# 第五节
# 多元复合函数的求导法则

一、链式法则

二、全微分形式的不变性

# 一元复合函数求导法则回顾:

$$y=f(u),\ u=\varphi(x) \quad \Rightarrow y=f(\varphi(x))$$

求导法则 $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$$

或 $$y'=f'(\varphi(x))\varphi'(x)$$

微分法则 $$\mathrm{d}y=f'(u)\,\mathrm{d}u=f'(u)\varphi'(x)\,\mathrm{d}x$$

多元复合函数举例:

$$1.\ z = f(u,v),\ \ u = \varphi(x,y), v = \psi(x,y)$$

$$\Rightarrow z = f(\varphi(x,y), \psi(x,y)).$$

$$2.\ z = f(u,v),\ \ u = \varphi(t), v = \psi(t)$$

$$\Rightarrow z = f(\varphi(t), \psi(t)).$$

$$3.\ z = f(x,u,v),\ \ u = \varphi(x,y), v = \psi(x,y)$$

$$\Rightarrow z = f(x, \varphi(x,y), \psi(x,y)).$$

等等…

## 一、链式法则

**定理1.** 若二元函数 $u=\varphi(x,y),\ v=\psi(x,y)$ 在点 $(x,y)$ 处可微，$z=f(u,v)$ 在对应点 $(u,v)$ 处可微，则复合函数

$$z=f(\varphi(x,y),\psi(x,y)).$$

在点 $(x,y)$ 处可微， 且有

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=f_1'\varphi_1'+f_2'\psi_1'$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=f_1'\varphi_2'+f_2'\psi_2'$$

此式称为求偏导**链式法则**

证：（在此仅证明求偏导数的链式法则）在

$$\Delta z = \frac{\partial z}{\partial u}\Delta u + \frac{\partial z}{\partial v}\Delta v + o(\rho)$$ (其中： $\rho = \sqrt{(\Delta u)^2 + (\Delta v)^2}$

$$\Delta u = \varphi(x+\Delta x, y+\Delta y) - \varphi(x, y),$$
$$\Delta v = \psi(x+\Delta x, y+\Delta y) - \psi(x, y)$$ )

中令 $\Delta y = 0$, 则有

$$\frac{\Delta_x z}{\Delta x} = \frac{\partial z}{\partial u}\frac{\Delta_x u}{\Delta x} + \frac{\partial z}{\partial v}\frac{\Delta_x v}{\Delta x} + \boxed{\underbrace{\frac{o(\rho)}{\rho}\frac{|\Delta x|}{\Delta x}\sqrt{\left(\frac{\Delta_x u}{\Delta x}\right)^2 + \left(\frac{\Delta_x v}{\Delta x}\right)^2}}_{\to 0}}$$

再令 $\Delta x \to 0$, 得

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x},$$ 同理 $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

特例. 若函数 $u=\varphi(t), v=\psi(t)$ 在点 $t$ 可导, $z=f(u,v)$ 在点 $(u,v)$ 处可微, 则复合函数 $z=f(\varphi(t),\psi(t))$ 在点 $t$ 可导, 且有求导**链式法则**

$$\frac{\mathrm{d}z}{\underline{\mathrm{d}t}}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\underline{\mathrm{d}t}}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\underline{\mathrm{d}t}}$$

**注1:** 由于 $u=\varphi(t), v=\psi(t), z=f(\varphi(t),\psi(t))$ 均为 $t$ 的一元函数，注意导数的记法。

**注2:** $\dfrac{\mathrm{d}z}{\mathrm{d}t}$ 称为**全导数**。

例1. 设 $z=e^u\sin v$, $u=xy$ , $v=x+y$, 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解: $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=e^u\sin v\cdot y+e^u\cos v\cdot 1$$

$$=e^{xy}[y\cdot\sin(x+y)+\cos(x+y)]$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=e^u\sin v\cdot x+e^u\cos v\cdot 1$$

$$=e^{xy}[x\cdot\sin(x+y)+\cos(x+y)]$$

例2. 设 $z = u\,v,\ u = e^t,\ v = \cos t$, 求全导数 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解法一: $$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$= v\,e^t - u\sin t$$

