UNSW SCHOOL OF PHYSICS

PHYS2111 – Quantum Mechanics

Tutorial 8 SOLUTIONS

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Question 1

A general wavefunction for a particle in a 2-dimensional Hilbert space is $\psi = \alpha |u\rangle + \beta |d\rangle$, where $|u\rangle$ and $|d\rangle$ are orthonormal basis vectors and α and β are complex numbers. Now consider a system with two such particles. In lectures we wrote down a product basis for this system (see also equations (1) below).

Write down the most general *factorisable* wavefunction for the joint system. That is, find the most general wavefunction for which the total wavefunction Ψ can be expressed as $\psi^{(1)} \otimes \psi^{(2)}$. Write down some wavefunctions that are *not* factorisable. For these wavefunctions, the two particles are called *entangled*.

The most general wavefunction possible in the 2-particle state can be written as

$$|\psi\rangle = \alpha_1 \left|u^1\right\rangle \otimes \left|u^2\right\rangle + \alpha_2 \left|u^1\right\rangle \otimes \left|d^2\right\rangle + \alpha_3 \left|d^1\right\rangle \otimes \left|u^2\right\rangle + \alpha_4 \left|d^1\right\rangle \otimes \left|d^2\right\rangle .$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are arbitrary complex numbers (subject to the normalization condition). This includes both factorizable and non-factorizable states. The most general factorizable wavefunction can be written as $\left|\psi^1\right\rangle \otimes \left|\psi^2\right\rangle$; that is, for arbitrary complex numbers $\beta_1, \beta_2, \beta_1', \beta_2'$ subject to the normalization condition, as

$$\begin{aligned}
|\psi\rangle_{fac} &= \left|\psi^1\right\rangle \otimes \left|\psi^2\right\rangle \\
&= (\beta_1 \left|u^1\right\rangle + \beta_2 \left|d^1\right\rangle) \otimes (\beta_1' \left|u^2\right\rangle + \beta_2' \left|d^2\right\rangle) \\
&= \beta_1\beta_1' \left|u^1\right\rangle \otimes \left|u^2\right\rangle + \beta_1\beta_2' \left|u^1\right\rangle \otimes \left|d^2\right\rangle + \beta_2\beta_1' \left|d^1\right\rangle \otimes \left|u^2\right\rangle + \beta_2\beta_2' \left|d^1\right\rangle \otimes \left|d^2\right\rangle
\end{aligned}$$

Now, while this may look equivalent to the general $|\psi\rangle$, it is not. This is particularly obvious when some terms are zero. For example, we can write an *entangled* state

$$|\psi\rangle_{ent} = \frac{1}{\sqrt{2}}[\left|u^1\right\rangle \otimes \left|d^2\right\rangle + \left|u^2\right\rangle \otimes \left|d^1\right\rangle].$$

I leave it to you to show that there's no combination of $\beta_1, \beta_2, \beta_1', \beta_2'$ that allows you to factorise this state. So, this is really a combination of two particles that cannot be written as the product of two single-particle states!

Question 2

An electron is trapped in a one-dimensional Harmonic potential to which a magnetic field B is also applied. The corresponding Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 z^2 + \mu_B \hat{\sigma}_z B$$

where μ_B is the Bohr magneton and $\hat{\sigma}_z$ is the Pauli matrix.

Find a suitable (separable) basis and hence write down the stationary states of $\hat{H}$.

Since the magnetic field system and the harmonic potential system are totally independent, our solution is just a separable solution. So, we can write

$$|\psi\rangle = |\phi\rangle \otimes |\alpha\rangle$$

where

$$\mu_B B\hat{\sigma}_z |\alpha\rangle = E_\alpha |\alpha\rangle$$
$$(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 z^2) |\phi\rangle = E_\phi |\phi\rangle$$

with overall energy

$$E = E_\alpha + E_\phi.$$

I'll let you solve both those eigenvector equations and find the answer. But don't be disturbed by the fact that $|\alpha\rangle$ is a 2D column vector and $|\phi\rangle$ is a function of z; they live in different vector spaces, and so their representation is allowed to be different!

