Group Structure on arbitrary sets: An algebraic application of the Axiom of Choice

by Oskar Emmerich

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Lund University

Faculty of Science
Department of Mathematics

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Thesis advisor: Anitha Thillaisundaram

Abstract

The thesis should include an abstract that summarises its contents; mathematical jargon can be utilised here. The typical length of an abstract is between 100 and 300 words.

Popular science description

test

Contents

Introduction

In 1902 Bertrand Russel showed with what is now known as *Russel's Paradox* that the previously used approach to set theory was inconsitent. Ernst Zermelo then created an axiomatic framework for set theory in 1905, motivated both by attempting to preserve results such as the thoery of infinities by Georg Cantor, as well as avoiding paradoxes. These axioms, later modified by Abraham Fraenkel, became known as the nine *Zermelo-Fraenkel Axioms* (ZF) as well as the *Axiom of Choice* (AC)[Gol17, pp.66-70, 75].

The axiom of choice in particular is of special interest in many areas of mathematics, especially in algebra and topology, ofthen in the form of the equivalent statement of *Zorn's Lemma*, which says that every non-emtpy partially ordered set with an upper bound has a maximal element [Jec78].

Finally in 1971 András Hajnal and Andor Kertész published a paper [HK72] which provided another equivalence to AC, namely that there exists a cancellative groupoid structure on every (uncountably infinite) set. This paper makes use of first-order model theory, an area of logic developed during the first half of the 20th century, which utalises models of formal languages to obtain results. Kertész later expanded on this, providing an alternative algebraic partial proof in a lecture series given at the University of Jyväskylä [Ker75].

Chapter 1

Preliminaries

The convention in this thesis will be to say **ZF** when talking about Zermelo-Fraenkel set theory *without* the axoim of choice. When talking about the axoim of choice on its own we will say **AC**, and when talking about Zermelo-Fraenkel set theory together with the axiom of choice use **ZFC**.

We assume that the reader has some familiarity with axiomatic set theory, but for convenience and constistency we restate some of the necessary basics here. For a more thorough review, see [Gol17].

1.1 Zermelo-Fraenkel Axioms

Taken unchanged from [Gol17].

1.1.1 Axiom of Extensionality

$$\forall x \forall y \left(x = y \iff \forall z \left(z \in x \iff z \in y \right) \right)$$

Two sets are equal if and only if they contain the same elements.

1.1.2 Empty Set Axiom

$$\exists x \forall y y \notin x$$

There is a set with no elements.

1.1.3 Axiom of Pairs

$$\forall x \forall y \exists z \forall w \left(w \in z \iff \left(w = x \vee w = y \right) \right)$$

For any two sets, there is a set whose elements are precisely these sets.

1.1.4 Axiom of Seperation

$$\forall x \exists y \forall z \left(z \in y \iff \left(z \in x \land \phi(z) \right) \right)$$

For any set x there is a set consiting of all z in x for which $\phi(z)$ holds.

1.1.5 Power Set Axiom

$$\forall x \exists y \forall z \left(z \in y \iff z \subseteq x \right)$$

For any set x there is a set consisting of all subsets of x.

1.1.6 Union Axiom

$$\forall x \exists y \forall z \left(z \in y \iff \exists w \left(z \in w \land w \in x \right) \right)$$

For any set x there is a set which is the union of all the elements of x.

1.1.7 Axiom of Infinity

$$\exists x \left(\in x \land \forall y \left(y \in x \implies y \cup \{y\} \in x \right) \right)$$

There is an inductive set.

1.1.8 Axiom of Replacement

$$\forall x \exists y \forall y' \left(y' \in y \iff \exists x' \left(x' \in x \land \phi(x', y') \right) \right),$$

where $phi(s, t)$ is a formula such that

$$\forall s \exists t \left(\phi(s, t) \land \forall t' \left(\phi(s, t') \implies t' = t \right) \right).$$

Of $\phi(s, t)$ is a class function, then when its domain is restricted to xm the resulting images form a set y.

1.1.9 Axiom of Foundation

$$\forall x \exists y \left(y \in x \land x \cap y = \right)$$

Every set is *well-founded*, i.e. contains an $\in$-minimal element.

Chapter 2

First chapter

Lemma 2.0.1. *[Har15] Let A be an arbitrary set. Then there exists an ordinal α, such that no subset of A can be injectively mapped on to α.*

Theorem 2.0.2. *[HK72] The following are equivalent in ZF:*

1. *Axiom of Choice*
2. *Every non-empty set admits a cancellative groupoid structure*

Bibliography

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