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UNIT-5

Series of complex curves

A series of the form.

A series of the form

$$a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \dots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n$$

is called a power series in $(z-a)$

Taylor series statement

If $f(z)$ is analytic inside a circle 'c' with centre at 'a', then $\forall z$ inside c_1, we have

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) \dots$$

NOTE put $a=0$ in the above equation

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) \dots \frac{z^n}{n!} f^n(0)$$

this series is called Maclurain series

Laurent's series

If $f(z)$ is analytic inside & on the boundary of ring shared region 'R' bounded by two concentric circles c_1 & c_2 of radii r_1 & r_2 $(r_1 > r_2)$ respectively, having centre at 'a' then for all 'z' in R we have

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2}$$

Problems

→ Expand $\frac{1}{z^2-3z+2}$ in the region

(i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$ (iv) $0<|z-1|<1$

Solution let $f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$Az - 2A + Bz - B = 1$$

$$z(A-B) \cancel{=} 2A - B = 1$$

$\cancel{A+B=0}$ $\cancel{A(z-2)}$

2A

$\Rightarrow A = 1;\ B = 1$

$$\boxed{\begin{array}{l}(1-x)^{-1} = 1+x+x^2\dots \\ (1+x)^{-1} = 1-x+x^2-x^3\dots \\ (1-x)^{-2} = 1+2x+3x^2\dots \\ (1+x)^{-2} = 1-2x+3x^2-4x^3\dots\end{array}}$$

$$\Rightarrow f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

(i) $|z|<1$

$$\frac{|z|}{2} < \frac{1}{2} \Rightarrow \frac{|z|}{2} < 1$$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2(1-\frac{z}{2})} - \frac{1}{(-1)(1-z)}$$

$$= -\frac{1}{2}\left[1-\frac{z}{2}\right]^{-1} + [1-z]^{-1}$$

$$= \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2\cdots\right]+\left[1+z+z^2\cdots\right]$$

(ii) $1<|z|<2 \Rightarrow 1<|z|$ & $|z|<2$

$$\frac{1}{|z|}<1 \quad \& \quad \frac{|z|}{2}<1$$

$$f(z) = \frac{1}{z-2}-\frac{1}{z-1} = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{-1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2\cdots\right]$$

(iii) $|z|>2 \Rightarrow$ $\cancel{\frac{|z|}{2}<1}$ $1>\frac{2}{|z|} \Rightarrow \boxed{\frac{2}{|z|}<1}$

$\cancel{f(z) = \frac{1}{(z-2)}}$

$$\frac{2}{|z|}\times\frac{1}{2}<\frac{1}{2}<1$$

$$\frac{1}{|z|}<1$$

$$f(z) = \frac{1}{z-2}-\frac{1}{z-1}$$

$$= \frac{1}{z\left[1-\frac{2}{z}\right]} - \frac{1}{z\left[1-\frac{1}{z}\right]}$$

$$= \frac{1}{z}\left[\left(1-\frac{2}{z}\right)^{-1} - \left(1-\frac{1}{z}\right)^{-1}\right]$$

$$= \frac{1}{z}\left[\left(1+\frac{2}{z}+\frac{4}{z^2}\cdots\right)-\left(1+\frac{1}{z}+\frac{1}{z^2}\cdots\right)\right]$$

(iv) $0<|z-1|<1$

$0<|z-1|$; $|z-1|<1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{(z-1)-1} - \frac{1}{(z-1)}$$

$$= -1[1-(z-1)]^{-1} - (z-1)^{-1}$$

2) Expand $\cos z$ in a taylor seriers about $z=\frac{\pi}{4}$

Sol: let $f(z)=\cos z$ & $z=a=\frac{\pi}{4}$

s c
c -s

the taylor series expansion is $f(z)= f(a)+(z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a)\cdots$

$f(a) = \frac{1}{\sqrt{2}}$

$\cancel{f(z)}$. $z-a = z-\frac{\pi}{4}$; $f'(z) = -\sin z = -\frac{1}{\sqrt{2}}$

$f''(z) = -\cos z \Rightarrow f''(a) = -\frac{1}{\sqrt{2}}$

$f'''(z) = +\sin z \Rightarrow f'''(a) = \frac{1}{\sqrt{2}}$

$$\Rightarrow \cos z = \frac{1}{\sqrt{2}} + \left(z-\frac{\pi}{4}\right)\cdot\left(-\frac{1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right)\cdots$$

