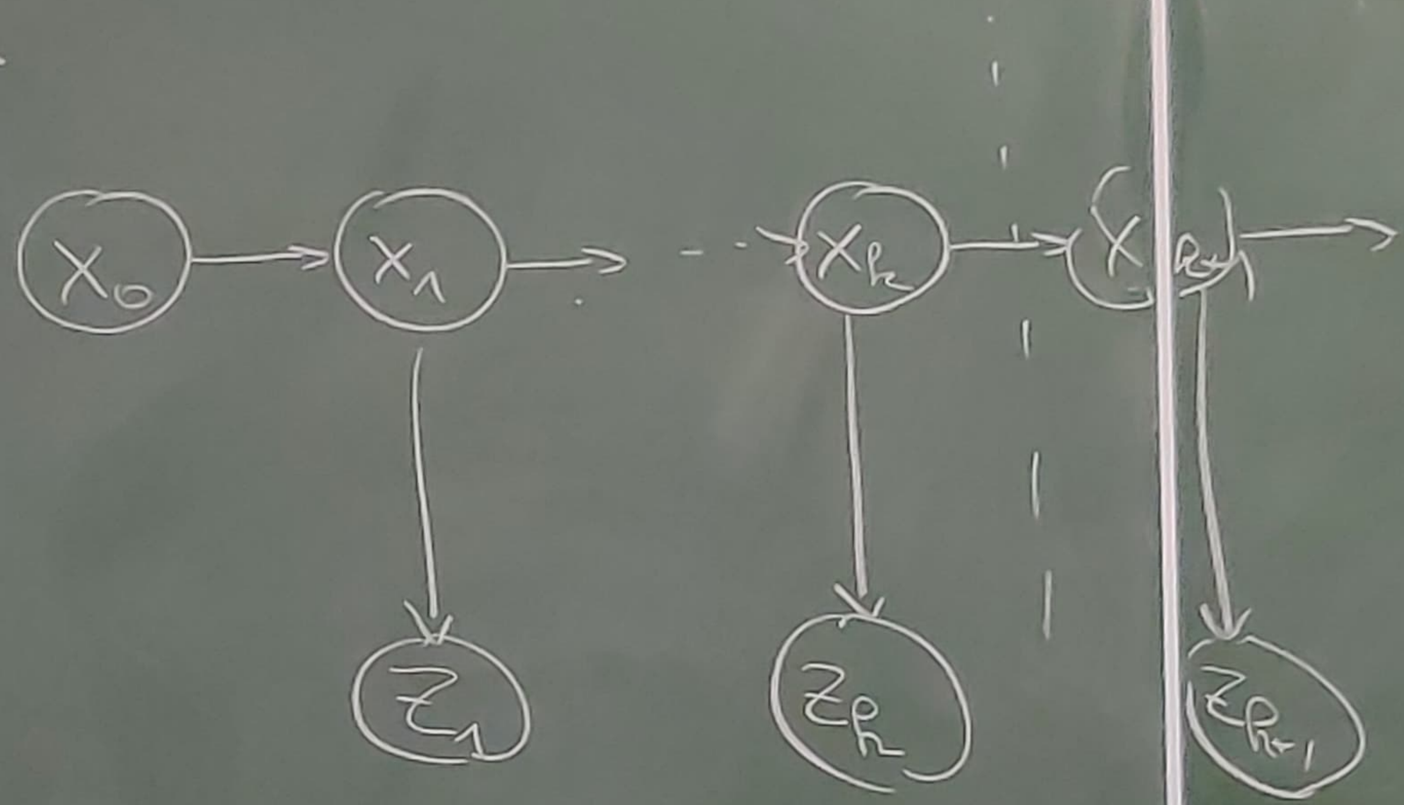
Filtre particulaire

10h

(Sequential Monte Carlo method(s))
(Particle filtering)

* Pourquoi ? • très utilisé pour filtrage Bayésien NL non Gaussien
• ∃ solution emblématique du SLAM (algos FASTSLAM / "gmapping")

1. Rappels : filtrage Bayésien



- séquence d'états cachés $x_{0:k} = x_0, x_1, \dots, x_k$
- Séquence d'observations : $z_{1:k} = z_1, z_2, \dots, z_k$

- Contexte incertain (approche probabiliste)

