$\rightarrow L\{\sin at\} = \frac{a}{s^2+a^2} \quad (s>0)$

Proof:

$$L\{\sin at\} = \int_0^\infty e^{-st} \cdot \sin at \, dt$$

$$\int \sin at \, e^{-st} dt = -\sin at \frac{e^{-st}}{s} + \frac{a}{s}\int e^{-st} \cos at \, dt$$

$$= -\sin at \frac{e^{-st}}{s} + \frac{a}{s}\left[\cos at \frac{e^{-st}}{-s} + \frac{a}{s}\int e^{-st} \sin at \, dt\right]$$

$$= -\frac{e^{-st}\sin at}{s} - \frac{a}{s^2}\cos at \, e^{-st} - \frac{a^2}{s^2}\int e^{-st} \sin at \, dt$$

$$\Rightarrow \left(1+\frac{a^2}{s^2}\right)\int e^{-st} \sin at \, dt = -\frac{e^{-st}\sin at}{s} - \frac{a}{s^2}\cos at \, e^{-st}$$

$$\Rightarrow \int e^{-st} \sin at \, dt = \frac{\cancel{1+s^2}}{a^2+s^2}\left[\frac{-s e^{-st}\sin at - a\cos at \, e^{-st}}{\cancel{s^2}}\right]$$

$$\Rightarrow \int_0^\infty e^{-st} \sin at \, dt = \frac{1}{a^2+s^2}[0+a] = \frac{a}{a^2+s^2}$$

$\rightarrow L\{\cos at\} = + \frac{\cancel{a}\, s}{a^2+s^2}$

Proof:

$$L\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$$

$$\int \cos at \, e^{-st} dt$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$\Rightarrow \int \cos at \, e^{-st} dt = \frac{e^{-st}}{s^2+a^2}[$$

$$= \frac{s}{a^2+s^2}$$

6th February 2017:

→ $L\{\sinh at\} = \frac{a}{s^2-a^2}; s>|a|$

Proof:

$L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$

$= \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$

$= \frac{1}{2}\left[\frac{1}{s-a}\right] - \frac{1}{2}\left[\frac{1}{s+a}\right] = \frac{a}{s^2-a^2}$

→ $L\{\cosh at\} = \frac{s}{s^2-a^2}$; $(s>|a|)$

Proof:

$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}$

→ $L\{\sin 6t \sin 4t\} =$

Proof:

$L\{\sin 6t \sin 4t\} = L\left\{\frac{1}{2}[\cos(\cancel{4t}) - \cos 10t]\right\}$

$= \frac{1}{2}L\{\cos 2t\} - \frac{1}{2}L\{\cos 10t\}$

$= \frac{1}{2}\cdot\frac{s}{s^2+4} - \frac{1}{2}\cdot\frac{s}{s^2+100} = \frac{s}{2}\left[\frac{s^2+100-s^2-4}{(s^2+4)(s^2+100)}\right]$

$= \frac{48s}{(s^2+4)(s^2+100)}$

→ $L\{\sin 2t \cos 3t\}$

$L\{\sin 2t \cos 3t\} = L\left\{\frac{1}{2}[\sin 5t - \sin t]\right\}$

$= \frac{1}{2}L\{\sin 5t\} - \frac{1}{2}L\{\sin t\}$

$= \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right] = \frac{5s^2+5-s^2-25}{2(s^2+25)(s^2+1)}$

$$= \frac{2s^2-10}{(s^2+2s)(s^2+1)}$$

* Change of scale property:

If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Proof:

We have $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$

Let $at = u \Rightarrow t = \frac{u}{a}$

$\Rightarrow dt = \frac{du}{a}$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$$

$$= \int_0^\infty e^{-s\cdot\frac{u}{a}} f(u)\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}u} f(u)\,du$$

$$= \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

* First shifting theorem:

If $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at}f(t)\} = \bar{f}(s-a)$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$$

$$L\{e^{at} f(t)\,dt\} = \int_0^\infty e^{-st}\cdot e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt = \bar{f}(s-a)$$

Similarly,

$\rightarrow L\{e^{-at}f(t)\} = \bar{f}(s+a)$

$\rightarrow L\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}$

$\rightarrow L\{e^{at}\cos bt\} = \frac{(s-a)}{(s-a)^2+b^2}$

$\rightarrow L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$

$\rightarrow L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

5.

