UNSW SCHOOL OF PHYSICS

**PHYS2111 – Quantum Mechanics**

**Tutorial 7 SOLUTIONS**

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**Question 1**

Prove the identity $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$.

This is really just a plug-and-chug-from-the-definition type question. Starting with the RHS, we have

$$\begin{aligned}
RHS &= A[B,C] + [A,C]B \\
&= ABC - ACB + ACB - CAB \\
&= ABC - CAB \\
&= (AB)C - C(AB) \\
&= [AB, C] \\
&= LHS
\end{aligned}$$

**Question 2**

Show that $\delta(ax) = \delta(x)/|a|$.

*Hint:* Consider $\int \delta(ax)\, d(ax)$ and remember that $\delta(x) = \delta(-x)$.

To answer this, we need to remember that, really, the $\delta$-function is a distribution that only fully makes sense under an integral sign. See the note under Question 9 for more details. But, essentially, what we need to show is that

$$\int_{-\infty}^{\infty} LHS \times f(x) dx = \int_{-\infty}^{\infty} RHS \times f(x) dx$$

for all (sufficiently well-behaved) functions $f(x)$.

Let's start with the LHS. We have, for arbitrary $f(x)$,

$$\int_{-\infty}^{\infty} LHS \times f(x) dx = \int_{-\infty}^{\infty} \delta(ax) f(x) dx.$$

As you likely guessed, we're going to do a change of variables $x \to x' \equiv ax$. we have $dx = adx'$, and for the bounds, either:

- $a > 0$: the infinite bounds remain unchanged, and

$$\int_{-\infty}^{\infty} LHS \times f(x) dx = \frac{1}{a} \int_{-\infty}^{\infty} \delta(x') f(x'/a) dx' = \frac{1}{a} f(0)$$

- $a < 0$: the infinite bounds swap order due to the change of variables, in which case an additional minus sign can be factored out from the $1/a$ out the front so that the bounds go back to the usual order, so we have

$$\int_{-\infty}^{\infty} LHS \times f(x) dx = \frac{1}{-a} \int_{-\infty}^{\infty} \delta(x') f(x'/a) dx' = \frac{1}{-a} f(0).$$

In both scenarios, then, we see that a factor of $\frac{1}{|a|}$ has been pulled out: if $a > 0$, then $|a| = a$; and if $a < 0$, then $|a| = -a$, so we have

$$\int_{-\infty}^{\infty} LHS \times f(x)dx = \frac{1}{|a|} f(0).$$

Finally, one can of course rewrite $f(0)$ as

$$f(0) = \int_{-\infty}^{\infty} \delta(x) f(x) dx$$

in which case

$$\int_{-\infty}^{\infty} LHS \times f(x)dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(x) f(x) dx = \int_{-\infty}^{\infty} RHS \times f(x)dx$$

as required.

**Question 3**

Show that the Hamiltonian operator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

is Hermitian.

To show that the Hamiltonian is Hermitian, we essentially need to show that

$$\langle H\psi_1 | \psi_2 \rangle = \langle \psi_1 | H\psi_2 \rangle$$

for all wavefunctions $\psi_1, \psi_2$.

To avoid having to write everything out, note that if $A, B$ are Hermitian, then $A + B$ is Hermitian, so we can consider the potential and kinetic parts of the Hamiltonian separately. Being a real function (and acting only with multiplication), $V(x)$ is obviously Hermitian; and in the kinetic term, the real factor of $-\frac{\hbar^2}{2m}$ evidently doesn't affect the Hermitianness of $\frac{d^2}{dx^2}$. So, to prove that $H$ is Hermitian, all that is left to prove is that $\frac{d^2}{dx^2}$ is Hermitian.

But this is actually easy enough. On the left hand side of our Hermitian identity, we have

$$\left\langle \frac{d^2}{dx^2}(\psi_1) | \psi_2 \right\rangle = \int_{-\infty}^{\infty} (\psi_1'')^* \psi_2 dx$$

where the dashes denote differentiation with respect to the spatial variable.

