$$\dot{x} = A(\delta)x + Bu$$
$$y = Cx + Du$$

$$P_A(s) = \det(sI - A) = s^n + \alpha_{n-1}(\delta)s^{n-1} + \dots + \alpha_0(\delta)$$

Kharitonov si 4 polynômes construits
avec les valeurs min max des α_i
sont stables $\Rightarrow$ syst. robustement
stable

Routh-Hurwitz

$A(\delta) \in \mathbb{R}^{n\times n}$

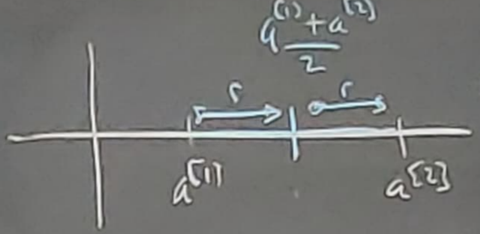
$a \in \mathbb{R}^{1\times 1}$ borné

$\underline{a} \le a \le \overline{a}$

$a = \theta \underline{a} + (1-\theta)\overline{a} \qquad 0 \le \theta \le 1$

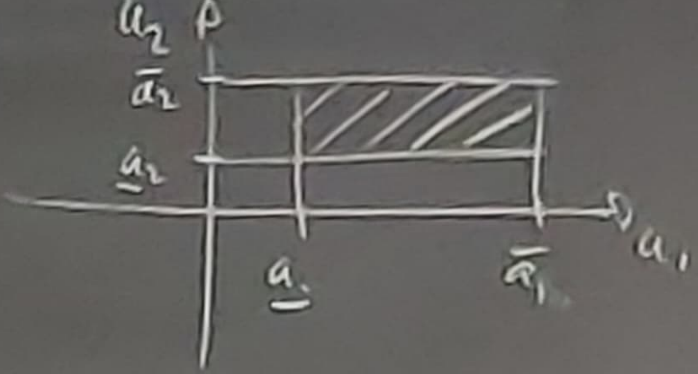
$= \xi_1 a^{[1]} + \xi_2 a^{[2]} \qquad a^{[1]} = \underline{a} \qquad a^{[2]} = \overline{a} \qquad \underset{\geq 0}{\xi_1 = \theta} \qquad \underset{\geq 0}{\xi_2 = (1-\theta)} \qquad \xi_1 + \xi_2 = 1$

$= \frac{a^{[1]}+a^{[2]}}{2} + \delta \left| \frac{a^{[1]}-a^{[2]}}{2} \right| \qquad -1 \le \delta \le 1$



