We now arrive at our main theorem:

Theorem 0.0.1. *[?] The following are equivalent in ZF:*

1. *The Axiom of Choice*

2. *Every non-empty set admits a cancellative groupoid structure*

Proof. The theorem is proven in two steps, deriving a single direction implication for each sentence.

1. **Groupoid Structure on arbitrary sets $\implies$ Axiom of Choice**

   We show that the existence of a groupoid structure on every non-empty set implies that every set can be well-ordered. By Theorem **??** this is equivalent to the Axiom of Choice.

   Let A be an arbitrary set and let α be an ordinal as described in Theorem **??** in Section **??**. This means that there exists no bijective mapping from α to any subset of A (including A itself). We then let $(B,\ R)$ be a well-ordered set of type α and such that $A \cap B = \emptyset$.

   Now let C be the set $C = A \cup B$, by assumption there exists some operation $+$, such that $(C,\ +)$ is a cancellative groupoid. We will show that for every $x \in A$ there exists $y \in B$, such that $x + y \in B$ holds.

   Let us assume for a contradiction that the above claim does not hold. This would imply that some $a \in A$ exists for which $a + y \in A$ holds for all $y \in B$. Let $f : B \longrightarrow A$ be the function defined by $f(y) = a + y$. We have that $+$ is a cancellative groupoid operation, hence f must be injective; a contradiction by Theorem **??**, since we had assumed that B is of type α.

   We let $D = B \times B$ be the well-ordered set with respect to the lexicographical ordering R' of R, and define a function $g : A \longrightarrow D$ by

   $$g(x) = \min_{R'} \{\langle u,\ v\rangle \in D \,|\, x + u = v\}\,.$$

   The function g maps every element x of A to the least pair $\langle u,\ v\rangle$ in $B \times B$ satisfying $x + u = v$. From earlier in the proof we know that such a pair must exists and that g must in fact be injective. This again follows from $+$ being cancellative, since if $x_1,\ x_2$ are two elements of A, having $f(x_1) = f(x_2)$ would imply that

   $$\begin{aligned} x_1 + u = \quad & v \quad = x_2 + u \\ \iff \quad x_1 = \ & v + u^{-1} = x_2 \end{aligned}$$

for some pair $\langle u, v \rangle \in D$. Since $\mathbf{Im}(g)$ is a subset of D it itself is a well-ordered set. As such we can define a well-order R'' on A by letting $x_i R'' x_j$ whenever $g(x_i) R' g(x_j)$.

2. **Axiom of Choice $\implies$ Groupoid Structure on arbitrary sets**

   ...

$\square$