Vector integration:-

Let f be a continuous diffeeable vector point function and $r = g(t)$ be a curve then the line integral of the function 'f' along the given curve is defined as $\boxed{\int_C f \cdot ds}$

If $F = f_1 i + f_2 j + f_3 k$

$\bar{r} = xi + yj + zk$

$dr = dx i + dy j + dz k$

$$\int_C F \cdot dr = \int_C (f_1 i + f_2 j + f_3 k) \cdot (dx i + dy j + dz k)$$

$$\int_C F \cdot dr = \int_C f_1 dx + f_2 dy + f_3 dz$$

⇒ The total work done by the force $\bar{F}$ during the displacement from A to B is given by

$$\boxed{\int_A^B F \cdot dr}$$

→ In fluid dynamics if $\bar{F}$ represent velocity of the fluid and C be a closed curve then

$\oint_C F \cdot dr$ is called circulation of F around the curve

→ Find $\int_C \bar{F}\cdot d\bar{r}$ where $F = xy\bar{i} + yz\bar{j} + xz\bar{k}$ and the curve c is $\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$ and t varying from -1 to 1.

$x = t \quad y = t^2 \quad z = t^3$

Sol: $\int_C \bar{F}\cdot d\bar{r} = \int_C (xy\bar{i} + yz\bar{j} + xz\bar{k})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k})$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C xy\,dx + yz\,dy + xz\,dz.$$

$dx = dt$
$dy = 2t\,dt$
$dz = 3t^2 dt$

$$= \int_C t^3 dt + t^5\, 2t\,dt + t^4\, 3t^2 dt.$$

$$= \int_{-1}^{1} (t^3 + 2t^6 + 3t^6)\,dt$$

$$= \left[\frac{t^4}{4} + \frac{2t^7}{7} + \frac{3t^7}{7}\right]_{-1}^{1}$$

$$= \cancel{\frac{1}{4}} + \frac{2}{7} + \frac{3}{7} - \cancel{\frac{1}{4}} + \frac{2}{7} + \frac{3}{7}$$

$$= \frac{10}{7} \quad //$$

(2) Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $F = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along

ⓐ the straight line from $(0,0,0)$ to $(2,1,3)$

ⓑ the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$

Scanned by CamScanner

Sol: $\int_C F\cdot dr = \int_C (3x^2 i + (2xz - y) j + z k)(dx\, i + dy\, j + dz\, k)$

$= \int_0^1 (3x^2 dx + (2xz - y) dy + z dz)$

$r = 2i + j + 3k$

$x = 2 \quad y = 1 \quad z = 3$

Eq'n of joining the points $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0} = t$

$x = 2t \quad y = t, \quad z = 3t$

$dx = 2dt \quad dy = dt, \quad dz = 3dt$

$= \int_0^2 12 t^2 \, 2dt + (12t^2 - t) dt + 3t\, 3dt$

$= \int_0^1 24t^2 dt + 12t^2 dt - t dt + 9t dt$

$= \left[\frac{24t^3}{3} + \frac{12t^3}{3} - \frac{t^2}{2} + \frac{9t^2}{2} \right]_0^1$

$= \frac{\cancel{24}^{8}}{\cancel{3}} + \frac{\cancel{12}^{4}}{\cancel{3}} - \frac{1}{2} + \frac{9}{2}$

$= 12 + \frac{\cancel{8}^{4}}{\cancel{2}} = 16$ //

for 2 prob) $x^2 = 4y$

$y = \frac{x^2}{4}$

$z = \frac{3x^3}{8}$

$\left(\frac{2x\,dx}{}\right)$

$dy = \frac{2x}{4_2} = \frac{x}{2}$

$dz = \frac{\cancel{3x^2}}{\cancel{8}_4} \quad \frac{9x^2}{8}$ //

$x^2=4y \quad 3x\cancel{4}y=8z$

$3\cancel{12}xy=8z$

$\int_0 \left(\frac{3x^2}{\cancel{12y}}y\,dx + \left(x(\cancel{4y})\left(\frac{3x^3}{8 \cdot 4}\right) - \frac{x^2}{4}\right)dy + \frac{3x^2}{8}dz\right)$

$\cancel{\int \cdot 12y\,dx +}$

$\int_0^2 3x^2 dx + \left(\frac{3x^4}{4} - \frac{x^2}{4}\right)dy + \frac{3x^2}{8}dz$

$= \int_0^2 3x^2 dx + \left(\frac{3x^4 - x^2}{4}\right)dy + \frac{3x^2}{8}dz$

$= \int_0^2 3x^2 dx + \frac{3x^4}{4}dy - \frac{x^2}{4}\frac{x\,dy}{2} + \frac{3x^2}{8}dz \quad \frac{9x^2}{8}$

$= \left(\frac{\cancel{3}x^3}{\cancel{3}} + \frac{\cancel{3}x^6}{\cancel{6}\times 8} - \frac{x^4}{8\times 4} + \frac{27x^5}{64\times 5}\right)_0^2$

$8+4-\frac{1}{2}+\frac{9}{2} = 16$

$F = 3xy\,i - y^2\,j$ then evaluate $\int_C F\cdot dr$ where

'c' is a curve in xy plane $y=2x^2$ from $(0,0)$ to $(1,2)$

$\frac{16}{4} - \frac{16}{6}$

$\int_C F\cdot dr = \int_C (3xy\,i - y^2\,j) \quad | \quad -\frac{7}{6}$ (answer)

) Evaluate the line integral $\int_C (x^2+xy)dx + (x^2+y^2)dy$

Where c is the square formed by the lines $y=\pm 1$ &

$x=\pm 1$

$$\int_C F\cdot dr = \int_{AB} F\cdot dr + \int_{BC} F\cdot dr + \int_{CD} F\cdot dr + \int_{DA} F\cdot dr \quad \text{— ①}$$

along AB $x = 1 \Rightarrow dx = 0$

$(y = -1 \text{ to } 1)$

$$\Rightarrow \int_C F\cdot dr = \int_C (1+y)0 + (1+y^2)dy$$

$$= \int_{-1}^{1} (1+y^2)dy$$

$$= \left[y + \frac{y^3}{3}\right]_{-1}^{1}$$

$1 + \frac{1}{3} + 1 + \frac{1}{3} = \frac{8}{3}$ $\cancel{1+\frac{1}{3}} + \cancel{1+\frac{1}{3}}$ = ⓐ

along BC x -1

$$\int_{1}^{-1} (x^2+x)dx$$

$$\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{1}^{-1}$$

$$\frac{-1}{3} + \frac{1}{2} - \frac{1}{3} - \frac{1}{2} = -\frac{2}{3}$$

along CD $x = -1 \Rightarrow dx = 0$ $y = (1 \text{ to } -1)$

$$\int_C F\cdot dr = \int_{1}^{-1} (1+y^2)dy$$

$$= \left[y + \frac{y^3}{3}\right]_{1}^{-1} = -1 - \frac{1}{3} - 1 - \frac{1}{3}$$

$$= -\frac{8}{3}$$

along DA $y=1$ $dy=0$ $x=(-1$ to $1)$

$$-\int_{1}^{-1}(x^2+x)\,dx$$

$$\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{1}$$

$$\frac{1}{3}+\cancel{\frac{1}{2}}+\frac{1}{3}-\cancel{\frac{1}{2}}=\frac{2}{3}\ //$$

$$=\frac{8}{3}-\frac{2}{3}-\frac{8}{3}+\frac{2}{3}=0\ //$$

Find the line integral of the vector function

$F=z\bar{i}+x\bar{j}+y\bar{k}$ $\quad x=a\cos t,\ y=a\sin t,\ z=\frac{t}{2\pi}$

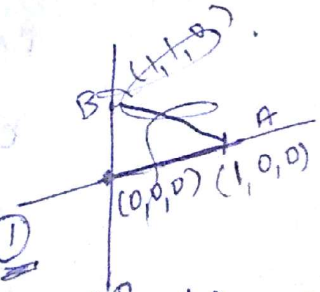
from $z=0$ to $z=1$. $\quad dx=-a\sin t\,dt$

