CHAPTER 1 Functions of A Complex Variable

1.1 INTRODUCTION

We are familiar with the concepts of limit, differentiation and integration concerning the functions of real variables. Similar concepts can be defined with reference to complex variables also and their study constitutes "complex analysis". Here we will define functions of a complex variable and then introduce the concepts of limits, continuity and differentiability for those functions.

A basic understanding of complex variable theory will be useful in diverse branches of science and engineering.

1.2 FUNCTIONS OF COMPLEX VARIABLE

Def. Suppose D is a set of complex numbers. A rule f defined on D which assigns to every z in D, a complex number w, is called a function f or mapping f on D and we write $w = f(z)$.

Here z is a complex variable and can be written as $z = x + iy$ where x, y, are real and $i = \sqrt{-1}$.

The set D is called **domain** of definition of f. The set of all $w = f(z)$ where $z \in D$ is called the range of f.

The image of z under the function f is $w = f(z)$ and as stated, this is also a complex number.

We write $w = f(z) = u + iv$ where u and v are real.

Since $z = x + iy$ and z depends on x and y, we notice that u and v also depend on x and y so that $u = u(x, y)$ and $v = v(x, y)$.

Hence, in general, we write $w = f(z) = u(x, y) + iv(x, y)$. These u and v are called the real and imaginary parts of $w = f(z)$.

SOLVED EXAMPLES

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Solution : $w = f(z) = z^2 + z = (x+iy)^2 + (x+iy)$

$$= \left(x^2 - y^2 + 2ixy\right) + (x+iy) = \left(x^2 - y^2 + x\right) + i(2xy + y) = u + iv \text{ (say)}$$

Then $u = x^2 - y^2 + x$ and $v = 2xy + y$

and $f(1+i) = (1+i)^2 + (1+i) = 1 + 3i$

Example 3 : Let $w = f(z) = iz + 3\bar{z}$. Find u and v and the value of $f(z)$ at $1 + 4i$.

Solution : $w = f(z) = i(x+iy) + 3(x-iy) \quad (\because \ \bar{z} = x - iy)$

$$= (3x - y) + i(x - 3y) = u + iv \text{ (say)}$$

Equating real and imaginary parts, we get $u = 3x - y$ and $v = x - 3y$

$\therefore \quad f(1+4i) = i(1+4i) + 3(1-4i) = i - 4 + 3 - 12i = -1 - 11i$

Note : The locus of z such that $|z - z_0| = R$ is a circle with centre z_0 and radius R in the z-plane

$\therefore \ \{z : |z - z_0| < R\} =$ The set of all points of z within the circle with centre z_0 and radius R in the z-plane.

1.3 ε - DISC AROUND $w = w_0$ IN THE COMPLEX w-PLANE:

$z \to z_0$

Let us represent all the complex numbers $w = u + iv$ where u, v are real on a [illegible] of cartesian (u, v) coordinate plane. This is called the w-plane or

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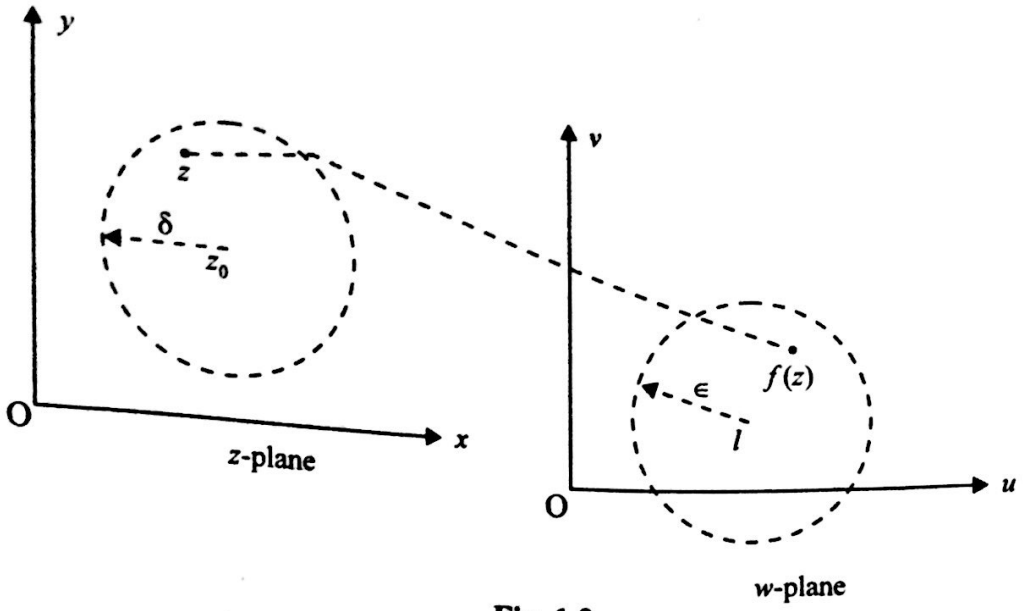

Fig. 1.2

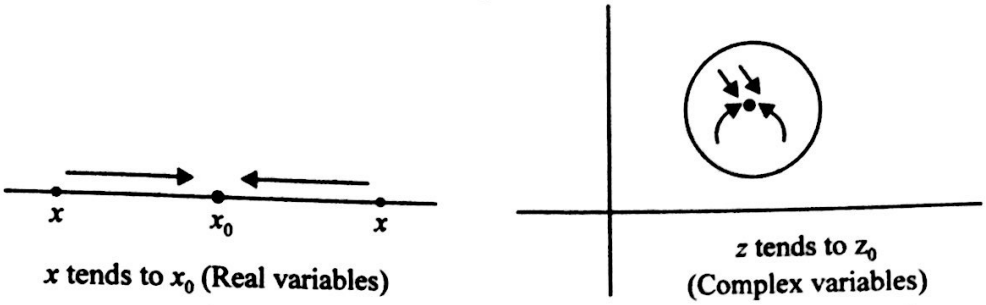

Fig. 1.3

Note : Though this definition is similar to the definition of limits, in the case of real number theory there is a big difference. In real case x can approach x_0 only along the real line either from its left or from its right, whereas in the above definition z may approach z_0 from any direction in the complex plane (fig. 1.3).

1.5 THEOREMS ON LIMITS

1. If the limit of a function exists as $z \to z_0$, then it is unique.

   Note : For $\underset{z \to z_0}{\text{Lt}} f(z) = l$, it is imperative that $f(z)$ must be defined at all points z in a disc around z_0. But it is not required that $f(z_0)$ is defined.

2. Let $f(z) = u(x, y) + i\,v(x, y)$, $z = x + iy$, $z_0 = x_0 + iy_0$, then the condition that limit of $f(z)$ exists at z_0, $\underset{z \to z_0}{Lt} f(z) = u_0 + iv_0$ is satisfied if and only if $\underset{\substack{x \to x_0 \\ y \to y_0}}{Lt} u(x, y) = u_0$ and $\underset{\substack{x \to x_0 \\ y \to y_0}}{Lt} v(x,y) = v_0$.

3. Let f, F be functions whose limit exist at z_0 and $\underset{z \to z_0}{Lt} f(z) = W_0$ and $\underset{z \to 0}{Lt} F(z) = w_0$.

   Then (*i*) $\underset{z \to z_0}{Lt} [f(z) + F(z)] = W_0 + w_0$ (*ii*) $\underset{z \to z_0}{Lt} [f(z) \cdot F(z)] = W_0 w_0$

   (*iii*) $\underset{z \to z_0}{Lt} \left[\dfrac{f(z)}{F(z)}\right] = \dfrac{W_0}{w_0}, w_0 \neq 0$ (*iv*) $\underset{z \to z_0}{Lt}$ C $f(z)$ = C.W_0.

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SOLVED EXAMPLES

Example 1 : Using the definition of the limit, prove that $\underset{z\to 1}{Lt}\dfrac{z^2-1}{z-1}=2$.

Solution : Let $f(z)=\dfrac{z^2-1}{z-1}$

This function is not defined when $z=1$.

When $z\neq 1, f(z)=z+1$. Thus $|f(z)-2|=|z+1-2|=|z-1|$

$\therefore\quad |f(z)-2|<\varepsilon$ whenever $|z-1|<\varepsilon$

Taking $\delta=\varepsilon$, the condition for limit is satisfied for every $\varepsilon>0$.

$$\therefore\quad \underset{z\to 1}{Lt}\frac{z^2-1}{z-1}=2$$

Example 2 : Let $f(z)=\begin{cases} z^2, & z\neq i \\ 0, & z=i\end{cases}$. Then

$\underset{z\to i}{Lt} f(z)=-1$. But this limit is not equal to $f(0)$.