$$= e^t(\cos t - \sin t)$$

解法二: $z = e^t\cos t$, 所以

$$\frac{dz}{dt} = (e^t)'\cos t + e^t(\cos t)'$$

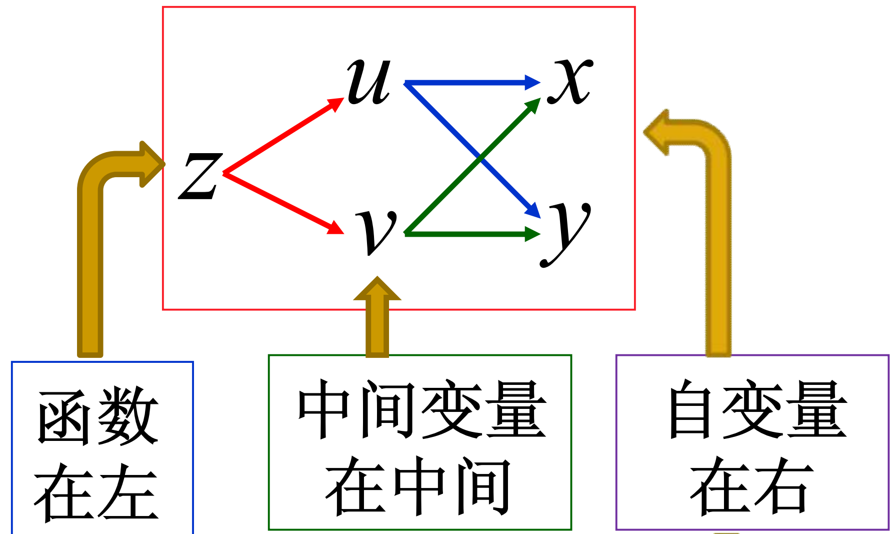
$$= e^t(\cos t - \sin t)$$

# 变量关系图：

**1.** $z = f(u,v)\quad u=\varphi(x,y), v=\psi(x,y)$，有

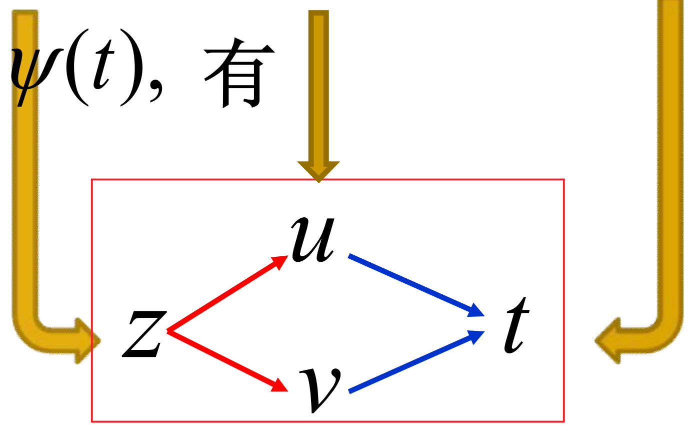
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$




**2.** $z = f(u,v), u=\varphi(t), v=\psi(t)$，有

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$



## 利用变量关系图求偏导数或全导数链式法则：

(1) 画出变量关系图；

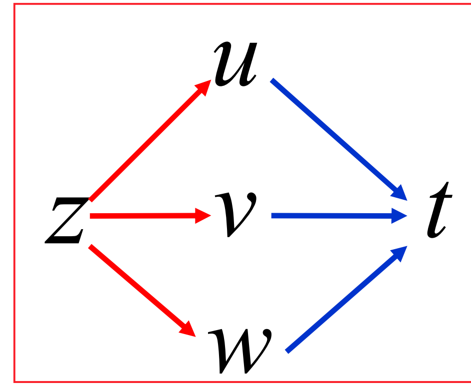
(2) 在变量关系图中，如果复合后的函数或因变量（如 $z$ ）到某自变量（如 $x$）的路径数目为 $i$，则表明复合函数（ $z$）对该自变量（ $x$）的**(偏)**导数为 $i$项之和；

(3) 在一条路径中，如果有 $j$ 条线段相连，则表明该路径对应的项为 $j$ 个**(偏)**导数的乘积，且每个 **(偏)** 导数为线段左边变量对右边变量的**(偏)**导数；

