UNSW SCHOOL OF PHYSICS

PHYS2111 – Quantum Mechanics

Tutorial 9 SOLUTIONS

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Question 1

(past exam question): The Schödinger Eqn is give by:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi \tag{1}$$

and the first eigenfunctions are given by,

$$\begin{aligned} \psi_0(x) &= A_0 e^{-u^2/2} \\ \psi_1(x) &= A_1 u e^{-u^2/2} \end{aligned} \tag{2}$$

where $u = (m\omega/\hbar)^{\frac{1}{2}}\, x$.

(a) Calculate the normalisation constants A_0 and A_1.

This is a very standard question. To find the normalization constants, all we have to do is demand

$$\int_{-\infty}^{\infty} \psi_0^* \psi_0 dx = \int_{-\infty}^{\infty} \psi_1^* \psi_1 dx = 1.$$

For simplicity let us assume A_0, A_1 to be real and positive. Then, for ψ_0 we have

$$\begin{aligned} 1 &= A_0^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} dx \\ &= A_0^2 \sqrt{\frac{\pi\hbar}{m\omega}} \end{aligned}$$

which gives

$$A_0 = (\frac{m\omega}{\pi\hbar})^{1/4}.$$

Similarly for ψ_1 we have

$$\begin{aligned} 1 &= A_1^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} dx \\ &= \frac{A_1^2}{2} \sqrt{\frac{\pi\hbar}{m\omega}} \end{aligned}$$

which gives

$$A_1 = (4\frac{m\omega}{\pi\hbar})^{1/4}.$$

(b) For $\psi_0(u)$ and $\psi_1(u)$: calculate the expectation values $\bar{x}$ and $\bar{p}$.

This question just requires us to follow the definitions and calculate. We have for ψ_0

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty} \psi_0 x \psi_0 \\ &= \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{\hbar}x^2} dx \\ &= 0\end{aligned}$$

since we're integrating an odd function of a symmetric domain, while

$$\begin{aligned}\langle p\rangle &= \int_{-\infty}^{\infty} \psi_0 (-i\hbar\frac{\partial}{\partial x}) \psi_0 \\ &= -i\hbar\sqrt{\frac{m\omega}{\pi\hbar}}(-\frac{m\omega}{\hbar}) \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{\hbar}x^2} dx \\ &= 0\end{aligned}$$

for the same reasons.

For ψ_1, we have

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty} \psi_0 x \psi_0 \\ &= 2\sqrt{\frac{m\omega}{\pi\hbar}}\frac{m\omega}{\hbar} \int_{-\infty}^{\infty} x^3 e^{-\frac{m\omega}{\hbar}x^2} dx \\ &= 0\end{aligned}$$

again by oddness. Finally, we also have

$$\begin{aligned}\langle p\rangle &= \int_{-\infty}^{\infty} \psi_1 (-i\hbar\frac{\partial}{\partial x}) \psi_1 \\ &= -2i\hbar\sqrt{\frac{m\omega}{\pi\hbar}}\frac{m\omega}{\hbar} \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{\hbar}x^2} [-x^2\frac{m\omega}{\hbar} + 1)] dx \\ &= 0\end{aligned}$$

since again, all the terms under the integral are odd, and we're integrating over a symmetric domain.

(c) Calculate the energy of the wave function ψ_0.

We know automatically that the energy of ψ_0 is $\hbar\omega/2$, since the energy of the n'th excited state is given by $(n+\frac{1}{2})\hbar\omega$.

If you didn't know it, then the way to find out would simply be to apply H to it. Since we're told it's an eigenstate, we know $H\psi_0 = E_0\psi_0$, so you can simply act on ψ_0 with H and read off the energy eigenvalue. I leave you to do this yourself.

(d) An atom of mass $m = 4.85 \times 10^{-23}$ g is vibrating. The energy of a vibrational level is $E = 7.2$ meV. What is the displacement amplitude in the classical limit?

> Assuming that the vibrations of the atom can be modelled as a harmonic oscillator, and that the state the system is in has a well defined energy (which it does; $E = 7.2$meV), then the displacement in the classical limit is zero. This is because in the classical limit, the state approaches its expectation value $\langle x \rangle$. And since the eigenfunctions for the harmonic oscillator have a definite parity (odd/even) because the potential is symmetric under $x \to -x$ (see the PhysSE answer by user28823 here for some more discussion), the expectation value of the position is always zero.

