Unit - IV

Theory of Complex Numbers:

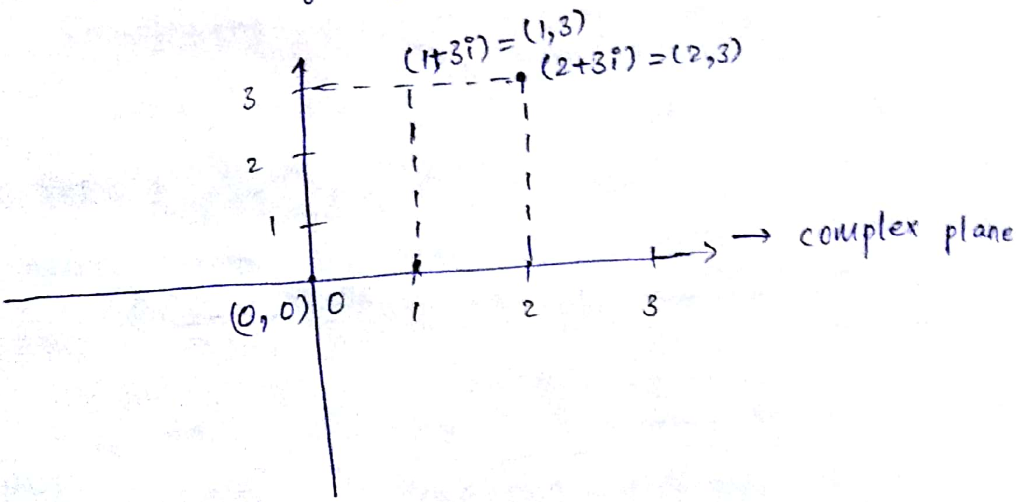
Complex number: An ordered pair (a, b) i.e $(a,b) = a+ib$ where a, b are real numbers and $i^2 = -1$ is called a complex number, where a is known as real part, b is imaginary part. The set of complex numbers is denoted by C and is defined as $C = \{a+ib / a, b \in R \ \& \ i^2 = -1\}$

The set of real numbers ~~are~~ 'R' is contained in 'C'.

Representation of a complex number

X-axis - Real axis

Y-axis - Imaginary axis



Modulus of a complex number:

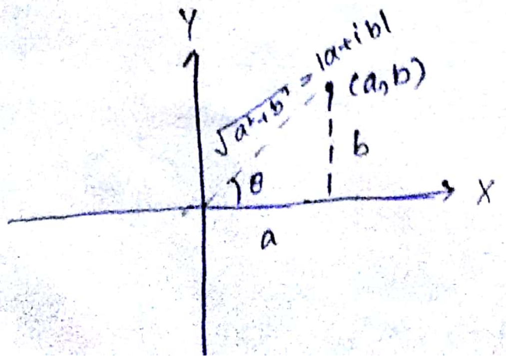
Modulus of a complex number (a+ib) is defined as

$|a+ib| = \sqrt{a^2+b^2}$

Argument of a complex number: (amplitude)

Argument of the complex number (a+ib) is defined as

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$



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Complex variable:

$z = x+iy$ where x, y are real variables is known as complex variable in cartesian coordinates

$z = x+iy \Rightarrow \bar{z} = \overline{x+iy} \Rightarrow \bar{z} = x-iy$ (is conjugate of z)

$\Rightarrow z+\bar{z} = 2x$ (Purely real)

$\Rightarrow z-\bar{z} = 2iy$ (Purely imaginary)

Complex variable in polar coordinates:

$x^2+y^2 = r^2$, $x = r\cos\theta$, $y = r\sin\theta$

$z = x+iy$

$z = r\cos\theta + ir\sin\theta$

$\Rightarrow z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z = re^{i\theta}$

$\Rightarrow |z| = |re^{i\theta}| = r|e^{i\theta}| = r$

$|z| = r \Rightarrow$ circle equation with $(0,0)$ as centre.