Now, we apply integration by parts, transfering one of the derivatives from $(\psi_1'')^*$ onto $\psi_2$ (and implicitly using the fact that $(f')^* = (f^*)'$. This gives

$$\left\langle \frac{d^2}{dx^2}(\psi_1) | \psi_2 \right\rangle = [(\psi_1')^* \psi_2]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (\psi_1')^* \psi_2' dx.$$

However, if we want any chance of our wavefunctions $\psi_1, \psi_2$ being normalizable, we need them (and their derivatives) to vanish at spatial infinity – otherwise, there's no way of integrals like $\psi^*\psi$ converging. So, our boundary term vanishes, and we have

$$\left\langle \frac{d^2}{dx^2}(\psi_1)|\psi_2 \right\rangle = -\int_{-\infty}^{\infty} (\psi_1')^* \psi_2' dx.$$

Integrating by parts again, we have

$$\left\langle \frac{d^2}{dx^2}(\psi_1)|\psi_2 \right\rangle = -[\psi_1^* \psi_2']_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (\psi_1)^* \psi_2'' dx.$$

Once again, we know that the boundary contribution has to vanish, and so all we're left with is the integral in the second term. But a closer inspection reveals that this is nothing but

$$\left\langle \frac{d^2}{dx^2}(\psi_1|\psi_2 \right\rangle = \left\langle \psi_1|\frac{d^2}{dx^2}(\psi_2) \right\rangle$$

which is precisely the statement that $\frac{d^2}{dx^2}$ is Hermitian. With our arguments at the beginning of the answer, this proves that $H$ is Hermitian!

**Question 4**

An anti-hermitian operator is equal to minus its Hermitian conjugate $\hat{L}^\dagger = -\hat{L}$.

(a) Show that the expectation value of an anti-hermitian operator is imaginary.

To prove this, all we need to do is to recall that, for an imaginary number $z$, one always has $z = -z^*$. With this, for some anti-hermitian operator $L$, we have

$$\begin{aligned}
[\langle L \rangle]^* &= [\langle \psi | L | \psi \rangle]^* \\
&= \langle \psi | L^\dagger | \psi \rangle \\
&= \langle \psi | (-L) | \psi \rangle \\
&= -\langle \psi | L | \psi \rangle \\
&= -\langle L \rangle
\end{aligned}$$

and so the expectation value $\langle L \rangle$ (in any arbitrary state $|\psi\rangle$) must be purely imaginary.

(b) Show that the commutator of two Hermitian operators is anti-hermitian. How about the commutator of two anti-hermitian operators?

This is a plug-and-chug type question. Let $H, M$ be Hermitian; then

$$\begin{aligned}
[H, M]^\dagger &= (HM - MH)^\dagger \\
&= (HM)^\dagger - (MH)^\dagger \\
&= M^\dagger H^\dagger - H^\dagger M^\dagger \\
&= MH - HM \\
&= -[H, M]
\end{aligned}$$

so $[H, M]$ is anti-Hermitian.

I leave it to you to repeat this proof for the anti-Hermitian case. Perhaps somewhat surprisingly, since the minus signs cancel each other out, the commutator of anti-Hermitian operators is still anti-Hermitian.

**Question 5**

Is the ground state of the infinite square potential well an eigenfunction of momentum? If so, what is its momentum? If not, why not

The easiest way to see that it is NOT an eigenfunction of momentum is to simply apply the momentum operator to it. We know that the ground state goes like $\psi_0 \sim A \sin(n\pi x/L)$, so obviously when we apply the derivative in $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ to it, we're not going to get some multiple of the same thing back.

As for why not, this is essentially because the boundary conditions prevent it from being a true plane wave $\sim Be^{ipx} = B(\cos px + i \sin px)$, which, as we know, *are* the eigenstates of momentum; the cosine component vanishes so that it can be zero at the boundaries.

**Question 6**

(Liboff 3.4) The displacement operator $\hat{\mathcal{D}}$ is defined by the equation

$$\hat{\mathcal{D}} f(x) = f(x + \zeta).$$

Show that the eigenfunctions of $\hat{\mathcal{D}}$ are of the form

$$\phi_\beta = e^{\beta x} g(x)$$

where

$$g(x + \zeta) = g(x)$$

and $\beta$ is any complex number. What is the eigenvalue corresponding to $\phi_\beta$?