3) Expand the function $\frac{\sin z}{z-\pi}$ about $z=\pi$

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a)$$

Sol. let $z-\pi = t \Rightarrow z = \pi+t$

$$\Rightarrow \frac{\sin z}{z-\pi} = \frac{\sin(\pi+t)}{t} = \frac{-\sin t}{t} = \frac{-1}{t}\left[t-\frac{t^3}{3!}+\frac{t^5}{5!}\cdots\right]$$

summit

$$= -1\left[1 - \frac{t^2}{3!} + \frac{t^4}{5!} \cdots\right]$$

$$t = z - \pi$$

$$\Rightarrow f(z) = -1\left[1 - \frac{(z-\pi)^2}{3!} + \frac{(z-\pi)^4}{5!} \cdots\right]$$

4) Expand $f(z) = \dfrac{z}{(z+1)(z+2)}$ about $z = -2$

Let ~~$z = -2$~~ $z+2 = t \Rightarrow z = t-2$

$$f(z) = \frac{z}{(z+1)(z+2)} = \frac{t-2}{t(t-1)}$$

$$= \frac{t-2}{t(-1)}\left[[1-t]^{-1}\right]$$

$$f(z) = \frac{t-2}{-t}\left[1 + t + t^2 + t^3 \cdots\right] = \frac{2-t}{t}\left[1+t+t^2+t^3 \cdots\right]$$

$$\frac{2-(z+2)}{z+2}\left[1+(z+2)+(z+2)^2+(z+3)^3 \cdots\right]$$

$$= \frac{-z}{z+2}\left[1+(z+2) + (z+2)^2 (z+3)^3 \cdots\right]$$

5) Expand by Laurent's series $\dfrac{1}{z^2-4z+3}$ for $1 < |z| < 3$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_2(z-a)$$

$$f(z) = \frac{\cancel{8}\,1}{(z-1)(z-3)} \Rightarrow f(z) = \frac{-1}{2(z-1)} + \frac{1}{2(z-3)}$$

$$= \frac{\cancel{8}\,1}{2(z-3)} - \frac{1}{2(z-1)}$$

$$1 < |z| < 3 \Rightarrow 1 < |z| \; ; \; |z| < 3$$

$$\frac{1}{\cancel{|z|}} \; |z| < 1 \quad \frac{|z|}{3} < 1$$

$$f(z) = \frac{1}{\frac{2\cancel{z}3}{\cancel{3}}\left(\frac{z}{3}-1\right)} - \frac{1}{2z\left(1-\frac{1}{z\cancel{3}}\right)} = \frac{1}{6}\left[\frac{z}{3}-1\right]^{-1} - \frac{1}{2z}\left[1-\frac{1}{z}\right]^{-1}$$

$$= \frac{1}{-6}\left[1+\frac{z}{3}\right]^{-1} - \frac{1}{2z}\left[1-\frac{1}{z}\right]^{-1}$$

$$= \frac{1}{2}\left[-\frac{1}{3}\left[1+\frac{z}{3}\right]^{-1} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}\right]$$

$$\frac{z^2-1}{(z+2)(z+3)} \quad \text{for } |z| > 3$$

$$\cancel{\frac{z^2-1}{(z+3)(z+3)} = \frac{Az+B}{z+3}} + \cancel{\frac{C}{(z+3)}}$$

$$\cancel{z^2-1} = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{(-5z-7)}{(z+2)(z+3)}$$

$$f(z) = 1 - \frac{5z+7}{(z+2)(z+3)}$$

~~$5z+7 =$~~

$$f(z) = 1 - \frac{3}{z+2} + \frac{8}{z+3}$$

$|z| > 3$

$3 < |z|$

$\frac{3}{|z|} < 1$

Def '0' of an analytic function

A zero of an analytic function $f(z)$ is a value of z such that $f(z)=0$ particularly a point 'a' is called a zero of an analytic function if $f(a)=0$

zero of mth order

If an analytic fun. $f(z)$ can be expressed in the form $(z-a)^m \phi(z)$, where $\phi(z)$ is analytic & $\phi(a) \neq$ then $z=a$ is called zero of mth order of $f(z)$

