Hence the required solution is

$$u = [K \sin x + K(1-e^{t})]\left(-\frac{1}{K}e^{-t} + \frac{1}{K}\right) = [\sin x + (1-e^{t})]\,(1-e^{-t})$$

It is obvious that, As $t \to \infty, u \to \sin x$.

EXERCISE 9 (B)

... the following equations by the method of separation of variables:

1. ... $+ \frac{\partial z}{\partial y} = 0$ [JNTU 2004 (Set No. 4)]

2. ... z

3. $4u_x + u_y = 3u$ and $u(0, y) = e^{-5y}$ [JNTU 2004S, (A) June 2010 (Set No. 2)]

4. $3u_x + 2u_y = 0$ with $u(x, 0) = 4e^{-x}$ [JNTU (K) Jan 2012 (Set No. 2)]

5. $u_x - 4u_y = 0$ and $u(0, y) = 8e^{-3y}$ [JNTU (A) June 2010 (Set No.4)]

6. $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, given $u = 3e^{-y} - e^{-5y}$ when $x = 0$ [JNTU 2001S, 2004 (Set No. 3)]

ANSWERS

1. $z = \left[Ae^{\left[1+\sqrt{1+k}\right]x} + Be^{\left[1-\sqrt{1+k}\right]x}\right]e^{-ky}$

2. $z = \left[Ae^{\left(\sqrt{2+k}\right)x} + Be^{\left(-\sqrt{2+k}\right)x}\right]e^{ky}$

3. $u = e^{2x-5y}$

4. $u = 4e^{-(2x-3y)/2}$

5. $u = 8e^{-3(4x+y)}$

6. $u = 3e^{x-y} - e^{2x-5y}$

9.3 ONE DIMENSIONAL WAVE EQUATION

Let OA be a stretched string of length l with fixed ends O and A. Let us take x-axis along OA and y-axis along OB perpendicular to OA, with O as origin. Let us assume that the tension T in the string is constant and large when compared with the weight of the string so that the effects of gravity are negligible. Let us pluck the string in the BOA plane and allow it to vibrate. Let P be any point of the string at time t. Let there be no external forces acting on the string. Let each point of the string make small vibrations at right angles to OA in the plane of BOA. Draw PP' perpendicular to OA. Let $OP' = x$ and $PP' = y$. Then y is a function of x and t. Under the assumptions, using Newton's second law of motion, it can be proved that $y(x, t)$ is governed by the equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad ...(1)$$

i.e. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

where $c^2 = T/m$ with T = tension in the string at any point and m is mass per unit length of the string.

Since the points O and A are not disturbed from their original positions for any time t we get ...(2)

$$y(0,t) = 0 \quad ...(3)$$

$$y(l,t) = 0$$

These are referred to as the end conditions or boundary conditions. Further it is possible that, we describe the initial position of the string (not necessarily along the equilibrium position OA) as well as the initial velocity at any point of the string at time $t = 0$ through the conditions ...(4)

$$y(x,0) = f(x), \quad 0 \le x \le l$$

...(5)

$$\frac{\partial y}{\partial t}(x,0) = g(x), \quad 0 \le x \le l$$

where $f(x)$ and $g(x)$ are functions such that $f(0) = f(l) = 0$; and $g(0) = g(l) = 0$. (These $f(x)$ and $g(x)$ may both be non-zero). Thus to study the subsequent motion of any point of the string (*i.e.* the subsequent motion of the string) we have to solve the following :

Determine $y(x, t)$ such that $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2}$ ...(1)

Subject to the conditions

$$\left.\begin{aligned} y(0,t) &= 0 \text{ for all } t && ...(2)\\ y(l,t) &= 0 \text{ for all } t && ...(3)\end{aligned}\right\} \text{End conditions}$$

$$\left.\begin{aligned} y(x,0) &= f(x),\ 0 \le x \le l && ...(4)\\ \left(\frac{\partial y}{\partial t}\right)_{at\, t=0} &= g(x),\ 0 \le x \le l && ...(5)\end{aligned}\right\} \text{Initial conditions}$$

The equation (1) is called one dimensional wave equation.

Note : In any typical problem on one dimensional wave equation describing the vibrations of a stretched string we have to solve equations (1) to (5).

If the initial displacement is given to be zero then $f(x) = 0$; any point of the string will be given an initial velocity $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = g(x)$.

If the string is released from rest and an initial non-zero displacement is given then $u(x, 0) = f(x) \neq 0$ and $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = g(x) = 0$.

It is also possible that $f(x) \neq 0,\ g(x) \neq 0$.

Solution of equation (1) Subject to the conditons (2) to (5): **[JNTU (H) May 2016]**

Consider the equation $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2}$ ...(1)

Let us use the method of separation of variables. Here $y = y(x,t)$. Let us take

$$y = X(x)T(t)$$

as solution of (1). Then

$$\frac{\partial y}{\partial x} = X'(x)\ T(t);\ \frac{\partial^2 y}{\partial x^2} = X''(x)\ T(t);$$

$$\frac{\partial y}{\partial t} = X(x)\ T'(t);\ \frac{\partial^2 y}{\partial t^2} = X(x)\ T''(t)$$

Using these in (1) we get

$$X''(x)\,T(t) = \frac{1}{c^2}X(x)\,T''(t)$$

$$\therefore \quad \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

Since the left hand side is a function of x and the right hand side is a function of t the equality is possible if and only if each side is equal to the same constant (say) λ. (This λ is called a separation constant.)

Hence we shall take

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = \lambda.$$

Let us take λ to be real. Then three cases are possible

$\lambda > 0$, $\lambda = 0$ or $\lambda < 0$.

Case 1. Let $\lambda > 0$. Then $\lambda = p^2$ $(p > 0)$.

$$\therefore \quad \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = p^2$$

Hence $X''(x) = p^2X(x)$ (*i.e.*) $X''(x) - p^2X(x) = 0$.

i.e. $$\frac{d^2X}{dx^2} - p^2X = 0 \Rightarrow X(x) = A_1\,e^{px} + B_1\,e^{-px}.$$

Also $T''(t) - p^2c^2T(t) = 0$

$\Rightarrow T(t) = C_1\,e^{pct} + D_1\,e^{-pct}$

Hence in this case, *a* typical solution is like

$$y(x,t) = \left(A_1\,e^{px} + B_1\,e^{-px}\right)\left(C_1\,e^{pct} + D_1\,e^{-pct}\right) \quad \text{... (S.1)}$$

where A_1, B_1, C_1, D_1 are arbitrary constants.

Case 2. Let $\lambda = 0$. Then

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2T(t)} = 0$$

$\Rightarrow \quad X''(x) = 0 \quad \Rightarrow X(x) = A_2 + B_2x$

and $\quad T''(t) = 0 \quad \Rightarrow T(t) = C_2 + D_2\,t$

$$\therefore \quad y(x,t) = (A_2 + B_2\,x)(C_2 + D_2\,t) \quad \text{... (S.2)}$$

where A_2, B_2, C_2, D_2 are arbitrary constants.

Case 3. Let $\lambda < 0$. Then we can write $\lambda = -p^2$ where $p > 0$. Then

$$\frac{X''(x)}{X(x)} = \frac{T''}{C^2\,T(t)} = -p^2$$

$$\Rightarrow \quad X''(x) + p^2 X(x) = 0 \Rightarrow X(x) = A_3 \cos px + B_3 \sin px$$

and $\quad T''(t) + p^2C^2\,T(t) = 0 \Rightarrow T(t) = C_3 \cos pct + D_3 \sin pct$

Hence a typical solution in this case is

$$y(x, t) = (A_3 \cos px + B_3 \sin px)(C_3 \cos pct + D_3 \sin pct).$$

Thus the possible solution forms of equation (1) are

$$y(x, t) = (A_1\, e^{px} + B_1\, e^{-px})(C_1\, e^{pct} + D_1\, e^{-pct}) \qquad \text{...(S.1)}$$

$$y(x, t) = (A_2 + B_2\, x)(c_2 + D_2\, t) \qquad \text{...(S.2)}$$

$$y(x, t) = (A_3 \cos px + B_3 \sin px)(C_3 \cos pct + D_3 \sin pct) \qquad \text{...(S.3)}$$

We have to note that here A's, B's, C's, D's are arbitrary constants to be determined. The constant p is also to be determined. The (A, B, C, D)'s depend on $f(x)$ and $g(x)$. The constant p will be determined using conditions (2) and (3).

