UNSW SCHOOL OF PHYSICS

PHYS2111 – Quantum Mechanics

Tutorial 7

Question 1

Prove the identity $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$.

Question 2

Show that $\delta(ax) = \delta(x)/|a|$.
Hint: Consider $\int \delta(ax)\, d(ax)$ and remember that $\delta(x) = \delta(-x)$.

Question 3

Show that the Hamiltonian operator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

is Hermitian.

Question 4

An anti-hermitian operator is equal to minus its Hermitian conjugate $\hat{L}^\dagger = -\hat{L}$.

(a) Show that the expectation value of an anti-hermitian operator is imaginary.

(b) Show that the commutator of two Hermitian operators is anti-hermitian. How about the commutator of two anti-hermitian operators?

Question 5

Is the ground state of the infinite square potential well an eigenfunction of momentum? If so, what is its momentum? If not, why not?

Question 6

(Liboff 3.4) The displacement operator $\hat{\mathcal{D}}$ is defined by the equation

$$\hat{\mathcal{D}}f(x) = f(x + \zeta).$$

Show that the eigenfunctions of $\hat{\mathcal{D}}$ are of the form

$$\phi_\beta = e^{\beta x} g(x)$$

where

$$g(x + \zeta) = g(x)$$

and β is any complex number. What is the eigenvalue corresponding to ϕ_β?

Question 7

(Liboff 3.17) Show that for an arbitrary "well-behaved" function $f(x)$,

$$\exp\left(\frac{i\zeta\hat{p}}{\hbar}\right) f(x) = f(x + \zeta)$$

where $\hat{p}$ is the momentum operator and the constant ζ represents a small displacement. In this problem, you should demonstrate that the left-hand side of the equation is the Taylor series expansion of the right-hand side about $\zeta = 0$.

Question 8

(a) Substitute the Fourier transform

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

into its own inverse to show that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(x'-x)} dk = \delta(x' - x) \tag{1}$$

(b) Alternatively, starting with the basis definition ($p = \hbar k$)

$$\langle k|k' \rangle = \delta(k - k')$$

and writing $|k\rangle = Ae^{ikx}$, insert the continuity formula for x

$$\int_{-\infty}^{\infty} |x\rangle \langle x| \, dx = 1$$

to obtain the same equation (1). What value of the normalisation constant A should be used?

Question 9

(Liboff 3.7) Show that the following are valid representation of $\delta(y)$:

(a)

$$2\pi\delta(y) = \int_{-\infty}^{\infty} e^{iky} dk$$

(b)

$$\pi\delta(y) = \lim_{\eta \to \infty} \frac{\sin \eta y}{y}$$

Note: In mathematics an object such as $\delta(y)$, which is defined in terms of its integral properties, is called a *distribution.* Consider all functions $\chi(y)$ defined on the interval $(-\infty, \infty)$ for which

$$\int_{-\infty}^{\infty} |\chi(y)|^2 dy < \infty.$$

Then two distributions δ_1 and δ_2 are *equivalent* if for all such $\chi(y)$,

$$\int_{-\infty}^{\infty} \chi(y)\delta_1 dy = \int_{-\infty}^{\infty} \chi(y)\delta_2 dy.$$

When one establishes that a mathematical form such as those shown in parts (a) and (b) above are a representation of $\delta(y)$, one is in effect demonstrating that these two object are equivalent as distributions.

Question 10

(past exam question) A particle with an energy E is tunneling through a potential barrier (see figure below). Note that such a tunneling process takes place for example at a metal-insulator-semiconductor thin film structure used in semiconducting industry.

(a) Sketch the form of the wave function expected in each of the 3 regions and give the functional forms of these wave functions. Explain the choice of the wave function for each region in a few words.

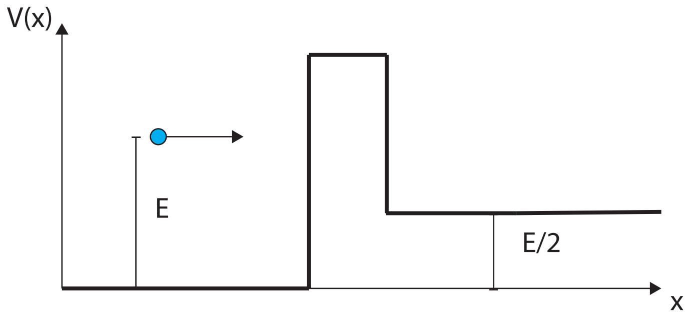

Figure 1:

(b) What are the mathematical boundary conditions for the waves when the particle enters and leaves the barrier?

(c) Derive the equations that must be solved to obtain the constants appearing in the wave functions

(d) If the potential energy in Region 3 is exactly E/2, calculate the ratio of the wavelength in Regions 1 and 3.

Question 11

Complete the analysis of the finite quantum well scattering states (i.e. $E > 0$ and $V = V_0$) from the notes (or otherwise) to derive an analytical expression for the reflectance (R) and transmittance (T). Find an expression for the energies at which the transmission is 100%.

Question 12

(adapted Griffiths 2.35) A particle of mass m and kinetic energy $E > 0$ approaches an abrupt potential drop V_0.

(a) What is the probability that it will "reflect" back, if $E = V_0/3$?

(b) When a free neutron enters a nucleus, it experiences a sudden drop in potential energy, from $V = 0$ to $V = -12\,\text{MeV}$ inside. Suppose a neutron, emitted with kinetic energy $V = 4\,\text{MeV}$ by a fission event, strikes such a nucleus. What is the probability it will be absorbed, initiating another fission? *Hint: You calculated the probability of reflection in part (a): use $T = 1 - R$ to get the probability of transmission through the surface.*