Complex function:

The function $w = f(z)$ is called complex function
w — depen, z — indep

$W = f(z) = z^2+4z+6$

$= (x+iy)^2 + 4(x+iy) + 6$

$= x^2-y^2+2xyi + 4x + 4iy + 6$

$= \underbrace{(x^2-y^2+4x+6)}_{u(x,y)} + i\underbrace{(2xy+4y)}_{v(x,y)}$

$u(x,y) + iv(x,y)$

$f(z) = u+iv$ is a complex function where u, v are functions of x, y

Neighbourhood (nbd)

Neighbourhood of a complex no 'z_0' is a set of all integers of the circle with centre of z_0 and radius 'δ'.

i.e., nbd of z_0 is set of points

$|z-z_0| < \delta$

where $z = x+iy$

let $z_0 = x_0+iy_0$

$z-z_0 = x-x_0 + i(y-y_0)$

$|z-z_0| < \delta$

$\Rightarrow |(x-x_0)+i(y-y_0)| < \delta$

$\Rightarrow (x-x_0)^2+(y-y_0)^2 < \delta^2$

The eq^n $|z-z_0| = r$ is circle with centre z_0 & radius r.

Analytic function (Regular function):

The function $f(z)$ is said to be analytic at a point z_0 if $f(z)$ is differentiable at all points of some neighbourhood of z_0.

The function $f(z)$ is said to be analytic at a point in a domain D if it is analytic at every point of the domain.

Eg: $f(z) = \dfrac{z^2}{z^2-1}$ is analytic everywhere except $z = \pm 1$

Entire function:

A function $f(z)$ is said to be entire function if it is analytic everywhere.

Eg: Every polynomial, exponential, trigonometric

Cauchy-Riemann (C-R) eqns:

let $f(z) = u+iv$ be any complex function then the equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$; $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ ($u_x = v_y$, $u_y = -v_x$) are known as cauchy-Riemann equations in cartesian coordinates.

The eq^{n}s $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$, $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ in polar coordinates

★ Necessary & sufficient conditions for a function to be analytic:

The necessary & sufficient conditions for the function $f(z) = u+iv$ to be analytic in a region 'R' are

i) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x, y in the region 'R'.

ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

Necessary condition:

Let $f(z) = u+iv$ be analytic function in the region 'R' and $f'(z)$ exists uniquely then we have to show that Cauchy-Riemann equations are true.

Let $\delta x, \delta y$ be the increments in x & y respectively then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

Now, $f(z) = f(x+iy)$

$= u(x,y) + iv(x,y)$

$$f(z+\delta z) = f(x+iy+\delta x+i\delta y)$$
$$= f(x+\delta x+i(y+\delta y))$$
$$= u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

Put these values in the above equation we have

$$f'(z) = \lim_{\delta z \to 0} \frac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - u(x,y) + iv(x,y)}{\delta x + i\delta y}$$

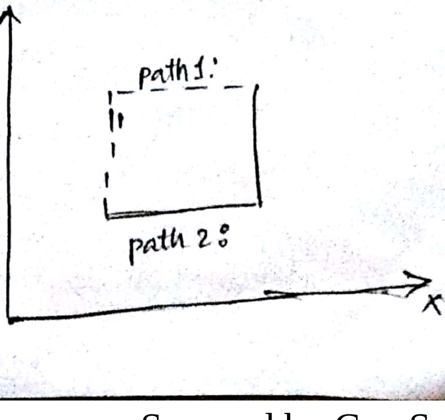
$$\Rightarrow f'(z) = \lim_{\delta x + i\delta y \to 0} \left(\frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y}\right) + i\left(\frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y}\right) \quad -(1)$$

Since $f(z)$ is differentiable

Now, taking the limit along path 1 $\delta x \to 0$, $\delta y \to 0$

$$f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \left(\frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y}\right) + i\left(\frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y}\right)$$

$$f'(z) = \lim_{\delta y \to 0} \left(\frac{u(x, y+\delta y) - u(x,y)}{i\delta y}\right) + \cancel{i} \lim_{\delta y \to 0} \left(\frac{v(x, y+\delta y) - v(x,y)}{\cancel{i}\delta y}\right)$$



$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h}$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(x,y+k) - u(x,y)}{k}$$

$$\Rightarrow f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Taking the limit along path 2 i.e $\delta y \to 0$ $\delta x \to 0$

$$f'(z) = \lim_{\substack{\delta y\to 0\\ \delta x\to 0}} \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y} + i\left(\frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y}\right)$$

$$= \lim_{\delta x\to 0}\left(\frac{u(x+\delta x,y) - u(x,y)}{\delta x} + i\lim_{\delta x\to 0}\frac{v(x+\delta x,y) - v(x,y)}{\delta x}\right)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad —(2)$$

$\therefore$ $f'(z)$ is unique

$$(1) = (2) \Rightarrow \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Equating real & imaginary parts on both sides we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$f(z+\delta z)$

$= u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$

$= u(x,y) + \left[\delta x \frac{\partial u}{\partial x} + \delta y \frac{\partial u}{\partial y}\right] + \cdots + iv(x,y) + \left[\delta x \frac{\partial v}{\partial x} + \delta y \frac{\partial v}{\partial y}\right] + \cdots$

since δx and δy are very small positive real numbers

Therefore 2nd & higher degree terms can be neglected.

