}{2}\right] \tag{10}$$

Here α_1 is used to normalise the wavefunction.

Rather than computing this from scratch with α, α_1, let's use this as an opportunity to see how we can exploit our theorems, since we calculated a very similar integral in Question 7 (c). Namely, for $f(x) = e^{-\frac{1}{2}x^2}$, we showed that $F(k) = e^{-\frac{1}{2}k^2}$.

First of all, consider the α_1 out the front. It's trivial to show from the definition that $\mathcal{F}\{\alpha_1 f(x)\} = \alpha_1 \mathcal{F}\{f(x)\}$.

Next, we have the shift theorem. Assuming $\alpha > 0$ so everything is nice and convergent, we've been given $\alpha_1 f(\sqrt{\alpha}x)$ (where we have to be careful to take the square root, since x appears as a square inside $f(x)$). So, our fourier transform $\tilde{\psi}(k)$ of $\psi(x)$ is given by

$$\tilde{\psi}(k) = \frac{\alpha_1}{\sqrt{\alpha}}e^{-\frac{k^2}{2\alpha}}.$$

Now, to measure our uncertainties in x and p, remember that we need to use

$\psi^*\psi$ and $\tilde{\psi}^*\tilde{\psi}$, not just the wavefunctions themselves. So, we have

$$\psi^*\psi \sim e^{-\alpha x^2}$$
$$\tilde{\psi}^*\tilde{\psi} \sim e^{-(1/\alpha)k^2}$$

where all we're looking for is the standard deviation, which is inside the exponential part. In particular, for a gaussian $\sim e^{-\frac{x^2}{2\sigma^2}}$, the uncertainty is given by σ. With this, we can read off

$$\sigma_x = \sqrt{\frac{1}{2\alpha}}$$
$$\sigma_k = \sqrt{\frac{\alpha}{2}}$$

such that

$$\sigma_x\sigma_k = \frac{1}{2}.$$

To compare this to the uncertainty principle, we recall that $p = \hbar k$, in which case $\sigma_p = \hbar\sigma_p$, and so

$$\sigma_x\sigma_p = \frac{\hbar}{2}$$

which does satisfy (and in fact reaches the maximal bound of) the uncertainty principle.

Question 11

Use the Gaussian wave function to check the equivalence of the x-space and k-space momentum operators in determining expectation value of the momentum.

We want to calculate $\langle p\rangle$ in some state $|\psi\rangle$, which we might as well take to be

$$\psi(x) = Ae^{-\frac{1}{2}x^2}$$
$$\psi(k) = Ae^{-\frac{1}{2}k^2}$$

since these have the exact same functional form, where A is just some normalization.

In position space,

$$\begin{aligned}\langle p\rangle &= \int_{-\infty}^{\infty} dx\psi(x)^*(-i\hbar\frac{\partial}{\partial x})\psi(x)\\ &= i\hbar A^2\int_{-\infty}^{\infty} dxxe^{-x^2}\\ &= 0\end{aligned}$$

since we're integrating against an odd $\times$ even $=$ odd function over a symmetric domain.

Meanwhile, in k space (with $p = \hbar k$), we have

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} dk \psi(k)^* (\hbar k) \psi(k) \\
&= \hbar A^2 \int_{-\infty}^{\infty} dk k e^{k^2} \\
&= 0
\end{aligned}$$

for the same reason. So, the two results line up, as expected.