$\rightarrow L\{\sin^3 2t\}$

$= L\left\{\frac{3\sin 2t - \sin 6t}{4}\right\}$

$= \frac{3}{4} L\{\sin 2t\} - \frac{1}{4} L\{\sin 6t\}$

$= \frac{3}{4} \cdot \frac{2}{s^2+4} - \frac{1}{4} \cdot \frac{6}{s^2+36} = \frac{6}{4}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right]$

$= \frac{48}{(s^2+4)(s^2+36)}$

$\rightarrow L\{t^2 \sin at\}$

we know, $L\{t^2\} = \frac{2!}{s^3}$

$L\{e^{iat} t^2\} = \frac{2!}{(s-ia)^3} = \frac{2}{(s-ia)^3} \times \frac{(s+ia)^3}{(s+ia)^3}$

$= \frac{2(s+ia)^3}{(s^2+a^2)^3} = \frac{2}{(s^2+a^2)^3}\left[s^3 + 3is^2a - 3a^2s - ia^3\right]$

$\Rightarrow L\{t^2 \sin at\} =$ Im part of $L\{e^{iat} t^2\}$

$= \frac{2}{(s^2+a^2)^3}(3s^2a - a^3) = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

1. Find $L\{t^3 e^{-3t}\}$

2. Find Laplace Transformation of $f(t)$ defined as

$$f(t) = \begin{cases} t & 0<t<4 \\ 5 & t>4 \end{cases}$$

3. Find $L\{f(t)\}$

if $f(t) = \begin{cases} \frac{t}{T} & 0<t<T \\ 1 & t>T \end{cases}$

4. Find $L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$

5. Find $L\{f(t)\}$ if

$$f(t) = \begin{cases} \cos & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}$$

2. $$f(t) = \begin{cases} t & 0 < t < 4 \\ 5 & t > 4 \end{cases}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^4 e^{-st} \cdot t\,dt + \int_4^\infty e^{-st}\, 5\,dt$$

$$= \left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^4 + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

$$= \left(\frac{-4e^{-4s}}{s} - \frac{e^{-4s}}{s^2}\right) - (0) + \frac{1}{s^2} + 5\left(0 + \frac{e^{-4s}}{s}\right)$$

$$= \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2} = \frac{se^{-4s} - e^{-4s} + 1}{s^2}$$

$$L\{t^3 e^{-3t}\} = \frac{3!}{(s+3)^4} = \frac{6}{(s+3)^4}$$

$$f(t) = \begin{cases} \frac{t}{T} & 0 < t < T \\ 1 & t > T \end{cases}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} \cdot \frac{t}{T}\,dt + \int_T^\infty e^{-st}\,dt$$

$$= \frac{1}{T}\left[\frac{-te^{-st}}{s} - \frac{1\,e^{-st}}{s^2}\right]_0^T + \left[\frac{e^{-st}}{-s}\right]_T^\infty$$

$$= \frac{1}{T}\left[\frac{-Te^{-sT}}{s} - \frac{e^{-sT}}{s^2} + \frac{1}{s^2}\right] + \frac{e^{-Ts}}{s}$$

$$= -\cancel{\frac{e^{-sT}}{s}} - \frac{e^{-sT}}{Ts} + \frac{1}{Ts^2} + \cancel{\frac{e^{-Ts}}{s}} = \frac{1}{Ts^2} - \frac{e^{-Ts}}{Ts^2} = \frac{1-e^{sT}}{Ts^2}$$

5. Find $L\{t e^{-t} \sin 3t\}$

$$L\{e^{-t}\sin 3t\} = \frac{3}{(s+1)^2+9}$$

$$L\{t e^{-t}\sin 3t\} = -1 \frac{d}{ds}\left[\frac{3}{(s+1)^2+9}\right] =$$

$$= \frac{+3}{[(s+1)^2+9]^2} 2(s+1) = \frac{6(s+1)}{(s^2+2s+10)^2}$$

6. Find $L\left\{\frac{1-e^t}{t}\right\}$

$$L\{1-e^t\} = \frac{1}{s} - \frac{1}{s-1} \Rightarrow L\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \frac{1}{s} - \frac{1}{s-1}\, ds$$

$$= \left[\log s - \log(s-1)\right]_s^\infty = \log\left(\frac{s-1}{s}\right)$$

7. $L\{t^2 \cos 2t\}$ 9. $L\{t(3\sin 2t - 2\cos 2t)\}$

8. $L\{t e^{-2t}\cos 3t\}$ 10. $L\{t \sin^2 t\}$ 11. $L\{t e^{-2t}\cosh t\}$

7. $L\{t^2 \cos 2t\}$

$$L\{\cos 2t\} = \frac{s}{s^2+4}$$

$$\Rightarrow L\{t^2\cos 2t\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$

$$= \frac{d}{ds} \cdot \frac{(s^2+4) - s(2s)}{(s^2+4)^2} = \frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{(s^2+4)^2(-2s) - (4-s^2)2(s^2+4)\,2s}{(s^2+4)^4} = \frac{-(s^2+4)2s + 4s(s^2-4)}{(s^2+4)^3}$$

$$= \frac{2s[-s^2-4+2s^2-8]}{(s^2+4)^3} = \frac{2s(s^2-12)}{(s^2+4)^3}$$

12. Evaluate $\int_0^\infty t e^{-2t} \sin t\, dt$

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8 $L\{t e^{-2t} \cos 3t\}$

$$L\{e^{-2t}\cos 3t\} = \frac{s}{(s+2)^2+9}$$

$$L\{t e^{-2t}\cos 3t\} = -1 \cdot \frac{d}{ds}\left[\frac{s}{s^2+4s+13}\right]$$

$$= -\frac{(s^2+4s+13)(1) - s(2s+4)}{(s^2+4s+13)^2} = \frac{s^2+\cancel{4s}+13-2s^2-\cancel{4s}}{(s^2+4s+13)^2}$$

$$= -\frac{13-s^2}{(s^2+4s+13)^2} = \frac{s^2-13}{(s^2+4s+13)^2}$$

12. $$\int_0^\infty t e^{-2t} \sin t \, dt = L\{t \sin t\} \text{ at } s=2$$

$$= L\{e^{-2t}\sin t \, dt\} = \frac{1}{(s+2)^2+1} = \frac{1}{s^2+4s+5}$$

$$\Rightarrow L\{t e^{-2t}\sin t \, dt\} = -\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)$$

$$: \frac{2s}{(s^2+1)^2} = \frac{4}{25}$$

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$$L\left\{\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y\right\} = 0$$

$$L\left\{\frac{d^3y}{dt^3}\right\} + 2L\left\{\frac{d^2y}{dt^2}\right\} - L\left\{\frac{dy}{dt}\right\} - 2L\{y\} = 0$$

$$\Rightarrow \bar{y} - s^2y(0) - sy'(0) - y''(0) + 2s^2\bar{y} - 2sy(0) - 2y'(0) - s\bar{y} + y(0) - 2\bar{y}$$

$$\times\left[\bar{y} - s^2y(0) - sy'(0) - y''(0) + (s+2)y'(0) + y''(0)\right.$$

$$\Rightarrow (2s^2-1)\bar{y} = (s^2+2s-1)y(0) + (s+2)y'(0) + y''(0)$$

$$\Rightarrow (2s^2-1)\bar{y} = s^2+2s-1+2s+4+2 \quad = s^2+4s+5 \left.\right]\times$$

$$\Rightarrow (2s^2-1)\bar{y} = s^2+2s-1+2s+4+2$$

$$\Rightarrow \times\left(\bar{y} = \frac{s^2+4s+5}{2s^2-1}\right), \quad \bar{y} = \frac{s^2+4s+5}{s^3+2s^2-s-2}$$

$$\bar{y} = \frac{s^2+4s+5}{(s-1)(s+1)(s+2)}$$

$$\Rightarrow \bar{y} \Rightarrow \frac{s^2+4s+5}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\Rightarrow s^2+4s+5 = A(s^2+3s+2) + B(s^2+s-2) + C(s^2-1)$$

$A+B+C=1$; $3A+B=4$; $2A-2B-C=5$

$2A+2B+2C=2$ $\qquad \Rightarrow 2A-2B+A+B-1=5 \Rightarrow 3A-B=6$

$$\Rightarrow 6A = 10 \Rightarrow A = 5/3 \Rightarrow B = -1$$

$$\Rightarrow C = 1/3$$

$$\Rightarrow \bar{y} = \frac{5}{3(s-1)} - \frac{1}{s+1} + \frac{1}{3(s+2)}$$

$$L^{-1}\{\bar{y}\} = y \Rightarrow y = \frac{5}{3}L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\Rightarrow y = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}$$

$$= e^{t} - e^{-t} + e^{-2t} - e^{-2t}$$

2. Solve the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{t} \text{ when } y(0)=0 \text{ and } y'(0)=0$$

$$L\left\{\frac{d^2y}{dt^2}\right\} + 4L\left\{\frac{dy}{dt}\right\} + 4L\{y\} = L\{e^{t}\}$$

$$\Rightarrow s^2\bar{y}(s) - sy(0) - y'(0) + 4s\bar{y}(s) - 4y(0) + 4\bar{y}(s) = \frac{1}{s-1}$$

$\Rightarrow (s^2+4s+4)\bar{y}$

$\Rightarrow \bar{y} \Rightarrow$

$\Rightarrow 1 = A($

$\Rightarrow 1 = A($

$\Rightarrow A+B=$

$\Rightarrow C=$

$\Rightarrow \bar{y} =$

$L^{-1}\{\bar{y}\} =$

$L\left\{\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y\right\} = 0$

$L\left\{\frac{d^3y}{dt^3}\right\} + 2L\left\{\frac{d^2y}{dt^2}\right\} - L\left\{\frac{dy}{dt}\right\} - 2L\{y\} = 0$

$\Rightarrow [\bar{y} - s^2y(0) - sy'(0) - y''(0) + 2s^2\bar{y} - 2sy(0) - 2y'(0) - s\bar{y} + y(0) - 2\bar{y}$

$\Rightarrow (2s^2-1)\bar{y} = (s^2+2s-1)y(0) + (s+2)y'(0) + y''(0)$

$\Rightarrow (2s^2-1)\bar{y} = s^2+2s-1+2s+4+2 = s^2+4s+5\,]\times$

$\Rightarrow \times\bar{y} = \frac{s^2+4s+5}{2s^2-1}$ $\qquad \bar{y} = \frac{s^2+4s+5}{s^3+2s^2-s-2}$

$\bar{y} = \frac{s^2+4s+5}{(s-1)(s+1)(s+2)}$

$\Rightarrow \bar{y} \Rightarrow \frac{s^2+4s+5}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$

$\Rightarrow s^2+4s+5 = A(s^2+3s+2) + B(s^2+s-2) + C(s^2-1)$

$A+B+C=1$; $\quad 3A+B=4$; $\quad 2A-2B-C=5$

$2A+2B+2C=2$ $\qquad \Rightarrow 2A-2B+A+B-1=5 \Rightarrow 3A-B=6$

$\Rightarrow 6A=10 \Rightarrow A=5/3 \Rightarrow B=-1$

$\Rightarrow C = 1/3$

$\Rightarrow \bar{y} = \frac{5}{3(s-1)} - \frac{1}{s+1} + \frac{1}{3(s+2)}$

$L^{-1}\{\bar{y}\} = y \Rightarrow y = \frac{5}{3}L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s+2}\right\}$

$\Rightarrow y = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}$

$= e^{t} + e^{-2t} - e^{-2t}$

2. Solve the differential equation

$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{t}$ when $y(0)=0$ and $y'(0)=0$

$L\left\{\frac{d^2y}{dt^2}\right\} + 4L\left\{\frac{dy}{dt}\right\} + 4L\{y\} = L\{e^{t}\}$

$\Rightarrow s^2\bar{y}(s) - sy(0) - y'(0) + 4s\bar{y}(s) - 4y(0) + 4\bar{y}(s) = \frac{1}{s-1}$

$\Rightarrow (s^2+4s$

$\Rightarrow$

$\Rightarrow \quad 1$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

L^{-1}

$\Rightarrow (s^2+4s+4)\bar{y}(s) = \frac{1}{s-1} + (s+4)y(0) + y'(0) = \frac{1}{s-1}$

$= \cancel{\frac{1}{s-1}} + \cancel{s} \Rightarrow \bar{y}(s) = \frac{1}{(s-1)(s+2)^2}$

$\Rightarrow \bar{y} \Rightarrow \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} = \frac{1}{(s-1)(s+2)^2}$

$\Rightarrow 1 = A(s+2)^2 + B(s+2)(s-1) + C(s-1)$

$\Rightarrow 1 = A(s^2+4s+4) + B(s^2+s-2) + C(s-1)$

$\Rightarrow A+B=0 ; 4A+B+C=0 ; 4A-2B-C=1$

$3A+C=0 \qquad 6A-C=01 \Rightarrow 9A=1 \Rightarrow A=\frac{1}{9} \Rightarrow B=\frac{-1}{9}$

$\Rightarrow C=-3A = \frac{-1}{3}$

$\Rightarrow \bar{y} = \frac{1}{9(s-1)} - \frac{1}{9(s+2)} - \frac{1}{3(s+2)^2}$

$L^{-1}\{\bar{y}\} = \frac{1}{9} e^{+t} - \frac{1}{9} e^{-2t}$

* Beta Function:

m and n: $\int_0^1 x^{m-1}(1-x)^{n-1}$ is called Beta function of

Euler's integer of first kind.

Beta function is denoted by $\beta(m,n)$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

→ Symmetry of β-function:

$$\int_0^a f(x) = \int_0^a f(a-x)\,dx$$

$$\beta(m,n) = \beta(n,m)$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}\left[1-(1-x)\right]^{n-1}dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= \beta(n,m)$$

→ $$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$$\Gamma m = \int_0^\infty e^{-t}\,t^{m-1}\,dt$$

let $t = x^2 \Rightarrow dt = 2x\,dx$

$$\Gamma m = \int_0^\infty e^{-x^2}(x^2)^{m-1}\,2x\,dx = 2\int_0^\infty e^{-x^2}x^{2m-1}\,dx \quad -①$$

Similarly,

$$\Gamma n = \int_0^\infty e^{-y^2}\,y^{2n-1}\,dy$$

$$\Gamma m\cdot\Gamma n = \left(2\int_0^\infty e^{-x^2}x^{2m-1}\,dx\right)\left(2\int_0^\infty e^{-y^2}y^{2n-1}\,dy\right)$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,x^{2m-1}\,y^{2n-1}\,dx\,dy$$

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$\Gamma m \, \Gamma n = 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} \, r \, dr \, d\theta$$

$$= \left(2 \int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta\right)\left(2 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr\right) \quad - (2)$$

From (1) we have

$$2 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr = \Gamma m+n \quad - (3)$$

consider,

$$2 \int_0^{\pi/2} (\cos\theta)^{2n-1} (\sin\theta)^{2n-1} d\theta = 2 \int_0^{\pi/2} (\sin^2\theta)^{n-1} (\cos^2\theta)^{m-1} (2\sin\theta\cos\theta) \, d\theta$$

Let $\sin^2\theta = t \Rightarrow 2\sin\theta\cos\theta \, d\theta = dt$

$$2 \int_0^{\pi/2} (\cos\theta)^{2m-1} (\sin\theta)^{2n-1} d\theta = \int_0^1 t^{n-1} (1-t)^{m-1} dt = \beta(m,n) \quad - (4)$$

Substitute (3) and (4) in (2)

$$\Gamma m \, \Gamma n = \beta(n,m) \, \Gamma m+n$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

1. Prove that $\displaystyle \int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{\frac{1}{2} \Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \int_0^{\pi/2} (\sin x)^{p-1} (\cos x)^{q-1} (2\sin x\cos x) \, dx$$

$$= \frac{1}{2} \int_0^{\pi/2} (\sin^2 x)^{\frac{p-1}{2}} (\cos^2 x)^{\frac{q-1}{2}} (2\sin x\cos x) \, dx$$

$\sin^2 x = t \Rightarrow 2\sin x\cos x \, dx = dt$

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2} \int_0^1 t^{\frac{p-1}{2}} (1-t)^{\frac{q-1}{2}} dt$$

$$= \frac{1}{2}\int_0^1 t^{\frac{p+1}{2}-1}(1-t)^{\frac{q+1}{2}-1}\,dt$$

$$= \frac{1}{2}\,\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2}\,\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

Let $p = q = 0$

$$\int_0^{\pi/2} dx = \frac{1}{2}\,\frac{\Gamma 1/2\,\Gamma 1/2}{\Gamma 1} \Rightarrow [x]_0^{\pi/2} = \frac{1}{2}\left(\Gamma 1/2\right)^2$$

$$\Rightarrow \frac{\pi}{2} = \frac{1}{2}\left(\Gamma\tfrac{1}{2}\right)^2 \Rightarrow \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

2. Show that $\displaystyle\int_0^{\pi/2} \sin^n x\,dx = \frac{\Gamma\frac{n+1}{2}\,\sqrt{\pi}}{2\,\Gamma\frac{n+2}{2}}$

$$\int_0^{\pi/2} \sin^p x \cos^q x\,dx = \frac{\frac{1}{2}\,\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

put $p = n$ and $q = 0$

$$\Rightarrow \int_0^{\pi/2} \sin^n x\,dx = \frac{\frac{1}{2}\,\Gamma\frac{n+1}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{n+2}{2}} = \frac{\frac{1}{2}\,\Gamma\frac{n+1}{2}\,\sqrt{\pi}}{\Gamma\frac{n+2}{2}}$$

3. Show that $\displaystyle\int_0^{\pi/2} \cos^n x\,dx = \frac{\Gamma\frac{n+1}{2}\,\sqrt{\pi}}{2\,\Gamma\frac{n+2}{2}}$

put $p = 0$ and $q = n$

$$\Rightarrow \int_0^{\pi/2} \cos^n x\,dx = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{n+2}{2}} = \frac{\Gamma\frac{n+1}{2}\,\sqrt{\pi}}{2\,\Gamma\frac{n+2}{2}}$$

4. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

$$= \int_0^1 x^m (1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^n\,dx$$

$$= \int_0^1 \left\{x^m (1-x)^{n-1} + x^{m-1}(1-x)^n\right\} dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x+1-x]\,dx = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

5. Prove that $\overline{|n+\frac{1}{2}} = \dfrac{\sqrt{\pi}\,\overline{|2n+1}}{2^{2n}\,\overline{|n+1}}$

$$\beta(m,n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$$

$$\beta\left(n+\tfrac{1}{2}, n+\tfrac{1}{2}\right) = \frac{\overline{|n+\frac{1}{2}}\,\overline{|n+\frac{1}{2}}}{\overline{|n+\frac{1}{2}+n+\frac{1}{2}}} = \frac{\left(\overline{|n+\frac{1}{2}}\right)^2}{\overline{|2n+1}} \quad -①$$

$$= \int_0^1 x^{n+\frac{1}{2}-1}(1-x)^{n+\frac{1}{2}-1}\,dx$$

let $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

$$\beta\left(n+\tfrac{1}{2}, n+\tfrac{1}{2}\right) = \int_0^{\pi/2} (\sin^2\theta)^{n-\frac{1}{2}}(\cos^2\theta)^{n-\frac{1}{2}}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n}\theta\cos^{2n}\theta\,d\theta = \frac{2}{2^n}\int_0^{\pi/2}(2\sin\theta\cos\theta)^{2n}\,d\theta$$

$$= \frac{2}{2^n}\int_0^{\pi}(\sin 2\theta)^{2n}\,d\theta = \frac{2}{2^n}\int_0^{\pi}\sin^{2n}\phi\,\frac{d\phi}{2}$$