Also, even $f(0)$ is not defined, we can still have $\underset{z\to i}{Lt} f(z)=-1$

Example 3 : Show that $\underset{z\to z_0}{Lt} z=z_0$.

Solution : Let $f(z)=z$

$|f(z)-z_0|=|z-z_0|$; Let $\varepsilon>0$.

If we take $\delta=\varepsilon$, we get that

$|f(z)-z_0|<\varepsilon$ when $|z-z_0|<\delta$.

Thus by definition the result follows.

Example 4 : Limit of a constant function is that constant.

Solution : Let $f(z)=K$, K is a constant. Then

$|f(z)-K|=|K-K|=0<\varepsilon$.

This is true for all $\delta>0$.

$\therefore\quad |z-K|<\delta\Rightarrow|f(z)-K|<\varepsilon$

$\therefore\quad \underset{z\to z_0}{Lt} f(z)=K.$

Example 5 : Show that $\underset{z\to 2i}{Lt}(2x+iy^2)=4i$.

Solution : Consider $f(z)=2x+iy^2$

$$\begin{aligned}|f(z)-4i|&=|2x+iy^2-4i|\\ &\le|2x|+|y^2-4|\\ &=2|x|+|y-2||y+2|\end{aligned}$$

Let $\varepsilon>0$ and $|x|<\varepsilon/4, |y-2|<1$

$$\Rightarrow\quad 2|x|<\frac{\varepsilon}{2}, |y+2|=|y-2+4|\le|y-2|+4<5$$

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Example 6 : Prove that $\underset{z \to z_0}{Lt} z^n = z_0^n$ where z_0 is any complex number.

Solution : We have from theorem (3), $\underset{z \to z_0}{Lt} z^n = z_0^n$ by induction.

Example 7 : The limit of a polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ is the value of the polynomial at z_0, $\forall z_0$. That is, $\underset{z \to z_0}{Lt} P(z) = P(z_0)$.

Solution : We know that $\underset{z \to z_0}{Lt} z^n = z_0^n$ and $\underset{z \to z_0}{Lt} K = K$.

Also, $\underset{z \to z_0}{Lt} [f(z) + F(z)] = \underset{z \to z_0}{Lt} f(z) + \underset{z \to z_0}{Lt} F(z)$

Using this theorem, we get $\underset{z \to z_0}{Lt} P(z) = P(z_0)$

1.6 CONTINUITY

A function $f(z)$ is said to be ***continuous at*** $z = z_0$, if $f(z_0)$ is defined and $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$.

We observe that for the function $f(z)$ to be continuous at z_0 the function must be defined in some neighbourhood of z_0 including z_0 and $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$.

(or) A function f is continuous at a point z_0, if, corresponding to each positive number t, a number δ exists, such that $|f(z) - f(z_0)| < t$ whenever $|z - z_0| < \delta$.

Def. $f(z)$ is said to be *continuous in a domain* if it is continuous at every point of that domain.

Theorem : If $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0 = x_0 + iy_0$, then u and v are also continuous at $z = z_0$, *i.e.*, at (x_0, y_0). Conversely, if $u(x, y)$ and $v(x, y)$ are both continuous at (x_0, y_0) then $f(z)$ will be continuous at $z = z_0$.

Properties of Continuous Functions :

In view of the above theorem, the properties of continuous functions of z can be deduced from the properties of continuous functions u and v of x and y.

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Example 1 : Show that the function $f(z) = \frac{z}{z}$ is not continuous at $z = 0$.

Solution : Let $z = x + iy$

Suppose $z \to 0$ along x –axis. Then we have $y = 0$, $z = x$ and $\bar{z} = x$.

$$\therefore \underset{z \to 0}{Lt} \frac{\bar{z}}{z} = \underset{x \to 0}{Lt} \frac{x}{x} = 1$$

Again suppose $z \to 0$ along y–axis. Then $x = 0$, $z = iy$ and $\bar{z} = -iy$

$$\text{Now } \underset{z \to 0}{Lt} \frac{\bar{z}}{z} = \underset{iy \to 0}{Lt} \left(\frac{-iy}{iy} \right) = -1$$

Thus $f(z)$ tends to different values as $z \to 0$ along different directions.

$$\therefore \underset{z \to 0}{Lt} \frac{\bar{z}}{z} \text{ does not exist.}$$

Example 2 : Show that the function $f(z) = \bar{z}$ is continuous over C.

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Example 5 : Is the function $f(z) = \begin{cases} \dfrac{z^2 + 3iz - 2}{z + i} & \text{for } z \neq i \\ 5, & \text{for } z = -i \end{cases}$ continuous. If not can the function be redefined to make it continuous at $z = -i$?

Solution : $f(z) = \dfrac{g(z)}{h(z)}$ is continuous when $g(z)$ and $h(z)$ are continuous except at $h(z) = 0$.

So $f(z)$ is continuous every where except at $z = -i$, since $g(z)$, $h(z)$ are continuous.

Continuity at $z = -i$

$$\underset{z \to -i}{Lt} f(z) = \underset{z \to 0}{Lt} \frac{(x+iy)^2 + 3i(x+iy) - 2}{}$$

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Example 2 : Find the derivative of $w = f(z) = z^3 - 2z$ at the point where (*i*) $z = z_0$ (*ii*) $z = 1$.

Solution : (*i*) We have

$$f'(z_0) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\Rightarrow f'(z_0) = \underset{\Delta z \to 0}{\text{Lt}} \frac{(z_0 + \Delta z)^3 - 2(z_0 + \Delta z) - (z_0^3 - 2z_0)}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \frac{z_0^3 + (\Delta z)^3 + 3z_0^2(\Delta z) + 3z_0(\Delta z)^2 - 2z_0 - 2\Delta z - z_0^3 + 2z_0}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} [3z_0^2 + 3z_0\,\Delta z + (\Delta z)^2 - 2] = 3z_0^2 - 2$$

In general, $f'(z) = 3z^2 - 2, \forall z$.

(*ii*) Substituting $z_0 = 1$, we get $f'(1) = 3 - 2 = 1$.

Example 3 : Show that $f(z) = \bar{z}$ is not differentiable any where.

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Example 4 : Show that the function $f(z) = z^n$, where n is a positive integer is differentiable for all values of z, where n is a positive integer. [JNTU 2002, 2003]

Solution : Using the definition of the derivative, we have

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \underset{\Delta z \to 0}{\text{Lt}} \frac{(z+\Delta z)^n - z^n}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \frac{z^n + nz^{n-1}\Delta z + \frac{n(n-1)}{2} z^{n-2} (\Delta z)^2 + \ldots + (\Delta z)^n - z^n}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \left[nz^{n-1} + \frac{n(n-1)}{2} z^{n-2} (\Delta z) + \ldots + (\Delta z)^{n-1} \right] = nz^{n-1}$$

Hence $f'(z)$ exists for all values of z.

1.8 THEOREM

If a function is differentiable at a point, then it is continuous there.

Proof. Let $f(z)$ be differentiable at z_0. Then

$$f'(z_0) = \underset{z \to z_0}{\text{Lt}} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

$\therefore$ $f(z_0)$ is well defined.

$$\text{Let } \underset{z \to z_0}{\text{Lt}} [f(z) - f(z_0)] = \underset{z \to z_0}{\text{Lt}} \left[\frac{f(z) - f(z_0)}{z - z_0} . (z - z_0) \right] = \underset{z \to z_0}{\text{Lt}} \left[\frac{f(z) - f(z_0)}{z - z_0} \right] \underset{z \to z_0}{\text{Lt}} (z - z_0)$$

$$= f'(z_0) . \underset{z \to z_0}{\text{Lt}} (z - z_0) = 0$$

$$\therefore \underset{z \to z_0}{\text{Lt}} f(z) = \underset{z \to z_0}{\text{Lt}} f(z_0) = f(z_0)$$

Hence $f(z)$ is continuous at z_0.

Note : The converse of the above theorem is not true. That is, a function can be continuous a point, but not differentiable at that point.

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Case I. Let $\Delta z \to 0$ along x-axis. Then $\Delta x = \Delta x$, $\Delta y = 0$ and $\Delta z = \Delta x$

$$\therefore \quad \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z} = \underset{\Delta x \to 0}{\text{Lt}} \frac{z_0 \Delta x + \Delta x\, \bar{z}_0 + \Delta x \cdot \Delta x}{\Delta x} = z_0 + \bar{z}_0 \qquad \text{...(1)}$$

Case II. Let $\Delta z \to 0$ along y-axis.

