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Applications of ~~Different~~ P.D.E

In physical problem we always see a solution of the DE which satisfy some specified conditio[ns] called Boundary conditions.

The differential equation together with these bound[ary] conditions constitute a boundary value prolem

Method of separation of Variables

Separation of Variable is a powerful technique to solve PDE, for a PDE in the function u of two independent variables x & y, assume that the req[uired] sol. is separable ie., $u(x,y) = X(x)\,Y(y)$ — ①

where, $X(x)$ is a function of 'x' alone & $Y(y)$ is a function of 'y' alone, the substitution of 'u' from ① & its derivatives reduces the PDE to the form $f(X, X', X'' \ldots) = g(Y, Y', Y'' \ldots)$ — ②

which is separable in X & Y

∵ the LHS of eq ② is a function of X alone & RHS of eq ② is a function of Y alone. Eq ② must be equal to common constant say 'k'

Thus, eq ② reduces $\underbrace{f(X, X', X'' \ldots)}_{=k \text{ — } ③},\ g(Y, Y', Y'' \ldots) = k$ — ④

Thus, the determination of solution to partial diff. eq reduces to the determination of sol. to two arbitrary sol. eqs. (with appropriate conditions)

PROBLEMS

Solve $U_{xx} - U_y = 0$ — (1) by separation of variables

assume that $U(x,y) = X(x)Y(y)$ — (2)

$$\frac{\partial U}{\partial x} = X'Y \; ; \; \frac{\partial U}{\partial x^2} = X''Y \; ; \; \frac{\partial U}{\partial y} = XY' \quad \text{— (3)}$$

Given $X''Y - XY' = 0 \Rightarrow \frac{X''}{X} = \frac{Y'}{Y} = k$

$$\Rightarrow \frac{X''}{X} = k \; ; \; \frac{Y'}{Y} = k \Rightarrow X'' = kX \;\&\; Y' = kY$$

$$X'' - kX = 0 \; ; \; Y' - kY = 0$$

The auxilary equation $f(m) = 0$

$$\Rightarrow m^2 - k = 0 \quad \& \quad n = k$$

$$\Rightarrow m = \pm\sqrt{k} \quad \& \quad n = k$$

$$\therefore X = c_1 e^{-x\sqrt{k}} + c_2 e^{x\sqrt{k}} \;\&\; Y = c_3 e^{ky}$$

$$U(x,y) = (c_1 e^{-x\sqrt{k}} + c_2 e^{x\sqrt{k}}) c_3 e^{ky}$$

Solve, $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$, where $U(x,0) = 6e^{-3x}$

assume that $U(x,t) = X(x)T(t)$

$$\frac{\partial U}{\partial x} = X'T \;\&\; \frac{\partial U}{\partial t} = XT'$$

$$\Rightarrow X'T = 2XT' + U \Rightarrow X'T - 2XT' = U$$

$X'T - 2XT' = XT$

$\Rightarrow \dfrac{X'T - 2XT'}{XT} = \cancel{k}\,0$

$\dfrac{X'}{X} - \dfrac{2T'}{T} = \cancel{k}\,0 \Rightarrow \dfrac{X'}{X} = \dfrac{2T'}{T}$

$X'T - 2XT' = XT \Rightarrow (X'-X)T = 2XT'$

$\Rightarrow \dfrac{(X'-X)}{X} = \dfrac{2T'}{T} = k$

$\Rightarrow X' = (1+k)X \;;\quad \cancel{2}T' = \dfrac{kT}{2}$

$X' - (1+k)X = 0 \;;\quad T' - \dfrac{k}{2}T = 0$

The auxilary equation $f(m)=0$ & $f(n)=0$

$m = 1+k \qquad n = \dfrac{k}{2}$

$\Rightarrow X = C_1 e^{(1+k)x} \;;\quad T = C_2 e^{\frac{k}{2}t}$

$U(x,t) = C_1 e^{(1+k)x} C_2 e^{\frac{k}{2}t} = C_1 C_2 e^{(1+k)x + \frac{kt}{2}}$