We now decide which solution is appropriate for the present problem.

Consider (S.1)

$$y(x, t) = (A\, e^{px} + B\, e^{-px})(C\, e^{pct} + D\, e^{-pct})$$

Using Conditions (2) (*viz*) $y(0, t) = 0$ for all t

$$(A + B)\,(C\, e^{pct} + D\, e^{-pct}) = 0 \text{ for all } t$$

$\therefore \quad A + B = 0$

Using condition (3), $y(l, t) = 0$ for all t

$\therefore \quad (A\, e^{pl} + B\, e^{-pl})\,(C\, e^{pct} + D\, e^{pct}) = 0$ for all t

$\therefore \quad A\, e^{pl} + B\, e^{-pl} = 0$

Solving $\quad A + B = 0$

and $\quad A\, e^{pl} + B\, e^{-pl} = 0$

we get $\quad A = B = 0$

Thus $\quad y(x, t) \equiv 0.$

This implies that there is no displacement for any x and for any t. This is impossible. Thus (S.1) is not an appropriate solution.

Consider (S.2) :

$$y(x, t) = (A + Bx)(C + Dt)$$

Using (2), $\quad y(0, t) = 0$ for all t

Hence $\quad A\,(C + Dt) = 0 \Rightarrow A = 0$

Using (3), $\quad y(l, t) = 0$ for all t

$\therefore \quad (A + Bl)\,(C + Dt) = 0$ for all t

$\therefore \quad Bl\,(C + Dt) = 0$ for all t since $A = 0$

Here $l \neq 0$; $C + Dt \neq 0$ for all t. Hence $B = 0$

Thus here again $y(x, t) \equiv 0$ for all x and t.

Thus as before, this solution also is not valid. *Hence (S.2) is also not appropriate for the present problem.*

Consider (S.3)

$y(x, t) = (A \cos px + B \sin px)(C \cos pct + D \sin pct)$.

Using condition (2), $y(x, t) = 0$ for all t, we get

$\Rightarrow \quad (A)(C \cos pct + D \sin pct) = 0$, for all t

$\Rightarrow \quad A = 0$

Using condition (3), $y(l, t) = 0$ for all t, we get

$\Rightarrow \quad B \sin pl(C \cos pct + D \sin pct) = 0$

If $B = 0$, $y(x, t) = 0$ and this is not valid.

Hence $\sin pl = 0$

$\therefore \quad pl = n\pi$ where $n = 1, 2, 3, \ldots$

Thus $p = \dfrac{n\pi}{l}$. $(n = 1, 2, 3, \ldots)$

[Note that the separation constant p is determined through conditions (2) and (3)].

Thus a typical solution of (1) satisfying conditions (2) and (3) is

$$y(x,t) = \sin\frac{n\pi x}{l}\left[C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right] \text{ for } n = 1, 2, 3, \ldots$$

(Here we have written C_n, D_n in place of C, D)

Since different solutions correspond to different positive integers n, *we shall make an important observation here.*

If $[y_n(x, t)]_{n=1}^{\infty}$ are functions satisfying (1) as well as conditions (2) and (3), as the equation (1) is linear, the most general solution of (1) here is

$$y(x, t) = \sum_{n=1}^{\infty} y_n(x, t)$$

Thus the most general solution of (1) satisfying (2) and (3) is

$$y(x, t) = \sum_{n=1}^{\infty}\left(C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \qquad \ldots(6)$$

where $[C_n]$ and $[D_n]$ are arbitrary constants to be determined (using conditions (4) and (5).

Let us use condition (4):

$$y(x, 0) = f(x), \; 0 \le x \le l$$

Thus putting $t = 0$ in (6), we obtain

$$\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = f(x), 0 \le x \le l$$

(L. H. S is the half range Fourier sine series of $f(x)$ in $[0, l]$)

$$\text{Hence } C_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx, \; n = 1, 2\ldots \qquad \ldots(7)$$

Thus C_n's are all determined.

Let us now use condition (5) :

$$\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = g(x) \text{ for } 0 \le x \le l.$$

Now $\dfrac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left[-C_n \sin\frac{n\pi ct}{l}\cdot\left(\frac{n\pi c}{l}\right) + D_n \cdot \cos\frac{n\pi ct}{l}\left(\frac{n\pi c}{l}\right)\right]\cdot \sin\frac{n\pi x}{l}$

$$\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = g(x) \text{ gives}$$

$$\sum_{n=1}^{\infty}\left(D_n \frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l} = g(x),\ 0 \le x \le l$$

Hence $D_n \cdot \dfrac{n\pi c}{l} = \dfrac{2}{l}\displaystyle\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$

Thus $D_n = \dfrac{2}{n\pi c}\displaystyle\int_0^l g(x)\ \sin\frac{n\pi x}{l}\,dx$, for $n = 1, 2.....$ ...(8)

Thus D_n are all determined.

Hence the displacement $y(x, t)$ at any point x and at any subsequent time t is given by

$$y(x,t) = \sum\left(C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \quad ...(6)$$

where $C_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\ \sin\frac{n\pi x}{l}\,dx$ ...(7)

$D_n = \dfrac{2}{n\pi c}\displaystyle\int_0^l g(x)\ \sin\frac{n\pi x}{l}\,dx.$ ...(8)

Note 1. If initial displacement is $0, f(x) = 0$ and in this case

$C_n = 0$ for all n.

If initial velocity is 0 at any point of the string, $g(x) = 0$ and in this case

$D_n = 0$ for all x.

There can be cases where neither $f(x)$ nor $g(x)$ is 0.

2. After the three solutions (S.1), (S.2), (S.3) are derived instead of using conditions (2), (3) to determine the appropriateness of the solution (S.3), one can argue as below:

As we are dealing with the vibrations of a string, the solution has to be periodic with respect to x and t. Hence the solution appropriate to the present problem is (S.3).

Now using conditions (2) and (3) we can determine p as before.

SOLVED EXAMPLES

Example 1 : Find the solution of the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \frac{2k}{l} x \text{ where } 0 < x < (l/2) \\ \frac{2k}{l}(l-x) \text{ where } (l/2) < x < l \end{cases}$$

and initial velocity equal to 0.

(or) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $f(x)$. The initial velocity is zero, where

$$f(x) = \begin{cases} \frac{2k}{l} x, & 0 \leq x \leq \frac{l}{2} \\ \frac{2k}{l}(l-x), & \frac{l}{2} < x < l \end{cases}$$

Obtain the displacement at any point x and any time t.

Solution : To find $u(x, t)$ we have to solve

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}. \quad \text{...(1)}$$

where $u(0, t) = 0$ for all t ...(2)

$u(l, t) = 0$ for all t ...(3)

$u(x, 0) = f(x) \; (0 \leq x \leq l)$ ...(4)

$$\left(\frac{\partial u}{\partial t}\right)_{at\, t=0} = g(x) = 0 \; (0 \leq x \leq l) \quad \text{...(5)}$$

As explained earlier, let us seek solution of (1) in the form

$u(x, t) = X(x)\, T(t)$

The three solutions of (1) are

$$u(x,t) = (A_1 e^{px} + B_1 e^{-px})(C_1 e^{pct} + D_1 e^{-pct}) \quad \text{...(S.1)}$$

$$u(x,t) = (A_2 + B_2 x)(C_2 + D_2 t) \quad \text{...(S.2)}$$

$$u(x,t) = (A_3 \cos px + B_3 \sin px)(C_3 \cos pct + D_3 \sin pct) \quad \text{...(S.3)}$$

The solution appropriate to the present problem is (S.3) as the required solution has to be periodic in x and t. Hence the required solution is of the form

$$u(x,t) = (A \cos px + B \sin px)\,(C \cos pct + D \sin pct)$$

Using conditions (2) and (3), we note, as explained earlier.

$A = 0;\; p = \frac{n\pi}{l}$ where $n = 1, 2, 3, \ldots..$

The most general solution of (1) satisfying (2) and (3) is

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}. \quad \text{...(6)}$$

Using condition (4)

$$u(x, 0) = f(x), 0 \le x \le l.$$

$$\therefore \quad \sum C_n \sin \frac{n\pi x}{l} = f(x) \quad , \quad 0 \le x \le l. \qquad \ldots(7)$$

Now we can expand the given function $f(x)$ in a half-range Fourier sine series for $0 < x < l$.