Then $\Delta x = 0$; $\Delta y = \Delta y$; $\Delta z = i\Delta y$

$$\underset{\Delta z \to 0}{\text{Lt}} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z} = \underset{\Delta y \to 0}{\text{Lt}} \frac{z_0(-i\Delta y) + i\Delta y\, \bar{z}_0 + (i\Delta y)(-i\Delta y)}{i\Delta y} = -z_0 + \bar{z}_0 \qquad \text{...(2)}$$

Thus, from (1) and (2), for $f'(z_0)$ to exist $z_0 + \bar{z}_0 = -z_0 + \bar{z}_0$.

i.e., $z_0 = -z_0$ *i.e.*, $2z_0 = 0$ *i.e.*, $z_0 \neq 0$.

$\therefore$ $f'(z)$ does not exist though $f(z) = |z|^2$ is continuous at all z.

1.9 RULES FOR DIFFERENTIATION

The rules of differentiation for functions of complex variables are similar to those of functions of real variables.

Theorem: If $f(z)$ and $g(z)$ are differentiable functions in a domain D, then their sum, product and quotient are also differentiable and we have

(*i*) $\dfrac{d}{dz}[f(z) \pm g(z)] = \dfrac{d}{dz}[f(z)] \pm \dfrac{d}{dz}[g(z)]$

(*ii*) $\dfrac{d}{dz}[cf(z)] = c \cdot \dfrac{d}{dz}[f(z)]$, where '$c$' is a constant.

(*iii*) $\dfrac{d}{dz}[f(z) \cdot g(z)] = f(z) \cdot \dfrac{d}{dz}[g(z)] + g(z) \cdot \dfrac{d}{dz}[f(z)]$

(*iv*) $\dfrac{d}{dz}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{\dfrac{d}{dz}[f(z)] \cdot g(z) - \dfrac{d}{dz}[g(z)] \cdot f(z)}{[g(z)]^2}$, $[g(z) \neq 0]$

(*v*) If $f(z) = F[g(z)]$, then $\dfrac{d}{dz}[f(z)] = F'[g(z)] \cdot g'(z)$ **(chain rule)**

1.10 ANALYTIC FUNCTIONS

[JNTU 2004, Nov. 2006, 2007S, Nov.2008S (Set No.2,3)]

Def. Let a function $f(z)$ be derivable at every point z in an ε neighbourhood of z_0.

i.e., $f'(z)$ exists for all z such that $|z - z_0| < \varepsilon$ where $\varepsilon > 0$. Then, $f(z)$ is said to be analytic at z_0.

Note : If $f(z)$ is analytic at z_0,

(*i*) $f'(z_0)$ exists and (*ii*) $f'(z)$ exists at every point z in a neighbourhood of z_0.

1. 11 DEFINITION

Let D be the domain of complex numbers.

If $f(z)$ is analytic at every $z \in D$, $f(z)$ is said to be analytic in the domain D.

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If $f(z)$ is analytic at every point z on the complex plane, $f(z)$ is said to be an ***entire function*** or ***integral function***.

That is, a point at which an analytic function ceases to have a derivative is called a **singular point**. An analytic function is also known as **regular** or **holomorphic**.

If $f'(z_0)$ does not exist then $z = z_0$ is called a **singular point** of $f(z)$.

If $f'(z)$ exists at every point in a neighbourhood of z_0 but $f'(z_0)$ does not exist, then z_0 is said to be an ***isolated singular point*** of $f(z)$.

e. g. $f(z) = \frac{1}{z}$ is analytic at every point $z \neq 0$.

In fact $f'(z) = -\frac{1}{z^2}$ if $z \neq 0$.

At $z = 0$, $f'(z)$ does not exist.

So $z = 0$ is an isolated singular point of $f(z)$.

1.12 POLYNOMIALS AND RATIONAL FUNCTIONS

We know that $f(z) = z^n$ is differentiable for all z.

Applying the above formula for $z, z^2, z^3, \ldots$, and the rules of differentiation, we can see that a "polynomial" $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ is an analytic function of z.

More generally, a "rational function" $f(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m}$ is an analytic function of z throughout any finite domain in the complex plane where the denominator is not zero.

Also "partial fractions" $\frac{c}{(z - z_0)^m} (c \neq 0)$ where c and z_0 are complex numbers, m is a positive integer are special rational functions. They are analytic every where except at z_0.

1.13 CAUCHY-RIEMANN (C-R) EQUATIONS [JNTU 2003, 2008]

Having defined analytical function, we will prove a very important result to test the analyticity of a complex function.

Theorem : A necessary and sufficient condition for the derivative of the function $f(z) = w = u(x, y) + iv(x, y)$ to exist for all values of z in domain R are

(*i*) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in R.

(*ii*) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ **[JNTU 2009S, (A) Nov. 2009 (Set No. 1)]**

The above relations are known as **Cauchy–Riemann equations.**

(or) Derive the necessary and sufficient condition for $f(z)$ to be analytic in cartesian co-ordinates.

Proof : Necessity.

[JNTU 1998, 1996, 2002S, Nov. 2006, 2007S, Nov. 2008S,Nov. 2008 (Set No. 4)]

If $f(z) = u + iv$ is analytic in domain R, then u and v satisfy the equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ provided the partial derivatives exist.

In order for $f(z)$ to be analytic, the limit

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$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \left[\frac{\quad}{\Delta x + i\Delta y} + i \frac{\quad}{\Delta x + i\Delta y} \right]$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right] = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

[u_x, v_x exist as $f'(z)$ exists]

Case II. Let $\Delta z \to 0$ along y-axis. Take $\Delta z = i\Delta y$.

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \left[\frac{u(x+\Delta x, y+\Delta y) - u(x, y)}{\Delta x + i\Delta y} + i \frac{v(x+\Delta x, y+\Delta y) - v(x, y)}{\Delta x + i\Delta y} \right]$$

$$= \text{Lt} \left[\frac{u(x, y+\Delta y) - u(x, y)}{\quad} + i \frac{v(x, y+\Delta y) - v(x, y)}{\quad} \right]$$

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$$= \{u(x+\Delta x,\ y+\Delta y) - u(x,\ y+\Delta y)\} + \{u(x,\ y+\Delta y) - u(x,\ y)\}$$

$$= \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1 \Delta x + \eta_1 \Delta y$$

where $\varepsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

Similarly, we have $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous.

$$\Delta v = \left(\frac{\partial v}{\partial x} + \varepsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_2 \Delta x + \eta_2\ \Delta y$$

where $\varepsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. Then

$$\Delta w = \Delta u + i\Delta v = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \varepsilon \Delta x + \eta \Delta y \qquad ...(2)$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

By Cauchy–Riemann equations, we can write (2) as

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \varepsilon \Delta x + \eta \Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \varepsilon \Delta x + \eta \Delta y$$

Then on dividing by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \to 0$, we see that

$$\frac{dw}{dz} = f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

So that the derivative exists and is unique.

i.e., $f(z)$ is analytic in R. Hence the theorem.

Remember :

1. If a function is analytic in a domain D, then u, v satisfy C-R equations at all points in D.

2. C - R conditions are necessary but not sufficient analyticity of $f(z)$.

3. For $f(z)$ $f(z) = u(x, y) + iv(x, y)$ to be analytic in a domain continuity of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ are sufficient conditions.

PROPERTIES OF ANALYTIC FUNCTIONS

1. If $f(z)$ and $g(z)$ are analytic functions, then $f \pm g$, $f.\ g$ and $\frac{f}{g}$ are also analytic functions, provided $g(z) \neq 0$.

2. Analytic function of an analytic function is analytic.

3. An entire function of an entire function is entire.

4. Derivative of an analytic function is itself analytic.

1.14 MAXIMUM - MINIMUM PRINCIPLE IN COMPLEX VARIABLE

(This is also referred to as maxi - mini principle)

Let $f(z)$ be a non constant function which is continuous everywhere in a closed bounded region R bounded by a simple closed curve C. Let $f(z)$ be analytic in the region R. Then the maximum value of $|f(z)|$ in R will occur at some point z_0 on the boundary C of R.

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Example 1 : Verify maximum - minimum principle for $f(z) = e^z$ for the region $|z| \le 1$.

Solution : $f(z) = e^z$ is analytic every where on the complex plane and hence, in particu within the curve $c : |z| = 1$. Further $e^z \neq 0$ for any z in the region.

Hence maximum - minimum principle can be applied for $f(z)$ in the region.

Consider $|f(z)| = |e^z| = |e^{x+iy}|$

$$= |e^x . e^{iy}| = |e^x||e^{iy}|$$

$$= e^x |\cos y + i \sin y|$$

$$= e^x . 1 = e^x$$

We know that $e^x > 0$ for all real x. Within the region, under consideration, e^x attains maximum value at $x = 1$ and minimum value at $x = -1$. At $z = (1,0)$, $|f(z)|$ attains its maximum val e and $(1,0)$ is on C. At $z = (-1,0)$, $|f(z)|$ attains its minimum value e^{-1} and $(-1,0)$ is on C.

