19-10-17

UNIT-III : APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS

➡ Solution Of Partial Differential Equations By The Method Of Separation Of Variables

In this method, we assume the solution to be the product of two functions in which each function contains only one variable.

The following example explains the method mentioned above.

* Problems :

1. Solve the differential equation : $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, given that $u(x,0) = 6e^{-3x}$

→ $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-3x}$

Dependent variable : u Independent Variables : x, t

In the given differential equation, dependent variable is 'u' and independent variables are 'x', 't'.

Therefore, assume the solution as: $u = XT$

where : $X = X(x)$, $T = T(t)$

Then, $\frac{\partial u}{\partial x} = X'T$, $\frac{\partial u}{\partial T} = XT'$

Substituting the values in the above equation, we have

$X'T = 2XT' + XT$

$\Rightarrow \frac{X'}{X} = \frac{(2T' + T)}{T}$

Since L.H.S. is function of 'x' and R.H.S. is function of 't' and both are equal, therefore

$\frac{X'}{X} = \left(\frac{2T' + T}{T}\right) = k$ (say)

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$$\frac{X'}{X} = \frac{(2T'+T)}{T} = k \text{ (say)}$$

$$\Rightarrow \frac{X'}{X} = k, \quad \frac{(2T'+T)}{T} = k$$

$$\Rightarrow X' = Xk, \quad (2T'+T) = kT$$

$$\Rightarrow X' - Xk = 0, \quad 2T' + T - kT = 0$$

$$\Rightarrow X' - kX = 0, \quad 2T' + T(1-k) = 0$$

$$\Rightarrow \frac{dX}{dx} - kX = 0, \quad 2\frac{dT}{dt} + (1-k)T = 0$$

The above two differential equations are ordinary differential equations whose auxillary equations are given by:

$$m - k = 0, \quad 2m + (1-k) = 0$$

$[\because (D-k)X = 0, \ (2D + (1-k))T = 0, \ D = \text{Derivation}]$

$$m = k, \quad m = \frac{(k-1)}{2}$$

$$X = c_1 e^{kx}, \quad T = c_2 e^{\left(\frac{k-1}{2}\right)t}$$

Then, its complete solution is given by:

$$u = c_1 e^{kx} + c_2 e^{(k-1)t/2}$$

$$\Rightarrow u = c_1 c_2 e^{kx + \frac{(k-1)}{2}t}$$

$$\Rightarrow u = A e^{\left(kx + \frac{(k-1)}{2}t\right)}$$

Substituting the given condition $u(x,0) = 6e^{-3x}$, i.e. when $t = 0$, $u = 6e^{-3x}$ $[\because u = u(x,t)]$ in the above equation, we have

$$6e^{-3x} = Ae^{kx}$$

$$A = 6, \quad k = -3$$

$$\Rightarrow u = Ae^{kx} = 6e^{(-3x-2t)} = 6e^{-(3x+2t)}$$

Ans:. $u = 6e^{-(3x+2t)}$

17-10-17

Solution of One Dimensional Wave Equation

The differential equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, 'c' is constant — (1)

is called one dimensional wave equation.

Solution:

In the above differential equation, u is a dependent variable; x, t are independent variables.

Therefore, $u = (x,t)$ is a solution of equation (1)

(It is known as displacement function)

Assume the solution of equation (1) as: $u = XT$ — (2)

Then, $\frac{\partial^2 u}{\partial x^2} = X''T$, $\frac{\partial^2 u}{\partial t^2} = XT''$

Substituting these values in equation (1), we have

$$XT'' = c^2 X''T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k \text{ (say)}$$

$$\Rightarrow X'' - kX = 0, \quad T'' - kc^2T = 0$$

i.e., $(D^2 - k)X = 0$; $(D^2 - kc^2)T = 0$

Its auxillary equations are given by:

$$m^2 - k = 0; \quad m^2 - k^2c^2 = 0.$$

Here, 'k' is a constant, it may be zero or may be positive or may be negative

Case (i): when $k = 0$

Auxillary equations are: $m = 0,0$ (for X); $m = 0,0$ (for T)

Solutions are:

$$X = c_1 + c_2x; \quad T = c_3 + c_4t$$

$m = m_1, m_2$, C.F $= c_1e^{m_1x} + c_2e^{m_2x}$

$m = m_1, m_2$ $(m_1 = m_2)$ C.F $= (c_1 + c_2x)e^{m_1x}$

$m = \alpha \pm \beta i$ C.F $= e^{\alpha x}(c_1\cos\beta x + c_2 \sin\beta x)$

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Case (ii): When K is positive