$dy=a\cos t\,dt$

Ans: $\boxed{a+\pi a^2}$ $\quad$ lim $\left(\frac{0 \text{ to } 1}{0 \text{ to } 2\pi}\right)$ $\quad dz=\frac{1}{2\pi}\,dt$

→ $F=(3x^2+6y)i-14yz\,j+20xz^2k$ then find $\int_C F\cdot d\bar{r}$ along the lines from $(0,0,0)$ to $(1,0,0)$ ① then to $(1,1,0)$ and then to $(1,1,1)$.

3D



$(0,0,0)$ $(1,0,0)$

$x=1$ $y=0$ $z=0$

$3x^2dx \Rightarrow \frac{3x^3}{3}=①$

$z=0$

$dy=\int_0$ $\quad 0$ $\quad \int_0^1 \frac{20z^3}{3}$

$dx=0$

$$\frac{20}{3}+1$$

$$\frac{23}{3}\ //$$

* $\bar{A} = (3x^2+6y)i - 14yz j + 20xz^2 k$ then Evaluate $\int_C A \cdot dr$ from $(0,0,0)$ to $(1,1,1)$ along the path $x=t$, $y=t^2$, $z=t^3$.

$x=t \Rightarrow dx = dt$ $\quad y=t^2 = dy = 2t\,dt$

$z=t^3 = 3t^2 dt$

$$\int_C A \cdot dr = \int_C (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz$$

$$= \int_0^1 (3t^2+6t^2)dt - 14t^2 \cdot t^3(2t\,dt) + 20 \cdot t^7 \cdot 3t^2 dt$$

$$= 5$$

* Find the work done by the force $F = 4xy i - 8y j + 2k$ along the curve $y=2x$ and $z=0$ from $(3,6,0)$ to $(0,0,0)$

$$\int_C F \cdot ds$$

$$= \int_C 4xy\,dx - 8y\,dy + 2dz$$

$$\int_C F \cdot dr = \int_C 4xy\,dx - 8y\,dy \quad \left(\because z=0,\ dz=0\right)$$

$(\because y = 2x \Rightarrow dy = 2dx)$

$$\Rightarrow \oint_C F\cdot dr = \int_{x=3}^{x=0} 4x\,(2x)dx - 8(2x)(2dx)$$

$$= \left(\frac{8x^3}{3} - \frac{32x^2}{2}\right)_3^0 = 0 - 0 - 72 + 144$$

$$= 72 \;//$$

→ Find the circulation of $F = y\,i + z\,j + x\,k$ or $\oint_C F\cdot dr$ where C is the circle $x^2+y^2=1$ & $z=0$

$$\oint_C F\cdot dr = \oint_C y\,dx + z\,dy + x\,dz$$

$(z=0$ & $dz=0.$

$$= \oint_C y\,dx$$

$y = \cos\theta$

$dy = -\sin\theta\, d\theta$

$x =$

$x = \cos\theta \;\Rightarrow\; dx = -\sin\theta\,d\theta$

$y = \sin\theta \;\Rightarrow$

$$\oint_C F\cdot dr = \oint_C y\,dx - \oint_C \sin\theta(-\sin\theta)d\theta$$

$$= -\int_0^{2\pi} \sin^2\theta\, d\theta = -\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = -\frac{1}{2}\left[2\pi - 0 - 0 - 0\right]$$

$$= -\pi \;//$$

* Evaluate $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$ where C is the curve enclosing the region $y=\sqrt{x}$ and $y=x^2$.

Sol: $\int_{0}^{1} c_1 + \int_{y=1}^{0} c_2$

$x=0$ $y=x^2$ $dy=2x\,dx$

$x=y^2$ $dx=2y\,dy$

$= \int_{0}^{1} 2x\,dx + 2\left(\cancel{\frac{y^2}{2}}\right)\int_{1}^{0} 2y\,dy$ Wrg

$= 2\left[\frac{x^2}{-2}\right]_0^1 + 2\left[\frac{y^2}{2}\right]_1^0 = 0$

⇒ Surface integrals:-

Let F is be a continuous differentiable vector point function then the surface integral 'F' over a surface is defined by

$$\boxed{\iint_S F\cdot N\,ds}$$

Where n is unit normal to the surfaces.

Note: ① Let R_1 be the projection of surface 's' on xy plane then double integral over R_1

$$\iint_S F\cdot N\,ds = \iint_{R_1} F\cdot N\,\frac{dx\,dy}{|N\cdot K|}$$

② Let R_2 be the projection of surface 's' on yz plane then double integral over R_2

$$\iint_S F\cdot N\,ds = \iint_{R_2} F\cdot N\,\frac{dy\,dz}{|N\cdot i|}$$

Let R_3 be the projection of surface 's' on XZ plane then double integral over R_3

$$\iint_{R_3} F\cdot N\, ds = \iint_{R_3} F\cdot N \frac{dx\,dz}{|N\cdot j|}.$$

Let $\bar{F}$ represent the velocity of fluid particle then the total outward flux of F across a closed surface S is a surface integral

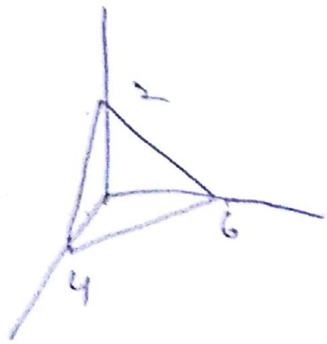
$$\boxed{\iint_S F\cdot N\, ds}$$

Q: Evaluate double integral $\iint_S F\cdot N\,ds$ where $\bar{F} = 6z\,i - 4j + yk$ and S is the portion of the plane $2x+3y+6z=12$ in the 1st octant

Sol:

$$\iint_S F\cdot N\, ds$$

$$\frac{x}{6} + \frac{y}{4} + \frac{z}{2} = 1.$$



$$N = \frac{\nabla S}{|\nabla S|} \qquad S = 2x+3y+6z-12 = 0$$

$$\nabla S = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(2x+3y+6z-12)$$

$$\nabla S = 2i + 3j + 6k$$

$$\bar{N} = \frac{\nabla S}{|\nabla S|} = \frac{2i+3j+6k}{\sqrt{4+9+36}} = \frac{2i+3j+6k}{7}$$