Example 2 : Verify maximum - minimum principle for $f(z) = e^z$ in the region $|z - 2 - 2i| \le 1$

(This is left as an exercise to the reader)

Example 3 : Verify maximum - minimum principle for $f(z) = z$ in the region $|z - 1 - i| \le 1$.

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The curve $C : |z-1-i| = 1$ is a circle with centre at A (1,1) and having radius 1.

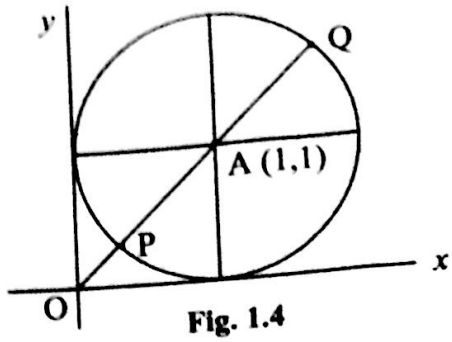

Fig. 1.4

$|f(z)| = |z| = \sqrt{x^2+y^2}$ where $z = (x, y)$.

Join OA. Let the line OA meet the circle C at P and Q (as in figure)

$|z|$ is the distance 'd' of the point (x, y) from the origin O (0,0). Geometrically, we note that, OP is minimum of $|z|$ and OQ is maximum of $|z|$ and hence minimum and maximum of $|z|$ are attained at points on C. Hence the principle is verified.

1.15 HARMONIC FUNCTIONS - LAPLACE EQUATION

Theorem : If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v satisfy Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, respectively in D, and have continuous second order partial derivatives in D.

Proof. We have by Cauchy – Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ...(1) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad ...(2)$$

Differentiating (1) with respect to x and (2) with respect to y partially, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad ...(3) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad ...(4)$$

Since the derivative of an analytic function is also analytic and second order derivatives

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

(3) + (4) gives $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *i.e.* $\nabla^2 u = 0$, where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and ∇^2 is called the **Laplacian operator.**

Again differentiating (1) with respect to y, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$ ...(5)

Again differentiating (2) with respect to x, $\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$ ...(6)

(5) + (6) gives $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ *i.e.* $\nabla^2 v = 0$

Hence the theorem.

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1.16 HARMONIC FUNCTIONS

Solutions of Laplace equations having continuous second order partial derivatives are called **Harmonic functions.** Their theory is called **Potential theory.** Hence, the real and imaginary parts of an analytic function are harmonic functions.

Thus the functions satisfying the Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0$$

are known as **Harmonic functions.**

1.17 CONJUGATE HARMONIC FUNCTION

If two harmonic functions u and v satisfy the Cauchy–Riemann equations in a domain D and they are the real and imaginary parts of an analytic function f in D then v is said to be a conjugate Harmonic function of u in D. Two harmonic functions, u and v which are such that $u + iv$ is an analytic function are called conjugate harmonic functions. In other words, if $f(z) = u + iv$ is analytic and if u and v satisfy Laplace's equation, then u and v are called conjugate harmonic functions.

1.18 POLAR FORM OF CAUCHY–RIEMANN EQUATIONS

[JNTU 1998 Sept., 2001S, 2003, 2005, (H) Nov. 2009 (Set No. 1)]

If $f(z)=f\left(re^{i\theta}\right)=u(r,\theta)+iv(r,\theta)$ and $f(z)$ is derivable at $z_0=r_0e^{i\theta_0}$ then

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof : Let $z=r\,e^{i\theta}$. Then $f(z)=u(r,\theta)+iv(r,\theta)$

Differentiating it with respect to r partially,

$$\frac{\partial}{\partial r}f(z)=f'(z)\frac{\partial z}{\partial r}=f'(z)\,e^{i\theta}$$

$$\therefore \quad f'(z)=\frac{1}{e^{i\theta}}\frac{\partial f}{\partial r}=\frac{1}{e^{i\theta}}\left(u_r+iv_r\right) \quad ...(1)$$

Similarly, differentiating partially with respect to 'θ',

$$\frac{\partial f}{\partial \theta}=f'(z)\frac{\partial z}{\partial \theta}=f'(z).\,r\,ie^{i\theta}$$

$$\therefore \quad f'(z)=\frac{1}{r\,ie^{i\theta}}\left(u_\theta+iv_\theta\right) \quad ...(2)$$

From (1) and (2), we have

$$\frac{1}{e^{i\theta}}\left(u_r+iv_r\right)=\frac{1}{r\,ie^{i\theta}}\left(u_\theta+iv_\theta\right)$$

$$\therefore \quad u_r+iv_r=\frac{1}{r}\frac{\partial v}{\partial \theta}-i\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Equating real and imaginary parts, we get

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$$\boxed{\begin{array}{c} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \hline \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{array}}$$

Hence, the result follows.

These are called **Cauchy–Riemann** equations in polar form.

Corollary. If $f''(z)$ exists

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

and $$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$ **[JNTU Aug. 2006S (Set No. 1)]**

Proof : We have $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ...(1)$$

and $$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \quad ...(2)$$

Diff. (1) partially w.r.t. 'r', we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta}$$

From (1), we have $\dfrac{1}{r}\dfrac{\partial u}{\partial r} = \dfrac{1}{r^2}\dfrac{\partial v}{\partial \theta}$

Diff. (2) partially w.r.t. 'θ' and multiplying with $\dfrac{1}{r}$, we get $\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = \dfrac{1}{r}\left(-\dfrac{\partial^2 v}{\partial r \partial \theta}\right)$

Adding the above three conditions, we get $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$

Similarly, we can prove that $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$

Note : If $f'(z)$ exists (and also $[f''(z)]$)

(*i*) u, v satisfy Laplace's equations.

1.19 ORTHOGONAL TRAJECTORIES

Two curves intersecting at a point P are said to intersect orthogonally, if their tangents at P are perpendicular to one another. (Two families of curves, or trajectories are said to be orthogonal to one [illegible] each curve of the other family orthogonally). Orthogo-

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Solution : $f(z_0) = x_0 y_0 + i y_0$ is well defined for any $z_0 = x_0 + i y_0$.

$$\underset{z \to z_0}{\text{Lt}} f(z) = \underset{z \to z_0}{\text{Lt}} (xy + iy)$$

$$= \underset{\substack{x \to x_0 \\ y \to y_0}}{\text{Lt}} (xy + iy) = x_0 y_0 + i y_0 = f(z_0)$$

$$= \underset{\substack{y \to y_0 \\ x \to x_0}}{\text{Lt}} (x_0 y_0 + i y_0) = f(z_0)$$

$\therefore$ f is continuous everywhere and $f(z) = u + iv = xy + iy$

So $u = xy,\ v = y$

Then $\quad u_x = y,\ u_y = x,\ v_x = 0,\ v_y = 1$

Since Cauchy – Riemann conditions are not satisfied

$\therefore$ f is not analytic.

Example 2 : Show that $f(z) = z^3$ is analytic for all z.

Solution : Let $z = x + iy \Rightarrow z^3 = (x+iy)^3 = \left(x^3 - 3xy^2\right) + i\left(3x^2 y - y^3\right)$

If $z^3 = u + iv$, then we have $u = x^3 - 3xy^2,\ v = 3x^2 y - y^3$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2,\ \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial y} = -6xy,\ \frac{\partial v}{\partial x} = 6xy$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy–Riemann equations are satisfied and the partial derivatives are continuous. Hence $f(z) = z^3$ is analytic.

Example 3 : Show that z^2 is analytic for all z.

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Example 4 : (i) If $w = \log z$, find $\frac{dw}{dz}$ and determine where w is non analytic.

(ii) Show that the real and imaginary parts of the function $w = \log z$ satisfy the C-R equations when z is not zero.

[JNTU (H) Nov. 2010, May 2011 (Set No. 4)]

Solution : Let $w = u + iv$

We have $z = x + iy = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$

$$\therefore \quad \log(x + iy) = \log(re^{i\theta}) = \log r + i\theta = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}(y/x)$$

$$\therefore \quad w = u + iv = \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}(y/x) \text{ so that}$$

$$u = \frac{1}{2}\log(x^2 + y^2),\ v = \tan^{-1}(y/x)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2},\ \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2},\ \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2} \text{ and } \frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2}$$

The C-R equations are satisfied. Also the partial derivatives are continuous except at (0, 0).

$\therefore \quad w = \log z$ is analytic everywhere except at $z = 0$.

