THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

PHYS2111 Quantum Mechanics

Final Exam

4th May 2023

2 hr Online Exam + 15 min for download/reading + 30 min for capture/upload.

Total marks: 80 (pro-rata to 60% of course assessment)

Please note:

- You are allowed to use your textbook and notes during the exam.
- Prepare two files, one with your responses to Q1 and Q2 and one with your responses to Q3 and Q4 for upload to Moodle by the end of the allotted exam time.
- When uploading, please use the following naming convention: PHYSXXXX zID Surname QxQy (e.g., PHYS2111 z5556789 Smith Q1Q2 and PHYS2111 z5556789 Smith Q3Q4).
- Please write your name and student number on the first page of your submission.
- Answer all questions concisely and legibly.
- Questions are not all worth the same marks; questions are worth the marks shown.
- You may use a university-approved calculator.

Formula Sheet

Planck's constant $h = 6.626 \times 10^{-34}$ Js

Fundamental charge unit $e = 1.60 \times 10^{-19}$ C

Mass of the electron = 9.11×10^{-31} kg

Boltzmann's constant = 1.380649×10^{-23} m^2 kg s^{-2} K^{-1}

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Pauli spin matrices: $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[A,B] = AB - BA$

Table of spin operator actions

$\sigma_x\|u\rangle = \|d\rangle$	$\sigma_x\|d\rangle = \|u\rangle$
$\sigma_y\|u\rangle = i\|d\rangle$	$\sigma_y\|d\rangle = -i\|u\rangle$
$\sigma_z\|u\rangle = \|u\rangle$	$\sigma_z\|d\rangle = -\|d\rangle$

Helpful Standard Integrals

Exponential Integral: $\int_0^\infty x^n e^{-x/a} dx = n!\, a^{n+1}$

Gaussian Integrals: $\int_0^\infty x^{2n} e^{-x^2/a^2} dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$

$\int_0^\infty x^{2n+1} e^{-x^2/a^2} dx = \frac{n!}{2} a^{2n+2}$

Fourier transform: $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$

Ladder Operators

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$$

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$$

$$a_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

Quantum Harmonic Oscillator

$$\psi_m = A_m H_m(u) e^{-u^2/2} \quad m = 0, 1, 2 \ldots \quad u = \sqrt{m\omega/\hbar}\, x$$

$$H_0 = 1, \quad H_1 = 2u, \quad H_2 = 4u^2 - 2, \quad H_3 = 8u^3 - 12u, \quad \ldots$$

Question 1: [20 marks total]

(a) What are the three features of measurement of a quantum system, e.g., angular momentum of an electron, compared to a classical system, e.g., angular momentum of a football. Explain each of them in a sentence or two. [3 marks]

(b) Suppose I write the following quantum state:

$$|\chi\rangle = \frac{i}{4}|u\rangle + \frac{3\sqrt{2}}{4}|o\rangle$$

Briefly comment on this state, e.g., is it a valid and perfectly good way to write a state, is it valid but not so good, is it invalid? What is right/wrong with this state? [1 marks]

(c) Using the definition:

$$|o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle$$

Obtain an expression for $|\chi\rangle$ that is entirely in the $|u\rangle/|d\rangle$ basis. [2 marks]

(d) Normalise the state that you obtain in (c). If the state you obtain is already normalised, demonstrate that this is indeed the case. [3 marks]

(e) Explain in one sentence what a projection operator is in quantum mechanics. Obtain the respective projection operators for the possible measurement outcomes you would obtain for a measurement of σ_z. [3 marks]

(f) Calculate the probability of obtaining each of the possible measurement outcomes you gave in (e) for a measurement of σ_z applied to the state $|\chi\rangle$? Show that the sum of probabilities over the possible measurement outcomes adds to 1, i.e., prove your normalisation from (d) is correct. [3 marks]

(g) Explain in one sentence what the expectation value means as a concept in quantum mechanics. Calculate the expectation value for σ_z for the state $|\chi\rangle$. [2 marks]

There is more than one way to do this, and either approach to obtaining the value will earn marks providing it is done correctly. One should obviously be faster than the other. 😊

(h) A fellow student who didn't show up to lectures asks you what completeness means. Give a brief explanation. As part of your answer, you should point out two key implications that completeness has in quantum mechanics. [3 marks]

Question 2: [20 marks total]

(a) Find an expression for the commutator $[X,YZ]$ in terms of the two simpler commutators $[X,Y]$ and $[X,Z]$. [4 marks]

(b) $[x,p_x] \geq \hbar/2$ yet $[x,p_y] \geq 0$. Explain why? Be concise. [2 marks]

(c) You are hanging out in the physoc common room when one of the first years, trying to sound cool, boasts that the dual-spin state |uu> is still an entangled state because whenever 'Alice measures an up, Bob is also guaranteed to get an up, and that's like correlation, still, right'.