$F\cdot N = (6z\,i - 4j + yk)\cdot\left(\frac{2i+3j+6k}{7}\right)$

$F\cdot N = \frac{1}{7}[12z - 12 + 6y]$

Let R be the projection of 's' on y-z plane then we have

$$\iint_S F\cdot N\,ds = \iint_S F\cdot N \frac{dy\,dz}{|N\cdot i|}$$

$$= \iint \frac{1}{7}(12z - 12 + 6y)\frac{dy\,dz}{\frac{2}{7}}$$

yz plane $x = 0$

$3y + 6z = 12$

$y = \frac{12-6z}{3}$

$\boxed{y = 4-2z}$

$4 = 2z$ $z = 2$ $y = 0$

$$\Rightarrow 3\int_0^2\int_0^{4-2z}(2z - 2 + y)\,dy\,dz$$

$$= 3\int_0^2 \left(2yz - 2y + \frac{y^2}{2}\right)_0^{4-2z} dz$$

$$= 3\int_0^2 \left\{2z(4-2z) - 2(4-2z) + \frac{(4-2z)^2}{2}\right\} dz$$

$$= 3\int_0^2 \left[8z - 4z^2 - 8 + 4z + \frac{16 + 4z^2 - 4z}{2}\right] dz$$

$$= \frac{3}{2}\int_0^2 16z - 8z^2 - 16 + 8z + 16 + 4z^2 - 4z\,dz$$

$$= \frac{3}{2}\int_0^2 (-4z^2 + 8z)\,dz$$

$$= \frac{3}{2}\times 2\int_0^2 (-2z^2 + 4z)\,dz$$

$$= 3\left(\frac{-2z^3}{3} + \frac{4z^2}{2}\right)_0^2$$

$$= 3\left(\frac{8}{3}\right) = 8$$

evaluate $\iint_S F\cdot N\,ds$ where $F = 18z\,i - 12\,j + 3y\,k$ & S is part of plane $2x+3y+6z=12$ locate in the first octane

$\frac{x}{6}+\frac{y}{4}+\frac{z}{2}=1$

$N = \frac{\nabla S}{|\nabla S|}$ $\quad S = 2x+3y+6z-12=0$

$\nabla S = 2i+3j+6k$

$N = \frac{\nabla S}{|\nabla S|} = \frac{2i+3j+6k}{7}$

$F\cdot N = (18z\,i - 12\,j + 3y\,k)\cdot\left(\frac{2i+3j+6k}{7}\right)$

$F\cdot N = \frac{1}{7}\left[36z - 36 + 18y\right]$

Same as above Problem

$\Rightarrow 8(3) = 24$

$F = 8i - 7x\,j + 9y\,k$

$F\cdot N = (8i - 7x\,j + 9y\,k)\cdot\left(\frac{2i+3j+6k}{7}\right)$

$= \frac{1}{7}\left(16 - 21x + 54y\right)$

$2x+3y=12$

$2x = 12-3y$

$\boxed{x = \frac{12-3y}{2}}$

$z = 0$

$\iint_S F\cdot N\,ds = \iint F\cdot N\,\frac{dy\,dz}{|N\cdot i|}$

$2x+3y=12$

$= \int_0^6 \int_0^{\frac{12-2x}{3}} \frac{1}{\cancel{7}}\left(16-21x+54y\right)\frac{dx\,dy}{\frac{6}{\cancel{7}}}$ $\quad 2x = \frac{12}{-3y}$

$= \frac{1}{6}\int_0^6 \left[16y - 21xy + \frac{54y^2}{2}\right]_0^{\frac{12-2x}{3}} dx$

$$= \frac{1}{6}\int_0^6 \left[16\left(\frac{12-2x}{3}\right) - 21x\left(\frac{12-2x}{3}\right) + 54\,\frac{\left(\frac{12-2x}{3}\right)^2}{2} \right] dx$$

$$= \frac{1}{6}\int_0^6 \frac{192-32x}{3} - \frac{252x + 42x^2}{3} + 54\left(1\right.$$

$\sqrt{4+9+36}$

$$N = \frac{\nabla S}{|\nabla S|} = \frac{(2x+3y+6z-12)}{} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$

$$\iint_S F\cdot N \frac{dx\,dy}{|\hat{k}\cdot\hat{N}|}$$

$\frac{1}{7}(16-21x+54y)$

$2x+3y+6z=12$

$2x+3y=12$

$2x=12-3y$

$x=\frac{12-3y}{2}$

$12=3y$

$$= \int_0^4\int_0^{\frac{12-3y}{2}} \frac{1}{7}(16-21x+54y)\frac{dx\,dy}{\frac{6}{7}}$$

$$= \int_0^4 \left[16 - \frac{21x^2}{2} + 54xy\right]_0^{\frac{12-3y}{2}}$$

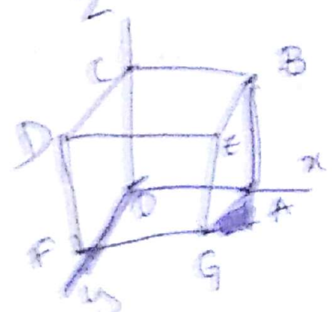
$$\int_0^4 \left[16 - \frac{21\left(\frac{12-3y}{2}\right)^2}{2} + 54\left(\frac{12-3y}{2}\right)y\right] dy$$

$\frac{24}{\frac{3}{2}}$

$$\int_0^4 \left[16 - \frac{21(144+9y^2-72y)}{2} + \frac{(54\times12 - 54\times3y)\,y}{2}\right] dy$$

Q. Evaluate the surface integral of $\bar{F} = 2xy\,i + yz^2 j + xz\,k$ over the surface of parallelopiped $x=0$; $y=0$; $z=0$; $x=2$; $y=1$; $z=3$

$$\iint_S F\cdot N\,ds = \iint_{S_1} F\cdot N\,ds + \iint_{S_2} F\cdot N\,ds + \iint_{S_3} F\cdot N\,ds$$
$$+ \iint_{S_4} F\cdot N\,ds + \iint_{S_5} F\cdot N\,ds + \iint_{S_6} F\cdot N\,ds$$



Along $S_1 \Rightarrow$ xy plane (OAGF)

xy plane at $z=0$ & $\bar{N} = -k$

$$\iint_{S_1} F\cdot N\,ds = \iint_{S_1} (2xy\,i + yz^2 j + xz\,k)(-k)\cdot \frac{dx\,dy}{|N\cdot k|}$$ Here $z=0$, $x=0$

$$= \iint -xz \cdot \frac{dx\,dy}{1}$$

Along $S_2 \longrightarrow$ BCDE at $z=3$ (xy plane) & $\bar{N}=k$

$$\iint_{S_2} F\cdot N\,ds = \iint_{S_2} xz\cdot\frac{dx\,dy}{1}$$

$$3\int_0^1\int_0^2 x\,dx\,dy \Rightarrow$$

$$= 3\left[\frac{x^2}{2}\right]_0^2 \int_0^1 dy$$

$$= 3\left(\frac{4}{2}\right) y\Big|_0^1 = 6(1)$$
$$= 6$$

along yz plane (OFDC)

yz plane $\bar{N} = -\hat{i}$ $x = 0$

$$\iint_S F \cdot N \, ds = \iint (2xy)\hat{i} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \frac{dy\,dz}{|N \cdot \hat{i}|}$$