Hence $\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{x + iy} = \frac{1}{z}$ $(z \neq 0)$

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Solution : Let $f(z) = u + iv = \dfrac{x}{x^2+y^2} + i\dfrac{-y}{x^2+y^2}$

So $u = \dfrac{x}{x^2+y^2}$ and $v = \dfrac{-y}{x^2+y^2}$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x^2+y^2).1 - x(2x+0)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$, using Quotient Rule

$$\frac{\partial v}{\partial x} = (-y).\frac{-1}{(x^2+y^2)^2}(2x) = \frac{2xy}{(x^2+y^2)^2} \quad [\because y \text{ is constant}]$$

$$\frac{\partial u}{\partial y} = x.\frac{-1}{(x^2+y^2)^2}(2y) = \frac{-2xy}{(x^2+y^2)^2} \quad [\because x \text{ is constant}]$$

and $$\frac{\partial v}{\partial y} = \frac{-[(x^2+y^2).1 - y(0+2y)]}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$, using Quotient Rule

Here we see that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x}$

These are Cauchy- Riemann equations and are satisfied.

Hence $f(z)$ is analytic.

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Similarly, $\dfrac{\partial^2}{\partial y^2}(u^2) = 2\left[\left(\dfrac{\partial u}{\partial y}\right)^2 + u\dfrac{\partial^2 u}{\partial y^2}\right]$ ... (3)

(2) + (3) gives

$$\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] \quad \text{... (4)}$$

Since u is real part of $f(z)$, therefore it satisfies the Laplace's equation.

Hence $\nabla^2 u = 0$ *i.e.* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ... (5)

From (4) and (5), we have

$$\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] = 2(u_x^2 + u_y^2) \text{ [by (1)]} = 2|f'(z)|^2$$

or $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2 = 2|f'(z)|^2$

or $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|\text{R } f(z)|^2 = 2|f'(z)|^2$

Hence the result.

Example 9 : Show that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$, where $f(z)$ is an analytic function.

[JNTU 2002, (A) Nov. 2010 (Set No. 4)]

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Example 10 : (*i*) Prove that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\,\partial \bar{z}}$

(*ii*) If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$

[JNTU (K) Nov. 2009 (Set No. 2), (H) Nov. 2012, (H) May 201

Solution : (*i*) Let $z=x+iy$. Then $\bar{z}=x-iy$

$$\therefore\ x=\frac{z+\bar{z}}{2} \text{ and } y=\frac{z-\bar{z}}{2i}=\frac{-i}{2}(z-\bar{z})$$

Now $\frac{\partial x}{\partial z}=\frac{1}{2}, \frac{\partial x}{\partial \bar{z}}=\frac{1}{2}$ and $\frac{\partial y}{\partial z}=\frac{-i}{2}, \frac{\partial y}{\partial \bar{z}}=\frac{i}{2}$

Let $f=f(x,y)$. Then $f=f(z,\bar{z})$

We know that $\frac{\partial f}{\partial z}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z}=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)f$

Similarly $\frac{\partial f}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)$

$$\therefore\ \frac{\partial^2 f}{\partial z\,\partial \bar{z}}=\frac{\partial}{\partial z}\frac{\partial f}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\cdot\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f$$

$$\Rightarrow\ \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

(*ii*) We know that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\,\partial \bar{z}}$

$$\therefore\ \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^2$$

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Solution : Let $f(z) = u(x, y) + iv(x, y)$ is a regular function. Then

$$|f(z)|^2 = u^2 + v^2 = \phi(x, y) \text{ (say)}$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} \text{ and } \frac{\partial \phi}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$$

and
$$\frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly,
$$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

Now
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right]$$
$$+ 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad \text{...(1)}$$

Since u and v have to satisfy Cauchy-Riemann equations and the Laplace equation, we have

$$\left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2, \left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2 \quad \text{... (2)}$$

and
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Thus,
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad \text{[using (2) in (1)]}$$

Hence,
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

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Solution : We know that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\,\partial \bar{z}}$ (Refer Ex. 10)

Also $|f(z)|^2=u^2+v^2=f(z)\overline{f(z)}$ ... (1)

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p=4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^p=4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)\overline{f(z)}\right]^{p/2}, \text{ [by (1)]}$$

$$=4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\{f(z)\}^{p/2}\{\overline{f(z)}\}^{p/2}\right]$$

Differentiating first w.r.t. $\bar{z}$ taking $f(z)$ as constant and then w.r.t. z taking $f(\bar{z})$ as const

we have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p=4\frac{\partial}{\partial z}\left[\{f(z)\}^{p/2}\cdot\frac{p}{2}\{f(\bar{z})\}^{\frac{p}{2}-1}\overline{f'(z)}\right]$$

$$=4\left[\frac{p}{2}\{f(z)\}^{\frac{p}{2}-1}f'(z)\cdot\frac{p}{2}\{f(\bar{z})\}^{\frac{p}{2}-1}\overline{f'(z)}\right]$$

$$=4\cdot\frac{p^2}{4}\left[\{f(z)\,f(\bar{z})\}^{\frac{p-2}{2}}f'(z)\,\overline{f'(z)}\right]$$

$$=p^2\left[\{|f(z)|^2\}^{\frac{p-2}{2}}|f'(z)|^2\right], \text{ [From (1)]}$$

$$=p^2|f(z)|^{p-2}|f'(z)|^2$$

Example 13 : If $f(z)=u+iv$ is an analytic function of $z=x+iy$, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|u|^p=p(p-1)|u|^{p-2}|f'(z)|^2$$

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Hence $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|u|^p = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left|\frac{f(z)+\overline{f(z)}}{2}\right|^p$

$$=\frac{4}{2^p}\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)+\overline{f(z)}\right]^p$$

$$=\frac{4}{2^p}\frac{\partial}{\partial z}\,p\left[f(z)+\overline{f(z)}\right]^{p-1}\overline{f'(z)}$$

$$=\frac{4}{2^p}\,p(p-1)\left[f(z)+\overline{f(z)}\right]^{p-2}f'(z)\,\overline{f'(z)}$$

$$=\frac{1}{2^{p-2}}\,p\,(p-1)\left[\{f(z)+\overline{f(z)}\}^2\right]^{\frac{p-2}{2}}|f'(z)|^2$$

$$=p\,(p-1)\left[\left|\frac{f(z)+\overline{f(z)}}{2}\right|^2\right]^{\frac{p-2}{2}}|f'(z)^2|$$

$$=p\,(p-1)\left[|u|^2\right]^{\frac{p-2}{2}}|f'(z)|^2$$

$$=p\,(p-1)\,|u|^{p-2}\,|f'(z)|^2$$

Example 14 : Find where the function

(*i*) $w=\frac{1}{z}$ (*ii*) $w=\frac{z}{z-1}$ (*iii*) $w=z^3-4z+1$ (*iv*) $w=\frac{z+2}{z(z^2+1)}$ (*v*) $z=e^{-v}(\cos u+i\sin u)$

ceases (fails) to be analytic. **[JNTU 2004, (A) Nov. 2009 (Set No. 2)]**

Solution : A function $w=f(z)$ ceases to be analytic if $\frac{dw}{dz}=0$.

(*i*) Given $w=\frac{1}{z}$

$$\therefore\quad \frac{dw}{dz}=-\frac{1}{z^2}\quad \text{if } z\neq 0$$

For $z=0$, $\frac{dw}{dz}$ does not exist. So, w is analytic everywhere except at the point $z=0$ which is a singular point of $f(z)$.

(*ii*) Given $w=\frac{z}{z-1}$

$$\therefore\quad \frac{dw}{dz}=\frac{(z-1)\cdot 1-z\cdot 1}{(z-1)^2}=\frac{-1}{(z-1)^2}\quad \text{if } z\neq 1$$

For $z=1$, $\frac{dw}{dz}$ does not exist. So, w is analytic everywhere except at the point $z=1$ which is a singular point of $f(z)$.

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(*iii*) Given $w = z^3 - 4z + 1$

$$\therefore \quad \frac{dw}{dz} = 3z^2 - 4$$

$\frac{dw}{dz}$ exists at all points of z in the complex plane.

Hence w is analytic everywhere in the complex plane.

(*iv*) Given $w = \dfrac{z+2}{z(z^2+1)}$

$$\therefore \quad \frac{dw}{dz} = \frac{z(z^2+1)(1)-(z+2)(3z^2+1)}{z^2(z^2+1)^2} = \frac{-2(z^3+3z^2+1)}{z^2(z^2+1)^2} \text{ if } z \neq 0 \text{ and } z^2+1 \neq 0$$

For $z = 0$ and $z = \pm i$, $\frac{dw}{dz}$ does not exist. So, w is analytic everywhere except at the poi

$z = 0$, $z = i$ and $z = -i$ which are the singular points of $f(z)$.