We know that $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$$\ldots(8)$$

where $b_n = \frac{2}{l}\int_0^l f(x). \sin \frac{n\pi x}{l}\, dx$

Comparing (7) and (8), we have $C_n = b_n$

$$\therefore \quad C_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2k}{l} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} \frac{2k}{l}(l-x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4k}{l^2}\left[\left\{x \frac{\left(-\cos \frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - 1 \frac{\left(-\sin \frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right\}_0^{l/2}\right.$$

$$\left. + \left\{(l-x)\frac{\left(-\cos \frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (-1)\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{4k}{l^2}\left[\frac{l}{2}\cdot\frac{l}{n\pi}\left(-\cos \frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right.$$

$$\left. - \left\{\frac{l}{2}\cdot\frac{l}{n\pi}\left(-\cos \frac{n\pi}{2}\right) - \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right\}\right]$$

$$= \frac{4k}{l^2}\cdot 2 \cdot \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$= \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}$$

If $n = 2m$ (an even number) $C_{2m} = 0.$

If $n = 2m + 1$ (an odd number), $C_{2m+1} = \frac{8k}{(2m+1)^2\, \pi^2}(-1)^m.$

Thus all C_n's are determined.

Using $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) = 0$ for $0 \le x \le l$, we have

$$D_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}dx = 0 \quad (\because g(x) = 0)$$

Hence $u(x,t) = \frac{8k}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\frac{(2m+1)\pi ct}{l}\sin\frac{(2m+1)\pi x}{l}$

Example 2 : A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3\frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.

[JNTU Dec. 2002 (Set No. 3)]

Solution : The displacement $y(x, t)$ is governed by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{... (1)}$$

Subject to

$$y(0,t) = 0 \text{ for all } t \quad \text{... (2)}$$

$$y(l,t) = 0 \text{ for all } t \quad \text{... (3)}$$

$$y(x,0) = y_0\sin^3\frac{\pi x}{l}, 0 \le x \le l \quad \text{...(4)}$$

and $\left(\frac{\partial y}{\partial t}\right)_{\text{at } t=0} = 0, 0 \le x \le l$ ...(5)

(The condition (5) is implied by the phrase "if it is released from rest from this position"). The most general solution of (1) satisfying (2) and (3) is given by

$$y(x,t) = \Sigma\left(C_n\cos\frac{n\pi ct}{l} + D_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \quad \text{...(6)}$$

(Refer Ex.1)

where C_n and D_n are to be determined using (4) and (5).

Differentiating (6) partially w.r.t. 't', we get

$$\frac{\partial y}{\partial t} = \sum\left(-c_n.\frac{n\pi c}{l}.\sin\frac{n\pi ct}{l} + D_n.\frac{n\pi c}{l}\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

Using condition (5) *i.e.,* $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$, we get

$$0 = \sum D_n.\frac{n\pi c}{l}\sin\frac{n\pi x}{l}$$

$$\Rightarrow D_n = 0 \left(\because \sin\frac{n\pi x}{l} \ne 0\right)$$

Substituting $D_n = 0$ in (6), we get

$$y(x,t) = \sum C_n\cos\frac{n\pi ct}{l}.\sin\frac{n\pi x}{l} \quad \text{...(7)}$$

Using condition (4) *i.e.*, $y(x, 0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$ in (7), we get,

$$\sum C_n \sin\frac{n\pi x}{l} = y_0 \sin^3\frac{\pi x}{l},\ (0 \le x \le l).$$

We have $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$(i.e.)\ \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$\therefore \quad \Sigma C_n \sin\frac{n\pi x}{l} = y_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right]$$

Comparing the coefficients of like terms (or by using uniqueness theorem of Fourier series)

$$C_1 = \frac{3y_0}{4},\ C_3 = \frac{-y_0}{4} \text{ and } C_2 = 0,\ C_4 = 0,\ C_5 = 0,\ \ldots$$

$$\text{Hence } y(x, t) = \frac{3y_0}{4}\cos\frac{\pi ct}{l}\sin\frac{\pi cx}{l} - \frac{y_0}{4}\cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}$$

Example 3 : A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set to vibrate by giving each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.

[JNTU 2000, (K) Jan, June 2012]

(OR) A string is stretched and fastened to two points at $x = 0$ and $x = l$. Motion is started by displacing the string into the form $y = k\,(lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t.

[JNTU (H) June 2014]

Solution : Using the notation explained earlier, the displacement $y\,(x, t)$ is given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad \ldots(1)$$

Subject to $y(0, t) = 0$ for all t ...(2)

$y(l, t) = 0$ for all t ...(3)

$y(x, 0) = 0$ for $0 \le x \le l$ ...(4)

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l - x)$ for $0 \le x \le l$ ...(5)

The most general solution of (1) subject to (2) and (3) is given by

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi ct}{l} + D_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \qquad \ldots(6)$$

Using condition (4), $\sum C_n \sin\frac{n\pi x}{l} = 0,\ 0 \le x \le l$

Hence $C_n = 0$ for all n.

$$D_n = \frac{2}{n\pi c}\int_0^l \lambda x(l - x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{n\pi c}\left[x(l-x)\frac{\left(-\cos\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (l-2x)\frac{\left(-\sin\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}} + (-2)\frac{\left(\cos\frac{n\pi x}{l}\right)}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$= \frac{2\lambda}{n\pi c}\left[\frac{-2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2\lambda}{n\pi c}\cdot\frac{2l^3}{n^3\pi^3}(1-\cos n\pi) \quad = \frac{4\lambda l^3}{n^4\pi^4 c}(1-\cos n\pi)$$

If n is even, $D_n = 0$

If n is odd, $n = 2m + 1$;

$$D_{2m+1} = \frac{4\lambda\, l^3}{(2m+1)^4\,\pi^4 c}\cdot 2 = \frac{8\lambda\, l^3}{(2m+1)^4\,\pi^4 c}.$$

Hence $y(x,t) = \dfrac{8\lambda\, l^3}{\pi^4 c}\displaystyle\sum_{m=0}^{\infty}\frac{\left[\sin\frac{(2m+1)\,\pi ct}{l}\sin\frac{(2m+1)\,\pi x}{l}\right]}{(2m+1)^2}$

Example 4 : If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity $V_0 \sin^3\frac{\pi x}{l}$, find the displacement $y(x, t)$.

Solution : With the explained notation, the displacement $y(x, t)$ is given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \ldots(1)$$

$$y(0, t) = 0 \text{ for all } t \quad \ldots(2)$$

$$y(l, t) = 0 \text{ for all } t \quad \ldots(3)$$

$$y(x, 0) = 0 \le x \le l \quad \ldots(4)$$

$$\left(\frac{\partial y}{\partial t}\right)_{at\, t=0} = V_0 \sin^3\frac{\pi x}{l} \quad \ldots(5)$$

The most general solution of (1) satisfying (2) and (3) is given by

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi\, ct}{l} + D_n\sin\frac{n\pi\, ct}{l}\right)\sin\frac{n\pi x}{l} \quad \ldots(6)$$

Using (4), we get

$\Sigma\, C_n \sin\dfrac{n\,\pi\, x}{l} = 0$ for $0 \le x \le l$ which implies $C_n = 0$ for all n.

Also using (5), we get

$$\Sigma D_n\frac{n\pi c}{l}\sin\frac{npx}{l} = V_0\sin^3\frac{\pi x}{l} = V_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right]$$

$$\Rightarrow \qquad D_1\frac{\pi c}{l} = \frac{3}{4}V_0 \qquad \therefore D_1 = \frac{3l}{4\pi c}V_0$$

and $D_3 \dfrac{3\pi c}{l} = -\dfrac{V_0}{4} \quad \therefore\ D_3 = -\dfrac{lV_0}{12\pi c}$

Hence $y(x,t) = \dfrac{3l}{4\pi c} V_0 \sin\dfrac{\pi ct}{l} \sin\dfrac{\pi x}{l} - \dfrac{lV_0}{12\pi c} \sin\dfrac{3\pi ct}{l} \sin\dfrac{3\pi x}{l}$.

