# Projet EKF-SLAM

## A Travail demandé

Objectif : implémenter le problème décrit par la figure ci-dessous, où

- 8 amers ponctuels sont placés dans l'environnement aux points d'intersection de deux droites horizontales et quatre droites verticales
- un robot mobile non holonome navigue au-dessus puis entre les amers en effectuant un mouvement en forme de U
- seuls les amers situés en deça d'une distance donnée sont simultanément perçus par le robot ; le robot est capable de mesurer les distances et azimuths des amers relativement à son repère propre ; dans un premier temps, le robot est supposé capable d'identifier sans ambiguïté les amers ; dans un deuxième temps, il pourra exister des ambiguïtés dans l'association des mesures aux identifiants des amers ;
- le mouvement du robot et les observations sont entâchés de bruits de dynamique et de mesure, respectivement, de statistiques « raisonnables »

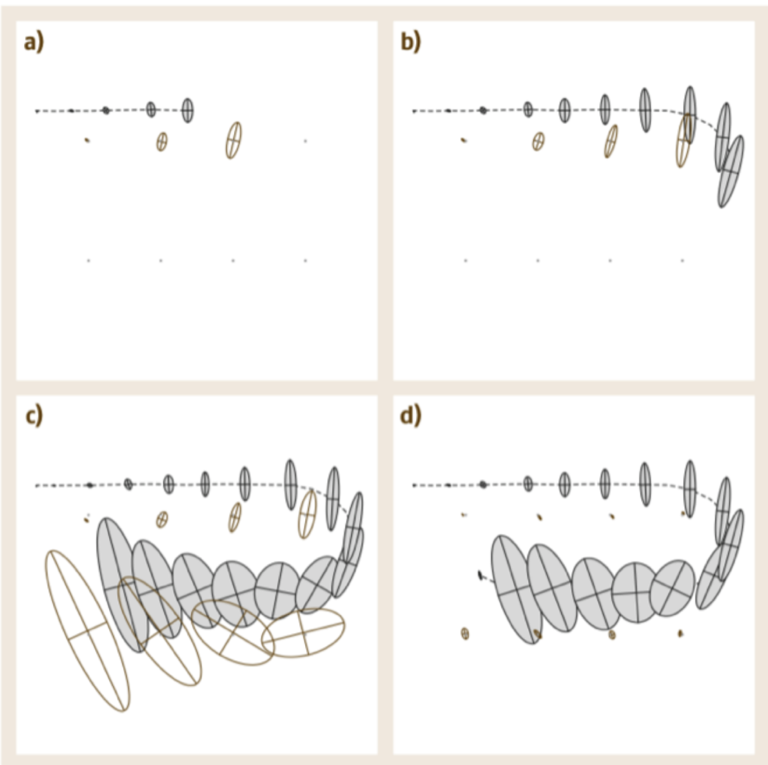


**Fig. 37.2a–d** EKF applied to the online SLAM problem. The robot's path is a *dotted line*, and its estimates of its own position are *shaded ellipses*. Eight distinguishable landmarks of unknown location are shown as *small dots*, and their location estimates are shown as *white ellipses*. In **(a–c)** the robot's positional uncertainty is increasing, as is its uncertainty about the landmarks it encounters. In **(d)** the robot senses the first landmark again, and the uncertainty of *all* landmarks decreases, as does the uncertainty of its current pose. (Image courtesy of Michael Montemerlo, Stanford University)

[Thrun, Leonard]