Question 3

Making a singlet state. Two particles are each in two-level systems with basis functions $|u\rangle$ and $|d\rangle$. The product wavefunction is given in the basis

$$\begin{aligned} |1\rangle &= \left|u^{(1)}\right\rangle \otimes \left|u^{(2)}\right\rangle \\ |2\rangle &= \left|u^{(1)}\right\rangle \otimes \left|d^{(2)}\right\rangle \\ |3\rangle &= \left|d^{(1)}\right\rangle \otimes \left|u^{(2)}\right\rangle \\ |4\rangle &= \left|d^{(1)}\right\rangle \otimes \left|d^{(2)}\right\rangle \end{aligned} \tag{1}$$

The Hamiltonian of the system is given in terms of Pauli matrices

$$\hat{H} = \hat{\sigma}_1^{(1)} \otimes \hat{\sigma}_1^{(2)} + \hat{\sigma}_2^{(1)} \otimes \hat{\sigma}_2^{(2)} + \hat{\sigma}_3^{(1)} \otimes \hat{\sigma}_3^{(2)}$$

where the superscript numbers refer to which particle the operators act on and

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Use the rules of direct products to find $\hat{H}|i\rangle$ for each of the basis vectors $i = 1$ to 4.

See here for a helpful guide on how to compute these direct products of operators. Going through the calculations, we have

$$\sigma_1^1 \otimes \sigma_1^2 = \begin{pmatrix} 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\sigma_2^1 \otimes \sigma_2^2 = \begin{pmatrix} 0\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\sigma_3^1 \otimes \sigma_3^2 = \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & -1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Hence write $\hat{H}$ as a matrix in this basis.
Answer:

$$\hat{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Summing together the three of the operators we computed in the previous section evidently gives the provided answer.

(c) Find the eigenvalues and eigenvectors of $\hat{H}$ and show that they form a singlet and triplet (three degenerate eigenstates). These are the stationary states for our Hamiltonian.

This is just an eigenvector and eigenvalue problem; I leave it to you to show that the eigenvectors and eigenvalues are of the form:

$$\lambda = -3: \quad |\lambda = -3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\lambda = 1: \quad |\lambda = -3\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where the $\lambda = 1$ space is spanned by three degenerate eigenvectors. To see that these really are the entangled triplet/singlet states, recall that we are in the basis of $|1\rangle, |2\rangle, |3\rangle, |4\rangle$ given above! Write out these column vectors explicitly in terms of the basis, and see if you can factorize it (in accordance to Question 1) – you can't!

(d) Express the singlet eigenvector in terms of the product basis.

As we saw, this is

$$|\lambda = -3\rangle = \frac{1}{\sqrt{2}}[|u^1\rangle |d^2\rangle - |d^1\rangle |u^2\rangle].$$

Question 4

(Liboff 3.20)

(a) The time-dependent Schrödinger equation is of the form

$$a\frac{\partial \psi}{\partial t} = \hat{H}\psi.$$

Consider a as an unspecified constant. Show that this equation has the following property. Let $\hat{H}$ be the Hamiltonian of a system composed of two independent parts so that

$$\hat{H}(x_1, x_2) = \hat{H}_1(x_1) + \hat{H}_2(x_2)$$

and let the stationary states of system 1 be $\psi_1(x_1, t)$ and those of system 2 be $\psi_2(x_2, t)$. Then the stationary states of the composite system are

$$\psi(x_1, x_2, t) = \psi_1(x_1, t)\psi_2(x_2, t)$$

That is, show that this product form is a solution to the Schrödinger-like equation for the given composite Hamiltonian.

Such a system might be two non-interacting particles moving on the same one-dimensional wire, where x_1 and x_2 are the coordinates of the particles.

This is essentially a separation of variables trick, which you've seen a few times now. First of all, we rewrite the Schrödinger like equation as

$$a\frac{\partial \psi}{\partial t} - \hat{H}\psi = 0.$$

To show that $\psi = \psi_1\psi_2$ solves this (for $H = H_1 + H_2$), we'll plug in that particular solution to the LHS and show that it equals zero, the RHS. Writing things out explicitly, applying the product rule on the time derivative, and applying the operators H_1, H_2 only to their respective spaces, we have

$$LHS = \alpha[\frac{\partial \psi_1}{\partial t}\psi_2 + \psi_1\frac{\partial \psi_2}{\partial t}] - (H_1\psi_1)\psi_2 - \psi_1(H_2\psi_2)$$

where I emphasize that H_2 does nothing to ψ_1, and H_1 does nothing to ψ_2, since they are in different spaces.

