Group Structure on arbitrary sets: An algebraic application of the Axiom of Choice

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Abstract

The thesis should include an abstract that summarizes its contents; mathematical jargon can be utilized here. The typical length of an abstract is between 100 and 300 words.

Contents

Introduction

Historical Background

In 1902 Bertrand Russell showed with what is now known as *Russell's Paradox* that the previously used approach to set theory was inconsistent. Ernst Zermelo then created an axiomatic framework for set theory in 1905, motivated both by attempting to preserve results such as the theory of infinities by Georg Cantor, as well as avoiding paradoxes. These axioms, later modified by Abraham Fraenkel, became known as the nine *Zermelo-Fraenkel Axioms* (ZF) as well as the *Axiom of Choice* (AC)[Gol98, pp.66-70, 75].

The axiom of choice in particular is of special interest in many areas of mathematics, especially in algebra and topology, often in the form of the equivalent statement of *Zorn's Lemma*, which says that every non-empty partially ordered set with an upper bound has a maximal element [Jec78].

Finally in 1971 András Hajnal and Andor Kertész published a paper [HK72] which provided another equivalence to AC, namely that there exists a cancellative groupoid structure on every (uncountably infinite) set. This paper makes use of first-order model theory, an area of logic developed during the first half of the 20th century, which utilizes models of formal languages to obtain results. Kertész later expanded on this, providing an alternative algebraic partial proof in a lecture series given at the University of Jyväskylä [Ker75].

Thesis Structure

The aim of this thesis is to provide context to the paper [HK72] and to derive the theory needed for the proof of its main theorem:

Theorem 0.0.1. *The following sentences are equivalent in ZF:*

1. *Axiom of Choice*
2. *Every non-empty set admits a cancellative groupoid structure*

We start in the first chapter by briefly giving an overview of some of the necessary background knowledge needed for the rest of the text. This includes stating the ZF axioms as well as the Axiom of Choice itself.

Then, in the second chapter, we will explore orderings and well-orderings in the context of axiomatic set theory. Of special importance here will be Zorn's Lemma, a well-known equivalence of AC. We will finish this chapter by giving a proof for a lemma by Hartogs [Har15], which is also found in [HK72]. This lemma states that for any arbitrary set, there always exists an ordinal which no subset of that set can be injectively mapped to.

In the third chapter we will move on to an introduction to model theory. This is done with the aim of proving the upwards Löwenheim-Skolem Theorem, which states that a language with a countable model also has a uncountable model. Model theory is a very useful tool for applying results from logic to non-logic areas of mathematics, especially abstract algebra as we will see later. As Chang and Heisler put it in [CK90] (a very good historical introduction to model theory and the first comprehensive textbook for the subject),

Model Theory = Universal Algebra + Logic.

Finally, in the fourth and final chapter, we will give a detailed proof of the aforementioned theorem by Hajnal and Kertész, In this, we will apply the previous results by Hartogs and Löwenheim and Skolem from chapters two and three.

Chapter 1

Preliminaries

The convention in this thesis will be to say **ZF** when talking about Zermelo-Fraenkel set theory *without* the axiom of choice. When talking about the axiom of choice on its own we will say **AC**, and when talking about Zermelo-Fraenkel set theory together with the axiom of choice we use **ZFC**.

We will use the convention of including 0 at the beginning of the natural numbers $\mathbb{N}$, i.e. $\mathbb{N} = \{0, 1, 2 \ldots\}$. This is a *natural* choice, since we then can use $\mathbb{N}$ to mean the set described by the Axiom of Infinity. If we want to talk about strictly positive integers we use the notation $\mathbb{Z}^+ = \mathbb{N} \setminus \{0\}$.

Lastly, whenever we deal with the negation of some symbol, we cross it out to mean this, for example $a \neq b$ means $\neg (a = b)$.

