The aim of this chapter is to give an introduction to the basic tools of model theory with which we will prove the Löwenheim-Skolem Theorem. This theorem, divided into two parts, is dependent on the Axiom of Choice, hence we will later use it to show that **AC** implies the existence of a group structure on all non-empty sets.

We were already concerned with axiomatizations in the previous chapters, however we will formalize these notions some more here. The set theory axioms outlined in Chapter **??** still hold true, but we will discover more about what is and is not true in for example group theory or other algebraic structures. We start the treatment of model theory with the motivating example of "Ehrenfeucht-Fraïsse games", exemplifying how the field relates to the linear orderings from Chapter **??**. The impatient or familiar reader however may skip this section without missing any necessary theory.

0.1 Ehrenfeucht-Fraïsse Games

Let us suppose we are given two linear orderings, based on which we define a two player game. Player I is called *spoiler* and starts by picking a point on one of the two orderings. If they pick an element of A we call this a_1 and if they pick an element of B we call it b_1. After the spoiler has picked their point, player II will pick a point on the other linear ordering. We call player II *duplicator*.[1] Say spoiler picked a_1 in A, duplicator then has to pick some element b_1 of B.

The spoiler and duplicator go back and forth picking points on the linear orderings for a predetermined amount of turns n. Spoiler always gets to choose the ordering they pick a point on and duplicator has to pick something on the other ordering. The second player, duplicator, wins if the elements $a_1, \ldots, a_n$ are in the same order with respect to A as the elements $b_1, \ldots, b_n$ are with respect to B. Spoiler wins if duplicator loses, if the elements are not in the same order. This is how a single play of an *Ehrenfeucht-Fraïsse game*, also called a *Back-and-forth game*, in n turns is played out. [**?**, §6.1]

Example 0.1.1. We will go through the play of a game with 3 steps. Let $A = \mathbb{Q}$ and $B = \mathbb{Z}$ under the usual order be the two linear orderings that are played on.

Spoiler starts and picks the point $a_1 = 0$ in $\mathbb{Z}$. Duplicator on their turn also picks the point $b_0 = 0$ but in $\mathbb{Z}$. On the next turn spoiler picks $a_2 = 1$ in $\mathbb{Z}$, duplicator again matches this and picks $b_2 = 1$ in $\mathbb{Q}$. Now however spoiler can and does pick b_3 in $\mathbb{Q}$ to be $\frac{1}{2}$. This is not possible for duplicator to match since they are now confined to picking a point a_3 in $\mathbb{Z}$. Hence no matter what point duplicator picks to be a_3, they will lose.

[1]The names "spoiler" and "duplicator" were coined by Joel Spencer.[**?**, §6] The literature [**?**] this section of the text is mainly based on uses "Player I" and "Player II" instead.

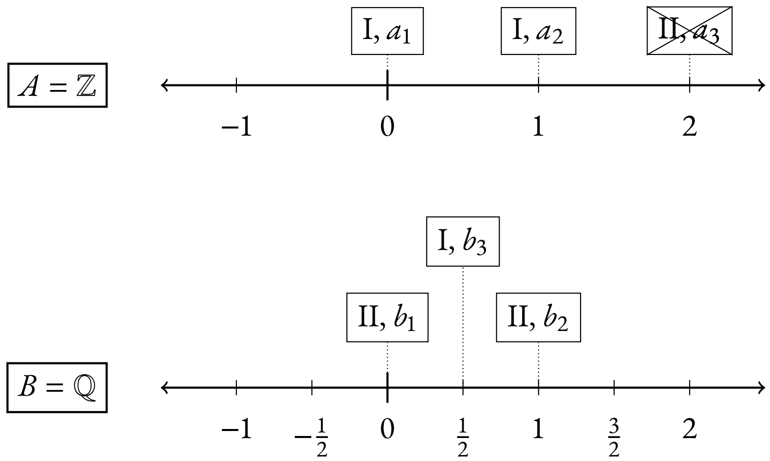

Figure 0.1: Example 0.1.1 illustrated. Points *spoiler* picked are prefixed by “I”, points *duplicator* picked are prefixed by “II”. It can be seen that duplicator is not able to pick a suitable element for b_3 in $\mathbb{Z}$ such that a_1, a_2 and a_3 have the same order as b_1, b_2 and b_3.

