0.1 Linear, Partial and Well-Orderings

We define the two types of partial orderings, *strong* and *weak* ones, as follows:

Definition 0.1.1 (Strong Partial Order)**.** [**?**, p.165] Let X be a set and $< \subseteq X \times X$ a binary relation on X. Then $<$ is called a *(strong) partial order of* X, and $(X, <)$ called a *(strongly) partially ordered set*, if it is

(i) **irreflexive**: $\forall x, y \in X \left((x < y) \vee (x = y) \vee (y < x)\right)$

(ii) **transitive**: $\forall x, y, z \in X \left((x < y \wedge y < z) \implies x < z\right)$

It is called *linear* if for all x, y in X, $x < y$ or $y < x$ or $x = y$.

Definition 0.1.2 (Weak Partial Order)**.** [**?**, p.164] Let X be a set and $\leq \subseteq X \times X$ a binary relation on X. Then $\leq$ is called a *weak partial order* of X, and $(X, \leq)$ called a *weakly partially ordered set*, if it is

(i) **reflexive**: $\forall x \in X \ (x \leq x)$

(ii) **transitive**: $\forall x, y, z \in X \left((x \leq y \wedge y \leq z) \implies x \leq z\right)$

(iii) **anti-symmetric**: $\forall x, y \in X \left((x \leq y) \wedge (y \leq x) \implies x = y\right)$

It is called *linear* if for all x, y in X, $x \leq y$ or $y \leq x$.

A partially ordered set $(X, <)$ is sometimes also referred to simply as X by some abuse of notation when the relation $<$ is known. Additionally, whenever we talk about partially or linearly ordered sets without specifying which type we are referring to strong ones. [**?**, p.12]

Definition 0.1.3 (Well-Order)**.** [**?**, p.13] A strong linear order $<$ of a set X is called a *well-ordering* if every subset of X has a least element.

0.2 Ordinals and Order Types

There are several ways to define the natural numbers $\mathbb{N}$, the way we do it here, and the way generally used in set theory, is to use the Axiom of Infinity. This states that $\mathbb{N}$ is an *inductive set*, meaning that it contains 0, defined as $\emptyset = \{\}$, as well as the successor of every element in it, including of course 0 itself. [**?**, p.39]

Definition 0.2.1. [**?**, p.38] The successor of a set α is $\alpha^+ = \alpha \cup \{\alpha\}$. The successor of $0 = \emptyset$ is called 1 and the successor of 1 is called 2, etc..

The consequence of this is that $\mathbb{N}$ is the set we are familiar with: $\{0, 1, 2, 3, \ldots\}$. It also means that any natural number is defined as the set of all of its predecessors. For example $3 = \{0, 1, 2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$ and $5 = \{0, 1, 2, 3, 4\}$.

Perhaps slightly more subtle then is that, under the usual ordering, $n < m \implies n \in m \wedge n \subset m$. This is an important property and the natural numbers, as well as the set $\mathbb{N}$ at large, are called *transitive* sets. The notion of a *transitive set* is not to be confused with that of a *transitive (binary) relation*, which is an unfortunate overlap in terminology.

Definition 0.2.2. [**?**, p.14] A set T is called *transitive* if

$$\forall x \left(x \in T \implies x \subseteq T\right).$$

Definition 0.2.3. [**?**, p.14] A set is called an *ordinal number* or *ordinal* if it is transitive and well-ordered by $\in$. We say $\alpha < \beta$ if and only if $\alpha \in \beta$.

Ordinals are denoted by lowercase greek letters: $\alpha, \beta, \gamma, \ldots$. The ordinal associated with $(\mathbb{N}, \in)$ specifically is denoted by ω. We know that ω is indeed an ordinal by construction. It follows from the following lemma that every natural number also is an ordinal with respect to set inclusion.

Lemma 0.2.4. *[**?**, Lemma 2.3, p.15]*

1. *The empty set $\emptyset$ is an ordinal.*
2. *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.*
3. *If α, β are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$.*
4. *If α, β are ordinals, then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$*

Proof. [**?**, Lemma 2.3, p.15]

1. The empty set has no non-empty subsets, hence it is transitive and well-ordered by $\in$.
2. If $\beta \in \alpha$, then $\beta \subseteq \alpha$ by definition. Since α is well-ordered and transitive, so is β.
3. Let γ be the least element of the set $\beta \setminus \alpha$. We show that $\alpha = \gamma$.

   The ordinal α is transitive by definition and from this it follows that there are no "gaps" in the order. Indeed α must be an initial segment of β. As an initial segment, we can describe α as the set $\left\{\xi \in \beta \mid \xi < \gamma\right\}$. Again by the definition of ordinals, this is the set γ itself and $\alpha = \beta$.