1.1 First-Order Logic and Classes

When talking about first order logic we mean the symbol

$$= \text{ (equals)}$$

in conjunction with the logical connectives

$$\neg \text{ (not)}, \ \wedge \text{ (and)}, \ \vee \text{ (or)}, \ \rightarrow \text{ (implies) and } \leftrightarrow \text{ (if and only if)}$$

as well as the quantifiers

$$\forall \text{ (for all) and } \exists \text{ (exists).}$$

Additionally, as we are talking about sets, we use the symbol $\in$ to denote set inclusion. [Jec78, pp.2-3]

In order to effectively talk about properties of set and set-like structures we need to somehow properly define formulas. We will go more into depth about formulas in Chapter 3, where we will introduce the notion of a formal language.

For now we are only concerned with the language of sets defined above.

Definition 1.1.1 (Formula of a Set)**.** An *atomic formula* in set theory is either

1. $x = y$, or
2. $x \in y$

A *formula* ϕ is an any combination of atomic formulas with logical connectives and quantifiers.

The symbols x and y above are called variables and for any two variables an atomic formula is either true or false for each x and y. A variable *occurs freely* inside of a formula if it does not appear inside of a $\exists$ or $\forall$ quantifier, otherwise the variable is *bound*. We write $\phi(x_1, \ldots, x_n)$ for a formula with $n \in \mathbb{Z}^+$ free variables. A formula where every variable is bound is called a *sentence*. [Mar02, pp.10-11]

A sentence is either true are false and a formula with free variables is true or false for each choice of the free variables. Each of the **ZF** axioms below are examples of sentences and as axioms we assume them to be inherently true (within the framework of our theory). An example of a formula with a free variable would be

$$\phi(x) = \exists y \left(y \in x \right).$$

This formula is only false for the empty set, since it is the unique set which does not contain any elements. In that sense we think of formulas with free variables describing a *property*, something we make use of in *classes*.

Definition 1.1.2 (Class)**.** Let $\phi\left(x, p_1, \ldots, p_n\right)$ be a formula in first order logic. Then a *class* $\mathbf{C}$ is defined as

$$\mathbf{C} = \left\{x \,|\, \phi\left(x, p_1, \ldots, p_n\right)\right\}.$$

The class $\mathbf{C}$ is called *definable from* $p_1, \ldots, p_n$. Furthermore, if x is the only free variable of ϕ, the class $\mathbf{C}$ is simply called *definable*. [Jec78, p.3]

In practice, we use classes as a tool to help us construct useful sets in **ZFC**, as elements of classes are always sets in the stricter sense. All sets are classes, but not all classes are sets, since if we have a fixed set s we can always construct the corresponding class would be $\mathbf{S} = \{x \,|\, x = s\}$. A class which is not a set is called a *proper class*.

We consider to two classes to be the same if they have the same elements. The familiar set operations of *inclusion*, *union*, *intersection*, and *difference* are definable using formulas. As such for classes $\mathbf{C}, \mathbf{D}$,

$$\begin{aligned}
\mathbf{C} \subseteq \mathbf{D} &\iff \forall x \,(x \in \mathbf{C} \implies x \in \mathbf{D}) \\
\mathbf{C} \cup \mathbf{D} &= \{x \,|\, x \in \mathbf{C} \lor x \in \mathbf{D}\} \\
\mathbf{C} \cap \mathbf{D} &= \{x \,|\, x \in \mathbf{C} \land x \in \mathbf{D}\} \\
\mathbf{C} \setminus \mathbf{D} &= \{x \,|\, x \in \mathbf{C} \land x \notin \mathbf{D}\} \\
\bigcup \mathbf{C} &= \{x \,|\, x \in S \text{ for some } S \in \mathbf{C}\}
\end{aligned}$$

[Jec78, pp.3-4]

For the use in this text, classes are in a sense "naive sets-like object"; they to help us describe collections of sets without worrying about paradoxes. Consider for example the class $\mathbf{V} = \{x \mid x = x\}$, which is the universe of all sets and does not exist in pure set theory. Another important class which we will make use of later is the *empty class* $\emptyset = \{x \mid x \neq x\}$ (although this is also a set as we will see).

1.2 Zermelo-Fraenkel Axioms of Set Theory

We assume that the reader has some familiarity with axiomatic set theory, but for convenience and consistency we restate some of the necessary basics here. For a more thorough introduction of the topic, see [Gol98, §§4.3-4.5], alternatively [Jec78, §1.1] gives a more technical overview. The formulation of the axioms below is based on both textbooks.