The easiest way – and what you should do if you're asked this under time pressure – to solve this problem would be through direct substitution. But, for fun, let's see if we can get at it directly. Our eigenvalue equation is given by

$$\hat{\mathcal{D}} \phi(x) = \lambda \phi(x)$$

or, in terms of $\zeta$, which we take to be some fixed complex number, it is

$$\phi(x + \zeta) = \lambda \phi(x).$$

Now, the intuition here is that, in Fourier space, translations can be represented just by additional phase factors (see last week's tutorial, and in particular the Shift Theorem of Question 9) – so, let's take a Fourier transform of both sides. We have

$$\mathcal{F}\{\phi(x + \zeta))\} = \mathcal{F}\{\lambda \phi(x)\}$$

which, after denoting $\mathcal{F}\{\phi(x)\}$ as $\tilde{\phi}(k)$ and using the Shift Theorem, gives

$$e^{ik\zeta} \tilde{\phi}(k) = \lambda \tilde{\phi}(k).$$

Rearranging, we have a curious equation given by

$$(e^{ik\zeta} - \lambda)\tilde{\phi}(k) = 0.$$

There are two possibilities here, depending on (for each $\lambda$) the value of $k$. Either $\tilde{\phi}(k) = 0$, or $\tilde{\phi}(k) \neq 0$ and

$$e^{ik\zeta} = \lambda.$$

Writing $\lambda = e^{i\alpha}$ for any complex number $\alpha$ (note that $\alpha$ is complex, so $\lambda$ does not have to be of unit length), we have then that

$$e^{ik\zeta} = e^{i\alpha}$$

and, solving for $k$, that

$$k = \frac{\alpha}{\zeta} + \frac{2\pi}{\zeta}n$$

where $n$ is some integer.

Interesting! So that means that the Fourier transform of $\phi(k)$ is zero everywhere, except at discrete values of $k$ given by $k = \frac{2\pi}{\zeta}n+$some complex constant, where we previously called the complex constant $\frac{\alpha}{\zeta}$, with $\alpha$ arbitrary.

But now, recall that for some arbitrary function $g(x)$ which is periodic in $\zeta$ (ie. $g(x) = g(x + \zeta)$, you usually represent it in the frequency domain as a Fourier series (with frequencies of $\frac{2\pi}{\zeta}n$ for integer $n$). If you want to write it as a Fourier transform, you need to essentially write is as a series of discrete spikes, peaked at $\frac{2\pi}{\zeta}n$ and zero everywhere else – which is almost exactly what we have! The only difference is that our spikes are shifted by some complex constant, which I'll now (for reasons you'll see in a moment) call $-i\beta$; then we can write

$$\tilde{\phi(k)} = \tilde{g}(k - i\beta)$$

where $\tilde{g}$ is the Fourier transform of our aforemention $\zeta$-periodic function $g(x)$. Doing an inverse Fourier transform to go back into $x$-space and again applying the Shift Theorem (but in reverse), we get

$$\phi(x) = e^{\beta x}g(x)$$

which is precisely the form suggested by the question, with $\beta$ and arbitrary complex number. Very nice!

The last thing to find is the eigenvalue $\lambda$, which we can find by direct substitution into our eigenvalue equation. We have

$$e^{\beta(x+\zeta)}g(x + \zeta) = \lambda e^{\beta x}g(x).$$

Using the fact that $g(x+\zeta) = g(x)$ and cancelling appropriate terms (assuming $g \neq 0$) we have

$$e^{\beta\zeta} = \lambda$$

which means essentially that $\lambda$ can be any complex number (including zero, with the trivial $g(x) = 0$).

**Question 7**

(Liboff 3.17) Show that for an arbitrary "well-behaved" function $f(x)$,

$$\exp\left(\frac{i\zeta\hat{p}}{\hbar}\right) f(x) = f(x+\zeta)$$

where $\hat{p}$ is the momentum operator and the constant $\zeta$ represents a small displacement. In this problem, you should demonstrate that the left-hand side of the equation is the Taylor series expansion of the right-hand side about $\zeta = 0$.