It should be clear that in order for spoiler to win, they need to exploit the differences of the linear orders in play. In the example above spoiler can alway pick their first two elements to be a_1 and a_2 in $\mathbb{Z}$ in a way where a_2 is the immediate successor of a_1. Then, because $\mathbb{Q}$ is dense, no matter which elements duplicator picks to be b_1 and b_2, spoiler can pick b_3 such that $b_1 < b_3 < b_2$. As a_2 is the immediate successor of a_1, there exists no element in $\mathbb{Z}$ satisfying $a_1 < a_3 < a_2$.

This is called a *winning strategy* for spoiler. A winning strategy for duplicator would be the converse, where duplicator can react according to every possible move by spoiler and still win. For duplicator to have a winning strategy, the linear orderings should be *similar* in some sort of way, in fact the existence of a winning strategy by duplicator is a type of equivalence in itself. Such a winning strategy shown in the example below, where we play a game with two dense orderings.

Example 0.1.2. We play an Ehrenfeucht-Fraïsse game in n rounds.

Let $A = \mathbb{R}$ and $B = \mathbb{Q}$ under the usual order. We can define a winning strategy for duplicator inductively. Duplicator then picks a point according to one of the following cases on an arbitrary turn i:

1. It is the first turn and spoiler picks $a_1 \in A$.

   Duplicator then picks $b_1 = 0$.

2. Spoiler picks $a_i \in A$ such that $a_j < a_i < a_k$ holds for some $a_j,\ a_k \in A$.

It can be assumed without loss of generality that a_j and a_k are the respectively largest and smallest points for which this property holds. Then there exist corresponding $b_j,\ b_k \in B$ with $b_j < b_k$. Hence duplicator can choose

$$b_i = \frac{b_k - b_j}{2}.$$

3. Spoiler picks $a_i \in A$ such that a_i is either larger or smaller than all a_k for $k < i$. Duplicator can pick

$$b_i = b_s - 1 \qquad \text{or} \qquad b_i = b_l + 1$$

in B where b_s and b_l are the smallest and largest elements of $\{b_k \mid k < n\}$ respectively.

Since $\mathbb{Q}$ is a proper subset of $\mathbb{R}$ the same strategy works if spoiler chooses an element b_i in $B = \mathbb{Q}$. The strategy for the second turn follows immediately from the third case.

With this example in mind we now formally a play of an Ehrenfeucht-Fraïsse game and what it means for duplicator to have a winning strategy.

Definition 0.1.3 (Ehrenfeucht-Fraïsse Game)**.** [**?**, Definition 6.2] Let A and B be linear orderings and let $n \in \mathbb{Z}^+$ be a fixed positive integer. Then a *play of an Ehrenfeucht-Fraïsse game* $EF_n(A,\ B)$ is defined as an ordered sequence with $2n$ elements. For each positive integer $k \leq n$ player I, *spoiler*, chooses an element in either A or B. Player II, *duplicator*, then chooses an element of the ordering spoiler did not pick. We denote an element at the kth turn as a_k if it was picked from A and b_k if it was picked from B.

We say that duplicator has won a play of the game $EF_n(A,\ B)$ if for all positive integers $i,\ j \leq n$

$$a_i <_A a_j \iff b_i <_B b_j$$

with respect to the linear orders $<_A$ and $<_B$ of A and B. We say that spoiler has won a play of the game if duplicator did not win.