Hence,

$$f(z+\delta z) = u(x,y) + iv(x,y) + \delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \delta y\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$\Rightarrow f(z+\delta z) = f(z) + \delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \delta y\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)$$

by C-R equations $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$.

$$f(z+\delta z) - f(z) = \delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\delta y\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$= (\delta x + i\delta y)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ exists & is unique.

$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ are continuous and unique. $f'(z)$ is differentiable at every point.

Hence $f(z)$ is analytic

C-R equations in polar coordinates:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Let $f(z) = u + iv$ be analytic function.

We know that $z = re^{i\theta}$ is complex variable in polar coordinates then $f(re^{i\theta}) = u + iv$ — (1)

Differentiating (1) wrt r on both sides.

$$f'(re^{i\theta})e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \quad \text{---} (3)$$

Again differentiating (1) w.r.t 'θ'

$$f'(re^{i\theta})re^{i\theta}.i = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$ir(f'(re^{i\theta})e^{i\theta}) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\Rightarrow ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\Rightarrow -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\Rightarrow -r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} \; ; \; r\cdot\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

Construction
~~Fdirection~~ of analytic function where one of u, v is given:

Method(1): Milne Thompsons Method:

Step(1): When u is given take $f'(z) = u_x - iu_y$

v is given take $f'(z) = v_y + iv_x$

Step(2): Then put these derivatives in the above equation

Replace x by z and y by '0' on RHS

Step(3): Integrate on both sides of the above equation we get required analytic function.

Find the analytic function $f(z)$ whose real part is

$u = x^2 - y^2 \Rightarrow \frac{\partial u}{\partial x} = 2x, \; \frac{\partial u}{\partial y} = -2y$ by milne method

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2x + 2y(i)$$

Replace x by 'z' and 'y' by '0' on RHS we have

$$f'(z) = 2z + i(0)$$

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→ Show that the func^n $e^x \cos y + 3x^2y - y^3$ is harmonic. Hence find its conjugate harmonic fun^n.

Sol : Let the given fun^n be $u = e^x \cos y + 3x^2y - y^3$

$$\frac{\partial u}{\partial x} = e^x \cos y + 6xy$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y + 6y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y + 3x^2 - 3y^2$$

$$\frac{\partial^2 u}{\partial y^2} = -e^x \cos y - 6y$$

$\because$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y + 6y - e^x \cos y - 6y = 0$$

$\therefore$ given fn is harmonic

To find its conjugate harmonic fun

We know that $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

i.e $$dv = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

$$\Rightarrow dv = \underbrace{(e^x \sin y - 3x^2 + 3y^2)}_{M} dx + \underbrace{(e^x \cos y + 6xy)}_{N} dy$$

Its Sol^n is given by

$$\int dv = \int M dx + \int (\text{the terms of } N \text{ which are not involving } x)\, dy + c$$

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$\Rightarrow v = \int (e^x \sin y - 3x^2 + 3y^2)\,dx + \int 0\,dy + c$
y is const

$\Rightarrow v = e^x \sin y - x^3 + 3xy^2 + c.$

→ **Complex Integration**

Complex line integral

Let $f(z) = u+iv$ be any complex fun, then the complex line integral along the curve 'c' from the point z_0 to z_1, is given by

$$\int_c f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz$$

where $f(z) = u+iv$

$$z = x+iy$$
$$dz = dx+idy$$

$$f(z)\,dz = (u+iv)\cdot(dx+idy)$$
$$= (u\,dx - v\,dy) + i(v\,dx + u\,dy)$$

→ Evaluate the line integral $\int_c z^2\,dz$ where c is straight line from $z=0$ to $z=2+i$