You might be unhappy with a derivative appearing inside an exponential here on the left hand side, but remember that we can define our exponential function in terms of a power series – a definition which extends even to any operator $A$ by

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

In particular, then, we have for $\exp(\frac{i\zeta p}{\hbar}) = \exp(\frac{\zeta\partial}{\partial x})$ that

$$\begin{aligned} LHS &= \exp(\frac{\zeta\partial}{\partial x}) f(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n (\frac{\partial^n f(x)}{\partial x}). \end{aligned}$$

Of course, with $\zeta$ taken to be infinitesimally small, we only need to account for the first non-trivial term in the series, with

$$LHS = f(x) + \zeta f'(x).$$

since all the other terms are suppressed by the higher powers of $\zeta$, when $\zeta$ is small enough. Now, with $\zeta$ infinitesimally small, we can also recall our definition of the derivative as

$$f'(x) = \frac{f(x+\zeta) - f(x)}{\zeta}$$

in which case

$$\begin{aligned} LHS &= f(x) + f(x+\zeta) - f(x) \\ &= f(x+\zeta) \\ &= RHS. \end{aligned}$$

So our proof is complete. As an aside, this is why momentum is often called the "generator of translations"; when you exponentiate the momentum operator (with the appropriate normalizing coefficients) and apply it to some function, you cause a slight shift in the position where the function is evaluated – you displace it. There's some wonderful group theory stuff going on here, which I leave you to look at in your own time.

**Question 8**

(a) Substitute the Fourier transform

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

into its own inverse to show that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(x'-x)} dk = \delta(x'-x) \tag{1}$$

The inverse Fourier transform is given (in these conventions) by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+ikx} f(k) dk.$$

So, plugging in our definition for $f(k)$ (and changing our variable of integration to $x'$, since $x$ is already used up on the LHS as a non-dummy variable), we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+ikx} [\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} f(x') dx'] dk.$$

We can now rearrange our integrals and regroup, which gives

$$f(x) = \int_{-\infty}^{\infty} [\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(x'-x)} dk] f(x') dx'.$$

Now, since this holds for arbitrary (well-behaved) $f(x)$, we see that the function

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(x'-x)} dk$$

selects out the value at $f(x'=x)$, and thus must be $\delta(x'-x)$ as required, since that's the very definition of the $\delta$-function.

(b) Alternatively, starting with the basis definition ($p = \hbar k$)

$$\langle k | k' \rangle = \delta(k - k')$$

and writing $|k\rangle = Ae^{ikx}$, insert the continuity formula for $x$

$$\int_{-\infty}^{\infty} |x\rangle \langle x| \, dx = 1$$

to obtain the same equation (1). What value of the normalisation constant $A$ should be used?

A sensible value of the normalization constant would be $A = \frac{1}{\sqrt{2\pi}}$, since that normalizes $\langle k|k \rangle$ over the period $[0, 2\pi]$ (as you can check).

Now, on the LHS, we have

$$\begin{aligned}\langle k|k'\rangle &= \langle k|\,(1)\,|k'\rangle \\ &= \langle k|\,[\int_{-\infty}^{\infty} |x\rangle\,\langle x|dx]\,|k\rangle \\ &= \int_{-\infty}^{\infty} dx\,\langle k|x\rangle\,\langle k'|x\rangle \\ &= \frac{1}{2\pi}\int e^{-ix(k-k')}dk\end{aligned}$$

which gives the same result as before, only with $x$ and $k$ swapped, when we equate it to the RHS.
As a bonus comment, note that although the question says $|k\rangle = Ae^{ikx}$, a more precise statement would be that it is the position-space representation of $|k\rangle$ that should be on the there; namely, that $\langle x|k\rangle = Ae^{ikx}$, which is what we used in the derivation above.

**Question 9**

(Liboff 3.7) Show that the following are valid representation of $\delta(y)$:

(a)

$$2\pi\delta(y) = \int_{-\infty}^{\infty} e^{iky}dk$$

We already showed this in Question 8, just with $x' = 0$.

(b)

$$\pi\delta(y) = \lim_{\eta\to\infty} \frac{\sin\eta y}{y}$$

This was a challenging one! I have to admit that I did resort to some sneaky inquiries for ChatGPT to answer this one.