## B Rappel : Mathematical statement and solution of landmark-based SLAM

- Variables (static environment)
  - State vector $x_k$ at time $k$, with $x_k = (r_k^T, m^T)^T$, $r_k$ = robot absolute pose and $m = (m_1^T, \ldots, m_M^T)^T$ absolute position of landmarks.
  - Control inputs $u_{0:k}$, given (by odometry, etc.)
  - Observations $z_{0:k}$, with $z_k$ measurement at time $k$ = stacking of measurements $\{z_{k,j}\}_j$ from $r_k$ of the set $\{m_j\}_j$ of visible landmarks.
- Models (static environment)
  - $r_{k+1} = f(r_k, u_k) + w_k$, $w_k \sim \mathcal{N}(0, Q_k)$ white Gaussian dynamic noise
    $\hookrightarrow$ $p(r_{k+1}|r_k) = p(r_{k+1}|r_k; u_k) = \mathcal{N}(r_{k+1}; f(r_k, u_k), Q_k)$.
  - ($m$ constant noise-free)
  - $z_k = h(r_k, m) + v_k$, $v_k \sim \mathcal{N}(0, R_k)$ white Gaussian measurement noise (noises on individual measurements are assumed mutually independent, $R_k$ diagonal)
    $\hookrightarrow$ $p(z_k|r_k, m) = \mathcal{N}(z_k; h(r_k, m), R_k)$
- The posterior marginal pdf of the robot pose together with the map $p(r_k, m|z_{0:k}) = p(r_k, m|z_{0:k}; u_{0:k-1})$ is approximated by a (huge-dimension) Gaussian pdf thanks to EKF approximate computations of posterior moments for nonlinear dynamics/observation model
  - Use of Taylor expansions for covariance and gain equations ($\rightsquigarrow$ Jacobian matrices of transition/measurement functions around last state estimates/predictions)
    $\hookrightarrow$ considering standard assumptions and notations of linear Kalman filter except the following nonlinear state and measurement stochastic equations

$$x_{k+1} = f(x_k) + w_k,\ w_k \sim \mathcal{N}(0, Q_k), \qquad \text{and} \qquad z_k = h(x_k) + v_k,\ v_k \sim \mathcal{N}(0, R_k),$$

    EKF equations write as

$$\begin{array}{rlrl}
\hat{x}_{0|0} &= m_{X_0} & P_{0|0} &= P_0 \\
\hat{x}_{k+1|k} &= f(\hat{x}_{k|k}) & P_{k+1|k} &= F_k P_{k|k} F_k^T + Q_k \\
\hat{z}_{k+1|k} &= h(\hat{x}_{k+1|k}) & S_{k+1|k} &= R_{k+1} + H_{k+1} P_{k+1|k} H_{k+1}^T \\
 & \multicolumn{2}{c}{K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1|k}^{-1}} & \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}(z_{k+1} - \hat{z}_{k+1|k}) & P_{k+1|k+1} &= P_{k+1|k} - K_{k+1} H_{k+1} P_{k+1|k}
\end{array}$$

$$\text{with } F_k = \left[\frac{\partial f(x)}{\partial x^T}\right]_{x=\hat{x}_{k|k}} \text{ and } H_{k+1} = \left[\frac{\partial h(x)}{\partial x^T}\right]_{x=\hat{x}_{k+1|k}}$$

- Outline of 2D range and/or bearing online EKF-SLAM
  - <u>State, control input and measurement output variables</u>
    - Planar robot $\mathcal{R}$ with associated frame $\mathcal{F}_k = (O_k, \vec{x}_k, \vec{y}_k)$ at time $k$. $M$ pointwise landmarks $\mathcal{L}_1, \ldots, \mathcal{L}_M$ at (unknown) loci $L_1, \ldots, L_M$, to be referenced in a map $\mathcal{M}$. World/Map frame is $\mathcal{F}_0$.
    - Hidden state vector $x_k = (r_k^T, m_1^T, \ldots, m_M^T)^T$ : absolute robot pose vector $r_k = (r_{x_k}, r_{y_k}, \theta_k)^T$ with $(r_{x_k}, r_{y_k})^T = \overrightarrow{O_0 O_k}_{(\mathcal{F}_0)}$ and $\theta_k = (\widehat{\vec{x}_0, \vec{x}_k})$; absolute landmark positions $m_m = (m_{x_m}, m_{y_m})^T = \overrightarrow{O_0 L_m}_{(\mathcal{F}_0)}$, gathered into vector $m = (m_1^T, \ldots, m_M^T)^T$.
    - Control input vector $u_k = (\tau_k^T, \rho_k)^T$ : robot translation $\tau_k = (\tau_{x_k}, \tau_{y_k})^T = \overrightarrow{O_k O_{k+1}}_{(\mathcal{F}_k)}$ and rotation $\rho_k = (\widehat{\vec{x}_k, \vec{x}_{k+1}})$ from $\mathcal{F}_k$ to $\mathcal{F}_{k+1}$ expressed in $\mathcal{F}_k$.