Now, dividing taking out a common factor of $\psi_1\psi_2$ (under the assumption without loss of generality that they're non-zero, since if they are zero, the equation holds anyways) and rearranging slightly, we get

$$LHS = \psi_1\psi_2[\frac{1}{\psi_1}(\alpha\frac{\partial \psi_1}{\partial t} - H_1\psi_1) + \frac{1}{\psi_2}(\alpha\frac{\partial \psi_2}{\partial t} - H_1\psi_2)].$$

But now, since ψ_1 and ψ_2 both solve their own respective Schrödinger like equations (with Hamiltonians H_1, H_2), we have

$$LHS = \psi_1\psi_2[\frac{1}{\psi_1} \times 0 + \frac{1}{\psi_2} \times 0] = 0 = RHS$$

and so the separable solution solves it!

(b) Show that this property is not obeyed by a wave equation that is second-order in time, such as

$$a^2\frac{\partial^2 \psi}{\partial t^2} = \hat{H}\psi.$$

We can try and go through the same logic as before, and see where things break down. Only now, we have to apply the product rule twice for the two time derivatives. So, plugging in $\psi = \psi_1\psi_2$ and $H = H_1 + H_2$ on our rearranged $LHS = \alpha^2\frac{\partial^2\psi}{\partial t^2} - H\psi$ we have

$$\begin{aligned}
LHS &= \alpha^2\frac{\partial}{\partial t}[\frac{\partial \psi_1}{\partial t}\psi_2 + \psi_1\frac{\partial \psi_2}{\partial t}] - (H_1\psi_1)\psi_2 - \psi_1(H_2\psi_2) \\
&= [\alpha^2\frac{\partial^2 \psi_1}{\partial t} - H_1\psi_1]\psi_2 + \psi_1[\alpha^2\frac{\partial^2 \psi_2}{\partial t} - H_2\psi_2]\psi_2 + 2\alpha^2\frac{\partial \psi_1}{\partial t}\frac{\partial \psi_2}{\partial t} \\
&= 2\alpha^2\frac{\partial \psi_1}{\partial t}\frac{\partial \psi_2}{\partial t}
\end{aligned}$$

where we have used the fact that ψ_1 and ψ_2 solve their own Schrödinger like equations in between lines two and three.

But for this to be equal to the RHS, namely zero, then either ψ_1 or ψ_2 (or both) must have $\frac{\partial \psi_{1,2}}{\partial t} = 0$. Thus, the only separable solutions possible are those for which either ψ_1 or ψ_2 are constant in time! Evidently, then, separable solutions are not very good at modelling the particle motion.

(c) Arguing from the Born postulate, show that the wavefunction for a system composed of two independent components must be in the product form, thereby disqualifying the wave equation in part (b) as a valid equation of motion for the wavefunction ψ.

Partial answer: The joint probability density must be $P_{1,2} = P_1 P_2$ in order to ensure that the probability density associated with component 1,

$$P_1(x_1) = \int P_{1,2}(x_1, x_2) dx_2$$

is independent of P_2 (and vice versa).

The Born postulate says that the probability density is proportional to the square of the wavefunction amplitude, $P_{12} = \psi^* \psi$. Here, we consider a system composed of two independent subsystems, labeled 1 and 2. However, subsystems 1 and 2 are themselves quantum systems, and as such, the probability density for 1 must be given by $P_1 = \psi_1^* \psi_1$ for some ψ_1, and the probability density for 2 must be given by $\psi_2^* \psi_2$ for some ψ_2. Now, since 1 and 2 are **independent**, we know that their probability densities simply multiply; that is,

$$P_{12} = P_1 P_2$$

But that means that

$$\psi^* \psi = \psi_1^* \psi_2^* \psi_1 \psi_2$$

which holds for all x only when

$$\psi = \psi_1 \psi_2 \text{ up to some complex phase}$$

which means precisely that it is a separable solution, since we don't care about that complex phase for physical observables.

Note crucially though that we needed the subsystems to be independent of each other for this to work; when events depend on each other, the probability $P_{12} \neq P_1 P_2$, and so the argument no longer works.

Question 5

I'm not going to write this question, but instead encourage those interested to look up Bell's theorem and its experimental verification (Wikipedia is good). The idea an advanced version of Bohm's thought experiment presented in Lectures and proved experimentally by Chien-Shiung Wu (side note, look up her amazing career while you're at it), where entangled particles are sent to different locations and correlations are measured. It shows that quantum correlations are different to classical, cannot be explained by hidden variables, and are non-local.