Example 5 : If a string of length l is initially at rest in equilibrium position and each of its points is given, the velocity

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\left(\frac{\pi x}{l}\right)$$

find the displacement $y(x, t)$. **[JNTU 2001]**

Solution : The displacement $y(x, t)$ is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

and boundary conditions : $y(l, t) = 0$

and the initial condition $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\left(\dfrac{\pi x}{l}\right)$

The general solution of (1) is given by

$$y(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

Since the string was at rest initially, we have $y(x, 0) = 0$

Hence $y(x,0) = \sum_{n=1}^{\infty} A_n \sin\dfrac{n\pi x}{l} = 0$

$\Rightarrow \quad A_n = 0$

$$\therefore \quad y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \qquad ...(2)$$

Differentiating w.r.t. 't', we get $\dfrac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\dfrac{n\pi c}{l} B_n \cos\dfrac{n\pi ct}{l}\sin\dfrac{n\pi x}{l}$

To determine B_n's use the initial condition

$$b\sin^3\left(\frac{\pi x}{l}\right) = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l}\sin\frac{n\pi x}{l}$$

or $\dfrac{b}{4}\left(3\sin\dfrac{\pi x}{l} - \sin\dfrac{3\pi x}{l}\right) = B_1 . \dfrac{\pi c}{l}\sin\dfrac{\pi x}{l} + B_2 . \dfrac{2\pi c}{l}\sin\dfrac{2\pi c}{l} + B_3 . \dfrac{3\pi c}{l}\sin\dfrac{3\pi c}{l} + ...$

$$\left[\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta \quad \therefore \sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)\right]$$

Equating the coefficients of $\sin\dfrac{\pi x}{l}, \sin\dfrac{2\pi x}{l}, \sin\dfrac{3\pi x}{l}, ...$ on both sides, we get

$$\frac{3b}{4} = B_1 . \frac{\pi c}{l}, 0 = B_2 . \frac{2\pi c}{l}, \frac{-b}{4} = B_3 . \frac{3\pi c}{l}, B_4 = 0, B_5 = 0, ...$$

$$\Rightarrow \quad B_1 = \frac{3bl}{4\pi c}, B_2 = 0, B_3 = \frac{-bl}{12\pi c}, B_4 = 0, B_5 = 0, ... \qquad ...(3)$$

Hence from (2) and (3), we have

$$y(x,t)=\frac{bl}{12\pi c}\left(9\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}-\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}\right)$$

Example 6 : Find the solution of the initial boundary value problem

$$\frac{\partial^2 y}{\partial t^2}=\frac{\partial^2 y}{\partial x^2},\ 0\le x\le 1,\ t\ge 0,\ \text{subject to}\ y(x,0)=\sin\pi x, 0\le x\le 1, y_t(x,0)=0.$$

Solution : The most general solution of $y(x, t)$ is given by

$$y(x,t)=\sum_{1}^{\infty}\left(C_n\cos\frac{n\pi ct}{l}+D_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

$\therefore$ The solution of the given problem is

$$y(x,t)=\sum_{n=1}^{\infty}C_n\cos n\pi t\sin n\pi x,$$

where $C_n=2\int_0^1\sin\pi x\sin n\pi\,dx=0$ for $n\neq 1$ and

$$C_1=2\int_0^1\sin^2\pi x\,dx=\int_0^1(1-\cos 2\pi x)\,dx=1$$

$\therefore\quad y(x,t)=\cos\pi t\sin\pi x.$

Example 7 : Solve the boundary value problem $u_{tt}=a^2u_{xx};\ 0<x<l;\ t>0$ with $u(0, t) = 0$; $u(l, t) = 0$ and $u(x, 0) = 0$, $u_t(x, 0) = \sin^3\left(\frac{\pi x}{l}\right)$.

[JNTU 2003S (Set No. 1)]

Solution : $u(x, t)$ is the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2} \qquad \dots (1)$$

Given conditions are

$u(0, t) = 0$ for all t ... (2)

$u(l, t) = 0$ for all t ... (3)

$u(x, 0) = 0$ for $o\le x\le l$ ... (4)

and $u_t(x, 0) = \sin^3\left(\frac{\pi x}{l}\right)$ for $0\le x\le l$ ... (5)

The required solution of (1) is of the form

$$u(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat) \qquad \dots (6)$$

Using conditions (2) and (3), we have

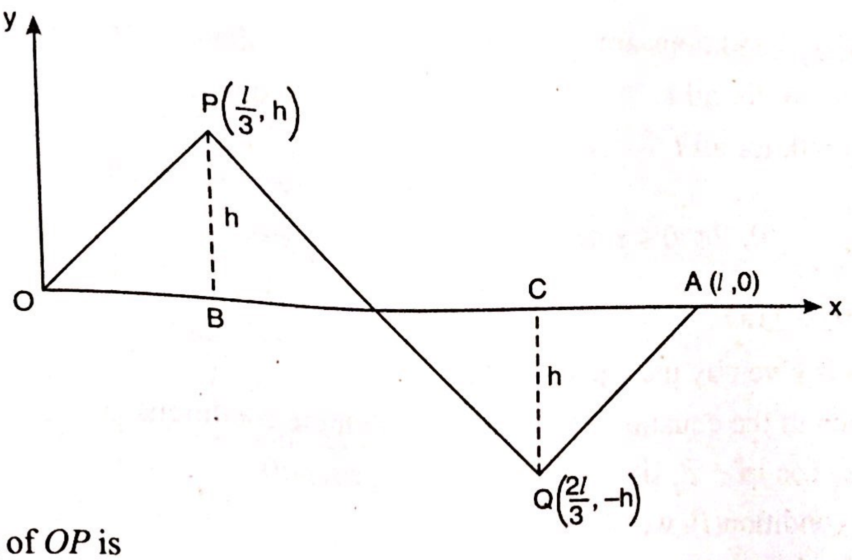
$c_1=0$ and $p=\frac{n\pi}{l}$ where n = 1, 2, 3,...

$\therefore$ The general solution of (1) satisfying (2) and (3) is

Example 9 : The points of trisection of a tightly stretched string of length l with fixed ends are pulled aside through a distance d on opposite sides of the position of equilibrium, and the string is released from rest. Obtain an expression for the displacement of the string at any subsequent time and show that the midpoint of the string is always at rest.

[JNTU 2003S (Set No.4)]

Solution : Let the points of trisection of the string OA be B and C, where O and A are fixed ends of the string. Let the two points of trisection are displaced by h. The initial position of the string is as shown in figure.



Equation of OP is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \quad i.e., \quad \frac{y-0}{h-0} = \frac{x-0}{\frac{l}{3}-0}$$

or $\quad y = \frac{3hx}{l}, 0 \le x \le \frac{l}{3}$

Equation of PQ is

$$\frac{y-h}{-h-h} = \frac{x-\frac{l}{3}}{\frac{2l}{3}-\frac{l}{3}}$$

$$i.e., \quad y = h - \frac{6h}{l}\left(x-\frac{l}{3}\right) = h\left[1-\frac{6}{l}\left(x-\frac{l}{3}\right)\right] = h\left(3-\frac{6}{l}x\right)$$

or $\quad y = \frac{3h}{l}(l-2x), \frac{l}{3} \le x \le \frac{2l}{3}$

Similarly, equation of QA is

$$y = \frac{3h}{l}(x-l), \frac{2l}{3} \le x \le l$$

$$\therefore \; f(x) = \begin{cases} \frac{3hx}{l}, 0 \le x \le \frac{l}{3}, \\ \frac{3h}{l}(l-2x), \frac{l}{3} \le x \le \frac{2l}{3} \\ \frac{3h}{l}(x-l), \frac{2l}{3} \le x \le l \end{cases} \quad \text{... (1)}$$

represents the initial position of the string.

The displacement $y(x, t)$ at any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{... (2)}$$

The boundary conditions are

(*i*) $y(0, t) = 0$, for all t

(*ii*) $y(l, t) = 0$, for all t

(*iii*) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$, for $0 \le x \le l$

(*iv*) $y(x,0) = f(x)$

where $f(x)$ is given by the equation (1).

