Group Structure on arbitrary sets: An algebraic application of the Axiom of Choice

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Abstract

The thesis should include an abstract that summarizes its contents; mathematical jargon can be utilized here. The typical length of an abstract is between 100 and 300 words.

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Introduction

Historical Background

In 1902 Bertrand Russell showed with what is now known as *Russel's Paradox* that the previously used approach to set theory was inconsistent. Ernst Zermelo then created an axiomatic framework for set theory in 1905, motivated both by attempting to preserve results such as the theory of infinities by Georg Cantor, as well as avoiding paradoxes. These axioms, later modified by Abraham Fraenkel, became known as the nine *Zermelo-Fraenkel Axioms* (ZF) as well as the *Axiom of Choice* (AC)[Gol98, pp.66-70, 75].

The axiom of choice in particular is of special interest in many areas of mathematics, especially in algebra and topology, often in the form of the equivalent statement of *Zorn's Lemma*, which says that every non-empty partially ordered set with an upper bound has a maximal element [Jec78].

Finally in 1971 András Hajnal and Andor Kertész published a paper [HK72] which provided another equivalence to AC, namely that there exists a cancellative groupoid structure on every (uncountably infinite) set. This paper makes use of first-order model theory, an area of logic developed during the first half of the 20th century, which utilizes models of formal languages to obtain results. Kertész later expanded on this, providing an alternative algebraic partial proof in a lecture series given at the University of Jyväskylä [Ker75].

Thesis Structure

The aim of this thesis is to provide context to the paper [HK72] and to derive the theory needed for the proof of its main theorem:

Theorem 0.0.1. *The following sentences are equivalent in ZF:*

1. *Axiom of Choice*

2. *Every non-empty set admits a cancellative groupoid structure*

We start in the first chapter by briefly giving an overview of some of the necessary background knowledge needed for the rest of the text. This includes stating the ZF axioms as well as the Axiom of Choice itself.

Then, in the second chapter, we will explore orderings and well-orderings in the context of axiomatic set theory. Of special importance here will be Zorn's Lemma, a well-known equivalence of AC. We will finish this chapter by giving a proof for a lemma by Hartogs [Har15], which is also found in [HK72]. This lemma states that for any arbitrary set, there always exists an ordinal which no subset of that set can be injectively mapped to.

In the third chapter we will move on to an introduction to model theory. This is done with the aim of proving the upwards Löwenheim-Skolem Theorem, which states that a language with a countable model also has a uncountable model. Model theory is a very useful tool for applying results from logic to non-logic areas of mathematics, especially abstract algebra as we will see later. As Chang and Heisler put it in [CK90] (a very good historical introduction to model theory and the first comprehensive textbook for the subject),

Model Theory = Universal Algebra + Logic.

Finally, in the fourth and final chapter, we will give a detailed proof of the aforementioned theorem by Hajnal and Kertész, In this, we will apply the previous results by Hartogs and Löwenheim and Skolem from chapters two and three.

Chapter 1

Preliminaries

The convention in this thesis will be to say **ZF** when talking about Zermelo-Fraenkel set theory *without* the axiom of choice. When talking about the axiom of choice on its own we will say **AC**, and when talking about Zermelo-Fraenkel set theory together with the axiom of choice use **ZFC**.

We will use the convention of including 0 at the beginning of the natural numbers $\mathbb{N}$, i.e. $\mathbb{N} = \{0, 1, 2 \ldots\}$. This is a *natural* choice, since we then can use $\mathbb{N}$ to mean the set described by the Axiom of Infinity.

1.1 First-order Logic and formal Languages

1.2 Zermelo-Fraenkel Axioms of Set Theory

We assume that the reader has some familiarity with axiomatic set theory, but for convenience and consistency we restate some of the necessary basics here. For a more thorough introduction of the topic, see [Gol98, 4.3-4.5], alternatively [Jec78, 1.1] gives a more technical overview. The formulation of the axioms below are based on both textbooks.

1.2.1 Axiom of Extensionality

$$\forall x \forall y \left(x = y \iff \forall z \left(z \in x \iff z \in y\right)\right)$$

Two sets are equal if and only if they contain the same elements.

1.2.2 Axiom of the Empty Set

$$\exists x \forall y\, y \notin x$$

There is a set with no elements.

