NOTES DE COURS
SLAM

M2 AURO/IAFA

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I Introduction

- Simultaneous Localization and Mapping : one of the most emblematic problems of mobile robotics
  - A robot evolving in an unknown environment concurrently builds a map of the environment and localizes itself into this map.
  - Huge literature - Huge number of special issues in Robotics Journal and of dedicated sessions in Robotics Conferences - Multiple industrial applications too !
  - Some references (cf. "so many books, so little time" –FZ)
    - H. Durrant-Whyte & T. Bailey, *Simultaneous Localization and Mapping : Part I & Part II*, IEEE RA Magazine 2006.
    - S. Thrun & J.J. Leonard, Springer Handbook of Robotics 2008.
    - G. Grisetti *et al.*, *A Tutorial on Graph-Based SLAM*, IEEE ITS Magazine 2010.
    - C. Cadena *et al.*, *Past, Present and Future of Simultaneous Localization and Mapping : Toward the Robust-Perception Age*, IEEE T-RO 2016.
    - F. Dellaert & M. Kaess, *Factor Graphs for Robot Perception*, Foundations and Trends in Robotics 2017.
    - S. Thrun, W. Burgard, D. Fox, Probabilistic Robotics, MIT Press.
    - Tutorials and MATLAB toolbox by J. Solá, `www.joansola.eu`.

- Some features of the problem
  - The robot has at its disposal : (i) control inputs / odometry / self motion ; (ii) relative observations of the environment from its embedded sensors.
  - Aim : incremental/recursive map building and localization.
  - "Chicken-and-Egg problem", see below.

- Pose graph
  - Robot trajectory (collection of poses) from relative pose measurements and incorporation of loop closure constraints.
  - Those relative pose measurements come from : self-motion (odometry, IMU), visual feature registration, ICP…

- Mapping
  - Synthesis of an environment model from sensor data given the robot pose, by incorporating (ii) above into a map.
  - Several types of map do exist (cf. M. Taïx's course)
    - Dense metric map of geometric cues (2D polygons, 3D polyedra,… $\longrightarrow$ obstacles) : hard to build from sensor data ; eases planning ; scalability $\Leftarrow$ # & complexity.
    - Sparse metric map of landmarks (reference points / descriptors $\longrightarrow$ salient features) : easy to build from sensor data ; eases localization ; scalability $\Leftarrow$ #.
    - Sparse point cloud map (3D points $\longrightarrow$ surfaces) : easy to build ; no planning ; scalability $\Leftarrow$ density, surface.
    - Dense metric occupancy grid (segmented space & occupancy posterior probability) : easy to build, eases planning, scalability $\Leftarrow$ surface—volume, resolution.
    - Dense metric elevation grid (terrain height) : easy to bulid, eases planning, scalability $\Leftarrow$ surface, resolution.
    - Topological and semantic maps : higher level.

- Localization
  - Given a known map, estimation of the robot pose from (i) and (ii) above.
  - Various forms of Bayes inference/filtering.
- *SLAM*
  - Neither the robot pose nor the map is given.
  - Two prominent approaches
    - Estimate the state vector of the problem (robot pose and map) with a Bayes filter : EKF/UKF, particle filter.
    - Optimize the robot trajectory and map.

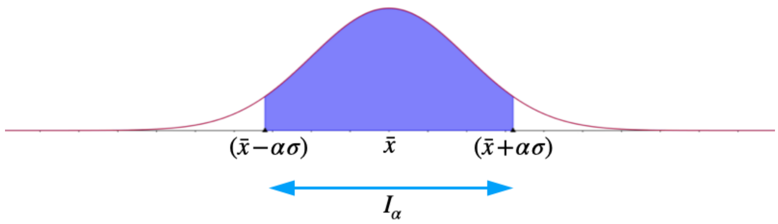
II Handling uncertainty in a probabilistic context : a bird's view

- Probabilities - Random variables - etc.
  - Random experiments - Outcome/Events - Probabilities - (Statistical) independence
  - Random variable/vector $X \in \mathbb{R}$ or $X \in \mathbb{R}^N$
    - x outcome of X ; $x = X(\omega)$
    - Elementary events $X \leq x$
    - Cumulative distribution function $\mathrm{P}_X(x) = \mathbb{P}(X \leq x)$
    - Probability density function $\mathrm{p}_X(x) = \left[\frac{\mathrm{dP}_X(\xi)}{\mathrm{d}\xi}\right]_{\xi=x} = \lim_{\mathrm{d}\xi\to 0} \frac{\mathbb{P}(x<X\leq x+\mathrm{d}\xi)}{\mathrm{d}\xi}$
      - $\mathrm{p}_X(x) \geq 0$
      - $\mathrm{P}_X(x) = \int_{-\infty}^{x} \mathrm{p}_X(\xi)\mathrm{d}\xi$; $\mathrm{P}_X(+\infty) = 1$; $\mathbb{P}(X \in I) = \int_I \mathrm{p}_X(\xi)\mathrm{d}\xi$
      - Mind the meading of $\mathrm{d}\xi$ in the multivariate case
  - Gaussian random variables/vectors
    - $X \in \mathbb{R}$, $X \sim \mathcal{N}(\bar{x}, \sigma^2)$ iff $\mathrm{p}_X(x) = \mathcal{N}(x; \bar{x}, \sigma^2) \triangleq \frac{1}{\sqrt{2\pi}\,\sigma} \exp -\frac{(x-\bar{x})^2}{2\sigma^2}$
    - $X \in \mathbb{R}^N$, $X \sim \mathcal{N}(\bar{x}, P)$ iff

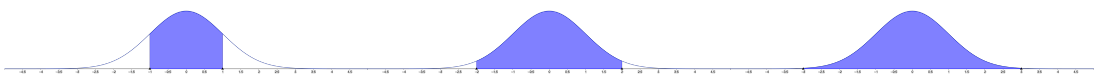
$$\mathrm{p}_X(x) = \mathcal{N}(x; \bar{x}, P) \triangleq \frac{1}{\sqrt{(2\pi)^N \det(P)}} \exp\left(-\frac{1}{2}(x-\bar{x})^T P^{-1}(x-\bar{x})\right),$$

$$\textit{i.e.}\text{, iff } \mathrm{p}_X(x) \triangleq \frac{1}{\sqrt{\det(2\pi P)}} \exp\left(-\frac{1}{2}(x-\bar{x})^T P^{-1}(x-\bar{x})\right)$$

    - Confidence sets
      - For $X \in \mathbb{R}$, $X \sim \mathcal{N}(\bar{x}, \sigma^2)$, I_α of minimum size such that $\mathbb{P}(X \in I_\alpha) = \text{(given)}p_\alpha$ is the confidence set $I_\alpha = [\bar{x} - \alpha\sigma; \bar{x} + \alpha\sigma] = \left\{\xi \,:\, \frac{(\xi-\bar{x})^2}{\sigma^2} \leq \alpha^2\right\}$



Hence, $Y = \frac{X-\bar{x}}{\sigma} \sim \mathcal{N}(0,1)$ and the confidence sets I_1, I_2, I_3 are associated to the respective probabilities $p_1 = 0.6827$, $p_2 = 0.9545$, $p_3 = 0.9973$



- For $X \in \mathbb{R}^N$, $X \sim \mathcal{N}(\bar{x}, P)$, $\mathcal{E}_\alpha$ of minimum size such that $\mathbb{P}(X \in \mathcal{E}_\alpha) = (\text{given})p_\alpha$ is the confidence ellipse/ellipsoid/hyperellipsoid $\mathcal{E}_\alpha = \left\{\xi \,:\, (\xi - \bar{x})^T P^{-1}(\xi - \bar{x}) \leq \alpha^2\right\}$
  $\hookrightarrow$ Centered on $\bar{x}$
  $\hookrightarrow$ Principal axes : (orthogonal) eigenvectors of P
  $\hookrightarrow$ Extent along eigenvector v_i : $\alpha\sqrt{\lambda_i}$, with λ_i the eigenvalue associated to v_i
  $\hookrightarrow$ $\mathbb{P}(X \in \mathcal{E}_\alpha)$ depends on α and N
- The central limit theorem : roughly speaking, a random vector which is the sum of a large number of independent random excitations is Gaussian

◦ Expectation operator - Mean/Expectation - Covariance
- $\mathbb{E}_X[g(X)] = \int g(x)\mathrm{p}_X(x)\mathrm{d}x$
- $m_X = \mathbb{E}_X[X] = \int x\mathrm{p}_X(x)\mathrm{d}x$
- $\mathrm{Cov}_X = \mathbb{E}_X\left[(X - m_X)(X - m_X)^T\right] = \mathbb{E}_X\left[XX^T\right] - m_X m_X^T$
  - Note : $\mathbb{E}_X\left[(X - m_X)^T(X - m_X)\right] = \mathrm{trace}(\mathrm{Cov}_X)$.
  - Expectation and covariance of $\mathcal{N}(\bar{x}, P)$ are $\bar{x}$ and P !
  - If $X \in \mathbb{R}^N$, $X \sim \mathcal{N}(\bar{x}, P)$, and A, b a matrix,vector pair of suitable dimensions, then $Y = AX + b \sim \mathcal{N}(A\bar{x} + b, APA^T)$

◦ Joint, marginal, conditional pdfs – Independence, correlation
  ◦ If $X = (X_1, \ldots, X_N)^T$, then $\mathrm{p}_X(x) = \mathrm{p}_{X_1,\ldots,X_N}(x_1, \ldots, x_N)$ is the joint pdf on $X_1, \ldots, X_N$
  ◦ $X_1, \ldots, X_N$ independent $\Leftrightarrow \forall x_1, \ldots, x_N,\ \mathrm{p}_{X_1 \ldots X_N}(x_1, \ldots, x_N) = \mathrm{p}_{X_1}(x_1) \ldots \mathrm{p}_{X_N}(x_N)$
  ◦ $X_1, \ldots, X_N$ independent, identically distributed (i.i.d.) $\Leftrightarrow X_1, \ldots, X_N$ independent and $\mathrm{p}_{X_1}(x_1) = \ldots = \mathrm{p}_{X_N}(x_N)$
  ◦ X, Y uncorrelated $\Leftrightarrow \mathbb{E}_{XY}\left[XY^T\right] = \mathbb{E}_X[X]\mathbb{E}_Y[Y]^T \Leftrightarrow C_{XY} = 0$
    - X, Y independent $\Rightarrow X, Y$ uncorrelated
    - $[X, Y$ independent iff uncorrelated$] \Leftarrow X, Y$ jointly Gaussian, i.e., $(X^T, Y^T)^T$ Gaussian
  ◦ Marginal pdfs

$$\mathrm{p}_X(x) = \int \mathrm{p}_{XY}(x, y)\mathrm{d}y$$

  ◦ Conditional pdfs

$$\mathrm{p}_{X|Y}(x|y) = \frac{\mathrm{p}_{XY}(x, y)}{\mathrm{p}_Y(y)} = \frac{\mathrm{p}_{XY}(x, y)}{\int \mathrm{p}_{XY}(x, y)\mathrm{d}x} \text{ if } \mathrm{p}_Y(y) \neq 0, \text{ (Bayes pdf)}$$

  - If X, Z jointly Gaussian then conditioned on the event $Z = z$, X is Gaussian !

$$\begin{pmatrix} X \\ Z \end{pmatrix} \text{ such that } \mathrm{p}_{X,Z}(x, z) = \mathcal{N}\left(\begin{pmatrix} x \\ z \end{pmatrix}; \begin{pmatrix} m_X \\ m_Z \end{pmatrix}, \begin{pmatrix} P_{XX} & P_{XZ} \\ P_{ZX} & P_{ZZ} \end{pmatrix}\right)$$
$$\Downarrow$$
$$\mathrm{p}_{X|Z}(x|z) = \mathcal{N}(x; m_{X|Z}, P_{X|Z})$$

  and we know $m_{X|Z}, P_{X|Z}$ analytically !

- Bayes estimation : general framework to stochastic estimation with prior knowledge
  - Static case : estimation of an hidden vector x from a measurement vector z
    - Statement
      - ω = outcome of ongoing experience ; $x = X(\omega)$ hidden parameter vector ; $z = Z(\omega)$ measured observation vector
      - $X \sim \mathrm{p}_X(\xi)$, $\mathrm{p}_X(\xi)$ (given) prior pdf
      - (Given) observation model $\mathrm{p}_{Z|X}(\zeta|\xi)$ expresses statistical relationship between any hidden ξ and possible consequent measurement ζ
      - Goal : synthesize an "estimator function" $g(.)$ such that $\hat{x} = g(z)$ is a (good) estimate of hidden x.
    - What is a "good" estimate ?
      - z is random $\Rightarrow$ $\hat{x}$ is random $\Rightarrow$ with ω the outcome of ongoing experience, $z = Z(\omega)$ and $\hat{x} = \hat{X}(\omega)$, with $\hat{X}$ the estimator (random vector)

    $\hookrightarrow$ $X - \hat{X}$ is the estimation error (random vector), $\mathbb{E}\Big[X - \hat{X}\Big]$ is the bias of $\hat{X}$, $\mathbb{E}\Big[\|X - \hat{X}\|^2\Big]$ is the mean square error of $\hat{X}$, $\mathbb{E}\Big[(X - \hat{X})(X - \hat{X})^T\Big]$ is the estimation error covariance matrix, etc.
    - "Holy Grail"
      - All the information on hidden X conveyed by measurement z (and prior knowledge) is captured by the posterior pdf $\mathrm{p}_{X|Z}(x|z)$

    $\hookrightarrow$ Expectation $\hat{x}_{\mathrm{MMSE}}$ of $\mathrm{p}_{X|Z}(x|z)$ is called the Minimum Mean Square Error (MMSE) Estimate as it minimises the expectation of $\mathbb{E}\Big[\|X - \mathring{X}\|\Big]^2$, over any estimator $\mathring{X} = \mathring{g}(Z)$

    $\hookrightarrow$ Peak of $\hat{x}_{\mathrm{MMSEE}}$ of $\mathrm{p}_{X|Z}(x|z)$, $\hat{x}_{\mathrm{MMSEE}} = \arg\max_x \mathrm{p}_{X|Z}(x|z) = \arg\min_x\Big(-\ln \mathrm{p}_{X|Z}(x|z)\Big)$, is called the Maximum a Posteriori (MAP) Estimate
    - Case of no prior knowledge
      - When $\mathrm{p}_X(x)$ is a flat prior ($\approx$ no prior knowledge), the MAP estimate boils down to the Maximum Likelihood Estimate (MLE) $\hat{x}_{\mathrm{MLE}} = \arg\max_x \mathrm{p}_{Z|X}(z|x) = \arg\min_x\Big(-\ln \mathrm{p}_{Z|X}(z|x)\Big)$

    $\hookrightarrow$ $\mathcal{L}(x;z) = \mathrm{p}_{Z|X}(z|x)$ (function of x parameterized by z) is called the likelihood of x w.r.t. z ; mind here : this is not a pdf...
    - "Linear Gaussian case"
      - If $Z = HX + V$, with $Z \in \mathbb{R}^N$, $X \in \mathbb{R}^M$, $V \in \mathbb{R}^N$, $\left(\begin{smallmatrix} X \\ V \end{smallmatrix}\right) \sim \mathcal{N}\Big(\left(\begin{smallmatrix} \hat{x}^- \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} P^- & 0 \\ 0 & R \end{smallmatrix}\right)\Big)$ and $R > 0$, then $\mathrm{p}_{X|Z}(x|z) = \mathcal{N}(x; \hat{x}^+, P^+)$, where $\hat{x}^+ = \mathbb{E}_{X|Z}[X|z]$ and $P^+ = \mathbb{E}_{X|Z}\Big[(X - \hat{x}^+)(X - \hat{x}^+)^T|z\Big]$ write as

$$\begin{cases} \hat{x}^+ = \hat{x}^- + K(z - H\hat{x}^-) \\ P^+ = P^- - KHP^- \\ K = P^-H^T(R + HP^-H^T)^{-1}. \end{cases}$$

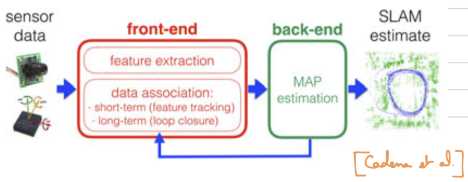
    $\hookrightarrow$ MMSE and MAP estimates are identical in this case.

    $\hookrightarrow$ MLE estimate is $\hat{x}_{\mathrm{MLE}} = (H^TR^{-1}H)^{-1}H^TR^{-1}z$. The covariance of the associated estimator $\hat{X}_{\mathrm{MLE}}$ is $\mathrm{Cov}_{\hat{X}_{\mathrm{MLE}}} = (H^TR^{-1}H)^{-1}$.

- ◦ Dynamic case : estimation of an hidden sequence $x_{0:k}$ from a measurement vector $z_{0:k}$
  - · Statement
    - · ω = outcome of ongoing experience ; $x_{0:k} = X_{0:k}(\omega)$ hidden state vector ; $z_{0:k} = Z_{0:k}(\omega)$ measured observation vector
    - · $X_0 \sim \mathrm{p}_{X_0}(\xi_0)$, $\mathrm{p}_{X_0}(\xi_0)$ (given) initial prior pdf
    - · $\{\mathrm{p}_{X_k|X_{k-1}}(\xi_k|\xi_{k-1})\}$ (given) prior dynamics pdf between times $k-1$ and k (Note : Markov state vector)
    - · (Given) observation model $\{\mathrm{p}_{Z_k|X_k}(\zeta_k|\xi_k)\}$ expresses statistical relationship between any hidden ξ_k and possible consequent measurement ζ_k at time k
  - · "Holy Grail"
    - · All the information on hidden $X_{0:k}$ conveyed by measurement $z_{0:k}$ (and prior knowledge) is captured by the joint posterior pdf $\mathrm{p}_{X_{0:k}|Z_{0:k}}(x_{0:k}|z_{0:k})$; Marginal posterior (filtering) pdf $\mathrm{p}_{X_k|Z_{0:k}}(x_k|z_{0:k})$ if often prefered ; another marginal posterior pdf $\mathrm{p}_{X_k|Z_{0:k-1}}(x_k|z_{0:k-1})$ has to do with prediction (and is named prediction pdf)
  - ↪ MMSE or MAP estimates can be sought for
    - · "Linear Gaussian case" : closed-form solution is given by Kalman filtering

III Variables, models and workflow of landmark-based SLAM

- Variables (static environment)
  - ◦ State vector x_k at time k, with $x_k = (r_k^T, m^T)^T$, r_k = robot absolute pose and $m = (m_1^T, \ldots, m_M^T)^T$ absolute position of landmarks
  - ◦ Control inputs $u_{0:k}$, given (by odometry, etc.)
  - ◦ Observations $z_{0:k}$, with z_k measurement at time k = stacking of measurements $\{z_{k,j}\}_j$ from r_k of the set $\{m_j\}_j$ of visible landmarks
- Models (static environment)
  - ◦ $r_{k+1} = f(r_k, u_k) + w_k$, $w_k \sim \mathcal{N}(0, Q_k)$ white Gaussian dynamic noise
    ↪ $p(r_{k+1}|r_k) = p(r_{k+1}|r_k; u_k) = \mathcal{N}(r_{k+1}; f(r_k, u_k), Q_k)$.
  - ◦ (m constant noise-free)
  - ◦ $z_k = h(r_k, m) + v_k$, $v_k \sim \mathcal{N}(0, R_k)$ white Gaussian measurement noise (noises on individual measurements are assumed mutually independent, R_k diagonal)
    ↪ $p(z_k|r_k, m) = \mathcal{N}(z_k; h(r_k, m), R_k)$
- Implementation workflow
  - ◦ Sensor data
  - ◦ Front-end
    - · Feature extraction
    - · Data association
      - ↪ short-term (feature tracking)
      - ↪ long-term (loop closure)
  - ◦ Back-end : estimation algorithm



IV Probabilistic SLAM

- Full SLAM : estimate the posterior joint pdf of the robot trajectory together with the map $p(r_{0:k}, m|z_{0:k}) = p(r_{0:k}, m|z_{0:k}; u_{0:k-1})$
- Online SLAM : estimate the posterior marginal pdf of the robot pose together with the map $p(r_k, m|z_{0:k}) = p(r_k, m|z_{0:k}; u_{0:k-1})$

IV.1 EKF solution

- First solution to SLAM : $p(r_k, m|z_{0:k})$ approximated by a (huge-dimension) Gaussian pdf thanks to EKF approximate computations of posterior moments for nonlinear dynamics/observation model
  - Use of Taylor expansions for covariance and gain equations ($\rightsquigarrow$ Jacobian matrices of transition/measurement functions around last state estimates/predictions)

    $\hookrightarrow$ considering standard assumptions and notations of linear Kalman filter except the following nonlinear state and measurement stochastic equations

$$x_{k+1} = f(x_k) + w_k,\ w_k \sim \mathcal{N}(0, Q_k), \qquad \text{and} \qquad z_k = h(x_k) + v_k,\ v_k \sim \mathcal{N}(0, R_k),$$

EKF equations write as

$$\begin{array}{rlrl}
\hat{x}_{0|0} &= m_{X_0} & P_{0|0} &= P_0 \\
\hat{x}_{k+1|k} &= f(\hat{x}_{k|k}) & P_{k+1|k} &= F_k P_{k|k} F_k^T + Q_k \\
\hat{z}_{k+1|k} &= h(\hat{x}_{k+1|k}) & S_{k+1|k} &= R_{k+1} + H_{k+1} P_{k+1|k} H_{k+1}^T \\
\multicolumn{4}{c}{K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1|k}^{-1}} \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}(z_{k+1} - \hat{z}_{k+1|k}) & P_{k+1|k+1} &= P_{k+1|k} - K_{k+1} H_{k+1} P_{k+1|k}
\end{array}$$

$$\text{with } F_k = \left[\frac{\partial f(x)}{\partial x^T}\right]_{x=\hat{x}_{k|k}} \text{ and } H_{k+1} = \left[\frac{\partial h(x)}{\partial x^T}\right]_{x=\hat{x}_{k+1|k}}$$

  - Robot Motion $\leftrightarrow$ EKF time update
  - Features observation $\leftrightarrow$
    - State augmentation if new landmark is detected and needs to be initialized in the map
    - EKF measurement update if known landmark is observed and needs to be corrected in the map
    - State reduction if a landmark must be detected in view of map corruption

- Some important facts
  - The robot moves $\Rightarrow$ pose uncertainty increases $\Rightarrow$ (as measurement uncertainty is fixed) growing uncertainty on nearby landmarks
  - As time increases, the covariance matrix of the filtering pdf gets "denser" as the cross-correlations between estimation errors on landmark locations increases

    $\hookrightarrow$ The relative location between any two m_i, m_j may be accurately known even when their absolute locations are uncertain
  - So, the observation of a landmark already observed in the very beginning of the mapping enables the decrease of the uncertainties in the robot pose and in the location of previously observed landmarks ! This is the loop closure phenomenon

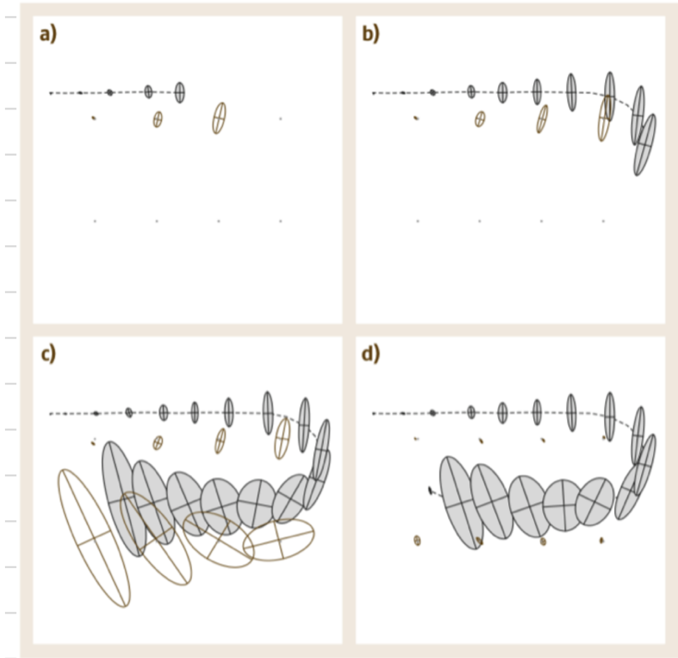


Fig. 37.2a–d EKF applied to the online SLAM problem. The robot's path is a *dotted line*, and its estimates of its own position are *shaded ellipses*. Eight distinguishable landmarks of unknown location are shown as *small dots*, and their location estimates are shown as *white ellipses*. In **(a–c)** the robot's positional uncertainty is increasing, as is its uncertainty about the landmarks it encounters. In **(d)** the robot senses the first landmark again, and the uncertainty of *all* landmarks decreases, as does the uncertainty of its current pose. (Image courtesy of Michael Montemerlo, Stanford University)

[Thrun, Leonard]

- Data association may have to be handled, so as to disambiguate which landmark have given rise to observations
- Mind the quadratic complexity of covariance computations. Exploit sparseness.
- Though interesting for small-scale problems and (mostly) for historical reasons, EKF-SLAM is fundamentally limited to scarce maps, involving appropriately selected (perceptually distinct) landmarks.

- Illustration on a (not-so-silly) toy-example : 2D range and/or bearing online EKF-SLAM
  - State, control input and measurement output variables
    - Planar robot $\mathcal{R}$ with associated frame $\mathcal{F}_k = (O_k, \vec{x}_k, \vec{y}_k)$ at time k. M pointwise landmarks $\mathcal{L}_1, \ldots, \mathcal{L}_M$ at (unknown) loci $L_1, \ldots, L_M$, to be referenced in a map $\mathcal{M}$. World/Map frame is $\mathcal{F}_0$.
    - Hidden state vector $x_k = (r_k^T, m_1^T, \ldots, m_M^T)^T$: absolute robot pose vector $r_k = (r_{x_k}, r_{y_k}, \theta_k)^T$ with $(r_{x_k}, r_{y_k})^T = \overrightarrow{O_0O_k}_{(\mathcal{F}_0)}$ and $\theta_k = (\widehat{\vec{x}_0, \vec{x}_k})$; absolute landmark positions $m_m = (m_{x_m}, m_{y_m})^T = \overrightarrow{O_0L_m}_{(\mathcal{F}_0)}$, gathered into vector $m = (m_1^T, \ldots, m_M^T)^T$.
    - Control input vector $u_k = (\tau_k^T, \rho_k)^T$: robot translation $\tau_k = (\tau_{x_k}, \tau_{y_k})^T = \overrightarrow{O_kO_{k+1}}_{(\mathcal{F}_k)}$ and rotation $\rho_k = (\widehat{\vec{x}_k, \vec{x}_{k+1}})$ from $\mathcal{F}_k$ to $\mathcal{F}_{k+1}$ expressed in $\mathcal{F}_k$.
    - Observation vector z_k : stacking of $\{z_{k,j}\}_{\{\text{visible } \mathcal{L}_j\}}$, where $z_{k,j}$ terms the range $\|O_kL_j\|$, bearing $(\widehat{\vec{x}_k, \overrightarrow{O_kL_j}}) = \Big(\mathrm{atan2}\big(\overrightarrow{O_kL_j}.\vec{y}_k, \overrightarrow{O_kL_j}.\vec{x}_k\big) = \mathrm{atan2}\big(\overrightarrow{O_kL_j}.\vec{y}_0, \overrightarrow{O_kL_j}.\vec{x}_0\big) - \theta_k\Big)$ or their stacking, at time k, of L_j relative to $\mathcal{F}_k$.
    - Note however—including in the lines below—that subvector m is built incrementally, *i.e.*, M is not known in advance…
  - Prior dynamics
    - Robot dynamics $r_{k+1} = f_{\mathcal{R}}(r_k, u_k) + w_k^{\mathcal{R}}$, $w_k^{\mathcal{R}} \sim \mathcal{N}(0, Q_k^{\mathcal{R}})$, $w_{0:k}^{\mathcal{R}}$ white, etc. where