The solution of the equation(2) consistent with these conditions is

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \quad \ldots (3)$$

Using the condition(*i*), we obtain

$$0 = c_1(c_3 \cos pat + c_4 \sin pat)$$

$$\Rightarrow c_1 = 0$$

$\therefore$ Equation (3) reduces to

$$y(x, t) = c_2 \sin px\,(c_3 \cos pat + c_4 \sin pat) \quad \ldots (4)$$

Using the condition (*ii*), we obtain

$$0 = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat)$$

$$\Rightarrow \sin pl = 0 \text{ or } pl = n\pi$$

$$\therefore \quad p = \frac{n\pi}{l}$$

Substituting this value of p in (4), we get

$$y(x, t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \quad \ldots (5)$$

Differentiating (5) partially w.r.t. 't', we get

$$\frac{\partial y}{\partial t} = c_2 \sin \frac{n\pi x}{l}\left(-c_3 \frac{n\pi a}{l} \sin \frac{n\pi at}{l} + c_4 \frac{n\pi a}{l} \cos \frac{n\pi at}{l}\right)$$

Using condition (*iii*), we obtain

$$\therefore \quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = c_2 \sin \frac{n\pi x}{l}\left(0 + c_4 \frac{n\pi a}{l}\right)$$

$$i.e., \quad 0 = c_2 c_4 \frac{n\pi a}{l} \sin \frac{n\pi x}{l}$$

$$\Rightarrow c_4 = 0 \quad \left(\because c_2 \ne 0, \sin \frac{n\pi x}{l} \ne 0\right)$$

Hence equation (5) becomes

$$y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

Putting $c_2 c_3 = c_n$, we get

$$y(x, t) = c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

The general solution is obtained by adding all such solutions so that

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \qquad \dots (6)$$

Using condition (*iv*) in (6), we obtain

$$y(x, 0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\text{i.e.,} \quad f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \qquad \dots (7)$$

To find c_n, expand $f(x)$ as a half-range sine series in (o, l)

$$\text{Let} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad \dots (8)$$

From (7) and (8), we have

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Comparing like coefficients,

$$c_n = b_n \qquad \dots (9)$$

$$\text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3hx}{l}\cdot\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/3}^{2l/3} \frac{3h}{l}(l-2x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{2l/3}^{l} \frac{3h}{l}(x-l)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{6h}{l^2}\left\{\left[x\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}\right) - 1\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}\right)\right]_0^{l/3}\right.$$

$$+\left[(l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - (-2)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}\right)\right]_{l/3}^{2l/3}$$

$$\left.+\left[(x-l)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - 1\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}\right)\right]_{2l/3}^{l}\right\}$$

$$= \frac{6h}{l^2}\left\{\left[-\left(\frac{l^2}{3n\pi}\right)\cos\frac{n\pi}{3}+\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{3}\right]\right.$$

$$+\left[\left(-\left(\frac{l^2}{3n\pi}\right)\cos\frac{2n\pi}{3}-2\left(\frac{l}{n\pi}\right)^2\sin\frac{2n\pi}{3}\right)\right.$$

$$\left.-\left[-\left(\frac{l^2}{3n\pi}\right)\cos\frac{n\pi}{3}-2\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{3}\right]\right]+\left.\left[0-\left(\frac{l^2}{3n\pi}\right)\cos\frac{2n\pi}{3}+\left(\frac{l}{n\pi}\right)^2\sin\frac{2n\pi}{3}\right]\right\}$$

$$= \frac{6h}{l^2}\left[3\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{3}-3\left(\frac{l}{n\pi}\right)^2\sin\frac{2n\pi}{3}\right]=\frac{18h\,l^2}{l^2n^2\pi^2}\left(\sin\frac{n\pi}{3}-\sin\frac{2n\pi}{3}\right)$$

$$= \frac{18h\,l^2}{l^2n^2\pi^2}\left[\sin\frac{n\pi}{3}-(-1)^n\sin\frac{n\pi}{3}\right]\left(\because \sin\frac{2n\pi}{3}=\sin\left(n\pi-\frac{n\pi}{3}\right)=(-1)^n\sin\frac{n\pi}{3}\right)$$

$$\therefore\quad b_n=\frac{18h}{n^2p^2}\sin\left(\frac{n\pi}{3}\right)\left[1-(-1)^n\right]$$

$$\text{Thus } b_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{36h}{n^2\pi^2}\sin\left(\dfrac{n\pi}{3}\right), & \text{if } n \text{ is even}\end{cases}$$

Substituting this value of b_n in (9), we get

$$c_n = \frac{36h}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)(n=1,2,3,\ldots)$$

Substituting c_n in (6) we get the required solution

$$y(x,t) = \sum_{n=2,4,6\ldots}^{\infty}\frac{36h}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$\text{or}\quad y(x,t) = \sum_{n=1}^{\infty}\frac{36h}{4n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right)\sin\left(\frac{2n\pi x}{l}\right)\cos\left(\frac{2n\pi at}{l}\right)$$

$$= \frac{9h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{2n\pi}{3}\right)\sin\left(\frac{2n\pi x}{l}\right)\cos\left(\frac{2n\pi at}{l}\right) \quad \ldots (10)$$

To obtain the displacement at the mid-point , put $x=\frac{l}{2}$ in (10).

$$\therefore\quad y\left(\frac{l}{2},t\right) = \frac{9h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{2n\pi}{3}\right)\sin(n\pi)\cos\left(\frac{2n\pi at}{l}\right)$$

$$= 0\ (\because \sin n\pi = 0)$$

Hence the mid-point of the string is at rest.

Note : While answering any typical question on one-dimensional wave equation one has to follow the procedure given below :

(1) Obtain (S.1), (S.2), (S.3).

(2) Provide argument to select the appropriate solution.

(3) Write the most general solution.

(4) Determine the arbitrary constants

(5) Write the final solution.

Example 10: A tightly stretched string of length l has its ends fastened at $x=0, x=l$. The mid-point of the string is then taken to height 'h' and then released from rest in that position. Find the lateral displacement of a point of the string at time t from the instant of release.

[JNTU (A) May 2012 (Set No. 3)]

Solution : Let $y(x,t)$ denote the displacement of the string.

The initial displacement is given by OAB.

Equation of OA is

$$y-0=\frac{h-0}{\frac{l}{2}-0}(x-0)$$

$$\Rightarrow y=\frac{2h}{l}x$$

Equation of AB is

$$y-h=\frac{0-h}{l-\frac{l}{2}}\left(x-\frac{l}{2}\right)\Rightarrow y-h=(-h)\frac{2}{l}\left(x-\frac{l}{2}\right)$$

$$\Rightarrow y=h-\frac{2h}{l}\left(x-\frac{l}{2}\right)=h\left[1-\frac{2}{l}\left(x-\frac{l}{2}\right)\right]$$

$$=h\left(1-\frac{2}{l}x+1\right)=h\left(2-\frac{2}{l}x\right)=2h\left(1-\frac{x}{l}\right)=\frac{2h}{l}(l-x)$$

Thus the problem is to solve the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2} \qquad \ldots (1)$$

with boundary conditions $y(0,t)=0, y(l,t)=0$ and with initial displacement

$$y(x,0)=f(x)=\begin{cases}\frac{2h}{l}x \text{ if } 0\le x\le\frac{l}{2}\\ \frac{2h}{l}(l-x) \text{ if } \frac{l}{2}\le x\le l\end{cases}$$

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

The solution of (1) satisfying the above boundary conditions and initial conditions is given by

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \quad \text{... (2)}$$

where

$$A_n = \frac{2}{l}\int_0^l f(x).\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2h}{l} x \sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} \frac{2h}{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{2}{l}.\frac{2h}{l}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - 1.\frac{\left(-\sin\frac{n\pi x}{l}\right)}{n^2\pi^2/l^2}\right\}_0^{l/2}\right.$$

$$\left. + \left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{4h}{l^2}\left[\left\{-\frac{l}{n\pi}x\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right\}_0^{l/2}\right.$$

$$\left. + \left\{-\frac{l}{n\pi}(l-x)\cos\left(\frac{n\pi x}{l}\right) - \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{4h}{l^2}\left[\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right.$$

$$\left. + \left\{(0+0) + \frac{l^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\}\right]$$

$$= \frac{4h}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right] = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Thus $A_n = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$

Substituting the values of A_n in (2), we get subsequent displacement of the string as

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

$$= \frac{8h}{\pi^2}\left[\frac{1}{1^2}\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi at}{l}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{3\pi at}{l}\right) + \ldots\right]$$

REVIEW QUESTIONS

1. Write the three possile solutions of $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

2. Find the general solution of the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ **[JNTU (H) May 2016]**

EXERCISE 9 (C)

1. (*i*) Obtain the general solution of the one dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ **[JNTU 2000 S]**

(*ii*) (*a*) Sol... $^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ subject to $y(0, t) = 0$ for all t, $y(l, t) = 0$ for all t,

... $= f(x)$ $(0 \le x \le l)$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x), 0 \le x \le l.$ **[JNTU 2002]**

... $g(x$ ... rite the solution. (*c*) If $f(x) = 0$, $g(x) \neq 0$, write the solution.