1.2.1 Axiom of Extensionality

$$\forall x \forall y \left(x = y \iff \forall z \left(z \in x \iff z \in y\right)\right)$$

Two sets are equal if and only if they contain the same elements.[Gol98, §4.3, p.76]

1.2.2 Axiom of Pairs

$$\forall x \forall y \exists z \forall w \left(w \in z \iff \left(w = x \lor w = y\right)\right)$$

For any two sets, there is a set whose elements are precisely these sets.

We define an ordered pair $\langle x, y \rangle$ to be the set $\left\{\{x\}, \{x, y\}\right\}$. Further, ordered n-tuples are defined recursively as $\langle x_1, x_2, x_3, \ldots, x_n \rangle = \langle x_1, \langle x_2, x_3, \ldots, x_n \rangle \rangle$.[Gol98, §4.3, pp.76, 79-80]

Ordered pairs satisfy the property that for any sets x, y, u, v, if $\langle x, y \rangle = \langle u, v \rangle$, then $x = u$ and $y = v$.[Gol98, Theorem 4.2]

1.2.3 Axiom Schema of Separation

Let $\phi(z, p)$ be a formula in first order logic with a free variable z. Then

$$\forall x \forall p \exists y \forall z \left(z \in y \iff \left(z \in x \land \phi(z, p)\right)\right). \tag{1.2.1}$$

For any sets x and p there is a unique set consisting of all z in x for which $\phi(z, p)$ holds. This is an axiom schema, meaning an infinite collection of axioms, since (1.2.1) is a separate axiom for every formula $\phi(z, p)$.[Jec78, pp.5-6]

1.2.4 Axiom of the Empty Set

$$\exists x \forall y\, y \notin x$$

There is a set with no elements. We call this set $\emptyset = \{\}$. [Gol98]

The Empty Set Axiom is not strictly required, the existence of the empty set also arises from the Axiom Schema of Seperation. Since we can define the empty class $\emptyset = \{u \mid u \neq u\}$, the empty set is also a set. However this follows from $\emptyset$ being a subset of all sets and hence only under the assumption that at least one other set exists. The existence of that set, in turn, follows from the Axiom of Infinity.[Jec78, p.6]

1.2.5 Axiom of Power Sets

$$\forall x \exists y \forall z \left(z \in y \iff z \subseteq x\right)$$

For any set x there is a set, denoted by $\mathcal{P}(x)$ and called the power set of x, consisting of all subsets of x.

1.2.6 Union Axiom

$$\forall x \exists y \forall z \left(z \in y \iff \exists w \left(z \in w \wedge w \in x\right)\right)$$

For any set x there is a set, denoted by $\bigcup x$, which is the union of all the elements of x.

1.2.7 Axiom of Infinity

$$\exists x \left(\emptyset \in x \wedge \forall y \left(y \in x \implies y \cup \{y\} \in x\right)\right)$$

There is an inductive set.

1.2.8 Axiom of Replacement

$$\forall x \exists y \forall y' \left(y' \in y \iff \exists x' \left(x' \in x \wedge \phi(x', y')\right)\right),$$

where $\phi(s, t)$ is a formula such that

$$\forall s \exists t \left(\phi(s, t) \wedge \forall t' \left(\phi(s, t') \implies t' = t\right)\right).$$

If $\phi(s, t)$ is a class function, then when its domain is restricted to a set x, the resulting images form a set y.

1.2.9 Axiom of Foundation

$$\forall x \exists y \left(y \in x \wedge x \cap y = \emptyset\right)$$

Every set contains an $\in$-minimal element, we call this being *well-founded*.[Gol98, p.92] This also means there exist no infinitely descending chains of sets, such as $x_0 \ni x_1 \ni x_2 \ni \cdots$.[Gol98, Theorem 4.3, p.95]

1.3 The Axiom of Choice

To talk about the axiom of choice we need to first define what a choice function is, the concept which the axiom is centered around.

Definition 1.3.1 (Choice Function)**.** [Jec78, p.38] Let S be a family of nonempty sets. A function $f : S \to \bigcup S$ is called a *choice function* of S if

$$f(X) \in X$$

holds for all sets $X \in S$.

