1 Definitions

- **Event**: An **event** is any collection of possible outcomes of an experiment.
- **Mutually Exclusive**: Two events A and B are **mutually exclusive** if they cannot both occur.
  - It is equivalent to say that A and B are **disjoint**.
  - It is equivalent to say that $A \cap B = \{\}$ (the set containing no outcomes, or "the empty set").
  - It is equivalent to say that $P(A \cap B) = 0$.
- **Independent**: Two events A and B are **independent** if whether or not one event occurs does not affect whether or not the other event occurs.
  - It is equivalent to say that $P(A \cap B) = P(A)P(B)$.
  - It is equivalent to say that $P(A|B) = P(A)$.
  - Intuitively, the above line is saying that knowing B occurs does not change the probability that A occurs.
  - If A and B are independent, then all of the following are true:
    * $P(B|A) = P(B)$
    * $P(B|A \text{ does not occur}) = P(B)$
    * $P(A|B \text{ does not occur}) = P(A)$

2 Examples

- **Experiment**: My experiment is to roll one die **and** flip one coin.
- **Events**: I will define three events:
  - $A =$ roll a 1
  - $B =$ roll greater than a 2
  - $C =$ flip a heads
- **A and B Are Mutually Exclusive**: The events A and B are mutually exclusive because it is impossible for A and B to occur in the one experiment. In probability, we might write $P(A \cap B) = 0$ (or that the probability of A and B occurring in the same experiment is 0).
- **A and C Are Not Mutually Exclusive**: The events A and C are **not** mutually exclusive because it is possible for A and C to occur in the one experiment. My experiment is to roll one die and flip one coin. I would expect that A and C would occur together about one in every twelve times. (That is, we might write $P(A \cap B) = \frac{1}{12}$).
- **A and B Are Not Independent**: The events A and B are **not** independent. Knowing whether or not one occurs affects the probability of the other.

$$\begin{aligned} P(A) &= P(\text{roll a } 1) = \frac{1}{6} \\ P(A|B) &= P(\text{roll a } 1|\text{roll greater than a } 2) = 0 \\ \Rightarrow P(A) &\neq P(A|B) \end{aligned}$$

Because $P(A) \neq P(A|B)$, the events A and B cannot be independent.

- A **and** C **Are Independent**: The events A and C are independent. Knowing whether or not one occurs has does not change change our understanding of if the other one occurs.

$$\begin{aligned} P(A) &= P(\text{roll a 1}) = \frac{1}{6} \\ P(A|C) &= P(\text{roll a 1}|\text{flip a heads}) = \frac{1}{6} \\ \Rightarrow P(A) &= P(A|C) \end{aligned}$$

  Because $P(A) = P(A|C)$, the events A and C are independent.

3 Proof

- If I have two events that are mutually exclusive, can they be independent? **The answer is no.** I'm going to show this by assuming two events are mutually exclusive and proving that they cannot be independent.
  - Let's assume that two events X and Y are mutually exclusive.
  - Let's also assume $P(X)$ and $P(Y)$ are both greater than 0. *(This is a really basic assumption... this just means that it's possible for X to occur and possible for Y to occur! Otherwise, these would be really silly events for us to care about.)*
  - Then, by the definition of mutually exclusive, $P(X \cap Y) = 0$.
  - One of our probability rules states that $P(X \cap Y) = P(X|Y)P(Y)$.
  - Thus, $P(X|Y)P(Y) = 0$.
  - The only way that $P(X|Y)P(Y) = 0$ is for $P(X|Y)$ to equal 0 or for $P(Y)$ to equal 0. *(Or both!)*
  - Remember that we assumed above that $P(Y) > 0$. So this can't be the case.
  - Therefore, it must be the case that $P(X|Y) = 0$.
  - So, is it possible for X and Y to be independent? Well, if so, then $P(X|Y) = P(X)$.
  - That would mean that $P(X) = 0$.
  - However, that violates our assumption above that $P(X) > 0$!
  - Therefore, $P(X|Y)P(Y)$ cannot equal 0.
  - However, X and Y are mutually exclusive, so $P(X|Y)P(Y)$ must equal 0.
  - This is a contradiction. We assumed that events X and Y were mutually exclusive and showed that it is impossible for them to be independent!
  - **Thus, if events X and Y are mutually exclusive, then they cannot be independent.**
- If I have two events that are independent, can they be mutually exclusive? **The answer is no.** I'm going to show this by assuming two events are independent and proving that they cannot be mutually exclusive.
  - Let's assume that two events X and Y are independent.
  - Let's also assume $P(X)$ and $P(Y)$ are both greater than 0. *(We'll do this for the same reasons as above.)*
  - Then, by the definition of independence, $P(X|Y) = P(X)$.
  - We can multiply both sides of the above equality by $P(Y)$ to get $P(X|Y)P(Y) = P(X)P(Y)$.
  - Recall our probability rule stating that $P(X|Y)P(Y) = P(X \cap Y)$.
  - Then, $P(X \cap Y) = P(X)P(Y)$.
  - So, is it possible for X and Y to be mutually exclusive? Well, if so, then $P(X \cap Y) = 0$.

- That implies that $P(X)P(Y) = 0$.
- However, this violates our assumption that $P(X)$ and $P(Y)$ are both greater than 0.
- This is a contradiction. We assumed that events X and Y were independent and showed that it is impossible for them to be mutually exclusive!
- **Thus, if events X and Y are independent, then they cannot be mutually exclusive.**