↳ $x_{0:k}$ réalisation de $X_{0:k}$

processus aléatoire <u>d'état</u>

séquence de V.A $X_0, X_1, \dots, X_k$

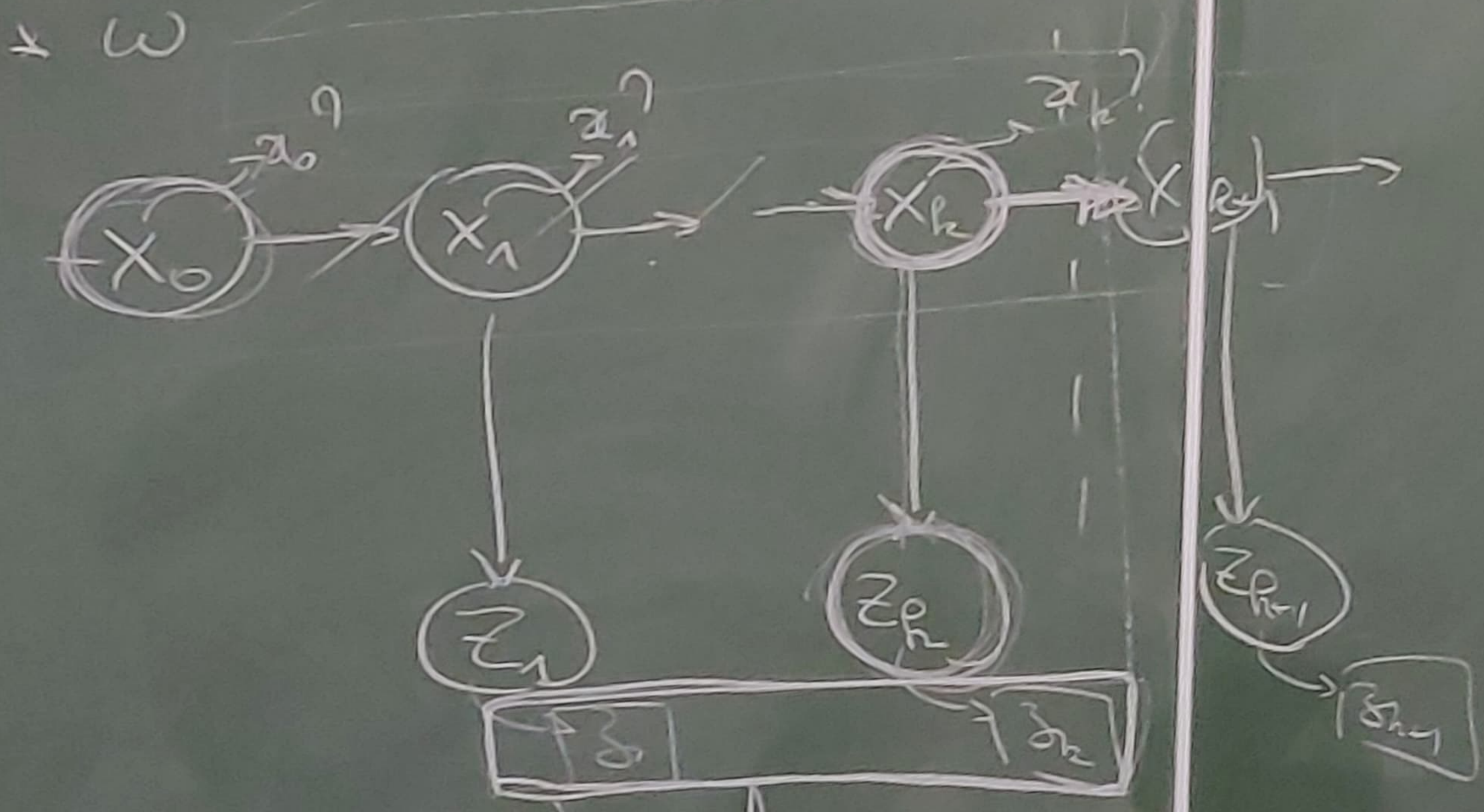
$x_{0:k} = X_{0:k}(\omega)$ <u>CACHÉ</u>