$2\theta = \phi$

$d\theta = \frac{d\phi}{2}$

$$= \frac{1}{2^n}\int_0^{\pi/2}\sin^{2n}\phi\,d\phi$$

$$\beta\left(n+\tfrac{1}{2}, n+\tfrac{1}{2}\right) = \frac{2}{2^n}\,\frac{\overline{|\frac{2n+1}{2}}\,\sqrt{\pi}}{2\,\overline{|\frac{2n+2}{2}}}$$

$$\beta\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{2^{2n}\,\Gamma(n+1)}$$

Substitute ③ in ①

$$\Rightarrow \frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{2^{2n}\,\Gamma(n+1)} = \frac{\left(\Gamma\left(n+\frac{1}{2}\right)\right)^2}{\Gamma(2n+1)}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}$$

14th March 2017

* Another form of Beta function:

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy \quad (m>0, n>0)$$

Hence show that

$$\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$$

By definition we have,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

let $x = \frac{1}{1+y} \Rightarrow 1+y = \frac{1}{x} \Rightarrow dx = \frac{-1}{(1+y)^2} dy$

$$= \int_\infty^0 \frac{1}{(1+y)^{m-1}} \left(1 - \frac{1}{1+y}\right)^{n-1} \frac{(-1)}{(1+y)^2}\, dy$$

$$\beta(m,n) = \int_0^\infty \frac{1}{(1+y)^{m+1}} \frac{(1+y-1)^{n-1}}{(1+y)^{n-1}}\, dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy \quad \left(\because \beta(m,n) = \beta(n,m)\right)$$

$$\beta(m,n) = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\, dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy \quad \text{--- (1)}$$

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$$\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_1^0 \frac{\left(\frac{1}{z}\right)^{m-1}}{\left(1+\frac{1}{z}\right)^{m+n}}\left(\frac{-1}{z^2}\right)dz$$

let $y = \frac{1}{z} \Rightarrow dy = \frac{-1}{z^2}dz$

*

$$\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{1}{z^{m-1+2}} \cdot \frac{z^{m+n}}{(z+1)^{m+n}}\,dz$$

$$= \int_0^1 \frac{z^{n-1}}{(1+z)^{m+n}}\,dz \quad — ②$$

Substitute ② in ①;

$$\beta(m,n) = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_0^1 \frac{z^{n-1}}{(1+z)^{m+n}}\,dz$$

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dz$$

* P.T. $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 \frac{1}{4}\frac{t^{-3/4}}{\sqrt{1-t}}\,dt$$

let $x^4 = t$, $x = t^{1/4}$, $dx = \frac{1}{4}t^{-3/4}dt$

$$= \frac{1}{4}\int_0^1 t^{\frac{1}{4}-1}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{4}\int_0^1 t^{\frac{1}{4}-1}(1-t)^{\frac{1}{2}-1}\,dt = \frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

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$$= \frac{1}{4} \frac{\Gamma\frac{1}{4} \cdot \Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$$

* Show that $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{B(m,n)}{a^n \cdot b^m}$

(m, n, a and b are positive)

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \int_0^\infty \frac{\left(\frac{at}{b}\right)^{m-1} \frac{a}{b} dt}{(a+at)^{m+n}}$$

$\left(\text{let } bx = at \Rightarrow x = \frac{at}{b} \Rightarrow dx = \frac{a}{b} dt\right)$

$$= \frac{\left(\frac{a}{b}\right)^m}{a^{m+n}} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$= \frac{1}{a^n b^m} \beta(m,n) \quad \left(\because \beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy\right)$$

* Show that $\frac{\beta(m+1,n)}{\beta(m,n)} = \frac{m}{m+n}$

$$\frac{\beta(m+1,n)}{\beta(m,n)} = \frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+1+n)} \cdot \frac{\Gamma(m+n)}{\Gamma m\,\Gamma n}$$

$$= \frac{m\,\Gamma m\,\Gamma n\,\Gamma(m+n)}{(m+n)\,\Gamma(m+n)\,\Gamma m\,\Gamma n} = \frac{m}{m+n}$$

z

dz

-②

—dz

$= t$

$= t^{1/4}$

$\frac{1}{4} t^{-3/4} dt$

$(1, \frac{1}{2})$

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* Show that $y \cdot \beta(x+1, y) = x\, \beta(x, y+1)$