Given that, $U(x,0) = 6e^{3x}$

$\Rightarrow C_1C_2 = 6,\ 1+k = -3 \Rightarrow k = -4$

$\Rightarrow U(x,t) = 6e^{-3x-2t}$

3) $\dfrac{\partial U}{\partial x} = 4\dfrac{\partial U}{\partial y}$ given that $U(0,y) = 8e^{-3y}$

$U(x,y) = X(x)\,Y(y)$

$\frac{\partial U}{\partial x} = X'Y$; $\frac{\partial U}{\partial y} = XY'$ $\Rightarrow$ $X'Y - 4XY' = 0$

$\Rightarrow \frac{X'}{X} = 4\frac{Y'}{Y} = k$

$\Rightarrow X' = Xk$; $Y' = \frac{kY}{4}$ $\Rightarrow$ $X' - Xk = 0$; $Y' - \frac{kY}{4} = 0$

The auxilary equation $f(m) = 0$; $f(n) = 0$

$\Rightarrow$ $m - k = 0$; $4n - k = 0 \Rightarrow n = \cancel{4k}\ k/4$

$X = c_1 e^{kx}$; $Y = c_2 e^{k/4}$

$\Rightarrow XY = c_1 c_2 e^{kx + k/4}$

$c_1 c_2 = 8$; $x = 0 \Rightarrow \frac{k}{4} = -3 \Rightarrow k = -12$

$\Rightarrow U(x,y) = 8e^{-12x - 3y}$

4) $3\frac{\partial U}{\partial x} + 2\frac{\partial U}{\partial y} = 0$; $U(x,0) = 4e^{-x}$

$U(x,y) = X(x)Y(y)$

$\Rightarrow \frac{\partial U}{\partial x} = X'Y$ & $\frac{\partial U}{\partial y} = XY'$

$3X'Y + 2XY' = 0 \Rightarrow 3X'Y = -2XY'$

$3\frac{X'}{X} = -2\frac{Y'}{Y} = k$

$\Rightarrow X' = X\frac{k}{3}$; $Y' = \frac{-k}{2}Y$

$f(m)=0$ & $f(n)=0$

$m-\frac{k}{3}=0 \Rightarrow m=\frac{k}{3}$; $n+\frac{k}{2}=0 \Rightarrow n=-\frac{k}{2}$

$x=c_1 e^{\frac{k}{3}x}$ $\quad y=c_2 e^{-\frac{k}{2}y}$

$U(x,y)=c_1 c_2 e^{\frac{k}{3}x-\frac{k}{2}y}$ $\quad \cancel{U(0,y)=8e^{-3y}}$ $U(x,0)$

$c_1 c_2=8$, $\cancel{y=}$ $x=0$

$\Rightarrow \quad -\frac{k}{2}=-3 \Rightarrow k=$

$\frac{\partial^2 U}{\partial x^2}=\frac{\partial U}{\partial y}+2u$; ∂

A no. of problems in engineering give rise to the following well known ~~fo~~ PDE

I WE $\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$

II One dimensional equation: $\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2}$

III Two dimensional heat flow equation which in steddy state becomes 2D Laplace equation

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

IV Transmission line equation

V Vibrating Membrane 2D WE

VI Laplace Eqn in 3D

Soln for Wave equation

The wave equation is $\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$ — (1)

Let $U = XT$ — (2)

$\frac{\partial U}{\partial x} = X'T$; $\frac{\partial^2 U}{\partial x^2} = X''T$; $\frac{\partial U}{\partial t} = XT'$; $\frac{\partial^2 U}{\partial t^2} = XT''$ — (3)

$\Rightarrow XT'' = c^2 X''T \Rightarrow \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X}$ — (4)

