MULTIPLICITY OF P-PARTITION GENERATING FUNCTION

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Theorem 1. *Removing/adding edges to labeled poset.*
1. Suppose (Q, τ) is obtained from (P, ω) by deleting edges. If (P, ω) has multiplicity, so does (Q, τ).
2. Suppose (P, ω) is obtained from (Q, τ) by adding more edges. If (Q, τ) is multiplicity-free, so is (P, ω).

Proof.
1. Let (Q, τ) inherit its labelling from (P, ω). Thus, every linear extension of (P, ω) is also a linear extension of (Q, τ). Hence, if (P, ω) has multiplicity (at least two linear extensions with the same F-function), (Q, τ) will also have such linear extensions, and thus has multiplicity.
2. Let (P, ω) inherit its labelling from (Q, τ). Thus, the set of linear extension of (P, ω) is a subset of the set of linear extensions of (Q, τ). If (Q, τ) is mult-free (having no two linear extensions that share the same F-function), then (P, ω) will also have no two linear extensions that share the same F-function, and thus is also mult-free.

□

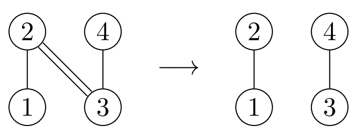

FIGURE 1. Example of deleting edges.

Theorem 2. *Involution*
We consider some natural involutions we can perform on a labeled poset (P, ω). First, we can switch strict and weak edges, denoting the result $\overline{(P, \omega)}$. Secondly, we can rotate the labeled poset 180°, preserving strictness and weakness of edges; we denote the resulting labeled poset $(P, \omega)^*$.

The involutions of (P, ω) will have the same multiplicity-ness as that of (P, ω).

Theorem 3. *Incomparable Elements Rule*
Suppose P contains a, b with a imcomparable to b $(a \| b)$.
In a linear extension of P either a comes before b or b comes before a. If a comes before b then we draw an edge from a upto b and vice versa.
Then the set of linear extensions of P equals the (disjoint) union of these sets of linear extensions from each of these 2 cases.

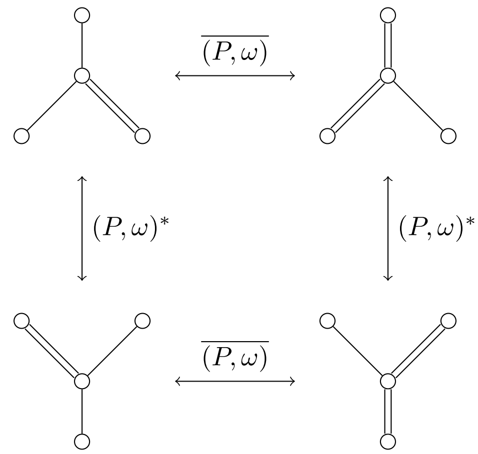

FIGURE 2. The bar and star involutions.

Let A, B, C be posets. We write $A = B + C$ when B and C are the two cases generated from a pair of incomparable elements in A.

Corollary 3.1. *If A is mult-free, then so are B and C.*

Corollary 3.2. *If B or C is not mult-free, then neiher is A.*

Corollary 3.3. *If B and C are mult-free, A can be either mult-free or not mult-free.*

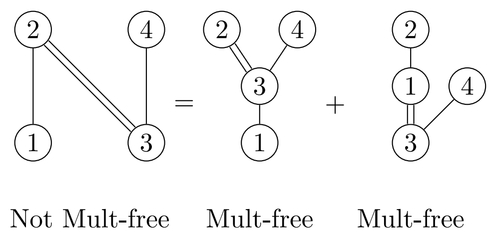

FIGURE 3. Example of incomparable elements rule.

Theorem 4. *(On page I1) Consider (P, ω). Suppose P has maximal element p such that $\omega(p)$ satisfies either*
(i) $\omega(p) < \omega(q)$ for all maximal elements of $(P - p, \omega)$
(ii) $\omega(p) > \omega(q)$ for all maximal elements of $(P - p, \omega)$
Then (P, ω) has multiplicity if $(P - p, \omega)$ does.

Theorem 5. *(Extending Theorem from I1, using idea of Bessenrodt and Willigenburg)*
If $(P - p, \omega)$ has multiplicity, so does (P, ω) with any other elements added, as long as none of these elements are below some elements of P.

Implication: As long as there exist 2 linear extensions that have the same F-function and have "common" last element (vice versa, first element). We can stick any other elements to it and we will always have a not mult-free poset.

Theorem 6. ***False Theorem***
Supposee (P, ω) contains (Q, τ) as a convex subposet. If (Q, τ) has multiplicity, then so does (P, ω).

Implication of theorem: **False**
If poset P contains a 3-element antichain, then P has multiplicity.
Counter-example:

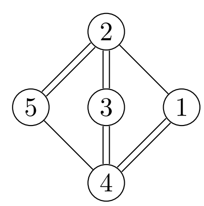

FIGURE 4. Mult-free labeled poset with a 3-element antichain.