0.1 Linear, Partial and Well-orderings

0.2 Ordinals and Order Types

0.3 The Well-ordering Theorem

The following, along with *Zorn's Lemma*, is one of the most fundamental results in set theory. There is a (bad) joke that goes:

> The *Axiom of Choice* is obviously true, the *Well-Ordering Theorem* obviously false, and who knows with *Zorn's Lemma*.

Definition 0.3.1 (Zermelo's Well-Ordering Theorem)**.** [**?**, Theorem 15] Every set can be well ordered.

We could provide a proof for definition 0.3.1 in **ZFC** here, and treat the well-ordering theorem as a regular theorem. This theorem, as it turn out, is not just another regular theorem, and we will therefore also not treat it as one.

Indeed, the well-ordering theorem is actually equivalent to the Axiom of Choice. This means that if either statement is assumed to be true (and it has to be assumed since we are talking about *axioms*), the other one can be proved from it. This is also how we will proceed with the proof, first showing that the well-ordering theorem is true in **ZFC**, then conversely proving **AC** in **ZF**, assuming that the well-ordering theorem holds true.

This is the same methodology we will use for proving our main result, theorem **??**, as well. There we will show equivalence of our main statement, that group structure exists on all arbitrary sets, with the Well-Ordering Theorem. As such by transitivity, this main statement is also equivalent to the Axiom of Choice.

Theorem 0.3.2. *The Well-ordering Theorem is equivalent to the Axiom of Choice.*

0.4 Hartogs' Lemma

We continue with the final result for this chapter, a lemma originally stated by Hartogs in 1915, restated in this form in our main paper [**?**].

Lemma 0.4.1. *[**?**] Let A be an arbitrary set. Then there exists an ordinal α, such that no injective map from any subset of A to α exists.*