- Observation vector $z_k$ : stacking of $\{z_{k,j}\}_{\{\text{visible } \mathcal{L}_j\}}$, where $z_{k,j}$ terms the range $\|O_kL_j\|$, bearing $(\widehat{\vec{x}_k, \overrightarrow{O_kL_j}}) = \Big(\text{atan2}\big(\overrightarrow{O_kL_j}.\vec{y}_k, \overrightarrow{O_kL_j}.\vec{x}_k\big) = \text{atan2}\big(\overrightarrow{O_kL_j}.\vec{y}_0, \overrightarrow{O_kL_j}.\vec{x}_0\big) - \theta_k\Big)$ or their stacking, at time $k$, of $L_j$ relative to $\mathcal{F}_k$.
- Note however—including in the lines below—that subvector $m$ is built incrementally, *i.e.*, $M$ is not known in advance...

∘ Prior dynamics

- Robot dynamics $r_{k+1} = f_\mathcal{R}(r_k, u_k) + w_k^\mathcal{R}$, $w_k^\mathcal{R} \sim \mathcal{N}(0, Q_k^\mathcal{R})$, $w_{0:k}^\mathcal{R}$ white, etc. where

$$f_\mathcal{R}(r_k, u_k) = r_k + g_\mathcal{R}(r_k, u_k) \text{ with } g_\mathcal{R}(r_k, u_k) = g_\mathcal{R}(\cancel{r_{x_k}}, \cancel{r_{y_k}}, \theta_k, u_k) = \begin{pmatrix} \tau_{x_k}\cos\theta_k - \tau_{y_k}\sin\theta_k \\ \tau_{x_k}\sin\theta_k + \tau_{y_k}\cos\theta_k \\ \rho_k \end{pmatrix}.$$

↪ Proof : show that the average noise-free model accounts for the rigid-body motion of the robot between times $k$ and $k+1$, described by

$$T_{0,k+1} = T_{0,k}T_{k,k+1}, \text{ with } T_{0,k} = \begin{pmatrix} \cos\theta_k & -\sin\theta_k & r_{x_k} \\ \sin\theta_k & \cos\theta_k & r_{y_k} \\ 0 & 0 & 1 \end{pmatrix} \text{ and } T_{k,k+1} = \begin{pmatrix} \cos\rho_k & -\sin\rho_k & \tau_{x_k} \\ \sin\rho_k & \cos\rho_k & \tau_{y_k} \\ 0 & 0 & 1 \end{pmatrix}.$$

- Incorporating $m_{k+1} = m_k$ leads to $x_{k+1} = f(x_k, u_k) + w_k$, with

$$f(x_k, u_k) = x_k + E_x^T g_\mathcal{R}(r_k, u_k) + w_k, \; w_k \sim \mathcal{N}(0, Q_k),$$
$$Q_k = \text{blkdiag}(Q_k^\mathcal{R}, \underbrace{\mathbb{O}_{2\times2}, \ldots, \mathbb{O}_{2\times2}}_{M \text{ times}}), \; E_x = \begin{pmatrix} \mathbb{I}_{3\times3} & \mathbb{O}_{3\times2M} \end{pmatrix}.$$

∘ Observation model

- $z_k = h_k(x_k) + v_k$, $v_k \sim \mathcal{N}(0, R_k)$, composed of measurements $z_{k,j} = h_{k,j}(r_k, m_j) + v_{k,j}$ of $\mathcal{L}_j$ w.r.t. $\mathcal{R}$, $v_{k,j} \sim \mathcal{N}(0, R_{k,j})$ mutually independent over $\{\mathcal{L}_j\}$ so that $R_k = \text{blkdiag}(\{R_{k,j}\})$, $v_{1:k}$ white, etc. with one of the following options :

$$h_{k,j}(r_k, m_j) = \sqrt{(m_{x_j} - r_{x_k})^2 + (m_{y_j} - r_{y_k})^2} \text{ or } \text{atan2}\big((m_{y_j} - r_{y_k}), (m_{x_j} - r_{x_k})\big) - \theta_k,$$
$$\text{or } h_{k,j}(r_k, m_j) = \begin{pmatrix} \sqrt{(m_{x_j} - r_{x_k})^2 + (m_{y_j} - r_{y_k})^2} \\ \text{atan2}\big((m_{y_j} - r_{y_k}), (m_{x_j} - r_{x_k})\big) - \theta_k \end{pmatrix}.$$