Question 2

(Park 4.24) A particle is oscillating in the state $n = 2$ when suddenly the spring loses half its elasticity, $K \to K/2$. Then the particle's energy is measured. What is the probability that

(a) the old ground-state energy is found?

> This problem is almost identical in its structure to Question 10 of tutorial 5; the particle is in the $n = 2$ groundstate given by (see Griffiths Table 2.1, for eg.)
>
> $$\psi_2^0 = (\frac{m\omega}{\pi\hbar})^{1/4}\frac{1}{\sqrt{2}}(\frac{m\omega}{\hbar}x^2 - 1)e^{-\frac{m\omega}{2\hbar}x^2}$$
>
> and we want to find the probability of measuring different energies – that is, of measuring ψ_2^0 as some eigenstates $\psi_0, \psi_1, \ldots$ of the new Hamiltonian.
>
> Also, note that $\omega \equiv \sqrt{k/m}$ so $k \to k/2$ means that $\omega \to w/\sqrt{2}$.
>
> Now, the energies of the old system were given by
>
> $$E_n^0 = (n + \frac{1}{2})\hbar\omega$$
>
> while the energies of the new system are given by
>
> $$E_n = (n + \frac{1}{2})\frac{\hbar\omega}{\sqrt{2}}.$$
>
> The old ground state, then, is $\hbar\omega$, while the new ground state is $\frac{\hbar\omega}{\sqrt{2}}$. Since $\sqrt{2}$ is irrational, there's no value of n that will make $E_n = (2n+1)\frac{\hbar\omega}{\sqrt{2}}$ equal to $E_0^0 = \hbar\omega$, which means that the probability of measuring the old ground state energy is zero. After all, you can only measure one of the energy eigenstates.

(b) the new ground-state energy is found?

> The probability of measuring the new ground state is essentially the prob-

ability of measuring ψ_2^0 as

$$\psi_0 = \frac{1}{2^{1/8}}(\frac{m\omega}{\pi\hbar})^{1/4}e^{-\frac{m\omega}{2\sqrt{2}\hbar}x^2}$$

that is, as the new ground state.
As usual, this probability is given by

$$P_0 = \left|\int_{-\infty}^{\infty} \psi_0^* \psi_2^0 dx\right|^2.$$

This overlap integral is given explicitly by

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_0^* \psi_2^0 dx &= \frac{1}{2^{1/8}}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2[1+1/\sqrt{2}]}dx \\ &= \frac{2^{1/8}}{\sqrt{\sqrt{2}+1}}\end{aligned}$$

in which case

$$P_0 = \frac{2^{1/4}}{\sqrt{2}+1} \approx 0.49$$

(c) the new energy corresponding to n=1 is found.

In exactly the same way as with the previous question, we have that

$$P_1 = \left|\int_{-\infty}^{\infty} \psi_1^* \psi_2^0 dx\right|^2$$

where the new first excited state is given by

$$\psi_1 = 2^{3/8}(\frac{m\omega}{\pi\hbar})^{1/4}(\frac{m\omega}{\hbar})^{1/2}xe^{-\frac{m\omega}{2\sqrt{2}}x^2}.$$

However, since ψ_1 is an odd (and real) function, ψ_0^2 is an even function, $\psi_1^*\psi_0^2 = \psi_1\psi_0^2$ is odd and thus integrates to zero, so $P_1 = |0|^2 = 0$.

Question 3

(a) Using the ladder operator method, calculate the $n = 3$ wave function and verify that it matches that predicted by the power series method.

I leave you to show this.

(b) Does the ladder operator preserve the normalisation condition when generating new eigenstates for the harmonic oscillator.

No. Check out the end of section 2.3.1 in Griffiths for a proof of the fact that you need to divide by a factor of $\frac{1}{\sqrt{n!}}$; that is,

$$\psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n \psi_0$$

to get the normalized solution.

Question 4

(Park 4.27) Starting with two operators $\hat{a}$ and $\hat{a}^\dagger$ that satisfy the $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, find the eigenvalues and eigenfunctions of $\hat{a}^\dagger \hat{a}$. Now do the same for operators $\hat{b}$ and $\hat{b}^\dagger$ that satisfy $\{\hat{b}, \hat{b}^\dagger\} = 1$, where $\{\hat{b}, \hat{b}^\dagger\}$ is the anti-commutator, $\hat{b}\hat{b}^\dagger + \hat{b}^\dagger \hat{b}$, and $\hat{b}^2 = \hat{b}^{\dagger 2} = 0$

See these notes for more details.

Consider the eigenvector/eigenvalue equation

$$a^\dagger a \left|\psi\right\rangle = \lambda \left|\psi\right\rangle$$

Now, drawing inspiration from the ladder operator method in the lectures, let's suppose we have one solution ψ, λ which we know this is true. Then, we conjecture that $a^\dagger \psi$ is also an eigenvector with eigenvalue $\lambda + 1$. Let's prove this. Applying $a^\dagger a$ to $a^\dagger \psi$ we have

$$\begin{aligned}
a^\dagger a(a^\dagger \left|\psi\right\rangle) &= a^\dagger (a a^\dagger) \left|\psi\right\rangle \\
&= a^\dagger (a a^\dagger - a^\dagger a + a^\dagger a) \left|\psi\right\rangle \\
&= a^\dagger ([a, a^\dagger] + a^\dagger a) \left|\psi\right\rangle \\
&= a^\dagger (1 + a^\dagger a) \left|\psi\right\rangle \\
&= a^\dagger (1 + \lambda) \left|\psi\right\rangle \\
&= (1 + \lambda)(a^\dagger \left|\psi\right\rangle).
\end{aligned}$$

So, we can call $a^\dagger$ a raising operator – if we have one eigenvector ψ with eigenvalue λ, then $a^\dagger \psi$ has eigenvalue $\lambda + 1$ (alternatively, you could also have that $a^\dagger \left|\psi\right\rangle = 0$). You can show similarly that a is a lowering operator – that is, $a\psi$ gives you eigenvalue $\lambda - 1$.

So, all that's left is to find one $\left|\psi\right\rangle$ and one λ for which this equation holds. Now, to do this, we need to prove a little lemma; namely that $\lambda \geq 0$. Why is this the case?

Well, any vector needs to have a positive norm. In particular, we know that

$$\left|a \left|\psi\right\rangle\right|^2 \geq 0$$

Writing out the (squared) norm explicitly, we have

$$\begin{aligned}
0 &\leq (a \left|\psi\right\rangle)^\dagger (a \left|\psi\right\rangle) \\
&= \left\langle\psi\right| a^\dagger a \left|\psi\right\rangle \\
&= \lambda \left\langle\psi|\psi\right\rangle \\
&= \lambda.
\end{aligned}$$

So, our lemma is proven!

But actually, if $\lambda \geq 0$, this means that necessarily there must be some eigenvector for which $\lambda = 0$. Why is this the case? Suppose that $\lambda = 0$ was not in the spectrum; then, there must be some $\lambda' \in (0, 1)$ (with $\lambda = 1$ excluded, since this solution could be lowered to a $\lambda = 0$ solution). But this means that $\lambda' - 1 < 0$ is also in the spectrum, which contradicts our lemma above!

So, there is a ground state $|\psi_0\rangle$ for which $a|\psi_0\rangle = 0$, in which case $a^\dagger a(a\psi_0) = a^\dagger a(0) = 0$; the $\lambda = 0$ solution.

Thus, the set of eigenvalues is given by the non-negative integers $\{0, 1, 2, \ldots\}$, and the set of eigenvectors is given by $(a^\dagger)^n |\psi_0\rangle$ for integer $n \geq 0$, where $|\psi_0\rangle$ is defined by $a|\psi_0\rangle = 0$.

For $b^\dagger b$, the idea is similar, but plays out quite differently. Again following the idea of ladder operators, let's suppose we have one state $|\phi\rangle$ and one eigenvalue ω for which we know

$$b^\dagger b |\phi\rangle = \omega |\phi]\rangle .$$

What happens if we apply $b^\dagger b$ to $b^\dagger |\phi\rangle$? We get

$$\begin{aligned} b^\dagger b(b^\dagger |\phi\rangle) &= b^\dagger(bb^\dagger) |\phi\rangle \\ &= b^\dagger(bb^\dagger + b^\dagger b - b^\dagger b) |\phi\rangle \\ &= b^\dagger(\{b, b^\dagger\} - b^\dagger b) |\phi\rangle \\ &= b^\dagger(1 - b^\dagger b) |\phi\rangle \\ &= b^\dagger |\phi\rangle - (b^\dagger)^2 b |\phi\rangle \\ &= b^\dagger |\phi\rangle . \end{aligned}$$

At a glance, this seems strange. It means that for any state $|\phi\rangle$, either $b^\dagger |\phi\rangle = 0$ or we have eigenvalue one. To investigate further, what about $b|\phi\rangle$? Well, we have

$$b^\dagger b(b|\phi\rangle) = b^\dagger(b^2)|\phi\rangle = 0.$$

So, this tells us that there are only two eigenvalues (which we can find using this method; see the final, final comment); $\omega = 0$ and $\omega = 1$. For any other eigenvalues, we would essentially need to construct powers of $b^\dagger$, but we know that $(b^\dagger)^n$ for $n \geq 2$ is zero.
Then, one can find $|\psi_0\rangle$ by insisting $b|\psi_0\rangle = 0$, as before.