Sol The given fun is $f(z) = z^2$ then

$$f(z)\,dz = z^2\,dz = (x+iy)^2\,(dx+idy)$$

c : s.line from $z=0$ to $z=2+i$

$$(0,0) \qquad (2,1)$$

$$c:\ x = 2y$$
$$dx = 2dy$$

Putting these values in abv eqn we have

$$f(z)\,dz = (2y+iy)^2\,(2dy+idy)$$
$$= (2+i)^3\,y^2\,dy$$

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Sol $|z| = 1$

$z = re^{i\theta}$

(i.e) $z = 1 \cdot e^{i\theta}$ $(r = 1)$

$z = e^{i\theta}$

$dz = e^{i\theta} \cdot i\,d\theta$

$(z^2 + z)\,dz = \left((e^{i\theta})^2 + e^{i\theta}\right)\left(e^{i\theta}\, i\,d\theta\right)$

$= \left(e^{3i\theta} + e^{2i\theta}\right) i\,d\theta$

$\theta = 0$ to π

Putting these values in the given integral we have $\int_C (z^2 + z)\,dz = \int_0^{\pi} \left(e^{3i\theta} + e^{2i\theta}\right) i\,d\theta$

$= i\left(\frac{e^{3i\theta}}{3i} + \frac{e^{2i\theta}}{2i}\right)_0^{\pi}$

$= \left(\frac{e^{3\pi i}}{3} + \frac{e^{2\pi i}}{2} - \left(\frac{1}{3} + \frac{1}{2}\right)\right)$

$= \left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{3} - \frac{1}{2}\right)$

$= -\frac{2}{3}$

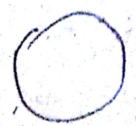
$e^{3\pi i} = \cos 3\pi + i \sin 3\pi$

$= -1$

$e^{4\pi i} = 1$

Simple closed curve (closed contour)

A closed curve without points of self intersection is called simple closed curve



is simple closed curve

is not simple closed curve.

Ex

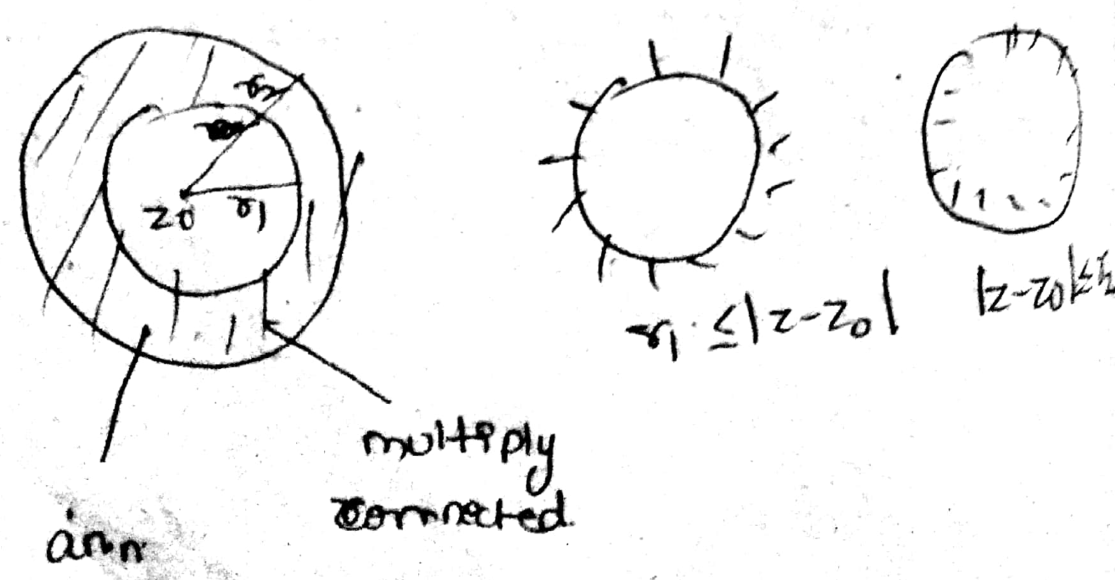
Boundaries of squares, circles, triangles and so on are closed contours

→ Simple / simply connected domain

A domain 'D' in which every closed path contains points of the same domain 'D' is known as simply connected domain.

