* Principe de maximisation de l'Hamiltonien (Pontryagine)

* Dans le cas continue :

trouver $u^*(x(t), t)$ qui maximise l'Hamiltonien

$$H(x(t), u, \lambda) = -L(x, u, t) + \overline{\lambda}' F(x, u, t) \text{ avec } \dot{x} = F(x, u, t)$$

* Dans le cas discret :

$$H_{k+1} = -L(x, u, k) + \overline{\lambda}'_{k+1} F(x, u, t) \text{ avec } x_{k+1} = F(x, u, k)$$

Dans le cas continu :

les conditions d'optimalité $\Longleftrightarrow$ minimiser le critère J

$\Longleftrightarrow$ maximiser l'Hamiltonien

① condition finale : $\lambda^{T}_{f} = -x^{T}_{f} P_{f}$

② $\dot{\lambda} = -\frac{\delta H}{\partial x}$

③ $\dot{x} = \frac{\delta H}{\partial \lambda}$

Recherche d'une commande optimale $\Leftrightarrow$ maximiser le H

$$\begin{cases} \left.\dfrac{\partial H}{\partial u}\right|_{u=u^*} = 0 \\ \left.\dfrac{\partial^2 H}{\partial u^2}\right|_{u=u^*} < 0 \end{cases}$$

$$\left.\frac{\partial H_{k+1}}{\partial u_k}\right|_{u=u^*} = 0$$

$$\left.\frac{\partial H_{k+1}}{\partial u_k}\right|_{u=u^*} < 0$$

* Dans le cas discret : les condition d'optimalité (les condition du premier ordre d'Hamilton)

① $x_{k+1} = \dfrac{\partial H_{k+1}}{\partial \lambda_{k+1}}$ | ② $\lambda_k = \dfrac{\partial H_{k+1}}{\partial x_k}$ | ③ la condition finale $\lambda_K^T = x_K^T P_K$

(*) Application P.M.P à la commande LQ

Soit un système : $x_{k+1} = A x_k + B u$

on recherche la commande optimale LQ qui minimise le critère suivant

$L(x, u, k) = \frac{1}{2} \{ x_k^T Q_k x_k + u_k^T R u_k \}$ avec $Q_j = Q_j^T \geq 0$, $R_j = R_j^T > 0$, $F_j = F_j^T \geq 0$

On pose l'Hamiltonien :

$$x_{k+1} = F(x, u, k)$$

$$H(x, u, k) = -\frac{1}{2}\left[x_k^T Q_k x_k + u_k^T R_k u_k\right] + \lambda_k^T \overbrace{\left(A x_k + B u_k\right)}$$

* La maximisation de H $\Longleftrightarrow$ conditions d'optimalité vérifiée

$* \; x_{k+1} = \dfrac{\delta H_{k+1}}{\delta \lambda_{k+1}}$

$* \; \lambda_k = \dfrac{\delta H_{k+1}}{\partial x_k}$

$\lambda_{T_f}^T = -x_{T_f}^T P_{T_f}$

$\left.\dfrac{\delta H_{k+1}}{\partial u_k}\right|_{u=u^*} = 0$

$\left.\dfrac{\delta^2 H_{k+1}}{\partial u_k^2}\right|_{u=u^*} < 0$

$$\boxed{x_{k+1} = \frac{\delta H_{k+1}}{\partial \lambda_{k+1}} = A x_k + B u_k}$$

$$\boxed{\lambda_k = \frac{\delta H_{k+1}}{\partial x_k} = A^T \lambda_{k+1} - Q_k x_k}$$

$$\left.\frac{\delta H_{k+1}}{\partial u_k}\right|_{u=u^*} = 0$$

$$\Rightarrow \left. -R_k u_k + B^T \lambda_{k+1} \right|_{u=u^*} = 0 \Rightarrow$$

$$\boxed{\vec{u}_k = R_k^{-1} B^T \lambda_{k+1}}$$

$$\boxed{\frac{\delta^2 H_{k+1}}{\partial u_k^2} = -R_k < 0}$$

on s'arrive d'avoir la commande optimale :