(4) 由(1)(2)(3)，正确写出求偏导数或全导数的链式法则，并由此求出偏导数或全导数。

口诀：路径用加,线段用乘；单出口求导，多出口偏导。

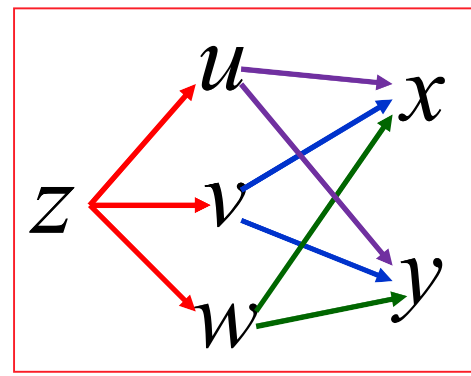
例3： $z=f(u,v,w)$ , $u=\varphi(t)$ , $v=\psi(t)$ , $w=\omega(t)$




$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\partial z}{\partial w}\cdot\frac{\mathrm{d}w}{\mathrm{d}t}$$

$$=f_1'\varphi'+f_2'\psi'+f_3'\omega'$$

例4: $z=f(u,v,w)$ , $u=\varphi(x,y)$ , $v=\psi(x,y)$ , $w=\omega(x,y)$



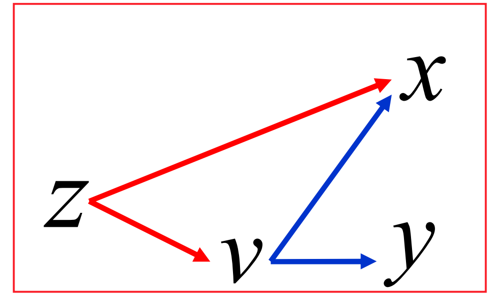

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$=f_1'\varphi_1'+f_2'\psi_1'+f_3'\omega_1'$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial y}=f_1'\varphi_2'+f_2'\psi_2'+f_3'\omega_2'$$

例5: $z = f(x, v)$, $v = \psi(x, y)$

$$\boxed{\frac{\partial z}{\partial x}} = \boxed{\frac{\partial f}{\partial x}} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = f_1' + f_2' \psi_1'$$




$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = f_2' \psi_2'$$

**注意:** 这里 $\frac{\partial z}{\partial x}$ 与 $\boldsymbol{\frac{\partial f}{\partial x}}$ 不同,

$\frac{\partial z}{\partial x}$ 表示在复合后的函数 $z = f(x, \psi(x, y))$ 中,

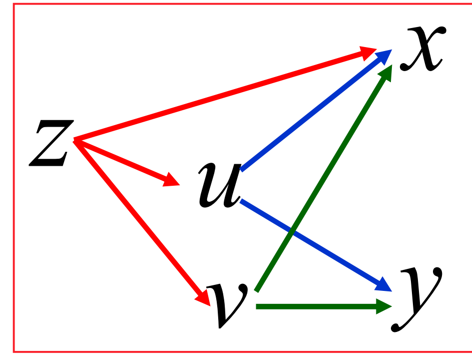
固定 $y$, 对 $x$ 求偏导;

$\boldsymbol{\frac{\partial f}{\partial x}}$ 表示在函数 $z = f(x, v)$ 中, 固定 $v$, 对 $x$求偏导。

例6: $z=f(x,u,v)$, $u=\varphi(x,y)$, $v=\psi(x,y)$

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

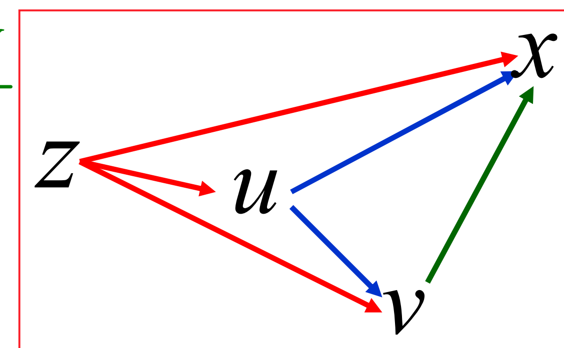
$$=f_1'+f_2'\varphi_1'+f_3'\psi_1'$$




$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=f_2'\varphi_2'+f_3'\psi_2'$$

例7: $z=f(x,u,v)$, $u=\varphi(x,v)$, $v=\psi(x)$

$$\frac{dz}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial\varphi}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{d\psi}{dx}+\frac{\partial f}{\partial u}\cdot\frac{\partial\varphi}{\partial v}\cdot\frac{d\psi}{dx}$$




$$=f_1'+f_2'\varphi_1'+f_3'\psi'+f_2'\varphi_2'\psi'$$

例8. 设 $z = uv + \sin t,\ u = e^t,\ v = \cos t$, 求全导数 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解法一: $z = e^t \cos t + \sin t$,

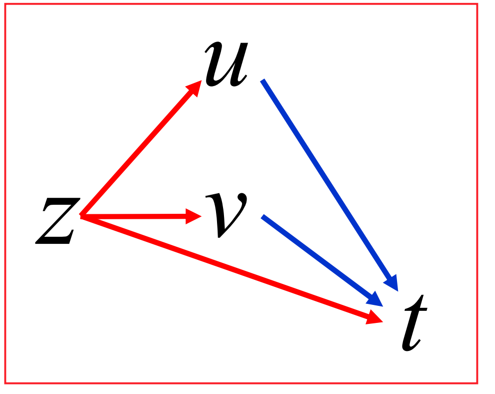
$$\frac{dz}{dt} = (e^t \cos t - e^t \sin t) + \cos t$$

$$= e^t (\cos t - \sin t) + \cos t$$

解法二: $$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial t}$$

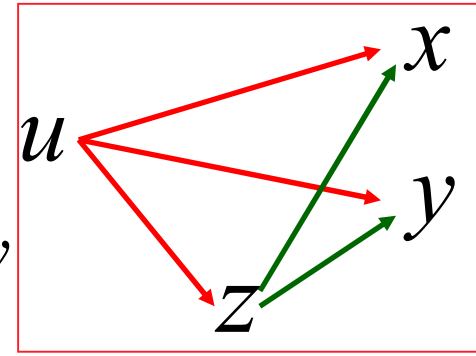
$$= v e^t - u \sin t + \cos t$$

$$= e^t (\cos t - \sin t) + \cos t$$



例9. $u=f(x,y,z)=e^{x^2+y^2+z^2}$, $z=x^2\sin y$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$

解: $$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$$




$$=2xe^{x^2+y^2+z^2}+2ze^{x^2+y^2+z^2}\cdot 2x\sin y$$

$$=2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$$

$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}$$

$$=2ye^{x^2+y^2+z^2}+2ze^{x^2+y^2+z^2}\cdot x^2\cos y$$

$$=2(y+x^4\sin y\cos y)e^{x^2+y^2+x^4\sin^2 y}$$

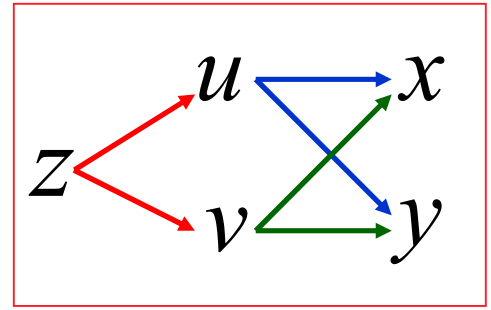
例10. 设 $z=f(x^2+y^2,xy)$，求$\dfrac{\partial^2 z}{\partial x\partial y}$，其中 $f$ 具有二阶连续偏导数。

提示： 把 $f(x^2+y^2,xy)$ 视为中间变量 $u=x^2+y^2$ 和 $v=xy$ 的二元函数 $f(u,v)$，再分别把 $u=x^2+y^2$ 和 $v=xy$ 视为 $x$ 和 $y$ 的二元函数。在解题过程中，符号 $u$ 和 $v$ 不应出现。

