THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

PHYS2111 Quantum Mechanics

Final Exam

10th May 2021

2 hr Online Exam + 15 min for download/reading + 30 min for capture/upload.

Total marks: 90 (pro-rata to 50% of course assessment)

Please note:

- You are allowed to use your textbook and notes during the exam.
- Prepare a single file with all of your responses to upload to Moodle by the end of the allotted exam time.
- When uploading, please use the following naming convention: PHYSXXXX zID Surname Name (e.g., PHYS2111 z5556789 Smith Jane).
- Please write your name and student number on the first page of your submission.
- Answer all questions concisely and legibly.
- Questions are not all worth the same marks; questions are worth the marks shown.
- You may use a university-approved calculator.

Formula Sheet

Planck's constant $h = 6.626 \times 10^{-34}$ Js

Fundamental charge unit $e = 1.60 \times 10^{-19}$ C

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Pauli spin matrices: $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[A,B] = AB - BA$

Table of spin operator actions

$\sigma_x\|u\rangle = \|d\rangle$	$\sigma_x\|d\rangle = \|u\rangle$
$\sigma_y\|u\rangle = i\|d\rangle$	$\sigma_y\|d\rangle = -i\|u\rangle$
$\sigma_z\|u\rangle = \|u\rangle$	$\sigma_z\|d\rangle = -\|d\rangle$

Helpful Standard Integrals

Exponential Integral: $\int_0^\infty x^n e^{-x/a} dx = n!\, a^{n+1}$

Gaussian Integrals: $\int_0^\infty x^{2n} e^{-x^2/a^2} dx = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$

$\int_0^\infty x^{2n+1} e^{-x^2/a^2} dx = \frac{n!}{2} a^{2n+2}$

Fourier transform: $F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx}$

Question 1: [20 marks total]

Consider an electron propagating in free space with the following initial ($t = 0$) wave function:

$$\Psi(x,0) = Ae^{-\frac{|x|}{\beta}}$$

where β is real and positive.

(a) Find the normalisation constant. [2 marks]

(b) Write the general expression for the time dependent wavefunction, $\Psi(x,t)$ and provide a physical interpretation of each term in terms of stationary states. [3 marks]

(c) Give a qualitative description of how the wave packet changes with the progress of time. [3 marks]

(d) Calculate the corresponding momentum space representation and plot the result. [6 marks]

The expectation value for $\langle x \rangle = 0$ and $\langle x^2 \rangle = \frac{1}{2}A^2\beta^3$

(e) By calculating appropriate expectation values, show that the Heisenberg position-momentum uncertainty relation holds for this wave packet. [6 marks]

Question 2: [20 marks total]

Consider the following 1D energy potential that is used to confine an electron to a certain region of space,

$$V(x) = \begin{cases} 0 & -a < x < a \\ \infty & \text{elsewhere} \end{cases}$$

(a) By solving Schrödinger's equation with appropriate boundary conditions (***showing all working***) provide an expression lowest three eigenstates and corresponding eigenvalues for the corresponding Hamiltonian, $H(x, p)$. [6 marks]

(b) Sketch the corresponding probability distribution for each. [6 marks]

(c) What is the average momentum for the 3rd stationary state? [2 marks]

An electron occupying the lower stationary state for this potential. The width of the potential is suddenly doubled such that the original state now represents an initial superposition of the eigenfunctions of the new potential.

(d) Comment on how the stationary states are modified by the change in size. [2 marks]

(e) Calculate the c_n coefficient for the new stationary ground state. [4 marks]

Question 3: [26 marks total]

(a) Operators are an important feature of quantum mechanics. Thinking in the broadest terms, why do operators exist? (Hint: don't overthink this, the answer is 1 sentence). [1 mark]

(b) For the subset of operators used for physical observables, we impose the requirement that the operator is Hermitian.

(i) Briefly explain what it means for an operator to be Hermitian. How does this differ between a matrix operator and an operator that is a function? [2 marks]

(ii) Why do we impose this requirement on observable operators? [1 mark]

(c) The completeness relation is also an important concept in quantum mechanics. In its most general form, it is written mathematically as:

$$\sum_{\alpha} |\alpha\rangle\langle\alpha| = 1$$

(i) What does the completeness relation mean and why is it important to quantum mechanics? [2 mark]

(ii) Write a simple expression for the completeness relation for a single quantum spin with basis states $|u\rangle$ and $|d\rangle$. Using $|u\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|d\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, briefly show that your expression makes sense. If you need to elaborate on what exactly 1 means above, do so. [3 marks]

(d) The projection operator $\hat{P} = |\alpha\rangle\langle\alpha|$ acts on an arbitrary state $|\beta\rangle$ to give the probability that a measurement of the observable A has the outcome α, where α is the corresponding eigenvalue for $|\alpha\rangle$.

(i) Is the projection operator an observable? Briefly justify your answer. [2 marks]

(ii) An interesting property of the projection operator is that $\hat{P}^2 = \hat{P}$. Using the projection operator as given above and the arbitrary state $|\beta\rangle$, show providing working that this is true as a general property. [5 marks]

(iii) The projection operator has eigenvalues ρ such that $\hat{P}|\rho\rangle = \rho|\rho\rangle$. By considering $\hat{P}^2|\rho\rangle$, show providing working that $\rho^2 = \rho$ holds for any ρ. What is required of $|\rho\rangle$ for this to hold? [3 marks]

(iv) Using the requirement $\rho^2 = \rho$ you can obtain the eigenvalues ρ. There are only two possibilities, what are they? Very briefly explain why Feel free to denote them ρ_1 and ρ_2. [1 mark]

(v) You will notice we never specified the dimension of the Hilbert space for $|\beta\rangle$. Notably, there's no requirement for it to be a 2D complex vector space. Yet in (iv) you find there are only two possible eigenvalues. Is this a problem? Explain why/why not? [1 mark]

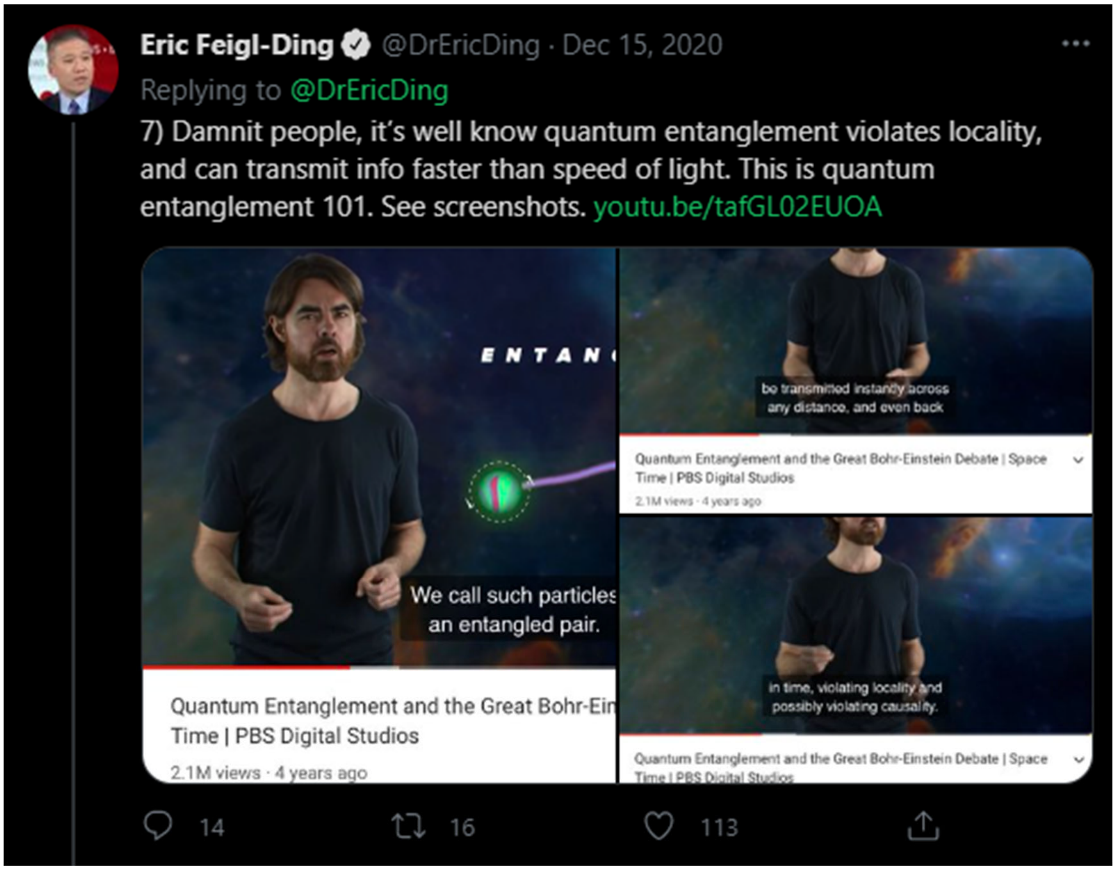
(vi) Following from (v), there will be n eigenvalues. How many of these n eigenvalues will take the value ρ_1 and how many will take the value ρ_2? Briefly explain why. [2 marks]

(vii) What are the corresponding eigenvectors for the eigenvalues ρ_1 and ρ_2? Do not try to solve for them, you will never complete the exam. Use your intuition and knowledge of inner products and basis vectors to arrive at and justify your answer. [2 marks]

(viii) Briefly comment on how the eigenvalues and eigenvectors in (v)-(vii) support your answer to (i) on the projection operator being an observable. [1 mark]

Question 4: [24 marks total]

(a) You are surfing your twitter feed and you see the following tweet:



Use your knowledge of correlation and entanglement to calmly, rationally and professionally demolish the argument about faster than light information transmission. You may ignore locality, we didn't deal with this aspect in the course. Arrogance & snark will not earn bonus marks. Be concise, twitter doesn't have time for your Ph.D. thesis. [8 marks]

(b) In quantum information, Charlie's job is to prepare spins to pass to Alice and Bob for measurement. Charlie was at a rock festival on the weekend, and he shows up to work on Monday with a sore head and starts producing states where:

$$|\psi\rangle = \sqrt{0.75}|ud\rangle + \sqrt{0.25}|du\rangle$$

rather than the usual singlet states.

(i) Calculate the correlation $C(\psi) = \langle\sigma_z\tau_z\rangle - \langle\sigma_z\rangle\langle\tau_z\rangle$ for the states Charlie is producing. [8 marks]

(ii) The spin-polarization principle states that $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = X$ where $0 < X < 1$. Find X for the state $|\psi\rangle$. [5 marks]

(iii) Noting that $|\psi\rangle$ is a superposition of $|ud\rangle$ and $|du\rangle$ with no components of $|uu\rangle$ and $|dd\rangle$, and your answers to (i) and (ii), are the states Charlie producing still maximally entangled? Briefly explain why? [3 marks]