1.2.3 Axiom of Pairs

$$\forall x \forall y \exists z \forall w \left(w \in z \iff \left(w = x \lor w = y \right) \right)$$

For any two sets, there is a set whose elements are precisely these sets.

1.2.4 Axiom of Separation

$$\forall x \exists y \forall z \left(z \in y \iff \left(z \in x \land \phi(z) \right) \right),$$

where $\phi(z)$ is any statement of the formal language with free variable z. For any set x there is a set consisting of all z in x for which $\phi(z)$ holds.

1.2.5 Axiom of Power Sets

$$\forall x \exists y \forall z \left(z \in y \iff z \subseteq x \right)$$

For any set x there is a set, denoted by $\mathcal{P}(x)$ and called the power set of x, consisting of all subsets of x.

1.2.6 Union Axiom

$$\forall x \exists y \forall z \left(z \in y \iff \exists w \left(z \in w \land w \in x \right) \right)$$

For any set x there is a set, denoted by $\bigcup x$, which is the union of all the elements of x.

1.2.7 Axiom of Infinity

$$\exists x \left(\varnothing \in x \land \forall y \left(y \in x \implies y \cup \{y\} \in x \right) \right)$$

There is an inductive set.

1.2.8 Axiom of Replacement

$$\forall x \exists y \forall y' \left(y' \in y \iff \exists x' \left(x' \in x \land \phi(x', y') \right) \right),$$

where $\phi(s, t)$ is a formula such that

$$\forall s \exists t \left(\phi(s, t) \land \forall t' \left(\phi(s, t') \implies t' = t \right) \right).$$

If $\phi(s, t)$ is a class function, then when its domain is restricted to a set x, the resulting images form a set y. Since no proper classes will be treated in this text, the relevant statement for us is that the image of any function, of which the domain is a proper set, always is a set.

1.2.9 Axiom of Foundation

$$\forall x \exists y \left(y \in x \land x \cap y = \varnothing \right)$$

Every set is *well-founded*, i.e. contains an $\in$-minimal element.

1.3 The Axiom of Choice

Chapter 2

Orderings and Well-Orderings

2.1 Linear, Partial and Well-orderings

2.2 Ordinals and Order Types

There are several ways to define the natural numbers $\mathbb{N}$, the way we do it here, and the way generally used in set theory, is to use the Axiom of Infinity. This states that $\mathbb{N}$ is an *inductive set*, meaning that it contains 0, defined as $\varnothing = \{\}$, as well as the successor of every element in it, including of course 0 itself. [Gol98, p.39]

Definition 2.2.1. [Gol98, p.38] The successor of a set α is $\alpha^+ = \alpha \cup \{\alpha\}$. The successor of $0 = \varnothing$ is called 1 and the successor of 1 is called 2, etc..

The consequence of this is that $\mathbb{N}$ is the set we are familiar with: $\{0, 1, 2, 3, \ldots\}$. It also means that any natural number is defined as the set of all of its predecessors. For example $3 = \{0, 1, 2\}$ and $5 = \{0, 1, 2, 3, 4\}$.

Perhaps slightly more subtle then is that, under the usual ordering, $n < m \implies n \in m \wedge n \subset m$. This is an important property and the natural numbers, as well as the set $\mathbb{N}$ at large, are *transitive* sets.

Definition 2.2.2. [Jec78, p.14] A set T is called *transitive* if

$$\forall x \, (x \in T \implies x \subseteq T) \, .$$

Definition 2.2.3. [Jec78, p.14] A set is called an *ordinal number* or *ordinal* if it is transitive and well-ordered by $\in$.

Ordinals are denoted by lowercase greek letter: $\alpha, \beta, \gamma, \ldots$. The ordinal associated with $(\mathbb{N}, \in)$ specifically is denoted by ω.

Lemma 2.2.4. *ω and every natural number $n \in \omega$ is an ordinal.*

Lemma 2.2.5. *[Jec78, Lemma 2.3, p.15]*

1. *If α is an ordinal abd $\beta \in \alpha$, then β is an ordinal.*
2. *If α, β are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$.*
3. *If α, β are ordinals, then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$*

Theorem 2.2.6. *[Jec78, Theorem 2, p.15] Every well-ordered set is isomorphic to a unique ordinal.*

2.3 The Well-ordering Theorem

The following, along with *Zorn's Lemma*, is one of the most fundamental results in set theory. There is a (bad) joke that goes:

> The *Axiom of Choice* is obviously true, the *Well-Ordering Theorem* obviously false, and who knows with *Zorn's Lemma*.