Simple zero is a zero of order one

Ex:- $f(z) = (z-1)^3$, $z=1$ is a ~~order~~ zero of order '3' of $f(z)$, $f(z) = 1 - \frac{1}{z}$, $z = \infty$ is a simple zero of $f(z)$

$f(z) = \sin z$, ~~$z = n\pi$~~ then $z = 0, \pm\pi, \pm 2\pi \ldots$ are simple zeros of $f(z)$

Singular points

a singular point (or singularity) of a fun. $f(z)$ is the point at which the function $f(z)$ ceases to be analytic

Different types of singularities

→ Isolated :- a pt $z=a$ is called an isolated singularity of an analytic function $f(z)$

if (a) $f(z)$ is not analytic at the point $z=a$

(b) analytic in the deleted neighbourhood of $z=a$ i.e., $\exists$ a neighbourhood of pt $z=a$ which contains no other singularity.

Ex:- If $f(z) = \frac{e^z}{z^2+1}$

then $z = \pm i$ two isolated ~~pt~~ singular pts

if $f(z) = \frac{2}{\sin z}$ $z = \pm\pi, \pm 2\pi, \pm 3\pi$ so on are infinte no. isolated singular pts of $f(z)$

Poles of an analytic function

If ~~g~~ $z=a$ is an isolated singular pt of an analytic fun. $f(z)$ then $f(z)$ can be expanded in laurents series about the pt $z=a$

i.e., $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n/(z-a)^n$

the series of negative integral powers of $z-a$ namely $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is known as the principle part of the laurent's series of $f(z)$

If the principle part contains a finite no. of terms say 'm' (i.e., $b_n = 0, \forall n \ni n > m$) then the singular pt $z=a$ is called a pole of order m of $f(z)$. Simple pole is a pole of order one

ex:- $f(z) = \frac{z^2}{(z-1)(z+2)^2}$

then $z=1$ is a simple pole &

$z=-2$ is a pole of order 2

Essential singularity :- If the principle part of $f(z)$ contains an infinte no. of terms i.e., the series $\sum_{n=1}^{\infty} b_n/(z-a)^n$ contains an infinte no. of terms then the point $z=a$ is called essential singularity of $f(z)$

Ex:- $z=0$ is an essential singularity of $e^{1/z}$ ∵ the principle part of $e^{1/z}$ contains ∞ no. of terms, containging -ve powers of $z-0$

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Removable singularity if the principle part of $f(z)$ contains no term i.e., $b_n = 0 \; \forall n$ then the singularity $z=a$ is called removable singularity of $f(z)$. In this case $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$

Ex:- $f(z) = \frac{1-\cos z}{z}$, $z=0$ is a removable singularity.

singularities at '∞'

taking $z = \frac{1}{t}$ in $f(z)$ we obtain $f(1/t) = F(t)$ then the nature of singularity at $z=\infty$ is defined to be the same as that of $F(t)$ at $t=0$.

Ex:- $f(z) = z^3$ as a pole of order '3' at $z=\infty$

$\because f(1/t) = \left(\frac{1}{t}\right)^3 = \frac{1}{t^3}$ has a pole of order '3' at $t=0$

$f(z) = e^z$ as an essential singularity at $z=\infty$

$\because f(\frac{1}{t}) = e^{(1/t)}$ has an essential singularity at '$t=0$'

Residues

The coff. of $\frac{1}{z-a}$ in the expansion of $f(z)$ about the isolated singularity $z=a$ is called the residue of $f(z)$ at that point

The residue of $f(z)$ at $z=a$ is 'b_1'

from laurent's series we know that the coff. 'b_1' is given by

$$b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$$

$$\Rightarrow \int_C f(z)\,dz = 2\pi i\, b_1$$

$$= 2\pi i\, \mathrm{Res}[f(z)]_a$$

where 'c' is a closed curve containing the pt $z=a$ &

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but by defination we have $\frac{1}{2\pi i}\int_{\gamma_j}$

NOTE: 1) ~~Residue~~ Res $[f(z) : z=a]$ which has a simplepole

$\lim_{z \to a} [(z-a) f(z)]$

2) Res$[f(z) : z=a]$ which is a pole of order 'm' then

$$\frac{1}{(m-1)!} \lim_{z \to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)\right]$$