$$= \iint -2xy \frac{dy\,dz}{(1)} = 0$$

yz - plane (BEGA) $N = \hat{i}$ $x = 0$ to 2

$$\iint_S F \cdot N \, ds = \iint 2xy \,(1) \frac{dy\,dz}{|N \cdot \hat{i}|}$$

$$= \iint 2xy \, dz \, dz$$

$$= 4 \int_0^3 \int_0^1 y \, dy \, dz$$

$$= 4 \left. \frac{y^2}{2} \right]_0^1 \int_0^3 dz$$

$$= \left(\frac{\cancel{4}^{2}}{\cancel{2}} \right)(3) = 6$$

along xz plane (OABC) $y = 0$ $\bar{N} = -\hat{j}$

$$\iint_S F \cdot N \, ds = \iint_S -yz^2 (-\hat{j}) \frac{dx\,dz}{|N \cdot J|}$$

$$= 0$$

along xz (EDFG) $y = 1$ $N = \hat{j}$

$$\iint_S F \cdot N \, ds = \iint_S yz^2 \, \hat{j} (1) \frac{dx\,dz}{|N \cdot \hat{j}|}$$

$$= \iint_S yz^2 \frac{dx\,dz}{1} \qquad y = 1$$

$$= \int_0^2 \int_0^3 z^2 \, dx \, dz$$

$$= \frac{z^3}{3}\Big]_0^3 \int_0^3 z \cdot dx$$

$$= \frac{27}{3} \times 2 = 18$$

$\Rightarrow 6+6+18 = 30$

If $\bar{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$ then $\iint_S F \cdot N\, ds$ when S is the surface of the parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$

Sol $S_1 = \iint F \cdot (-k)\, dx\, dy$

$$= \iint -(z^2 - xy)\, dx\, dy$$

at $z = 0$

$$\int_0^b \int_0^a xy\, dx\, dy$$

$$\int_0^b \left[\frac{x^2 y}{2}\right]_0^a$$

$$\frac{a^2}{2}\, dy$$

$$\frac{a^2}{2} \frac{y^2}{2}\Big|_0^b$$

$$= \frac{a^2 b^2}{4}$$

$S_2 = \iint (z^2 - xy)\, dx\, dy$

$$= \int_0^b \int_0^a (c^2 - xy)\, dx)\, dy$$

$$c^2 x - y\frac{x^2}{2}\Big|_0^a$$

$$\int_0^b c^2 a - y\frac{a^2}{2} = c^2 a y - \frac{y^2}{2}\frac{a^2}{2}\Big|_0^b$$

$$c^2 ab - \frac{a^2 b^2}{4}$$

$S_1 = \frac{a^2b^2}{4}$

$S_2 = c^2ab - \frac{a^2b^2}{4}$

$S_3 = \frac{b^2c^2}{4}$

$S_4 = a^2bc - \frac{b^2c^2}{4}$

$S_5 = \frac{a^2c^2}{4}$

$S_6 = b^2ac - \frac{a^2c^2}{4}$

$S_1 + S_2 + S_3 + \cdots + S_6 = a^2bc + ab^2c + abc^2$

$abc(a+b+c)$

→ $\bar{F} = 4xz\,\hat{i} + yz\,\bar{k} - y^2\hat{j}$ Evaluate $\iint_S F \cdot N\,ds$ where 's' is the surface of the cube bounded by ~~x=0~~ $x=0$ $x=1$, $y=0$ $y=1$, $z=0$ $z=1$.

Sol: ∯

→ Evalute $\iint_S F \cdot N\,ds$ Where $\bar{F} = z\hat{i} + x\hat{j} - 3y^2 z\,k$ and S is surface $x^2+y^2=16$ included in the first octant b/w $z=0$ to $z=5$

1st octant so -4 shd be ...

Sol: $\phi \Rightarrow x^2+y^2=16$

$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2-16)$

$= 2x\hat{i} + 2y\hat{j} + 0$

$\nabla\phi = 2x\hat{i} + 2y\hat{j}$

$$\bar{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x i + 2y j}{\sqrt{4x^2+4y^2}} = \frac{2x i + 2y j}{2\sqrt{16}} = \frac{xi + yj}{4}$$

$$\iint_S F \cdot N \, ds = \iint_S (z i + x j - 3y^2 z k) \cdot \left(\frac{xi+yj}{4}\right) \frac{dx\,dz}{|N \cdot j|}$$

$$= \iint_S \left(\frac{xz + xy}{4}\right) \frac{dx\,dz}{\frac{y}{4}}$$ (It is big process) so take as dy dz

$$\iint_S F \cdot N ds = \iint_S (z i + x j - 3y^2 z k) \cdot \left(\frac{xi + yj}{4}\right) \frac{dy\,dz}{|N \cdot i|}$$

$$= \iint_S \left(\frac{xz + xy}{4}\right) \frac{dy\,dz}{\frac{x}{4}}$$

$$= \iint_S \frac{\cancel{x}}{\cancel{4}}(z+y) \frac{dy\,dz}{\cancel{\frac{x}{4}}}$$

yz plane

$x = 0$

$x^2 + y^2 = 16$

$y^2 = 16 - x^2$

$y = 4 - x$

$y = 4$

$$= \iint_0 (z+y)\, dy\, dz$$

yz - plane $\Rightarrow x = 0$

$x^2 + y^2 = 16 \Rightarrow y^2 = 16 \Rightarrow \boxed{y = -4 \text{ to } 4}$

$z = 0$ to 5 (Given)

$$= \int_0^5 \int_0^4 z + y \, dy\, dz$$

$$= \int \left[zy + \frac{y^2}{2}\right]_0^4 dz$$

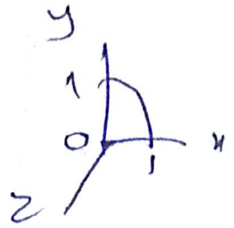
$$= \int \left[4z + \frac{\cancel{16}^{8}}{\cancel{2}}\right] dz$$

$$\int_0^5 [4z + 8]\, dz = \left[\frac{4z^2}{2} + 8z\right]_0^5$$

$$= \frac{\cancel{4}^{2} \times 25}{\cancel{2}} + 8 \times 5$$

$= 50 + 40 = 90$ //

4\. Evaluate $\iint_S F\cdot N\,ds$ where $F = yz\,i + xz\,j + xy\,k$ & 's' is the part of the surface of sphere $x^2+y^2+z^2=1$ which lies in the 1st octant



Sol: $\iint_S F\cdot N\,ds$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-1)$$

$$= 2xi + 2yj + 2zk$$

$$\bar{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2xi+2yj+2zk}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2xi+2yj+2zk}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{2xi+2yj+2zk}{2} \Rightarrow xi+yj+zk$$

$$\iint F\cdot N\,ds = \iint (yzi + xzj + xyk)(xi+yj+zk)\frac{dx\,dy}{|N\cdot K|}$$

$x=0$ to 1

$\cancel{y=0 \text{ to } 1}$

$$= \iint (xyz + xyz + xyz)\frac{dx\,dy}{z}$$

$x^2+y^2+\cancel{z^2}^{0}=1$

$x^2+y^2=1$

$y=\sqrt{1-x^2}$

0 to $\sqrt{1-x^2}$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} 3xy\,dx\,dy$$

$$= \int_0^1 \left[\frac{3xy^2}{2}\right]_0^{\sqrt{1-x^2}} dy$$

$$= \int_0^1 \frac{3x(\sqrt{1-x^2})^2}{2} = \left[\frac{3x - 3x^3}{2}\right]_0^1$$

$$= \frac{3x^2}{4} - \frac{3x^4}{8} = \frac{3}{4} - \frac{3}{8} = \frac{6-3}{8}$$

$$= \frac{3}{8}$$ //

Volume integrals:-

If a closed surface ~~on~~ closed volume 'v' then the ~~b.~~ $\iiint_V F \cdot dv$ is called volume integrals.

