**Example 17.8.** *Solve* $(mz - ny)\dfrac{\partial z}{\partial x} + (nx - lz)\dfrac{\partial z}{\partial y} = ly - mx.$

(*Rohtak, 2011 ; V.T.U., 2010 ; C.S.V.T.U., 2009*)

**Solution.** Here the subsidiary equations are $\dfrac{dx}{mz - ny} = \dfrac{dy}{mx - lz} = \dfrac{dz}{ly - mx}$

Using multipliers $x$, $y$, and $z$, we get each fraction $= \dfrac{xdx + ydy + zdz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$ which on integration gives $x^2 + y^2 + z^2 = a$ ...(*i*)

Again using multipliers $l$, $m$ and $n$, we get each fraction $= \dfrac{ldx + mdy + ndz}{0}$

$\therefore$ $ldx + mdy + ndx = 0$ which on integration gives $lx + my + nz = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the required solution is $x^2 + y^2 + z^2 = f(lx + my + nz)$.

**Example 17.9.** *Solve* $(x^2 - y^2 - z^2)\,p + 2xyq = 2xz.$ (*V.T.U., 2013 ; Anna, 2009 ; C.S.V.T.U., 2008*)

**Solution.** Here the subsidiary equations are $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$.

From the last two fractions, we have $\dfrac{dy}{y} = \dfrac{dz}{z}$

hich on integration gives $\log y = \log z + \log a$ or $y/z = a$ ...(*i*)

Using multipliers $x$, $y$ and $z$, we have

$$\text{each fraction} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} \quad \therefore \quad \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

Example 17.11. *Solve* $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy.$ (V.T.U., 2012 ; Bhopal, 2008)

dx-dy, dy dz -1

**Solution.** Here the subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \qquad \ldots(i)$$

Each of these equations $= \dfrac{dx - dy}{x^2 - y^2 - (y - x)z} = \dfrac{dy - dz}{y^2 - z^2 - x(z - y)}$

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)} \quad \text{or} \quad \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, $\log(x - y) = \log(y - z) + \log c$ or $\dfrac{x-y}{y-z} = c \qquad \ldots(ii)$

Each of the subsidiary equations (*i*) $= \dfrac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz}$

$$= \frac{xdx + ydy + zdz}{(x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy)} \qquad \ldots(iii)$$

Also each of the subsidiary equations $= \dfrac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy} \qquad \ldots(iv)$

Equating (*iii*) and (*iv*) and cancelling the common factor, we get

$$\frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

or $$\int (xdx + ydy + zdz) = \int (x + y + z)d\,(x + y + z) + c'$$

or $$x^2 + y^2 + z^2 = (x + y + z)^2 + 2c' \quad \text{or} \quad xy + yz + zx + c' = 0 \quad ...(v)$$

Combining (*ii*) and (*v*), the general solution is

$$\frac{x - y}{y - z} = f\,(xy + yz + zx).$$

**Example 17.12.** *Solve $p - q = 1$.* (*Anna, 2009*)

**Solution.** The complete solution is $z = ax + by + c$ where $a - b = 1$

Hence $z = ax + (a - 1)y + c$ is the desired solution.

**Example 17.13.** *Solve $x^2p^2 + y^2q^2 = z^2$.* (*Rohtak, 2011 ; Anna, 2008 ; Bhopal, 2008 ; Kerala, 2005*)

**Solution.** Given equation can be reduced to the above form by writing it as

$$\left(\frac{x}{z} \cdot \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{...(i)}$$

and setting $\dfrac{dx}{x} = du, \dfrac{dy}{y} = dv, \dfrac{dz}{z} = dw$ so that $u = \log x, v = \log y, w = \log z$

Then (*i*) becomes $\left(\dfrac{\partial w}{\partial u}\right)^2 + \left(\dfrac{\partial w}{\partial v}\right)^2 = 1$

$$P^2 + Q^2 = 1 \quad \text{where} \quad P = \frac{\partial w}{\partial u} \quad \text{and} \quad Q = \frac{\partial w}{\partial v}.$$

Its complete solution is $w = au + bv + c$

where $a^2 + b^2 = 1$ or $b = \sqrt{(1 - a^2)}$. ...(ii)

∴ (ii) becomes $w = au + \sqrt{(1 - a^2)}\, v + c$

$$\log z = a \log x + \sqrt{(1 - a^2)} \log y + c$$ which is the required solution.

a$^{xy}$

**Form II. f(z, p, q) = 0,** *i.e., equations not containing x and y.*

As a trial solution, assume that $z$ is a function of $u = x + ay$, where $a$ is an arbitrary constant.

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting the values of $p$ and $q$ in $f(z, p, q) = 0$, we get

$$f\left(z, \frac{\partial z}{\partial u}, a\frac{dz}{du}\right) = 0$$ which is an ordinary differential equation of the first order.

Rewriting it as $\dfrac{dz}{du} = \phi(z, a)$ it can be easily integrated giving

$F(z, a) = u + b$, or $x + ay + b = F(z, a)$ which is the desired complete solution.

*Thus to solve $f(z, p, q) = 0$,*

*(i) assume $u = x + ay$ and substitute $p = dz/du$, $q = a\, dz/du$ in the given equation;*

*(ii) solve the resulting ordinary differential equation in z and u;*

*(iii) replace u by $x + ay$.*

**Example 17.14.** *Solve $p(1 + q) = qz$.* *(Rohtak, 2012)*

**Solution.** Let $u = x + ay$, so that $p = dz/du$ and $q = a\, dz/du$.

Substituting these values of $p$ and $q$ in the given equation, we have

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du} \quad \text{or} \quad a\frac{dz}{du} = az - 1 \quad \text{or} \quad \int \frac{a\,dz}{az - 1} = \int du + b$$

$$\log(az - 1) = u + b \text{ or } \log(az - 1) = x + ay + b$$

This is the required complete solution.

u = x + ay

**Example 17.15.** *Solve $q^2 = z^2p^2(1 - p^2)$.* *(J.N.T.U., 2005 ; Kerala, 2005)*

**Solution.** Setting $u = y + ax$ and $z = f(u)$, we get

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = a\frac{dz}{du} \text{ and } q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}$$

∴ The given equation becomes $\left(\dfrac{dz}{du}\right)^2 = a^2z^2\left(\dfrac{dz}{du}\right)^2\left\{1 - a^2\left(\dfrac{dz}{du}\right)^2\right\}$ ...(i)

$$a^4z^2\left(\frac{dz}{du}\right)^2 = a^2z^2 - 1 \quad \text{or} \quad \frac{dz}{du} = \frac{\sqrt{(a^2z^2 - 1)}}{a^2z}$$

Integrating, $$\int \frac{a^2z}{\sqrt{(a^2z^2 - 1)}}dz = \int du + c \quad \text{or} \quad (a^2z^2 - 1)^{1/2} = u + c$$

$$a^2z^2 = (y + ax + c)^2 + 1 \qquad [\because u = y + ax]$$

The second factor in (i) is $dz/du = 0$. Its solution is $z = c'$.

**Example 17.16.** *Solve $z^2(p^2x^2 + q^2) = 1$.* *(Bhopal, 2008 S)*