UNSW SCHOOL OF PHYSICS

PHYS2111 – Quantum Mechanics

Tutorial 8

Question 1

A general wavefunction for a particle in a 2-dimensional Hilbert space is $\psi = \alpha |u\rangle + \beta |d\rangle$, where $|u\rangle$ and $|d\rangle$ are orthonormal basis vectors and α and β are complex numbers. Now consider a system with two such particles. In lectures we wrote down a product basis for this system (see also equations (1) below).

Write down the most general *factorisable* wavefunction for the joint system. That is, find the most general wavefunction for which the total wavefunction Ψ can be expressed as $\psi^{(1)} \otimes \psi^{(2)}$. Write down some wavefunctions that are *not* factorisable. For these wavefunctions, the two particles are called *entangled*.

Question 2

An electron is trapped in a one-dimensional Harmonic potential to which a magnetic field B is also applied. The corresponding Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 z^2 + \mu_B \hat{\sigma}_z B$$

where μ_B is the Bohr magneton and $\hat{\sigma}_z$ is the Pauli matrix.

Find a suitable (separable) basis and hence write down the stationary states of $\hat{H}$.

Question 3

Making a singlet state. Two particles are each in two-level systems with basis functions $|u\rangle$ and $|d\rangle$. The product wavefunction is given in the basis

$$\begin{aligned}
|1\rangle &= \left|u^{(1)}\right\rangle \otimes \left|u^{(2)}\right\rangle \\
|2\rangle &= \left|u^{(1)}\right\rangle \otimes \left|d^{(2)}\right\rangle \\
|3\rangle &= \left|d^{(1)}\right\rangle \otimes \left|u^{(2)}\right\rangle \\
|4\rangle &= \left|d^{(1)}\right\rangle \otimes \left|d^{(2)}\right\rangle
\end{aligned} \tag{1}$$

The Hamiltonian of the system is given in terms of Pauli matrices

$$\hat{H} = \hat{\sigma}_1^{(1)} \otimes \hat{\sigma}_1^{(2)} + \hat{\sigma}_2^{(1)} \otimes \hat{\sigma}_2^{(2)} + \hat{\sigma}_3^{(1)} \otimes \hat{\sigma}_3^{(2)}$$

where the superscript numbers refer to which particle the operators act on and

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Use the rules of direct products to find $\hat{H}|i\rangle$ for each of the basis vectors $i = 1$ to 4.

(b) Hence write $\hat{H}$ as a matrix in this basis.
Answer:

$$\hat{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) Find the eigenvalues and eigenvectors of $\hat{H}$ and show that they form a singlet and triplet (three degenerate eigenstates). These are the stationary states for our Hamiltonian.

(d) Express the singlet eigenvector in terms of the product basis.

Question 4

(Liboff 3.20)

(a) The time-dependent Schrödinger equation is of the form

$$a\frac{\partial\psi}{\partial t} = \hat{H}\psi.$$

Consider a as an unspecified constant. Show that this equation has the following property. Let $\hat{H}$ be the Hamiltonian of a system composed of two independent parts so that

$$\hat{H}(x_1, x_2) = \hat{H}_1(x_1) + \hat{H}_2(x_2)$$

and let the stationary states of system 1 be $\psi_1(x_1, t)$ and those of system 2 be $\psi_2(x_2, t)$. Then the stationary states of the composite system are

$$\psi(x_1, x_2, t) = \psi_1(x_1, t)\psi_2(x_2, t)$$

That is, show that this product form is a solution to the Schrödinger-like equation for the given composite Hamiltonian.

Such a system might be two non-interacting particles moving on the same one-dimensional wire, where x_1 and x_2 are the coordinates of the particles.

(b) Show that this property is not obeyed by a wave equation that is second-order in time, such as

$$a^2\frac{\partial^2\psi}{\partial t^2} = \hat{H}\psi.$$

(c) Arguing from the Born postulate, show that the wavefunction for a system composed of two independent components must be in the product form, thereby disqualifying the wave equation in part (b) as a valid equation of motion for the wavefunction ψ.

Partial answer: The joint probability density must be $P_{1,2} = P_1P_2$ in order to ensure that the probability density associated with component 1,

$$P_1(x_1) = \int P_{1,2}(x_1, x_2)dx_2$$

is independent of P_2 (and vice versa).

Question 5

I'm not going to write this question, but instead encourage those interested to look up Bell's theorem and its experimental verification (Wikipedia is good). The idea an advanced version of Bohm's thought experiment presented in Lectures and proved experimentally by Chien-Shiung Wu (side note, look up her amazing career while you're at it), where entangled particles are sent to different locations and correlations are measured. It shows that quantum correlations are different to classical, cannot be explained by hidden variables, and are non-local.

Question 6

(Griffiths 2.21: warning - this is an extension question) A free particle has the initial wave function

$$\psi(x,0) = Ae^{-\alpha x^2}$$

where A and α are (real and positive) constants.

(a) Normalise $\Psi(x,0)$

(b) Find $\Psi(x,t)$. *Hint*: Integrals of the form

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)}dx$$

can be handled by "completing the squares". *Answer:*

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{1}{\gamma} e^{-ax^2/\gamma^2}, \quad \text{where} \quad \gamma \equiv \sqrt{1+(2i\hbar at/m)}$$

(c) Find $|\Psi(x,t)|^2$. Express your answer in terms of the quantity

$$w \equiv \sqrt{a/[1+(2\hbar at/m)^2]}.$$

Sketch $|\Psi|^2$ (as a function of x) at $t = 0$, and again for some very large t. Qualitatively, what happens to $|\Psi|^2$ as time goes on?

(d) Find $< x >$, $< p >$, $< x^2 >$, $< p^2 >$, σ_x, and σ_p. *Partial answer:* $< p^2 >= a\hbar^2$

(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit.