(10R1)

**Green's Theorem:**

*If $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous and single valued functions in a regon R enclosed by the curve C, then* $\int_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$

*Note:*

1. *If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the value of the integral $\int_C (Mdx + Ndy)$ is indipendent of the path of integration.*
2. *If R is a region bounded by a simple closed curve 'C' then the area R is given by*

$$\frac{1}{2}\int_C (xdy - ydx)$$

**Problems on Green's Theorem:**

**Example: 1.** Verify Greens theorem in the plane for $\int_C (x^2dx + xydy)$, where C is the curve in the xy-plane given by x = 0, y = 0, x=a and y = a (a>0).

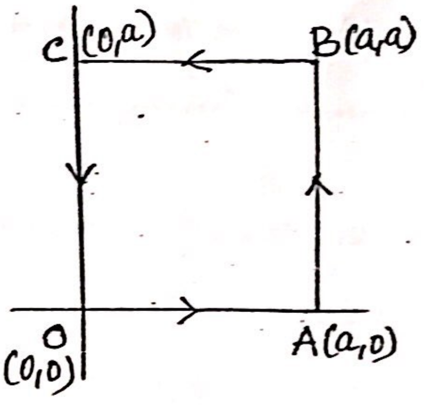
*Solution: by Green's theorem we have* $\int_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

*comparing, we have* $M = x^2$ *and* $N = xy$ *and* $\frac{\partial M}{\partial y} = 0$ *and* $\frac{\partial N}{\partial x} = y$ *and* $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = y$

$\int_C (x^2dx + xydy) = \iint_R ydxdy$ *we need to evaluate both integrals.*

Let us evaluate the RHS integral, for the given region x and y varies from 0 to 1

*therefore,* $\iint_R ydxdy = \int_0^a \int_0^a ydydx = \int_0^a \left[\frac{y^2}{2}\right]_0^a dx = \int_0^a \frac{a^2}{2} dx = \frac{a^3}{2}$




*(i) along the line OA, $y = 0$ and $dy = 0$*

$$\int_C (x^2dx + xydy) = \int_0^a x^2dx = \frac{a^3}{3}$$

*(ii) along AB $x = a$ and $dx = 0$*

$$\int_C (x^2dx + xydy) = \int_0^a aydy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

*(iii) along BC $y = a$ and $dy = 0$*

$$\int_C (x^2dx + xydy) = \int_{-a}^0 x^2\, dx = \left[\frac{x^3}{3}\right]_{-a}^0 = -\frac{a^3}{3}$$

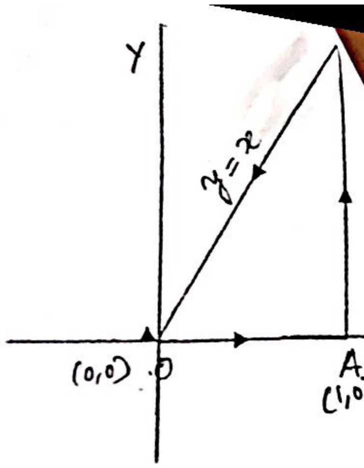
*(iv) along CO $x = 0$ and $dx = 0$ and* $\int_C (x^2dx + xydy) = 0$

Summing up all the integrals, we get $\frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$

*Thus,* $\int_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \frac{a^3}{2}$, *hence verified.*

Example:2. Verify Green's Theorem for

$\int_C (x^2y\,dx + x^2dy)$ *where C is the triangle bounded by the vertices* (0,0), (1,0) *and* (1,1).




*Solution: Here* $M = x^2y$ *and* $N = x^2$ *and* $\frac{\partial M}{\partial y} = x^2$ *and* $\frac{\partial N}{\partial x} = 2x$

*Therefore by Greens theorem* $\int_C Mdx + Ndy = \int\int \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dxdy$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1\int_0^x (2x - x^2)\, dydx = \int_0^1 (2x - x^2)\,[y]_0^x dx = \int_0^1 [2x^2 - x^3]\, dx = \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{12}.$$

To evaluate the LHS we have three paths along OA, then along AB and finally along BO

*(i)Along OA* $y = 0$ *and* $dy = 0$ *and* $\int_{OA} (x^2(0)dx + x^2(0)) = 0$

*(ii)Along AB* $x = 1$ *and* $dx = 0$ *and* $\int_C (x^2y\,dx + x^2dy) = \int_0^1 dy = 1$

*(iii) Along BO* $y = x$ *and* $dy = dx$ *and* $\int_C (x^2y\,dx + x^2dy) = \int_1^0 (x^3 + x^2)dx = \left[\frac{x^4}{4} + \frac{x^3}{3}\right]_1^0 = -\left(\frac{1}{4} + \frac{1}{3}\right) = -\frac{7}{12}$

*The sum of all the three paths* $= 0 + 1 - \frac{7}{12} = \frac{5}{12}$ *; thus LHS = RHS , hence verified.*

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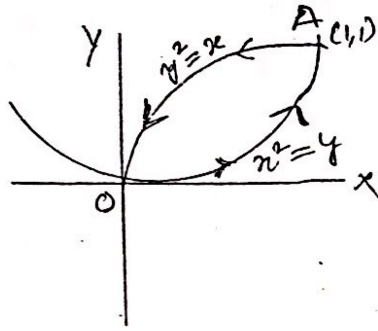
*Example: 3.Verify Green'stheorem in a plane for* $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)dy$ *, where C is the region bounded by* $y = \sqrt{x}$ *and* $y = x^2$.




*Solution: Solution: Here* $M = 3x^2 - 8y^2$ *and* $N = 4y - 6xy$

*and* $\frac{\partial M}{\partial y} = -16y$ ; $\frac{\partial N}{\partial x} = -6y$ *and* $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y = 10y$

*Therefore by Greens theorem* $\int_C Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1\int_{x^2}^{\sqrt{x}} 10y\,dy\,dx = \frac{10}{2}\int_0^1 [y^2]_{x^2}^{\sqrt{x}}\,dx = 5\int_0^1 (x - x^4)\,dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{5.3}{10} = \frac{3}{2}$$

To evaluate the LHS we have two paths along OA, then along AO

*(i)Along OA* $y = x^2$ *and* $dy = 2xdx$ *and* $\int_{OA} Mdx + Ndy = \int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)dy$

$$\int_{OA} (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)2xdx$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx = [x^3 + 2x^4 - 4x^5]_0^1 = 1 + 2 - 4 = -1$$

*(ii)Along BO* $y = \sqrt{x} \Rightarrow y^2 = x$ *and* $dx = 0$ *and* $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)dy$

$$\int_C (3y^4 - 8y^2)2ydy + (4y - 6y^3)dy = \int_1^0 [6y^5 - 16y^3 + 4y - 6y^3]dy = \int_1^0 [6y^5 - 22y^3 + 4y]dy$$

$$= \left[y^6 - \frac{11y^4}{2} + 2y^2\right]_1^0 = -\left[1 - \frac{11}{2} + 2\right] = -\left[\frac{2 - 11 + 4}{2}\right] = -\left[\frac{6 - 11}{2}\right] = \frac{5}{2}$$

*Along OA + Along BO* $= -1 + \frac{5}{2} = \frac{3}{2}$, *Hence verified.*

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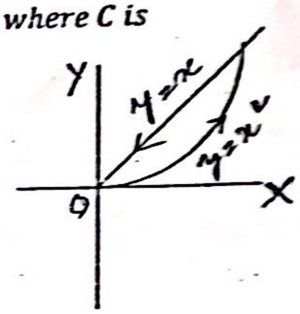
*Example: 4. Verify Green's theorem in xy – plane for* $\int_C (xy + y^2)\,dx + (x^2)dy$ *, where C is the region bounded by* $y = x$ *and* $y = x^2$.




*Solution: Solution: Here* $M = xy + y^2$ *and* $N = x^2$

*and* $\frac{\partial M}{\partial y} = x - 2y$; $\frac{\partial N}{\partial x} = 2x$ *and* $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x + 2y = x - 2y$

*Therefore by Greens theorem* $\int_C Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\int_C (xy + y^2)\,dx + (x^2)dy = \int_0^1 \int_{x^2}^{x} (x - 2y)\,dydx = \int_0^1 [xy - y^2]_{x^2}^{x}\,dx = \int_0^1 [(x^2 - x^2) - (x^3 - x^4)]\,dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

To evaluate the LHS we have two paths along OA, then along AO

$(i)$*Along OA* $y = x^2$ *and* $dy = 2xdx$ *and* $\int_{OA} Mdx + Ndy$

$$= \int_C (xy + y^2)\, dx + (x^2)dy$$

$$= \int_C (x^3 + x^4)\, dx + (x^2)2xdx = \int_0^1 (3x^3 + x^4)\, dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

$(ii)$*Along BO* $y = x \Rightarrow dy = dx$ *and* $\int_C (x^2 + x^2)\, dx + (x^2)dx = \int_1^0 3x^2 dx = [x^3]_1^0 = -1$

*The sum of the integrals* [*Along OA* + *Along BO*] $= -1 + \frac{19}{20} = -\frac{1}{20}$ ; *Hence verified.*

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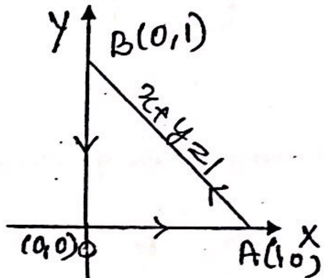
*Examp;e:* 5. *Verify Green'stheorem in a plane for* $\int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)dy$ *, where C is the triangle region bounded by* $x = 0, y = 0$ *and* $x + y = 1$

y




*Solution: Solution: Here* $M = 3x^2 - 8y^2$ *and* $N = 4y - 6xy$

*and* $\frac{\partial M}{\partial y} = -16y$ ; $\frac{\partial N}{\partial x} = -6y$ *and* $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y = 10y$

*Therefore by Greens theorem* $\int_C Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)dy = \int_0^1 \int_0^{1-x} 10y\, dydx = 10 \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = 5 \int_0^1 (x^2 - 2x + 1)\, dx$$

$$= 5\left[\frac{x^3}{3} - x^2 + x\right]_0^1 = 5\left[\frac{1}{3} - 1 + 1\right] = \frac{5}{3}$$

To evaluate the LHS we have three paths along OA, AB and BO

$(i)$*Along OA* $y = 0$ *and* $dy = 0$ *and* $\int_{OA} Mdx + Ndy = \int_0^1 3x^2\, dx = \left[\frac{3x^3}{3}\right]_0^1 = 1$

$(ii)$*Along AB* $y = 1 - x \Rightarrow dy = -dx$, *and* $\int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)dy$

$$= \int_C (3x^2 - 8(1 - x)^2)\, dx + (6x - 4)(1 - x)dx = \int_1^0 [3x^2 - 8(x^2 - 2x + 1) + (6x - 4 - 6x^2 + 4x)]dx$$

$$= \int_1^0 (3x^2 - 8x^2 + 16x - 8 + 6x - 4 - 6x^2 + 4x)\,dx = \int_1^0 [-11x^2 + 26x - 12]\,dx = \left[-11\frac{x^3}{3} + 26\frac{x^2}{2} - 12x\right]_1^0$$

$$= \frac{11}{3} - 13 + 12 = \frac{8}{3}$$

*(iii) Along BO* $x = 0;\ dx = 0 \Rightarrow \int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)dy$

$$= \int_1^0 4y\,dy = 4\left[\frac{y^2}{2}\right]_1^0 = -2$$

*The sum of the integrals* [*Along OA + Along AB + Along BO*] $= 1 + \frac{8}{3} - 2 = \frac{5}{3}$ ; *Hence verified.*

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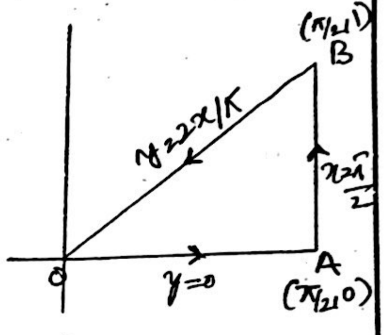
***Example : 6. Using Green's Theorem evaluate***




$\int_C (y - \sin x)\,dx + \cos x\,dy$, *where C is the triangle*

*bonded by* $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$ *& the point of intersection is* $\left(\frac{\pi}{2}, 1\right)$

*Solution: By Green's Theorem we have* $\int_C M dx + N dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

*Here* $M = y - \sin x$ *and* $N = \cos x$

*and* $\frac{\partial M}{\partial y} = 1$ ; $\frac{\partial N}{\partial x} = -\sin x$ *and* $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -\sin x - 1$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_{\frac{\pi y}{2}}^{\pi/2} (-\sin x - 1)\,dy\,dx = \int_0^1 [\cos x - x]_{\frac{\pi y}{2}}^{\frac{\pi}{2}} dy = \int_0^1 \left[\left(0 - \frac{\pi}{2}\right) - \left(\cos\frac{\pi y}{2} - \frac{\pi y}{2}\right)\right] dy$$

$$= \left[-\frac{\pi}{2}y - \frac{2}{\pi}\sin\frac{\pi y}{2} + \frac{\pi y^2}{4}\right]_0^1 = -\frac{\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4} = \frac{-2\pi^2 - 8 + \pi^2}{4\pi} = -\frac{(\pi^2 + 8)}{4\pi}$$

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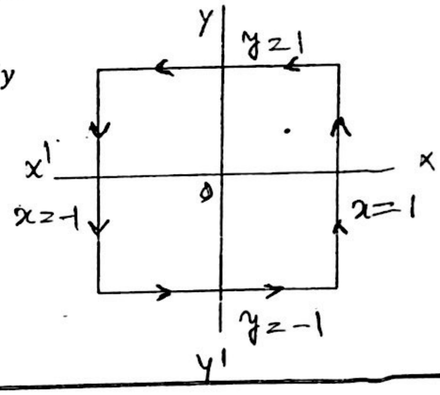
*Example: 7. Evaluate by Green's Theorem* $\int_C (xy + x^2)\,dx + (x^2 + y^2)dy$




*where C is the square formed by*

*the lines* $x = -1, x = 1, y = -1$ *and* $y = 1$.

*Solution: By Green's Theorem we have*

$$\int_C M dx + N dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Here $M = xy + x^2$ and $N = x^2 + y^2$

and $\frac{\partial M}{\partial y} = x$ ; $\frac{\partial N}{\partial x} = 2x$ and $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{-1}^{1}\int_{-1}^{1} x\, dxdy = 0 \text{ [since the integrand being an odd function]}$$

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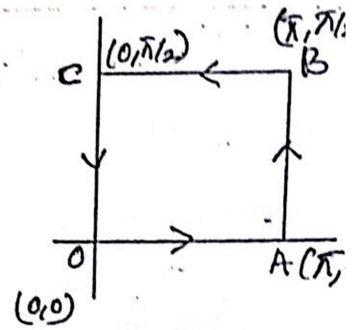
*Example*: 8. Evaluate by Green's Theorem $\int_C e^{-x}(\sin y dx + \cos y dy)$, where $C$ is rectangle bounded by the vertices $(0,0)$, $(\pi, 0)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$

*Solution*: By Green's Theorem we have




$$\int_C M dx + N dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Here $M = e^{-x}\sin y$ and $N = e^{-x}\cos y$

and $\frac{\partial M}{\partial y} = e^{-x}\cos y$ ; $\frac{\partial N}{\partial x} = -e^{-x}\cos y$ and $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -e^{-x}\cos y - e^{-x}\cos y = -2e^{-x}\cos y$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = -2\int_0^{\frac{\pi}{2}}\int_0^{\pi} e^{-x}\cos y\, dxdy = -2\int_0^{\frac{\pi}{2}} [-e^{-x}]_0^{\pi} \cos y dy = -2\int_0^{\frac{\pi}{2}} (-e^{-\pi} + 1) \cos y dy$$

$$= 2\int_0^{\frac{\pi}{2}} (e^{-\pi} - 1) \cos y\, dy = 2(e^{-\pi} - 1)[\sin y]_0^{\frac{\pi}{2}} = 2(e^{-\pi} - 1)$$

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*Example*: 9. Evaluate by Green's Theorem $\int_C (2x^2 - y^2)\, dx + (x^2 + y^2)\, dy$, where $C$ is the boundary in the $xy$ – plane of the area enclosed by the $x$ – axis and the semi – circle $x^2 + y^2 = 1$ above $x$ – axis.

*Solution*: By Green's Theorem we have

$$\oint_C Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$


Here $M = 2x^2 - y^2$ and $N = x^2 + y^2$

and $\frac{\partial M}{\partial y} = -2y$, $\frac{\partial N}{\partial x} = 2x$ and $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2(x+y)$

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R 2(x+y)dxdy = 2\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x+y)\,dydx = \int_{-1}^{1}\left[xy + \frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \left[\int_{-1}^{1} x\sqrt{1-x^2} + \frac{1-x^2}{2}\right]dx = \int_{-1}^{1} x\sqrt{1-x^2}\,dx + \frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx = I_1 + I_2$$

$$I_1 = \int_{\pi}^{0} x\sqrt{1-x^2}\,dx = \int_{\pi}^{0} x\sqrt{1-x^2}\,dx = \int_{\pi}^{0} \sin^2 t \cos t\,dt = \int_{\pi}^{0} \sin^2 t\, d(\sin t) = \left[-\frac{\sin^3 t}{3}\right]_{\pi}^{0} = 0$$

got by putting $x = \cos t \Rightarrow dx = -\sin t\,dt$; also when $x = -1, t = \pi$ and when $x = 1, t = 0$

$$I_2 = \frac{1}{2}\int_{-1}^{1} 1 - x^2\,dx = \int_{0}^{\frac{\pi}{2}} \cos^2 t\,dt = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{4}{2}\right)} = \frac{\frac{\sqrt{\pi}}{2}}{2\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}} = \frac{4}{3}$$

$$I_1 + I_2 = \frac{4}{3}$$

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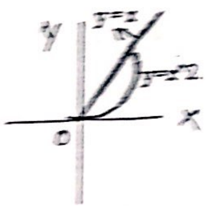
**Example: 10** Evaluate by Green's Theorem $\int_C (2y^2)\,dx + (3x)\,dy$, where $C$ is the boundary in the $xy$-plane of the area enclosed by $y = x$ and $y = x^2$.




**Solution:** By Green's Theorem we have

$$\oint_C Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Here $M = 2y^2$ and $N = 3x$

and $\frac{\partial M}{\partial y} = 4y$; $\frac{\partial N}{\partial x} = 3$ and $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3 - 4y$

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (3 - 4y)dxdy = 2\int_{0}^{1}\int_{x^2}^{x} 3 - 4y\,dydx = \int_{0}^{1} [3y - 2y^2]_{x^2}^{x}\,dx$$

$$= \int_0^1 [(3x - 2x^2) - (3x^2 - 2x^4)]\, dx = \int_0^1 (2x^4 - 5x^2 + 3x)\, dx = \left[\frac{2}{5}x^5 - \frac{5}{3}x^3 + \frac{3}{2}x^2\right]_0^1$$

$$= \frac{2}{5} - \frac{5}{3} + \frac{3}{2} = \frac{12 - 50 + 45}{30} = \frac{7}{30}$$

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**Assignment Questions:**

1. *Verify Green's Theorem in the xy – plane for the integral* $\int_C (x^2 + y)\, dx - xy^2 dy$ *taken around the boundary of a squre with vertices* $(0,0), (1,0), (1,1), (0,1)$.

2. *Verify Green's Theorem in the xy – plane for the integral* $\int_C (x - y)\, dx + (x + y)dy$ *taken around the boundary ofthe area in the first quadrant between the curves* $y = x^2$ *and* $y^2 = x$.

3. *Verify Green's Theorem in the xy – plane for the integral* $\int_C (x - 2y)\, dx + (x)dy$ *taken around the boundary ofthe circle* $x^2 + y^2 = 1$.

4. *Verify Green's Theorem in the xy – plane for the integral* $\int_C (2x - y^3)\, dx + (xy)dy$ *; where C is the boundary of the region enclosed by the circles* $x^2 + y^2 = 1$ *and* $x^2 + y^2 = 9$.

5. *Show that the integral* $\int_{(0,0)}^{(1,1)} (x^2 + y^2)\, dx - 2xy\, dy$ *is indipendent of the integration.*

6. *Evaluate by Green's Theorem* $\int_C (x^2 + y^2)\, dx - 2xy\, dy$ *, where C is the rectangle bounded by* $y = 0, x = 0, x = a$ *and* $y = b$.

7. *Evaluate by Green's Theorem* $\int_C (\cos x \sin y - xy)\, dx + (\sin x \cos y)\, dy$ *, where C is the circle* $x^2 + y^2 = 1$

8. *Evaluate by Green's Theorem* $\int_C (x^2 - \cosh y)\, dx + (y + \sin x)\, dy$ *, where C is the rectangle bounded by the vertices* $(0,0), (\pi, 0), (\pi, 1)$ *and* $(0,1)$.

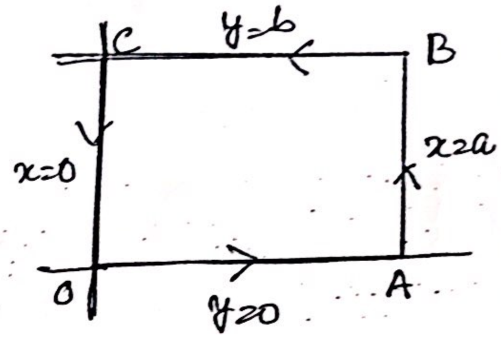
STOKE'S THEOREM:

Line integral of the tangential component of a vector function $\vec{F}$ which is finite and differentiable , around a simple closed curve C is equal to the surface integral of the normal component of $\vec{F}$ over any surface of S having its boundary

$$i.e. \int_C \vec{F}.d\vec{r} = \iint_S (\nabla X \vec{F}).\hat{n}\, ds \quad or \quad \int_C \vec{F}.d\vec{r} = \iint_S Curl\, \vec{F}.\hat{n}\, ds$$

Example:1. Verify stoke's Theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{\imath} + 2xy\vec{\jmath}$ in the rectangular region in the xy-plane bounded by $x = 0, x = a, y = 0$ $and$ $y = b$.

*Solution: By Stoke'stheorem we have* $\int_C \vec{F}.d\vec{r} = \iint_S Curl\, \vec{F}.\hat{n}\, ds$




First let us evaluate the RHS

*given* $\vec{F} = (x^2 - y^2)\vec{\imath} + 2xy\vec{\jmath}$

$$Curl\, \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = i(0-0) - j(0-0) + k(2y+2y)$$

$Curl\, \vec{F} = 4yk$ , $\hat{n} = k$,

$$\vec{F}.d\vec{r} = [(x^2 - y^2)\vec{\imath} + 2xy\vec{\jmath}].[dxi + dyj]$$
$$= (x^2 - y^2)dx - 2xy\, dy$$

$$\iint_S Curl\, \vec{F}.\hat{n}\, ds = \int_0^a \int_0^b 4y\; dydx = \int_0^a 2b^2 dx = 2ab^2$$

Now to evaluate the LHS we have four paths OA,AB,BC and CO

(i) *Along* $OA, y = 0$ *and* $dy = 0$ *and* $x$ *varies from* 0 *to* $a$.

$$Therefore, \int_C \vec{F}.d\vec{r} = \int_0^a x^2\, dx = \frac{a^3}{3}$$

(ii) *Along* $AB, x = a$ *and* $dx = 0$ *and* $x$ *varies from* 0 *to* $b$.

$$Therefore, \int_C \vec{F}.d\vec{r} = \int_0^b (a^2 - y^2)dx - (2ay)\, dy = [-2ay^2/2]_0^b = -ab^2$$

(iii) *Along* $BC, y = b$ *and* $dy = 0$ *and* $x$ *varies from* $a$ *to* 0.

$$Therefore, \int_C \vec{F}.d\vec{r} = \int_a^0 (x^2 - b^2)dx - (0)\, dy = \left[\frac{x^3}{3} - b^2x\right]_a^0 = \left[0 - \left(\frac{a^3}{3} + b^2a\right)\right]$$

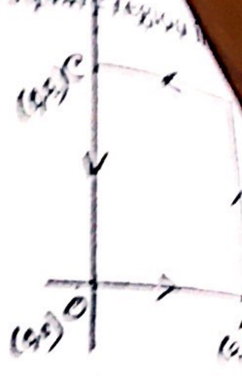
$$= -\frac{a^3}{3} - b^2a$$

(iv) *Along* $CO, x = 0$ *and* $dx = 0$ *and* $y$ *varies from* $b$ *to* 0.

$$Therefore, \int_C \vec{F}.d\vec{r} = \int_b^0 (x^2 - y^2)dx - 2xy\, dy = \int_b^0 (0)\, dy = 0$$

$$Sum\ of\ the\ four\ paths = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - b^2a = 2ab^2$$

**Example:2.** Verify stoke's Theorem for a vector field defined by $\vec{F} = (x^2)\hat{i} + xy\hat{j}$ in the square region in the xy plane bounded by $x = 0, x = a, y = 0$ and $y = b$



*Solution:* By Stoke's theorem we have $\int_C \vec{F}\cdot d\vec{r} = \iint_S Curl\,\vec{F}\cdot\hat{n}\,ds$

First let us evaluate the RHS $\iint_S Curl\,\vec{F}\cdot\hat{n}\,ds$

Given $\vec{F} = (x^2)\hat{i} + xy\hat{j}$ and $d\vec{r} = dx\hat{i} + dy\hat{j}$

$$Curl\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = (0-0)\hat{i} - (0-0) + k(y) = y\hat{k} \text{ and } \hat{n} = \hat{k}$$

$$\iint_S Curl\,\vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a y\,dy\,dx = \int_0^a \left[\frac{y^2}{2}\right]_0^a dx = \int_0^a \frac{a^2}{2}dx = \frac{a^3}{2}.$$

Now to evaluate the LHS we have four paths OA, AB,BC and CO

$\vec{F}\cdot dr = x^2dx + xydy$

(i) Along $OA, y = 0$ and $dy = 0$ and $x$ varies from 0 to $a$.

$$Therefore, \int_C \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx = \frac{a^3}{3}$$

(ii) Along $AB, x = a$ and $dx = 0$ and $y$ varies from 0 to $a$.

$$Therefore, \int_C \vec{F}\cdot d\vec{r} = \int_0^a (ay)dy = \left[\frac{ay^2}{2}\right]_0^a = \frac{a^3}{2}$$

(iii) Along $BC, y = a$ and $dy = 0$ and $x$ varies from $a$ to 0.

$$Therefore, \int_C \vec{F}\cdot d\vec{r} = \int_a^0 (x^2)dx = \left[\frac{x^3}{3}\right]_a^0 = \left[0-\left(\frac{a^3}{3}\right)\right] = -\frac{a^3}{3}$$

(iv) Along $CO, x = 0$ and $dx = 0$ and $y$ varies from $a$ to 0.

$$Therefore, \int_C \vec{F}\cdot d\vec{r} = \int_a^0 0 = 0$$

Sum of the four paths $= \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$ ,hence verified.

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*Example:3.* Evaluate by Stoke's Theorem $\int_C (2x-y)dx - yz^2dy - y^2zdz$, where $C$ is the circle $x^2 + y^2 = 1$, corrodponding to the surface of the sphere of unit radius.

*Solution:* By Stoke's theorem we have $\int_C \vec{F}\cdot d\vec{r} = \iint_S Curl\,\vec{F}\cdot\hat{n}\,ds$

let us evaluate $\iint_S Curl\,\vec{F}\cdot\hat{n}\,ds$

Given $\vec{F} = (2x-y)\vec{i} - yz^2\,\vec{j} - y^2z\,\vec{k}$ and $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

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$$\vec F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = (-2yz+2yz)\vec{\imath} - (0-0)\vec{\jmath} + \vec k = \vec k$$

$$\iint_S Curl\, \vec F.\hat n\, ds = \iint_S \vec k.\vec k \frac{dxdy}{|\hat n.\vec k|} = \iint_S dxdy = \pi \text{ (area of the unit circle)}$$

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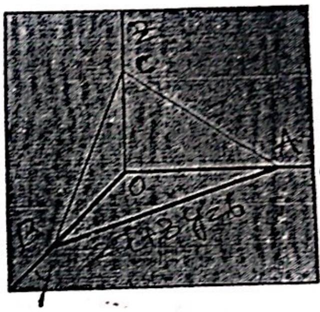
*Example: 4. Evaluate by Stoke's Theorem the integral*

$\int_C (x+y)dx + (2x-z)dy + (y+z)dz$, *where C is the trianglewith the vertices* (2,0,0), (0,3,0), (0,0,6)

*Solution: By Stoke'stheorem we have* $\int_C \vec F.d\vec r = \iint_S Curl\, \vec F.\hat n\, ds$



*let us evaluate* $\iint_S Curl\, \vec F.\hat n\, ds$

$$Curl\, \vec F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = i(1+1) - j(0-0) + k(2-1) = 2i + k$$

*The given plane is* $3x+2y+z=6$ *so Let* $\varphi = 3x+2y+z-6$

$$\hat n = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{3i+2j+k}{\sqrt{14}};\quad Curl\, \vec F.\hat n = (2i+k).\frac{3i+2j+k}{\sqrt{14}} = \frac{6+1}{\sqrt{14}} = \frac{7}{\sqrt{14}} \text{ and } ds = \frac{dxdy}{|\hat n.k|} = \frac{dxdy}{1/\sqrt{14}}$$

$$\iint_S Curl\, \vec F.\hat n\, ds = \iint_S \frac{7}{\sqrt{14}}.\sqrt{14}\, dydx = 7\iint_S dxdy = 7(\text{area of the triangle}) = 7.\left(\frac{1}{2}\right)2.3 = 21.$$

*Note: Area of the triangle* $= \frac{1}{2} xbxh = \frac{1}{2}.2.3 = 3$

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Example: 5. Evaluate by Stoke's Theorem $\int_C (e^x dx + 2ydy - dz)$, *where C is the curve* $x^2+y^2=4$

$z = 2.$

*Solution: By Stoke'stheorem we have* $\int_C \vec F.d\vec r = \iint_S Curl\, \vec F.\hat n\, ds$

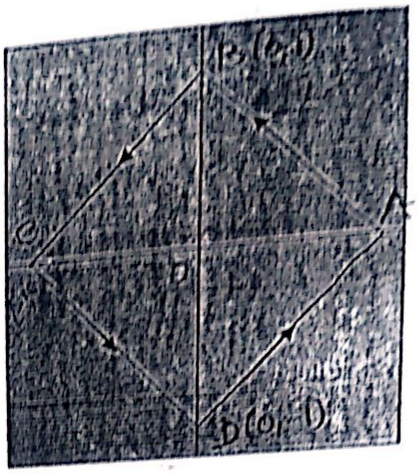
*let us evaluate* $\iint_S Curl\, \vec F.\hat n\, ds$, *where* $\hat n$ *is the unit normal vector to the surface S*

$$Curl\, \vec F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = i(0-0) - j(0-0) + k(0-0) = 0. \text{since, } Curl\, \vec F = 0, \iint_S Curl\, \vec F.\hat n\, ds = 0$$

Example: 5. Evaluate by Stoke's Theorem $\int_C (xydx + xy^2dy)$, where $C$ is the square in the $xy$ – plane with the vertices $(1,0), (-1,0), (0,1)$ and $(0,-1)$.

Solution: By Stoke's theorem we have $\int_C \vec{F}.d\vec{r} = \iint_S Curl\,\vec{F}.\hat{n}\,ds$

let us evaluate $\iint_S Curl\,\vec{F}.\hat{n}\,ds$, where $\hat{n}$ is the unit normal vector to the surface $S$



$$Curl\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = i(0-0) - j(0-0) + k(y^2 - x)$$

$Curl\,\vec{F} = k(y^2 - x)$ and $\hat{n} = k$ and $ds = \frac{dxdy}{|\hat{n}.k|} = dxdy$

$$\iint_S Curl\,\vec{F}.\hat{n}\,ds = \iint_S (y^2 - x)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} (y^2 - x)dxdy$$

$$= \int_{-1}^{1}\left[y^2x - \frac{x^2}{2}\right]_{-1}^{1} dx = \int_{-1}^{1}\left\{\left(y^2 - \frac{1}{2}\right) - y^2 - \frac{1}{2}\right) dx$$

$$= 2\int_{-1}^{1} y^2 dy = \left[\frac{2}{3}y^3\right]_{-1}^{1} = 2\left[\frac{1}{3} + \frac{1}{3}\right] = \frac{4}{3}$$

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Example: 6. Using Stoke's Theorem Prove that (i) $\int_C \vec{r}.d\vec{r} = 0$

and (ii) $\int_C \varphi\,\nabla\psi.d\vec{r} = -\int_C \psi\,\nabla\varphi.d\vec{r}$

Solution: (i) $\int_C \vec{r}.d\vec{r} = \iint_S Curl\,\vec{r}.d\vec{r} = 0.$ since $Curl\,\vec{r} = 0.$

(ii) $\int_C \varphi\,\nabla\psi.d\vec{r} = -\int_C \psi\,\nabla\varphi.d\vec{r}$

we know that from the vector Identities $\nabla(\varphi\psi) = \varphi(\nabla\psi) + \psi(\nabla\varphi)$

$$\int_C \nabla(\varphi\psi).d\vec{r} = \iint_S \nabla\,X\,\nabla(\varphi\psi).\hat{n}\,ds = 0 \text{ since curl grad}(\varphi\psi) = 0$$

Therefore, $\int_C \nabla(\varphi\psi).d\vec{r} = 0 \Longrightarrow \int_C [\varphi(\nabla\psi) + \psi(\nabla\varphi)].d\vec{r} = 0$

$$\int_C \varphi(\nabla\psi).d\vec{r} + \int_C \psi(\nabla\varphi).d\vec{r} = 0 \Rightarrow \int_C \varphi(\nabla\psi).d\vec{r} = -\int \psi(\nabla\varphi).d\vec{r}, \text{hence proved.}$$

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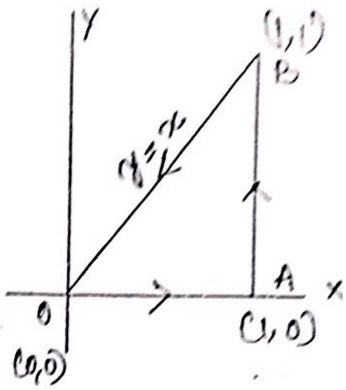
*Example: 7. Evaluate by Stoke's theorem* $\int_C \vec{F}.dr$, *where* $\vec{F} = y^2 i + x^2 j - (x+z)k$ *and C is the boundary of the triangle with vertices* $(0,0,0), (1,0,0)$ *and* $(1,1,0)$




*Solution: By Stoke'stheorem we have* $\int_C \vec{F}.d\vec{r} = \iint_S Curl\,\vec{F}.\hat{n}\,ds$

$$Curl\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = i(0-0) - j(-1-0) + k(2x-2y)$$

and $\hat{n} = \vec{k}$ ; $Curl\,\vec{F}.\hat{n} = 2(x-y)$ and $ds = \frac{dxdy}{|\hat{n}.\vec{k}|} = dxdy$ as $\vec{k}.\vec{k} = 1$

$$\iint_S Curl\,\vec{F}.\hat{n}\,ds = \iint_S 2(x-y)\,dydx = 2\int_0^1\int_0^x (x-y)\,dydx = 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x = 2\int_0^1 \left(x^2 - \frac{x^2}{2}\right)dx$$

$$= 2\int_0^1 \frac{1}{2}x^2dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

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Example:2. Verify stoke's Theorem for a vector field defined by $\vec{A} = (xz)\vec{i} - y\vec{j} + x^2y\vec{k}$ where S is the surface of the region bounded by $x = 0,, y = 0, z = 0$ and $2x + y + 2z = 8$ which is not included in the xz-plane.