→ Multiple / Multiply connected domain

A domain which is not simply connected is known as multiply connected domain multiply connect domain is combination of 2 or more than 2 domains



Green's Theorem:

let M,N are continuous functions having continuous first order partial derivatives in the region 'R' bounded by closed curve 'C'

then $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Now

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

$$= I_1 + iI_2 \quad \text{—— (1)}$$

$$I_1 = \oint_C u\,dx - v\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dx\,dy = 0 \quad \text{(by C-R equations)}$$

$$I_2 = \oint_C (v\,dx + u\,dy) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy$$

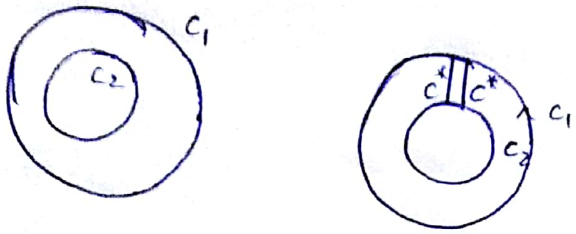
$$= \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} \right) dx\,dy = 0$$

(by C-R equations)

$$\Longrightarrow \oint_C f(z)\,dz = 0$$

Cauchy's Theorem for Multiply Connected Domains:

A doubly connected domain can be made as simply connected domain ~~where~~ with one cut



$B = C_1 + C^* + C_2 + C^*$

Let $f(z)$ be analytic in a simply connected domain 'D' bounded by B a combination of C_1, C_2, C^* & C^*

Then by Cauchy's theorem $\oint_B f(z)\,dz = 0$

$$\Rightarrow \oint_{C_1 + C^* + C_2 + C^*} f(z)\,dz = 0$$

$$\Rightarrow \oint_{C_1} f(z)\,dz + \cancel{\int_{C^*} f(z)\,dz} + \oint_{C_2} f(z)\,dz + \cancel{\oint_{C^*} f(z)\,dz} = 0$$

C^*, C^* are in opposite direction.

One is +ve and other is −ve

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

The cursor is traversing in clockwise direcn along c_2.

By neglecting -ve sign we can change direcn of cursor from clockwise to anti-clockwise direcn along curve c_2

$$\oint_{C_1} f(z)\,dz = -\oint_{C_2} f(z)\,dz$$

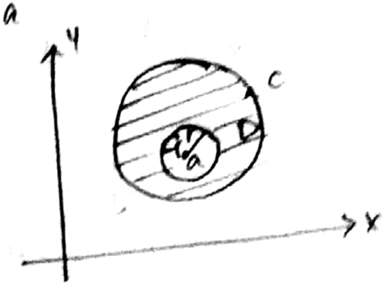
$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

in anticlock wise direcn

Cauchy's Integral Formula:

Let $f(z)$ be analytic with in and on a simple closed curve 'c' enclosing a point 'A' then $\oint_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

Draw a circle C_1 with centre at 'A' and radius is 'r' as shown in the figure. Now the domain is doubly connected domain and $\frac{f(z)}{z-a}$ is analytic in this domain. Then by Cauchy's theorem for doubly connected domain we have $\oint_c \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz$



Put $z - a = re^{i\theta}$

$z = a + re^{i\theta}$

$dz = re^{i\theta}\, i\, d\theta$

$$\Rightarrow \oint_c \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(a+re^{i\theta}).\cancel{re^{i\theta}}\, i\, d\theta}{\cancel{re^{i\theta}}}$$

$$= i\oint_{C_1} f(a+re^{i\theta})\,d\theta \quad \text{---(1)}$$

as $r \to 0$ the circle $C_1 \to a$

$$\Rightarrow \oint_c \frac{f(z)}{z-a}\,dz = i\oint_{C_1} f(a)\,d\theta$$

$$= i f(a) \oint_{C_1} d\theta$$

$$= i f(a)\, \theta\Big|_0^{2\pi} = 2\pi i f(a)$$

Generalized Cauchy's Integral Formula

Let $f(z)$ be an analytic function within and on a simple closed curve 'c' enclosing the point 'a' then

$$\oint_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a) \rightarrow \left(\frac{d^n}{dx^n} f(z)\right)_{z=a}$$

By cauchy's integral formula we have

$$\oint_c \frac{f(z)}{(z-a)} dz = 2\pi i f(a)$$

Differentiating the above equation on both sides partially wrt 'a' we have

$$\oint_c f(z) \left(\frac{-1}{(z-a)^2}(-1)\right) = 2\pi i f'(a)$$

$$\Rightarrow \oint_c \frac{f(z)}{(z-a)^{1+1}} dz = \frac{2\pi i}{1!} f'(a)$$

$\therefore$ Theorem is true for $n=1$

Again differentiating the above eqn partially wrt 'a'

$$\oint_c f(z) \left(\frac{-2}{(z-a)^3} (-1)\right) dz = 2\pi i f''(a)$$

$$\Rightarrow \oint_c \frac{f(z)}{(z-a)^{2+1}} dz = \frac{2\pi i}{2!} f''(a)$$

Suppose the theorem is true for $n=k$.