(*v*) Given $z = e^{-v}(\cos u + i \sin u)$

$$= e^{-v} . e^{iu} = e^{iu-v} = e^{iu+i^2 v}$$

$$= e^{i(u+iv)} = e^{iw}, \text{ where } w = u + iv$$

$\therefore \log z = iw$. Hence $\dfrac{dw}{dz} = \dfrac{1}{iz}$.

Clearly w is not analytic when $iz = 0$ i.e. when $z = 0$

Example 15 : Find all values of k, such that $f(z) = e^x(\cos ky + i \sin ky)$ is analytic

[JNTU 2003 (Set No. 4), Reg-2008 (Set No.

Solution : Let $f(z) = u + iv = e^x(\cos ky + i \sin ky)$. Then

$$u(x, y) = e^x \cos ky \text{ and } v(x, y) = e^x \sin ky.$$

$$\therefore \quad u_x = e^x \cos ky, \; u_y = -k e^x \sin ky$$

and $\quad v_x = e^x \sin ky, \; v_y = k e^x \cos ky$

The Cauchy – Riemann equations are satisfied if $u_x = v_y$ *i.e.* $e^x \cos ky = k e^x \cos ky$ which is true for $k = 1$

and $u_y = -v_x$ *i.e.* $-k e^x \sin ky = -e^x \sin ky$ which is true for $k = 1$.

Thus $f(z)$ is analytic when $k = 1$.

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Example 17 : Prove that the function $f(z)$ defined by

$$f(z)=\begin{cases}\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & (z\neq 0)\\ 0, (z=0)\end{cases}$$

is continuous and the Cauchy–Riemann equations are satisfied at the origin , yet $f'(0)$ *does not exist.* **[JNTU 99, Nov. 2006, 2007S, Nov. 2008S, 2008, (K) 2009, (A) 2010, (H) Dec. 2011, May 2012 (Set No. 4), May 2013]**

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$$= \underset{x \to 0}{\text{Lt}} \frac{x\left[1+i-m^3(1-i)\right]}{1+m^2} = 0$$

$\therefore$ Whatever is the manner in which $z \to 0$, we have $\underset{z \to 0}{\text{Lt}} f(z) = 0 = f(0)$

$\therefore$ The function is continuous at $z = 0$.

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \underset{x \to 0}{\text{Lt}} \frac{u(x,0)-u(0,0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \underset{y \to 0}{\text{Lt}} \frac{u(0,y)-u(0,0)}{y} = \underset{y \to 0}{\text{Lt}} -\frac{y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \underset{x \to 0}{\text{Lt}} \frac{v(x,0)-v(0,0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \underset{y \to 0}{\text{Lt}} \frac{v(0,y)-v(0,0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{y}{y} = 1$$

Thus we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Hence Cauchy–Riemann equations are satisfied at the origin.

Now $f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z)-f(0)}{z} = \underset{z \to 0}{\text{Lt}} \frac{f(z)}{z} = \underset{z \to 0}{\text{Lt}} \frac{\left(x^3-y^3\right)+i\left(x^3+y^3\right)}{\left(x^2+y^2\right)(x+iy)}$

Let $z \to 0$ along the path $y = mx$. Then $f'(0) = \frac{1-m^3+i\left(1+m^3\right)}{\left(1+m^2\right)(1+im)}$

which depends upon m and hence is not unique. Thus, $f'(z)$ does not exist at (0, 0).

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Example 21 : Show that $f(z)=\begin{cases}\dfrac{x\,y^2\,(x+i\,y)}{x^2+y^4}, & z\neq 0\\ 0, \quad \text{if} \quad z=0\end{cases}$

is not analytic at $z = 0$ although C – R equations are satisfied at the origin.

[JNTU 2003, 2007S, Nov. 2008, (K) Nov. 2010 (Set No. 3)]

Solution : $\dfrac{f(z)-f(0)}{z-0}=\dfrac{f(z)-0}{z}=\dfrac{f(z)}{z}$

$$=\frac{x\,y^2\,(x+i\,y)}{(x^2+y^4)\cdot z}=\frac{x\,y^2(z)}{(x^2+y^4)\cdot z}=\frac{x\,y^2}{x^2+y^4}$$

Clearly $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}}\frac{x\,y^2}{x^2+y^4}=\lim_{\substack{y\to 0\\ x\to 0}}\frac{x\,y^2}{x^2+y^4}=0$

Along the path $y = mx$,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{x\to 0}\frac{x\,(m^2\cdot x^2)}{x^2+m^4\cdot x^4}=\lim_{x\to 0}\frac{m^2\,x}{1+x^2m^4}=0$$

Also along the path $x = m\,.\,y^2$,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{y\to 0}\frac{(m\,y^2)\,y^2}{m^2\,y^4+y^4}=\lim_{y\to 0}\frac{m}{m^2+1}\neq 0$$

Limit value depends on m *i.e.* on the path of approach and is different for the different paths followed and therefore limit does not exist. Hence $f(z)$ is not differentiable at $z = 0$. Thus $f(z)$ is not analytic at $z = 0$.

To prove that $C - R$ conditions are satisfied at the origin.

Let $f(z)=u+i\,v=\dfrac{x\,y^2(x+i\,y)}{x^2+y^4}$

Then $u\,(x, y)=\dfrac{x^2\,y^2}{x^2+y^4}$ and $v\,(x, y)=\dfrac{x\,y^3}{x^2+y^4}$, for $z\neq 0$

Also $u\,(0, 0) = 0$ and $v\,(0, 0) = 0$ $\quad[\because\ f(z)=0\ \ at\ \ z=0]$

Now $\dfrac{\partial u}{\partial x}=\lim\limits_{x\to 0}\dfrac{u\,(x,0)-u\,(0,0)}{x}=\lim\limits_{x\to 0}\dfrac{0-0}{x}=0$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u\,(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v\,(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

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Example 22 : Show that the function

$$f(z) = e^{-z^{-4}}, \; z \neq 0 \quad \text{and} \quad f(0) = 0$$

is not analytic at $z = 0$, although Cauchy – Riemann equations are satisfied at this point.

[JNTU (A) Nov. 2010S]

Solution : We have

$f(z) = e^{-z^{-4}} = e^{-(x+iy)^{-4}} \quad (z \neq 0)$ or $f(z) = u(x, y) + i\,v(x, y) = e^{-(x+iy)^{-4}}$

$\Rightarrow \quad u(x, 0) + i\,v(x, 0) = e^{-x^{-4}}$ and $u(0, y) + i\,v(0, y) = e^{-y^{-4}}$

$\Rightarrow \quad u(x, 0) = e^{-x^{-4}}, \; v(x, 0) = 0$ and $u(0, y) = e^{-y^{-4}}, \; v(0, y) = 0$

Also we have $f(0) = 0$

i.e. $f(0) = u(0, 0) + i\,v(0, 0) = 0$

$\Rightarrow \quad u(0, 0) = 0$ and $v(0, 0) = 0$

Now at $z = 0$, $\dfrac{\partial u}{\partial x} = \left(\dfrac{\partial u}{\partial x}\right)_{x=0,\, y=0} = \underset{x \to 0}{lt} \dfrac{u(x, 0) - u(0, 0)}{x}$

$$= \underset{x \to 0}{lt} \frac{e^{-x^{-4}} - 0}{x} = \underset{x \to 0}{lt} \frac{1}{x\, e^{x^{-4}}} = \underset{x \to 0}{lt} \frac{1}{x} \cdot \frac{1}{x\, e^{1/x^4}}$$

$$= \underset{x \to 0}{lt}\; x \frac{1}{\left[1 + \dfrac{1}{x^4} + \dfrac{1}{x^8\, 2!} + \ldots\right]} \quad \left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)$$

$$= \underset{x \to 0}{lt} \frac{1}{x + \dfrac{1}{x^3} + \dfrac{1}{x^7\, 2!} + \ldots} = 0$$

Similarly, $\left(\dfrac{\partial u}{\partial y}\right)_{x=0,\, y=0} = \underset{y \to 0}{lt} \dfrac{u(0, y) - u(0, 0)}{y}$

$$= \underset{y \to 0}{lt} \frac{1}{y} \cdot \frac{1}{e^{1/y^4}} = 0$$

At $z = 0$, $\dfrac{\partial v}{\partial x} = \left(\dfrac{\partial v}{\partial x}\right)_{x=0,\, y=0} = \underset{x \to 0}{lt} \dfrac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{lt} \dfrac{0 - 0}{x} = 0$

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Solution : (*i*) Let $w = f(z) = e^z = e^{x+iy} = e^x e^{iy}$

or $\quad w = u + i\,v = e^x (\cos y + i \sin y)$

$\Rightarrow \quad u = e^x \cos y$ and $v = e^x \sin y$

$$\therefore \quad \frac{\partial u}{\partial x} = e^x \cos y, \ \frac{\partial u}{\partial y} = -e^x \sin y \quad \text{and} \quad \frac{\partial v}{\partial x} = e^x \sin y, \ \frac{\partial v}{\partial y} = e^x \cos y$$

Hence $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Thus the Cauchy – Riemann equations are satisfied. Also the partial derivatives are continuous.

