0.1 Models of Formal Languages

Definition 0.1.1. [?, Definition 1.1.1] A *formal language* $\mathcal{L}$ in first order logic is given by the following:

1. A set $\mathcal{F}$ of functions f of n_f variables, with $n_f \in \mathbb{Z}^+$ a positive integer,
2. a set $\mathcal{R}$ of n_R-ary relations R, with $n_R \in \mathbb{Z}^+$ a positive integer,
3. a set $\mathcal{C}$ of constants.

Since we utilize set theory in the definition of languages, we consider the set inclusion symbol $\in$ to be to be part of first-order logic here (in the same vein as =.) As such, when we dealt with set theoretic constructions we used the language of given by $\mathcal{L}_{Set} = \varnothing$. Some other examples of languages are those of groups or of rings,

$$\mathcal{L}_g = \{\cdot,\ e\} \qquad\qquad \text{and} \qquad\qquad \mathcal{L}_r = \{+,\ \cdot,\ 0,\ 1\},$$

where +, and $\cdot$ are binary functions and e, 0, 1 are constants.

These are just *languages* however, they do not dictate any additional structure related to the objects they contain. The axioms of groups and rings are not part of the language itself, for groups, say, we only know the parity of $\cdot$ and that some distinguished element e exists. As such the language of rings is also the same as the language of fields, since fields are just a special case of rings.

The existence of constant symbols also means we cannot just reuse the definition of a formula we used for sets. As such we first define *terms*, a notion of chaining together operations.

Definition 0.1.2. [?, Definition 1.1.4] We say that T is the set of $\mathcal{L}$-terms, if T is the smallest set such that

1. $c \in T$ for each constant $c \in \mathcal{C}$
2. $v_i \in T$ for variable symbols v_i, where $i = 1,\ 2, \ldots,$
3. $f(t_1, \ldots,\ t_n) \in T$ for $f \in \mathcal{F}$ and terms $t_1, \ldots,\ t_n \in T$.

Definition 0.1.3 (Formula of a Set)**.** An *atomic formula* in a language $\mathcal{L}$ is either

1. $t_1 = t_2$ for terms $t_1,\ t_2 \in T$, or
2. $R(t_1, \ldots,\ t_{n_R})$ for $R \in \mathcal{R}$ and terms $t_1, \ldots,\ t_{n_R} \in T$.

A *formula* ϕ in $\mathcal{L}$ is an any combination of atomic formulas with logical connectives and quantifiers.

In order to actually utilize languages we also define the namesake of model theory, models.

Definition 0.1.4. [?, Definition 1.1.2] We $\mathcal{L}$*-structure* or *model* $\mathcal{M}$ is given by the following:

1. A nonempty set M,
2. a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$,
3. a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$,
4. an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$.

The set M is referred to as the *universe*, *domain* or *underlying set* of $\mathcal{M}$ and $f^{\mathcal{M}}$, $R^{\mathcal{M}}$ and $c^{\mathcal{M}}$ are called the *interpretations* of $\mathcal{M}$. We sometimes also identify a model $\mathcal{M}$ with the pair $\langle M, \mathcal{I}\rangle$, where $\mathcal{I}$ is the function mapping $\mathcal{F}$, $\mathcal{R}$ and $\mathcal{C}$ to their respective interpretations in M.

0.2 The Löwenheim-Skolem Theorem

Lemma 0.2.1. *[?, Lemma 2.1.1] Let T be a consistent set of sentences of $\mathcal{L}$. Let C be a set of new constant symbols of power $|C| = \|\mathcal{L}\|$ and let $\bar{\mathcal{L}} = \mathcal{L} \cup C$ be the simple expansion of $\mathcal{L}$ formed by adding C.*

Then T can be expanded to a consistent set of sentences $\bar{T}$ in $\bar{\mathcal{L}}$, which has C as a set of witnesses in $\bar{\mathcal{L}}$.

Lemma 0.2.2. *[?, Lemma 2.1.2] Let T be a set of sentences and let C be a set of witnesses of T in $\mathcal{L}$. Then T has a model $\mathfrak{U}$, such that every element of $\mathfrak{U}$ is an interpretation of a constant $c \in C$.*

Theorem 0.2.3 (Extended Completeness Theorem)**.** *[?, Theorem 1.3.21] Let Σ be a set of sentences in $\mathcal{L}$. Then Σ is consistent if and only if Σ has a model.*

Theorem 0.2.4 (Downward Löwenheim-Skolem Theorem)**.** *[?, Corollary 2.1.4] Every consistent theory T in $\mathcal{L}$ has a model of power at most $\|\mathcal{L}\|$.*

Theorem 0.2.5 (Compactness Theorem)**.** *[?, Theorem 1.3.22] A set of sentences Σ has a model if and only if every finite subset of Σ has a model.*

Theorem 0.2.6 (Upward Löwenheim-Skolem Theorem)**.** *[?, Corollary 2.1.6] If T has infinite models, then it has infinite models of any given power $\alpha \geq \|\mathcal{L}\|$.*