4. We know that the intersection $\alpha \cap \beta = \gamma$ must be an ordinal, since not least the empty set also is an ordinal. However anything other than $\alpha = \gamma$ or $\beta = \gamma$ results in a contradiction:

   Assume for this contradiction that $\gamma \in \alpha$. Then $\gamma \in \beta$ by the second point of the lemma. Because γ is defined as the intersection of α and β, this means that $\gamma \in \gamma$. Since γ is an ordinal, strongly linearly ordered, this is not possible. □

Theorem 0.2.5. *[?, Theorem 2, p.15] Every well-ordered set is order isomorphic to a unique ordinal.*

0.3 The Well-Ordering Theorem

The following, along with *Zorn's Lemma*, is one of the most fundamental results in set theory. There is a (bad) joke that goes:

> The *Axiom of Choice* is obviously true, the *Well-Ordering Theorem* obviously false, and who knows with *Zorn's Lemma*.

Definition 0.3.1 (Zermelo's Well-Ordering Theorem)**.** [?, Theorem 15, p.39] Every set can be well ordered.

We could provide a proof for definition 0.3.1 in **ZFC** here directly. This theorem, as it turn out, is not just another regular theorem, and we will therefore also not treat it as one.

Indeed, the Well-Ordering Theorem is actually equivalent to the Axiom of Choice. This means that if either statement is assumed to be true (and it has to be assumed since we are talking about *axioms*), the other one can be proved from it. This is the same methodology we will use for proving our main result, theorem **??**, as well. There we will show equivalence of our main statement, that a group structure exists on all arbitrary sets, with the Well-Ordering Theorem. As such by transitivity, this main statement is also equivalent to the Axiom of Choice.

Theorem 0.3.2. *The Well-ordering Theorem is equivalent to the Axiom of Choice.*

Proof. [?, Theorem 15, p.39] We provide a proof in two parts; first showing that the Well-Ordering Theorem is true in **ZFC**. Then, conversely, we prove **AC** in **ZF**, assuming that the Well-Ordering Theorem holds true.

1. **Axiom of Choice $\implies$ Well-Ordering Theorem**

   We proceed by transfinite induction.

Let A be an arbitrary set and let $S = \mathcal{P}(A) \setminus \emptyset$ be the collection of all non-empty subsets of A. Let $f : S \to A$ be a choice function (as specified by the Axiom of Choice). We then define an ordinal sequence $(a_\alpha \mid \alpha < \theta)$ the following way:

$$\begin{aligned} a_0 &= f(A) \\ a_\alpha &= f\left(A \setminus \left\{a_\xi \mid \xi < \alpha\right\}\right) \qquad \text{if } A \setminus \left\{a_\xi \mid \xi < \alpha\right\} \text{ is non-empty.} \end{aligned}$$

Now let θ be the smallest ordinal such that $A = \left\{a_\xi \mid \xi < \theta\right\}$.

We know that such an ordinal must exist, since the sequence $(a_\alpha \mid \alpha < \theta)$ is entirely defined by the choice function f. The function f maps every non-empty subset of A, i.e. members of S, to an element of that subset (in A).

By defining the ordinal sequence the way we did, it is not possible for any element of A to occur in the sequence twice. Any subset of A, which is the input of the choice function for some element a_γ in the sequence, does not contain any elements a_α for $\alpha < \gamma$, and by definition f cannot map to any of these members.

As such $\mathbf{Im}\left((a_\alpha \mid \alpha < \theta)\right) = A$ and $(a_\alpha \mid \alpha < \theta)$ enumerates A, meaning the sequence is a bijection.[1] Hence A can be well-ordered, the least element of any subset being the one which corresponds to the smallest ordinal in the sequence.

2. **Well-Ordering Theorem $\implies$ Axiom of Choice**

   Let S be a set of non-empty sets.

   The union $\bigcup S$ can be well-ordered by assumption and clearly $s \in S$ implies $s \subseteq \bigcup S$. We can then define the function $f : S \to \bigcup S$ to map any elements of S to its least element, according to the well-order of $\bigcup S$.

   Evidently f is a choice function and since the set S was arbitrary the Axiom of Choice holds. $\square$

0.4 Hartogs' Lemma

We continue with the final result for this chapter, a lemma originally stated by Hartogs in 1915, restated and proven in this form in our main paper by Hajnal and Kertész.

Lemma 0.4.1. *[?] Let A be an arbitrary set and let $S = \mathcal{P}(A)$ be the collection of subsets of A. Then there exists an ordinal α, such that no no mapping $f_s : s \to \alpha$ from any subset $s \in S$ of A to α is an order isomorphisms.*

[1] Recall that a sequence is just a function from $\mathbb{N}$, respectively an ordinal, to the set of its elements

Proof. [?, Lemma] We let the ordinal α take the following value:

$$\alpha = \cup \left\{ \text{type}\,(X, R) + 1 \,|\, X \subseteq A, R \subseteq A \times A \wedge R \text{ well-orders } A \right\}.$$

We will show that α is such that it satisfies the lemma's statement.

□