$$y \cdot \beta(x+1, y) = \frac{y\,\Gamma(x+1) \cdot \Gamma y}{\Gamma(x+y+1)}$$

$$= \frac{x\,\Gamma x \cdot y\,\Gamma y}{\Gamma(x+y+1)}$$

$$= \frac{x\,\Gamma x \cdot \Gamma(y+1)}{\Gamma(x+y+1)}$$

$$= x\, \beta(x, y+1)$$

* Show that $\int_0^a x^{n-1}(a-x)^{m-1}\,dx = a^{m+n-1}\beta(m,n)$

$$\int_0^a x^{n-1}(a-x)^{m-1}\,dx = \int_0^1 (at)^{n-1}(a-at)^{m-1}\, a\,dt$$

$$\boxed{\begin{array}{l}\text{let}\\ x = at\\ dx = a\cdot dt\end{array}}$$

$$= a^{n-1+m-1+1}\int_0^1 t^{n-1}(1-t)^{m-1}\,dt$$

$$= a^{m+n-1}\,\beta(m,n)$$

* Show that $\frac{\beta(p, q+1)}{q} = \frac{\beta(p+1, q)}{p} = \frac{\beta(p,q)}{p+q}$

$$\frac{\beta(p, q+1)}{q} = \frac{\Gamma p\,\Gamma(q+1)}{\Gamma(p+q+1)} \times \frac{1}{q} = \frac{\Gamma p\, q\,\Gamma q}{\Gamma(p+q+1)} \times \frac{1}{q}$$

$$= \frac{\Gamma p\,\Gamma q}{\Gamma(p+q+1)} = \frac{\Gamma p\,\Gamma q}{(p+q)\,\Gamma(p+q)} = \frac{\beta(p,q)}{p+q}$$

— ①

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$$\frac{\beta(p+1, q)}{p} = \frac{\Gamma p+1 \; \Gamma q}{\Gamma p+1+q} \times \frac{1}{p}$$

$$= \frac{\cancel{p}\,\Gamma p\,\Gamma q}{p+q\,\Gamma p+q} \times \frac{1}{\cancel{p}}$$

$$= \frac{\beta(p, q)}{p+q} \quad -②$$

From ① & ② :

$$\frac{\beta(p, q+1)}{q} = \frac{\beta(p+1, q)}{p} = \frac{\beta(p, q)}{p+q}$$

* Find $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$ $\left(\Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi} \right)$

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Evaluate $\int_C (3x^2-8y^2)\,dx+(4y-6xy)\,dy$ where C is the region enclosed by the regions bounded by $y=\sqrt{x}$ & $y=x^2$.

$$\int_{y=1}^{y=0}$$

$y=1$, $x=y^2$, $dx=2y\,dy$

$$\int (3x^2-8y^2)\,dx+(4y-6xy)\,dy + \int_{y=1}^{y=0} (3x^2-8y^2)\,dx+(4y-6xy)\,dy$$

$y=1$, $x=y^2$, $dx=2y\,dy$

$$=\int (3x^2-8x^4)\,dx+(4x^2-6x^3)\,2x\,dx + \int_{y=1}^{y=0} (3y^4-8y^2)\,2y\,dy+(4y-6y^3)\,dy$$

$$=\int_0^1 3x^2+8x^3-20x^4\,dx + \int_{y=1}^{y=0} (4y-22y^3+6y^5)\,dy$$

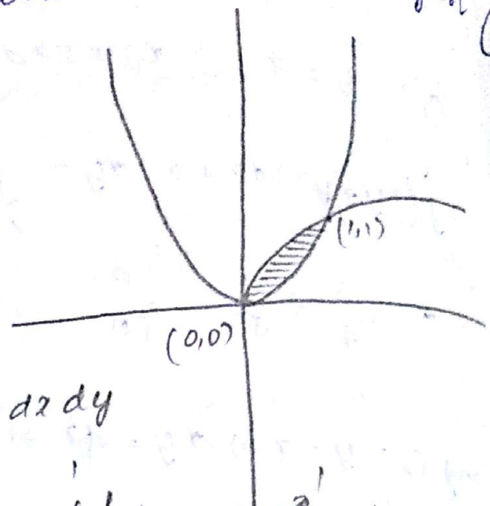
$$=\left[x^3+2x^4-4x^5\right]_0^1 + \left[2y^2-\frac{11}{2}y^4+\frac{4}{3}y^6\right]_1^0$$

$$=(1+2-4)+\left(0-\left(2-\frac{11}{2}+\frac{4}{3}\right)\right)$$

$$= -1-2+\frac{11}{2}-\frac{4}{3} = -3-\frac{4}{3}+\frac{11}{2} = \frac{-13}{3}+\frac{11}{2}$$

$$=\frac{-26+33}{6}=\frac{7}{6}$$

3: Verify Green's theorem in the plane for $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$ where 'c' is closed region bounded by $y=x^2$, $y=\sqrt{x}$



