03-07-17 # 1. Fourier Series

1) $f(x) = f(x \pm T) = f(x \pm 2T)$

Periodic function :- A function $f(x)$ is said to be a periodic function with period (T) if

$$f(x) = f(x \pm T) = f(x \pm 2T) = f(x \pm 3T) = \cdots$$

* Sinx & Cosx are periodic function with period 2π

→ Periodic functions are of common occurence in many physical & engineering problems.

For Ex: In conduction of heat & mechanical Vibration

→ It is useful to express these function in terms of sine & cosine

→ Most of the single valued function which occur in applied mathematics can be expressed as

$a_0/2 + a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots + b_1 \cos x + b_2 \cos 2x + \cdots$ within a defined range of value of the variable such series are called Fourier Series.

Fourier Series

The fourier series of the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Where $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$; $a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx \, dx$$

<u>Note</u>:- 1) If $f(x)$ is odd function

$$\int_{-c}^{c} f(x)\,dx = 0.$$

2) If $f(x)$ is an even function

$$\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx.$$

<u>Note</u>:- 1> If $\alpha = 0$ then

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \; ; \; a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \; ; \; b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

→ These are called as Euler's Concept.

2> If $\alpha = -\pi$ then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \; ; \; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \; ; \; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

* Fourier series of even function $f(x)$ in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \; ; \; b_n = 0 \; ; \; a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

* Fourier Series of odd function $f(x)$ in $(-\pi, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \; ; \; a_n = 0 \; ; \; a_0 = 0 \; ;$$

* $\int uv = uv_1 - u'v_2 + u''v_3 + \cdots$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$v_1 = \int v \qquad u' = d(u)$

$v_2 = \int v_1 \qquad u'' = d(u')$

$v_3 = \int v_2 \qquad u''' = d(u'')$

Q) Obtain Fourier Series exp^n of the function $f(x) = \frac{1}{4}(\pi - x)^2$ in $0<x<2\pi$

Sol^n

$$f(x) = \frac{1}{4}(\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 dx = \frac{1}{4\pi} \left. \frac{-(\pi - x)^3}{3} \right|_0^{2\pi}$$

$$= \frac{-1}{12\pi}\left[(-\pi^3) - (\pi^3)\right] = \frac{2\pi^3}{12\pi} = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \cos nx\, dx = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} \cancel{\pi^2 \cos nx + x^2 \cos nx} = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - 2(\pi - x)\frac{(\cos nx)}{n^2} - 2(-1)\frac{(-\sin nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left(0 - 0 + \frac{2(-\pi)(-1)^n + 2(\pi)}{n^2} - 0\right)$$

$$= \frac{1}{4\pi}\left(\frac{+2\pi\left(1 + (+1)^n\right)}{n^2}\right) = \frac{+1}{n^2}$$ //

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi-x)^2 \sin nx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2 \frac{(-\cos nx)}{n} - 2(\pi-x)\frac{(-\sin nx)}{n^2} - 2(-1)\frac{(\cos nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{(\pi)^2(-1)}{n} - \frac{\pi^2(-1)}{n} - 0 - 0 + \frac{2}{n^3}\left[1-1\right]\right] = 0$$

$$f(x) = \frac{(\pi-x)^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx + 0$$

Q) find the fourier series to represent $(x-x^2)$ from $-\pi$ to π
Hence S.T $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - - - = \frac{\pi^2}{12}$

Sol:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,dx - \int_{-\pi}^{\pi} x^2 dx\right]$$

$$= \frac{1}{\pi}\left[0 - 2\int_0^{\pi} x^2 dx\right] = -\frac{2}{\pi}\,\frac{\pi^3}{3} = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx - \int_{-\pi}^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx - 2\int_0^{\pi} x^2\cos nx\,dx$$

$$= -2\left[x^2\left(+\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) - 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[0 - 2\pi\frac{(-(-1)^n)}{n^2} + 0\right] = \frac{-4\pi(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx - x^2 \sin nx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\left[\cancel{\pi}\frac{(-(-1)^n)}{n} + 0\right] = \frac{-2(-1)^n}{n}$$

$$\therefore f(x) = (x-x^2) = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4\pi(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin nx$$

$$\rightarrow (x-x^2) = -\frac{\pi^2}{3} + \left(\frac{4}{1^2}\cos x - \frac{4}{2^2}\cos 2x + \frac{4}{3^2}\cos x\right) + \cdots \left(\frac{2}{1}\sin x - \frac{2}{2}\sin nx + \cdots\right)$$

<u>Put $x=0$</u>

$$0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}\right) + (0)$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{12}$$

Q) Obtain fourier series $f(x) = e^{-x}$ in $0 < x < 2\pi$

Sol

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}dx = \frac{1}{\pi}\left.\frac{e^{-x}}{-1}\right|_0^{2\pi} = \frac{-1}{\pi}\left[e^{-2\pi}-1\right]$$

$$= \frac{1-e^{-2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-1\cos nx + n\sin\right.\right.$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}\left[0-1\right] - \frac{1}{1+n^2}\left[-1+0\right]\right]$$

$$= \frac{1}{\pi}\left[-\frac{e^{-2\pi}}{1+n^2} + \frac{1}{1+n^2}\right] = \frac{1-e^{-2\pi}}{(1+n^2)\pi}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx = \frac{1}{\pi}\left(\frac{e^{-x}}{1+n^2}\left(-n\cos nx +\right.\right.$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}\left(-1-0\right) - \frac{1}{1+n^2}(-1)\right]$$

$$= \frac{n}{\pi(1+n^2)}\left[-e^{-2\pi}+1\right] = \frac{(1-e^{-2\pi})n}{(1+n^2)\pi}$$

$$f(x) = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{1-e^{-2\pi}}{(1+n^2)\pi}\cos nx + \sum_{n=1}^{\infty}\frac{(1-e^{-2\pi})n}{(1+n^2)\pi}\sin$$

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4Q) Expand $f(x) = x\sin x$ in the interval $-\pi < x < \pi$; $(n \neq 1)$

Sol^n ~~f(x)~~ $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \, dx = \frac{2}{\pi} \int_{0}^{\pi} x \sin x \, dx$

$= \frac{2}{\pi} \quad x(-\cos x) -$

5) $f(x) = e^{ax}$ in $-\pi < x < \pi$.

6) $f(x) = x^2$ in $-\pi < x < \pi$ in fouri. series & S.T that

i) $1/1^2 + 1/2^2 + 1/3^2 + \cdots = \pi^2/6$ ii) $1/1^2 - 1/2^2 + 1/3^2 - \cdots = \pi^2/12$

iii) $1/1^2 + 1/3^2 + 1/5^2 + \cdots = \pi^2/8$

Sol^n $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi} \left.\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{\pi^3 - (-\pi^3)}{3\pi} = \frac{2\pi^2}{3}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \frac{(\sin nx)}{n} - 2x \cdot \frac{(-\cos nx)}{n^2} \ldots$

$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} + \frac{2x \cos nx}{n^2} + \frac{2 \sin nx}{n^3} \right]_0^{\pi}$ $[\cos n\pi = (-1)^n]$

$= \frac{2}{\cancel{\pi}}\left[0 + \frac{2\cancel{\pi}(-1)^n}{n^2} + 0 - 0 + 0 + 0 \right] = \frac{4(-1)^n}{n^2}$

$b_n = 0;$

$f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx = x^2$

$\frac{2\pi^2}{3} + 4 \left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \cdots \right] = x^2$

let $x = 0$ $\frac{2\pi^3}{3} + 4\left[-1/1^2 + 1/2^2 - 1/3^2 \cdots \right] = 0$

$\pi^3/6 = 1/1^2 - 1/2^2 + 1/3^2 - \cdots$

Let $x = \pi$ $\quad \pi^2 = \frac{2\pi^3}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$

$$\frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

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Let $x =$ $\frac{\pi^2}{12} + \frac{\pi^3}{6} = \frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$ Ans

Fourier Series Of discontinuous Function

* If $f(x)$ has finitely many finite points of finite discontinuity even then it can be expressed as fourier series

> The integrals for a_0, a_n, b_n are to be evaluated by breaking the range of integration.

> Let $f(x)$ be defined by $f(x) = \begin{cases} f_1(x) & c < x < x_0 \\ f_2(x) & x_0 < x < c+2\pi \end{cases}$

* Where x_0 is the pt. of finite discontinuity in the interval $(c, c+2\pi)$ then the values of eulers constants are given by -

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\,dx + \int_{x_0}^{c+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\cos n(x)\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\sin nx\,dx\right]$$

5 Q) find fourier series expansion for the function

6 $$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Sol. $$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right] = \frac{1}{\pi}\left[-\pi\left[0+\pi\right] + \frac{\pi^2}{2}\right]$$

S $$= \frac{-\pi^2}{2\pi} = -\pi/2$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)\Big|_{-\pi}^{0} + \left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \cancel{\pi(-(-1)^n)}\ \left[0 + \frac{(-1)^n}{n^2}\right] - \left[0 + \frac{1}{n^2}\right]\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[+\pi\left(\frac{\cos nx}{n}\right)\Big|_{-\pi}^{0} + \left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left[\frac{1-(-1)^n}{n}\right] + \frac{\pi(-(-1)^n)}{n}\right]$$

Q) $f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < \pi < \pi \end{cases}$

Sol^n

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} -k\,dx = \frac{-k(0+\pi)}{\pi} = -k + \frac{1}{\pi}\int_{0}^{\pi} k\,dx = \cancel{-k} + \cancel{k} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -k\cos nx + \int_{0}^{\pi} k\cos nx\,dx = \frac{1}{\pi}\left[-k.\frac{\sin nx}{n}\Big|_{-\pi}^{0} + k\frac{\sin nx}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-k(0-0) + k(0-0)\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -k\sin nx + \int_{0}^{+\pi} k\sin nx = \frac{1}{\pi}\left[-k\frac{(-\cos nx)}{n}\Big|_{-\pi}^{0} + k\frac{(-\cos nx)}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{k}{n}(1-(-1)^n) - \frac{k}{n}((-1)^n - 1)\right] = \frac{1}{\pi}\left[\frac{k}{n}\left(1-(-1)^n - (-1)^n + 1\right)\right]$$

$$= \frac{k}{\pi n}(2 - 2(-1)^n) = \frac{2k(1-(-1)^n)}{\pi n}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2k(1-(-1)^n)}{\pi n}\sin nx = \frac{2k}{\pi}\left[\frac{2\sin x}{1} + 0 + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x + \cdots\right]$$