复合关系： $z=f(u,v),\ u=x^2+y^2,\ v=xy$

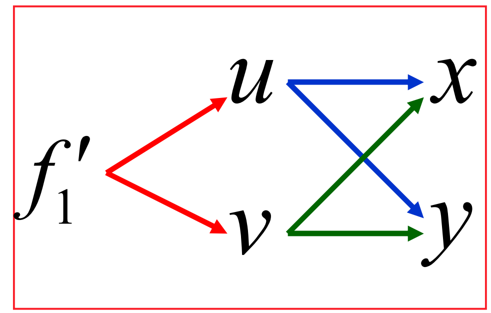
此处 $f$ 是没有给出具体表达式的“抽象函数”。

解： $\dfrac{\partial z}{\partial x} = f_1'\cdot 2x + f_2'\cdot y$, 故




$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial f_1'}{\partial y}\cdot 2x + \frac{\partial f_2'}{\partial y}\cdot y + f_2'$$

$f_1' = f_1'(x^2+y^2, xy)$



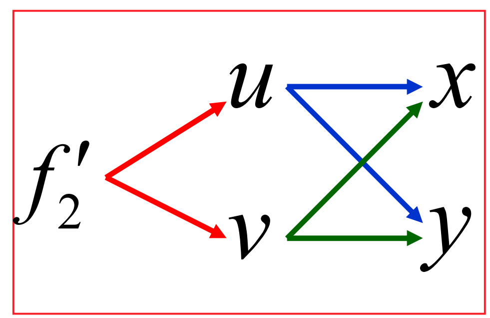

$$= (f_{11}''\cdot 2y + f_{12}''\cdot x)\cdot 2x$$

$$+(f_{21}''\cdot 2y + f_{22}''\cdot x)\cdot y + f_2'$$

$$= 4xyf_{11}'' + 2(x^2+y^2)f_{12}''$$

$$+xyf_{22}'' + f_2'.$$

$f_2' = f_2'(x^2+y^2, xy)$



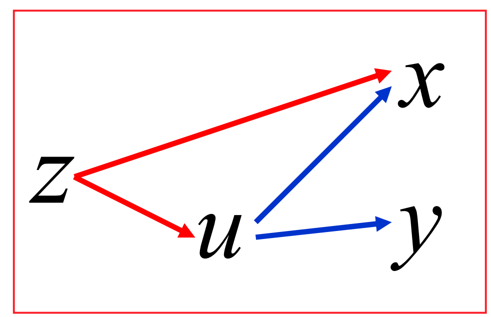

注： 由题意知 $f_{12}'' = f_{21}''$. $\longrightarrow$ 切记！

例11. 设 $z=\frac{1}{x}f(\frac{x}{y})$, 求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},\frac{\partial^2 z}{\partial x\partial y}$, 其中 $f$ 二阶可导。

**解题思路：** 此处应把 $f(\frac{x}{y})$ 视为中间变量 $u=\frac{x}{y}$ 的一元函数 $f(u)$，再把 $u=\frac{x}{y}$ 视为 $x$ 和 $y$ 的二元函数。在求(偏)导过程中，不应该出现 $u$。

解：$$\frac{\partial z}{\partial x}=\frac{-1}{x^2}f(\frac{x}{y})+\frac{1}{x}f'(\frac{x}{y})\cdot\frac{1}{y},$$

$$\frac{\partial z}{\partial y}=\frac{1}{x}f'(\frac{x}{y})\cdot\frac{-x}{y^2},$$



简记为 $\dfrac{\partial z}{\partial x}=\dfrac{-1}{x^2}f+\dfrac{1}{xy}f',\ \dfrac{\partial z}{\partial y}=\dfrac{-1}{y^2}f'$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{-1}{x^2}f+\frac{1}{xy}f'\right)$$

$$=\frac{-1}{x^2}\frac{\partial f}{\partial y}+\left(-\frac{1}{xy^2}f'+\frac{1}{xy}\frac{\partial f'}{\partial y}\right)$$

$$=\cancel{-\frac{1}{x^2}f'\cdot\left(\frac{-x}{y^2}\right)-\frac{1}{xy^2}f'}+\frac{1}{xy}f''\cdot\left(\frac{-x}{y^2}\right)$$

$$=-\frac{1}{y^3}f''.$$

$$\begin{tikzcd}[row sep=small] & & x \\ f \arrow[r] & u \arrow[ur] \arrow[dr] & \\ & & y \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & & x \\ f' \arrow[r] & u \arrow[ur] \arrow[dr] & \\ & & y \end{tikzcd}$$