Now, as a final comment, this might seem like a whole lot of mathematical mumbo-jumbo, but I love this question because it gets at some really deep physics. Namely, in quantum field theory, it turns out that $a^\dagger$ and a are intimately related to states involving multiple *bosons*, with $a^\dagger$ adding a boson, a removing a boson, and the operator $a^\dagger a$ counting the number of bosons in a particular state. Meanwhile $b^\dagger$ and b are intimately related to states involving multiple *fermions*, with $b^\dagger$ adding a fermion, b removing a fermion, and $b^\dagger b$ counting the number of fermions in a particular state. The fact that $b^\dagger b$ only has eigenvalues $0, 1$ is essentially a statement of the Pauli exclusion principle; you can have either 0 or 1 fermions in a particular state, but none more. Meanwhile, you can put as many bosons in a particular state as you like.

(As a final, final comment; you might be wondering: what about completeness? Could we have missed some eigenvectors/eigenvalues? Check out this answer on PhysSE by Valter Moretti.)

Question 5

Solve Schrödinger's Eqn for the infinitely thin potential barrier $V(x) = +\alpha\delta(x)$ and calculate the corresponding reflectance and transmission.

This is the same as the discussion in the lecture, only with $\alpha \to -\alpha$; the reflectance and transmission, however, depend only on the square of α, so they remain the same.

Note however that the bound state cannot exist with $V(x) = +\alpha\delta(x)$ for $\alpha > 0$; from the relation

$$\kappa = \frac{m\alpha}{\hbar^2}$$

with $V(x) = -\alpha\delta(x)$, then taking $\alpha \to -\alpha$ makes κ negative, which leads to a non-normalizable solution.

Question 6

(Griffiths 2.24) Check the uncertainty principle for the wave function described by the bound infinite delta function potential.

This question looks like another plug and chug type question, but actually has some subtleties; see this PhysSE answer and this Wikipedia page about derivatives of the absolute value function . We need to calculate σ_x and σ_p, and show that $\sigma_x\sigma_p \geq \frac{\hbar}{2}$. For simplicity, let's define

$$z \equiv \frac{\sqrt{m\alpha}}{\hbar}$$

in which case

$$\psi_b = ze^{-z^2|x|}.$$

Now we just have lots of integrals to do. But, by symmetry, we know that

$$\langle x \rangle = \langle p \rangle = 0$$

(where one way to see that $\langle p \rangle = 0$ is to notice that, since ψ_b is even, so is its Fourier transform, in which case $\hbar k\times$ the k-space wavefunction will be odd and so the expectation value of p evaluated using the k space wavefunction will vanishes), so... only two integrals left!

This gives

$$\begin{aligned}
\left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} z^2 x^2 e^{-2z^2|x|} dx \\
&= 2\int_{0}^{\infty} z^2 x^2 e^{-2z^2 x} dx \quad \text{ by even-ness} \\
&= \frac{1}{2z^4}
\end{aligned}$$

and

$$\left\langle p^2 \right\rangle = -\hbar^2 z^2 \int_{-\infty}^{\infty} e^{-z^2|x|} \frac{\partial^2}{\partial x^2} e^{-z^2|x|}$$

so we need to know how to take derivatives involving the absolute value function. In particular,

$$\frac{\partial}{\partial x} e^{-z^2|x|} = -z^2 \mathrm{sgn}(x) e^{-z^2|x|}$$

where $\mathrm{sgn}(x)$ is the sign function, equal to $+1$ for $x > 0$ and -1 for $x < 0$, and technically undefined at $x = 0$. Differentiating again, we get