At $x = \pi/2$; $f(x) = k$

$$\cancel{k} = \frac{4\cancel{k}}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right] = \frac{\pi}{4}$$

Q) 9) $f(x) = \begin{cases} 1 & 0<x<\pi \\ 2 & \pi<x<2\pi \end{cases}$

Soln

$$a_0 = \frac{1}{\pi}\int_0^{\pi} dx + \int_{\pi}^{2\pi} 2\,dx = \frac{1}{\pi}\left[\pi + 2\pi\right] = 3$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos nx\,dx + \int_{\pi}^{2\pi} 2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\Big|_0^{\pi} + 2\left(\frac{+\sin nx}{n}\right)\Big|_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi n}\left[0+0 \right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin nx + \int_{\pi}^{2\pi} 2\sin nx = \frac{1}{\pi}\left[-\frac{\cos \pi x}{n}\Big|_0^{\pi} + 2\frac{(-\cos}{n} \right.$$

$$= \frac{1}{\pi n}\left[-(-1)^n + 1 + 2\left(-1+(-1)^n\right) \right] = \frac{1}{\pi n}\left[-(-1)^n + 1 - 2 + \right.$$

$$= \frac{(-1)^n - 1}{\pi n}$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n} \sin x$$

Q) Find Fourier series of the function $f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$

Sol:

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 + \int_{0}^{\pi} \sin x \, dx\right] = \frac{1}{\pi}(-\cos x)\Big|_0^{\pi} = \frac{1+1}{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx \, dx = \frac{1}{2\pi}\int_0^{\pi}\left[\sin(1+n)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(1+n)x}{(1+n)} - \frac{\cos(1-n)x}{(1-n)}\right]_0^{\pi} = \frac{1}{2\pi}\left[\frac{-(-1)^n}{(1+n)} - \frac{(-1)^n}{(1-n)} - \left(-\frac{1}{1+n} - \frac{1}{1-n}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1-(-1)^{n+1}}{1+n} + \frac{1-(-1)^{(1-n)}}{1-n}\right] = \frac{1-(-1)^n}{2\pi}\left[\frac{\cancel{1}-\cancel{n}+\cancel{1}+\cancel{n}}{(1+n)(1-n)}\right]$$

$$= \left[\frac{1-(-1)^n}{1+n^2} + \frac{(-1)^{1-n}-1}{n-1}\right]\frac{1}{2\pi}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx = \frac{1}{2\pi}\int_0^{\pi}\left(\cos(1-n)x - \cos(1+n)x\right)dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)x}{(1-n)} - \frac{\sin(1+n)x}{(1+n)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[0 - 0 + 0 - 0\right] = 0$$

$$\cancel{f(x) = \sum_{n=1}^{\infty}\frac{1-(-1)^n \cos nx}{(1-n^2)} + \frac{2}{\pi}}$$

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left\{\frac{(1-(-1)^{n+1}}{n+1} + \frac{(-1)^{1-n}-1}{n-1}\right\}\cos nx$$

$$a_n = \begin{cases} 0 & ; n \text{ is odd} \\ \frac{2n}{\pi(n^2-1)} & ; n \text{ is even} \end{cases} \quad ; n \neq 1$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{0} 0 + \frac{1}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{2\pi} \frac{-\cos 2x}{2} \Big|_0^{\pi} = \frac{-(+1)-(-1)}{2}$$

$$a_1 = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx = \int_0^{\pi} \frac{1-\cos 2x}{2\pi} dx = \frac{x}{2\pi} - \frac{\sin 2x}{4\pi} \Big|$$

$$= \frac{\pi}{2\pi} - 0 - 0 - 0 = \frac{1}{2}$$

$$b_1 = \frac{1}{2}$$

$$\therefore f(x) = \frac{1}{\pi} + \sum_{n=2}^{\infty} \frac{1}{2\pi}\left[\frac{1-(-1)^n}{1+n} + \frac{(-1)^{1-n}-1}{1-n}\right] + \frac{1}{2}\sin nx$$

$$= \frac{1}{\pi} + \sum_{n=2,4,6}^{\infty} \frac{2}{\pi(1-n^2)} \cos nx + \sum_{n=1,3,}^{\infty} (0) + \frac{b}{2} \sin nx$$

$$= \frac{1}{\pi} + \sum_{n=\text{even}}^{\infty} \frac{2}{\pi(1-n^2)} \cos nx + \frac{b}{2} \sin nx$$

Q) Find F.S to represent the function $f(x) = \begin{cases} -K & -\pi < x < 0 \\ K & 0 < x < \pi \end{cases}$
and S.T $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - - - -$

Q) $f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \\ 2 & \text{for } \pi < x < 2\pi \end{cases}$

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1) If $f(x) = \begin{cases} x & 0 \leq x \leq \pi \\ 2\pi - x & \pi \leq x \leq 2\pi \end{cases}$ then find fourier series & deduce that

i) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

sol^n

$$a_0 = \frac{1}{\pi} \int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\Big|_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)\Big|_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right) \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\cancel{4}^{2}\pi^2}{\cancel{2}} - \frac{3\pi^2}{2} \right] = \frac{1}{\pi}\left[\pi^2\right] = \pi$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx\,dx \right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{+\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\Big|_0^{\pi} + \left[(2\pi - x)\left(\frac{+\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi}\left[0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2} \right] + \left[0 + \frac{(-(1))}{n^2} - \left[0 - \frac{(-1)^n}{n^2}\right] \right]$$

$$= \frac{(-1)^n - 1 - 1 + (-1)^n}{\pi n^2} = \frac{2\left((-1)^n - 1\right)}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x \sin nx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[\left[x\frac{(-\cos nx)}{n} - \frac{(-\sin nx)}{n^2}\right]_0^{\pi} + \left[(2\pi - x)\frac{(-\cos nx)}{n} - (-1)\frac{(-\sin nx)}{n^2}\right.\right.$$

$$= \frac{1}{\pi}\left[\left(\pi\frac{(-(-1)^n)}{n} - 0\right) - (0-0)\right] + \left[0 - 0 - \left[\pi\frac{(-(-1)^n)}{n} - 0\right]\right]$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cancel{(-1)^n}}{\cancel{n}} + \pi\frac{\cancel{(-1)^n}}{\cancel{n}}\right] = 0$$

$$f(x) = \pi/2 + \sum_{n=1}^{\infty}\frac{2((-1)^n - 1)}{\pi n^2}\cos nx = \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2\cos x}{1^2} + \frac{0}{2^2} + \frac{-2\cos 3x}{3^2} + \cdots\right]$$

$$\frac{\pi}{2} + \frac{(-4)}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right] = f(x)$$

Put $x=0$ $\quad f(x) = \begin{cases} x & ; 0 < x < \pi \end{cases} = x$

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[1/1^2 + 1/3^2 + \cdots\right]$$

$$\pi^2/8 = 1/1^2 + 1/3^2 + \cdots$$

Note*:- Let $x = x_0$ is the pt of discontinuity and left limit and right limit at x_0 exist but are not equal then the value of $f(x)$ at pt x_0 is

$$f(x) = \frac{1}{2}\left[f(x_0 - 0) + f(x_0 + 0)\right] \qquad f(x) = \begin{cases} f_1(x) & 0 \le x \le x_0 \\ f_2(x) & x_0 < x \le 2\pi \end{cases}$$

$$= 1/2\left[f_1(x) + f_2(x)\right]$$

Change of Interval

In many applications we require an expansion of a given function $f(x)$ over an interval of length diff. from 2π.

Let $f(x)$ be a periodic function defined in the interval $(c, c+2l)$ then

$$* \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx \; ; \quad a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Q) $f(t) = 1 - t^2$ as fourier series in $-1 \le t \le 1$

Sol^n ~~$a_0 = \frac{1}{\pi}\int_{-1}^{1}$~~

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx \quad \& \quad a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\,dx$$

$$= 2\int_0^1 (\cos n\pi t - t^2\cos n\pi t)$$

$$= 2\int_0^1 (1-t^2)\,dt$$

$$= 2\left(t - \frac{t^3}{3}\right)\Big|_0^1$$

$$= 2\left[(1-0) - \left(\frac{1-0}{3}\right)\right]$$

$$a_n = 2\int_0^1 (\cos n\pi t - t^2 \cos n\pi t)\, dt$$

$$= 2\left[\frac{\sin n\pi t}{n\pi}\Big|_0^1 - \left(t^2\left(\frac{\sin n\pi t}{n\pi}\right) - 2t\left(-\frac{\cos n\pi t}{(n\pi)^2}\right) - 2 \right.\right.$$

$$= 2\left[(0-0) - \left(0 - \frac{2(-(-1)^n)}{n\pi^2} - 2(0)\right) - 0 - 0 - 0\right]$$

$$= 2\left[-\frac{2(-1)^n}{n\pi^2}\right] = -\frac{4(-1)^n}{n^2\pi^2}$$

$$f(x) = \frac{4}{6} + \sum_{n=1}^{\infty} \frac{(-4(-1)^n)}{n^2\pi^2} \cos n\pi t$$

(2) $f(x) = x - x^2$ in $(-1, 1)$

(3) $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$

(4) $f(x) = e^{-x}$ in $(-l, l)$

(5) $f(x) = x^2 - 2$ in $-2 < x < 2$

(6) $f(x) = \begin{cases} x\pi & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\left[1-\cancel{(-1)^n}+\cancel{(-1)^n}\right]\right] = \frac{1}{n}$$

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}\left((-1)^n-1\right)\cos nx + \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