résultat de l'expérience en cours

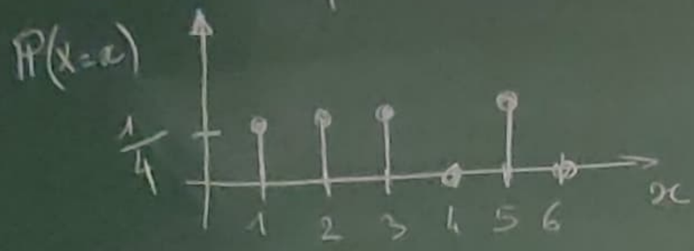
↳ $z_{1:k}$ réalisation de $Z_{1:k}$

PA de mesure

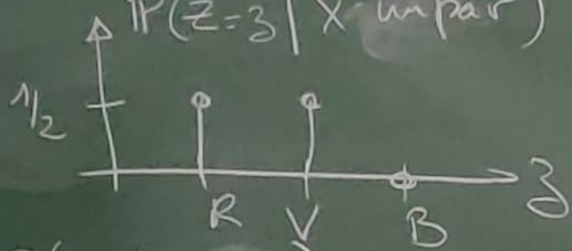
$z_{1:k} = Z_{1:k}(\omega)$ <u>ACCESSIBLE</u>

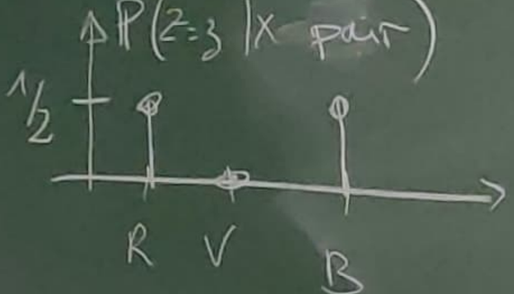

ω
x0
X0
x1
X1
xk
Xk
Xk+1
Z1
Zk
Zk+1
z1
zk
zk+1

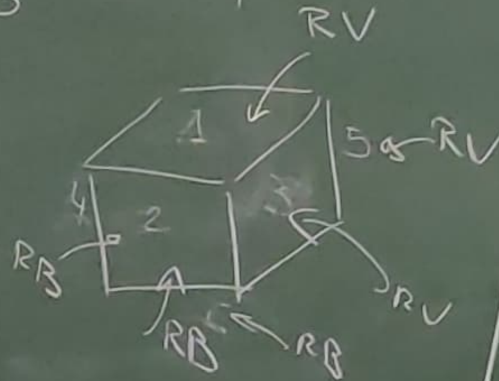
• loi a priori

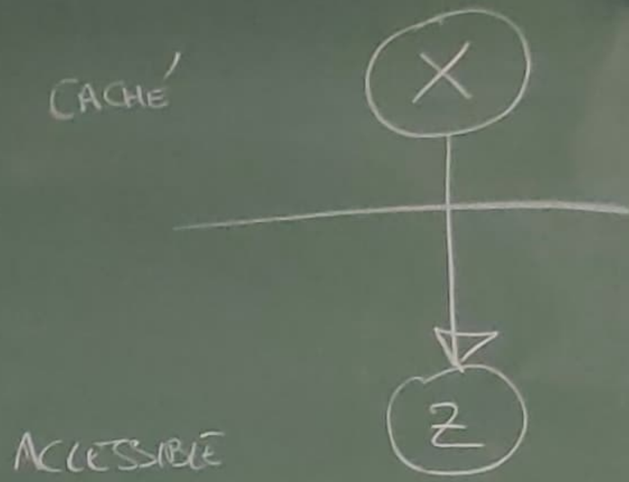


• modèle d'observation







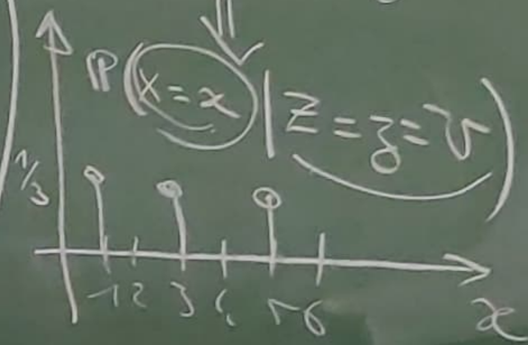


• Estimation

$\rightarrow \omega : Z(\omega) = z = V$

$\Downarrow$

$\mathbb{P}(X=x \mid Z=z=V)$



$$\int x_k \, p_{X_k|Z_{1:k}}(x_k|z_{1:k}) \, dx_k$$

observation

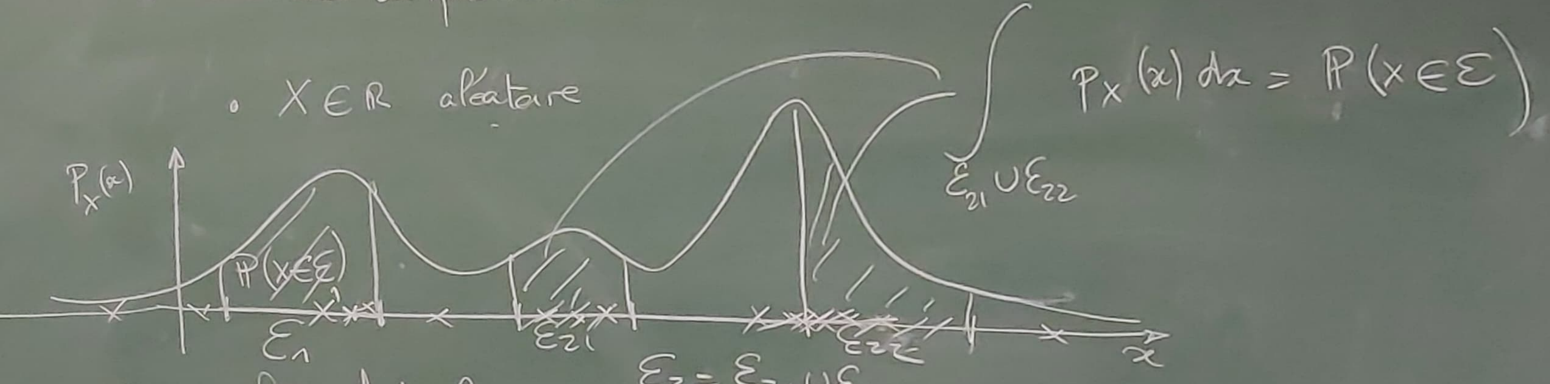
ESTIME

$$\mathbb{E}\left[X_k \mid Z_{1:k} = z_{1:k}\right]$$

MMSE estimate

* Densités de probabilités

• $X \in \mathbb{R}$ aléatoire



$P_X(x)$ = f° dont la variable est x, réalisation possible de X

↳ ≥ 0

↳ $\int_{\mathbb{R}} P_X(x)\,dx = \mathbb{P}(X\in\mathbb{R}) = 1$

↳ $\int_{\mathcal{E}} P_X(x)\,dx = \mathbb{P}(X\in\mathcal{E})$

→ quelle est l'espérance de X ? (valeur moyenne des issues possibles de X) ?

$$\mathbb{E}[X] = \mathbb{E}_{P_X(x)}[X] = \int_{\mathbb{R}} x \, P_X(x) \, dx \text{ noté } \bar{x}$$

→ comment se disperse X autour de $\mathbb{E}[X]$?

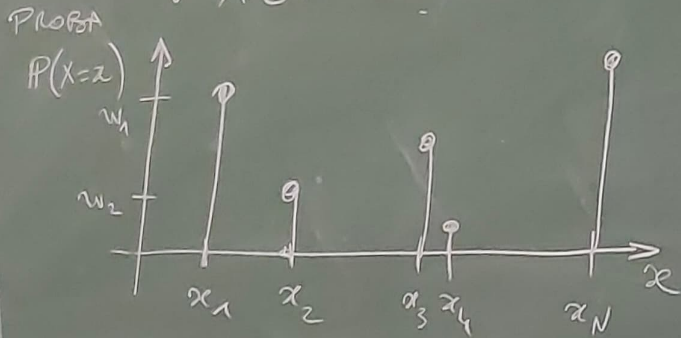
$$\mathrm{Var}[X] = \mathbb{E}_{P_X(x)}\left[\left(X - \mathbb{E}[X]\right)^2\right] = \dots = \text{aussi } \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2$$

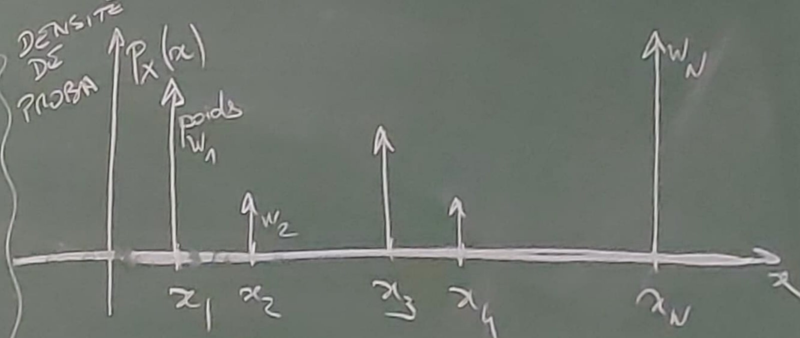
noté σ_X^2

avec $\sigma_X^2 = \displaystyle\int_{\mathbb{R}} (x - \bar{x})^2 \, P_X(x) \, dx$