Is this student correct? Briefly justify your answer. [3 marks]

(d) Is the statement "Superposition is a necessary but not sufficient condition for entanglement" true or false? Why? If it helps to give an example, please feel free to. Be careful to not massively over-answer this question, there are only really two key ideas needed to lock this down. [6 marks]

(e) You learn about ladder operators in the context of the quantum harmonic oscillator in Peter's half of the course. A fellow student pitches a clever idea, which is a hypothetical pair of ladder operators σ_+ and σ_- for quantum spin, that looks something like:

$$\sigma_\pm = \sigma_x \pm i\sigma_y$$

Obtain the corresponding matrices for the two ladder operators σ_+ and σ_- and show using four examples that they work on the states $|u\rangle$ and $|d\rangle$ as intended. If you need to add any numerical prefactors or other 'corrections' to the hypothetical σ_+ and σ_- to get things to work properly, e.g., have unitarity or conserve normalisation, please do so. [4 marks]

(f) The ladder operators σ_+ and σ_- that you get in (e) are obviously not Hermitian. Is this a problem? Briefly explain why or why not. [1 mark]

Question 3: [20 marks total]

A particle of mass, m, is subject to an energy-conserving force that generates a one-dimensional confining potential. The potential is infinite for $x < x_0$ and experiences a force,

$$F = -\beta(x - x_0) \quad x \geq x_0$$

a) Determine the corresponding potential, $V(x)$ for $x \geq x_0$. [3 marks]

b) Write an expression for the time independent Schrödinger Eqn for this potential. [3 marks]

c) Using the solutions to the quantum harmonic oscillator and appropriate boundary conditions, determine the wave functions, $\psi(x)$, in terms of position for the lowest two energy eigenstates described by this potential. Note: you do not need to normalise at this stage. [6 marks]

d) Make a sketch of the lowest two eigenstates as a function position marking the important features. [2 marks]

e) Calculate normalisation coefficient for the lowest energy eigenstate. [2 marks]

f) Calculate the expectation value for the position of the lowest energy eigenstate. [4 marks]

Question 4: [20 marks total]

A series of thermalised neutrons (T = 300K), of mass 1.674×10^{-27} kg, are directed towards a particular experiment a distance 10 m from the source.

(a) What is the classical average velocity of the neutron? [2 mark]

(b) If the emitted neutron has an initial uncertainty in position of 1 nm, determine the corresponding uncertainty in momentum. [2 marks]

(c) Estimate the position uncertainty when the neutron arrives at the detector and comment on the change to the position – momentum uncertainty relation with time. [4 marks]

(d) If the initial probability distribution of the neutron is described as a Gaussian,

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

write down an expression for wave function in terms of its wave vector components (i.e. momentum representation). Remember the average momentum of the neutron is not equal to zero. [6 marks]

(e) The time dependent wave function for free-space propagation can be described by,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx-\omega(k)t)} dk.$$

Use this to explain why the position-momentum uncertainty relation evolves in time as predicted in (c) [3 marks]

(f) The apparatus depicted in the figure below is used as an energy selective filter for the neutron beam. It consists of two connected neutron absorbing disks, separated by a distance, d, spinning synchronously. Each disk has a radial slot cut out, such that the slots are off-set by an angular displacement of $\delta\theta$. Derive an expression for the energy of the transmitted neutrons as a function of angular frequency of the spinning disk. [3 marks]

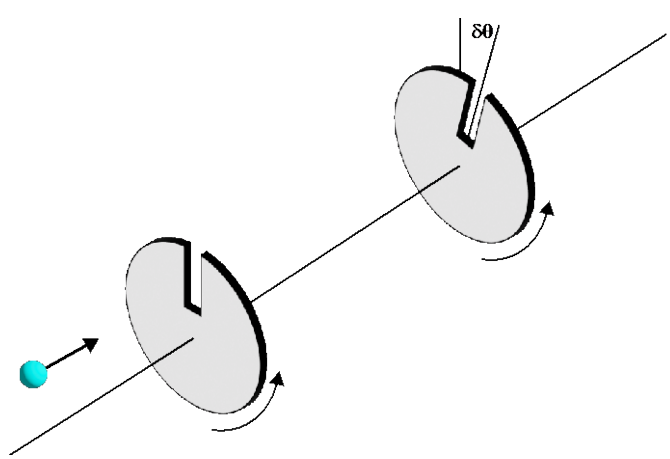

Figure: spinning disk apparatus