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} e^{-z^2|x|} &= -z^2 \frac{\partial}{\partial x} \mathrm{sgn}(x) e^{-z^2|x|} \\
&= -z^2 [-z^2 \mathrm{sgn}^2(x) e^{-z^2|x|} - e^{-z^2|x|} \frac{\partial}{\partial x} \mathrm{sgn}(x)] \\
&= z^2 [z^2 - 2\delta(x)] e^{-z^2|x|}
\end{aligned}$$

where I leave you to peruse the linked Wikipedia article above for an explanation on why the derivative of the sgn function gives $2\delta(x)$. With this, then, we have

$$\begin{aligned}
\left\langle p^2 \right\rangle &= -\hbar^2 z^4 \int_{-\infty}^{\infty} e^{-2z^2|x|} [z^2 - 2\delta(x)] \\
&= -\hbar^2 z^4 \int_{-\infty}^{\infty} \psi_b^* \psi_b - 2\delta(x) e^{-2z^2|x|} \\
&= -\hbar^2 z^4 (1 - 2) \\
&= \hbar^2 z^4
\end{aligned}$$

where we used the fact that ψ_b is a normalized wavefunction. Thus is the subtlety of this question; if you didn't account for that δ-function lurking about, you'd get a negative value for $\left\langle p^2 \right\rangle$, which is evidently non-sensical. Perhaps it would have been more sensible to just do the Fourier transform into k-space, and evaluate all the momentum expectation values there in the first place...

Putting it together, we thus have

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{2z^4} - 0} = \frac{1}{\sqrt{2}} \frac{1}{z^2} \\
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\hbar^2 z^4} = \hbar z^2
\end{aligned}$$

in which case

$$\sigma_x \sigma_p = \frac{\hbar}{\sqrt{2}} = \sqrt{2} \times \frac{\hbar}{2} \geq \frac{\hbar}{2}$$

as required!

Question 7

(Griffiths 2.25) Check that the bound state of the delta function well is orthogonal to the scattering state.

From the lectures, the bound state is given by

$$\psi_b = \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha}{\hbar^2}|x|}$$

while the scattering state is given by

$$\psi_s = \begin{cases} Ae^{ikx} + \frac{Ai\beta}{1-i\beta}e^{-ikx} & x \leq 0 \\ \frac{A}{1-i\beta}e^{ikx} & x \geq 0 \end{cases}$$

for k some positive constant (related to the energy of the scattering state), and $\beta = \frac{m\alpha}{\hbar^2 k}$.

So, to show that they're orthogonal, we just need to calculate the overlap integral. We can break it up into two regions

$$\int_{-\infty}^{\infty} \psi_b^* \psi_s dx = I_1 + I_2$$

where

$$I_1 \equiv A \int_{-\infty}^{0} e^{(l+ik)x} + \frac{i\beta}{1-i\beta} e^{(l-ik))x}$$
$$I_2 \equiv A \int_{0}^{\infty} \frac{1}{1-i\beta} e^{(-l+ik)x}$$

where we've split up the absolute value function into the positive and negative regions, and defined $l \equiv m\omega/\hbar^2$. Doing these integrals explicitly, we have

$$I_1 = A[\frac{1}{l+ik} + \frac{i\beta}{1-i\beta}\frac{1}{l-ik}]$$
$$I_2 = A[\frac{1}{1-i\beta}\frac{1}{l-ik}].$$

Adding these together, one has

$$\begin{aligned} I_1 + I_2 &= A[\frac{1}{l+ik} + \frac{1+i\beta}{(1-i\beta)(l-ik)}] \\ &= \frac{l-ik-il\beta-k\beta+l+ik+il\beta-k\beta}{(1-i\beta)(l^2+k^2)} \\ &= \frac{2(l-k\beta)}{(1-i\beta)(l^2+k^2)}. \end{aligned}$$

However, looking at our definitions for k and β, it's clear that $k\beta = l$! So, almost magically, $I_1 + I_2 = 0$ and the overlap integral vanishes, which means the states are orthogonal. How beautiful!

Question 8

(Griffiths 2.27 - harder) Consider the double delta-function potential

$$V(x) = -\alpha\left[\delta(x+a) + \delta(x-a)\right] \tag{3}$$

where α and a are positive constants.

(a) Sketch the potential

> The potential essentially looks like two infinite spikes pointing downwards at $x = a$ and $x = -a$.

(b) How many bound states does it possess? Find the allowed energies, for $\alpha = \hbar^2/ma$ and for $\alpha = \hbar^2/4ma$ and sketch the wave functions

(c) What are are the bound state energies in the limiting cases of (i) $a \to 0$ and (ii) $a \to \infty$

> There is an excellent solution set uploaded online here; I suggest you have a look through!