LHS of eq (4) is function of 't' only & the RHS of eq (4) is function of 'x' only $\because$ x & t are independent variables can hold only when both the sides reduced to constant k

Then eq ④ leads to ODE i.e.,

$$\frac{1}{c^2}\frac{T''}{T} = k \quad \& \quad \frac{X''}{X} = k$$

$$\Rightarrow T'' = c^2Tk \quad \& \quad X'' = kX$$

$$T'' - kc^2T = 0 \quad \& \quad X'' - kX = 0 \quad \text{—— ⑤}$$

$$\Rightarrow m^2 - c^2k = 0 \quad \& \quad n^2 - k = 0$$

$$\Rightarrow m = \pm c\sqrt{k} \qquad n = \pm\sqrt{k}$$

1) when $k = +ve$ & say p^2 $\Rightarrow X = c_1e^{px} + c_2e^{-px}$; $T = c_3e^{cpt} + c_4e^{-cpt}$

2) when $k = -ve$ & $-p^2 \Rightarrow X = c_1\cos px + c_2\sin px$; $T = c_3\cos(cpt) + c_4\sin(cpt)$

3) when $k = 0$ $\Rightarrow X = c_1x + c$; $T = c_3t + c_4$

$$U = XT = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-(cpt)})$$

$$U = (c_1\cos px + c_2\sin px)(c_3\cos(cpt) + c_4\cos(cpt))$$

$$U = (c_1x + c_2)(c_3t + c_4)$$

Of these 3 solutions are to choose that sol. which is consistent the physical nature.

U must perodic function $\therefore$ $U = (c_1\cos px + c_2\sin px) + (c_3\cos(cpt) + c_4\sin(cpt))$

A string is stretched & fastned to two points l apart motion is started by displacing in the form $y = a \sin\left(\frac{\pi x}{l}\right)$

S.T. disp. of any pt at a distance 'x' from one end at time is given by $y(x,t) = a \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi c t}{l}\right)$

Sol. The vibration of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—(1)}$$

$$y(0,t) = 0 = y(l,t) \quad \text{—(2)}$$

Since, the intial transverse velocity of any pt of the string is zero ie, $\left[\frac{\partial y}{\partial t}\right]_{(t=0)} = 0$

also $y(x,0) = a \sin\left(\frac{\pi x}{l}\right)$ —(3)

$\because$ the vibrations is perodic $\therefore$ sol. of eq (1) is of form $y(x,t) = (c_1 \cos(px) + c_2 \sin(px))(c_3 \cos(cpt) + c_4 \sin(cpt))$ —(4)

from (2) $y(0,t) = 0$ substitute in (4)

$$y(0,t) = c_1 (c_3 \cos(cpt) + c_4 \sin(cpt)) = 0$$

$$\Rightarrow c_1 = 0 \text{ —(5) substitute (5) in (4)}$$

$$\Rightarrow y(x,t) = c_2 \sin px \, (c_3 \cos(cpt) + c_4 \sin(cpt)) \quad \text{—(6)}$$

$$y(l,t) = c_2 \sin pl \, (c_3 \cos(cpt) + c_4 \sin(cpt)) = 0$$

$$\Rightarrow c_2 \sin pl = 0 \Rightarrow c_2 \sin pl = \sin(n\pi)$$

$$\Rightarrow p = \frac{n\pi}{l} \quad \text{—(7)}$$

$$y(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_3 \cos\left(\frac{n\pi ct}{l}\right) + c_4 \sin\left(\frac{n\pi ct}{l}\right)\right] \quad \text{—} (8)$$

Substitute eq (3) i.e., $\frac{\partial y}{\partial t} = 0$ in eq (8)

$$\Rightarrow \frac{\partial y}{\partial t} = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[-c_3 \sin\left(\frac{n\pi ct}{l}\right)\cdot\left(\frac{\pi c n}{l}\right) + c_4 \cos\left(\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right)\right] = 0$$

$$\Rightarrow 0 = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_4 \frac{n\pi c}{l}\right] = c_2 c_4 \sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi c}{l}\right)$$

if $c_2 = 0 \Rightarrow$ eq (8) will leads to a trivial sol.