The Axiom of Choice is then defined as follows:

Definition 1.3.2 (Axiom of Choice)**.** [Jec78, p.38] There exists a choice function for every family of nonempty sets.

The Axiom of Choice is not always needed for showing that a choice function exist. Take for example $S = \mathcal{P}(\mathbb{N})$, under the usual order $<$ every subset of $\mathbb{N}$ has a least element. We can therefore construct a choice function $f : S \to \mathbb{N}$ by letting $f(N)$ be the unique least element of N for $N \in S$. This is however not possible for a family of possibly infinite subsets of $\mathbb{R}$; for example the open interval $(0, 1)$ does not contain a least element.

In general there does not always exist an external structure for sets which we can utilize to construct a choice function. The Axiom of Choice ensures that we can, but not how that choice function might look like. In fact **AC** is the only axiom of **ZFC** which states the existence of a mathematical object without explicitly defining it. This is a powerful tool, but can lead lead to fairly unintuitive results. As such, with **AC** there exists a way to order the real numbers where every subset has a least element (including open intervals like $(0, 1)$)!

Chapter 2

Orderings and Well-Orderings

2.1 Linear, Partial and Well-Orderings

We start by defining the two types of partial orderings, *strict* and *weak* ones. To give a more intuitive understanding of how these differ we use the notation $<$ and $\leq$ respectively, but R is also commonly used to denote a relation.

Definition 2.1.1 (Strict Partial Order)**.** [Gol98, p.165] Let X be a set and $< \subseteq X \times X$ a binary relation on X. Then $<$ is called a *(strict) partial order of* X, and $(X, <)$ called a *(strictly) partially ordered set*, if it is

(i) **irreflexive**: $\forall x \in X\ (x \not< x)$

(ii) **transitive**: $\forall x, y, z \in X\ \big((x < y \wedge y < z) \implies x < z\big)$

It is called *linear* if for all x, y in X, $x < y$ or $y < x$ or $x = y$.

Definition 2.1.2 (Weak Partial Order)**.** [Gol98, p.164] Let X be a set and $\leq \subseteq X \times X$ a binary relation on X. Then $\leq$ is called a *weak partial order* of X, and $(X, \leq)$ called a *weakly partially ordered set*, if it is

(i) **reflexive**: $\forall x \in X\ (x \leq x)$

(ii) **transitive**: $\forall x, y, z \in X\ \big((x \leq y \wedge y \leq z) \implies x \leq z\big)$

(iii) **anti-symmetric**: $\forall x, y \in X\ \big((x \leq y) \wedge (y \leq x) \implies x = y\big)$

It is called *linear* if for all x, y in X, $x \leq y$ or $y \leq x$.

Definition 2.1.3. [Jec78, p.13] If $(X, <_X)$ and $(Y, <_Y)$ are two partially ordered sets, we call a function $f : X \to Y$ *order-preserving* if

$$x_1 <_X x_2 \iff f(x_1) <_Y f(x_2).$$

If X and Y are both linearly ordered, an order-preserving function is also said to be *increasing*.

The function f is called an *order-isomorphism* if f is both order-preserving and a bijective. Whenever it is clear from context that we are talking about ordered sets we simply call f an *isomorphism*. If f is order-preserving and injective, it is called an *order-embedding*. [Gol98, p.167]

A partially ordered set $(X, <)$ is sometimes also referred to simply as X by some abuse of notation when the relation $<$ is known. Additionally, whenever we talk about partially or linearly ordered sets without specifying which type, and where the type of partial order matters, we are referring to strict ones. [Jec78, p.12] Out of convenience, when talking about a strict partial order $<$, we sometimes refer to the term $(a < b \vee a = b)$ as $a \leq b$.

Clearly it is straightforward to define a weak partial order R' from a strict partial order R, letting $\langle x, y\rangle \in R'$ whenever $\langle x, y\rangle \in R'$ or $x = y$.

Definition 2.1.4. [Jec78, p.12] An element a of an ordered set $(X, <)$ is the *least element* of X with respect to $<$, if $\forall x \in X\ (a < x \vee x = a)$. Similarly, an element z is called the *greatest element* of X if $\forall x \in X\ (x < a \vee x = a)$.