Residues at each pole $\frac{3z}{z^2+2z+5}$

Sol. $f(z) = \frac{3z}{z^2+2z+5} = \frac{3z}{(z+1)^2+4} = \frac{3z}{(z+1)^2-(2i)^2} = \frac{3z}{(z+2i+1)(z+1-2i)}$

poles $= (z+1+2i)(z+1-2i) = 0 \Rightarrow z+1+2i = 0 \Rightarrow z = -(1-2i)$

$z+1-2i \Rightarrow z = 2i-1$

Res$[f(z)$ ~~at~~$, z = -1-2i]$

$$\lim_{z \to a} [(z-a) f(z)] \Rightarrow \lim_{z \to -1-2i} \left[\cancel{(z+1+2i)} \frac{3z}{\cancel{(z+1+2i)}(z+1-2i)}\right]$$

$$\Rightarrow \frac{3(-1-2i)}{\cancel{-1}-2i\cancel{+1}-2i} = \frac{-3-6i}{-4i} = \frac{3+6i}{4i}$$

$$\frac{3i-6}{-4} = \frac{6-3i}{4}$$

$$\lim_{z\to 2i-1}\left[\cancel{(z+1-2i)}\cdot\frac{3z}{(z+1+2i)\cancel{(z+1-2i)}}\right]$$

$$\frac{3(2i-1)}{2i\cancel{-1}\cancel{+1}+2i} = \frac{6i-3}{4i} = \frac{6+3i}{4}$$

2) $\frac{4z-3}{z(z-1)(z-2)} \Rightarrow z=0;\ z=1;\ z=2$

$$\lim_{z\to 0}\left[\cancel{z}-0\ \frac{4z-3}{\cancel{z}(z-1)(z-2)}\right] = \cancel{0} - 3/2$$

$$\lim_{z\to 1}\left[\cancel{(z-1)}\ \frac{4z-3}{z\cancel{(z-1)}(z-2)}\right] = \cancel{\frac{1}{-2}}\quad -1$$

$$\lim_{z\to 2}\left[\cancel{(z-2)}\ \frac{4z-3}{z(z-1)\cancel{(z-2)}}\right] = \frac{5}{2}$$

3) $\frac{z^2}{(z-1)^2(z+2)}$ $\quad z^2=1 \Rightarrow z=\pm 1$

$z=-2$

$\frac{u}{v} = \frac{\cancel{(\ )}v\ldots}{v^2}$

$$\frac{1}{(2-1)!}\lim_{z\to 1}\left[\frac{d}{dz}\left[\frac{\cancel{(z-1)^2}\, z^2}{\cancel{(z-1)^2}(z+2)}\right]\right]$$

$$= \lim_{z\to 1}\frac{(z+2)(2z) - z(z^2)}{(z+2)^2}$$

$$= \cancel{\ } \frac{(1+2)(2) - \cancel{z}(2^2)}{9} = \frac{5}{9}$$

$4/9$

4) $\frac{z^2}{z^4-1} = \frac{z^2}{(z^2)^2-1^2} = \frac{z^2}{(z^2+1)(z^2-1)} = \frac{z^2}{(z+1)(z-1)(z+i)(z-i)}$

$\boxed{z = 1, -1, i, -i}$

$\lim_{z\to 1} \cancel{(z-1)} \frac{z^2}{(z+1)\cancel{(z-1)}(z+i)(z-i)} = \frac{1}{4}$

$\frac{-1}{4}, \frac{i}{4}, \frac{-i}{4}$

5) $\frac{2z+1}{z^2-z-2} =$

$z^2-z-2 = z^2-2z+z-2)$

$= z(z-2) + 1(z-1)$

$=(z+1)(z-2)$

$\frac{2z+1}{(z+1)(z-2)} \quad z=-1; z=2)$

$\lim_{z\to -1} (\cancel{z+1}) \left(\frac{2z+1}{\cancel{(z+1)}(z-2)}\right) = \frac{-1}{-3} \quad 1/3$

$5/3$

$\frac{2z+1}{z^2(z-2)} \left[-3/4, 3/4\right] \quad \frac{z^2}{(z-1)(z-2)^2} \quad [1, 0]$

$\frac{e^z}{z^2+\pi^2} \left[\frac{i}{2\pi}, \frac{-i}{2\pi}\right]$

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Evaluate $\oint \frac{3z+4}{z(z-1)(z-2)} dz$, $|z| = \frac{3}{2}$