Say $K = p^2$

Then, auxillary equations are:

$$m^2 - p^2 = 0; \quad m^2 - p^2c^2 = 0$$

$$m = \pm p; \quad m = \pm pc$$

Solutions are: $X = c_1 e^{px} + c_2 e^{-px}$, $T = c_3 e^{pct} + c_4 e^{-pct}$

Case (iii): When K is negative

Say $K = -p^2$

Then, auxillary equations are:

$$m^2 + p^2 = 0; \quad m^2 + p^2c^2 = 0$$

$$m = \pm pi; \quad m = \pm pci$$

Solutions are: $X = c_1 \cos px + c_2 \sin px$, $T = c_3 \cos cpt + c_4 \sin cpt$

Substituting these values in equation (2), we have:

$$u = (c_1 + c_2 x)(c_3 + c_4 t) \quad -(3)$$

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pct} + c_4 e^{-pct}) \quad -(4)$$

$$u = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad -(5)$$

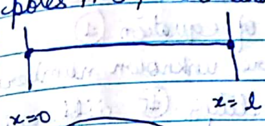
From these three solution, we have to choose a solution which is suitable to the physical nature of the problem.

Since we are dealing with a problem on vibrations, i.e., one dimensional wave equation, therefore the solution must contain periodic functions, i.e., sine and cosine terms. Hence, the solution of one dimensional wave equation is:

$$\boxed{u = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_p \sin cpt)}$$

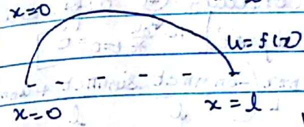
Initial and Boundary Conditions:

Fix a string having elastic property between two poles, i.e., $x = 0$ and $x = l$. At $x = 0, x = l$, displacement $= 0$



For the given string

$$\left.\begin{aligned} u(0,t) &= 0 \quad -(7) \\ u(l,t) &= 0 \quad -(8) \end{aligned}\right\} \text{one boundary conditions}$$



$$\left.\begin{aligned} u(x,0) &= f(x) \quad -(9) \\ \text{(velocity)} \left(\frac{\partial u}{\partial t}\right)_{t=0} &= 0 \quad -(10) \end{aligned}\right\} \text{are known as initial conditions}$$

Applying boundary condition (7), i.e., $u(0,t) = 0$, i.e., when $x = 0$, $u = 0$, ~~one~~ equation (6), we have

$$u = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

$$\Rightarrow \cancel{u = c_2 \sin px (c_3 \cos} \quad \Rightarrow 0 = (c_3 \cos cpt + c_4 \sin cpt) c_1$$

$$\Rightarrow c_1 = 0.$$

$$u = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \quad -(11)$$

Applying boundary conditions (8), i.e., $u(l,t) = 0$, i.e., $u = 0$ when $x = l$ on equation (11) is:

$$0 = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$$

If $c_2 = 0$, then the solution is $u = 0$, which is known as trivial solution.

We need to calculate a non-trivial solution.

Therefore, $c_2 \neq 0$

$$\Rightarrow pl = 0 \Rightarrow \sin n\pi \Rightarrow pl = n\pi \Rightarrow p = n\pi / l$$

Substituting this value in equation (11), we have

$$u = c_2 \sin \frac{n\pi x}{l} \left[c_3 \cos \frac{n\pi c t}{l} + c_4 \sin \frac{n\pi c t}{l} \right]$$

Since c_1, c_2, c_3, c_4 are arbitary constants, Replacing $c_2 c_3$ by b_n and $c_2 c_4$ by a_n, we have

$$u = \sin \frac{n\pi x}{l} \left[b_n \cos \frac{n\pi c t}{l} + a_n \sin \frac{n\pi c t}{l} \right]$$

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Adding all these solutions for different values of 'n', we have:

$$\boxed{u = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(b_n \cos\frac{n\pi c t}{l} + a_n \sin\frac{n\pi c t}{l}\right)} \quad - (12)$$

is also a solution of equation (1)

Here, a_n and b_n are unknown numbers.

To apply initial condition (10), differentiating equation (12) partially w.r.t 't' $\left[\text{i.e. } \frac{\partial u}{\partial t}_{t=0} = 0\right]$

we have: $\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(-b_n \frac{n\pi c}{l} \sin\frac{n\pi c t}{l} + a_n \frac{n\pi c}{l} \cos\frac{n\pi c t}{l}\right)$

Substituting $\frac{\partial u}{\partial t} = 0, t = 0$ in the above equation, we have:

$$\Rightarrow 0 = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(\frac{a_n n\pi c}{l}\right)$$

$$\Rightarrow a_n = 0 \ \forall n$$

Substituting this value in equation (1), we have

$$\boxed{u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c t}{l}} \quad - (13)$$

Applying another initial condition, $u(x,0) = f(x)$; i.e; $u = f(x)$, $t = 0$ in equation (13), we have:

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \ ; \ [0, l]}$$

which is half range sine series, in the interval $[0, l]$. Therefore, $\boxed{b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{L}\, dx}$

Hence, solution of equation (1) is:

$$u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c t}{l}$$

where $b_n = \frac{2}{\pi}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$

v.v.imp * **Problems:**

1. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by: $u = u_0 \sin^3\frac{\pi x}{l}$. Then, it is released from rest. Then, find the displacement of the string at any point 'x' at any time 't'.