We say that there exists *winning strategy of duplicator* if there exists a sequence of functions $f_1, \ldots, f_n$, which fulfills the following requirements:

(i) The domain of each f_k is the set of all ordered k-tuples with elements in $A \cup B$;

(ii) For elements $c_1, \ldots,\ c_k \in A \cup B$, representing the first k moves by spoiler, f_k satisfies

$$f_k(c_1, \ldots,\ c_k) \in A \text{ if } c_k \in B \text{ and}$$
$$f_k(c_1, \ldots,\ c_k) \in B \text{ if } c_k \in A.$$

(iii) For elements $c_1, \ldots, c_k \in A \cup B, f_k, a_k$ and b_k are defined as

$$a_k = \begin{cases} c_k & \text{if } c_k \in A \\ f_k(c_1, \ldots, c_k) & \text{if } c_k \in B \end{cases} \quad \text{and} \quad b_k = \begin{cases} c_k & \text{if } c_k \in B \\ f_k(c_1, \ldots, c_k) & \text{if } c_k \in A \end{cases}$$

for positive integers $k \leq n$. These terms, based on $f_1, \ldots, f_n$, then satisfy

$$a_i <_A a_j \iff b_i <_B b_j$$

for all $i, j \leq n$, where $i, j \in \mathbb{Z}^+$.

If no such sequence of functions exists, we say that there exists a *winning strategy of spoiler*.

The proper definition of a winning strategy lets us define the following type of equivalence:

Definition 0.1.4 (EF_n-equivalence)**.** [**?**, Definition 6.8] We say that a linearly ordered set A is EF_n-equivalent to a linearly ordered set B, if there exists a winning strategy of duplicator in $EF_n(A, B)$.

The orderings A and B are called EF-equivalent if duplicator has a winning strategy in $EF_n(A, B)$ for all $n \in \mathbb{N}$.

This is indeed a proper type of equivalence, since by the Gale-Stewart Theorem from game theory all Ehrenfeucht-Fraïsse games are determined.[**?**, §6] This means that one, and only one, of the players will always have a winning strategy.

We saw in Example 0.1.2 that $\mathbb{R}$ and $\mathbb{Q}$ are EF-equivalent. We were able to find a winning strategy for duplicator because both ordered sets were dense. As such we could always find a point between any other two points in the rational numbers that could match the order of the possibly irrational points in $\mathbb{R}$. The sets $\mathbb{R}$ and $\mathbb{Q}$ are of course different, but they still *satisfy the same property* of densely linear orders.

This notion of what structure satisfy which properties is at the core of model theory. And its no coincidence that Ehrenfeucht-Fraïsse games seem give rise so naturally to the field. We do indeed have a theorem relating them to model theory proper.

Theorem 0.1.5. *[**?**, Theorem 2.4.6] Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures, where $\mathcal{L}$ is a finite language without any function symbols. Then duplicator has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ for all $n \in \mathbb{N}$ if and only if $\mathcal{M}$ and $\mathcal{N}$ satisfy exactly the same theories.*

We will not prove Theorem 0.1.5 here, but rather include it as an appetizer; a means show the usefulness of model theory applied to problems we may already be interested in. The reader interested in the proof of the theorem may read it in the subsubsection titled “Ehrenfeucht-Fraïsse Games” in [**?**, §2.4]. The rest of this chapter should provide an adequate background for understanding it.

0.2 Models of Formal Languages

Definition 0.2.1. [?, Definition 1.1.1] A *formal language* $\mathcal{L}$ in first order logic is given by the following:

1. A set $\mathcal{F}$ of functions f of n_f variables, with $n_f \in \mathbb{Z}^+$ a positive integer,
2. a set $\mathcal{R}$ of n_R-ary relations R, with $n_R \in \mathbb{Z}^+$ a positive integer,
3. a set $\mathcal{C}$ of constants.

Since we utilize set theory in the definition of languages, we consider the set inclusion symbol $\in$ to be to be part of first-order logic here (in the same vein as =.) As such, when we dealt with set theoretic constructions we used the language of given by $\mathcal{L}_{Set} = \varnothing$. Some other examples of languages are those of groups or of rings,

$$\mathcal{L}_g = \{\cdot,\ e\} \qquad \text{and} \qquad \mathcal{L}_r = \{+,\ \cdot,\ 0,\ 1\},$$

where +, and $\cdot$ are binary functions and e, 0, 1 are constants.