$z(z-1)(z-2) = 0 \Rightarrow z = 0, 1, 2$

$\Rightarrow |x+iy| = \frac{3}{2}$

$\Rightarrow (x-0)^2 + (y-0)^2 = \left(\frac{3}{2}\right)^2$

$\int_C f(z)\, dz = 2\pi i (R_1+R_2)$

$R_1 = \lim_{z\to a} [f(z)(z-a)] \Rightarrow R_1 = 2;\ R_2 = -7$

$\Rightarrow \int_C f(z)\, dz = 2\pi i(2-7) = -10\pi i$

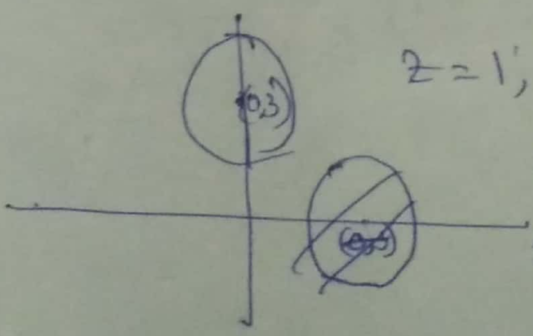
Evaluate $\int \frac{\sin z}{(z-1)^2(z^2+9)}$; where ~~at~~ circle is

$|z-3i| = 1$

$(x-0)^2+(y-3)^2=1$

$C = (0, 3);\ r = 1$

$(z-1)^2(z^2+9)=0$



$z=1,\ z=\pm 3i$

$z=1$ is a pole of order

$z=\pm 3i$ is simple pole

$$\frac{1}{(m-1)!} \lim_{z \to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) \right]$$

$$1 \lim_{z \to 1} \left[\frac{d}{dz} \cancel{(z-1)} \frac{\sin z}{(z-1)^2(z^2+9)} \right]$$

$$\frac{U}{\cancel{8}V}$$

$$\frac{VU' - UV'}{V^2}$$

$$\lim_{z \to 1} \left[\right.$$

$z = 3i$ $\qquad$ $z = -3i$

$x + iy = 3i$ $\qquad$ $x + yi = -3i$

$\Rightarrow y = 3$ $\qquad$ $\Rightarrow y = -3$

$$\oint_c f(z)\,dz = 2\pi i\,[R_1 + R_2]$$

$R_2 \not\exists$

$\cancel{R_1 = \lim}$ $\qquad$ $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$$\Rightarrow \sin(3i) = \frac{e^{-3} - e^{3}}{2i} = \frac{-1}{i}\left[\frac{e^{3} - e^{-3}}{2}\right] = \frac{-1}{i} \sin(3h)$$

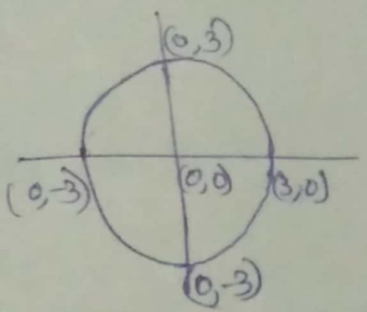
$$\cancel{\lim_{z \to 3i}} \left[\frac{-\sin(3h)}{6(8+6i)} \right] 2\pi i$$

3) Evaluate $\int_C \frac{\cos\pi z^2 + \sin\pi z^2}{(z-1)(z-2)}$ $|z| = 3$

$$(x-0)^2 + (y-0)^2 = (3)^2$$

$$c = (0,0) \ ; \ r = 3$$

Poles are $(z-1)(z-2) = 0 \Rightarrow z = 1; \ z = 2$



$$\oint_C f(z)\,dz = 2\pi i [R_1 + R_2]$$

$$R_1 = \lim_{z \to a} [f(z)(z-a)]$$

$$= \lim_{z \to 1} \left[\cancel{(z-1)} \frac{\sin\pi z^2 + \cos\pi z^2}{\cancel{(z-1)}(z-2)} \right]$$

$$= \frac{-1}{-1} = \boxed{1 = R_1}$$

$$R_1 = \lim_{z \to 2} \left[\frac{\sin 4\pi + \cos 4\pi}{1} \right] = \boxed{1 = R_2}$$

$$\oint_C f(z)\,dz = 4\pi i$$

$|z|=3$

$z=1;\ y\ z=2$

$+\cos\pi z^2$
$z-2)$

π] $= \boxed{1=R_2}$

→ Evaluate $\int_C \frac{e^z}{z^2+1}\,dz$, where $|z|=2$ $[2\pi i \sin 1]$

→ find sum of residues of the function $f(z) = \frac{\sin z}{z\cdot\cos z}$ $|z|=2$ $[0]$