2sol: $f(x) = x - x^2$ in $(-1, 1)$

$$a_0 = \frac{1}{1}\int_{-1}^{1}(x-x^2)\,dx = \left.\frac{x^2}{2}-\frac{x^3}{3}\right|_{-1}^{1} = \left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{2}+\frac{1}{3}\right)$$

$$= -\frac{2}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1}(x-x^2)\cos n\pi x\,dx$$

$$= \left[x\left(+\frac{\sin n\pi x}{n\pi}\right) - \frac{(-\cos n\pi x)}{(n\pi)^2}\right]_{-1}^{1} - \left[x^2\frac{(\sin n\pi x)}{n\pi} - 2x\frac{(-\cos n\pi x)}{(n\pi)^2} - 2\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_{-1}^{1}$$

$$= \left(\left[0 - \cancel{\frac{(-(-1)^n)}{(n\pi)^2}}\right] - \left[0 - \cancel{\frac{(-(-1)^n)}{(n\pi)^2}}\right]\right) - \left(\left[0 - \frac{(2(-(-1)^n)}{(n\pi)^2} - 0\right]\right.$$

$$\left. - \left(0 - \frac{2(-1)(-(-1)^n)}{(n\pi)^2} - 0\right)\right)$$

$$= -\left[\frac{2(-1)^n + 2(-1)^n}{(n\pi)^2}\right] = \frac{-4(-1)^n}{(n\pi)^2}$$

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$$b_n = \int_{-1}^{1} (x \sin n\pi x - x^2 \sin n\pi x)\,dx$$

$$= \left[x\left(\frac{-\cos n\pi x}{n\pi}\right) - \left(\frac{-\sin n\pi x}{(n\pi)^2}\right) \right]_{-1}^{1} -$$

$$\left[x^2 \frac{(-\cos n\pi x)}{n\pi} - 2x \frac{(-\sin n\pi x)}{(n\pi)^2} - 2 \frac{(\cos n\pi x)}{(n\pi)^3} \right]_{-1}^{1}$$

$$= \left[\left(1\frac{(-(-1)^n)}{n\pi} - 0 \right) + \left(\frac{(-1)(-(-1)^n)}{n\pi} \right) \right] -$$

$$\left(\left[(1)\cancel{\frac{(-(-1)^n)}{n\pi}} - 2(0) - 2\cancel{\frac{(-1)^n}{(n\pi)^3}} \right] - \left[(1)\cancel{\frac{(-(-1)^n)}{n\pi}} - 2(0) - 2\cancel{\frac{(-1)^n}{(n\pi)^3}} \right] \right)$$

$$= -\frac{2(-1)^n}{n\pi} - [0] = \frac{-2(-1)^n}{n\pi}$$

Q) $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$

$a_0 = \frac{1}{l}\int f(x)$ $0 < x$

$c + 2l$

-2 $= 0$ $2l = $

$l = 1$

Sol $a_0 = \frac{1}{\cancel{2}}\left[\int_0^2 dx\right] = \frac{1}{\cancel{2}}(2) = 2$

$$a_1 = \frac{1}{\cancel{2}}\int_0^2 (1)\cos n\pi x\,dx = \frac{\sin n\pi x}{n\pi} = 0$$

$$b_n = (1)\int_{-2}^{0} 0 \, dx + \int_{0}^{2} (1) \sin n\pi x \, dx = \int_{0}^{2} \sin n\pi x \, dx$$

$$= \left[\frac{-\cos n\pi x}{n\pi}\right]_0^2 = -1/n\pi + 1/n\pi = 0$$

$$f(x) = \frac{a_0}{2} + 0 + 0 = 2/2 = 1$$

(4) $f(x) = e^{-x} \quad (-l, l)$

Sol $a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x} dx = \frac{1}{l} \quad -e^{-x}\Big|_{-l}^{l}$

$$= \frac{1}{l}\left(-e^{-l} - (-e^{l})\right) = \frac{e^{l} - e^{-l}}{l}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} e^{-x} \cos\frac{n\pi x}{l} \, dx \qquad = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left((-1)\cos\frac{n\pi}{l}x + \frac{n\pi}{l}\sin\frac{n\pi}{l}x\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{e^{-l}}{1+\left(\frac{n\pi}{l}\right)^2}\left((-1)(-1)^n + (0)\right) - \frac{e^{l}}{1+\left(\frac{n\pi}{l}\right)^2}\left((-1)(-1)^n + 0\right)\right]$$

$$= \frac{1}{l}\left[\frac{e^{-l}(-(-1)^n) + e^{l}(-1)^n}{1+\left(\frac{n\pi}{l}\right)^2}\right] = \frac{(e^{l} - e^{-l})(-1)^n}{1+\left(\frac{n\pi}{l}\right)^2}$$

$(b\cos bx - a\sin ...)$

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(\frac{n\pi}{l}\cos\frac{n\pi}{l}x ...\right.\right.$$

$\sin\frac{n\pi}{l}$

$$= \frac{1}{l}\left[\frac{e^{-l}}{1+\left(\frac{n\pi}{l}\right)^2}\left(\frac{n\pi}{l}(-1)^n+0\right) - \frac{e^{l}}{1+\left(\frac{n\pi}{l}\right)^2}\left(\frac{n\pi}{l}(-1 ...\right.\right.$$

$$= \frac{1}{l}\left[\frac{e^{-l}-e^{l}}{1+\left(\frac{n\pi}{l}\right)^2}\right]\left[\frac{n\pi}{l}(-1)^n\right] = \frac{(e^{-l}-e^{l})(n\pi ...)}{l^2+(n\pi)^2}$$

8) $f(x) = x^2-2 \;;\; -2 \le x \le 2$

Sol: $b_n = 0$; even function

$$a_0 = \frac{\cancel{2}}{\cancel{2}}\int_0^2 (x^2-2)\,dx = \frac{x^3}{3} - 2x\Big|_0^2 = \frac{8}{3} - 4 = -\frac{4}{...}$$

$$a_n = \int_0^2 (x^2-2)\cos\frac{n\pi x}{2} = \int_0^2 x^2\cos\frac{n\pi}{2}x - 2\cos\frac{n\pi}{2} ...$$

$$= \left[\frac{x^2\left(-\sin\frac{n\pi}{2}x\right)}{n\pi/2} - \frac{2x\left(-\cos\frac{n\pi}{2}x\right)}{(n\pi/2)^2} - \frac{2\left(\sin\frac{n\pi}{2}x\right)}{(n\pi/2)^3} ...\right.$$

$$\left.\frac{2\sin\frac{n\pi}{2}x}{(n\pi/2)}\right|_0^2$$

$$= \left[\left(0 + \frac{4(-1)^n}{\left(\frac{n\pi}{2}\right)^2} - 0 \right) - \right.$$

Half range series

* Sometimes it is required to represent a function $f(x)$ by a fourier series in the interval $(0,\pi)$ not in the interval $(-\pi,\pi)$ since the function may be not defined in $(-\pi,0)$

* We can choose $f(-x) = f(x)$ in the interval $(-\pi,0)$ in such cases $f(x)$ behaves as a even function for which $b_n = 0$ and which is known as Half range cosine series

* Half range cosine ~~function~~ series of the fun $f(x)$ in the range $(0,\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{where}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \; ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

* We choose $f(-x) = -f(x)$ in the interval $(-\pi, 0)$ then $f(x)$ behaves as an odd function. hence a_0, a_n are 0's. then
* The half range sine series of the function $f(x)$ in the range $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx$$

18-07-17 :-

* Half range sine series of the function $f(x)$ in $(0,l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \quad \text{where} \quad b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi}{l}x$$

* Half range cosine series of the function $f(x)$ in the range $(0,l)$ is.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x$$

Where $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$ and $a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi}{l}x\,dx$

Q) Express $f(x) = x$ as H.R.S.S in $0 < x < 2$

Sol: $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{\cancel{2}}{\cancel{2}}\int_0^{2} x\sin\frac{n\pi}{2}x\,dx = \left[x\frac{\left(-\cos\frac{n\pi}{2}x\right)}{(n\pi/2)} - \frac{\left(-\sin\frac{n\pi}{2}x\right)}{(n\pi/2)^2}\right]_0^2$$

$$= \frac{2\left(-(-1)^n - 0\right)}{n\pi/2} = \frac{-4(-1)^n}{n\pi}; \quad f(x) = \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n\pi}\sin\frac{n\pi}{2}x$$

Q) $f(x) = \begin{cases} \frac{1}{4} - x & 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & \frac{1}{2} < x < 1 \end{cases}$ Express as Sine term $(l = \frac{1}{2})$

Sol $b_n = 2\left[\int_0^{1/2} (\frac{1}{4} - x) \sin 2n\pi x + \int_{1/2}^{1} (x - \frac{3}{4}) \sin 2n\pi x\right]$

$= 2\left[\frac{1}{4}\frac{(-\cos\frac{n\pi}{2}x)}{\frac{n\pi}{2}} - \left[x\frac{(-\cos\frac{n\pi}{2}x)}{\frac{n\pi}{2}} - \frac{(-\sin\frac{n\pi}{2}x)}{(\frac{n\pi}{2})^2}\right]\right]_0^{1/2} +$

$\left[x\frac{(-\cos n\pi x)}{n\pi} - \frac{(-\sin n\pi x)}{(n\pi)^2} - \frac{3}{4}\left(-\frac{\cos n\pi x}{n\pi}\right)\right]_{1/2}^{1}$

$= 2\left[\frac{1}{4}(0) + 0 - \frac{(-1)^n}{(n\pi)}\right] = \frac{1}{2n\pi}\left[(-1)^n - 1\right]$

find H.R.C.S in $0 < x < l$.

Q) $f(x) = (x-1)^n$; $0 < x < 1$;

Q) $f(x) = e^x$; find H.R.S.S in $0 < x < 1$

Q) $f(x) = \begin{cases} x^2 & 0 \le x \le 2 \\ 4 & 2 \le x \le 4 \end{cases}$ find Cosine Series

Q) $f(x) = x^2$. find H.R.C.C in $0 < x < \pi$

~~Q~~ Hence S.T $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Q) find H.R.C.S of $f(x) = \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x < \pi \end{cases}$

Q) Obtain Cosine Series of $x \sin x$ in $(0, \pi)$ and

Hence S.T $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \cdots = \frac{\pi - 2}{4}$

Q) H.R.C.S of $f(x) = \begin{cases} kx & 0 < x < \frac{l}{2} \\ k(l - x) & \frac{l}{2} < x < l \end{cases}$

8) $f(x) = (x-1)^2 ; \quad 0 < x < 1$

Sol^n $a_0 = \frac{2}{1}\int_0^1 (x-1)^2 dx = 2 \left. \frac{(x-1)^3}{3} \right|_0^1 = 2\left[0 - \left(\frac{-1}{3}\right)\right] = \frac{2}{3}$

$a_n = \frac{2}{1}\int_0^1 (x-1)^2 \cos n\pi x = 2\left[(x-1)^2 \frac{(-\sin n\pi x)}{n\pi} - (2(x-1))\frac{(-\cos n\pi x)}{(n\pi)^2}\right.$

$\left.\left. - 2\left(+\frac{\sin n\pi x}{(n\pi)^3}\right)\right]\right|_0^1$

$= 2\left[0 - 0 - 0 - \left(0 - 2\frac{(-1)(-1)}{(n\pi)^2}\right)\right] = \frac{4}{(n\pi)^2}$

9) $f(x) = e^x ; \quad 0 < x < 1$

Sol $b_n = \frac{2}{1}\int_0^1 e^x \sin n\pi x \, dx$

$= 2\left[\left. \frac{e^x}{1+(n\pi)^2}\left((n\pi)\cos n\pi x - \sin n\pi x\right)\right|_0^1\right]$

$= \frac{2e}{1+(n\pi)^2}\left(n\pi(-1)^n - 0\right) - \frac{2}{(1+n\pi)^2}\left(n\pi - 0\right)$

$= \frac{2n\pi}{1+(n\pi)^2}\left[(-1)^n e - 1\right]$

5Q) $f(x) = \begin{cases} x^2 & 0 \le x \le 2 \\ 4 & 2 \le x \le 4 \end{cases}$ $\quad l = 4$

Sol:

$$a_0 = \frac{2}{4}\left[\int_0^2 x^2 dx + \int_2^4 4\,dx\right] = \frac{1}{2}\left[\frac{x^3}{3}\Big|_0^2 + 4x\Big|_2^4\right]$$

$$= \frac{1}{2}\left[\frac{8}{3} + 8\right] = \frac{16}{3}$$

$$a_n = \frac{2}{4}\left[\int_0^2 x^2 \cos\frac{n\pi}{4}x + \int_2^4 4\cos\frac{n\pi}{4}x\right]$$

$$= \frac{1}{2}\left[x^2\frac{\left(-\sin\frac{n\pi}{4}x\right)}{(n\pi/4)} - 2x\frac{\left(-\cos\frac{n\pi}{4}x\right)}{(n\pi/4)^2} - 2\frac{\left(\sin\frac{n\pi}{4}x\right)}{(n\pi/4)^3}\right]_0^2 +$$

$$\left[4\frac{\left(-\sin\frac{n\pi}{4}x\right)}{n\pi/4}\right]_2^4$$

$$= \frac{1}{2}\left[\cancel{4\frac{\left(-\sin\frac{n\pi}{2}\right)}{(n\pi/4)}} - 4\frac{\left(-\cos\frac{n\pi}{2}\right)}{(n\pi/4)^2} - 2\frac{\left(\sin\frac{n\pi}{2}\right)}{(n\pi/4)^3}\right] +$$

$$\left[4(0) - \cancel{4\frac{\left(-\sin\frac{n\pi}{2}\right)}{(n\pi/4)}}\right]$$

$$= \frac{1}{2}\left[\frac{4\cos n\pi/2}{(n\pi/4)^2} - \frac{2\sin n\pi/2}{(n\pi/4)^3}\right]$$

68) $f(x) = x^2$; $0 < x < \pi$ H.R.C.C

Sol:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\cancel{\pi}} \frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos \frac{n\cancel{\pi}}{\cancel{\pi}} x = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(-\frac{\cos nx}{n^2} \right) - 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(0 - 2\pi \frac{(-(-1)^n)}{n^2} - 0 \right) - (0 - 0 - 0) \right]$$

$$= \frac{4(-1)^n}{n^2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{2\pi^2}{3} + 4 \left[\frac{-1}{1} \cos x + \frac{1}{2^2} \cos 2x - \frac{1}{3^2} \cos 3x - - - \right]$$

$$x^2 = \frac{2\pi^2}{3} + 4 \left[-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} \right]$$

$$\frac{\cancel{2}\pi^2}{3} = \cancel{4}^{2} \left[-\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - - \right] = \frac{\pi^2}{6}$$

Partial Differential Equations

→ Let 'z' is a function of x & y where x & y are independent variable. Then partial derivatives of z are denoted as follows : ; $z = f(x,y)$

$$p = \frac{\partial z}{\partial x} \; ; \; q = \frac{\partial z}{\partial y} \; ; \; r = \frac{\partial^2 z}{\partial x^2} \; ; \; s = \frac{\partial^2 z}{\partial x \partial y} \; ; \; t = \frac{\partial^2 z}{\partial y^2}$$

→ Partial diff. eq^n :-

> An equation involving partial derivatives is called an partial diff. eq^n.

* Formation of partial diff.^{n} eq^n:-

Ex:- $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + 2\frac{\partial^2 z}{\partial x \partial y} = 0$

→ Partial diff. eq^n can be formed by elimination of
1. Arbitary Constant
2. Arbitary functions

1> Elimination of arbitory Constant

* If the no. of arbitary Constant eliminated are equal to the no. of independent variables then the resulting partial diff. eq^n will be of first order.

* If the no. of arbitary constant eliminated are more than the no. of independent variable.

then the resulting partial diff. eqn will be of higher order.

2> Elimination of Arbitrary function

The number of arbitrary function eliminated should be equal to the order of the resulting partial diff. eqn.

→ Form the partial differential equation from the following equations by eliminating arbitrary constants:

① $z = (x^2+a)(y^2+b)$ Where a & b are arbitrary constant

Sol: $\frac{\partial z}{\partial x} = (2x)(y^2+b) = 2xy^2+2xb$ —①

$\frac{\partial z}{\partial y} = (x^2+a)(2y) = 2x^2y + 2ya$ —②

$\cancel{\frac{\partial z}{\partial}}$ $\frac{p}{2x} - y^2 = b$; $\frac{q}{2y} - x^2 = a$

$z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right)$.

② $2z = x^2/a^2 + y^2/b^2$

③ $(x-a)^2+(y-b)^2+z^2 = c^2$

Where a, b, are arbitrary constant.

4) $z = xy + y\sqrt{a^2+x^2} + b$

5) $z = a(x+y) + b(x-y) + abt + c$

6) $z = ax + by + a^2 + b^2$

7) $z = axe^y + \frac{1}{2}a^2e^{2y} + b$

8) $az + b = a^2x + y$

2 Sol^n $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$\cancel{2}p = \frac{\cancel{2}x}{a^2}$; $\cancel{2}q = \frac{\cancel{2}y}{b^2}$

$a = \sqrt{x/p}$ $b = \sqrt{y/q}$

$2z = xp + yq$

3 Sol^n $(x-a)^2 + (y-b)^2 - c^2 = -z^2$

$-\cancel{2}zp = \cancel{2}(x-a)$; $-\cancel{2}zq = \cancel{2}(y-b)$

$a = x + pz$ — ① $b = y + zq$ — ②

$\Rightarrow (-pz)^2 + (-zq)^2 - c^2 = -z^2$

$p^2z^2 + z^2q^2 + z^2 = c^2$

4 sol. $z = xy + y\sqrt{a^2+x^2} + b$

$$\frac{\partial z}{\partial x} = p = y + y\frac{\cancel{2}x}{\cancel{2}\sqrt{a^2+x^2}} = y\left(1+\frac{x}{\sqrt{a^2+x^2}}\right)$$

$$\Rightarrow \frac{p}{y} - 1 = \frac{x}{\sqrt{a^2+x^2}} \Rightarrow \frac{(p-y)^2}{y^2} = \frac{x^2}{a^2+x^2}$$

$$\Rightarrow \frac{x^2y^2}{(p-y)^2} = a^2+x^2 \Rightarrow a = \sqrt{\left(\frac{x^2y^2}{(p-y)^2} - x^2\right)} \quad —(1)$$

$$\Rightarrow \frac{\partial z}{\partial y} = q = x + \sqrt{a^2+x^2}$$

$$\sqrt{(q-x)^2 - x^2} = a \quad —(2)$$

$$\Rightarrow (1) = (2)$$

$$\frac{x^2y^2}{(p-y)^2} - \cancel{x^2} = (q-x)^2 - \cancel{x^2}$$

$$\Rightarrow \frac{xy}{p-y} = q - x \Rightarrow xy = (p-y)(q-x) //$$

5 sol. $z = a(x+y) + b(x-y) + abt + c$

~~$p = a(1+$~~

$p = a + b$ —(1) ; $q = a - b$ —(2) ; $r = ab$ —(3)

$a = \frac{p+q}{2}$; $b = \frac{p-q}{2}$; $\Rightarrow p^2 = q^2 + 4r$

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6) $z = ax + by + a^2 + b^2$