$$f_{\mathcal{R}}(r_k, u_k) = r_k + g_{\mathcal{R}}(r_k, u_k) \text{ with } g_{\mathcal{R}}(r_k, u_k) = g_{\mathcal{R}}(\cancel{r_{x_k}}, \cancel{r_{y_k}}, \theta_k, u_k) = \begin{pmatrix} \tau_{x_k}\cos\theta_k - \tau_{y_k}\sin\theta_k \\ \tau_{x_k}\sin\theta_k + \tau_{y_k}\cos\theta_k \\ \rho_k \end{pmatrix}.$$

$\hookrightarrow$ Proof : show that the average noise-free model accounts for the rigid-body motion of the robot between times k and $k+1$, described by

$$T_{0,k+1} = T_{0,k}T_{k,k+1}, \text{ with } T_{0,k} = \begin{pmatrix} \cos\theta_k & -\sin\theta_k & r_{x_k} \\ \sin\theta_k & \cos\theta_k & r_{y_k} \\ 0 & 0 & 1 \end{pmatrix} \text{ and } T_{k,k+1} = \begin{pmatrix} \cos\rho_k & -\sin\rho_k & \tau_{x_k} \\ \sin\rho_k & \cos\rho_k & \tau_{y_k} \\ 0 & 0 & 1 \end{pmatrix}.$$

· Incorporating $m_{k+1} = m_k$ leads to $x_{k+1} = f(x_k, u_k) + w_k$, with

$$f(x_k, u_k) = x_k + E_x^T g_{\mathcal{R}}(r_k, u_k) + w_k,\ w_k \sim \mathcal{N}(0, Q_k),$$
$$Q_k = \text{blkdiag}(Q_k^{\mathcal{R}}, \underbrace{\mathbb{O}_{2\times2}, \ldots, \mathbb{O}_{2\times2}}_{M \text{ times}}),\ E_x = \begin{pmatrix} \mathbb{I}_{3\times3} & \mathbb{O}_{3\times2M} \end{pmatrix}.$$

- Observation model

· $z_k = h_k(x_k) + v_k$, $v_k \sim \mathcal{N}(0, R_k)$, composed of measurements $z_{k,j} = h_{k,j}(r_k, m_j) + v_{k,j}$ of $\mathcal{L}_j$ w.r.t. $\mathcal{R}$, $v_{k,j} \sim \mathcal{N}(0, R_{k,j})$ mutually independent over $\{\mathcal{L}_j\}$ so that $R_k = \text{blkdiag}(\{R_{k,j}\})$, $v_{1:k}$ white, etc. with one of the following options :

$$h_{k,j}(r_k, m_j) = \sqrt{(m_{x_j} - r_{x_k})^2 + (m_{y_j} - r_{y_k})^2} \text{ or } \operatorname{atan2}\big((m_{y_j} - r_{y_k}), (m_{x_j} - r_{x_k})\big) - \theta_k,$$

$$\text{or } h_{k,j}(r_k, m_j) = \begin{pmatrix} \sqrt{(m_{x_j} - r_{x_k})^2 + (m_{y_j} - r_{y_k})^2} \\ \operatorname{atan2}\big((m_{y_j} - r_{y_k}), (m_{x_j} - r_{x_k})\big) - \theta_k \end{pmatrix}.$$

$\hookrightarrow$ Proof : rewrite $\begin{pmatrix} \overrightarrow{O_k L_j}_{(\mathcal{F}_k)} \\ 1 \end{pmatrix} = T_{0,k}^{-1} \begin{pmatrix} \overrightarrow{O_0 L_j}_{(\mathcal{F}_0)} \\ 1 \end{pmatrix} \Leftrightarrow \overrightarrow{O_k L_j}_{(\mathcal{F}_k)} = R_{0,k}^T \overrightarrow{O_0 L_j}_{(\mathcal{F}_0)} - R_{0,k}^T P_{0,k}$, with $R_{0,k} = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}$, $P_{0,k} = \begin{pmatrix} r_{x_k} \\ r_{y_k} \end{pmatrix}$, $\overrightarrow{O_0 L_j}_{(\mathcal{F}_0)} = \begin{pmatrix} m_{x_j} \\ m_{y_j} \end{pmatrix}$, as $\overrightarrow{O_k L_j}_{(\mathcal{F}_k)} = \begin{pmatrix} \cos\theta_k(m_{x_j} - r_{x_k}) + \sin\theta_k(m_{y_j} - r_{y_k}) \\ -\sin\theta_k(m_{x_j} - r_{x_k}) + \cos\theta_k(m_{y_j} - r_{y_k}) \end{pmatrix}$; then, show that the first line of $h_{k,j}(r_k, m_j)$ is the norm of $\overrightarrow{O_k L_j}_{(\mathcal{F}_k)}$ and that the second line of $h_{k,j}(r_k, m_j)$ is its angle, *e.g.*, resorting to the expansion of $\tan\big(\operatorname{atan2}(\ldots) - \theta_k\big)$.

- SLAM Initialization

· The initial pose of the robot is the origin of the map, which contains no landmarks, hence

$$\hat{x}_{0|0} = \hat{r}_{0|0} = \mathbb{O}_{3\times1},\ P_{0|0} = P_{\mathcal{R}\mathcal{R}_{0|0}} = \mathbb{O}_{3\times3}.$$

- SLAM Time update between times k and $k+1$

· Standard EKF equations apply, leading to sparse $F_k = \left[\frac{\partial f(x, u_k)}{\partial x^T}\right]_{x=\hat{x}_{k|k}}$ so that predicted covariance $2M \times 2M$ submatrix $P_{\mathcal{M}\mathcal{M}_{k+1|k}} = P_{\mathcal{M}\mathcal{M}_{k|k}}$ is unchanged.

- SLAM Measurement update at time $k+1$ for already seen landmarks

· Standard EKF equations, with $H_{k+1} = \left[\frac{\partial h_{k+1}(x)}{\partial x^T}\right]_{x=\hat{x}_{k+1|k}}$, and each individual Jacobian matrix $H_{k+1,j} = \left[\frac{\partial h_{k+1,j}(x)}{\partial x^T}\right]_{x=\hat{x}_{k+1|k}}$ also features a sparse structure.

· Interesting sparsity properties also hold in the computation of the residual $z_{k+1,j} - \hat{z}_{k+1|k,j}$ and its covariance matrix.

- For unseen landmarks, additional stage to be inserted before SLAM Measurement update

· If $\mathcal{R}$ discovers a landmark, say $\mathcal{L}_m$, which has not yet been mapped, then the current state vector must be augmented with m_m. Even if, for given r_{k+1}, the output function $h_{k+1,m}(r_{k+1}, m_m)$ is bijective w.r.t. m_m, it is necessary to define a prediction $\hat{m}_{m_{k+1|k}}$ of m_m in order to compute a meaningful individual Jacobian $H_{k+1,m}$. Otherwise this Jacobian may not be a valid approximation of the derivative of $h_{k+1,m}(.,.)$ over the regions where the state vector prediction pdf is high, and EKF may fail!

- ◦ <u>Data association handling</u>
  - · The easiest way to decide whether an already seen landmark in $\{\mathcal{L}_1, \dots, \mathcal{L}_{m-1}\}$ or a new $\mathcal{L}_m$ gives rise to the observation z_{k+1} is "gating".

$$\text{Compute } \delta_{k+1,j} \triangleq (z_{k+1} - \hat{z}_{k+1|k,j})^T S_{k+1|k,j}^{-1} (z_{k+1} - \hat{z}_{k+1|k,j}),\ j \in \{1, \dots, m-1\}\,;$$

$$\text{Detect most likely } \mathcal{L}_{j^\star} \text{ by } j^\star = \arg \min_{j \in \{1,\dots,m\}} \delta_{k+1,j}.$$

IV.2 Particle Filter solution

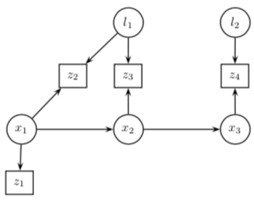
- An interesting (and subtle) independence property in the Bayes network of the SLAM problem enables an efficient Rao-Blackwellized Particle Filter : FastSLAM algorithm.
  - ◦ The joint posterior pdf is approximated by

$$p(r_{0:k}, m_{0:k}|z_{0:k}) \approx \sum_{i=1}^N \bar{w}_k^{(i)} \Big[\prod_{j=1}^M \mathcal{N}(m_j|\hat{m}_{j\,k|k}^{(i)}, P_{j\,k|k}^{(i)})\Big] \delta(r_{0:k} - r_{0:k}^{(i)}).$$

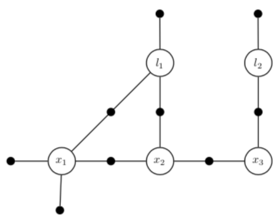
- Some important facts
  - ◦ The SLAM posterior pdf is factored into low-dimensional estimation problems : a weighted particle cloud (with stochastic support) for the robot pose/trajectory ; for each particle, an associated conditional map posterior pdf through M low-dimensional independent EKFs (one per landmark). Two versions (increasing efficiency) : FastSLAM 1.0, FastSLAM 2.0.
  - ◦ Inner computations complexities : $\mathcal{O}(N)$ and $\mathcal{O}(N \log M)$ $\longrightarrow$ FastSLAM scales up to problems with $M = 10^5 - 10^6$ landmarks.
  - ◦ Efficient data association mechanism.
  - ◦ EKF landmark map can be replaced with occupancy grid map.

