0.1 Models of Formal Languages

Definition 0.1.1. [?, Definition 1.1.1] A *formal language* $\mathcal{L}$ in first order logic is given by the following:

1. A set $\mathcal{F}$ of functions f of n_f variables, with $n_f \in \mathbb{Z}^+$ a positive integer,
2. A set $\mathcal{R}$ of n_r-ary relations r, with $n_r \in \mathbb{Z}^+$ a positive integer,
3. A set $\mathcal{C}$ of constants.

0.2 The Löwenheim-Skolem Theorem

Lemma 0.2.1. *[?, Lemma 2.1.1] Let T be a consistent set of sentences of $\mathcal{L}$. Let C be a set of new constant symbols of power $|C| = \|\mathcal{L}\|$ and let $\bar{\mathcal{L}} = \mathcal{L} \cup C$ be the simple expansion of $\mathcal{L}$ formed by adding C.*

Then T can be expanded to a consistent set of sentences $\bar{T}$ in $\bar{\mathcal{L}}$, which has C as a set of witnesses in $\bar{\mathcal{L}}$.

Lemma 0.2.2. *[?, Lemma 2.1.2] Let T be a set of sentences and let C be a set of witnesses of T in $\mathcal{L}$. Then T has a model $\mathfrak{A}$, such that every element of $\mathfrak{A}$ is an interpretation of a constant $c \in C$.*

Theorem 0.2.3 (Extended Completeness Theorem)**.** *[?, Theorem 1.3.21] Let Σ be a set of sentences in $\mathcal{L}$. Then Σ is consistent if and only if Σ has a model.*

Theorem 0.2.4 (Downward Löwenheim-Skolem Theorem)**.** *[?, Corollary 2.1.4] Every consistent theory T in $\mathcal{L}$ has a model of power at most $\|\mathcal{L}\|$.*

Theorem 0.2.5 (Compactness Theorem)**.** *[?, Theorem 1.3.22] A set of sentences Σ has a model if and only if every finite subset of Σ has a model.*

Theorem 0.2.6 (Upward Löwenheim-Skolem Theorem)**.** *[?, Corollary 2.1.6] If T has infinite models, then it has infinite models of any given power $\alpha \geq \|\mathcal{L}\|$.*