Solⁿ $\frac{\partial z}{\partial x} = p = a$; $\frac{\partial z}{\partial y} = q = b$;

$$z = px + qy + p^2 + q^2$$

7) $z = axe^y + \frac{1}{2}a^2e^{2y} + b$

Solⁿ $p = ae^y + 0$; $q = axe^y + \frac{a^2 2e^{2y}}{2} + 0$

$\boxed{a = \frac{p}{e^y}}$ — (1) $\quad q = \frac{p}{\cancel{e^y}} x \cancel{e^y} + \frac{p^2}{\cancel{e^{2y}}} \cancel{e^{2y}}$

$$q = p^2 + px \quad //$$

8) $az + b = a^2x + y$

Solⁿ $ap = a^2$; $aq = 1$

$p = a$ $\quad q = 1/a$

$p = 1/q \implies \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1$

Problem eliminating Arbitary function

* Form the partial diff. eqⁿ

1) $z = f(x^2 - y^2)$ — (1)

D.W to 'x'	D.W to 'y'
$p = 2x f'(x^2 - y^2)$	$q = -2y f'(x^2 - y^2)$

$\frac{p}{q} = -\frac{x}{y} \implies py + qx = 0$

2) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

3) $z = f(\sin x + \cos y)$

4) $F(x+y+z,\ x^2+y^2+z^2) = 0$

(or)

$x+y+z = f(x^2+y^2+z^2)$

(or)

$x^2+y^2+z^2 = f(x+y+z)$

2sol^n $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

$p = \underline{2f'\left(\frac{1}{x} + \log y\right)}\left(-\frac{1}{x^2}\right)$ — (1)

$q = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$ — (2)

$q = 2y - \frac{x^2 p}{y}$

3sol^n $z = f(\sin x + \cos y)$

$p = f'(\sin x + \cos y)(\cos x)$ — (1)

$q = f'(\sin x + \cos y)(-\sin y)$ — (2)

$\frac{p}{q} = -\frac{\cos x}{\sin y} \Rightarrow$ $p\sin y + q\cos x = 0$

4 Sol: $f(x^2+y^2+z^2) = x+y+z$ $\frac{\partial y}{\partial x}$

$$f'(x^2+y^2+z^2)(2x+2zP) = 1+P \quad \text{--- (1)}$$

$$f'(x^2+y^2+z^2)(2y+2zq) = 1+q \quad \text{--- (2)}$$

$$\frac{(1)}{(2)} \quad \frac{(2x+2zP)}{(2y+2zq)} = \frac{(1+P)}{(1+q)} \Rightarrow x+zp = (y+zq)\left(\frac{1+p}{1+q}\right)$$

$\cancel{\Rightarrow (x-y) = z(q-p)}$

$$\Rightarrow (x+zp)(1+q) = (y+zq)(1+p)$$

$$\Rightarrow (x+zp)(1+q) = (y+zq)(1+p)$$

5Q) $xyz = \phi(x+y+z)$

6Q) $F(x^2+y^2, z-xy) = 0$

7Q) $F(xy+z^2, x+y+z) = 0$

8Q) $z = f_1(x) f_2(y)$

9Q) $z = f(x+at) + g(x-at)$

10Q) $z = x f_1(x+t) + f_2(x+t)$

11Q) $z = y f(x) + x g(y)$

12Q) $z = f(x+4t) + g(x-4t)$

13Q) $z = f_1(y+2x) + f_2(y-3x)$

14Q) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

5 Sol: $xyz = \phi(x+y+z)$

W.r.t. 'x' $(xyP+yz) = \phi'(x+y+z)(1+P)$ --- (1)

'y' $(xz+xyq) = \phi'(x+y+z)(1+q)$ --- (2)

$$\frac{(1)}{(2)} \quad \frac{y(xP+z)}{x(yq+z)} = \frac{(1+P)}{(1+q)}$$

$$(y+yq)(xP+z)=(x+Px)(yq+z)$$

$$\cancel{xyP}+yz+\cancel{xyPq}+yzq=\cancel{xyq}+xz+\cancel{xyPq}+Pxz$$

$$xy(P-q)=z(x(1+P)-y(1+q))$$

6 Sol^n $F(x^2+y^2,\ z-xy)=0$

$$x^2+y^2=f(z-xy)$$

w.r.to 'x' : $2x=f'(x^2+y^2)(P-y)$ —①

w.r.to 'y' $2y=f'(z-xy)(q-x)$ —②

$\frac{①}{②}$ $\frac{x}{y}=\frac{P-y}{q-x}$

$$\Rightarrow qx-x^2=Py-y^2$$

$$\boxed{x^2-y^2=qx-Py}$$

7 Sol^n $x+y+z=f(xy+z^2)$

w.r.to 'x' $1+P=f'(\ )(y+2zP)$ —①

w.r.to 'y' $1+q=f'(\ )(x+2zq)$ —②

$\frac{①}{②}$ $\frac{1+P}{1+q}=\frac{y+2zP}{x+2zq}$

$$\underline{x}+\underline{2zq}+xP+2z\cancel{Pq}=\underline{y}+\underline{2zp}+yq+2z\cancel{Pq}$$

$$(x-y)+2z(q-P)=yq-Px$$

8Sol: $z = f_1(x) f_2(y)$.

W.r.to 'x' $\quad p = f_1'(x) f_2(y) \quad$ —①

W.r.to 'y' $\quad q = f_1(x) f_2'(y) \quad$ —②

W.r.to x to ③ $\quad r = f_1'(x) f_2'(y) \quad$ —③

① × ② $\quad p \times q = f_1'(x) f_2(y) \times f_1(x) f_2'(y) \quad$ —④

divide $\quad \boxed{pq = rz}$

9Sol: $z = f(x+at) + g(x-at)$

W.r.to 'x' $\quad p = f'(x+at) + g'(x-at) \quad$ —①

W.r.to 'y' $\quad q = f'(x+at)(a) + g'(x-at)(-a)$

$= a(f'(x+at) - g'(x-at)) \quad$ —②

~~② W.r.to 'x'~~ $\cancel{r = a(f'(x}$

$\cancel{q = a\sqrt{p^2}}$

① W.r.to 'x' $\quad \dfrac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at)$

$\dfrac{\partial^2 z}{\partial t^2} = f''(x+at)(a^2) + g''(x-at)(a)^2$

$= a^2 \dfrac{\partial^2 z}{\partial x^2}$

(11) $z = y f(x) + x g(y)$

Sol w.r.to 'x' $p = y f'(x) + g(y)$ —(1) $\times x$

w.r.to 'y' $q = f(x) + x g'(y)$ —(2) $\times y$

(2) w.r.to 'x' $r = f'(x) + g'(y)$ —(3)

$(1) \times x + (2) \times y \Rightarrow px + qy = yx(f'(x) + g'(y)) + (x(g(y)) + y(f(x)))$

$px + qy = xy(r) + z.$

(12) $z = f(x+4t) + g(x-4t)$

Sol w.r.to 'x' $p = f'(x+4t) + g'(x-4t)$ —(1)

w.r.to 't' $q = f'(x+4t)4 - g'(x-4t)4$

$= 4(f'(x+4t) - g'(x-4t))$ —(2)

(2) w.r.to 'x' $\cancel{\frac{\partial^2 z}{\partial x \partial t^2} = 4(f''(x+4t))}$

(1) w.r.to 'x' $\frac{\partial^2 z}{\partial x^2} = f''(x+4t) + g''(x-4t)$

(2) w.r.to 'y' $\frac{\partial^2 z}{\partial t^2} = 16(f''(x+4t) + g''(x-4t))$

$= 16 \frac{\partial^2 z}{\partial x^2}$ //

Zia

Partial Linear Differential equation of Ist Order

A partial diff. eq^n of 1^{st} order is said to be linear partial diff. eq^n of I order if $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ occur in first degree only and are not multiplied together.

Langranges Linear Equation

A partial differential equation of the form $Pp + Qq = R$ is called Langranges Linear Equation where P, Q, R are function of x, y, z.

Note: 1> $Pp + Qq = R$ is a standard form of a linear partial diff. equation of I^{st} order.

2> The langranges linear equation is obtained by eliminating an arbitrary function $F(u,v) = 0$ where u & v are functions of x, y, z.

Langranges Auxillary / Subsidary Equations

The equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ are called langranges auxillary equations.

Working rule to solve langranges linear eq

To solve the equation of the form $Pp + Qq = R$

1> Form the subsidary eq^n $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

2> Solve the subsidary eq^n by the method of grouping or by the method of multiplier or both to get two independent solution. $u = a$ & $v = b$ where a & b are arbitory constant.

3> The complete solution is $\phi(u, v) = 0$ (or) $u = f(v)$ (or) $v = f(u)$

<u>Note</u>:- 1> To solve $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, suppose one of the variable is either absent or cancels out from any two fraction of given equations. Thus the integral can be obtained by using usual methods. Same procedure can be repeated for another set of fraction.

2> To solve $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ suppose one of the solⁿ is known by note 1. and also suppose that another solⁿ cannot be obtained by using note 1.

→ In such case one solⁿ known to us can be used to find another solution.

3> To solve $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ —① let P_1, Q_1, R_1 be the functions of x, y, z are constants then each fraction of eq ① will be equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} \quad \text{—②}$$

if $P_1 P + Q_1 Q + R_1 R = 0$ then numerator of eq ② is also

'0'. This gives $P_1 dx + Q_1 dy + R_1 dz = 0$ which can be integrated to get $u(x, y, z) = C_1$

→ This method can be repeated to get another s

$$v(x, y, z) = c_2$$

Where P_1, Q_1, R_1 are called multiplier.