$r = \left| \frac{a^{[1]}-a^{[2]}}{2} \right|$

$$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^{2\times 1}$$



$$\underline{a}_1 \leq a_1 \leq \bar{a}_1$$

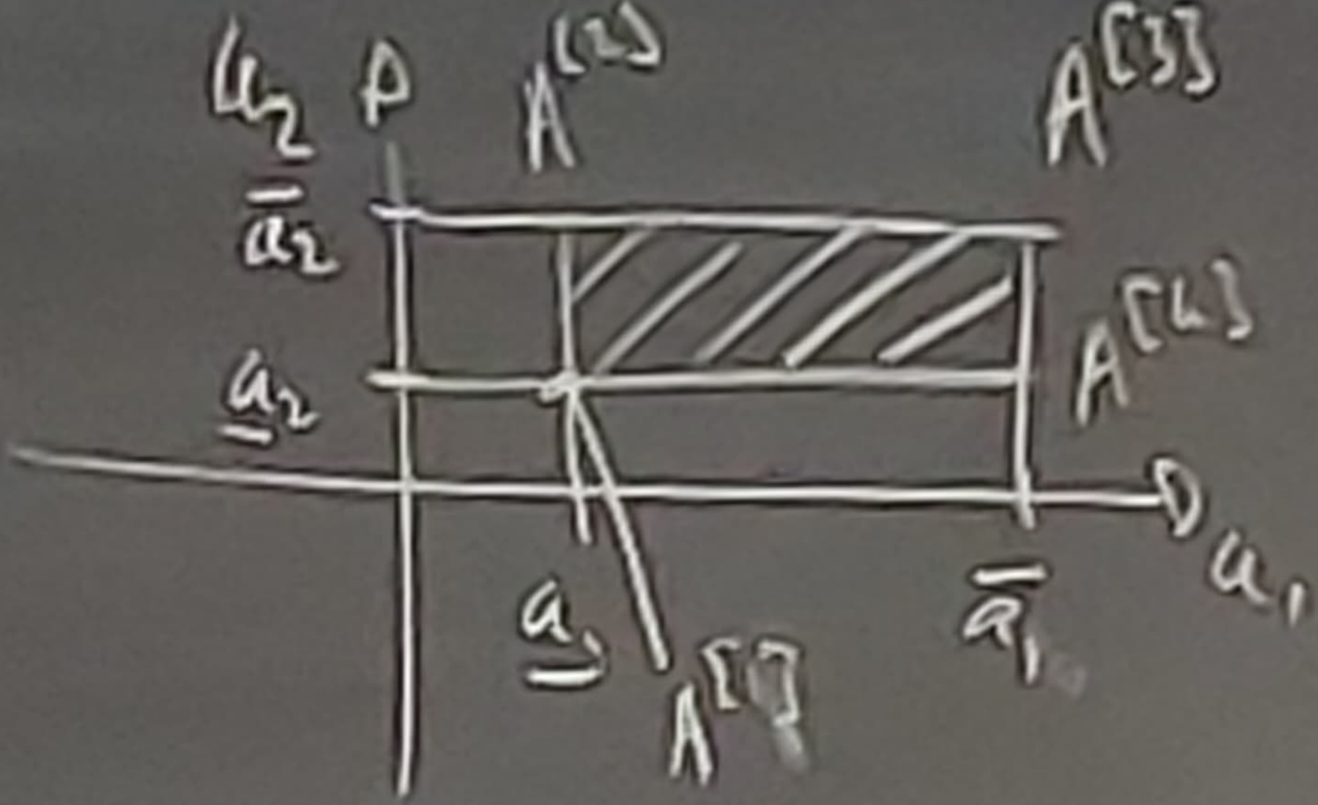
$$\underline{a}_2 \leq a_2 \leq \bar{a}_2$$

$$A = \begin{bmatrix} \theta_1 \underline{a}_1 + (1-\theta_1)\bar{a}_1 \\ \theta_2 \underline{a}_2 + (1-\theta_2)\bar{a}_2 \end{bmatrix}$$

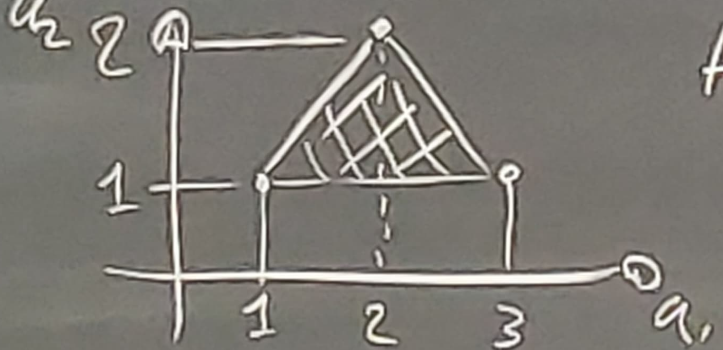
$$A = \begin{bmatrix} (\theta_2 + (1-\theta_2))(\theta_1 \underline{a}_1 + (1-\theta_1)\bar{a}_1) \\ (\theta_1 + (1-\theta_1))(\theta_2 \underline{a}_2 + (1-\theta_2)\bar{a}_2) \end{bmatrix}$$

$$= \underbrace{\theta_1\theta_2}_{\xi_1} \underbrace{\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \end{bmatrix}}_{A^{[1]}} + \underbrace{\theta_1(1-\theta_2)}_{\xi_2} \underbrace{\begin{bmatrix} \underline{a}_1 \\ \bar{a}_2 \end{bmatrix}}_{= A^{[2]}} + \underbrace{(1-\theta_1)(1-\theta_2)}_{\xi_3} \underbrace{\begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{bmatrix}}_{A^{[3]}} + \underbrace{\theta_2(1-\theta_1)}_{\xi_4} \underbrace{\begin{bmatrix} \bar{a}_1 \\ \underline{a}_2 \end{bmatrix}}_{A^{[4]}}$$

$$\xi_v \geq 0 \qquad \sum_{v=1}^{\bar{v}=4} \xi_v = \theta_1 \underbrace{(\theta_2 + (1-\theta_2))}_{=1} + (1-\theta_1)\underbrace{(\theta_2 + (1-\theta_2))}_{=1} = \theta_1 + (1-\theta_1) = 1$$



u_2
$A^{[2]}$
$A^{[3]}$
$\bar{a}_2$
$A^{[4]}$
$\underline{a}_2$
u_1
$\underline{a}_1$
$\bar{a}_1$
$A^{[1]}$



$$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in Co\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

$$= \xi_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \xi_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \xi_3 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\xi_v \geq 0 \quad \sum \xi_v = 1$$

$A(\delta) \in \mathbb{R}^{n\times n}$

Représentation polytopique

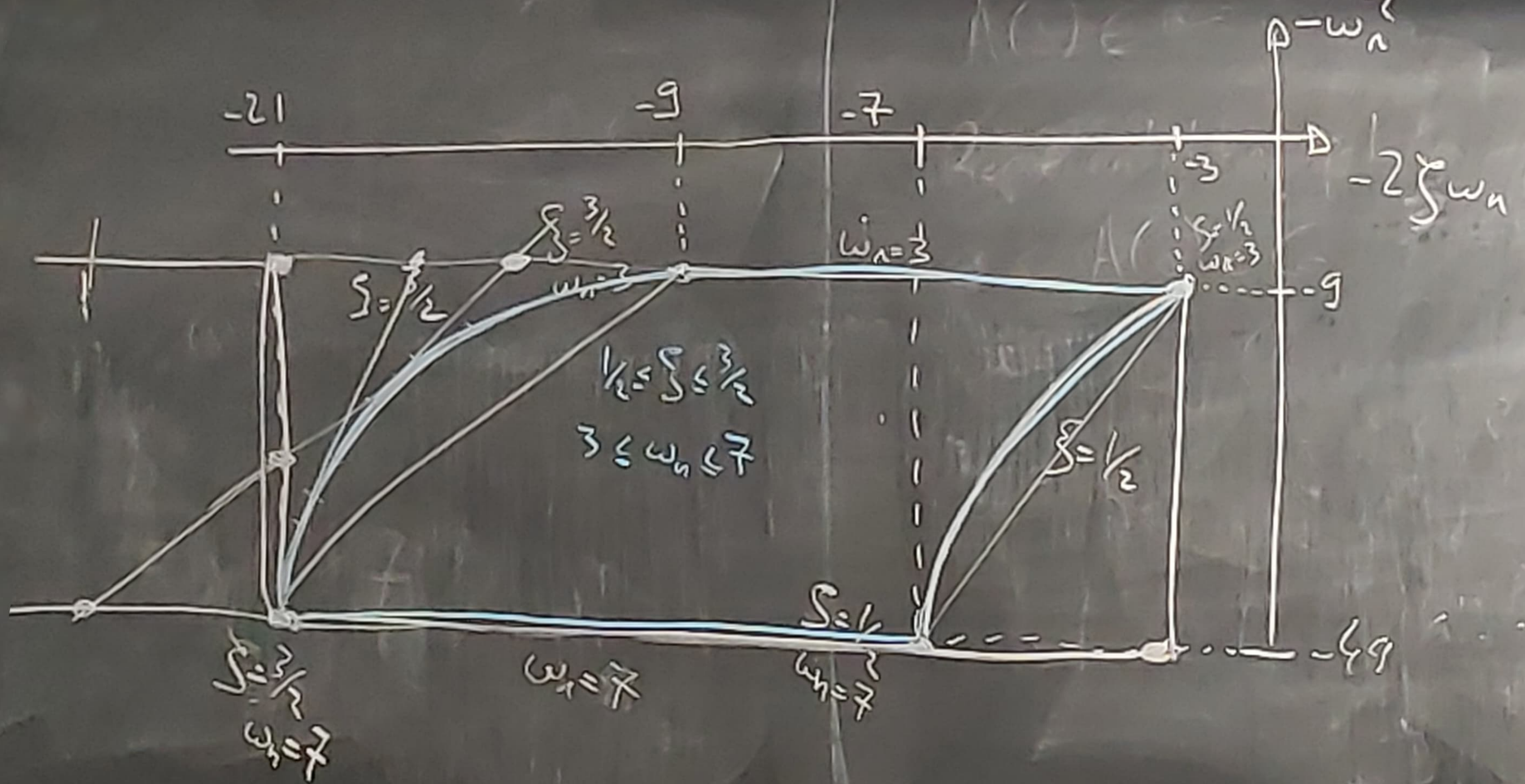
$$A(\delta) \in \mathrm{Co}\left\{ A^{[1]}, A^{[2]} \dots A^{[\bar{v}]} \right\} \qquad A^{[v]} \in \mathbb{R}^{n\times m}$$

$$= \xi_1 A^{[1]} + \xi_2 A^{(2)} + \dots + \xi_{\bar{v}} A^{[\bar{v}]}$$

$$\xi_v \geq 0, \quad \sum_{v=1}^{\bar{v}} \xi_v = 1$$

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = u \qquad 1$$

$$X = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} \left\{ \dot{X} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right. \qquad !$$



$-\omega_n^2$
$-2\zeta\omega_n$
-21
-9
-7
-3
-9
-49
$\zeta = 3/2$
$\zeta = 3/2$
$\omega_n = 3$
$\omega_n = 3$
$\zeta = 1/2$
$\omega_n = 3$
$\frac{1}{2} \leq \zeta \leq \frac{3}{2}$
$3 \leq \omega_n \leq 7$
$\zeta = 1/2$
$\zeta = 1/2$
$\omega_n = 7$
$\zeta = 3/2$
$\omega_n = 7$
$\omega_n = 7$

$$-21 \le -2\zeta\omega_n \le -3$$

$$-49 \le -\omega_n^2 \le -9$$

$$A(\zeta,\omega_n) \in P_1 = Co\left\{ \begin{bmatrix} 0 & 1 \\ -49 & -21 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -9 & -21 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -9 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -49 & -3 \end{bmatrix} \right\}$$

$$A(\zeta\,\omega_n) \in P_2 = Co\left\{ \begin{bmatrix} 0 & 1 \\ -49 & -21 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -9 & -21 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -9 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -49 & -7 \end{bmatrix} \right\}$$

$$\left\{ A(\zeta\,\omega_n) \begin{array}{l} 1/2 \le \zeta \le 3/2 \\ 3 \le \omega_n \le 7 \end{array} \right\} \subset P_2 \subset P_1$$

$$A = \begin{bmatrix} \alpha & 1+\omega\alpha \\ 0 & 1+\omega^2 \end{bmatrix} \qquad \begin{matrix} -1 \leq \alpha \leq 1 \\ -1 \leq \omega \leq 1 \end{matrix}$$

Construire un polytope tq $A(\alpha\omega) \in Co\{[\ ][\ ][\ ][\ ]\}$

$$\dot{x} = \begin{bmatrix} \delta - 1 & 0 \\ \frac{\delta}{1+\delta} & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \qquad -\frac{1}{2} \leq \delta \leq \frac{1}{2}$$

$$\begin{cases} \ddot{q} = u + w + \delta_1 \dot{q} + \frac{\delta_1}{1+\delta_2} q \\ z = q + \delta_2 \dot{q} \end{cases} \qquad \begin{matrix} -\frac{1}{2} \leq \delta_1 \leq \frac{1}{2} \\ -\frac{1}{2} \leq \delta_2 \leq \frac{1}{2} \end{matrix}$$

$$\begin{cases} \dot{x} = \begin{bmatrix} -1 & 1+\delta \\ \frac{-1}{1+\delta} & -1 \end{bmatrix} x + \begin{bmatrix} 1{,}2+\delta & 0 \\ 0 & \frac{1{,}2}{1+\delta} \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ z = x \\ y = x \end{cases} \qquad -1 \leq \delta \leq 1$$

$\dot{x} = A(t)x \qquad A(t) \in Co\{A^{[1]}, A^{[2]} \dots A^{[\bar{v}]}\}$

parmi les réalisations possibles $\dot{x} = A^{[v]}x$

pour avoir la stabilité = condition nécessaire $\forall v = 1 \dots \bar{v}$: $A^{[v]}$ est stable.

ce n'est pas une condition suffisante.

Contre exemple

$$A^{(1)} = \begin{bmatrix} -1 & 10 \\ -1 & -1 \end{bmatrix}$$

$$-\mathrm{Tr}\, A^{(1)} = -(-1-1) = 2 > 0$$

$$\det A^{(1)} = 1 + 10 = 11 > 0$$

$$A^{(2)} = \begin{bmatrix} -1 & -1 \\ 10 & -1 \end{bmatrix}$$

$$-\mathrm{Tr}\, A^{(2)} = 2 > 0$$

$$\det A^{(2)} = 11 > 0$$

$$\tfrac{1}{2} A^{(1)} + \tfrac{1}{2} A^{(2)} = \begin{bmatrix} -1 & 9/2 \\ 9/2 & -1 \end{bmatrix}$$

$$-\mathrm{Tr}\, A = 2 > 0$$

$$\det A = 1 - (9/2)^2 < 0$$

Barmish

Bernussou

Stabilité Quadratique

déf. nie positive

définie négative

THM : Si $\exists P \in \mathbb{R}^{n\times n}$ $P = P^T \succ 0$

telle que $\forall v = 1 \dots \bar{v}$ $\quad A^{[v]^T} P + P A^{[v]} \prec 0$

Alors le système polytopique $\dot{x}(t) = A(t)x(t)$

$A(t) \in Co\{A^{[1]} \dots A^{[\bar{v}]}\}$

est robustement stable.

$$P \succ 0 \iff \exists\, \varepsilon_1 > 0 \quad \forall x \neq 0,\ x \in \mathbb{R}^n \quad x^T P x > \varepsilon \|x\|^2$$

$$P = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \qquad x^T P x = \frac{1}{2} x_1^2 + 2 x_2^2 > \frac{1}{3}\left(x_1^2 + x_2^2\right)$$

$$\frac{1}{3}\left(x_1^2 + x_2^2\right) = \frac{1}{3}\, x^T x = \frac{1}{3}\, \|x\|^2$$

$$\forall_{v=1\ldots\bar{V}} \quad A^{[v]^T} P + P A^{[v]} < -\varepsilon_2^{[v]} \mathbb{I}_n < -\underbrace{\min \varepsilon_2^{[v]}}_{\varepsilon_2} \mathbb{I}$$

$$\exists\, \varepsilon_2^{[v]}$$

$$\sum_{v=1}^{\bar{V}} \zeta_v \left(A^{[v]^T} P + P A^{[v]} + \varepsilon_2 \mathbb{I} \right) < 0$$

