The convention in this thesis will be to say **ZF** when talking about Zermelo-Fraenkel set theory *without* the axiom of choice. When talking about the axiom of choice on its own we will say **AC**, and when talking about Zermelo-Fraenkel set theory together with the axiom of choice we use **ZFC**.

We will use the convention of including 0 at the beginning of the natural numbers $\mathbb{N}$, i.e. $\mathbb{N} = \{0, 1, 2 \ldots\}$. This is a *natural* choice, since we then can use $\mathbb{N}$ to mean the set described by the Axiom of Infinity. If we want to talk about strictly positive integers we use the notation $\mathbb{Z}^+ = \mathbb{N} \setminus \{0\}$.

Lastly, whenever we deal with the negation of some symbol, we cross it out to mean this, for example $a \neq b$ means $\neg(a = b)$.

0.1 First-Order Logic and Classes

When talking about first order logic we mean the symbol

$$= \text{ (equals)}$$

in conjunction with the logical connectives

$$\neg \text{ (not)}, \ \wedge \text{ (and)}, \ \vee \text{ (or)}, \ \rightarrow \text{ (implies) and } \ \leftrightarrow \text{ (if and only if)}$$

as well as the quantifiers

$$\forall \text{ (for all) and } \exists \text{ (exists)}.$$

Additionally, as we are talking about sets, we use the symbol $\in$ to denote set inclusion. [**?**, pp.2-3]

In order to effectively talk about properties of set and set-like structures we need to somehow properly define formulas. We will go more into depth about formulas in Chapter **??**, where we will introduce the notion of a formal language.

For now we are only concerned with the language of sets defined above.

Definition 0.1.1 (Formula of a Set)**.** An *atomic formula* in set theory is either

1. $x = y$, or
2. $x \in y$

A *formula* ϕ is an any combination of atomic formulas with logical connectives and quantifiers.

The symbols x and y above are called variables and for any two variables an atomic formula is either true or false for each x and y. A variable *occurs freely* inside of a formula if it does not appear inside of a $\exists$ or $\forall$ quantifier, otherwise the variable is *bound*.

We write $\phi\left(x_1, \ldots, x_n\right)$ for a formula with $n \in \mathbb{Z}^+$ free variables. A formula where every variable is bound is called a *sentence*. [**?**, pp.10-11]

A sentence is either true are false and a formula with free variables is true or false for each choice of the free variables. Each of the **ZF** axioms below are examples of sentences and as axioms we assume them to be inherently true (within the framework of our theory). An example of a formula with a free variable would be

$$\phi(x) = \exists y \ \left(y \in x\right).$$

This formula is only false for the empty set, since it is the unique set which does not contain any elements. In that sense we think of formulas with free variables describing a *property*, something we make use of in *classes*.

Definition 0.1.2 (Class)**.** Let $\phi\left(x, p_1, \ldots, p_n\right)$ be a formula in first order logic. Then a *class* $\mathbf{C}$ is defined as

$$\mathbf{C} = \left\{x \mid \phi\left(x, p_1, \ldots, p_n\right)\right\}.$$

The class $\mathbf{C}$ is called *definable from* $p_1, \ldots, p_n$. Furthermore, if x is the only free variable of ϕ, the class $\mathbf{C}$ is simply called *definable*. [**?**, p.3]

In practice, we use classes as a tool to help us construct useful sets in **ZFC**, as elements of classes are always sets in the stricter sense. All sets are classes, but not all classes are sets, since if we have a fixed set s we can always construct the corresponding class would be $\mathbf{S} = \{x \mid x = s\}$. A class which is not a set is called a *proper class*.

We consider to two classes to be the same if they have the same elements. The familiar set operations of *inclusion*, *union*, *intersection*, and *difference* are definable using formulas. As such for classes $\mathbf{C}$, $\mathbf{D}$,

$$\begin{aligned}
\mathbf{C} \subseteq \mathbf{D} &\iff \forall x \left(x \in \mathbf{C} \implies x \in \mathbf{D}\right) \\
\mathbf{C} \cup \mathbf{D} &= \{x \mid x \in \mathbf{C} \lor x \in \mathbf{D}\} \\
\mathbf{C} \cap \mathbf{D} &= \{x \mid x \in \mathbf{C} \land x \in \mathbf{D}\} \\
\mathbf{C} \setminus \mathbf{D} &= \{x \mid x \in \mathbf{C} \land x \notin \mathbf{D}\} \\
\bigcup \mathbf{C} &= \{x \mid x \in S \text{ for some } S \in \mathbf{C}\}
\end{aligned}$$

[**?**, pp.3-4]

For the use in this text, classes are in a sense "naive sets-like object"; they to help us describe collections of sets without worrying about paradoxes. Consider for example the class $\mathbf{V} = \{x \mid x = x\}$, which is the universe of all sets and does not exist in pure set theory. Another important class which we will make use of later is the *empty class* $\varnothing = \{x \mid x \neq x\}$ (although this is also a set as we will see).

0.2 Zermelo-Fraenkel Axioms of Set Theory

We assume that the reader has some familiarity with axiomatic set theory, but for convenience and consistency we restate some of the necessary basics here. For a more thorough introduction of the topic, see [?, §§4.3-4.5], alternatively [?, §1.1] gives a more technical overview. The formulation of the axioms below is based on both textbooks.