① $2p + 3q = 1$

Sol:
$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1}$$

$$\frac{dx}{dz} = 2 \quad —① \qquad ; \qquad \frac{dy}{dz} = 3 \quad —②$$

$$x = 2z + c_1 \qquad\qquad y = 3z + c_2$$

$$\phi(x - 2z, y - 3z) = 0$$

② $y^2 z p + x^2 z q = xy^2$

Sol:
$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

$$\frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}}$$

$$\frac{dx}{dy} = \frac{y^2}{x^2}$$

$$\boxed{\frac{x^3}{3} - \frac{y^3}{3} = c_1}$$

$$\frac{dy}{x^2 z} = \frac{dz}{xy^2} \qquad \frac{dx}{y^2 z} = \frac{dz}{xy^2}$$

$$\boxed{\frac{x^2}{2} - \frac{z^2}{2} = c_2}$$

$$\phi\left(\frac{x^3 - y^3}{3}, \frac{x^2 - z^2}{2}\right) = 0$$

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③ $yzp + xzq = xy$

④ $p + q = x + y + z$

⑤ $pz - qz = z^2 + (x+y)^2$

⑥ $p - q = x + y + z$

⑦ $y^2 p - xyq = x(z - 2y)$

③ sol: $\dfrac{dx}{yz} = \dfrac{dy}{xz} = \dfrac{dz}{xy}$

$\dfrac{dx}{y\cancel{z}} = \dfrac{dy}{\cancel{x}\cancel{z}}$

$\dfrac{x^2}{2} - \dfrac{y^2}{2} = c_1$

$\dfrac{dy}{\cancel{x}z} = \dfrac{dz}{\cancel{x}y}$

$\dfrac{y^2}{2} - \dfrac{z^2}{2} = c_2$

$\phi\left(\dfrac{x^2 - y^2}{\cancel{2}}, \dfrac{y^2 - z^2}{\cancel{2}}\right) = 0$

4 sol: $p + q = (x + y + z)$

$dx = dy = \dfrac{dz}{(x+y+z)}$

$\Rightarrow dx = dy$

$x - y = c_1$

$dy = \dfrac{dz}{(2y + z + c_1)}$

$\dfrac{dz}{dy} = (2y + z + c_1)$

$\cancel{dz = 2ydy + zdy + c_1 dy}$

$\cancel{\int dz - zdy = y^2 + c_1 y + c_2}$

$\cancel{dz} = \cancel{ydy} + \cancel{ydy} + \cancel{zdy} + c_1 y$

$\dfrac{dz}{dy} - z = (2y + c_1)$

$$ye^{-\int dx} = \int (2y+c_1)\, e^{-\int dx} + c$$

$$\frac{dy}{dx} + Py = Q$$

$$ye^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$ye^{-x} = \int (2y+c_1) e^{-x} + c$$

$$ye^{-x} = (2y+c_1) e^{-x}(-1) - (2) e^{-x}(+1) + c$$

$$ye^{-x} = -2ye^{-x} - c_1 e^{-x} - 2e^{-x} + c$$

$$\cancel{3ye^{-x} = (c_1+2)e^{-x} + \frac{c}{e^{-x}}}$$

$$\boxed{(z+y+x+2)e^{-y} = c_1}$$

$$\cancel{3y = c_2}$$

$$\cancel{y = c_2/3}$$

$$\phi(x-y,\ (z+y+x+2)e^{-y}) = 0$$

⑤ $pz - qz = z^2 + (x+y)^2$

Sol^n $\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$

$$x+y = c \quad \text{— ①}$$

$$\Rightarrow \frac{dz}{z^2+(2y+c)^2} = \frac{dy}{-z} \Rightarrow \int -z\,dz = \int (z^2 + (2y-c)^2)\, dy$$

$$\Rightarrow \cancel{\int z\,dz = \int z^2 dy + \int (2y-c)^2 dy + c}$$

$$\Rightarrow \frac{dz}{z^2+c^2} = \frac{dy}{-z} \Rightarrow -z\,dz = z^2 dy + c^2 dy$$

$$\Rightarrow \int \frac{dz}{dy} = \int -\left(z + \frac{c^2}{z}\right) \quad ; \quad \frac{-z\,dz}{2(z^2+c^2)} = dy$$

$$\Rightarrow -\frac{1}{2}\log(z^2+c^2) = y + c \quad \text{— ②}$$

$$\Rightarrow \phi(x+y,\ 2y + \log(z^2+c^2)) = 0$$

(6) $p - q = x + y + z$

S.f. $\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{x+y+z}$

$x + y = c_1$ — (1)

$\frac{dz}{(c_1 - y) + y + z} = \frac{dy}{-1}$

$-dy = \frac{dz}{c_1 + z}$

$\int dy = -\log(z + c_1) + c$

$y = \log\frac{1}{z+c_1} + c_2$ — (2)

$\phi\left(x+y, y - \log\frac{1}{z+c_1}\right) = 0;$

(7) $y^2 p - xyq = x(z - 2y)$

S.f. $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

$\frac{dx}{y} = \frac{dy}{-x}$

$\frac{x^2}{2} + \frac{y^2}{2} = c_1$ — (1)

$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

$\frac{dy}{dz} = \frac{-y}{(z-2y)}$

$z\,dy - 2y\,dy = -y\,dz$

$z\,dy - 2y\,dy + y\,dz = 0$

$yz - y^2 = c_2$ — (2)

$\boxed{\phi(x^2 + y^2, yz - y^2) = 0}$

$\frac{x^2}{y} = \frac{2xydx - x^2}{y}$

8) $(x^2-y^2-z^2)p + 2xyq = 2xz$

Sol^n $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2x}$

$\frac{dy}{y} = dz$

$\log y - z = c$ — (1)

$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy}$

$\frac{dx}{x^2-y^2-(\log y)^2} = \frac{dy}{2xy}$

$2xydx = x^2dy - y^2dy - (\log y)^2 dy$

$\frac{2xydx - x^2dy}{y^2} = -\left(\frac{y^2dy}{y^2} + \frac{(\log y)^2 dy}{y^2}\right)$

$\frac{x^2}{y} = -\left(y + (\log y)^2\left(-\frac{1}{y}\right) - \int 2\log y\left(\frac{1}{y}\right)\left(-\log y\right)\right.$

$\frac{x^2}{y} = -\left(y - \frac{(\log y)^2}{y} + \int 2(\log y)^2 \frac{1}{y} + c\right)$

$\frac{x^2}{y} = -\left(y - \left(\frac{\log y}{y}\right)^2 + \frac{2(\log y)^3}{3} + c\right)$

$\frac{x^2}{y} + y - \left(\frac{\log y}{y}\right)^2 + \frac{2(\log y)^3}{3} = c$

$\phi\left(\log y - z,\ \frac{x^2}{y} + y - \left(\frac{\log y}{y}\right)^2 + \frac{2(\log y)^3}{3}\right) = 0$

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8) $(x^2-y^2-z^2)p + 2xyq = 2xz$

Sol^n $\dfrac{dx}{x^2-y^2+z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$

$dy = y\,dz$

$\log y - z = c$ — ①

$\dfrac{dx}{x^2-y^2-(\log y-c)^2} = \dfrac{dy}{2xy}$

$\dfrac{dx}{x^2-y^2-(\log y-c)^2} = \dfrac{dy}{2xy}$

$2xy\,dx = x^2dy - y^2dy - (\log y-c)^2dy$

$\dfrac{2xy\,dx - x^2dy}{y^2} = -\left(dy + \dfrac{(\log y-c)^2}{y^2}dy\right)$

$\dfrac{x^2}{y} = -\left[y + (\log y-c)^2\left(\dfrac{-1}{y}\right) - \int\dfrac{2(\log y-c)}{y}dx\right]$ $(\log y)$

$\Rightarrow \dfrac{x^2}{y} + y = -\left[-\dfrac{(\log y-c)^2}{y} - \int\dfrac{2(\log y)^2 - c\log y}{y}dx\right]$

$\Rightarrow \dfrac{x^2}{y} + y = \dfrac{(\log y-c)^2}{y} + \dfrac{2(\log y)^3}{3} - c\dfrac{(\log y)^2}{2} + C_2$ — ②

Sol^n $\dfrac{dx}{x^2-y^2-z} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$

$y = az$

Let $P_1 = x$, $Q_1 = y$, $R_1 = z$ be multipl

$\dfrac{x\,dx + y\,dy + z\,dz}{(x^2-y^2-z)x + 2xy^2 + 2xz^2} = \dfrac{dy}{2xy}$

$2\dfrac{(x\,dx + y\,dy + z\,dz)}{(x^2+y^2+z)^2} = \dfrac{dy}{y}$

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$$\log(x^2+y^2+z^2) = \log y + c;$$

$$\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{y}\right) = 0;$$

(9) $(z^2 - 2yz - y^2)P + (xz + xy)q = xy - xz$

Soln

$$\frac{dx}{\cancel{(z^2-y)^2}\;(z^2-2yz-y^2)} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

$$\Rightarrow \frac{dy}{y+z} = \frac{dz}{y-z}$$

$$y\,dy - z\,dy = y\,dz + z\,dz$$

$$\frac{y^2}{2} = yz + \frac{z^2}{2} + c$$

$$y^2 - z^2 - 2yz = c_1$$

Let $P_1 = x, Q_1 = y; R_1 = z$

$$\frac{x\,dx + y\,dy + z\,dz}{x(z^2-2yz-y^2) + xy(y+z) + xz(y-z)}$$

$$\frac{x\,dx + y\,dy + z\,dx}{0}$$

$$x\,dy + y\,dy + z\,dz = 0$$

$$x^2 + y^2 + z^2 = c_2$$

$$\phi\left(y^2 - z^2 - 2yz,\; x^2+y^2+z^2\right) = 0$$

Q) $(mz - ny)P + (nx - lz)q = (ly - mx)$

Sol

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

$$P_1 = x;\; Q_1 = y;\; R_1 = z$$

$$\rightarrow \frac{x\,dx + y\,dy + z\,dy}{\cancel{mxz} - \cancel{nxy} + \cancel{nxy} - \cancel{lyz} + \cancel{lyz} - \cancel{mxz}} = \frac{dx}{x(y+z)}$$

$$\longrightarrow x^2 + y^2 + z^2 = 2c;$$

$\rightarrow P_1 = l,\ Q_1 = m;\ R_1 = n;$

$$\frac{l\,dx + m\,dy + n\,dy}{l(mz-ny)+m(nx-lz)+n(ly-mx)} = \frac{dy}{nx-lz}$$

$$lx + my + nx = c_2$$

$$\phi(lx+my+nz,\ x^2+y^2+z^2) = 0$$

(11) $(x^2-yz)p + (y^2-xz)q = z^2-xy$

(12) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

(13) $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

(14) $x(y-z)p + y(z-x)q = z(x-y)$

(15) $(z-y)p + (x-z)q = (y-x)$

(16) $p\tan x + q\tan y = \tan z$

(18) $p - q = \log(x+y)$

12 Sol. $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

$P_1 = 1/x;\ Q_1 = 1/y;\ R_1 = 1/z$

$\cancel{P_1 = Q_1 = R_1 = l_i}$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{x^2}{x}(y-z) + \frac{y^2}{y}(z-x) + \frac{z^2}{z}(x-y)} = c_1$$