Hence $\dfrac{dw}{dz} = f'(z)$ exists at all points in the complex plane

i.e. $f(z)$ is analytic everywhere.

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i\, e^x \sin y$$
$$= e^x (\cos y + i \sin y) = e^x e^{iy} = e^{x+iy} = e^z$$

(*ii*) This is left as an exercise to the student.

Note : If a complex function is analytic, then it can be differentiated just in the ordinary way.

Example 34 : Prove that z^n (n is a positive integer) is analytic and hence find its derivative.
[JNTU 2003S, 2004, Aug. 2007S, (H) Nov. 2009, (A) Nov. 2010 (Set No. 3)]

Solution : Let $f(z) = z^n$.

Using polar co-ordinates, let $f(z) = u(r, \theta) + i\,v(r, \theta)$

Then as, $z = r e^{i\theta}$,

$$f(z) = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta) = u + iv$$

$\Rightarrow \quad u = r^n \cos n\theta$ and $v = r^n \sin n\theta$

$$\therefore \quad \frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta, \ \frac{\partial v}{\partial r} = nr^{n-1} \sin n\theta \ \text{and} \ \frac{\partial u}{\partial \theta} = -nr^n \sin n\theta, \ \frac{\partial v}{\partial \theta} = nr^n \cos n\theta$$

Now $\dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = \dfrac{1}{r} \cdot n\, r^n \cos n\theta = n\, r^{n-1} \cos n\theta = \dfrac{\partial u}{\partial r}$

and $\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = \dfrac{1}{r}(-n\, r^n \sin n\theta) = -n\, r^{n-1} \sin n\theta = -\dfrac{\partial v}{\partial r}$

Since the Cauchy – Riemann equations are satisfied and the partial derivatives are continuous, therefore $f(z) = z^n$ is analytic.

Differentiating $f(z) = u(r, \theta) + i\,v(r, \theta)$ partially with respect to 'r', we get

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Example 35 : Prove that the function $f(z) = \bar{z}$ is not analytic at any point.

[JNTU 2003, 2005, (A) Dec. 2009 (Set No. 3), (H) Dec. 2014, Nov. 2015

Solution : Let $f(z) = u + i\,v = \bar{z} = x - i\,y$

Then $u = x$ and $v = -y$

$$\therefore \quad \frac{\partial u}{\partial x} = 1, \ \frac{\partial u}{\partial y} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} = 0, \ \frac{\partial v}{\partial y} = -1$$

Clearly $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Hence Cauchy – Riemann conditions are not satisfied. Therefore $f(z)$ is not analytic at any point.

Example 36 : Determine whether the function $2xy + i\,(x^2 - y^2)$ is analytic

[JNTU 2003 (Set No. 4)

Solution : Let $f(z) = u + i\,v = 2xy + i\,(x^2 - y^2)$

So $u(x, y) = 2xy$ and $v(x, y) = x^2 - y^2$

$$\therefore \quad \frac{\partial u}{\partial x} = 2y, \ \frac{\partial u}{\partial y} = 2x \quad \text{and} \quad \frac{\partial v}{\partial x} = 2x, \ \frac{\partial v}{\partial y} = -2y$$

Hence Cauchy – Riemann conditions are not satisfied.

Therefore $f(z)$ is not analytic.

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Solution : Since u and v are harmonic functions, we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \Rightarrow \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 u}{\partial x^2} \qquad \text{... (1)}$$

and $$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \Rightarrow \frac{\partial^2 v}{\partial y^2}=-\frac{\partial^2 v}{\partial x^2} \qquad \text{... (2)}$$

Let $U=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}$ **... (3)** and $V=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$ ... (4)

Differentiating (3) and (4) partially with respect to x, we get

$$\frac{\partial U}{\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}-\frac{\partial^2 v}{\partial x^2} \quad \textbf{... (5)} \quad \text{and} \quad \frac{\partial V}{\partial x}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x \partial y} \qquad \text{... (6)}$$

Differentiating (3) and (4) partially with respect to y, we get

$$\frac{\partial U}{\partial y}=\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial y \partial x} \quad \textbf{... (7)} \quad \text{and} \quad \frac{\partial V}{\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial^2 v}{\partial y^2} \qquad \text{... (8)}$$

From (1) and (7), we have

$$\frac{\partial U}{} \quad \left(\frac{\partial^2 u}{} \quad \frac{\partial^2 v}{}\right) \quad \left(\frac{\partial^2 u}{} \quad \frac{\partial^2 v}{}\right) \quad \frac{\partial V}{}$$

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Example 49 : Show that $u(x, y) = x^3 - 3x\,y^2$ is harmonic and find its harmonic conjugate. Find also the corresponding analytic function $f(z)$ in terms of z.

Solution : Given $u = x^3 - 3x\,y^2$

$$\therefore \quad \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \frac{\partial u}{\partial y} = -6x\,y \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = 6x, \frac{\partial^2 u}{\partial y^2} = -6x$$

Clearly $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Hence u is harmonic.

Let v be the conjugate of u. Then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{(Using } C - R \text{ equations)}$$

or $dv = 6\,x\,y\,dx + (3x^2 - 3y^2)\,dy$ ... (1)

Taking $M = 6xy$ and $N = 3x^2 - 3y^2$, we get

$dv = M dx + N dy$

Now $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = 6x$

Hence (1) is exact differential equation.

Integrating, $v = \underset{(y \text{ constant})}{\int M dx} + \underset{\substack{\text{(only those terms} \\ \text{which do not contain } x)}}{\int N dy} + c$

$$= \underset{(y \text{ constant})}{\int 6\,x\,y\,dx} + \int (-3\,y^2)\,dy + c$$

or $\quad v = 6y\left(\dfrac{x^2}{2}\right) - 3\left(\dfrac{y^3}{3}\right) + c = 3x^2\,y - y^3 + c$

$$\therefore \quad f(z) = u + i\,v = x^3 - 3\,x\,y^2 + i\,(3x^2 y - y^3) + i\,c$$

$$= x^3 + 3x^2(i\,y) + 3x(i\,y)^2 + (i\,y)^3 + i\,c$$

or $f(z) = (x + i\,y)^3 + k$ where $k = i\,c$.

Note : The above result may be obtained by using Milne – Thomson's method.

Example 50 : Find k such that $f(x, y) = x^3 + 3k\,x\,y^2$ may be harmonic and find its conjugate.

[JNTU 2004 (Set No. 4), 2004S, (K) Nov. 2010, (H) May 2011 (Set No. 3)]

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i.e. $6x + 6kx = 0$ *i.e.* $x(1+k) = 0$

i.e. $1 + k = 0$ $(\because x \neq 0)$ or $k = -1$

Hence $f(x, y) = x^3 - 3xy^2$

Let $g(x, y)$ be the conjugate of $f(x, y)$. Then

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = -\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy \quad \text{(Using } C \text{ ; } R \text{ equations)}$$

$$= -6kxy\,dx + (3x^2 + 3ky^2)\,dy$$

or $dg = 6xy\,dx + (3x^2 - 3y^2)\,dy$ $(\because k = -1)$

This is exact differential equation.

Integrating, $g = \int 6xy\,dx + \int -3y^2\,dy + c$

(y constant) (only those terms which do not contain x)

$$= 6y\left(\frac{x^2}{2}\right) - 3\left(\frac{y^3}{3}\right) + c = 3x^2y - y^3 + c$$

Example 51 : If u is a harmonic function, show that $w = u^2$ is not a harmonic function unless u is a constant. **[JNTU 2005S, Aug. 2006S (Set N**

Solution : Given u is a harmonic function.

By definition, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(1)}$$

Also given $w = u^2$

$$\therefore \frac{\partial w}{\partial x} = 2u\frac{\partial u}{\partial x} \text{ and } \frac{\partial w}{\partial y} = 2u\frac{\partial u}{\partial y}$$

Now $\dfrac{\partial^2 w}{\partial x^2} = 2\left[u\dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\partial u}{\partial x}\right)^2\right]$ and $\dfrac{\partial^2 w}{\partial y^2} = 2\left[u\dfrac{\partial^2 u}{\partial y^2} + \left(\dfrac{\partial u}{\partial y}\right)^2\right]$

$$\therefore \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[u(0) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right], \text{ using (1)}$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

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(2) + (3) gives, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

$\therefore$ u is harmonic.]