Let $F = f_1 i + f_2 j + f_3 k$ then

$$\iiint_V F \cdot dv = \iiint (f_1 i + f_2 j + f_3 k)\, dx\,dy\,dz$$

$$= i \iiint f_1\, dx\,dy\,dz + j \iiint f_2\, dx\,dy\,dz + k \iiint f_3\, dx\,dy\,dz.$$

Q If $\bar{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ then evaluate $\iiint_V \operatorname{div} F\, dv$ where d is bounded by the planes $x=0$, $y=0$, $z=0$, & $2x+2y+z=4$

Sol: $\operatorname{div} F = \nabla \cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left((2x^2-3z)i - 2xyj - 4xk\right)$

$$= 4x - 2x + 0 = 2x$$

$$\iiint_V \operatorname{div} F\, dv = \iiint_V 2x\, dx\,dy\,dz.$$

to get y ~~z~~=0 we get y-limits

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} x\, dz\,dy\,dx$$

$$= 2\int_0^2\int_0^{2-x}\left[xz\right]_0^{4-2x-2y} dy\,dx$$

$$= 2\int_0^2\int_0^{2-x} (x(4-2x-2y))\, dy\,dx.$$

$$= 2\int_0^2\int_0^{2-x} \cancel{x+} \; 4x - 2x^2 - 2xy \; dy\, dx$$

$$= 4\int_0^2\int_0^{2-x} 2x - x^2 - xy \; dy\, dx$$

$$= 4\int_0^2 \int \left[2xy - x^2y - \frac{xy^2}{2}\right]_0^{2-x} dx$$

$$= 4\int_0^2 \left[2x(2-x) - x^2(2-x) - \frac{x(2-x)^2}{2}\right] dx$$

$$= 4\int_0^2 \left[4x - 2x^2 - 2x^2 + x^3 - \frac{x(4+x^2-4x)}{2}\right] dx$$

$$= 4\int_0^2 \left[4x - 4x^2 + x^3 - \frac{4x - x^3 + 4x^2}{2}\right] dx$$

$$= 4\int_0^2 8x - 8x^2 + 2x^3 - 4x - x^3 + 4x^2 \; dx$$

$$= 4\int_0^2 4x - 4x^2 + x^3 \; dx$$

$$= 4\left[\frac{4x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4}\right]_0^2$$

2^4

$$= 4\left(\frac{\cancel{4}^{2}\times 4}{\cancel{2}} - \frac{4\times 8}{3} + \frac{\cancel{16}^{4}}{\cancel{4}}\right]$$

24
12
36

$$= 4\left(8 - \frac{32}{3} + 4\right]$$

Some Where mistake

$$= 12(24 - 32 + 12)$$

$$= \cancel{48} \,//$$ Ans $\left(\frac{8}{3}\right)$

→ Evaluate $\iiint_V F\,dv$ where $f=2x+y$ 'v' is closed region bounded by the cyclinder $z=4-x^2$ & planes $x=0,\ y=0,\ y=2\ \&\ z=0$

limits

$z = 0$ to 4

$x = 0$ to $\sqrt{4-z}$

→ Evaluate $\iiint_V F\,dv$ where $f=45x^2y$ and 'v' denotes the closed region bounded by the planes $4x+2y+z=8$ & $x=0,\ y=0,\ \&\ z=0$

As :- $\underline{\underline{128}}$

→ Evaluate $\iiint_V \nabla\cdot F\,dv$ where $f=4x\bar{i}-2y^2\bar{j}+z^2\bar{k}$ and S is the surface of cylinder $x^2+y^2=4$ and the planes $z=0,3$.

→ If $\bar{a}=2xz\bar{i}-x\bar{j}+y^2\bar{k}$ then find $\iiint_V a\,dv$ where b is the region bounded by surface $x=0,\ y=0,\ x=2,\ y=6,\ z=x^2\ \&\ z=4$

$\underline{Sol:}$ $$\int_0^2\int_0^{\sqrt{4-z}}\int_0^4 2x+y\ dz\,dx\,dy$$

$$\int_0^2\int_0^4\int_0^{\sqrt{4-z}} 2x+y\ dx\,dz\,dy$$

$$\int_0^2\int_0^4 \left(\frac{\cancel{2}x^2}{\cancel{2}}+yx\right)dz\,dy$$

Scanned by CamScanner

$$\int_0^2\int_0^4 (x^2+yx)\Big|_0^{\sqrt{4-z}} \, dz\, dy$$

$$\int_0^2\int_0^4 \left((\sqrt{4-z})^2 + y(\sqrt{4-z})\right) dz\, dy$$

$$= \int_0^2\int_0^4 (4-z+y\sqrt{4-z})\, dz\, dy$$

$$\cancel{\int_0^2 \left(\sqrt{4-z}^2\right.}$$

$$= \int_0^4 \left(4y - zy + \frac{y^2}{2}\sqrt{4-z}\right)_0^2 dz$$

$$= \int_0^4 \left(8 - \frac{8z^2}{2} + \frac{4}{2}\sqrt{4-z}\right) dz$$

$2(4-2)^{1/2}$

$$= \int_0^4 \left(8 - 8z + 2\sqrt{4-z}\right) dz$$

$$\frac{(4-z)^{3/2}}{\frac{3}{2}}$$

$$= \cancel{\frac{8z - 8z^2}{2}} + \cancel{2(4-z)^{1/2}}$$

$4-4$

$2\left(4z - \frac{z^2}{2}\right)_0$

$\frac{6\times 8}{7}$

$$= 2\int_0^4 4 - 4z + \sqrt{4-z}\, dz$$

$\sqrt{8}$

$$= 2\left[4z - \frac{4z^2}{2} + \frac{(4-z)^{3/2}}{\frac{3}{2}}\right]_0^4$$

$$= 2\left\{16 - 32 + \frac{2^3 \times 2}{3}\right\}$$

Scanned by CamScanner

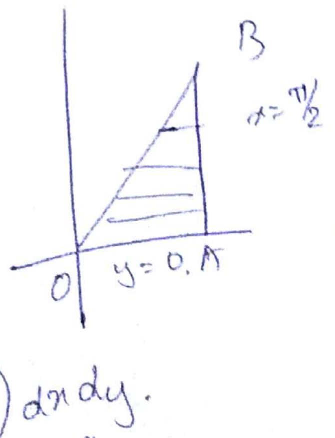
$$2\left[+16+\frac{16}{3}\right] = \frac{48+32}{3} = \frac{80}{3}$$ //

Green's Theorem:-

Let $P(x,y)$ & $Q(x,y)$, $\frac{\partial Q}{\partial x}$ & $\frac{\partial P}{\partial y}$ in a region 'R' of XY-plane bounded by a closed curve 'C' then $\int_C P\,dx + Q\,dy$