$P = 3x^2-8y^2 \Rightarrow \frac{\partial P}{\partial y} = -16y$

$Q = 4y-6xy \Rightarrow \frac{\partial Q}{\partial x} = -6y$

$$\Rightarrow \int_C (3x^2-8y^2)dx + (4y-6xy)dy = \iint_R 10y\,dx\,dy$$

$$= \int_{x=0}^{x=1}\int_{y=x^2}^{y=\sqrt{x}} 10y\,dy\,dx = \int_0^1 [5y^2]_{x^2}^{\sqrt{x}} dx = \int_0^1 \cancel{5x^4dx} = [x^5]_0^1 = 1$$

$$= \int_0^1 (5x - 5x^4)dx = \frac{5}{2} - 1 = \frac{3}{2}$$

Along C_1

$y = x^2 \Rightarrow dy = 2x\,dx$

$$\int_{C_1} (3x^2-8y^2)dx + (4y-6xy)dy = \int_0^1 (3x^2-8x^4)dx + (4x^2-6x^3)2x\,dx$$

$$= \int_0^1 (3x^2-8x^4+8x^3-12x^4)dx = \int_0^1 (3x^2+8x^3-20x^4)dx$$

$$= [x^3+2x^4-4x^5]_0^1 = 1+2-4 = -1$$

$$\int_{C_2} (3x^2-8y^2)dx + (4y-6xy)dy$$

$y=\sqrt{x} \Rightarrow dy\,2y = dx$

$y^2 = x \Rightarrow dy = \frac{1}{2\sqrt{x}}dx$

$$\int_1^0 \left(3x^2-8x + \frac{4\sqrt{x}-6x\sqrt{x}}{2\sqrt{x}}\right)dx = \int_1^0 3x^2-11x+2\,dx$$

$$= \left[x^3 - \frac{11x^2}{2} + 2x\right]_1^0 = -\left(1-\frac{11}{2}+2\right) = \frac{11}{2} - 3 = \frac{5}{2}$$

$$\Rightarrow \int_C (3x^2-8y^2)dx + (4y-6xy)dy = \frac{5}{2} - 1 = \frac{3}{2}$$

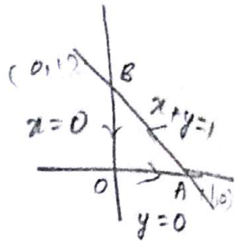
5. Evaluate $\int_C (x^2+xy)d\ldots$ Ans: 2

formed by $x = \pm 1;\ y = \pm 1$

6. Valify Greens' theorem for $\int_C (3x-8y^2)\,dx + (4y-6xy)\,dy$ wh...

'c' is the boundary of the region bounded by $x=0,\ y=0$ and $x+y=1$

5 $\quad P = x^2+xy \Rightarrow \frac{\partial P}{\partial y} = x \quad Q = x^2+y^2 \Rightarrow \frac{\partial Q}{\partial x} = 2x$



$$\Rightarrow \int_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \int_{x=-1}^{x=1}\int_{y=-1}^{y=1} x\,dy\,dx$$

$$= \int_{-1}^{1} 2x\,dx = \left[x^2\right]_{-1}^{1} = 1+1 = 2$$

6. $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$

$P = 3x^2-8y^2 \Rightarrow \frac{\partial P}{\partial y} = -16y \ ; \ Q = 4y-6xy \Rightarrow \frac{\partial Q}{\partial x} = -6y$

$$\Rightarrow \int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \iint_R 10y\,dx\,dy$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} 10y\,dy\,dx = \int_0^1 \left[5y^2\right]_0^{1-x} dx = 5\int_0^1 (1-x)^2\,dx$$

$$= -5\left[\frac{(1-x)^3}{3}\right]_0^1 = -5\left(0-\frac{1}{3}\right) = 5/3$$

Along OA:

$y=0 \Rightarrow dy=0$

$$\Rightarrow \int_{OA} (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \int_0^1 3x^2\,dx = \left[x^3\right]_0^1 = 1$$

Along AB:

$x+y=1 \Rightarrow dx+dy=0 \Rightarrow dx=-dy$

$\Rightarrow y = 1-x$

$$\int_{AB} (3x^2-8y^2)dx + (4y-6xy)dy$$

$$= \int_1^0 [3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x)]\,dx$$

$$= \int_1^0 (3x^2 - 8(1+x^2-2x) - 4 + 4x + 6x - 6x^2)\,dx$$

$$= \int_1^0 (-11x^2 + 26x - 12)\,dx = \int_0^1 (11x^2 - 26x + 12)\,dx$$

$$= \left[\frac{11}{3}x^3 - 13x^2 + 12x\right]_0^1 = \frac{11}{3} - 13 + 12 = \frac{11}{3} - 1 = \frac{8}{3}$$

Along BO:

$x = 0 \Rightarrow dx = 0$

$$\int_{BO} (3x^2-8y^2)dx + (4y-6xy)dy$$

$$= \int_1^0 4y\,dy = [2y^2]_1^0 = 0 - 2 = -2$$

$$\Rightarrow \int_C (3x^2-8y^2)dx + (4y-6xy)dy = 1 + \frac{8}{3} - 2 = \frac{8}{3} - 1 = \frac{5}{3}$$

$\Rightarrow$ Greens' theorem is verified.