I'll let you check out the Wikipedia page for more details. The basic idea is that, if hidden local variables were to exist (along with some additional assumptions, like realism – that these variables exist independently of measurement; and some degree of 'free will' – that the measurement apparatus is uncorrelated with the system being measured), then the correlations have to obey certain mathematical inequalities. However, in QM, it is possible to construct (entangled) states explicitly which violate these mathematical inequalities!

Question 6

(Griffiths 2.21: warning - this is an extension question) A free particle has the initial wave function

$$\psi(x,0) = Ae^{-\alpha x^2}$$

where A and α are (real and positive) constants.

(a) Normalise $\Psi(x,0)$

(b) Find $\Psi(x,t)$. *Hint*: Integrals of the form

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)}dx$$

can be handled by "completing the squares". *Answer:*

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{1}{\gamma} e^{-ax^2/\gamma^2}, \quad \text{where} \quad \gamma \equiv \sqrt{1+(2i\hbar at/m)}$$

(c) Find $|\Psi(x,t)|^2$. Express your answer in terms of the quantity

$$w \equiv \sqrt{a/[1+(2\hbar at/m)^2]}.$$

Sketch $|\Psi|^2$ (as a function of x) at $t = 0$, and again for some very large t. Qualitatively, what happens to $|\Psi|^2$ as time goes on?

(d) Find $< x >$, $< p >$, $< x^2 >$, $< p^2 >$, σ_x, and σ_p. *Partial answer:* $< p^2 >= a\hbar^2$

(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit.

A great solution to this question has already been written up and is available online here, so I'll only provide a rough outline of the key steps and leave the details of the calculation to that solution.

- For part (a), this is a very standard question; you just have to demand that

$$\int_{-\infty}^{\infty} \psi(x,0)^* \psi(x,0) dx = 1$$

which gives you a condition on A; namely, that

$$A = (\frac{2a}{\pi})^{1/4}$$

assuming for simplicity that A is real and positive. As a remark, since time evolution in quantum mechanics is unitary (a consequence of the Hamiltonian H being Hermitian), we know that if it's normalized at $t = 0$, it'll remain normalized for all times.

- For part (b), this is conceptually quite similar to Question 9 of Tutorial 5; we have a state which is *not* an eigenstate of the free Hamiltonian (you can check this), and we're trying to find its time evolution. Since time evolution in the Hamiltonian is trivial – you just need to tack on an $e^{-i\frac{E_k}{\hbar}t}$ factor to your wavefunction – the easiest way to get the time evolution is to express $\psi(x,0)$ in a basis of Hamiltonian eigenstates. For a free particle, the eigenstates of the Hamiltonian correspond to plane waves, expressing $\psi(x,0)$ in a basis of Hamiltonian eigenstates really means... you guessed it... doing a Fourier transform, since a Fourier transform tells you how to express the wavefunction in a basis of plane waves. Then, after tacking on your time dependence, with $E_k = \frac{\hbar k^2}{2m}$, you need to Fourier transform back to get it in position space.

Mathematically, another way of putting this all is that

$$\Phi(x,t) = \mathcal{F}^{-1}\{\mathcal{F}\{\Phi(x,0\}e^{-i\frac{\hbar k^2}{2m}t}\}$$

where Fourier transforms into the k free-Hamiltonian eigenbasis are given by

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}f(x)dx$$
$$\mathcal{F}^{-1}\{f(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{+ikx}f(k)dk.$$

The attached file uses the convolution theorem to simplify this result.

One final comment to make here is that this only applies to the free-particle case – the Fourier transform is just a convenient way of mapping into the basis of plane waves, which is the basis of free-Hamiltonian eigenstates. If your Hamiltonian had a potential, then the eigenstates would no longer necessarily be plane waves, so you'd have to transform into that basis instead.

- Part (c) is really just algebra.
- Part (d) also just involves various integrals. The answers, as outlined in the solution document, are given by

$$\langle x\rangle = 0$$
$$\langle x^2\rangle = \frac{m^2 + 4\hbar^2a^2t}{4m^2a}$$
$$\langle p\rangle = 0$$
$$\langle p^2\rangle = a\hbar^2$$

- Part (e) can be read off from the solution to the previous part; we have

$$\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}} \geq \frac{\hbar}{2}$$

so the uncertainty principle holds at all times t, with the bound saturated at $t=0$.