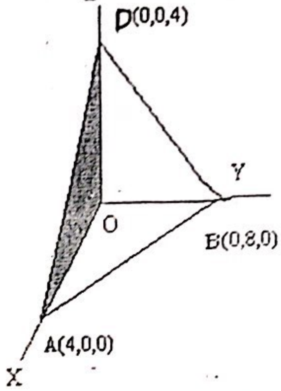
*Solution: By Stoke'stheorem we have* $\int_C \vec{A}.d\vec{r} = \iint_S Curl\,\vec{A}.\hat{n}\,ds$

First let us evaluate the RHS $\iint_S Curl\,\vec{A}.\hat{n}\,ds$

*Given* $\vec{A} = (xz)\vec{i} - y\vec{j} + x^2y\vec{k}$ and $dr = dxi + dyj + dzk$

$$Curl\,\vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y & x^2y \end{vmatrix} = x^2 i - x(1-2y)j$$

There are three surfaces $S_1: OAB, S_2: OBD$ and $S_3: ABD$




$$RHS = \iint_S Curl\,\vec{A}.\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = SI_1 + SI_2 + SI_3$$

On the surface $S_1: OAB: z = 0, \hat{n} = -\vec{k}, (Curl\,\vec{A}).(-\vec{k}) = 0$

$$Therefore, \iint_{S_1} Curl\,\vec{A}.\hat{n}\,ds = 0$$

On the surface $S_2$: OBD, plane $x = 0, \hat{n} = -\hat{\imath}$ and $Curl\ \vec{A}.\hat{n} = 0$

Therefore, $\iint_{S_2} Curl\ \vec{A}.\hat{n}\, ds = 0$

On the surface,

$S_3$: ABD, $2x + y + 2z = 8$ ; $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2i + j + 2}{3}$ and $Curl\ \vec{A}.\hat{n} = [x^2 i - x(1-2y)j].\left[\frac{2i + j + 2}{3}\right]$

$= \frac{2x^2 - x(1-2y)}{3} = \frac{2}{3}x^2 - \frac{1}{3}x(1-2y)$; $ds = \frac{dxdz}{|\hat{n}.\hat{j}|} = \frac{dxdz}{\frac{1}{3}}$

Therefore, $\iint_{S_3} Curl\ \vec{A}.\hat{n}\, ds = \int\int \left[\frac{2}{3}x^2 + \frac{1}{3}x(1-2y)\right] 3dxdz$

$$\iint_{OAD} Curl\ \vec{A}.\hat{n}\, ds = \int\int_{OAD} [2x^2 + x(1-2y)]dxdz = \int\int_{OAD} [2x^2 + x(1-2y)]dxdz$$

To evaluate the surface integral on $S_3$: $ABD$, we project the surface $S_3$ on xz-plane,i.e. projection of ABD on xz-plane is AOD.
since the region AOD is covered by varing z fron 0 to 4-x while x varies from 0 to 4. Using the equation surface
$S_3$, i.e. $2x + y + 2z = 8$ *we have* $y = 8 - 2x - 2z$. *substituting this in the above integral we get*

$$= \int_{x=0}^{4}\int_{z=0}^{4-x} [2x^2 + x - 2x(8 - 2x - 2z)]dzdx = \int_{x=0}^{4}\int_{z=0}^{4-x} [2x^2 + x - 16x + 4x^2 + 4xz]dzdx$$

$$= \int_{x=0}^{4}\int_{z=0}^{4-x} [6x^2 - 15x + 4xz]dzdx$$

$$= \int_0^4 [6x^2 z - 15xz + 2xz^2]_0^{4-x} dx = \int_0^4 [23x^2 - 4x^3 - 28x]dx = \left[23.\frac{x^3}{3} - x^4 - 14x^2\right]_0^4 = \frac{32}{3}$$

Here C consiste of three lines AO,OD and DA

*Now let us evaluate the LHS* $\int_C \vec{A}.d\vec{r} = \int_{AO} \vec{A}.d\vec{r} + \int_{OD} \vec{A}.d\vec{r} + \int_{DA} \vec{A}.d\vec{r}$.

(i) *On the St. line AO*: $y = 0, z = 0$ *and* $\vec{A} = 0$ *so* $\int_{AO} \vec{A}.d\vec{r} = 0$

(ii) *On the St. line OD*: $x = 0, y = 0$ *and* $\vec{A} = 0$ *so* $\int_{OD} \vec{A}.d\vec{r} = 0$

(iii) *On the St. line DA*: $x + z = 4$ & $y = 0$ *and* $\vec{A} = xzi = x(4 - x) = (4x - x^2)i$