0.2.1 Axiom of Extensionality

$$\forall x \forall y \left(x = y \iff \forall z \left(z \in x \iff z \in y \right) \right)$$

Two sets are equal if and only if they contain the same elements.[?, §4.3, p.76]

0.2.2 Axiom of Pairs

$$\forall x \forall y \exists z \forall w \left(w \in z \iff \left(w = x \lor w = y \right) \right)$$

For any two sets, there is a set whose elements are precisely these sets.

We define an ordered pair $\langle x,\ y \rangle$ to be the set $\left\{ \{x\},\ \{x,\ y\} \right\}$. Further, ordered n-tuples are defined recursively as $\langle x_1,\ x_2,\ x_3, \dots,\ x_n \rangle = \langle x_1,\ \langle x_2,\ x_3, \dots,\ x_n \rangle \rangle$.[?, §4.3, pp.76, 79-80]

Ordered pairs satisfy the property that for any sets x, y, u, v, if $\langle x,\ y \rangle = \langle u,\ v \rangle$, then $x = u$ and $y = v$.[?, Theorem 4.2]

0.2.3 Axiom Schema of Separation

Let $\phi(z, p)$ be a formula in first order logic with a free variable z. Then

$$\forall x \forall p \exists y \forall z \left(z \in y \iff \left(z \in x \land \phi(z, p) \right) \right). \tag{0.2.1}$$

For any sets x and p there is a unique set consisting of all z in x for which $\phi(z, p)$ holds. This is an axiom schema, meaning an infinite collection of axioms, since (0.2.1) is a separate axiom for every formula $\phi(z, p)$.[?, pp.5-6]

0.2.4 Axiom of the Empty Set

$$\exists x \forall y\, y \notin x$$

There is a set with no elements. We call this set $\varnothing = \{\}$. [?]

The Empty Set Axiom is not strictly required, the existence of the empty set also arises from the Axiom Schema of Seperation. Since we can define the empty class $\varnothing = \{u \mid u \neq u\}$, the empty set is also a set. However this follows from $\varnothing$ being a subset of all sets and hence only under the assumption that at least one other set exists. The existence of that set, in turn, follows from the Axiom of Infinity.[?, p.6]

0.2.5 Axiom of Power Sets

$$\forall x \exists y \forall z \left(z \in y \iff z \subseteq x\right)$$

For any set x there is a set, denoted by $\mathcal{P}(x)$ and called the power set of x, consisting of all subsets of x.

0.2.6 Union Axiom

$$\forall x \exists y \forall z \left(z \in y \iff \exists w \left(z \in w \land w \in x\right)\right)$$

For any set x there is a set, denoted by $\bigcup x$, which is the union of all the elements of x.

0.2.7 Axiom of Infinity

$$\exists x \left(\varnothing \in x \land \forall y \left(y \in x \implies y \cup \{y\} \in x\right)\right)$$

There is an inductive set.

0.2.8 Axiom of Replacement

$$\forall x \exists y \forall y' \left(y' \in y \iff \exists x' \left(x' \in x \land \phi(x', y')\right)\right),$$

where $\phi(s, t)$ is a formula such that

$$\forall s \exists t \left(\phi(s, t) \land \forall t' \left(\phi(s, t') \implies t' = t\right)\right).$$

If $\phi(s, t)$ is a class function, then when its domain is restricted to a set x, the resulting images form a set y.

0.2.9 Axiom of Foundation

$$\forall x \exists y \left(y \in x \land x \cap y = \varnothing\right)$$

Every set contains an $\in$-minimal element, we call this being *well-founded*.[**?**, p.92] This also means there exist no infinitely descending chains of sets, such as $x_0 \ni x_1 \ni x_2 \ni \cdots$.[**?**, Theorem 4.3, p.95]

0.3 The Axiom of Choice

To talk about the axiom of choice we need to first define what a choice function is, the concept which the axiom is centered around.

Definition 0.3.1 (Choice Function)**.** [**?**, p.38] Let S be a family of nonempty sets. A function $f : S \to \bigcup S$ is called a *choice function* of S if

$$f(X) \in X$$

holds for all sets $X \in S$.

The Axiom of Choice is then defined as follows:

Definition 0.3.2 (Axiom of Choice)**.** [**?**, p.38] There exists a choice function for every family of nonempty sets.

The Axiom of Choice is not always needed for showing that a choice function exist. Take for example $S = \mathcal{P}(\mathbb{N})$, under the usual order $<$ every subset of $\mathbb{N}$ has a least element. We can therefore construct a choice function $f : S \to \mathbb{N}$ by letting $f(N)$ be the unique least element of N for $N \in S$. This is however not possible for a family of possibly infinite subsets of $\mathbb{R}$; for example the open interval $(0, 1)$ does not contain a least element.

In general there does not always exist an external structure for sets which we can utilize to construct a choice function. The Axiom of Choice ensures that we can, but not how that choice function might look like. In fact **AC** is the only axiom of **ZFC** which states the existence of a mathematical object without explicitly defining it. This is a powerful tool, but can lead lead to fairly unintuitive results. As such, with **AC** there exists a way to order the real numbers where every subset has a least element (including open intervals like $(0, 1)$)!