... ngth l has its ends fixed at $x = 0$, $x = l$. The mid-point of

... ight h and then released from rest in that position. Find

... oint of the string at time t from the instant of release.

... nitial displacement is as in the figure above.]

by displacing the string in ... from one end at time t is given by time $t = 0$. Show that the displacement at distance x from one end at time t is given by

$$y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}.$$

4. A tightly stretched elastic string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $f(x) = bx\,(l - x)$ where b is a constant and then released. Find the displacement of any point x of the string at any time $t > 0$.

5. A tightly stretched string with fixed end-points $x = 0$ and $x = 10$ cm is initially at rest in its equilibrium position and at every point x on it is given an initial velocity $g(x) = \lambda x\,(10 - x)$. Find the displacement of the string at any distance x from one end at any time t.

$$\left[\textbf{Ans. } y = \frac{8\lambda l^3}{cn^4} \sum_{n=1,3,5,....}^{\infty} \left(\frac{1}{n^4} \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}\right), \text{where } l = 10\right]$$

6. A string of length 100 cm is tightly stretched between $x = 0$ and $x = 100$ and is displaced from its equilibrium position by imparting to each of its points an initial velocity

$$g(x) = \begin{cases} x, 0 \le x \le 50 \\ 100 - x, 50 \le x \le 100 \end{cases}$$

Determine the displacement at any subsequent time.

7. Solve $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t \ge 0$ with conditions $u(x, 0) = f(x)$ and at $(x, 0)$ $\left(\dfrac{\partial u}{\partial t}\right)_{(x,0)} = g(x)$, assuming u, $\dfrac{\partial u}{\partial x} \to 0$ as $x \to \infty$. **[JNTU 2004S (Set No. 4)]**

9.4 ONE DIMENSIONAL HEAT CONDUCTION EQUATION (OR DIFFUSION EQUATION)

Let OA be a homogeneous bar of uniform cross-section. Let the surface of the rod be laterally insulated with a material impervious to heat. Let the stream lines of heat flow be parallel to one another and perpendicular to the cross sectional area. Let O be the origin and OA as the positive x axis. Let ρ be the density (gm/cm³), s the specific heat (cal./gr.deg) and k the thermal conductivity (cal./cm.deg.sec). Let $u(x, t)$ be the temperature at a distance x from O. It can be shown that $u(x, t)$ is governed by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \quad \text{or} \quad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

where $c^2 = \dfrac{k}{\rho s}$, and c^2 is called the diffusivity of the substance.

The equation (1) is called **one dimensional heat flow equation or diffusion equation.**

Solution of (1) using method of Separation of Variables :

Let us seek a solution of (1) in the form $u(x, t) = X(x)\,T(t)$

Substituting this in (1), we have

$$X''(x)\,T(t) = \frac{1}{c^2} X(x)\,T'(t)$$

Hence $\dfrac{X''(x)}{X(x)} = \dfrac{T'(t)}{c^2 T(t)}$.

Since L.H.S. is a function of x and R.H.S is a function of t, the equality is possible if and only if each is equal to the same constant say λ. (This λ is called separation constant). Thus

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{c^2 T(t)} = \lambda \text{ (a real constant)}$$

Here three cases are possible : $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

Case 1 : Let $\lambda > 0$. We can write $\lambda = p^2$ $(p > 0)$. We get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{c^2 T(t)} = p^2$$

$\Rightarrow$ $X''(x) - p^2 X(x) = 0 \Rightarrow X(x) = (A_1 e^{px} + B_1 e^{-px})$

and $T'(t) - p^2c^2 T(t) = 0 \Rightarrow T(t) = C_1 e^{p^2c^2t}$

Thus $u(x, t) = (A_1 e^{px} + B_1 e^{-px}) e^{p^2c^2t}$ ... (S.1)

where constant C_1 is absorbed into A_1 and B_1.

Case 2 : Let $\lambda = 0$. We get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{C^2 T(t)} = 0$$

$\Rightarrow$ $X''(x) = 0 \Rightarrow X(x) = (A_2^* + B_2^* x)$

and $T'(t) = 0 \Rightarrow T(t) = C_2$

$\therefore$ $u(x,t) = (A_2^* + B_2^* x).C_2 \cong A_2 + B_2 x.$ ... (S.2)

where C_2 is absorbed into A_2^* and B_2^*

Case 3 : Let $\lambda < 0$. We can write $\lambda = -p^2$ where $p > 0$. Then we get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{C^2 T(t)} = -p^2$$

Hence $X''(x) + p^2 X(x) = 0 \Rightarrow X(x) = (A_3^* \cos px + B_3^* \sin px)$

$T'(t) + p^2c^2 T(t) = 0 \Rightarrow T(t) = C_3 e^{-p^2c^2t}$

Hence $u(x,t) = (A_3 \cos px + B_3 \sin px) e^{-p^2c^2t}$ ... (S.3)

Thus there are three different forms of solutions of (1) given by

$u(x, t) = (A_1 e^{px} + B_1 e^{-px}) e^{p^2c^2t}$ ... (S.1)

$u(x, t) = A_2 + B_2 x$ ... (S.2)

$u(x, t) = (A_3 \cos px + B_3 \sin px) e^{-p^2c^2t}$ ... (S.3)

Depending on the physical conditions of the problem, we choose an appropriate form of solution. Here as we are dealing with a heat conduction problem, the solution $u(x, t)$ has *to depend on time and it cannot diverge to ∞ as t tends to ∞. Hence the appropriate form of solution is (S.3).*

We shall solve two specific problems giving a general formulation and proceed to solve some problems as illustrations.

SOLVED EXAMPLES

Example 1 : Find the temperature $u(x, t)$ in a bar OA of length l which is perfectly insulated laterally and whose ends O and A are kept at 0°C, given that the initial temperature at any point P of the rod (where $OP = x$) is given as $u(x, 0) = f(x)$ $(0 \le x \le l)$.

[JNTU (K) June 2012 (Set No. 2)]

Solution : The temperature distribution $u(x, t)$ is governed by the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t} \qquad \text{...(1)}$$

Subject to $u(0, t) = 0$ for all t ...(2)

$u(l, t) = 0$ for all t ...(3)

and $u(x, 0) = f(x)$ for $0 \le x \le l$ ...(4)

The solution appropriate to the problem is of the form

$$u(x,t) = (A\cos px + B\sin px)e^{-p^2c^2t} \qquad \text{... (S.3)}$$

Using condition (2)

$$u(0,t) = 0 \Rightarrow A\,e^{-p^2c^2t} = 0 \text{ for all } t \quad \therefore A = 0$$

$$\therefore \quad u(0,t) = B\sin px.e^{-p^2c^2t}.$$

Now using condition (3),

$$u(l,t) = 0 \;\Rightarrow\; B\cdot\sin pl\cdot e^{-p^2c^2t} = 0$$

$$\Rightarrow \sin pl = 0 \;(\because B \text{ cannot be } 0 \text{ as this will mean } u = 0)$$

$$\Rightarrow pl = n\pi \text{ where } n \text{ is a +ve integer}$$

Thus $p = \dfrac{n\pi}{l}$ where $n = 1, 2, 3, \ldots\ldots$

Thus a typical solution of (1) satisfying conditions (2) and (3) is given by

$$u(x, t) = B_n \sin\frac{n\pi x}{l}\; e^{-\frac{n^2\pi^2c^2}{l^2}.t} \quad \text{for } n = 1, 2, 3, \ldots..$$

We note that equation (1) is linear. Hence if $u_1(x, t), u_2(x, t), u_3(x, t), \ldots\ldots$ are solutions of (1) satisfying the homogeneous conditions (2), (3), the most general solution of (1) satisfying (2) and (3) is $\sum_{n=1}^{\infty} u_n(x, t)$.

Hence the most general solution of (1) satisfying conditions (2) and (3) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2c^2}{l^2}.t} \qquad \text{...(5)}$$

where B_n's are arbitrary constants to be determined using condition (4).