Definition 2.3.1 (Zermelo's Well-Ordering Theorem)**.** [Jec78, Theorem 15, p.39] Every set can be well ordered.

We could provide a proof for definition 2.3.1 in **ZFC** here, and treat the well-ordering theorem as a regular theorem. This theorem, as it turn out, is not just another regular theorem, and we will therefore also not treat it as one.

Indeed, the well-ordering theorem is actually equivalent to the Axiom of Choice. This means that if either statement is assumed to be true (and it has to be assumed since we are talking about *axioms*), the other one can be proved from it. This is the same methodology we will use for proving our main result, theorem 4.0.1, as well. There we will show equivalence of our main statement, that group structure exists on all arbitrary sets, with the Well-Ordering Theorem. As such by transitivity, this main statement is also equivalent to the Axiom of Choice.

Theorem 2.3.2. *The Well-ordering Theorem is equivalent to the Axiom of Choice.*

Proof. [Jec78, p.39] We provide a proof in two parts; first showing that the Well-Ordering Theorem is true in **ZFC**. Then, conversely, we prove **AC** in **ZF**, assuming that the well-ordering theorem holds true.

1. **Axiom of Choice $\implies$ Well-Ordering Theorem**

   We proceed by transfinite induction.

Let A be an arbitrary set and let $S = \mathcal{P}(A) \setminus \varnothing$ be the collection of all non-empty subsets of A. Let $f : S \to A$ be a choice function (as specified by the Axiom of Choice). We then define an ordinal sequence $\langle a_\alpha \mid \alpha < \theta \rangle$ the following way:

$$\begin{aligned} a_0 &= f(A) \\ a_\alpha &= f\left(A \setminus \left\{a_\xi \mid \xi < \alpha\right\}\right) \qquad \text{if } A \setminus \left\{a_\xi \mid \xi < \alpha\right\} \text{ is non-empty.} \end{aligned}$$

Now let θ be the smallest ordinal such that $A = \left\{a_\xi \mid \xi < \theta\right\}$.

We know that such an ordinal must exist, since the sequence $\langle a_\alpha \mid \alpha < \theta \rangle$ is entirely defined by the choice function f. f maps every non-empty subset of A, i.e. members of S, to a element of that subset (in A).

By defining the ordinal sequence the way we did, it is not possible for any element of A to occur in the sequence twice. Any subset of A, which is the input of the choice function for some element a_γ in the sequence, does not contain any elements a_α for $\alpha < \gamma$, and by definition f can not map to any of these members.

As such $\mathbf{Im}\left(\langle a_\alpha \mid \alpha < \theta \rangle\right) = A$ and $\langle a_\alpha \mid \alpha < \theta \rangle$ enumerates A, i.e. is a bijection[1]. Hence A can be well-ordered, the least element of any subset being the one which corresponds to the smallest ordinal in the sequence.

2. **Well-Ordering Theorem $\implies$ Axiom of Choice**

   Let S be a set of non-empty sets.

   $\bigcup S$ can be well-ordered by assumption and clearly $s \in S \implies s \subseteq \bigcup S$. We can then define the function $f : S \to \bigcup S$ to map any elements of S to its least element, according to the well-order of $\bigcup S$.

   Evidently f is a choice function and since the set S was arbitrary the Axiom of Choice holds.

□

2.4 Hartogs' Lemma

We continue with the final result for this chapter, a lemma originally stated by Hartogs in 1915, restated in this form in our main paper [HK72].

Lemma 2.4.1. *[Har15] Let A be an arbitrary set. Then there exists an ordinal α, such that no injective map from any subset of A to α exists.*

[1] Recall that a sequence is just a function from $\mathbb{N}$, resp. an ordinal, to the set of its elements

Chapter 3

Model Theory

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Chapter 4

Hajnal's and Kertész's Theorem

Theorem 4.0.1. *[HK72] The following are equivalent in ZF:*

1. *Axiom of Choice*
2. *Every non-empty set admits a cancellative groupoid structure*

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