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

→ Apply Green's theorem to evaluate $\int_C (y-\sin x)\,dx + \cos x\,dy$ where C is the plane triangle enclosed by lines $y=0, x=\pi/2$ & $y=\frac{2x}{\pi}$



$P = y - \sin x \qquad Q = \cos x.$

$$\frac{\partial Q}{\partial x} = -\sin x \quad \& \quad \frac{\partial P}{\partial y} = 1$$

$$\int P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

$$\int (y-\sin x)\,dx + \cos x\,dy = \int_0^{\pi/2}\int_0^{\frac{2x}{\pi}} (-\sin x - 1)\,dy\,dx$$

$$= -\int_0^{\pi/2} (1+\sin x)\,(y)_0^{\frac{2x}{\pi}}\,dx$$

$$= \int_0^{\pi/2} (1+\sin x)\frac{2x}{\pi}\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} (x+x\sin x)\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2} + x(-\cos x) - (1)(-\sin x)\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{8} - 0 + 1\right) = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

Verify green's theorem for $\int_C (xy+y^2)\,dx + x^2 dy$

Where C is bounded by $y=x$ & $y=x^2$

Sol: $\int_C (xy+y^2)\,dx + x^2 dy = \int_{C_1} (xy+y^2)\,dx + x^2 dy + \int_{C_2} (xy+y^2)\,dx + x^2 dy$

Along C_1 $y=x^2 \Rightarrow dy = 2x\,dx$ & $x = 0$ to 1

$$\int_C (xy+y^2)\,dx + x^2 dy = \int_0^1 (x^3 + x^4)\,dx + x^2(2x\,dx)$$

$$= \int_0^1 (x^3+x^4+2x^3)\,dx$$

$$= \int_0^1 (3x^3+x^4)\,dx$$

$$= \left(\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along C_2 : $y = x \Rightarrow dy = dx$ & $x : 1$ to 0

$$\int_C (xy+y^2)\,dx + x^2 dy = \int_1^0 (x^2+x^2)\,dx + x^2 dx$$

$$= \left(\frac{3x^3}{3}\right)_1^0 = -1$$

from ①

$$\int_C (xy+y^2)\,dx + x^2 dy = \frac{19}{20} - 1 = \frac{-1}{20}$$

$P = xy + y^2 \quad Q = x^2 \qquad \frac{\partial Q}{\partial x} = 2x$

$\cancel{\frac{\partial P}{\partial x} = y+y^2} \quad \cancel{\frac{\partial Q}{\partial y} = 0} \quad \frac{\partial P}{\partial y} = x+2y$

$$\iint_R \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx\,dy = \iint (2x - x - 2y)\,dx\,dy$$

$$\int P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int_C (xy+y^2)\,dx + x^2 dy = \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \int_0^1 \int_{y=x^2}^{x} (x-2y)\,dy\,dx = \int_0^1 (xy - y^2)_{x^2}^{x}\,dx$$

$$= \int_0^1 (\cancel{x^2} - \cancel{x^2} - x^3 + x^4)\,dx = \left(-\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1 = -\frac{1}{4} + \frac{1}{5} = \frac{-1}{20}$$

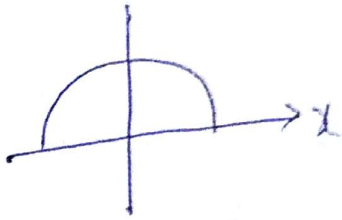
③ Verify Green's theorem in the plane for

$$\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy . \quad C \text{ is closed region}$$

$y = \sqrt{x}$ & $y = x^2$

① ⇒ Apply Green's theorem to evaluate $\int_C (2x^2-y^2)dx + (x^2+y^2)dy$ 'c' is the boundary of the area enclosed by x axis and upper half of circle $x^2+y^2=a^2$

Ans ⇒ $\frac{4a^3}{3}$



② ⇒ Evaluate $\int_C (x^2+xy)dx + (x^2+y^2)dy$ where c is the square formed by $x=\pm 1$, $y=\pm 1$

③ ⇒ Verify Green's theorem $\int_C (3x-8y^2)dx + (4y-6xy)dy$ where 'c' is boundary of region bounded by $x=0$, $y=0$, $x+y=1$

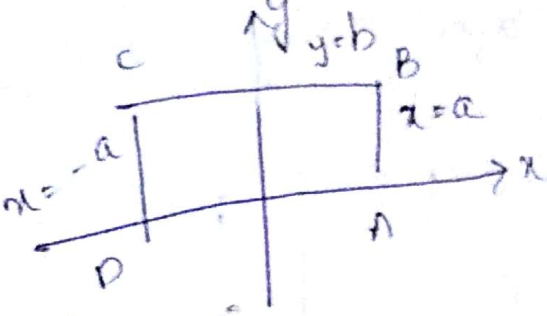
⇒ <u>Stokes theorem</u> :- If 's' be an opaque surface bounded by a closed curve 'c'

$F = f_1 i + f_2 j + f_3 k$ be a continuous differentiable vector function then

$$\int_C F\cdot dr = \iint_S \text{curl } F \cdot N \, ds$$ where N is outward ~~from~~ drawn normal to surface.

Verify Stokes theorem $F = (x^2+y^2)i - 2xy j$ taken around the rectangle bounded by lines $x = \pm a$, $y = 0$ & $y = b$

$\int F\cdot dr = \int_{AB} F\cdot dr$



$\frac{1}{41}$

$+\int_{BC} F\cdot dr + \int_{CD} F\cdot dr + \int_{DA} F\cdot dr$

Along AB: $(x = a \Rightarrow dx = 0$ & $(y = 0$ to $b)$

$\int F\cdot dr = \int_{AB} \left[(x^2+y^2)\hat{i} - 2xy\hat{j}\right]\cdot(dx\hat{i} + dy\hat{j})$

$= \int (x^2+y^2)\,dx - 2xy\,dy \Rightarrow \int_{y=0}^{y=b} 0 - 2ay\,dy = \left(-\frac{\cancel{2}ay^2}{\cancel{2}}\right)_0^b$

$= -ab^2$

Along BC:- $(y = b \Rightarrow dy = 0)$ $(x = a$ to $-a)$

$\int_{BC} F\cdot dr = \int_{BC} (x^2+y^2)\,dx - 2xy\,dy$

$= \int_a^{-a} (x^2+b^2)\,dx = \left(\frac{x^3}{3} + b^2x\right)_a^{-a} = -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$

$= -\frac{2a^3}{3} - 2ab^2$

Along CD: $x = -a \Rightarrow dx = 0$ $(y = b$ to $0)$

$\int_C F\cdot dr = \int_C (x^2+y^2)\,dx - 2xy\,dy$

$= \int_b^0 2ay\,dy = \left(\frac{\cancel{2}ay^2}{\cancel{2}}\right)_b^0 = -ab^2$

Along DA $y = 0 \Rightarrow dy = 0$ & $(x = -a$ to a

$\int_C F\cdot dr = \int_C (x^2+y^2)\,dx - 2xy\,dy$

$= \int_{-a}^{a} x^2\,dx = \left.\frac{x^3}{3}\right]_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3}$