Consider some function $\chi(y)$; let's integrate our representation on the RHS multiplied by $\chi$ to see if we get $\pi\chi(0)$ as the LHS suggests! We have

$$\int_{-\infty}^{\infty} RHS \times \chi(y)dy = \lim_{\eta\to\infty}\int_{-\infty}^{\infty} \frac{\sin\eta y}{y}\chi(y)dy$$

where we have commuted the limit with the integral (assuming all the nice convergence properties and so on).
Now, let's rewrite this in terms of the sinc function $\text{sinc(x)} = \sin x/x$. This gives

$$\int_{-\infty}^{\infty} RHS \times \chi(y)dy = \lim_{\eta\to\infty}\int_{-\infty}^{\infty} \eta\text{sinc}(\eta y)\chi(y)dy.$$

The trick here is to now do a change of variables $y \to y' \equiv \eta y$. Then (assuming $\eta > 0$, since we take the limit to infinity), we get

$$\int_{-\infty}^{\infty} RHS \times \chi(y)dy = \lim_{\eta \to \infty} \int_{-\infty}^{\infty} \text{sinc}(y')\chi(\frac{y'}{\eta})dy'$$
$$= \chi(0) \int_{-\infty}^{\infty} \text{sinc}(y')dy'$$

where we have used the fact that $\lim_{\eta \to \infty} \chi(y/\eta) = \chi(0)$, and taken it out the integral. However, this final integral is a standard integral which can be evaluated using the so-called "Feynman's trick" (I'll let you search this one up"), which gives $\int_{-\infty}^{\infty} \text{sinc}(y')dy' = \pi$ (note that others define sinc with different conventions involving $\pi$, so be careful if you're looking at a different result). So, we have

$$\int_{-\infty}^{\infty} RHS \times \chi(y)dy = \pi\chi(0)$$
$$= \int_{-\infty}^{\infty} \pi\delta(y)\chi(y)dy$$
$$= \int_{-\infty}^{\infty} LHS \times \chi(y)dy$$

and so our result is proven.

*Note:* In mathematics an object such as $\delta(y)$, which is defined in terms of its integral properties, is called a *distribution*. Consider all functions $\chi(y)$ defined on the interval $(-\infty, \infty)$ for which

$$\int_{-\infty}^{\infty} |\chi(y)|^2 dy < \infty.$$

Then two distributions $\delta_1$ and $\delta_2$ are *equivalent* if for all such $\chi(y)$,

$$\int_{-\infty}^{\infty} \chi(y)\delta_1 dy = \int_{-\infty}^{\infty} \chi(y)\delta_2 dy.$$

When one establishes that a mathematical form such as those shown in parts (a) and (b) above are a representation of $\delta(y)$, one is in effect demonstrating that these two object are equivalent as distributions.

**Question 10**

(past exam question) A particle with an energy E is tunneling through a potential barrier (see figure below). Note that such a tunneling process takes place for example at a metal-insulator-semiconductor thin film structure used in semiconducting industry.

(a) Sketch the form of the wave function expected in each of the 3 regions and give the functional forms of these wave functions. Explain the choice of the wave function for each region in a few words.

For this problem, since we're told explicitly that the wavefunction tunnels through the barrier, we'll assume that $E$ is sufficiently high so that the wavefunction can tunnel through it, without landing inside the second step barrier and exponentially decaying away.

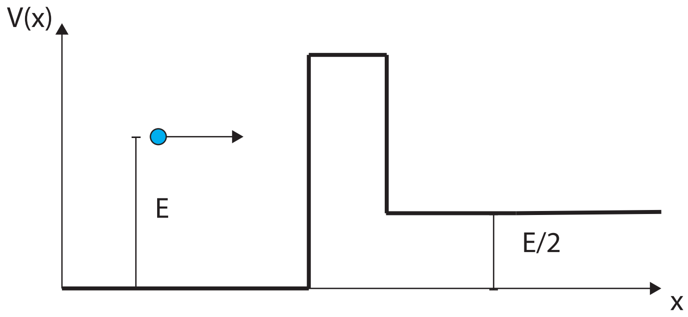


Figure 1:

I'll leave you to sketch, but essentially the wave function is a propagating plane wave before the barrier (including both the incoming and reflected components), then inside the barrier, it becomes an exponential decaying function (including an incoming and reflected component), and then under our assumptions, it resumes its plane wave oscillation (including only an outgoing wave, since there's no reflection after the particle exits the barrier). If $E$ were lower than the height of the second barrier, it would continue to decay into the second barrier, and continue decaying forever onwards. As mentioned at the start, we'll assume that this is not the case.

To get a mathematical form for these regions, let's assume the first step in Region 2 is at height $V$, while the second step in Region 3 is at height $V'$ (with $V' = E/2$ shown in the Figure).