V Optimization SLAM

- SLAM can be viewed as a sparse graph of constraints $\longrightarrow$ optimization of the robot trajectory and the map
  $\hookrightarrow$ this has become the modern paradigm to solve full SLAM !
- Factor graph approach : Graph SLAM
  - ◦ Factor graph = bipartite graph : variable/states and factors/constraints
    - · Rewriting of a conditional distribution as a product of factors



$$\begin{aligned} p(X, Z) = {} & p(x_1)p(x_2|x_1)p(x_3|x_2) \\ & \times p(l_1)p(l_2) \\ & \times p(z_1|x_1) \\ & \times p(z_2|x_1, l_1)p(z_3|x_2, l_1)p(z_4|x_3, l_2). \end{aligned}$$



$$\begin{aligned} p(X|Z) \propto {} & p(x_1)p(x_2|x_1)p(x_3|x_2) \\ & \times p(l_1)p(l_2) \\ & \times l(x_1; z_1) \\ & \times l(x_1, l_1; z_2)l(x_2, l_1; z_3)l(x_3, l_2; z_4). \end{aligned}$$

- Application to SLAM
  - Nodes = robot poses & landmark locations
  - Arcs = "soft constraints"
    - information conveyed by odometry (between consecutive robot poses)
    - information from observations (between robot poses and landmark locations)
    - loop closure constraints
    - priors
  - The factor graph is built online – Very sparse as constraints number sublinear w.r.t. time and nodes number
- Statement
  - $\hat{r}_{0:k}, \hat{m} = \arg\max_{r_{0:k},m} \ln p(r_{0:k}, m|z_{0:k}) = \arg\max_{r_{0:k},m} \ln p(r_{0:k}, m|z_{0:k}; u_{0:k-1})$
  $= \arg\max_{r_{0:k},m} \ln p(r_0) + \sum_{t=1}^{k} \ln p(r_t|r_{t-1}; u_{t-1}) + \sum_{t=0}^{k} \ln p(z_t|r_t, m)$
  $\Leftrightarrow \hat{r}_{1:k}, \hat{m} = \arg\min_{r_{1:k},m} \frac{1}{2} \sum_{i=1}^{k} (r_i - f(r_{i-1}, u_{i-1}))^T Q_i^{-1} (r_i - f(r_{i-1}, u_{i-1}))$
  $+ \frac{1}{2} \sum_{j=0}^{k} (z_j - h(r_j, m))^T R_j^{-1} (z_j - h(r_j, m))$
  $= \arg\min_{r_{1:k},m} \frac{1}{2} \sum_p (r_p - f(r_{p-1}, u_{p-1}))^T \Omega_p (r_p - f(r_{p-1}, u_{p-1}))$
  $+ \frac{1}{2} \sum_q (z_q - h(r_q, m))^T \Omega_q (z_q - h(r_q, m))$
  with Ω_p and Ω_q the dynamic and measurement noise information matrices
  $\Leftrightarrow \hat{x} = \left(\hat{r}_{1:k}^T, \hat{m}^T\right)^T = \arg\min_{x=(r_{1:k}^T, m^T)^T} \frac{1}{2} \sum_{l=1}^{L} e_l^T(x_i, x_j) \Omega_l e_l^T(x_i, x_j)$
  with L the total number of factors, *i.e.*, the number of soft constraints implied by the odometry/landmarks measurements, each of them uniting a pair of unknowns
  $\Leftrightarrow \hat{x} = \left(\hat{r}_{1:k}^T, \hat{m}^T\right)^T = \arg\min_x \frac{1}{2} e^T(x) \Omega e(x)$
- Sparse factor graph $\longrightarrow$ Sparse criterion $\longrightarrow$ Solution by sparse optimization
  - Reminder : Newton algorithm to compute the minimum $\hat{x} = \arg\min_x f(x)$ of the real scalar-valued multivariate function $f(.)$
    - Select an initial guess $\mathring{x}$
    - Write the Taylor expansion $f(\mathring{x} + \Delta x) \approx \hat{f}(\mathring{x} + \Delta x) \triangleq f(\mathring{x}) + g_f^T(\mathring{x}) \Delta x + \frac{1}{2} \Delta x^T H_f(\mathring{x}) \Delta x$, with $g_f(\mathring{x}) = \left[\frac{\partial f(x)}{\partial x}\right]_{\mathring{x}}$ the gradient (column) vector and $H_f(\mathring{x}) = \left[\frac{\partial^2 f(x)}{\partial x \partial x^T}\right]_{\mathring{x}}$ the Hessian (square symmetric) matrix of $f(.)$ at $\mathring{x}$
    - Compute $\Delta x^\star = \arg\min_{\Delta x} \hat{f}(\mathring{x} + \Delta x) = -H_f^{-1}(\mathring{x}) g_f(\mathring{x})$
    - Set $\mathring{x} \leftarrow \mathring{x} + \Delta x^\star$ and iterate until convergence
  - Gauss-Newton minimization of $f(x) = e^T(x) \Omega e(x)$ with $e(.)$ a real vector-valued multivariate function
    - Set $J_e(\mathring{x}) = \left[\frac{\partial e(x)}{\partial x^T}\right]_{\mathring{x}}$ the Jacobian of $e(.)$ at $\mathring{x}$, and deduce $g_f(\mathring{x}) = J_e^T(\mathring{x}) \Omega e(\mathring{x})$
    - Neglect (hence "Gauss-Newton" instead of "Newton" minimization) the second derivatives of $e(.)$ in the second-order Taylor expansion of $f(.)$, so that $H_f(\mathring{x}) \approx J_e^T(\mathring{x}) \Omega J_e(\mathring{x})$
    - Deduce $\Delta x^\star = -\left(J_e^T(\mathring{x}) \Omega J_e(\mathring{x})\right)^{-1} J_e^T(\mathring{x}) \Omega e(\mathring{x})$, and iterate
  - $J_e(\mathring{x})$ is sparse! Only two nonzero blocs per group of lines corresponding to $e_k(.)$
    $\hookrightarrow$ The problem can be solved with very high computational performance by matrix QR or Cholesky factorization, *provided good initial guesses can be defined* $\longrightarrow$ see open-source solvers `g2o`, `ceres`, `gtsam`
  - Around the MAP $x^\star$, the joint posterior pdf on x is locally approximated by $\mathcal{N}(x; x^\star, J_e^T(x^\star) \Omega J_e(x^\star))$