$\cancel{x+y+z}$

$\log x + \log y + \log z = \log c_1$

$xyz = c_1.$

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$P_1 = 1/x^2$; $Q_1 = 1/y^2$; $R_1 = 1/z^2$

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{(y-z)+(z-x)+(x-y)} = \frac{dx}{x(y-z)}$$

$$1/x + 1/y + 1/z = c_2$$

$$\phi\left(xyz,\ 1/x + 1/y + 1/z\right) = 0$$

11Q) $(x^2-yz)p + (y^2-xz)q = (z^2-xy)$

Sol

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-xz} = \frac{dz}{z^2-xy}$$

$x^2yz - y^2z^2 + y^2xz - x^2z^2 - z^2xy$

$\frac{x^3-xyz}{xyz}$

$\frac{x^2}{yz} - 1 - \frac{y^2}{xz} - 1 - \frac{z^2}{xy} - 1$

$\frac{x^3-y^3-z^2}{x^2y^2} - 3$

$$\frac{dx-dy}{x^2-yz-y^2+xz} = \frac{dy-dz}{y^2-xz-z^2+xy} = \frac{dz-dx}{z^2-xy-x^2+yz}$$

$$\frac{dx-dy}{x^2-yz-y^2+xz} = \frac{dy-dz}{y^2-xz-z^2+xy}$$

~~$\int d(\log(x-y))=$~~

$$\frac{dx-dy}{(x-y)\cancel{(x+y+z)}} = \frac{dy-dz}{(y-z)\cancel{(x+y+z)}}$$

$$\log(x-y) = \log(y-z) + \log c$$

$$\frac{x-y}{y-z} = c_1$$

$$\frac{dy-dz}{(y-z)\cancel{(x+y+z)}} = \frac{dz-dx}{(z-x)\cancel{(x+y+z)}}$$

$$\log(y-z) = \log(z-x) + \log c$$

$$\frac{y-z}{z-x} = c_2$$

$$\phi\left(\frac{x-y}{y-z},\ \frac{y-z}{z-x}\right) = 0$$

13 Sol^n $\sqrt{x}\,p + \sqrt{y}\,q = \sqrt{z}$

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

$$\frac{\sqrt{x}}{2} - \frac{\sqrt{y}}{2} = c_1 \quad ; \quad \frac{\sqrt{x}}{2} - \frac{\sqrt{z}}{2} = c_2$$

$$\phi\left(\sqrt{x} - \sqrt{y}, \sqrt{x} - \sqrt{z}\right) = 0$$

14 Sol^n $x(y-z)p + y(z-x)q = z(x-y)$

$$\frac{dx}{xy - xz} = \frac{dy}{yz - xy} = \frac{dz}{xz - yz}$$

* $P_1 = Q_1 = R_1 = 1$;

$$\frac{dx + dy + dz}{\cancel{xy} - \cancel{xz} + yz - \cancel{xy} + \cancel{xz} - yz} = \frac{dx}{xy - xz}$$

$$x + y + z = c_1$$

* $(P_1, Q_1, R_1) : \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y - z + z - x + x - y} = \frac{dx}{xy - xz}$$

$$\log x + \log y + \log z = \log c_2$$

$$xyz = c_2$$

$$\phi(xyz, x + y + z) = 0$$

15 sol $(z-y)p+(x-z)q=y-x$

Sol- $\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{y-x}$

$P_1=Q_1=R_1=1$

$\frac{dx+dy+dz}{z-y+x-z+y-x}=\frac{dx}{z-y}$

$x+y+z=c_1$

$(P_1, Q_1, R_1)=(x, y, z)$

$\frac{xdx+ydy+zdz}{xz-xy+xy-zy+yz-xz}=\frac{dx}{z-y}$

$x^2+y^2+z^2=c_2$

$\phi(x+y+z, x^2+y^2+z^2)=0$

16) $p\tan x+q\tan y=\tan z$

Sol $\frac{dx}{\tan x}=\frac{dy}{\tan y}=\frac{dz}{\tan z}$

$\cot x\, dx=\cot y\, dy$

$\log(\sin x)=\log(\sin y)+c_1$

$\cot y\, dy=\cot z\, dz$

$\log(\sin y)=\log(\sin z)+c_2$

$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right)=0$

18) $p - q = \log(x+y)$.

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

$dx = -dy$

$x = -y + c_1$

$x + y = c_1$

$$dx = \frac{dz}{\log(x+y)}$$

$$dx = \frac{dz}{\log c_1}$$

$x \log c_1 = z + c_2$

$x \log c_1 - z = c_2$

$V = x\log(x+y) - z = c_2$

$\phi(x+y,\, x\log c_1 - z) = 0$.

$\phi(x+y,\, x\log(x+y) - z) = 0$.

Non-Linear Partial differential Equation

→ N.L.P.D. Eq^n for the 1st order :-

A partial diff. eq^n which involves 1st order partial derivatives p & q with degree higher than one and the product of p & q is called la non linear partial diff. eq^n.

We have the foll. four methods which can be solved easily by method other than the general method.

1. Equations of the form $f(p,q)=0$.
(i.e Equations containing p & q only)

Let the given equation is $f(p,q)=0$ — ①

The complete solution of given eq^n ① is

$z = ax+by+c$ — ②

Where a, b are connected by the relation

$f(a,b)=0$; — ③

from ③ we get $b=\phi(a)$ — ④

Subs ④ in eq ②

$z = ax+\phi(a)y+c$

Q) Find the sol^n of $p^2-q^2=4$ — ①

Sl^n Let the sol^n of be $ax+bx+c=z$; — ②

Where $f(a,b)=a^2-b^2-4=0$

$a^2=b^2+4$

& $a=\sqrt{b^2+4}$

∴ $z=\sqrt{b^2+4}\,x+by+c$

Q) $pq + p + q = 0$ — ①

S.n $f(a,b) = ab + a + b = 0$

$$a(1+b) = -b$$

$$a = \frac{-b}{1+b}$$

Let it solⁿ be $z = ax + by + c$;

$$z = \frac{-bx}{1+b} + by + c$$

3Q) $\sqrt{p} + \sqrt{q} = 1$ 4) $p = e^q$ 5) $3p^2 - 2q^2 = 4pq$

3S) $\sqrt{p} + \sqrt{q} = 1$

$f(a,b) = \sqrt{a} + \sqrt{b} - 1 = 0$

$$\sqrt{a} = 1 - \sqrt{b}$$

$$a = (1 - \sqrt{b})^2$$

Let Solⁿ be $z = (1 - \sqrt{b})^2 x + by + c$;

4S) $p = e^q$

$f(a,b) = a = e^b$;

Solⁿ be $z = e^b x + by + c$

5S)

Sol: $3p^2 - 2q^2 = 4pq$

$f(a,b) = 3a^2 - 2b^2 - 4ab = 0$

$a = \cancel{\mp 4ab}$

$= 3a^2 - 4b(a) - 2b^2 = 0$

$$a = \frac{+4b \pm \sqrt{16b^2 + 24b^2}}{6} = \frac{4b \pm 2b\sqrt{10}}{6}$$

$$a = \frac{2b \pm b\sqrt{10}}{3}$$

The solⁿ be $z = \left(\frac{2b + b\sqrt{10}}{3}\right)x + by + c$

$$z = \left(\frac{2b - b\sqrt{10}}{3}\right)x + by + c$$

Case 2:- Equations of the form $f(z,p,q) = 0$ i;e eqⁿ containing z & p, q only. ~~are~~ (or) eqⁿ not containing x & y.

Working rule:-

1) assume the $z = \phi(u)$ is the solution where $u = x + ay$ so that $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$

2) Subs p & q value in given equation we get the ordinary differential equation.

3) Solve the resulting ordinary diff. eqⁿ in z & u

4) replace u by $x+ay$

Q) Find the solⁿ of $p(1+q) = qz$ — ①

Sol: Let $z = \phi(u) \Rightarrow u = x+ay$; $p = \frac{dz}{du}$; $q = a\frac{dz}{du}$

Subs in ①

$$\cancel{\frac{dz}{du}}\left(1 + a\frac{dz}{du}\right) = a\cancel{\frac{dz}{du}} \cdot z$$

$$1 + a\frac{dz}{du} = az$$

$$\frac{dz}{du} - z = -1/a$$

$\longrightarrow$ Solⁿ

$$\int \frac{a\,dz}{az-1} = \int du$$

$$\log(az-1) = u + c$$

$$\log(az-1) = (x+ay) + c$$

1Q) $q^2 = z^2p^2(1-p^2)$ 2Q) $z = p^2+q^2$ 3Q). $z^2 = 1+p^2+q^2$

Sol: Let $z = \phi(u)$; $u = x+ay$; $p = \frac{dz}{du}$; $q = a\frac{dz}{du}$

$$a^2\cancel{\left(\frac{dz}{du}\right)^2} = z^2\cancel{\left(\frac{dz}{du}\right)^2}\left(1 - \left(\frac{dz}{du}\right)^2\right)$$ → continue

$$a^2 = z^2 - z^2\left(\frac{dz}{du}\right)^2$$

$$z^2\left(\frac{dz}{du}\right)^2 = z^2 - a^2$$

$$\left(\frac{z^2}{z^2-a^2}\right)(dz)^2 = (du)^2$$

$$\frac{z\,dz}{\sqrt{z^2-a^2}} = du$$

$$u = \sqrt{z^2-a^2} + c$$

$$(x+ay) = \sqrt{z^2-a^2} + c$$

a

III method :-

Equations of the form $\boxed{z = px+qy+f(p,q)}$

replace 'p' by 'a' & 'q' by 'b'

G. Solⁿ of above eqⁿ $\boxed{z = ax+by+f(a,b)}$: — ①

Complete

Note :- Singular Solution is obtained by eliminating a & b b/w complete solution and equations obtained by ~~eliminating~~ differentiate equation ① with 'a' & 'b'.