Then, we generically expect three different functional forms for the three different regions, following the lectures:

$$\begin{aligned}
&\text{Region 1: } \psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \\
&\text{Region 2: } \psi(x) = Ce^{-\kappa_2x} + De^{\kappa_2x} \\
&\text{Region 3: } \psi(x) = Fe^{ik_3x}
\end{aligned}$$

where we define

$$\begin{aligned}
\hbar k_1 &= \sqrt{2mE} \\
\hbar \kappa_2 &= \sqrt{2m(V-E)} \\
\hbar k_3 &= \sqrt{2m(E-V')}
\end{aligned}$$

(b) What are the mathematical boundary conditions for the waves when the particle enters and leaves the barrier?

Let's set the beginning of the Region 2 barrier at $x = 0$, and then the end of the Region 2 barrier at $x = a$. As boundary conditions, we can use the fact that:

- $\psi(x)$ should be continuous at $x = 0$ and $x = a$.

- $\psi'(x)$ should be continuous at $x = 0$ and $x = a$.

This enforces the matchings of the different coefficients. As this article points out, we would like $\psi$ to be continuous, since momentum is associated with a derivative of $\psi$, and a discontinuity in $\psi$ would thus appear as an infinite spike in the momentum; similarly, we would like $\psi'$ to be continuous, since energy is associated with the second derivative (via $p^2/2m$), and so a discontinuity in $\psi'$ would thus appear as an infinite spike in energy. In both cases, it seems rather unphysical.

(c) Derive the equations that must be solved to obtain the constants appearing in the wave functions

From the matchings of the wavefunctions at each interface we have

$$\begin{aligned} \psi(0): &\quad A + B = C + D \\ \psi(a): &\quad Ce^{-\kappa_2 a} + De^{\kappa_2 a} = Fe^{ika} \end{aligned}$$

and from the matchings of the derivatives at each interface we have

$$\begin{aligned} \psi'(0): &\quad ik_1(A - B) = \kappa_2(D - C) \\ \psi'(a): &\quad \kappa_2(De^{\kappa_2 a} - Ce^{-\kappa_2 a}) = i\kappa_3 Fe^{ika} \end{aligned}$$

(d) If the potential energy in Region 3 is exactly E/2, calculate the ratio of the wavelength in Regions 1 and 3.

If $V' = E/2$, then from our answer to part $A$ we have

$$\frac{k_1}{k_3} = \frac{\sqrt{2mE}}{\sqrt{2mE/2}} = \sqrt{2}.$$

**Question 11**

Complete the analysis of the finite quantum well scattering states (i.e. $E > 0$ and $V = V_0$) from the notes (or otherwise) to derive an analytical expression for the reflectance ($R$) and transmittance ($T$). Find an expression for the energies at which the transmission is 100%.

At the end of Lecture 8, we have (adjusting some typos from the notes).

$$\begin{pmatrix} Ae^{ik_1x} \\ Be^{-ik_1x} \end{pmatrix} = D_1^{-1} D_2 P_2 D_2^{-1} D_1 \begin{pmatrix} Fe^{ik_2x} \\ 0 \end{pmatrix}$$

where we note that the frequency of the outgoing wave is $k_1$ (so $D_3 = D_1$ from the lecture notes). Using the definitions from the lectures, we have

$$\begin{aligned} D_1^{-1} D_2 P_2 D_2^{-1} D_1 = &\begin{pmatrix} 1/2 & 1/(2k_1) \\ 1/2 & -1/(2k_1) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ k_2 & -k_2 \end{pmatrix} \begin{pmatrix} e^{-ik_2\delta_x} & 0 \\ 0 & e^{ik_2\delta_x} \end{pmatrix} \\ &\times \begin{pmatrix} 1/2 & 1/(2k_2) \\ 1/2 & -1/(2k_2) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ k_1 & -k_1 \end{pmatrix} \end{aligned}$$

which we can simplify after some algebra to

$$= \begin{pmatrix} \cos(\delta_x k_2) - \frac{i\left(k_1^2+k_2^2\right)\sin(\delta_x k_2)}{2k_1k_2} & \frac{i(k_1-k_2)(k_1+k_2)\sin(\delta_x k_2)}{2k_1k_2} \\ -\frac{i(k_1-k_2)(k_1+k_2)\sin(\delta_x k_2)}{2k_1k_2} & \cos(\delta_x k_2) + \frac{i\left(k_1^2+k_2^2\right)\sin(\delta_x k_2)}{2k_1k_2} \end{pmatrix}.$$