$$\zeta_v \geq 0$$

$$\underbrace{\left(\sum_{v=1}^{\bar{V}} \zeta_v A^{[v]^T} \right)}_{} P + P \left(\sum_{v=1}^{\bar{V}} \zeta_v A^{[v]} \right) < -\left(\sum_{v=1}^{\bar{V}} \zeta_v \right) \varepsilon_2 \mathbb{I}$$

$$A(t)^T P + P A(t) < -\varepsilon_2 \mathbb{I}$$

$$x^T(t)\left(A^T(t)P + PA(t)\right)x(t) < x^T(t)\left(-\varepsilon_2 \mathbb{I}\right)x(t) \qquad \forall\, x(t) \neq 0$$

$$x^T(t)A^T(t)Px(t) + x^T(t)PA(t)x(t) < -\varepsilon_2 \underbrace{x^T(t)x(t)}_{\|x(t)\|^2}$$

$$\dot{x}(t) = A(t)x(t)$$

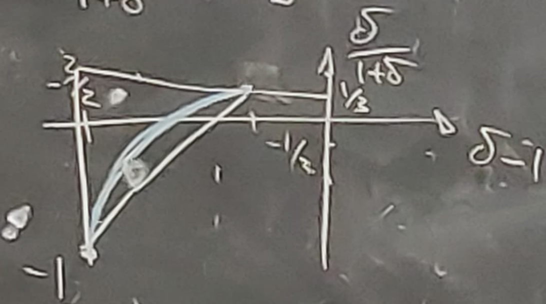
$$\dot{x}^T(t)Px(t) + x(t)^T P\dot{x}(t) < -\varepsilon_2 \|x(t)\|^2$$

$$\frac{d\, x^T(t)Px(t)}{dt}$$

$$\|$$

$$\dot{V}(x) < -c\|x\|$$

$$A = \begin{bmatrix} \delta - 1 & 0 \\ \frac{\delta}{1+\delta} & -1 \end{bmatrix} \qquad -\tfrac{1}{2} \le \delta \le \tfrac{1}{2}$$



$$A \in \mathrm{Co}\left\{ \begin{bmatrix} -3/2 & 0 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -3/2 & 0 \\ 1/3 & -1 \end{bmatrix}, \begin{bmatrix} -1/2 & 0 \\ 1/3 & -1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{I}$$

Symétrique définie négative

$$A^{[1]^T} P + P A^{[1]} = \begin{bmatrix} -3/2 & -1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -3/2 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix}$$

$Tr = -5$

$\det = 5$

$$A^{[2]^T} P + P A^{[2]} = \begin{bmatrix} -3/2 & 1/3 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -3/2 & 0 \\ 1/3 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 1/3 \\ 1/3 & -2 \end{bmatrix}$$

$Tr = -5 < 0$

$\det = 6 - 1/9 > 0$

$$A^{[3]^T} P + P A^{[3]} = \begin{bmatrix} -1/2 & 1/3 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1/2 & 0 \\ 1/3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1/3 \\ 1/3 & -2 \end{bmatrix}$$

$Tr = -3 < 0$

$\det = 2 - 1/3 > 0$

THM : Si $\forall \xi_v \geq 0$ $\sum \xi_v = 1$ $\exists P(\xi) > 0$

telle que $A(\xi)^T P(\xi) + P(\xi) A(\xi) < 0$

alors : $\dot{x} = A(\xi) x$ est robustement stable

: $\forall A(\xi) = \text{cste} = \sum \xi_v A^{(v)}$

THM. Si $\exists X > 0$ $\in \mathbb{R}^{n\times n}$ $\exists S \in \mathbb{R}^{m\times n}$

$\forall v = 1 \dots \bar{v}$ $\quad A^{[v]} X + BS + S^T B^T + X A^{[v]^T} < 0$

alors $K = S X^{-1}$ est un retour d'état qui stabilise robustement

$$\dot{x} = A(t) x + Bu$$

$$A(t) \in \mathrm{Co}\{A^{[1]} \dots A^{[\bar{v}]}\}$$

$$V(x) = x^T P x \text{ avec } P = X^{-1}$$