→ The given problem is one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad - (1)$$

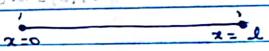
Then, its solution is given by:

$$u = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad - (2)$$

For the given problem, $u = u_0 \sin^3\frac{\pi x}{l}$,

boundary conditions are:

$u(0,t) = 0 \ - (3)$, $u(l,t) = 0 \ - (4)$



Applying the boundary condition (3) on equation (2), we have:

$0 = c_1(c_3 \cos cpt + c_4 \sin cpt) \Rightarrow c_1 = 0$

Substituting this value in equation (2), we have:

$$u = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \quad - (5)$$

Again applying another boundary condition (4) on (5), we have

$0 = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) \Rightarrow \sin pl = 0 = \sin n\pi$

$$\Rightarrow p = \frac{n\pi}{l}$$

Substituting this value in equation (5) and replacing $c_2 c_3$ by b_n and $c_2 c_4$ by a_n, we have:

$$u = \sin\frac{n\pi x}{l}\left(b_n \cos\frac{n\pi c t}{l} + a_n \sin\frac{n\pi c t}{l}\right)$$

Adding all these solutions for different values of 'n', we have: $u = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(b_n \cos\frac{n\pi c t}{l} + a_n \sin\frac{n\pi c t}{l}\right)$ — (6)

For the given problem, the initial conditions are:

$$u(x,0) = u_0 \sin^3 \frac{\pi x}{l} \quad \text{—(7)}, \qquad \left(\frac{\partial u}{\partial t}\right)_{(t=0)} = 0 \quad \text{—(8)}$$

Differentiating equation (6) partially w.r.t. 't'

$$\Rightarrow \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(-b_n \sin\frac{n\pi c}{l} \sin\frac{n\pi ct}{l} + a_n \frac{n\pi c}{l} \cos\frac{n\pi ct}{l}\right)$$

Applying the initial condition $\frac{\partial u}{\partial t}_{(t=0)} = 0$ in above equation

$$0 = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(a_n \frac{n\pi c}{l}\right)$$

$$\Rightarrow a_n = 0$$

Substituting this value in equation (6), we have

$$u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \quad \text{—(9)}$$

Applying another initial condition ~~(4, x, 0)~~ $u(x,0) = u_0 \sin^3 \frac{\pi x}{l}$ on equation (9), we have:

$$u_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\sin 3A = 3\sin A - 4\sin^3 A \Rightarrow \sin^3 A = (3\sin A - \sin 3A)/4$$

$$\Rightarrow \frac{u_0\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]}{4} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\ldots$$

Equating co-efficients of like terms on both sides,

$$b_1 = \frac{3u_0}{4}; \quad b_2 = 0; \quad b_3 = -\frac{u_0}{4}; \quad b_4 = b_5 = b_6 = \cdots$$

Substituting these values in equation (9), we have:

$$u = \frac{3u_0}{4} \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l} - \frac{u_0}{4} \sin\frac{3\pi x}{l} \cos\frac{3\pi ct}{l}$$

Ans:. $u = \frac{3u_0}{4} \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l} - \frac{u_0}{4} \sin\frac{3\pi x}{l} \cos\frac{3\pi ct}{l}$

2. Solve the boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad u(0,t) = 0, \; u(l,t) = 0, \; u(x,0) = x, \; \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

$$\left[\text{ANS: } u(x,t) = \sum_{n=1}^{\infty} \frac{2l}{n\pi} (-1)^{n+1} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}\right]$$

3. A tightly stretched flexible string, as fixed between $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape $f(x) = \mu x(l-x)$ where μ is constant and then released. Then, find the displacement of the string at any point 'x' at any time 't'.

$$\left[\begin{array}{l} u = f(x) = \mu x(l-x) \\ u(x,t) = \dfrac{4\mu l^2}{\pi^3} \displaystyle\sum_{n=1}^{\infty} \dfrac{(1-(-1)^n)}{n^3} \sin\dfrac{n\pi x}{l} \cos\dfrac{n\pi ct}{l} \end{array}\right]$$

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Case (ii): When k is positive