Using condition (4) (*i.e.*) $u(x, 0) = f(x)$. Putting $t = 0$ in (5), we get

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x),\; 0 \le x \le l$$

$$\therefore \quad B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx,\ n = 1, 2, 3, ... \qquad ...(6)$$

Hence the solution is given by (5) and (6).

Example 2 : Solve the one dimensional heat flow equation $\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2}$ given that $u(0,t) = 0,\ u(L,t) = 0,\ t > 0$ and $u(x,0) = 3\sin\left(\frac{\pi x}{L}\right), 0 < x < L$. **[JNTU (A) Dec. 2011]**

Solution : Let $u(x,t) = X(x)\,T(t)$ be the solution of the equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad ... (1)$$

Substituting $u = XT$ in (1), we get

$$XT' = c^2 X''T$$

$$\frac{X''}{X} = \frac{1}{c^2}\cdot\frac{T'}{T} = \lambda \text{ (say), where } \lambda \text{ is a constant.}$$

$$X'' - \lambda X = 0 \qquad .... (2)$$

$$T' - c^2\lambda T = 0 \qquad .... (3)$$

Three cases arise according as λ is positive, zero or negative.

Case I : $\lambda > 0$. Let $\lambda = p^2$

Then (2) and (3) becomes $X'' - p^2X = 0$ and $T' - c^2p^2T = 0$

Solving these differential equations, we get

$$X = A_1e^{px} + B_1e^{-px} \text{ and } T = c_1e^{p^2c^2t} \qquad .... (4)$$

Case II : Let $\lambda = 0$

Then (2) and (3) becomes $X'' = 0$ and $T' = 0$

Solving these differential equations, we get

$$X = A_2^* + B_2^*x \text{ and } T = c_2 \qquad .... (5)$$

Case III : $\lambda < 0$. Let $\lambda = -p^2$

Then (2) and (3) becomes $X'' + p^2X = 0$ and $T' + c^2p^2T = 0$

Solving these differential equations, we get

$$X = A_3\cos px + B_3\sin px \text{ and } T = c_3e^{-p^2c^2t} \qquad .... (6)$$

Thus depending upon the values of λ, we have from (4), (5) and (6), the solution of (1) in the form

$$u(x,t) = (A_1 e^{px} + B_1 e^{-px})\, e^{p^2c^2t} \quad .... (7)$$

$$u(x,t) = (A_2^* + B_2^* x)\, C_2 = A_2 + B_2 x \quad .... (8)$$

$$u(x,t) = (A_3 \cos px + B_3 \sin px)\, e^{-p^2c^2t} \quad .... (9)$$

Since $u(x,t)$ decreases as time increases, we find that solution (9) only is consistent with the physical nature of the problem. Hence (9) is the required solution.

Equation (9) may be written as

$$u(x,t) = (A \cos px + B \sin px)\, e^{-p^2c^2t}$$

Now $u(0,t) = 0$ gives $A = 0$

Hence $u(x,t) = B \sin px \cdot e^{-p^2c^2t}$

Also $u(L,t) = 0$

$\therefore\ 0 = B \sin pL\, e^{-p^2c^2t} \Rightarrow \sin pL = 0 \quad (\because\ B \neq 0)$

$\therefore\ pL = n\pi$ or $p = \dfrac{n\pi}{L}$ for $n = 1, 2, 3, ...$

Hence the solution is $u(x,t) = B \sin\left(\dfrac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2c^2}{L^2}t}$

Since this is a solution for $n = 1, 2, 3, ...$, the sum of these solutions (by superposition principle) is also a solution.

Hence $u(x,t) = \displaystyle\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2c^2t/L^2}$

Since $u(x,0) = 3\sin\left(\dfrac{\pi x}{L}\right)$, we get

$$3\sin\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Comparing coefficients of different terms on both sides,

$B_1 = 3, B_2 = B_3 = B_4 = ... = 0$

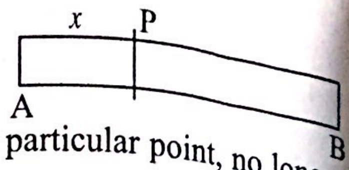
Hence $u(x,t) = 3\sin\left(\dfrac{\pi x}{L}\right) e^{-\pi^2c^2t/L^2}$, which is the required solution.

This is the required solution.

Example 7 : An insulated rod of length L has its ends A and B maintained at 0°C and 100°C ...ctively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained ...°C, find the temperature at a distance x from A at time t. **[JNTU 2003 (Set No. 4)]**

Solution : Let $u(x, t)$ be the temperature at P, at a distance x from the end A at time t. The ...tion for conduction of heat is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$



First we will find the temperature distribution at any distance x, before the end B is reduced to 0°C. Previous to the temperature change at the end B, when $t = 0$, the heat flow was independent of time. (Steady-state condition means that the temperature at any particular point, no longer varies with time.) When u depends only on x, (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0 \text{ i.e., } \frac{d^2 u}{dx^2} = 0 \qquad \ldots(2)$$

$$\ldots(3)$$

Its general solution is $u = ax + b$

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = L$, therefore (3) gives $b = 0$ and $a = \frac{100}{L}$.

So (3) becomes $u = \frac{100x}{L}$ which is the temperature at any point distant x, at time $t = 0$ *i.e.*, before the temperature at B is reduced.

Thus the initial condition is expressed by

$$u(x, 0) = \frac{100x}{L} \qquad \ldots(4)$$

Also the boundary conditions for the subsequent unsteady flow are

$$u(0, t) = 0 \text{ for all values of } t \qquad \ldots(5)$$

and $$u(L, t) = 0 \text{ for all values of } t \qquad \ldots(6)$$

Thus we have to find a temperature function $u(x, t)$ satisfying the equation (1) and the boundary conditions (5) and (6) and the initial condition (4).

Now the solution of (1) is of the form

$$u(x, t) = (c_1 \cos px + c_2 \sin px)e^{-c^2p^2t} \qquad \ldots(7)$$

Substituting (5) in (7), we have

$$u(0, t) = c_1 e^{-c^2p^2t} = 0, \text{ for all values of } t.$$

Hence $c_1 = 0$ and (7) reduces to

$$u(x, t) = c_2 \sin px\, e^{-c^2p^2t} \qquad \ldots(8)$$

Applying the boundary condition (6) in (8), we get

$$u(L, t) = c_2 \sin pL\, e^{-c^2p^2t} = 0, \text{ for all values of } t.$$

i.e., $c_2 \sin pL = 0$

If $c_2 = 0$, (8) will give a trivial solution $u(x, t) = 0$ and hence we must have $\sin pL = 0$

i.e., $pL = n\pi$ where n is any integer

or $p = \frac{n\pi}{L}$

Hence (8) reduces to

$$u(x, t) = c_2 \sin\frac{n\pi x}{L} . e^{-\frac{c^2n^2\pi^2}{L^2}t} \qquad \ldots(9)$$

Since equation (1) linear, its most general solution is got by a linear combination of solutions of the form (9).

Hence, consider the infinite series

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-\frac{c^2 n^2 \pi^2 t}{L^2}}, \text{ where } b_n = c_2 \qquad \ldots(10)$$

Putting $t = 0$ in (10), we have

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Hence in order that the condition (4) may be satisfied, we must have

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = u(x, 0) = \frac{100x}{L} \qquad \ldots(11)$$

We now expand $\frac{100x}{L}$ in a half-range Fourier sine series in $(0, L)$. We know that

$$\frac{100x}{L} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \qquad \ldots(12)$$

where $a_n = \frac{2}{L}\int_0^L \frac{100x}{L} \sin\frac{n\pi x}{L}\, dx$

Comparing (11) and (12), we have $b_n = a_n$

Now $a_n = \frac{200}{L^2}\int_0^L x \sin\frac{n\pi x}{L}\, dx$

$$= \frac{200}{L^2}\left[x\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_0^L$$

$$= \frac{200}{L^2}\left[\frac{-L^2}{n\pi}\cos n\pi\right] = \frac{200}{n\pi}(-1)^{n+1}$$

Hence $b_n = \frac{200}{n\pi}(-1)^{n+1}$

$\therefore$ From (10), we have

$$u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{L} e^{-\frac{c^2 n^2 \pi^2 t}{L^2}}$$

Example 8 : An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If the ends A and B are changed to 40°C and 60°C and maintained at these values, find the transient distribution of the rod.