Q) Find the general & Singular solution of the following

i) $z = px+qy+pq$

G. Solⁿ $z = ax+by+ab$ — ①

D.W.to 'a' $\frac{\partial z}{\partial a} = x+b$ — ② ; D.W.r.to 'b' $\frac{\partial z}{\partial b} = y+a$ — ③

$b = \partial z/\partial a - x$ $\qquad$ $a = \partial z/\partial b - y$

$b = -x$ $\qquad$ $-a = y$

Sub ② & ③ in ①

$z = -yx - xy + xy$

$z + yx = 0$ // Singular Solution

2Q) $z = px+qy-pq$

3Q) $z = px+qy+\log pq$

4Q) $z = px+qy+\sqrt{1+p^2+q^2}$

48° G.S $z = ax + by + \sqrt{1+a^2+b^2}$

S.S $\frac{\partial z}{\partial a} = x + \frac{a}{\sqrt{1+a^2+b^2}}$ | $\frac{\partial z}{\partial b} = y + \frac{b}{\sqrt{1+a^2+b^2}}$

$x = \frac{-a}{\sqrt{1+a^2+b^2}}$ | $y = \frac{-b}{\sqrt{1+a^2+b^2}}$

~~$z \neq f(ax)$~~ $z = -x\sqrt{1+a^2+b^2}\,x - y\sqrt{1+a^2+b^2}\,y + \sqrt{1+(x^2+y^2)(1+a^2+b^2)}$

$z = \cancel{\sqrt{1+a^2+b^2}}\left[\frac{\sqrt{1+(x^2+y^2)(1+a^2+b^2)} - \sqrt{1+a^2+b^2}\,x^2 -}{\sqrt{1+a^2+b}}\right.$

45^{n} $z = ax + by + \sqrt{1+a^2+b^2}$ — G.S

$\frac{\partial z}{\partial a} = x + \frac{a}{\sqrt{1+a^2+b^2}}$ | $\frac{\partial z}{\partial b} = y + \frac{b}{\sqrt{1+a^2+b^2}}$

$x^2 = \frac{a^2}{1+a^2+b^2}$ | $y^2 + a^2y^2 + y^2b^2 = b^2$ — 3

$x^2 + a^2x^2 + b^2x^2 = a^2$ — 2 | $y^2(1+a^2+b^2) = b^2$

$x^2(1+a^2+b^2) = a^2$

2 + 3 $x^2 + y^2 = \frac{a^2+b^2+1-1}{1+a^2+b^2} = 1 - \frac{1}{1+a^2+b^2}$

$1-(x^2+y^2) = \frac{1}{1+a^2+b^2}$; $\left[1+a^2+b^2 = \frac{1}{1-(x^2+y^2)}\right]$ — 4

Sub in 1

$$\frac{x^2}{1-x^2-y^2} = a^2 \quad (5)$$

$$a = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$\frac{y^2}{1-x^2-y^2} = b^2 \quad (6)$$

$$b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{-x^2 - y^2}{\sqrt{1-x^2-y^2}} + 1/\sqrt{1-x^2-y^2} = \frac{-(x^2+y^2)+1}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2} \quad //$$

✱ **Case – IV**: The equation in which z is absent and of the form $f_1(x, P) = f_2(y, q)$

→ The eqⁿ in which z is absent and the terms involving x & P can be separated from those containing y & q.

Consider $f_1(x, P) = a$ & $f_2(y, q) = a$

⟹ Solving these eqⁿ for p & q we get $p = F_1(x)$, $q = F_2(y)$

substitute P & q in $dz = P\,dx + q\,dy$ and on integrating gives the required solution.

$$dz = F_1(x)\,dx + F_2(y)\,dy$$

① $p^2 - q^2 = x - y$

Sol $p^2 - x = q^2 - y = a$

$p^2 = x + a$	$q^2 = y + a$
$p = \sqrt{x+a}$	$q = \sqrt{y+a}$

$$dz = p\,dx + q\,dy = \sqrt{x+a}\,dx + \sqrt{y+a}\,dy$$

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2}$$

② $p + q = \sin x + \sin y$

Sol^n $z = ax - \cos x - ay - \cos y + b$

③ $\sqrt{p} + \sqrt{q} = 2x$

Sol $z = a^2x + \frac{4}{3}x^2 + 2ax^2 + a^2y + b$

④ $\sqrt{p} + \sqrt{q} = x + y$

Sol^n $3z = (x+a)^3 + (y-a)^3 + b$

⑤ $py + xq + pq = 0$

Sol^n $2z = \frac{y^2}{a} - \frac{x^2}{a+1} + b$

⑥ $yp = 2xy + \log q$

Sol^n $z = ax + x^2 + \frac{e^{ay}}{a} + b$.

16-Aug-17 **Charpite's method**

→ This is the general method to obtain the solution of non linear partial differential equation of I^{st} order.

→ If the given non linear equation cannot be reduce to any of the standard forms then we use charpits method as it is lengthy

Charpit's Subsidary Equation

→ Let the given eq^n is of the form $f(x,y,z,p,q)=0$ then the charpits auxillary equation are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

Working rule for Charpits method

1. Transfer all the term of the given equation to left hand side and denote the entire exp^n by f.

2. Write the charpits Subs eq^n by using partial derivative of 'f' from step 1.

3. From step 2 select two proper fraction so that the resulting may come out to be simplest relation involving atleast p (or) q

④ The simplest relation of step ③ is solved with the given equation to determine the values of p & q. and put this value of p and q in $dz = pdx + qdy$ which on integrating gives complete solution of given equation.

Q) Find the solⁿ of $(p^2+q^2)y = qz$

Sol $f = (p^2+q^2)y - qz = 0$ —①

$$\frac{\partial f}{\partial x} = 0 ; \quad \frac{\partial f}{\partial y} = p^2+q^2 ; \quad \frac{\partial f}{\partial z} = -q ; \quad \frac{\partial f}{\partial p} = 2py$$

$$\frac{\partial f}{\partial q} = 2qy - z$$

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dx}{-2py} = \frac{dy}{z-2qy} \quad \text{—②}$$

$$\Rightarrow \frac{dp}{-\cancel{p}q} = \frac{dq}{p^{\cancel{2}}}$$

$$\Rightarrow \frac{p^2}{2} = \frac{-q^2}{2} + c$$

$$\Rightarrow \boxed{p^2+q^2 = c} \Rightarrow p^2+q^2 = c \text{ subs in ①}$$

$(p^2+q^2)y - qz = 0$ —①

$(q^2+q^2+c)y - qz = 0$

$\cancel{(2q^2+c)y - qz = 0}$ $\qquad c_1 y - q z = 0$

$\cancel{2q^2(2y) + q(-z) + cy = 0}$ $\qquad \boxed{\dfrac{c_1 y}{z} = q}$

$\cancel{q = \dfrac{z \pm \sqrt{z^2 - 8cy^2}}{4y}}$; $\quad p = \sqrt{q^2 + c_1} = \sqrt{c_1 - \dfrac{c_1^2 y^2}{z^2}}$

Step 4 $\quad dz = p\,dx + q\,dy = \dfrac{c_1 y}{z} dy + \sqrt{\dfrac{c_1 z^2 - c_1^2 y^2}{z}}\, dx$

$$z\,dz = \sqrt{c_1 z^2 - c_1^2 y^2}\,dx + c_1 y\,dy$$

$$\int \frac{2c_1 z\,dz - 2c_1^2 y\,dy}{2c_1\sqrt{c_1 z^2 - c_1^2 y^2}} = \int dx$$

$$x = \frac{\sqrt{c_1 z^2 - c_1^2 y^2}}{c_1} + c_2$$

(2) $\quad pxy + pq + qy = yz$

Sol $\quad f = pxy + pq + qy - yz = 0 \quad -(1)$

$$\frac{\partial f}{\partial x} = py \;;\; \frac{\partial f}{\partial y} = px + q - z \;;\; \frac{\partial f}{\partial z} = -y$$

$$\frac{\partial f}{\partial p} = xy + q \;;\; \frac{\partial f}{\partial q} = p + y.$$

$$\frac{dP}{Py-Py} = \frac{dq}{Px+q-z+q(-y)} = \frac{dz}{-Pxy-Pq-Pq-qy} = \frac{dx}{-xy-q} = \frac{dy}{-P-y}$$

$\Rightarrow dP = 0$

$\boxed{P = C_1}$ subs in ①

$$c_1xy + c_1q + qy - yz = 0$$

$$q = \frac{yz - c_1xy}{c_1 + y}\ ; \quad P = c_1$$

$\Rightarrow dz = Pdx + qdy$

$$dz = \frac{yz - c_1xy}{c_1+y}dy + c_1dx$$

$$dz = \frac{yz\,dy}{c_1+y} - \frac{c_1xy}{c_1+y}dy + c_1dx$$

$$\int \frac{dz - c_1dx}{(z - xc_1)} = \int \frac{y}{c_1+y}dy$$

$$\log(z - c_1x) = y - c_1\log(y+c_1) + C_2$$

③ $2z + p^2 + qy + 2y^2 = 0$

$\longrightarrow y^2z = bx - \frac{y^4}{2} + \frac{b^2}{2y^2} + c$

④ $p(1+q^2)+(b-z)q=0$

$\longrightarrow 2\sqrt{z-b-a} = \sqrt{ax} + \frac{1}{\sqrt{a}}y + c$

⑤ $q + xp = p^2$

$\longrightarrow z = axe^{-y} - \frac{a^2}{2}e^{-2y} + b$

⑥ $z^2 = pqxy$

$\longrightarrow z = x^a y^{1/a} b$