Let v be its harmonic conjugate.

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=2x+i2y$$

Using Milne–Thomson method, $f'(z)=2z$

Integrating, we get $f(z)=z^2+c$

$\Rightarrow$ $u+iv=(x+iy)+ik$ where $c=ik$

$$=x^2-y^2+i\,2xy+ik=x^2-y^2+i\,(2xy+k)$$

Equating imaginary parts, $v = 2xy + k$ is the required form.

Example 18 : Show that $u(x, y)=e^{2x}\,(x\cos 2y-y\sin 2y)$ is harmonic and find its harmonic conjugate. **[JNTU 1998]**

(or) Find the analytic function whose real part is $u=e^{2x}(x\cos 2y-y\sin 2y)$.

[JNTU (H) Nov. 2009 (Set No. 3), (A) Nov. 2010 (Set No. 2), (H) May 2013, Nov. 2015]

Solution : Given $u(x, y)=e^{2x}\,(x\cos 2y-y\sin 2y)$ ...(1)

Differentiating (1) partially w.r.t. x, $\frac{\partial u}{\partial x}=e^{2x}\left[(2x+1)\cos 2y-2y\sin 2y\right]$

Again differentiating w.r.t. x, $\frac{\partial^2 u}{\partial x^2}=e^{2x}\left[(4x+4)\cos 2y-4y\sin 2y\right]$ ...(2)

Differentiating (1) partially w.r.t. y, $\frac{\partial u}{\partial y}=e^{2x}\left[-2x\sin 2y-2y\cos 2y-\sin 2y\right]$

Again differentiating w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2}=e^{2x}\left[-(4x+4)\cos 2y+4y\sin 2y\right]=-\frac{\partial^2 u}{\partial x^2}\ \text{[using (2)]}$$

$$\therefore\ \frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}=0$$

Hence u is harmonic.

Let v be its harmonic conjugate. Then

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y},\ \text{using Cauchy-Riemann equations}$$

$$=e^{2x}\left[(2x+1)\cos 2y-2y\sin 2y\right]-i\,e^{2x}\left[-2x\sin 2y-2y\cos 2y-\sin 2y\right]$$

Using Milne–Thomson method,

$$f'(z)=e^{2z}\left[(2z+1)\cos 0-0\right]-i\,e^{2z}\,(0)=e^{2z}\,(2z+1)=e^{2z}\cdot 2z+e^{2z}$$

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Solution : Given $u = 4xy - 3x + 2$ ...(1)

Differentiating (1) partially w.r.t. x, $\frac{\partial u}{\partial x} = 4y - 3$

Again differentiating, $\frac{\partial^2 u}{\partial x^2} = 0$

Differentiating (1) partially w.r.t. y, $\frac{\partial u}{\partial y} = 4x$

Again differentiating, $\frac{\partial^2 u}{\partial y^2} = 0$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u is harmonic.

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \Rightarrow f'(z) = 4y - 3 - i \cdot 4x$

Using Milne–Thomson method,

$f'(z) = -3 - i\,4z$ (putting $x = z$ and $y = 0$)

Integrating, $f(z) = -3z - i\,2z^2 + c.$

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Example 21 : If $f(z) = u+iv$ is an analytic function of z and if $u - v = e^x (\cos y - \sin y)$, find $f(z)$ in terms of z. **[JNTU 1995S, June 2009S, (K) Nov. 2009, (H) Dec. 2011 (Set No. 3)]**

(or) Find the analytic function $f(z) = u + iv$ if $u - v = e^x(\cos y - \sin y)$

[JNTU 2005S (Set No.2)]

Solution : Given $\quad u - v = e^x (\cos y - \sin y)$ ...(1)

Differentiating partially w.r.t. x and y,

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x (\cos y - \sin y) \quad ...(2)$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^x (-\sin y - \cos y)$$

But $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$ $\left[\because f(z) \text{ is analytic}\right]$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = e^x (\sin y + \cos y) \quad ...(3)$$

(2) + (3) gives, $2\dfrac{\partial u}{\partial x} = 2e^x \cos y$ or $\dfrac{\partial u}{\partial x} = e^x \cos y$ ...(4)

(3) – (2) gives, $\dfrac{\partial v}{\partial x} = e^x \sin y.$ ...(5)

Now $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y$

$$= e^x(\cos y + i\sin y) = e^x . e^{iy} = e^{x+iy} = e^z$$

Integrating, $f(z) = e^z + c$

Aliter : Integrating (4) and (5) keeping y as constant gives

$$u = e^x \cos y + \psi_1(y) \text{ and } v = e^x \sin y + \psi_2(y)$$

$$\therefore \quad u - v = e^x (\cos y - \sin y) + \psi_1(y) - \psi_2(y) \quad ...(6)$$

From (1) and (6), $\psi_1(y) - \psi_2(y) = 0 \Rightarrow \psi_1(y) = \psi_2(y)$

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We have $u = e^x \cos y + \psi_1(y)$

Differentiating partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = -e^x \sin y + \psi_1'(y)$$

$$-\frac{\partial v}{\partial x} = -e^x \sin y + \psi_1'(y) \qquad \text{[From Cauchy – Riemann equations]}$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} = e^x \sin y - \psi_1'(y) \qquad \text{...(7)}$$

From (7) and (5), $\psi_1'(y) = 0$

Integrating, $\psi_1(y) = c$

Hence $u = e^x \cos y + c$ and $v = e^x \sin y + c$

$$\therefore \quad f(z) = u + iv$$

$$= e^x \cos y + c + i\left(e^x \sin y + c\right)$$

$$= e^x (\cos y + i \sin y) + (1+i)c$$

$$= e^x \cdot e^{iy} + k = e^{x+iy} + k$$

or $f(z) = e^z + k$.

Example 22 : If $f(z) = u + iv$ be an analytic function of z and if $u - v = (x-y)\left(x^2 + 4xy + y^2\right)$, find $f(z)$ in terms of z. **[JNTU**

(or) Find the Analytic function $f(z) = u + iv$ in terms of z, where $u - v = (x-y)(x^2 + 4$

[JNTU (H) Ma

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Example 27 : Show that the function $f(x, y) = x^3y - xy^3 + xy + x + y$ is the imaginary part of an analytic function of $z = x + iy$. [JNTU 2004, 2004S, 2005, 2008S (Set No. Determine the real part and also the complex function.

Solution : Let $\phi(z)$ be an analytic function of z.

Given $f(x, y) = x^3y - xy^3 + xy + x + y$

$f(x, y)$ will be the imaginary part of $\phi(z)$ if $\nabla^2 f = 0$

Now $\dfrac{\partial f}{\partial x} = 3x^2y - y^3 + y + 1$, $\dfrac{\partial f}{\partial y} = x^3 - 3xy^2 + x + 1$ and $\dfrac{\partial^2 f}{\partial x^2} = 6xy$, $\dfrac{\partial^2 f}{\partial y^2} = -6x$

Clearly $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ *i. e.*, $\nabla^2 f = 0$

So $f(x, y)$ is the imaginary part of an analytic function of z.

To determine real part and complex function.

Let $g(x, y)$ be the real part of $\phi(z)$. Then

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = -\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dx \quad \text{(Using Cauchy – Riemann equations)}$$

or $dg = -(x^3 - 3xy^2 + x + 1)\,dx + (3x^2y - y^3 + y + 1)\,dy$

The above equation is exact differential equation.

Integrating, $g = -\int (x^3 - 3xy^2 + x + 1)\,dx + \int (-y^3 + y + 1)\,dy + c$

(y constant) (only those terms which do not contain x)

$$= -\left(\frac{x^4}{4} - \frac{3x^2y^2}{2} + \frac{x^2}{2} + x\right) + \left(\frac{-y^4}{4} + \frac{y^2}{2} + y\right) + c$$

$$= -\frac{1}{4}(x^4 + y^4 + 6x^2y^2) + \frac{1}{2}(y^2 - x^2) + y - x + c$$

Now $\phi(z) = g(x, y) + i\,f(x, y)$ where $f(x, y) = x^3y - xy^3 + xy + x + y$

$$\therefore \quad \phi'(z) = \frac{\partial g}{\partial x} + i\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y} + i\frac{\partial f}{\partial x} \quad \text{(Using } C - R \text{ equations)}$$

$$= -(x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y + 1)$$

By Milne – Thomson method, $\phi'(z)$ is expressed in terms of z by replacing x by z by 0.

Hence $\phi'(z) = -(z^3 + z + 1) + i$

Integrating, $\phi(z) = -\left(\dfrac{z^4}{4} + \dfrac{z^2}{2} + z\right) + iz + A$ where A is a complex constant

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