Plugging this into our original definition we get

$$\begin{pmatrix} Ae^{ik_1x} \\ Be^{-ik_1x} \end{pmatrix} = Fe^{ik_2x} \begin{pmatrix} \cos(\delta_x k_2) - \frac{i\left(k_1^2+k_2^2\right)\sin(\delta_x k_2)}{2k_1k_2} \\ -\frac{i(k_1-k_2)(k_1+k_2)\sin(\delta_x k_2)}{2k_1k_2} \end{pmatrix}$$

With this, we can calculate our reflectance as

$$R = \left|\frac{B}{A}\right|^2 = \left|\frac{Be^{-ik_1x}}{Ae^{ik_1x}}\right|^2$$

since the absolute value squared cancels out the complex phases. So, with some algebra using the result above, this becomes

$$R = \frac{\left(k_1^2 - k_2^2\right)^2 \sin^2(\delta k_2)}{\left(k_1^2 + k_2^2\right)^2 \sin^2(\delta_x k_2) + 4k_1^2k_2^2\cos^2(\delta_x k_2)}.$$

For transmittance, we need only use that $T = 1 - R$, which gives (after some simplifying)

$$T = \frac{4k_1^2k_2^2}{4k_1^2k_2^2\cos(\delta_x k_2)^2 + (k_1^2 + k_2^2)\sin^2(\delta_x k_2)}$$

**Question 12**

(adapted Griffiths 2.35) A particle of mass $m$ and kinetic energy $E > 0$ approaches an abrupt potential drop $V_0$.

(a) What is the probability that it will "reflect" back, if $E = V_0/3$?

This is again just a scattering problem. Incoming to the drop $V = -V_0$ at, say, $x = 0$, we have plane waves

$$\psi(x) = Ae^{ik_1x} + Be^{-ik_1x}$$

where $A$ corresponds to the forward-propagating, incoming wave, and $B$ corresponds to the backwards-propagating, reflected wave. Then, on the other side of the barrier, we have

$$\psi(x) = Ce^{ik_2x}$$

with no backwards propagating reflection, since this is on the other side of the barrier. Our wavevectors are given by

$$\hbar k_1 = \sqrt{2mE}$$
$$\hbar k_2 = \sqrt{2m(E + V_0)}$$

seen by solving the Schrodinger equation in these regions.

Now, as usual, we demand that the wavefunction and its first derivative are continuous at $x = 0$. This gives

$$A + B = C$$
$$k_1(A - B) = k_2 C$$

Dividing our second equation by $k_1$ on the both sides, we have

$$A + B = C$$
$$A - B = \frac{k_2}{k_1} C.$$

Adding these equations gives

$$A = \frac{C}{2}(1 + k_2/k_1)$$

while subtracting them gives

$$B = \frac{C}{2}(1 - k_2/k_1).$$

Specialising specifically for $E = V_0/3$, we have

$$\frac{k_2}{k_1} = \sqrt{\frac{2m(4V_0/3)}{2m(V_0/3)}} = 2$$

such that

$$A = \frac{3C}{2}$$
$$B = -\frac{C}{2}$$

The reflectance $R$ is defined as $\left|\frac{B}{A}\right|^2 = \frac{1}{9}$, so there is a 1 in 9 chance that it will reflect back.

(b) When a free neutron enters a nucleus, it experiences a sudden drop in potential energy, from $V = 0$ to $V = -12\,\mathrm{MeV}$ inside. Suppose a neutron, emitted with kinetic energy $V = 4\,\mathrm{MeV}$ by a fission event, strikes such a nucleus. What is the probability it will be absorbed, initiating another fission? *Hint: You calculated the probability of reflection in part (a): use $T = 1 - R$ to get the probability of transmission through the surface.*

Note that this is precisely the scenario provided in the previous part; the incoming wave has energy $E = V/3$, where here $V = 12$ MeV is the energy of the drop. The probability of absorption is then the probability of the particle being transmitted through the potential – which is the transmittance, $T$. Of course, it either transmits or reflects, so $T + R = 1$, and thus from part (a) we have $T = 1 - 1/9 = 8/9$.