$\therefore$ Only possibility is $c_4 = 0$ — (9)

(9) in (8)

$$\Rightarrow y(x,t) = c_2 c_3 \sin\left(\frac{n\pi x}{l}\right)\left(\cos\left(\frac{n\pi ct}{l}\right)\right) \quad \text{—} (10)$$

Substitute $y(x,0)$ in (10)

$$y(x,0) = c_2 c_3 \sin\left(\frac{n\pi x}{l}\right)\cdot 1 = a \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore \boxed{y(x,t) = a \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{\pi ct}{l}\right)}$$

2) A tightly stretched string of length 'l' with fixed end is initially in equilibrium position it is set in vibrating by setting each pt with velocity $v_0 \sin^3\left(\frac{\pi x}{l}\right)$, find '$y$' displacement

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$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad -①$$

$$y(x,t) = (c_1\cos(px) + c_2\sin(px))(c_3\cos(cpt) + c_4\sin(cpt)) - ②$$

$y(0,t) = 0 = y(l,t)$ - ③ (boundary conditions)

$$\left.\begin{aligned}\left[\frac{\partial y}{\partial t}\right]_{t=0} &= V_0 \sin^3\left(\frac{\pi x}{l}\right) \\ y(x,0) &= 0\end{aligned}\right\} - ④$$

④ in ③ $\quad y(0,t) = 0$

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3) A tightly stretched fo

fixed at $x=0$ & $x=l$ at time ($t=0$) the string is given a shape defined by $f(x) = ux(l-x)$, where u is a constant & then released find the disp. of any point 'x' of the string at any time $t>0$

e

Sol. The equation of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad -①$$

the sol. of ① is $y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos(cpt) + c_4\sin(cpt))$ -②

$y(0,t) = 0 = y(l,t)$ -③

Initial conditions
$$\left.\begin{aligned} y(x,0) &= Vx(l-x) \\ \left[\frac{\partial y}{\partial t}\right]_{t=0} &= 0\end{aligned}\right\} - ④$$

Put $(y(0,t)) = 0$ in ②

$y(0,t) = c_1 (c_3 \cos cpt + c_4 \sin(cpt)) = 0$

but $0 = c_1 (c_3 \cos(cpt) + c_4 \sin(cpt)) \Rightarrow c_1 = 0$ — (5)

(5) in (2)

$$y(x,t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \quad \text{— (6)}$$

$y(l,t) = 0$ in (6)

$\Rightarrow 0 = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$

$\Rightarrow c_2 \sin pt = 0 \Rightarrow \sin pl = 0 \Rightarrow p = \frac{n\pi}{l}, \forall n \in N$ — (7)

substitute (7) in (6)

$$y(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_3 \cos\left(\frac{n\pi ct}{l}\right) + c_4 \sin\left(\frac{n\pi ct}{l}\right)\right]$$

$$\Rightarrow y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[\underset{c_2 c_3}{\underset{\downarrow}{a_n}} \cos\left(\frac{n\pi ct}{l}\right) + \underset{c_2 c_4}{\underset{\downarrow}{b_n}} \sin\left(\frac{n\pi ct}{l}\right)\right] \quad \text{— (8)}$$

diff. eq (8) w.r.t 't'

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left[a_n\left(\frac{n\pi c}{l}\right)\left(-\sin\left(\frac{n\pi ct}{l}\right)\right) + b_n \frac{n\pi c}{l}\left(\cos\left(\frac{n\pi ct}{l}\right)\right)\right] \sin\left(\frac{n\pi x}{l}\right) \quad \text{— (9)}$$

$$\left[\frac{\partial y}{\partial t}\right]_{t=0} = 0 = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} \cdot \sin\left(\frac{n\pi x}{l}\right) \Rightarrow b_n = 0 \quad \text{— (10)}$$

substitute (10) in (8)