This notion of a least element lets us define a special kind of linearly ordered set:

Definition 2.1.5 (Well-Order)**.** [Jec78, p.13] A strict linear order $<$ of a set X is called a *well-ordering* if every subset of X has a least element.

Well-ordered sets are central to the axiomatic set theory at hand. In fact, one of the most important results we will treat here, is that every set can we well-ordered (with the Axiom of Choice).

Further, we will introduce the concept of ordinals as a way effectively classify all well-ordered sets. The next two lemmata are needed for the proof of Theorem 2.2.5, an important result with regards to ordinals. For this we need to define the initial section of an ordered set first:

Definition 2.1.6. If X is a well-ordered set and $s \in X$, we call the set $\{x \in X \mid x < s\}$ an *initial segment* of X.

Lemma 2.1.7. *[Jec78, Lemma 2.1, p.13] If $(W, <)$ is a well-ordered set and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$.*

Proof. [Jec78, Lemma 2.1, p.13] In order to contrive a contradiction, we assume that $X = \{x \in W \mid f(x) < x\}$, the collection of elements of W not satisfying the lemma, is a non-empty set. We then let z be the least element of X and $w = f(z)$ its preimage in f. By the definition of X we this means that $f(w) < w$, contradicting the initial assumption that f is an increasing function. □

Lemma 2.1.8. *[Jec78, Lemma 2.2, p.13] No well-ordered set is isomorphic to an initial segment of itself.*

Proof. [Jec78, Lemma 2.2, p.13] Assume for a contradiction that f is an order isomorphism from an ordered set $(X, <)$ to an initial segment $(S, <) = \{x \in X \mid x < s\}$, for some $s \in X$ of itself. The image of f is then $\mathbf{Im}\left(f\right) = \{x \in X \mid x < s\} = S$, but we know this is not possible by Lemma 2.1.7. □

We are now ready to define ordinal numbers, motivated by the way we describe the natural numbers in set theory.

2.2 Ordinals and Order Types

There are several ways to define the natural numbers $\mathbb{N}$, the way we do it here, and the way generally used in set theory, is to use the Axiom of Infinity. This states that $\mathbb{N}$ is an *inductive set*, meaning that it contains 0, defined as $\emptyset = \{\}$, as well as the successor of every element in it, including of course 0 itself. [Gol98, p.39]

Definition 2.2.1. [Gol98, p.38] The successor of a set α is $\alpha^+ = \alpha \cup \{\alpha\}$. The successor of $0 = \emptyset$ is called 1 and the successor of 1 is called 2, etc.

The consequence of this is that $\mathbb{N}$ is the set we are familiar with: $\{0, 1, 2, 3, \ldots\}$. It also means that any natural number is defined as the set of all of its predecessors. For example $3 = \{0, 1, 2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$ and $5 = \{0, 1, 2, 3, 4\}$.

Perhaps slightly more subtle then is that, under the usual ordering, $n < m \implies n \in m \wedge n \subset m$. This is an important property and the natural numbers, as well as the set $\mathbb{N}$ at large, are called *transitive* sets. The notion of a *transitive set* is not to be confused with that of a *transitive (binary) relation*, which is an unfortunate overlap in terminology.

Definition 2.2.2. [Jec78, p.14] A set T is called *transitive* if

$$\forall x \, (x \in T \implies x \subseteq T) .$$

Definition 2.2.3. [Jec78, p.14] A set is called an *ordinal number* or *ordinal* if it is transitive and well-ordered by $\in$. We say $\alpha < \beta$ if and only if $\alpha \in \beta$.

Ordinals are denoted by lowercase Greek letters: $\alpha, \beta, \gamma, \ldots$. The ordinal associated with $(\mathbb{N}, \in)$ specifically is denoted by ω. We know that ω is indeed an ordinal by construction. It follows from the following lemma that every natural number also is an ordinal with respect to set inclusion.