$= 2a^3/3$

$F = 0$ after adding $\Rightarrow -4ab^2$ — ①

$$\text{curl} F = \nabla \times F \Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= i(0) - j(0) + k(-2y-2y) = -4yk$$

$$\iint_C \text{curl} F \cdot N ds = \iint (-4yk) \cdot k \, dx \, dy$$

$$= -4 \int_0^b \int_{-a}^{a} y \, dx \, dy = -4 \int_0^b y(x)_{-a}^{a} dy$$

$$= -4 \int_0^b 2ay \, dy = -4ab^2 \rightarrow ②$$

from ① & ② stokes is verified

5) For a vector $F = (2x-y)i - yz^2 j - y^2 z k$ over the upper half surface of $x^2+y^2+z^2=1$ rounded by its projection of xy plane

in xy plane $z = 0$

$dz = 0$

$$\int_C F \cdot dr = \int_C (2x-y)i - yz^2 j - y^2 z k$$

$$= \int_C (2x-y) dx \quad \left(\begin{array}{l} x = 1\cos\theta \\ y = 1\sin\theta \\ dx = -\sin\theta d\theta \end{array} \right)$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta) d\theta$$

$$\int_0^{2\pi} (-\sin 2\theta + \sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \left(-\sin 2\theta + \frac{(1-\cos 2\theta)}{2}\right) d\theta$$

$$= \left(\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_0^{2\pi}$$

$$= 1/2 + \frac{2\pi}{2} - 0 - 1/2 - 0 - 0$$

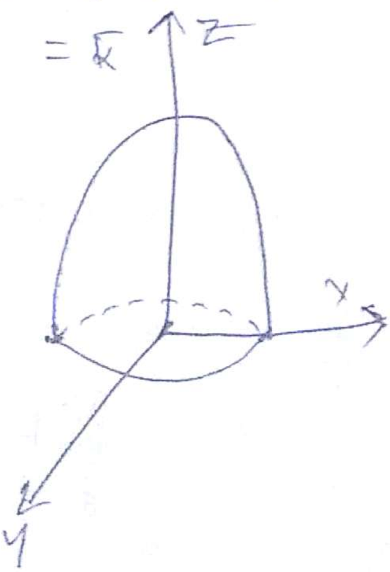
$$\int_C F \cdot dr = \pi \quad \text{--- (1)}$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= i(-2yz + 2zy) - j(0) + k(1)$$

$$= \bar{k}$$

$$\bar{N} = \frac{\nabla\phi}{|\nabla\phi|}$$



If $\phi = x^2 + y^2 + z^2 - 1 = 0$

$$\nabla\phi = 2xi + 2yj + 2zk$$

$$\bar{N} = \frac{2xi + 2yj + 2zk}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$\boxed{\bar{N} = xi + yj + zk}$$

$$\iint_S \text{curl } F \cdot \bar{n}\, ds$$

$$= \iint_S k \cdot (xi + yj + zk) \frac{dx\,dy}{|N \cdot K|}$$

$$\iint_S \cancel{z} \frac{dx\,dy}{\cancel{z}}$$

$$\iint_S dx\,dy = \pi(1)^2 = \pi$$

<u>Gauss divergence theorem:</u>

Let 'F' be a V.P.F Let V be the volume enclosed by surface 'S' $\iint_S F \cdot N\, ds$

$$= \iiint_V \operatorname{div} F\, dv.$$

Where 'N' is unit normal of the surface.

→ Verify divergence theorem for $(x^2 - yz)i + (y^2 - xz)j + (z^2 - xy)k$; $0 \le x \le a$, $0 \le y \le b$ & $0 \le z \le c$ taken over rectangular parallelopiped

Prove same as before.

$$\iint_S F \cdot N\, ds = abc(a+b+c) \longrightarrow (1)$$

$$\nabla \cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left((x^2 - yz)i + (y^2 - xz)j + (z^2 - xy)k\right)$$

$$\nabla \cdot F = 2x + 2y + 2z = 2(x+y+z)$$

$$= \iiint_V \operatorname{div} F \cdot dv = \int_0^c\int_0^b\int_0^a 2(x+y+z)\, dx\,dy\,dz$$

$$\iint 2\left(\frac{x^2}{2}+x(y+z)\right]_0^a dy\,dz$$

$$\iint \left[\frac{a^2}{2}+a(y+z)\right\} dy\,dz = 2\int \left(\frac{a^2}{2}y+\frac{ay^2}{2}+azy\right)_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2b}{2}+\frac{ab^2}{2}+abz\right) dz$$

$$= 2\left(\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]$$

$$= abc(a+b+c) \longrightarrow (2)$$

from (1) & (2)

$$\iint_S F\cdot N\,ds = \iiint_V \operatorname{div} F\cdot dv$$

Evaluate $\iint_S F\cdot N\,ds$ where $F = 4xi - 2y^2j + z^2k$

where 'S' is surface bounding region

$x^2+y^2=4$ & $z=0$ to $z=3$

S)

Differentiation under integral sign:-

If $f(x,\alpha)$ is a function of two variables x,α (called a parameter) be integrated with respective to x b/w the limits a and b then

$\int_a^b f(x,\alpha)dx$ is a function of α ie

$f(\alpha)$ say to find the derivative of $f(\alpha)$ when it exist it is n't always possible to first evaluate this integral & then to find the derivative such problems are solved

(1) **Leibnitz rule**: If $f(x,\alpha)$ be continuous function of x,α then

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}dx$$

(1) $\int_0^1 \frac{x^\alpha - 1}{\log x}\,dx \quad \alpha \geq 0$

let $F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x}\,dx$ —(1)

$$F'(\alpha) = \frac{d}{d\alpha}\left[\int_0^1 \frac{x^\alpha - 1}{\log x}\,dx\right]$$

$$= \int_0^1 \frac{d}{d\alpha}\left(\frac{x^\alpha - 1}{\log x}\right)dx$$

$$= \int_0^1 \frac{1}{\log x}\,\frac{\partial}{\partial\alpha}(x^\alpha - 1)\,dx$$

$$= \int_0^1 \frac{1}{\cancel{\log x}} \cdot x^{\alpha} \cdot \cancel{\log x}\, dx$$

$$F'(\alpha) = \int_0^1 x^{\alpha}\, dx \qquad F'(\alpha) = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

integrating w.r.t to α we get

$$F(\alpha) = \log(\alpha+1) + c. \quad \text{---} \quad (2)$$

Put $\alpha = 0$ in Eq (1) we get.

$$F(0) = 0$$

sub Eq-(2) i.e $\alpha = 0$ & $F(0) = 0$ we get

$$F(0) = \log(1) + c$$

$$0 = 0 + c \quad \Rightarrow \boxed{c = 0}$$

sub $c = 0$ in Eq (2) we get

$$\boxed{F(\alpha) = \log(\alpha+1)}$$

(c) ~~s.t.a~~ prove that $\int_0^\infty \frac{e^{-x}}{x}(1 - e^{-ax})\,dx = \log(1+a)$ $(a > -1)$

Let $F(a) = \int_0^\infty \frac{e^{-x}}{x}(1 - e^{-ax})\,dx.$

$$F'(a) = \frac{d}{da}\left[\int_0^\infty \frac{e^{-x}}{x}(1 - e^{-ax})\,dx\right]$$

$$= \int_0^\infty \frac{d}{da} \frac{e^{-x}}{x}(1 - e^{-ax})\,dx$$

$$= \int_0^\infty \frac{e^{-x}}{\cancel{x}} \cdot \cancel{x}\, e^{-ax}\,dx$$

$$= \int_0^\infty e^{-x} e^{-ax}\,dx. = \int_0^\infty e^{-x(a+1)}\,dx.$$

$$\left[\frac{e^{-x(a+1)}}{-(a+1)}\right]_0^\infty = \frac{-1}{a+1}\left(e^{-x(a+1)}\right)_0^\infty$$

$$= \frac{1}{a+1} \;//$$

$$F'(a) = \frac{1}{a+1} \Rightarrow F(a) = \log(1+a) \;//$$

<u>Note</u>:- (1) Leibnitz rule enables is to derive from the value of a simple definite integral The value of another definite integral when it may otherwise be different (or) Even impossible to evaluate.