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \quad \text{— (11)}$$

$y(x,0) = ux(l-x) \rightarrow$ in (11)

$$ux(l-x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_0^l \mu x(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$\mu(x(l-x)$

$$\Rightarrow a_n = \frac{2\mu}{l}\left[\int_0^l lx\sin\left(\frac{n\pi x}{l}\right)dx - \int_0^l x^2\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

sin cos
cos −sin

$$= \frac{2\mu}{l}\left[\int\left[\frac{-lx\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - \right.\right.$$

$$\Rightarrow a_n = -\frac{4\mu l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$y(x,t) = \sum_{n=1}^{\infty} \frac{-4\mu l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

Solution of heat equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

sol. of eq (1) $u(x,t) = X(x)\,T(t)$ where, X $f(x)$ alone

& T $f(t)$ alone

$$\frac{\partial u}{\partial x} = X'T \;;\; \frac{\partial^2 u}{\partial x^2} = X''T \;;\; \frac{\partial u}{\partial t} = XT' \quad \text{—— (3)}$$

(3) in (1) $\Rightarrow$ $\frac{X''}{X} = \frac{1}{c^2}\cdot\frac{T'}{T} = k$

$$\Rightarrow \frac{X''}{X} = k \Rightarrow X'' - kX = 0$$

$$\frac{1}{c^2}\cdot\frac{T'}{T} = k \Rightarrow T' - c^2Tk = 0 \quad (5)$$

—— (4)

solving eq (4) we get

when $k>0$, $=p^2$ (say) we get $X = c_1 e^{px} + c_2 e^{-px}$

$T = c_3 e^{c^2p^2t}$

when $k<0$, $=-p^2$ we get $X = c_1 \cos px + c_2 \sin px$

$T = c_3 e^{-c^2p^2t}$

when $k=0$, $X = c_1 x + c_2$; $T = c_3$

Thus the various possible solutions of the heat eq are

$$U = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{c^2p^2t}$$

$$U = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2p^2t}$$

$$U = (c_1 x + c_2) c_3.$$

of these three solutions we have to choose that sol. which is consistent with the physical nature of problem, as we are dealing with problems on heat conduction it must be a trancent sol. i.e, U is to decrease with increase of time accordingly, the sol. is $U = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2p^2t}$ is the only suitable sol. of the heat eq.

1) solve the equation $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, with boundary conditions $U(x,0) = 3\sin(n\pi x)$, ~~&~~ $U(0,t)=0$ & $U(1,t)=0$

$0<x<1$, $t>0$

<u>solution</u> The sol. of eq is $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ — (1)

$U(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-p^2t}$ — (2)

when $x=0$ we have $u(0,t)=0$ — (3) , (3) in (2)

$\Rightarrow u(0,t)=0=c_1 c_3 e^{-p^2t} \Rightarrow c_1=0$ — (4)

(4) in (2)

$\Rightarrow u(x,t)=(c_2 \sin px)\, c_3 e^{-p^2t} = c_2c_3 \sin px\, e^{-p^2t}$ — (5)

$u(1,t)=0$ in (5)

$\Rightarrow 0 = c_2c_3 \sin p\, e^{-p^2t} \Rightarrow \sin p=0 \Rightarrow p=n\pi, \forall n\in N$ — (6)

(6) in (5) $\Rightarrow u(x,t)=c_2c_3\sin(n\pi x)e^{-n^2\pi^2t}$

$\therefore u(x,t)=\sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-n^2\pi^2t}$ — (7)

substitute $u(x,0)=3\sin\cancel{\sin}(3n\pi)$ in (7)