注：由于 $f$ 为一元函数，因此在解题过程中，

决不能出现 $f'_x, f'_y, f''_{xy}$ 等记号。

例12. 设 $u=f(x,y)$的偏导数连续, 求 $(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2$

在极坐标系下的形式。

解: 已知 $x=r\cos\theta,\ y=r\sin\theta$, 则

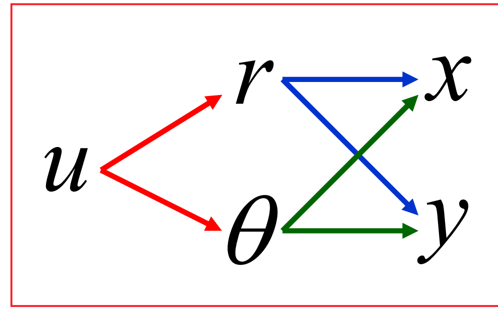
$$r=\sqrt{x^2+y^2},\quad \theta=\arctan\frac{y}{x}$$

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{r},\ \frac{\partial \theta}{\partial x}=\frac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2}=\frac{-y}{r^2}$$

同理, $$\frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial \theta}{\partial y}=\frac{x}{r^2}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}=\frac{\partial u}{\partial r}\frac{x}{r}-\frac{\partial u}{\partial \theta}\frac{y}{r^2}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}=\frac{\partial u}{\partial r}\frac{y}{r}+\frac{\partial u}{\partial \theta}\frac{x}{r^2}$$




$$\therefore \quad (\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial r}\frac{x}{r}-\frac{\partial u}{\partial \theta}\frac{y}{r^2})^2+(\frac{\partial u}{\partial r}\frac{y}{r}+\frac{\partial u}{\partial \theta}\frac{x}{r^2})^2$$

$$=(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

## 二、全微分形式的不变性

设函数 $z = f(u,v)$ 可微, 则

$$\mathrm{d}z = \frac{\partial z}{\partial u}\,\mathrm{d}u + \frac{\partial z}{\partial v}\,\mathrm{d}v \qquad ①$$

此处 $u$, $v$ 为自变量。

又设 $u = \varphi(x,y), v = \psi(x,y)$ 都可微, 则

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y,\ \mathrm{d}v = \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y$$

此处 $u$, $v$ 为中间变量。

此时复合函数 $z = f(\varphi(x,y), \psi(x,y))$ 的全微分为

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y$$

$$= \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}y$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right)$$

$$= \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v \qquad ②$$

由①和②可见，无论 $u, v$ 是自变量还是中间变量,其全微分表达形式都一样，这种性质叫做

**全微分形式不变性.**

例13. 利用全微分形式不变性再解例1.

$$\mathrm{d}z = \mathrm{d}(\, e^u \sin v\,)$$

$$= e^u \sin v\,\mathrm{d}u + e^u \cos v\,\mathrm{d}v$$

$$= e^{xy}[\sin(x+y)\,\mathrm{d}(xy) + \cos(x+y)\,\mathrm{d}(x+y)\,]$$

$$= e^{xy}[\sin(x+y)(y\mathrm{d}x + x\mathrm{d}y) + \cos(x+y)\,(\mathrm{d}x+\mathrm{d}y)\,]$$

$$= e^{xy}[\,y\sin(x+y)+\cos(x+y)]\mathrm{d}x$$

$$+ e^{xy}[x\sin(x+y)+\cos(x+y)\,]\mathrm{d}y$$

所以 $$\frac{\partial z}{\partial x} = e^{xy}[y\cdot\sin(x+y)+\cos(x+y)]$$

$$\frac{\partial z}{\partial y} = e^{xy}[x\cdot\sin(x+y)+\cos(x+y)]$$

全微分形式不变性可以推广到更多元函数上去.

例14. 利用全微分形式不变性再解例8.