(2) The rule for differentation under the integral sign of an infinite integral is the same as for a definite integral.

(3) Leibnitz rule for variable limits of integration

If $f(x,\alpha)$ and $\frac{\partial f(x,\alpha)}{\partial \alpha}$ be continuous function of 'x' & 'α' then

$$\frac{d}{d\alpha}\left[\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right] = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx + \frac{d\psi}{d\alpha}\,f(\psi(\alpha),\alpha) - \frac{d\phi}{d\alpha}\,f(\phi(\alpha),\alpha)$$

Provided $\phi(\alpha)$ & $\psi(\alpha)$ posses continuous first order derivative w.r.t 'α'.

(1) Evaluate $\int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2}\,dx$ & S.T $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}$

Sol: let $f(\alpha) = \int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2}\,dx$ —— (1)

diffe Eq (1) w.r.t α we get

$$f'(\alpha) = \int_0^{\alpha} \frac{\partial}{\partial\alpha}\left(\frac{\log(1+\alpha x)}{1+x^2}\right)dx + \frac{d\alpha}{d\alpha}\,\frac{\log(1+\alpha^2)}{1+\alpha^2}$$

$$= \int_0^{\alpha} \frac{1}{1+x^2}\cdot\frac{x}{1+\alpha x}\,dx + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

Partial Fractions:

$$\frac{1}{1+x^2}\cdot\frac{x}{1+\alpha x} = \frac{A}{1+\alpha x} + \frac{Bx+C}{1+x^2}$$

$\Rightarrow x = A(1+x^2) + (Bx+C)(1+\alpha x)$

$\Rightarrow x = A + Ax^2 + Bx + B\alpha x^2 + C + C\alpha x$ $\qquad C\alpha + B = 1$

$\Rightarrow A + B\alpha = 0$; $B + C\alpha = 1$; ~~A = 0~~ $\qquad C = 0$

$A + C = 0$ $\qquad B = 1$

$B\alpha = C \Rightarrow \alpha = \frac{C}{B} \Rightarrow B + \frac{C^2}{B} = 1$

$C = B\alpha = \frac{\alpha}{1+\alpha^2}$ $\Rightarrow A = \frac{-\alpha}{1+\alpha^2}$

$$= \int_0^{\alpha} \frac{A}{1+\alpha x}\,dx + \int_0^{\alpha} \frac{Bx+C}{1+x^2}\,dx$$

$$= \frac{-\alpha}{1+\alpha^2}\int_0^{\alpha} \frac{1}{1+\alpha x}\,dx + \int_0^{\alpha} \frac{1x + (\alpha)}{(1+\alpha^2)(1+x^2)}\,dx$$

$$= \frac{-\alpha}{1+\alpha^2}\int_0^{\alpha} \frac{1}{1+\alpha x}\,dx + \frac{1}{1+\alpha^2}\int \frac{x}{(1+x^2)}\,dx$$

$$+ \int_0^{\alpha} \frac{\alpha}{1+x^2}\,dx \cdot \frac{1}{1+\alpha^2}$$

$$= \frac{-1}{1+\alpha^2}\left[\log(1+\alpha x)\right]_0^\alpha + \frac{1}{2(1+\alpha^2)}\left(\log(1+x^2)\right]_0^\alpha +$$

$$\frac{\alpha}{1+\alpha^2}\left(\tan^{-1}x\right)_0^\alpha$$

$$= \frac{-\log(1+\alpha^2)}{1+\alpha^2} + \frac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \frac{\tan^{-1}\alpha}{1+\alpha^2}$$

$$F(\alpha) = \int \frac{\log(1+\alpha^2)}{2(1+\alpha^2)}\,d\alpha + \int \frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}\,d\alpha$$

$$= \frac{1}{2}\left[\log(1+\alpha^2)\int\frac{1}{1+\alpha^2}\,d\alpha - \int\frac{1}{1+\alpha^2}\left(\int\frac{1}{1+\alpha^2}\,d\alpha\right)d\alpha\right] + \int\frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}\,d\alpha$$

$$= \frac{1}{2}\left[\log(1+\alpha^2)\tan^{-1}\alpha - \int\frac{2\alpha\tan^{-1}\alpha}{1+\alpha^2}\,d\alpha\right] + \int\frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}\,d\alpha$$

$$= \frac{1}{2}\log(1+\alpha^2)\tan^{-1}\alpha + C$$

(1) ~~By~~

<u>Scalar potential</u> : In an irrotational field for which $\nabla\times F = 0$. The vector F can always be expressed as the gradient the scalar function ϕ.

i.e., $F = \nabla\phi$ Such a scalar function ϕ is called potential function SP

(1) A vector field is given by $F = (x^2 - y^2 + x)i - (2xy + y)j$ ~~where~~ Show that field is irrotational, & find its Scalar potential.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2+x & -2xy-y & 0 \end{vmatrix}$$

$$= i(0) - j(0) + k(-2y+2y) = \underline{\underline{0}}$$

Thus F is irrotational.

$$F = \nabla\phi \Rightarrow (x^2-y^2+x)i - (2xy+y)j = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = x^2-y^2+x$$

$$\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} + g(y) \quad — ①$$

$$\frac{\partial\phi}{\partial y} = -2xy - y$$

$$\phi = -\frac{\cancel{2}xy^2}{\cancel{2}} - \frac{y^2}{2} + f(x)$$

$$= -xy^2 - \frac{y^2}{2} + f(x) \quad — ②$$

$$\frac{x^3}{3} - \cancel{xy^2} + \frac{x^2}{2} + g(y) = -\cancel{xy^2} - \frac{y^2}{2} + f(x)$$

$$\frac{x^3}{3} + \frac{x^2}{2} + g(y) = -\frac{y^2}{2} + f(x)$$

$$g(y) = -\frac{y^2}{2} \qquad f(x) = \frac{x^3}{3} + \frac{x^2}{2}$$

$$\boxed{\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2}}$$

sub in ① or ②
We get same

① Determine whether $\bar{F} = (y^2\cos x + z^3)i + (2y\sin x - 4)j + (3xz^2 + 2)k$

is conservative vector field if so find scalar potential

Sol:- F is conservative when $\nabla \times F = 0$.

② If S is an closed surface enclosing in volume V

and $F = axi + byj + czk$ then prove that $\iint_S F \cdot n \, ds = (a+b+c)V$.

(3)