Evaluation of Real Integrals

We consider the evaluation of Real definte integral, to evalute these we apply 'R' theorem, which is simpler than the usual method of integration

Integration round the unit circle

We consider the evalution of integrals of type $\int_0^{2\pi} F(\cos\theta,\sin\theta)\,d\theta$; F = real rational fun. of $\sin\theta$ & $\cos\theta$.

We now ~~st~~ write, $z=re^{i\theta}$ but $r=1$

$$z=e^{i\theta}$$

$$\Rightarrow dz = ie^{i\theta}d\theta \Rightarrow \boxed{\frac{dz}{iz}=d\theta}$$

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$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \left(\frac{z + \frac{1}{z}}{2}\right) \text{ \&}$$

$$\sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i} = \left(\frac{z - \frac{1}{z}}{2i}\right)$$

$$\theta \in [0, 2\pi]$$

'θ' on entire unit circle & $|z| = |e^{i\theta}| = 1$

$$\therefore \int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta = \int_C F\left(\left(z+\frac{1}{z}\right)\times\frac{1}{2}, \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{dz}{iz}$$

$$= \int_C f(z)\, dz$$

$f(z)$ = rational fun.

By 'R' theorem, $\int_C f(z)\, dz = 2\pi i\,[R_1 + R_2 + \dots R_n]$

S.T. $\displaystyle\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$ $(a > b > 0)$

Sol.

$$\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} \quad \text{—(1)}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$$

$|z| = 1 \Rightarrow z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta$

$\Rightarrow \dfrac{dz}{iz} = d\theta$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

3) $\cancel{z+1/z}$

$$\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} = \int_c \frac{dz}{iz\left(a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right)}$$

$$= \frac{1}{i}\int_c \frac{dz}{z\frac{(2az+bz^2+b)}{2z}} = \frac{2}{i}\int \frac{dz}{2az+bz^2+b}$$

$$= \frac{2}{i}\int f(z)\,dz \qquad \cdot \quad bz^2+2az+b=0$$

the poles of $f(z) \Rightarrow a=b;\ b=2a;\ c=b$

$$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2a\pm\sqrt{4a^2-4b^2}}{2b}$$

$$z = \frac{-a\pm\sqrt{a^2-b^2}}{b} = \frac{-a\pm\sqrt{a^2-b^2}}{b}$$

Let $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ & $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

let $a=2;\ b=1$

$$|\alpha| = \left|\frac{-2+\sqrt{3}}{1}\right| < 1 \Rightarrow \beta > 1$$

$\because a>b>0$ we have $|\beta|>1$

but $|\alpha\beta|>1$ so that $|\alpha|<1$

thus $z=\alpha$ is only simple lies inside c

$$\therefore f(z) = \frac{1}{b(z-\alpha)(z-\beta)}$$

$$\Rightarrow R = \lim_{z \to \alpha} \left[f(z) \cdot (z-\alpha) \right]$$

$$\Rightarrow R = \frac{1}{b(\alpha-\beta)}$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{i} \int_c f(z)\,dz = \frac{1}{b(\alpha-\beta)} \times 4\pi$$

$$= \frac{4\pi}{\cancel{b}\left[\frac{-\alpha+\sqrt{a^2-b^2}}{\cancel{b}} + \frac{\alpha+\sqrt{a^2-b^2}}{\cancel{b}}\right]} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\Rightarrow \int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$$

→ Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$ $\quad a \in [0,1] \Rightarrow \boxed{0<a<1}$

$$z = e^{i\theta} \Rightarrow \frac{dz}{iz} = d\theta$$

$$\cos\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{\left[z+\frac{1}{z}\right]}{2}$$

$$\Rightarrow \int_0^{2\pi} \frac{dz}{iz\left(1-2a\frac{\left(z+\frac{1}{z}\right)}{2}+a^2\right)}$$