解法3: $dz = d(u\,v + \sin t) = d(u\,v) + d\sin t$

$$= (v du + u\,dv) + \cos t dt$$

$$= (\cos t de^t + e^t\,d\cos t) + \cos t dt$$

$$= [\cos t \cdot e^t dt + e^t \cdot (-\sin t) dt] + \cos t dt$$

$$= [e^t(\cos t - \sin t) + \cos t] dt$$

所以

$$\frac{dz}{dt} = e^t(\cos t - \sin t) + \cos t$$

# 内容小结

## 1. 复合函数求导的链式法则

● 画出变量关系图；

● 口诀：

路径用加, 路段用乘； 单出口求导, 多出口偏导。

## 2. 全微分形式不变性

对 $z = f(u,v)$, 不论 $u$, $v$是自变量还是因变量,

$$\mathrm{d}z = f_u'(u,v)\,\mathrm{d}u + f_v'(u,v)\,\mathrm{d}v$$

## 思考与练习

例 1：设函数 $F(x,y)=\int_0^{xy}\frac{\sin t}{1+t^2}dt$，则 $\left.\frac{\partial^2 F}{\partial x^2}\right|_{\substack{x=0\\y=2}}=$ ______.

解： $\frac{\partial F}{\partial x}=y\frac{\sin xy}{1+(xy)^2}$，

$$\frac{\partial^2 F}{\partial x^2}=y\cdot\frac{y[1+(xy)^2]\cos xy-2xy^2\sin xy}{[1+(xy)^2]^2},$$

于是 $\left.\frac{\partial^2 F}{\partial x^2}\right|_{\substack{x=0\\y=2}}=4$.

【答案】填 4 .

例 2：已知 $f(x+y,\frac{y}{x})=x^2-y^2$，求 $f_x'(x,y)$ 及 $f_y'(x,y)$．

解：令 $\begin{cases} x+y=u, \\ \frac{y}{x}=v, \end{cases}$ 则 $x=\frac{u}{1+v}$， $y=\frac{uv}{1+v}$， 于是

$$f(u,v)=\frac{u^2}{(1+v)^2}-\frac{u^2v^2}{(1+v)^2}=\frac{u^2(1-v)}{1+v},$$

得 $f(x,y)=\frac{x^2(1-y)}{1+y}$，所以

$$f_x'(x,y)=\frac{2x(1-y)}{1+y},\quad f_y'(x,y)=-\frac{2x^2}{(1+y)^2}.$$

例3. 设函数 $z=f(x,y)$ 在点 $(1,1)$ 处可微，且

$$f(1,1)=1,\quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)}=2,\quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)}=3,$$

$\varphi(x)=f(\underline{x},\underline{f(x,x)})$，求 $\left.\dfrac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}$.

解: 由题设 $\varphi(1)=f(1,f(1,1))=f(1,1)=1$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}=3\varphi^2(x)\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=1}$$

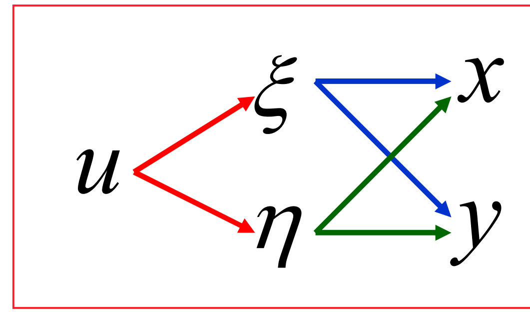
$$\begin{aligned}&=3\Big[f_1'(x,f(x,x))\\&\quad\left.+f_2'(x,f(x,x))\big(\underline{f_1'(x,x)+f_2'(x,x)\cdot 1}\big)\Big]\right|_{x=1}\end{aligned}$$

$$=3\cdot\big[2+3\cdot(2+3)\big]=51$$

**例 4：** 求线性变换 $\xi = x + ay$， $\eta = x - y$， 将方程

$$2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$ 化为方程 $$\frac{\partial u}{\partial \eta} = 0$$, $a$ 为常数.