$$u_k^* = R^{-1} B^T \lambda_{k+1} \quad ; \quad \lambda_{T_f} = -P_{T_f} x_{T_f}$$

on pose : $\lambda_k = -P_k x_k$

$$u_k^* = R^{-1} B^T \lambda_{k+1}$$

$$= -R^{-1} B^T P_{k+1} x_{k+1}$$

$$R u_k^* = -B^T P_{k+1} x_{k+1} = -B^T P_{k+1} (A x_k + B u_k^*)$$

$$R u_k^* = -B^T P_{k+1}(A x_k + B u_k^*)$$

$$(R + B^T P_{k+1} B) u_k^* = -B^T P_{k+1} A x_k$$

$$\Rightarrow u_k^* = (R + B^T P_{k+1} B)^{-1} \times -B^T P_{k+1} A x_k$$

$$u_k^* = -\underbrace{(R + B^T P_{k+1} B)^{-1} \times B^T \times P_{k+1} A}_{L_{k+1}} x_k$$

$$\therefore \boxed{u_k^* = -L_{k+1} x_k}$$

$$\lambda_k = A^T \lambda_{k+1} - Q_k x_k$$

$$\checkmark \quad \text{et} \begin{cases} \lambda_k \overset{\text{on pose}}{=} -P_k x_k \\ \lambda_{T_f} = -P_{T_f} x_{T_f} \end{cases}$$

$$-P_k x_k = A^T \underline{\lambda_{k+1}} - Q_k x_k$$

$$-P_k x_k = A^T \times -P_{k+1} x_{k+1} - Q_k x_k$$

$$-P_k x_k = -A^T P_{k+1} (A x_k + B u_k^*) - Q_k x_k$$

$$-P_k x_k = -A^T P_{k+1}(A x_k - B L_{k+1} x_k) - Q_k x_k$$

$$-P_k x_k = \left[-A^T P_{k+1} A + A^T P_{k+1} B L_{k+1} - Q_k\right] x_k$$

$$\underbrace{-P_k}\, x_k = \underbrace{-\left[A^T P_{k+1} A - A^T P_{k+1} B L_{k+1} + Q_k\right]} x_k$$

$$\boxed{P_k = A^T P_{k+1} A - A^T P_{k+1} B L_{k+1} + Q_k}$$

avec : $\lambda_{t_f} = -P_{t_f} x_{t_f}$

Exercice : Soit $\dot{x}_1 = -x_1 + u$

(*) Déterminer la commande optimale qui minimise :

$$J = \frac{1}{2}\int_0^1 (3x_1^2 + u^2)\,dt$$

sous la contrainte de la dynamique du système sachant que système évolue de $x_1 = 0$ à $t_0 \longrightarrow x_1 = 2$ à $t = 1$

Solution: $A = -1,\ B = 1,\ Q = 3 > 0,\ R = 1 > 0$

$$\dot{\lambda}(t) = 3x_1 + \lambda$$

$$\dot{x}_1 = -x_1 + \lambda \Rightarrow \ddot{x}_1 = -\dot{x}_1 + \dot{\lambda}$$

$$\ddot{x}_1 = -\dot{x}_1 + 3x_1 + \lambda$$

$$\ddot{x}_1 = -\dot{x}_1 + 3x_1 + \lambda \quad \left(\boxed{\lambda = \dot{x}_1 + x_1}\right)$$

$$\ddot{x}_1 = -\cancel{\dot{x}_1} + 3x_1 + \cancel{\dot{x}_1} + x_1$$

$$\boxed{\ddot{x}_1 = 4x_1}$$

$$\ddot{x}_1 = u\, x_1 \Rightarrow x_1(t) = K \sin(\alpha t + B)$$

$$\lambda = x_1 + x_1$$

$$\lambda = \alpha K \cosh(\alpha t + B) + K \sinh(\alpha t + B) \Rightarrow u^*(t) \cdot + \lambda$$

$$\Leftrightarrow u^* = \alpha K \cosh(\alpha t + B) + K \sinh(\alpha t + B)$$