Lemma 2.2.4. *[Jec78, Lemma 2.3, p.15]*

1. *The empty set $\emptyset$ is an ordinal.*

2. *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.*

3. *If α, β are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$.*

4. *If α, β are ordinals, then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

Proof. [Jec78, Lemma 2.3, p.15]

1. The empty set has no non-empty subsets, hence it is transitive and well-ordered by $\in$.

2. If $\beta \in \alpha$, then $\beta \subseteq \alpha$ by definition. Since α is well-ordered and transitive, so is β.

3. Let γ be the least element of the set $\beta \setminus \alpha$. We show that $\alpha = \gamma$.

   The ordinal α is transitive by definition and from this it follows that there are no "gaps" in the order. Indeed α must be an initial segment of β. As an initial segment, we can describe α as the set $\{\xi \in \beta \mid \xi < \gamma\}$. Again by the definition of ordinals, this is the set γ itself and $\alpha = \beta$.

4. We know that the intersection $\alpha \cap \beta = \gamma$ must be an ordinal, since not least the empty set also is an ordinal. However anything other than $\alpha = \gamma$ or $\beta = \gamma$ results in a contradiction:

   Assume for this contradiction that $\gamma \in \alpha$. Then $\gamma \in \beta$ by the second point of the lemma. Because γ is defined as the intersection of α and β, this means that $\gamma \in \gamma$. Since γ is an ordinal, strictly linearly ordered, this is not possible. □

Theorem 2.2.5. *[Jec78, Theorem 2, p.15] Every well-ordered set is order isomorphic to a unique ordinal.*

2.3 The Well-Ordering Theorem

The following, along with *Zorn's Lemma*, is one of the most fundamental results in set theory. There is a (bad) joke that goes:

> The *Axiom of Choice* is obviously true, the *Well-Ordering Theorem* obviously false, and who knows with *Zorn's Lemma*.

Definition 2.3.1 (Zermelo's Well-Ordering Theorem)**.** [Jec78, Theorem 15, p.39] Every set can be well ordered.

We could provide a proof for Definition 2.3.1 in **ZFC** here directly. This theorem, as it turn out, is not just another regular theorem, and we will therefore also not treat it as one.

Indeed, the Well-Ordering Theorem is actually equivalent to the Axiom of Choice. This means that if either statement is assumed to be true (and it has to be assumed since we are talking about *axioms*), the other one can be proved from it. This is the same methodology we will use for proving our main result, theorem 4.0.1, as well. There we will show equivalence of our main statement, that a group structure exists on all arbitrary sets, with the Well-Ordering Theorem. As such by transitivity, this main statement is also equivalent to the Axiom of Choice.

Theorem 2.3.2. *The Well-ordering Theorem is equivalent to the Axiom of Choice.*

Proof. [Jec78, Theorem 15, p.39] We provide a proof in two parts; first showing that the Well-Ordering Theorem is true in **ZFC**. Then, conversely, we prove **AC** in **ZF**, assuming that the Well-Ordering Theorem holds true.

1. **Axiom of Choice $\implies$ Well-Ordering Theorem**

   We proceed by transfinite induction.

   Let A be an arbitrary set and let $S = \mathcal{P}(A) \setminus \emptyset$ be the collection of all non-empty subsets of A. Let $f : S \to A$ be a choice function (as specified by the Axiom of Choice). We then define an ordinal sequence $(a_\alpha \mid \alpha < \theta)$ the following way:

   $$\begin{aligned} a_0 &= f(A) \\ a_\alpha &= f\left(A \setminus \left\{a_\xi \mid \xi < \alpha\right\}\right) \qquad \text{if } A \setminus \left\{a_\xi \mid \xi < \alpha\right\} \text{ is non-empty.} \end{aligned}$$

   Now let θ be the smallest ordinal such that $A = \left\{a_\xi \mid \xi < \theta\right\}$.

   We know that such an ordinal must exist, since the sequence $(a_\alpha \mid \alpha < \theta)$ is entirely defined by the choice function f. The function f maps every non-empty subset of A, i.e. members of S, to an element of that subset (in A).

   By defining the ordinal sequence the way we did, it is not possible for any element of A to occur in the sequence twice. Any subset of A, which is the input of the choice function for some element a_γ in the sequence, does not contain any elements a_α for $\alpha < \gamma$, and by definition f cannot map to any of these members.