$\Rightarrow 3\sin(n\pi x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

$\cos 2\theta = 1-2\sin^2\theta$

$b_n = \frac{2}{l}\int_0^l 3\sin(n\pi x).\sin\left(\frac{n\pi x}{l}\right)dx$

$c(A\mp B)=cAcB\pm sAsiB$

$C(A+B)=CACB-SASiB$

$\sin A\sin B = \cos(A-B)-\cos(A+B)$

$= \frac{6}{l}\int_0^l \sin(n\pi x).\sin\left(\frac{n\pi x}{l}\right)dx$

$= \frac{3}{l}\int_0^l\left[\cos\left(n\pi x-\frac{n\pi x}{l}\right)-\cos\left(n\pi x+\frac{n\pi x}{l}\right)\right]dx$

$= \frac{3}{l}\left[\left[\frac{\sin\left(n\pi x-\frac{n\pi x}{l}\right)}{n\pi-\frac{n\pi}{l}}\right]_0^l - \left[\frac{\sin\left(n\pi x+\frac{n\pi x}{l}\right)}{n\pi+\frac{n\pi}{l}}\right]_0^l\right]$

$= \frac{3}{l}\left[\; 6\cdot 3\int_0^1 \sin^2 n\pi x\, dx \right.$

$= 6\int_0^1\left(\frac{1-\cos 2n\pi x}{2}\right)dx$

$= \frac{6}{2}\int_0^1 dx - \frac{6}{2}\int_0^1 \cos 2n\pi x\, dx$

$$= 3[x]_0^1 - 3\left[\frac{\sin 2m\pi x}{2n\pi}\right]_0$$

$b_n = 3 - \frac{\cancel{2}}{\cancel{2}n\pi}$

$$U(x,t) = \sum_{n=1}^{\infty} 3\sin(n\pi x)\, e^{-n^2\pi^2 t}$$

Solve the D.E. $\frac{\partial U}{\partial t} = \alpha^2 \frac{\partial^2 U}{\partial x^2}$ for the conduction of heat along a rod without radiation, subject to following conditions (i) U is not ∞ for $t \to \infty$

(ii) $\frac{\partial U}{\partial x} = 0$, for $x = 0$ & $x = l$ (iii) $u = lx - x^2$ for $t = 0$

(iv) between $x=0$ & $x = l$

$$U = XT \Rightarrow \frac{\partial U}{\partial t} = XT' ; \frac{\partial U}{\partial x} = X'T ; \frac{\partial^2 U}{\partial x^2} = X''T$$

$$\Rightarrow XT' = \alpha^2 \frac{\partial^2 U}{\partial t^2} \Rightarrow XT' = \alpha^2 X''T = -k^2$$

$$\Rightarrow \frac{X''}{X} = -k ; \frac{T'}{T} \cdot \frac{1}{\alpha^2} = -k^2 \Rightarrow X'' + k^2 X = 0 ; T' + k^2 \alpha^2 T = 0$$

$$\Rightarrow \frac{d^2X}{dx^2} + k^2 X = 0 \quad \& \quad \frac{\partial T}{dt} + k^2\alpha^2 T = 0 \quad \text{--- (1)}$$

when $k^2 = k^2$

$\therefore$ the solutions are $X = C_1 \cos kx + C_2 \sin kx$; $T = C_3 e^{-k^2\alpha^2 t}$ --- (2)

when $k^2 = -k^2$, $X = C_4 e^{kx} + C_5 e^{-kx}$; $T = C_6 e^{k^2\alpha^2 t}$ --- (3)

when $-k^2 = 0$; $X = C_7 x + C_8$; $T = C_9$ --- (4)