Say $k = p^2$

Then, the auxillary equations are:

$$m^2 - p^2 = 0; \quad m - p^2c^2 = 0$$

$$m = \pm p; \quad m = p^2c^2$$

Then, the solutions are: $X = c_1 e^{px} + c_2 e^{-px}$, $T = c_3 e^{c^2p^2t}$

Case (iii): When k is negative

Say $k = -p^2$

~~Sou~~ Then, the auxillary equations are:

$$m^2 + p^2 = 0; \quad m^2 + p^2c^2 = 0$$

$$m = \pm ip; \quad m = -p^2c^2$$

Then, the solutions are: $X = (c_1 \cos px + c_2 \sin px)$, $T = c_3 e^{-p^2c^2t}$

Substituting all these solutions in equation (2), we have

$u = (c_1 + c_2 x)\, c_3$ – (3)

$u = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{p^2c^2t})$ – (4)

$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{-p^2c^2t})$ – (5) [Time↑ Temp↓]

are the possible solutions of one dimensional heat equation.

From these solutions, we need to choose a solution which is suitable to the physical nature of the problem.

Since the problem is one dimensional heat equation, therefore as the time increases, temperature of the body should decrease.

The suitable solution for one dimensional heat equation is:

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{-p^2c^2t})$$

Since c_1, c_2, c_3 are arbitrary constants,

* ~~Problems:~~ therefore replacing c_1c_3 by c_1 and c_2c_3 by c_2:

the solution of one dimensional heat equation is:

$$u = (c_1 \cos px + c_2 \sin px)(e^{-p^2c^2t})$$

* Problems:

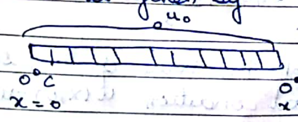
1. A rod of length 'l' with insulated sides is initially at a uniform temperature 'u_0'. Its ends are suddenly cooled to 0°c and are kept at that temperature. Find the temperature function $u(x,t)$.

→ The problem is on one dimensional heat equation.

Consider one dimensional heat equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{– (1)}$$

Its solution is given by: $u = (c_1 \cos px + c_2 \sin px)(e^{-p^2c^2t})$ – (2)



Since the ends of the rod $x=0$ and $x=l$ are cooled to 0°c and kept at that temperature throughout, therefore the boundary conditions are;

$u(0,t) = 0°c$ —③, $u(l,t) = 0°c$ —④

When time 't' is 0, temperature of the rod is '$u_0°c$'

Therefore, the initial condition is: $u(x,0) = u_0$ ⑤

Applying the boundary condition ③, i.e, $u(0,t)=0°$ on equation ②, we have: $[x=0 \Rightarrow u=0]$

$$0 = c_1 e^{-p^2c^2t} \Rightarrow c_1 = 0 \quad [\because \text{if } p^2c^2t=0 \Rightarrow \text{soln}=0, \text{ it is trivial soln, we need non trivial soln}]$$

Substituting this value in equation ①, we have:

$$u = c_2 \sin px \, e^{-c^2p^2t} \quad —⑥$$

Applying another boundary condition ④ i.e, $u(l,t)=0$ on equation ⑥, we have: $[x=l \Rightarrow u=0]$

$$0 = c_2 \sin pl \, e^{-c^2p^2t} \Rightarrow \sin pl = 0 = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l} \quad [\because c_2 \neq 0, \text{ if } c_2=0 \text{ trivial soln}]$$

~~Substitution~~

Substituting this value in equation ⑥ and replacing c_2 by b_n, we have

$$u = b_n \sin\frac{n\pi x}{l} e^{-\frac{n^2c^2\pi^2t}{l^2}}$$

Adding all the solutions for different values of 'n' we have:

$$u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-(n^2c^2\pi^2t)/l^2} \quad —⑦$$

To get the value of b_n, substituting the initial condition $u(x,0)=u_0 \Rightarrow u=u_0, t=0$ in ⑦

$$\underset{f(x)}{u_0} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

which is Half Range Sine Series in the interval $[0,l]$.

Therefore: $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx$

$$= \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l} dx = \frac{2u_0}{\cancel{l}}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{\cancel{l}}}\right]_0^l$$

$$= \frac{2u_0}{n\pi}[1-\cos n\pi] = \frac{2u_0}{n\pi}[1-(-1)^n]$$

Substituting this value in equation ⑦, we have:

$$u = \sum_{n=1}^{\infty} \frac{2u_0}{n\pi}[1-(-1)^n] \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2c^2t}{l^2}}$$

Ans: The required solution is:

$$u = \sum_{n=1}^{\infty} \frac{2u_0}{n\pi}[1-(-1)^n] \sin\frac{n\pi x}{l} e^{-(n^2\pi^2c^2t)/l^2}$$

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