Differentiating the above eqn partially wrt 'a'

we have $\oint_c \frac{f(z)}{(z-a)^{k+1}} dz = \frac{2\pi i}{k!} f^k(a)$

$$\oint \frac{f(z)}{(z-a)^{(k+1)+1}} (k+1) dz = \frac{2\pi i}{k!} f^{k+1}(a)$$

$$\Rightarrow \oint_c \frac{f(z)}{(z-a)^{(k+1)+1}} = \frac{2\pi i}{(k+1)k!} f^{(k+1)}(a)$$

Evaluate the integral $\oint_C \frac{z^2+4z+1}{z-2} dz$ where 'c' is the circle $|z|=3$

The given function $\frac{z^2+4z+1}{z-2}$ is not analytic at $z=2$

Curve is $C: |z|=3$ — (1)

put $z=2$ in (1)

$\Rightarrow |z|=2<3$ (radius)

$\therefore z=2$ is within C

Now $\frac{z^2+4z+1}{z-2} = \frac{f(z)}{z-a}$ say

then $f(z) = z^2+4z+1$

$\Rightarrow a=2$.

$$\oint_C \frac{z^2+4z+1}{z-2} dz = \oint_C 2\pi i (z^2+4z+1)\Big|_{z=2}$$

$$= 2\pi i (4+8+1)$$

$$= 26\pi i$$

Evaluate $\oint_C \frac{\cos 2z}{z^2-3z+2} dz$ where $c: |z+i|=2$

$z^2-3z+2=0$ $\qquad |z+i|=2$ — ①

$\Rightarrow$ ~~at~~ $z=2, z=1$

The func^n $\frac{\cos 2z}{z^2-3z+1}$ is not analytic at $z=1, z=2$

Put $z=1$ in ①

$\Rightarrow |1+i| = \sqrt{1^2+1^2} = \sqrt{2} < 2$

$z=1$ is within c

$\Rightarrow |2+i| = \sqrt{2^2+1^2} = \sqrt{5} < 2$ X

$\sqrt{5} > 2$

$z=2$ is outside 'c'.

$\downarrow$

$z-2$ is factor.

Now, $\frac{\cos 2z}{z^2-3z+2} = \frac{\cos 2z/(z-2)}{(z-1)} = \frac{f(z)}{(z-a)}$ where $f(z) = \frac{\cos 2z}{z-2}$; $a=1$

By Cauchy's integral formula we have

$$\oint \frac{\cos 2z}{z^2-3z+2}\,dz = \oint_c \frac{\cos 2z/z-2}{z-1}\,dz$$

$$= 2\pi i\left(\frac{\cos 2z}{z-2}\right)\bigg|_{z=1}$$

$$= -2\pi i \cos 2$$

Evaluate the integral $\oint \frac{z^3+2}{(z-2)^3}\,dz$ where $C: |z+i| = 3$

$(z-2)^3 = 0 \quad \Rightarrow z = 2$

$\frac{z^3+2}{(z-2)^3}$ is not analytic at $z=2$

$|z+i| = 3$ —(1)

$\Rightarrow$ Put $z=2$ in (1)

$\Rightarrow |2+i| = 3$

$\Rightarrow \sqrt{5} < 3$ ✓ True

$\therefore$ $z=2$ is within the region 'C'

Let $\frac{z^3+2}{(z-2)^3} = \frac{f(z)}{z-a}$ say

then $f(z) = z^3+2/(z-2)^2$

$\Rightarrow a=2$

$$\oint_c \frac{z^3+2}{(z-2)^2} = \cancel{2\pi i\left(z^3+2/(z-2)^2\right)_{a=2}}$$

$$= \cancel{2\pi i}\,($$

$$= \frac{2\pi i\, f''(a)}{2!}$$

$f(z) = \frac{z^3+2}{(z-2)^2}$

$\Rightarrow f'(z) = \frac{(3z^2)(z-2)^2 - (z^3+2)\,2(z-2)}{(z-2)^4}$