These are just *languages* however, they do not dictate any additional structure related to the objects they contain. The axioms of groups and rings are not part of the language itself, for groups, say, we only know the parity of $\cdot$ and that some distinguished element e exists. As such the language of rings is also the same as the language of fields, since fields are just a special case of rings.

The existence of constant symbols also means we cannot just reuse the definition of a formula we used for sets. As such we first define *terms*, a notion of chaining together operations.

Definition 0.2.2. [?, Definition 1.1.4] We say that T is the set of $\mathcal{L}$-terms, if T is the smallest set such that

1. $c \in T$ for each constant $c \in \mathcal{C}$
2. $v_i \in T$ for variable symbols v_i, where $i = 1,\ 2, \ldots,$
3. $f(t_1, \ldots,\ t_n) \in T$ for $f \in \mathcal{F}$ and terms $t_1, \ldots,\ t_n \in T$.

Definition 0.2.3 (Formula of a Set)**.** An *atomic formula* in a language $\mathcal{L}$ is either

1. $t_1 = t_2$ for terms $t_1,\ t_2 \in T$, or
2. $R(t_1, \ldots,\ t_{n_R})$ for $R \in \mathcal{R}$ and terms $t_1, \ldots,\ t_{n_R} \in T$.

A *formula* ϕ in $\mathcal{L}$ is an any combination of atomic formulas with logical connectives and quantifiers.

In order to actually utilize languages we also define the namesake of model theory, models.

Definition 0.2.4. [?, Definition 1.1.2] We $\mathcal{L}$-*structure* or *model* $\mathcal{M}$ is given by the following:

1. A nonempty set M,
2. a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$,
3. a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$,
4. an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$.

The set M is referred to as the *universe*, *domain* or *underlying set* of $\mathcal{M}$ and $f^{\mathcal{M}}$, $R^{\mathcal{M}}$ and $c^{\mathcal{M}}$ are called the *interpretations* of $\mathcal{M}$. We sometimes also identify a model $\mathcal{M}$ with the pair $\langle M, \mathcal{I} \rangle$, where $\mathcal{I}$ is the function mapping $\mathcal{F}$, $\mathcal{R}$ and $\mathcal{C}$ to their respective interpretations in M.

0.3 The Löwenheim-Skolem Theorem

Lemma 0.3.1. *[?, Lemma 2.1.1] Let T be a consistent set of sentences of $\mathcal{L}$. Let C be a set of new constant symbols of power $|C| = \|\mathcal{L}\|$ and let $\bar{\mathcal{L}} = \mathcal{L} \cup C$ be the simple expansion of $\mathcal{L}$ formed by adding C.*

Then T can be expanded to a consistent set of sentences $\bar{T}$ in $\bar{\mathcal{L}}$, which has C as a set of witnesses in $\bar{\mathcal{L}}$.

Lemma 0.3.2. *[?, Lemma 2.1.2] Let T be a set of sentences and let C be a set of witnesses of T in $\mathcal{L}$. Then T has a model $\mathfrak{U}$, such that every element of $\mathfrak{U}$ is an interpretation of a constant $c \in C$.*

Theorem 0.3.3 (Extended Completeness Theorem)**.** *[?, Theorem 1.3.21] Let Σ be a set of sentences in $\mathcal{L}$. Then Σ is consistent if and only if Σ has a model.*

Theorem 0.3.4 (Downward Löwenheim-Skolem Theorem)**.** *[?, Corollary 2.1.4] Every consistent theory T in $\mathcal{L}$ has a model of power at most $\|\mathcal{L}\|$.*

Theorem 0.3.5 (Compactness Theorem)**.** *[?, Theorem 1.3.22] A set of sentences Σ has a model if and only if every finite subset of Σ has a model.*

Theorem 0.3.6 (Upward Löwenheim-Skolem Theorem)**.** *[?, Corollary 2.1.6] If T has infinite models, then it has infinite models of any given power $\alpha \geq \|\mathcal{L}\|$.*