线性代数

第一章　行列式

★主要内容

§1　行列式的概念

§2　行列式的性质

§3　克拉默法则 —— 线性方程组的求解

§1　行列式的概念

一、二元线性方程组与二阶行列式

二元线性方程组$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$

由消元法，得 $(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1a_{22} - a_{12}b_2$

$(a_{11}a_{22} - a_{12}a_{21})x_2 = a_{11}b_2 - b_1a_{21}$

当$a_{11}a_{22} - a_{12}a_{21} \neq 0$时，该方程组有唯一解

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \qquad x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$ 求解公式为 $$\begin{cases} x_1 = \dfrac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \\ x_2 = \dfrac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \end{cases}$$

请观察，此公式有何特点?

① 分母相同，由方程组的四个系数确定.

② 分子、分母都是四个数分成两对相乘再相减而得.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

$$\begin{cases} x_1 = \dfrac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}} \\ x_2 = \dfrac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \end{cases}$$

原则：横行竖列

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \quad \text{记为} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

数表

表达式 $a_{11}a_{22} - a_{12}a_{21}$ 称为由该数表所确定的二阶行列式，即

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

其中 $a_{ij}(i = 1,2; j = 1,2)$ 称为元素.

i 为行标，表明元素位于第 i 行；

j 为列标，表明元素位于第 j 列.

■ 二阶行列式的计算 —对角线法则

$\boxed{\text{主对角线}}$

$\boxed{\text{副对角线}}$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

即：主对角线上两元素之积－副对角线上两元素之积

■ $$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

若令 $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ （方程组的系数行列式）

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \qquad D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

则上述二元线性方程组的解可表示为

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}} = \frac{D_1}{D}$$

$$x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{D_2}{D}$$

例1 求解二元线性方程组 $\begin{cases} 3x_1 - 2x_2 = 12 \\ 2x_1 + x_2 = 1 \end{cases}$

解 因为 $D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3-(-4) = 7 \neq 0$

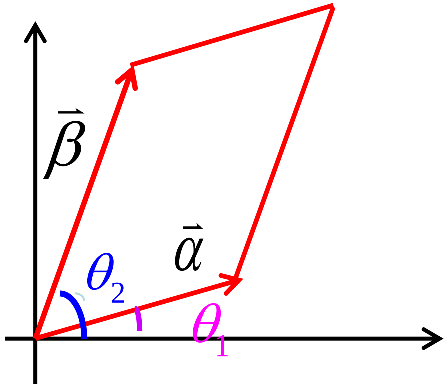
$$D_1 = \begin{vmatrix} 12 & -2 \\ 1 & 1 \end{vmatrix} = 12-(-2) = 14$$

$$D_2 = \begin{vmatrix} 3 & 12 \\ 2 & 1 \end{vmatrix} = 3-24 = -21$$

所以 $x_1 = \dfrac{D_1}{D} = \dfrac{14}{7} = 2,\qquad x_2 = \dfrac{D_2}{D} = \dfrac{-21}{7} = -3$

二阶行列式的几何意义

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \overset{\Delta}{=} \begin{vmatrix} \vec{\alpha} \\ \vec{\beta} \end{vmatrix}$$



二阶行列式可看作以 $\vec{\alpha},\vec{\beta}$ 为邻边的平行四边形的**有向面积**.

其中 $\sin(\theta_2-\theta_1)=\sin\theta_2\cos\theta_1-\cos\theta_2\sin\theta_1$

$$=\frac{a_{22}}{|\vec{\beta}|}\cdot\frac{a_{11}}{|\vec{\alpha}|}-\frac{a_{21}}{|\vec{\beta}|}\cdot\frac{a_{12}}{|\vec{\alpha}|}=\frac{a_{22}a_{11}-a_{21}a_{12}}{|\vec{\alpha}|\cdot|\vec{\beta}|}$$

$$S_{\square}=\frac{1}{2}|\vec{\alpha}|\cdot|\vec{\beta}|\cdot\sin(\theta_2-\theta_1)\cdot 2=a_{11}a_{22}-a_{12}a_{21}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\overset{\Delta}{=}\begin{vmatrix} \vec{\alpha} \\ \vec{\beta} \end{vmatrix}$$

二、三阶行列式

定义 设有9个数排成3行3列的数表

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}$$

引进记号

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

称为三阶行列式.

■ 三阶行列式的计算 —对角线法则

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

实线上的三个元素的乘积冠正号,
虚线上的三个元素的乘积冠负号.

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

注意：对角线法则只适用于二阶与三阶行列式.

例2 计算行列式 $D=\begin{vmatrix} 1 & 2 & -4 \\ -2 & 2 & 1 \\ -3 & 4 & -2 \end{vmatrix}$

解 按对角线法则，有

$$D = 1\times2\times(-2)+2\times1\times(-3)+(-4)\times(-2)\times4$$

$$-1\times1\times4-2\times(-2)\times(-2)-(-4)\times2\times(-3)$$

$$=-4-6+32-4-8-24$$

$$=-14.$$

例3 求解方程 $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{vmatrix} = 0.$

解 方程左端

$$D = 3x^2 + 4x + 18 - 9x - 2x^2 - 12$$

$$= x^2 - 5x + 6,$$

由 $x^2 - 5x + 6 = 0$ 得

$$x = 2 \text{ 或 } x = 3.$$

三、n 阶行列式的定义

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = ?$$

对于三阶行列式，容易验证：

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

问题：一个 n 阶行列式是否可以转化为若干个n-1阶行列式来计算？

在n阶行列式中，把元素a_{ij}所在的第i行和第j列划去后，留下来的n-1阶行列式称为元素a_{ij}的余子式，记作M_{ij}.

将$A_{ij}=(-1)^{i+j}M_{ij}$称为元素a_{ij}的代数余子式.

例如

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \qquad M_{23}=\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

$$A_{23}=(-1)^{2+3}M_{23}=-M_{23}$$

结论　因为行标和列标可唯一标识行列式的元素，所以行列式中每一个元素都分别对应着一个余子式和一个代数余子式.

行列式定义 n阶行列式等于它的第一行的各元素与其代数余子式的乘积之和，即

$$D_n=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$=a_{11}A_{11}+a_{12}A_{12}+\cdots+a_{1n}A_{1n}$$

注：当$n=1$时，一阶行列式$|a|=a$，注意不要与绝对值的记号相混淆．例如：一阶行列式$|-1|=-1$．

例4 计算三角形行列式 （主对角线上侧元素都为0）

$$D=\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}=a_{11}a_{22}\cdots a_{nn}$$

$$D=\begin{vmatrix} 0 & 0 & \cdots & 0 & a_{1n} \\ 0 & 0 & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{n,n} \end{vmatrix}=(-1)^{\frac{n(n-1)}{2}}a_{1n}a_{2,n-1}\cdots a_{n1}$$

结论：

(1) 对角行列式

$$D=\begin{vmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{vmatrix}=a_{11}a_{22}\cdots a_{nn}$$

$$D=\begin{vmatrix} & & & a_{1n} \\ & & a_{2,n-1} & \\ & \iddots & & \\ a_{n1} & & & \end{vmatrix}=\boxed{(-1)^{\frac{n(n-1)}{2}}}a_{1n}a_{2,n-1}\cdots a_{n1}$$

(2) 上三角形行列式（主对角线下侧元素都为0）

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}=a_{11}a_{22}\cdots a_{nn}$$

(3) 下三角形行列式（主对角线上侧元素都为0）

$$D=\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}=a_{11}a_{22}\cdots a_{nn}$$