In eq ③ $T \to \infty$ for $t \to \infty$ also $U \to \infty$, the given condition is not satisfied

applying the condition ② to eq ④ we get $\frac{\partial U}{\partial x} = 0$ for $x = 0$ & $x = l$

$$U = XT = (C_1 \cos kx + C_2 \sin kx) e^{-k^2 \alpha^2 t}$$

$$\frac{\partial U}{\partial x} = 0 = (-C_1 k \sin kx + C_2 k \cos kx) C_3 e^{-k^2 \alpha^2 t}$$

$$x = 0, \quad \frac{\partial U}{\partial x} = 0 \Rightarrow C_2 = 0$$

again $\frac{\partial U}{\partial x} = 0$ when $x = l \Rightarrow 0 = (-C_1 \sin kl + C_2 \cos kl) k C_3 e^{-k^2 \alpha^2 t}$

$$\Rightarrow k = \frac{n\pi}{\alpha l}, \quad \forall n \in N$$

$$U = C_1 C_3 \cos\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2 \pi^2 \alpha^2 t}{l^2}}$$

$$\Rightarrow U = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2 \pi^2 \alpha^2 t}{l^2}}$$

$$U(x,0) = lx - x^2 \Rightarrow lx - x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{l^2}{3}, \quad a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx = -4l^2$$

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An insolated rod of length 'l' has its ends 'A' & 'B' maintained at 0°C & 100°C respectively. Untill steady state conditions preval, if B is suddenly reduced to 0°C & maintained at 0°C then find the temp. at a distance 'x' from A at time 't'

Let the eq for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—①}$$

Prior to the temp. change at the end 'B' wh
't = 0' the heat flow was independent of time
(steady state conditions) when 'u' depends only on 'x'

eq ① ⇒ $\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial u}{\partial x} = a \Rightarrow u = ax + b$ —②

$\because$ $u = 0$ for $x = 0$ & $u = 100$ for $x = l$

$\therefore$ $u = 0;\ x = 0 \Rightarrow b = 0 \Rightarrow u = ax$

$100 = a l \Rightarrow a = \frac{100}{l}$ in ②

$u(x,0) = \frac{100x}{l}$ —③ also boundary condition
~~subsi~~ for subsequent flow are

$$u(0,t) = 0 = u(l,t)\ \forall t \quad \text{—④}$$

thus, we have to find temp fun. $u(x,t)$ satisfy
the diff. eq ① subject to intial condition
③ & boundary condition ④

Sol. of eq ① is of form

$$u(x,t) = (c_1 \cos px + c_2 \sin px)e^{-c^2p^2t} \quad \text{—⑤}$$

substitute $u(0,t) = 0$ in ⑤

$\Rightarrow u(0,t) = c_1 e^{-c^2p^2t} = 0 \Rightarrow c_1 = 0$ —⑥

substitute ⑥ in ⑤

$$v(x,t) = C_2 \sin px\, e^{-c^2p^2t} \quad \text{—} ⑦$$

$$v(l,t) = C_2 \sin pl\, e^{-c^2p^2t} = 0 \Rightarrow \sin pl = 0 \Rightarrow \sin pl = \sin(n\pi)$$

$$\Rightarrow p = \frac{n\pi}{l}, \quad \forall n \in N \text{ — } ⑧ \qquad ⑧ \text{ in } ⑦$$

$$\Rightarrow v(x,t) = C_2 \sin\left(\frac{n\pi x}{l}\right) e^{-c^2p^2t}$$

$$v(x,t) = \Sigma b_n \sin\left(\frac{n\pi x}{l}\right) e^{-c^2p^2t} \quad \text{—} ⑨$$

③ in ⑨

$$v(x,0) = \Sigma b_n \sin\left(\frac{n\pi x}{l}\right) = \frac{100x}{l}$$

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$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l^2}\int_0^l 100x \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\frac{2}{l^2}\left[\frac{-100x\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - \frac{100\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]_0^l$$

$$b_n = \frac{200}{n\pi}(-1)^{n+1} \Rightarrow$$

Solution of Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

$\frac{\partial u}{\partial x} = X'Y$; $\frac{\partial^2 u}{\partial x^2} = X''Y$

$\frac{\partial u}{\partial y} = XY'$; $\frac{\partial u}{\partial y} = XY''$

Let $u = X(x)\,Y(y)$ be a solution of eq (1) ~~subs~~ (2)

