Group Structure on arbitrary sets: An algebraic application of the Axiom of Choice

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Abstract

The thesis should include an abstract that summarizes its contents; mathematical jargon can be utilized here. The typical length of an abstract is between 100 and 300 words.

Popular science description

test

Contents

Introduction

Historical Background

In 1902 Bertrand Russell showed with what is now known as *Russel's Paradox* that the previously used approach to set theory was inconsistent. Ernst Zermelo then created an axiomatic framework for set theory in 1905, motivated both by attempting to preserve results such as the theory of infinities by Georg Cantor, as well as avoiding paradoxes. These axioms, later modified by Abraham Fraenkel, became known as the nine *Zermelo-Fraenkel Axioms* (ZF) as well as the *Axiom of Choice* (AC)[Gol98, pp.66-70, 75].

The axiom of choice in particular is of special interest in many areas of mathematics, especially in algebra and topology, often in the form of the equivalent statement of *Zorn's Lemma*, which says that every non-empty partially ordered set with an upper bound has a maximal element [Jec78].

Finally in 1971 András Hajnal and Andor Kertész published a paper [HK72] which provided another equivalence to AC, namely that there exists a cancellative groupoid structure on every (uncountably infinite) set. This paper makes use of first-order model theory, an area of logic developed during the first half of the 20th century, which utilizes models of formal languages to obtain results. Kertész later expanded on this, providing an alternative algebraic partial proof in a lecture series given at the University of Jyväskylä [Ker75].

Thesis Structure

The aim of this thesis is to provide context to the paper [HK72] and to derive the theory needed for the proof of its main theorem:

Theorem 0.0.1. *The following sentences are equivalent in ZF:*

1. *Axiom of Choice*
2. *Every non-empty set admits a cancellative groupoid structure*

We will first explore orderings and well-orderings in the context of axiomatic set theory. Of special importance here will be Zorn's Lemma, a well-known equivalence of AC. We will finish by giving a proof for a lemma by Hartogs [Har15], which states that there always exists an ordinal which no subset of an arbitrary set can be injectively mapped to.

We will then move on to an introduction to model theory, with the aim of proving the upwards Löwenheim-Skolem Theorem. Model theory is a very useful tool for applying results from logic to non-logic areas of mathematics, especially abstract algebra as we will see later. As Chang and Heisler put it in [CK90] (a very good historical introduction to model theory and the first comprehensive textbook for the subject),

Model Theory = Universal Algebra + Logic.

Finally we will give a detailed proof of the aforementioned theorem by Hajnal and Kertész, applying the results by Hartogs and Löwenheim and Skolem.

Chapter 1

Preliminaries

The convention in this thesis will be to say **ZF** when talking about Zermelo-Fraenkel set theory *without* the axiom of choice. When talking about the axiom of choice on its own we will say **AC**, and when talking about Zermelo-Fraenkel set theory together with the axiom of choice use **ZFC**.

We will use the convention of including 0 at the beginning of the natural numbers $\mathbb{N}$, i.e. $\mathbb{N} = \{0, 1, 2 \ldots\}$. This is a *natural* choice, since we then can use $\mathbb{N}$ to mean the set described by the Axiom of Infinity.

1.1 Zermelo-Fraenkel Axioms of Set Theory

We assume that the reader has some familiarity with axiomatic set theory, but for convenience and consistency we restate some of the necessary basics here. For a more thorough review, see [Gol98], from which the formulations below are used as well.

1.1.1 Axiom of Extensionality

$$\forall x \forall y \left(x = y \iff \forall z \left(z \in x \iff z \in y\right)\right)$$

Two sets are equal if and only if they contain the same elements.

1.1.2 Axiom of the Empty Set

$$\exists x \forall y \, y \notin x$$

There is a set with no elements.

1.1.3 Axiom of Pairs

$$\forall x \forall y \exists z \forall w \left(w \in z \iff \left(w = x \vee w = y\right)\right)$$

For any two sets, there is a set whose elements are precisely these sets.

1.1.4 Axiom of Separation

$$\forall x \exists y \forall z \left(z \in y \iff \left(z \in x \land \phi(z)\right)\right),$$

where $\phi(z)$ is any statement of the formal language with free variable z. For any set x there is a set consisting of all z in x for which $\phi(z)$ holds.

1.1.5 Axiom of Power Sets

$$\forall x \exists y \forall z \left(z \in y \iff z \subseteq x\right)$$

For any set x there is a set, denoted by $\mathcal{P}(x)$ and called the power set of x, consisting of all subsets of x.

1.1.6 Union Axiom

$$\forall x \exists y \forall z \left(z \in y \iff \exists w \left(z \in w \land w \in x\right)\right)$$

For any set x there is a set, denoted by $\bigcup x$, which is the union of all the elements of x.

1.1.7 Axiom of Infinity

$$\exists x \left(\varnothing \in x \land \forall y \left(y \in x \implies y \cup \{y\} \in x\right)\right)$$

There is an inductive set.

1.1.8 Axiom of Replacement

$$\forall x \exists y \forall y' \left(y' \in y \iff \exists x' \left(x' \in x \land \phi(x', y')\right)\right),$$

where $\phi(s, t)$ is a formula such that

$$\forall s \exists t \left(\phi(s, t) \land \forall t' \left(\phi(s, t') \implies t' = t\right)\right).$$

If $\phi(s, t)$ is a class function, then when its domain is restricted to a set x, the resulting images form a set y. Since no proper classes will be treated in this text, the relevant statement for us is that the image of any function, of which the domain is a proper set, always is a set.

1.1.9 Axiom of Foundation

$$\forall x \exists y \left(y \in x \land x \cap y = \varnothing\right)$$

Every set is *well-founded*, i.e. contains an $\in$-minimal element.

Chapter 2

First chapter

Lemma 2.0.1. *[Har15] Let A be an arbitrary set. Then there exists an ordinal α, such that no subset of A can be injectively mapped on to α.*

Theorem 2.0.2. *[HK72] The following are equivalent in ZF:*

1. *Axiom of Choice*
2. *Every non-empty set admits a cancellative groupoid structure*

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