$$\frac{i\cancel{z}\left(z-az^2-a+a^2z\right)}{\cancel{z}}$$

$$= \frac{1}{i}\int_0^{2\pi} \frac{dz}{-az^2 + (a^2+1)z - a}$$

$x^2 - (a+b)x + ab$

$-az^2 + (a^2+1)z + 1$

$a = -a \quad b = a^2+1 \quad c = -a$

$\cancel{z = \frac{-a^2-1 \pm \sqrt{(a^2+1)^2+4}}{-2a}}$

$\cancel{= \frac{-(a^2+1) \pm \sqrt{a^4+2a^2+5}}{}}$

$$z = \frac{-(a^2+1) \pm \sqrt{(a^2+1)^2 - 4(-a)(a)}}{-2a} = \frac{-(a^2+1) \pm \sqrt{a^4+2a^2+1-4a^2}}{-2a}$$

$$= \frac{-(a^2+1) \pm \sqrt{a^4-2a^2+1}}{-2a} = \frac{-(a^2+1) \pm (a^2-1)}{-2a}$$

$$\alpha = \frac{-\cancel{a^2} + \cancel{a^2} - 2}{-2a} = \frac{1}{a}; \quad \beta = \frac{-a^2-1-a^2+1}{-2a} = \frac{-2a^2}{-2a} = a$$

$$f(z) = \cancel{a(z)} \qquad -az^2\cdot - z^2a + (a^2+1)z - a = 0$$

$$= -a\left[z^2 - \left(\frac{a^2+1}{a}\right)z + 1\right]$$

$$f(z) = \frac{i}{-a(z-a)\left(z-\frac{1}{a}\right)}$$

$$\Rightarrow \frac{1}{-ai}\int_0^{2\pi} \frac{dz}{(z-a)\left(z-\frac{1}{a}\right)}$$

$$R = \lim_{z \to a} \left[\frac{\cancel{z-a}}{(\cancel{z-a})(z-\frac{1}{a}) \cdot -a} \right] = \frac{1}{\frac{a^2-1}{\cancel{a}}(-\cancel{a})}$$

$$R = \frac{-1}{a^2-1}$$

$$\frac{2\pi \cancel{i} \cdot R}{\cancel{i}} = \frac{-2\pi}{a^2-1} = \frac{2\pi}{1-a^2}$$

$$1 \cdot x^2 - (\alpha+\beta)x + \alpha\beta$$

$$b z^2 + 2az + b$$

$$b\left(z^2 + \frac{2a}{b} z + 1\right) = b(z-\alpha)$$

Integrals of the type $-\infty \to +\infty$

Integration around semi circle

where 'C' is the close contour consisting of the semi circle ~~CR~~ $C_R : |z| = R$ togther with the real axis $-R \to +R$

If $f(z)$ has nosingular point on the real axis by 'R' theorem we have over region C_R

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \left[R_1 + R_2 \ldots R_n\right]$$

So we find the value of $\int_{-\infty}^{\infty} f(x)\,dx$ provided $\int_{C_R} f(z)\,dz = 0$ making $R \to \infty$

P.T $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx = \frac{5\pi}{12}$

Sol. To evaluate the given $\int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\,dz$

$$= \int_C \frac{z^2 - z + 2}{(z^2+1)(z^2+9)}\,dz = f(z)\,dz$$

where C is the contour consisting of the semi circle of radius 'R' together with the part of the real axis from $-R \to +R$ observe that the integrand as simple poles at $z = \pm i$ & $\pm 3i$ but $z = i$ & $z = 3i$ are the only two poles lie inside semicircle of contour 'C' ∴ by residue theorem

$$\int_C f(z)\,dz = 2\pi i[R_1 + R_2] = 2\pi i\left[\lim_{z \to i} (z-i) f(z) + \lim_{z \to 3i} (z-3i) f(z)\right]$$

$$R_1 = \frac{z^2 - z + 2}{\cancel{(z-i)}(z+i)(z+3i)(z-3i)} \cancel{(z-i)} = \frac{-1 - i + 2}{\cancel{2i} + 4i - \cancel{7i}} = \frac{1-i}{4i}$$

$$R_2 = \frac{z^2 - z + 2}{(z+i)(z-i)(z+3i)} = \frac{-9 - 3i + 2}{4i}$$

2) $\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^6+1} \Rightarrow \int_C \frac{dz}{z^6+1} = \int_C f(z)\,dz$$

$$z^6+1=0 \Rightarrow z^6=-1 \Rightarrow z^6 = e^{\pi i}$$