   As such $\mathbf{Im}\left((a_\alpha \mid \alpha < \theta)\right) = A$ and $(a_\alpha \mid \alpha < \theta)$ enumerates A, meaning the sequence is a bijection.[1] Hence A can be well-ordered, the least element of any subset being the one which corresponds to the smallest ordinal in the sequence.

[1] Recall that a sequence is just a function from $\mathbb{N}$, respectively an ordinal, to the set of its elements

2. **Well-Ordering Theorem $\implies$ Axiom of Choice**

   Let S be a set of non-empty sets.

   The union $\bigcup S$ can be well-ordered by assumption and clearly $s \in S$ implies $s \subseteq \bigcup S$. We can then define the function $f : S \to \bigcup S$ to map any elements of S to its least element, according to the well-order of $\bigcup S$.

   Evidently f is a choice function and since the set S was arbitrary the Axiom of Choice holds. □

2.4 Hartogs' Lemma

We continue with the final result for this chapter, a lemma originally stated by Hartogs in 1915, restated and proven in this form in our main paper by Hajnal and Kertész.

Lemma 2.4.1. *[Har15] Let A be an arbitrary set and let $S = \mathcal{P}(A)$ be the collection of subsets of A. Then there exists an ordinal α, such that no no mapping $f_s : s \to \alpha$ from any subset $s \in S$ of A to α is an order isomorphisms.*

Proof. [HK72, Lemma] We let the ordinal α take the following value:

$$\alpha = \cup \left\{ \text{type}\,(X, R) + 1 \mid X \subseteq A, R \subseteq A \times A \wedge R \text{ well-orders } A \right\}.$$

We will show that α is such that it satisfies the lemma's statement.

□

Chapter 3

Model Theory

3.1 Models of Formal Languages

Definition 3.1.1. [Mar02, Definition 1.1.1] A *formal language* $\mathcal{L}$ in first order logic is given by the following:

1. A set $\mathcal{F}$ of functions f of n_f variables, with $n_f \in \mathbb{Z}^+$ a positive integer,
2. A set $\mathcal{R}$ of n_r-ary relations r, with $n_r \in \mathbb{Z}^+$ a positive integer,
3. A set $\mathcal{C}$ of constants.

3.2 The Löwenheim-Skolem Theorem

Lemma 3.2.1. *[CK90, Lemma 2.1.1] Let T be a consistent set of sentences of $\mathcal{L}$. Let C be a set of new constant symbols of power $|C| = \|\mathcal{L}\|$ and let $\bar{\mathcal{L}} = \mathcal{L} \cup C$ be the simple expansion of $\mathcal{L}$ formed by adding C.*

Then T can be expanded to a consistent set of sentences $\bar{T}$ in $\bar{\mathcal{L}}$, which has C as a set of witnesses in $\bar{\mathcal{L}}$.

Lemma 3.2.2. *[CK90, Lemma 2.1.2] Let T be a set of sentences and let C be a set of witnesses of T in $\mathcal{L}$. Then T has a model $\mathfrak{U}$, such that every element of $\mathfrak{U}$ is an interpretation of a constant $c \in C$.*

Theorem 3.2.3 (Extended Completeness Theorem)**.** *[CK90, Theorem 1.3.21] Let Σ be a set of sentences in $\mathcal{L}$. Then Σ is consistent if and only if Σ has a model.*

Theorem 3.2.4 (Downward Löwenheim-Skolem Theorem)**.** *[CK90, Corollary 2.1.4] Every consistent theory T in $\mathcal{L}$ has a model of power at most $\|\mathcal{L}\|$.*

Theorem 3.2.5 (Compactness Theorem)**.** *[CK90, Theorem 1.3.22] A set of sentences Σ has a model if and only if every finite subset of Σ has a model.*

Theorem 3.2.6 (Upward Löwenheim-Skolem Theorem)**.** *[CK90, Corollary 2.1.6] If T has infinite models, then it has infinite models of any given power $\alpha \geq \|\mathcal{L}\|$.*

Chapter 4

Hajnal's and Kertész's Theorem

Theorem 4.0.1. *[HK72] The following are equivalent in ZF:*

1. *Axiom of Choice*
2. *Every non-empty set admits a cancellative groupoid structure*

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