* Densités de probabilités

- $X \in \mathbb{R}$ aléatoire À VALEURS DISCRÈTES $(x_1, \dots, x_N)$





notons $\mathbb{P}(X=x_i) =: w_i \geqslant 0$

proba

où $\sum_{i=1}^{N} w_i = 1$

$$\bar{x} := \mathbb{E}[X] = \sum_{i=1}^{N} w_i \, x_i$$

$$\sigma_X^2 = \sum_{i=1}^{N} w_i (x_i - \bar{x})^2 = \dots = \left(\sum_{i=1}^{N} x_i^2\right) - \bar{x}^2$$

$$P_X(x) = \sum_{i=1}^{N} w_i \, \delta(x - x_i)$$

où $\int_{\mathbb{R}} P_X(x)\,dx = \int_{\mathbb{R}} \sum_{i=1}^{N} w_i \, \delta(x-x_i)\,dx$

$$= \sum_{i=1}^{N} w_i \int_{\mathbb{R}} \delta(x - x_i)\,dx = \sum_{i=1}^{N} w_i = 1$$

→ quelle est l'espérance de X ? (valeur moyenne des issues possibles de X) ?

$$\bar{x} := \mathbb{E}[X] = \mathbb{E}_{P_X(x)}[X] = \int_{\mathbb{R}} x \left[\sum_{i=1}^{N} w_i \delta(x-x_i)\right] dx = \sum_{i=1}^{N} w_i \underbrace{\int_{\mathbb{R}} x\, \delta(x-x_i)\, dx}_{= x_i} = \sum_i w_i x_i$$

→ comment se disperse X autour de $\mathbb{E}[X]$?

$$\mathrm{Var}[X] = \mathbb{E}_{P_X(x)}\left[\left(X - \mathbb{E}[X]\right)^2\right] = \dots = \text{avec } \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2$$

noté σ_X^2

$$\text{avec } \sigma_X^2 = \int_{\mathbb{R}} (x-\bar{x})^2 P_X(x)\, dx = \sum_{i=1}^{N} w_i (x_i - \bar{x})^2$$

$$\int_{\mathbb{R}} \underbrace{f(x)\,\delta(x-x_i)}_{= f(x_i)\,\delta(x-x_i)}\, dx = f(x_i)$$

$$\boxed{f(x) * \delta(x-x_i)}$$

- $X \in \mathbb{R}^M$

$$p_X : \mathbb{R}^M \longrightarrow \mathbb{R}$$
$$x \longmapsto p_X(x)$$

$$\mathbb{P}(X \in \mathcal{E}) = \underbrace{\iiint}_{\mathcal{E}} p_X(x_1, \dots, x_M)\, dx_1 \dots x_M$$

$\mathcal{E} \in \mathbb{R}^M$

noté $\displaystyle\int_{\mathcal{E}} p_X(x)\, dx$

(dans le cas discret (N issues possibles) multivarié (dans $\mathbb{R}^M$)

→ (quelle est l'espérance de X ? (valeur moyenne des issues possibles de X)?

si les valeurs possibles sont $x^{(1)} \dots x^{(N)} \in \mathbb{R}^M$ avec les poids (probas) respectifs $w^{(1)} — w^{(N)}$

$$\bar{x} = \mathbb{E}[X] = \int_{\mathbb{R}^M} \underbrace{\begin{pmatrix} x_1 \\ | \\ x_M \end{pmatrix}}_{x} \underbrace{\sum_{i=1}^{N} w^{(i)} \delta\left(\begin{pmatrix} x_1 \\ | \\ x_M \end{pmatrix} - \begin{pmatrix} x_1^{(i)} \\ \vdots \\ x_M^{(i)} \end{pmatrix}\right)} \underbrace{dx_1 \dots dx_M}_{dx} = \sum_{i=1}^{N} w^{(i)} x^{(i)}$$

$$P_X(x) = \underbrace{\sum_{i=1}^{N} w^{(i)} \delta(x - x^{(i)})} = \sum_{i=1}^{N} w^{(i)} \delta(x_1 - x_1^{(i)}) \dots \delta(x_M - x_M^{(i)})$$

j^e composante

$$\times \int_{\mathbb{R}^M} x_j \sum_{i=1}^{N} w^{(i)} \underbrace{\delta\left(x_1 - x_1^{(i)}\right)}_{1} \cdots \delta\left(x_j - x_j^{(i)}\right) \cdots \underbrace{\delta\left(x_M - x_M^{(i)}\right)} \, \underbrace{dx_1}_{1} \cdots dx_j \cdots \underbrace{dx_M}$$