Solution : The problem at hand, is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \ 0 \le x \le l$$

$$u(0, t) = 40°C, \ u(l, t) = 60°C, \ t > 0$$

$$u(x, 0) = \frac{100x}{l}$$

$u_s(x) = \frac{60-40}{l}x + 40 = \frac{20x}{l} + 40$ is the straight line passing through (0, 40) and (L, 60).

We have

$$u(x, t) = v(x, t) + u_s(x) = u_s(x) + \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{2} e^{-n^2\pi^2c^2t/l^2}$$

where $D_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx + \frac{l}{n\pi}\left[(-1)^n u_l - u_0\right]$

$$\therefore \ u(x,t) = \frac{20x}{l} + 40 + \sum_1^{\infty} D_n \sin\frac{n\pi x}{l} e^{-n^2\pi^2c^2t/l^2}$$

where $D_n = \frac{2}{n\pi}\left[(-1)^n 60 - 40\right] + \frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}dx$

$$= \frac{2}{n\pi}\left[(-1)^n 60 - 40\right] + \frac{200}{n\pi}(-1)^{n+1}$$

Example 9 : If the ends *A* and *B* of a rod 20 cm long have the temperature at 30°C and 80°C until steady state prevail. If the temperatures of the ends are changed at 40°C and 60°C respectively, find the temperature distribution in the rod at time *t*.

[JNTU 2002S, 2004S (Set No. 4)]

Hence from (3), we get

$$30 = a\,.\,0 + b \Rightarrow b = 30$$

and $$80 = a\,.\,l + b = al + 30 \Rightarrow a = \frac{50}{l}$$

So (3) becomes

$$u(x, 0) = \frac{50}{l}x + 30 \qquad \ldots(4)$$

which is the initial temperature distribution in the bar.

When the steady-state condition is reached, the temperatures at the ends A and B have been changed to 40°C and 60°C respectively. So the boundary conditions are

$$u(0, t) = 40 \text{ for all values of } t \qquad \ldots(5)$$

$$u(l, t) = 60 \text{ for all values of } t \qquad \ldots(6)$$

which are non-homogeneous. Therefore assume the solution as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad \ldots(7)$$

where $u_s(x)$ is a solution of (1), involving x only and satisfying the boundary conditions (5) and (6). $u_t(x, t)$ is then a function defined by (7) satisyfing (1). Thus $u_s(x)$ is then steady state solution of (1) and $u_t(x, t)$ may therefore be regarded as a transient solution which decreases with increase of t.

To get $u_s(x)$, we have to solve the equation

$$\frac{d^2u}{dx^2} = 0$$

Its general solution is

$$u_s(x) = ax + b \qquad \ldots(8)$$

Now $u_s(x)$ satisfies the boundary conditions (5) and (6).

i.e., $u_s(0) = 40$ and $u_s(l) = 60$

Applying these in (3), we get

$$b = 40 \text{ and } a = \frac{20}{l}$$

Hence $u_s(x) = \frac{20x}{l} + 40 \qquad \ldots(9)$

Now from (7), we have

$$u_t(x, t) = u(x, t) - u_s(x)$$

$$\therefore \quad u_t(0, t) = u(0, t) - u_s(0) = 40 - 40 = 0$$

and $$u_t(l, t) = u(l, t) - u_s(l) = 60 - \left(\frac{20l}{l} + 40\right) = 0$$

Also $$u_t(x, 0) = u(x, 0) - u_s(x)$$

$$= \frac{50x}{l} + 30 - \left(\frac{20x}{l} + 40\right) \quad \text{[From (4) and (9)]}$$

$$= \frac{30x}{l} - 10$$

Hence the boundary conditions relative to the transient solution $u_t(x, t)$ are

$$u_t(0, t) = 0 \quad \ldots(10)$$
$$u_t(l, t) = 0 \quad \ldots(11)$$
and $$u_t(x, 0) = \frac{30x}{l} - 10 \quad \ldots(12)$$

Since the boundary values are now zero, we use the same procedure to get $u_t(x, t)$.

The transient solution is given by

$$u_t(x, t) = (A \cos px + B \sin px)\, e^{-c^2p^2t} \quad \ldots(13)$$

Substituting (10) in (13), we have

$$u_t(x, t) = A\ e^{-c^2p^2t} = 0 \text{ for all values of } t.$$

Hence $A = 0$

Now equation (13) reduces to

$$u_t(x, t) = B \sin px \,.\, e^{-c^2p^2t} \quad \ldots(14)$$

Applying the boundary condition (11) in (14), we get

$$u_t(l, t) = B \sin pl.\, e^{-c^2p^2t} = 0$$
$$\Rightarrow \quad B \sin pl = 0$$

It $B = 0$, (14) will give the trivial solution $u_t(x, t) = 0$

$\therefore\ \sin pl = 0 \quad i.e.\ pl = n\pi$

or $p = \frac{n\pi}{l}$ where n is any integer.

Hence equation (14) reduces to

$$u_t(x, t) = B \sin \frac{n\pi x}{l}\, e^{-c^2p^2t}, \text{ where } p = \frac{n\pi}{l} \quad \ldots(15)$$

The sum of a finite number of terms of the form (15) will also satisfy the p.d.e. and the boundary conditions (10) and (11) but it will not satisfy the other condition given by (12).

Hence the most general solution is given by the infinite series

$$u_t(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-c^2p^2t}, \text{ where } p = \frac{n\pi}{l} \quad \ldots(16)$$

Putting $t = 0$ in (16), we get

$$u_t(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

Hence in order that the boundary condition (12) may be satisfied, we must have

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = u_t(x, 0) = \frac{30x}{l} - 10 \quad \ldots(17)$$

We now express $\frac{30x}{l} - 10$ in a half-range Fourier sine series in $(0, l)$.

We know that

$$\frac{30x}{l} - 10 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \quad \ldots(18)$$

where $A_n = \frac{2}{l}\int_0^l \left(\frac{30x}{l} - 10\right) \sin\frac{n\pi x}{l} dx$

Comparing (17) and (18), we have $B_n = A_n$

$$\therefore \quad B_n = \frac{2}{l}\int_0^l \left(\frac{30x}{l} - 10\right) \sin\frac{n\pi x}{l} dx$$

$$= \frac{2}{l}\left[\left(\frac{30x}{l} - 10\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \frac{30}{l}\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{-2}{l}\left[\left(\frac{30x}{l} - 10\right)\cos\left(\frac{n\pi x}{l}\right).\frac{l}{n\pi} - \frac{30}{l}\sin\left(\frac{n\pi x}{l}\right).\frac{l^2}{n^2\pi^2}\right]_0^l$$

$$= \frac{-2}{l}\left[20\cos n\pi.\frac{l}{n\pi} - \frac{30}{l}\sin n\pi.\frac{l^2}{n^2\pi^2} + \frac{10l}{n\pi}\right]$$

$$= -2\left[20.\frac{(-1)^n}{n\pi} + \frac{10}{n\pi}\right] = \frac{-20}{n\pi}\left[1 + 2(-1)^n\right]$$

Substituting in (16),

$$u_t(x, t) = \sum_{n=1}^{\infty} \frac{-20}{n\pi}[1 + 2(-1)^n]\sin\left(\frac{n\pi x}{l}\right)e^{-c^2p^2t}$$

$$= -\frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1 + 2(-1)^n]\sin\left(\frac{n\pi x}{l}\right)e^{\frac{-c^2p^2t}{l^2}} \qquad \ldots(19)$$

From (7), (9) and (19), we have

$$u(x, t) = \frac{20x}{l} + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1 + 2(-1)^n]\sin\left(\frac{n\pi x}{l}\right)e^{\frac{-c^2n^2\pi^2t}{l^2}} \qquad \ldots(20)$$

It is given that length of the rod = 20 cm.

$\therefore$ Substituting $l = 20$ in (20), we have

$$u(x, t) = x + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1 + 2(-1)^n]\sin\left(\frac{n\pi x}{20}\right)e^{\frac{-c^2n^2\pi^2t}{400}}$$

which is the required solution.

Example 10 : A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady-state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution. **[JNTU 2004S (Set No. 2)]**

Solution : For convenience, we shall take l to be the length of the bar and put $l = 100$ wherever necessary.