~~subst~~ substituting

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} \quad \text{--- (3)}$$

$\because$ x & y are independent variables then eq (3)

can hold good only if each side of eq (3) is eq(3) =

$$\frac{X''}{X} = \frac{-Y''}{Y} = k \Rightarrow \frac{X''}{X} = k \; ; \; \frac{-Y''}{Y} = k$$

$$\Rightarrow X'' - kX = 0 \; ; \; Y'' + kY = 0$$

$$\frac{d^2(X)}{dx^2} + kX = 0 \; ; \; \frac{d^2Y}{dy^2} + kY = 0$$

solving these equations we get,

I $k > 0$ & $= p^2$ (say) $\Rightarrow X = c_1 e^{px} + c_2 e^{-px}$; $Y = c_3 \cos py + c_4$

II $k < 0$ & $= -p^2$ (say) $\Rightarrow X = c_5 \cos px + c_6 \sin px$; $Y = c_7 e^{-py} + c_8 e$

III $k = 0$ $\Rightarrow X = c_9 x + c_{10}$; $Y = c_{11} y + c_{12}$

thus, the various possible solution are

$$U = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

$$U = (c_5 \cos px + c_6 \sin px)(c_7 e^{-py} + c_8 e^{py})$$

$$U = (c_9 x + c_{10})(c_{11} y + c_{12})$$

Problems

1) solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which satisfies the conditions $u(0,y) = u(l,y) = u(x,0) = 0$ & $u(x,a) = \sin\left(\frac{n\pi x}{l}\right)$

Sol: The given eq is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

the three possible sol. of eq (1) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \text{--- (2)}$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{-py} + c_8 e^{py}) \quad \text{--- (3)}$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \text{--- (4)}$$

keeping in view the given B.C. the only possible sol. is eq (3) i.e.,

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{-py} + c_4 e^{py}) \quad \text{--- (5)}$$

substitute $u(0,y) = 0$ in (5)

$$\Rightarrow c_1 = 0 \Rightarrow u(x,y) = c_2 \sin px (c_3 e^{-py} + c_4 e^{py}) \quad \text{--- (6)}$$

$$u(l,y) = 0 \Rightarrow c_2 \sin pl (c_3 e^{-py} + c_4 e^{py}) = 0$$

$$\Rightarrow p = \frac{n\pi}{l} \quad \text{--- (7)}, \quad (7) \text{ in } (5)$$

$$u(x,y) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left(c_3 e^{-\frac{n\pi y}{l}} + c_4 e^{\frac{n\pi y}{l}}\right)$$

$$u(x,y) = \left[c_2 c_3 e^{-\frac{n\pi y}{l}} + c_2 c_4 e^{\frac{n\pi y}{l}}\right] \sin\left(\frac{n\pi x}{l}\right)$$

$U(x,0)=0$

$\Rightarrow \quad (\underbrace{c_2c_3}_{A}+\underbrace{c_2c_4}_{B})\sin\left(\frac{n\pi x}{\lambda}\right)=0 \Rightarrow A=-B$

$\cancel{\Rightarrow c_2c_3 = -c_2c_4}$

$\Rightarrow U(x,y) = c_2 \sin\left(\frac{n\pi x}{\lambda}\right)\left(c_3 e^{\frac{-n\pi y}{\lambda}} - c_3 e^{\frac{n\pi y}{\lambda}}\right)$

$U(x,y) = A\sin\left(\frac{n\pi x}{\lambda}\right)\left[e^{\frac{-n\pi y}{\lambda}} - e^{\frac{n\pi y}{\lambda}}\right]$

$\Rightarrow U(x,y) = \Sigma b_n \sin\left(\frac{n\pi x}{\lambda}\right)\left[e^{\frac{-n\pi y}{\lambda}} - e^{\frac{n\pi y}{\lambda}}\right]$