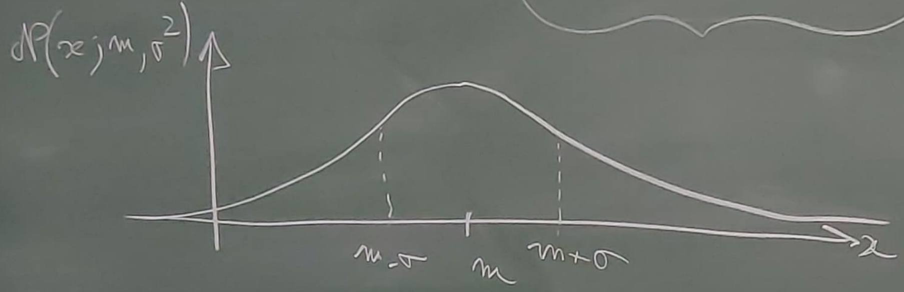
$$= \sum_{i=1}^{N} w^{(i)} \int_{\mathbb{R}} x_j \, \delta\left(x_j - x_j^{(i)}\right) = \sum_{i=1}^{N} w^{(i)} x_j^{(i)}$$

• Rappel sur les lois Gaussiennes

$$X \in \mathbb{R} \quad ; \quad X \sim \mathcal{N}(m, \sigma^2)$$

$$\Updownarrow$$

$$P_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] =: \mathcal{N}(x\,;\, m, \sigma^2)$$



$$\mathbb{P}\left(X \in [a,b[\right) = \int_a^b \mathcal{N}(x\,;\,m,\sigma^2)\,dx$$

$$= \Pi(b) - \Pi(a)$$

$\longrightarrow \mathbb{E}_{\mathcal{N}(x;m,\sigma^2)}[X] = \int_{\mathbb{R}} x \, \mathcal{N}(x;m,\sigma^2)\,dx = \dots = m$

$\mathbb{E}_{\mathcal{N}}\left[(X-m)^2\right] = \int \dots = \sigma^2$

$\longrightarrow$ Intervalles de confiance I_α de taille min tq $\mathbb{P}(X \in I_\alpha) = P_\alpha$ donné

$I_\alpha = [m - \alpha\sigma ; m + \alpha\sigma] = \left\{ x \in \mathbb{R}, \frac{(x-m)^2}{\sigma^2} \leq \alpha^2 \right\}$ et

$\alpha = 1 \longleftrightarrow P_\alpha = 0.6827$

$\alpha = 2 \longleftrightarrow P_\alpha = 0.95{\scriptstyle xx}$

$\alpha = 3 \longleftrightarrow P_\alpha = 0.997{\scriptstyle x}$

$$X \in \mathbb{R}^M \sim \mathcal{N}(\underset{\in \mathbb{R}^M}{m}, \underset{\in \mathbb{R}^{M\times M}}{P})$$

$$\Updownarrow$$

$$p_X(x) = \frac{1}{\sqrt{\det(2\pi P)}} \exp\left[-\frac{1}{2}(x-m)^T P^{-1}(x-m)\right]$$

$$\text{et } \sqrt{\det(2\pi P)} = \sqrt{(2\pi)^M \det(P)} = \sqrt{(2\pi)^M}\sqrt{\det(P)}$$

$$\rightarrow \bar{x} := \mathbb{E}[X] = \int_{\mathbb{R}^M} x\, \mathcal{N}(x; m, P)\, dx = \dots = m \,!$$

$$\hookrightarrow \mathrm{Cov}[X] = \mathbb{E}\left[(X-\bar{x})(X-\bar{x})^T\right] = \int_{\mathbb{R}^M} (x-\bar{x})(x-\bar{x})^T \mathcal{N}(x; m, P)\, dx = \dots = P \,!$$

$$\hookrightarrow \mathcal{E}_\alpha = \left\{ x,\ (x-m)^T P^{-1}(x-m) \leq \alpha^2 \right\}$$