解： $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \xi} + \dfrac{\partial u}{\partial \eta}$， $\dfrac{\partial u}{\partial y} = a\dfrac{\partial u}{\partial \xi} - \dfrac{\partial u}{\partial \eta}$，




由 $2\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$ 得

$$2\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \left(a\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) = 0$$， 即 $$(2+a)\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} = 0$$

得 $a = -2$， 所以所求线性变换为 $\xi = x - 2y$， $\eta = x - y$。

例 5：设 $f(u,v)$ 具有二阶连续偏导数，且满足 $\dfrac{\partial^2 f}{\partial u^2}+\dfrac{\partial^2 f}{\partial v^2}=1$，

又 $g(x,y)=f(xy,\dfrac{1}{2}(x^2-y^2))$，求 $\dfrac{\partial^2 g}{\partial x^2}+\dfrac{\partial^2 g}{\partial y^2}$.

解：由于 $\dfrac{\partial g}{\partial x}=y\dfrac{\partial f}{\partial u}+x\dfrac{\partial f}{\partial v}$，$\dfrac{\partial g}{\partial y}=x\dfrac{\partial f}{\partial u}-y\dfrac{\partial f}{\partial v}$，进而得

$$\frac{\partial^2 g}{\partial x^2}=y^2\frac{\partial^2 f}{\partial u^2}+2xy\frac{\partial^2 f}{\partial u\partial v}+x^2\frac{\partial^2 f}{\partial v^2}+\frac{\partial f}{\partial v},$$

$$\frac{\partial^2 g}{\partial y^2}=x^2\frac{\partial^2 f}{\partial u^2}-2xy\frac{\partial^2 f}{\partial v\partial u}+y^2\frac{\partial^2 f}{\partial v^2}-\frac{\partial f}{\partial v},$$

所以 $\dfrac{\partial^2 g}{\partial x^2}+\dfrac{\partial^2 g}{\partial y^2}=(x^2+y^2)(\dfrac{\partial^2 f}{\partial u^2}+\dfrac{\partial^2 f}{\partial v^2})=x^2+y^2$.

**例6.** 已知 $f(x,y)\big|_{y=x^2}=1$, $f_1'(x,y)\big|_{y=x^2}=2x$, 求 $f_2'(x,y)\big|_{y=x^2}$.

解: 由 $f(x,x^2)=1$ 两边对 $\boldsymbol{x}$ 求导, 得

$$f_1'(x,x^2)+f_2'(x,x^2)\cdot 2x=0$$

$$f_1'(x,x^2)=2x$$

$$f_2'(x,x^2)=-1$$

例7. 已知 $f(x,y)\big|_{y=2x}=x$ ，$f_1'(x,y)\big|_{y=2x}=x^2$ ，求 $f_2'(x,y)\big|_{y=2x}$ .

解: 由 $f(x,2x)=x$ 两边对 $x$ 求导, 得

$$f_1'(x,2x)+f_2'(x,2x)\cdot 2=1$$

$$f_1'(x,2x)=x^2$$

$$f_2'(x,2x)=\frac{1-x^2}{2}\ .$$

例8. 设 $z=f(u)$,方程 $u=\varphi(u)+\int_y^x p(t)\,dt$ 确定

**$u$是$x, y$的函数，其中$f(u),\varphi(u)$**可微，$p(t),\varphi'(u)$

连续, 且 $\varphi'(u)\neq 1$, 求 $p(y)\dfrac{\partial z}{\partial x}+p(x)\dfrac{\partial z}{\partial y}$.

解: $\dfrac{\partial z}{\partial x}=f'(u)\dfrac{\partial u}{\partial x}$, $\dfrac{\partial z}{\partial y}=f'(u)\dfrac{\partial u}{\partial y}$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\varphi'(u)\frac{\partial u}{\partial x}+p(x)\\ \frac{\partial u}{\partial y}&=\varphi'(u)\frac{\partial u}{\partial y}-p(y)\end{aligned}\right\}\Longrightarrow\left\{\begin{aligned}\frac{\partial u}{\partial x}&=\frac{p(x)}{1-\varphi'(u)}\\ \frac{\partial u}{\partial y}&=\frac{-p(y)}{1-\varphi'(u)}\end{aligned}\right.$$

$$\therefore\ p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}=f'(u)\left[\,p(y)\frac{\partial u}{\partial x}+p(x)\frac{\partial u}{\partial y}\,\right]=0$$