↪ Proof : rewrite $\begin{pmatrix} \overrightarrow{O_kL_j}_{(\mathcal{F}_k)} \\ 1 \end{pmatrix} = T_{0,k}^{-1} \begin{pmatrix} \overrightarrow{O_0L_j}_{(\mathcal{F}_0)} \\ 1 \end{pmatrix} \Leftrightarrow \overrightarrow{O_kL_j}_{(\mathcal{F}_k)} = R_{0,k}^T \overrightarrow{O_0L_j}_{(\mathcal{F}_0)} - R_{0,k}^T P_{0,k}$, with $R_{0,k} = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}$, $P_{0,k} = \begin{pmatrix} r_{x_k} \\ r_{y_k} \end{pmatrix}$, $\overrightarrow{O_0L_j}_{(\mathcal{F}_0)} = \begin{pmatrix} m_{x_j} \\ m_{y_j} \end{pmatrix}$, as $\overrightarrow{O_kL_j}_{(\mathcal{F}_k)} = \begin{pmatrix} \cos\theta_k(m_{x_j} - r_{x_k}) + \sin\theta_k(m_{y_j} - r_{y_k}) \\ -\sin\theta_k(m_{x_j} - r_{x_k}) + \cos\theta_k(m_{y_j} - r_{y_k}) \end{pmatrix}$ ; then, show that the first line of $h_{k,j}(r_k, m_j)$ is the norm of $\overrightarrow{O_kL_j}_{(\mathcal{F}_k)}$ and that the second line of $h_{k,j}(r_k, m_j)$ is its angle, *e.g.*, resorting to the expansion of $\tan\big(\text{atan2}(\ldots) - \theta_k\big)$.

∘ SLAM Initialization

- The initial pose of the robot is the origin of the map, which contains no landmarks, hence

$$\hat{x}_{0|0} = \hat{r}_{0|0} = \mathbb{O}_{3\times1}, \; P_{0|0} = P_{\mathcal{R}\mathcal{R}_{0|0}} = \mathbb{O}_{3\times3}.$$

∘ SLAM Time update between times $k$ and $k+1$

- Standard EKF equations, with $F_k = \left[\frac{\partial f(x, u_k)}{\partial x^T}\right]_{x = \hat{x}_{k|k}}$, leads to ($M$ being the total number of perceived landmarks)

$$F_k = \mathbb{I}_{(3+2M)\times(3+2M)} + E_x^T \begin{pmatrix} \left[\frac{\partial g_\mathcal{R}(r, u_k)}{\partial r^T}\right]_{r = \hat{r}_{k|k}} & \mathbb{O}_{3\times2M} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_{3\times3} + \left[\frac{\partial g_\mathcal{R}(r, u_k)}{\partial r^T}\right]_{r = \hat{r}_{k|k}} & \mathbb{O}_{3\times2M} \\ \mathbb{O}_{2M\times3} & \mathbb{I}_{2M\times2M} \end{pmatrix},$$

so that predicted covariance $2M \times 2M$ submatrix $P_{\mathcal{M}\mathcal{M}_{k+1|k}} = P_{\mathcal{M}\mathcal{M}_{k|k}}$ is unchanged.

· In addition,

$$\left[\frac{\partial g_{\mathcal{R}}(r, u_k)}{\partial r^T}\right]_{r=\hat{r}_{k|k}} = \begin{pmatrix} \mathbb{O}_{2\times 2} & \left[\frac{\partial((\mathbb{I}_{2\times 2}\ \mathbb{O}_{2\times 1})g_{\mathcal{R}}(\theta,u_k))}{\partial\theta}\right]_{\theta=\hat{\theta}_{k|k}} \\ \mathbb{O}_{2\times 1} & 0 \end{pmatrix},$$

so that $F_k$ is sparse.

◦ SLAM Measurement update at time $k+1$ for already seen landmarks

· Standard EKF equations, with $H_{k+1} = \left[\frac{\partial h_{k+1}(x)}{\partial x^T}\right]_{x=\hat{x}_{k+1|k}}$, and each individual Jacobian matrix $H_{k+1,j} = \left[\frac{\partial h_{k+1,j}(x)}{\partial x^T}\right]_{x=\hat{x}_{k+1|k}}$ also features the sparse structure

$$H_{k+1,j} = \begin{pmatrix} H_{\mathcal{R}} & \mathbb{O} & \ldots & \mathbb{O} & H_{\mathcal{L}_j} & \mathbb{O} & \ldots & \mathbb{O} \end{pmatrix}, \text{ with } H_{\mathcal{R}} = \left[\frac{\partial h_{k+1,j}(x)}{\partial r^T}\right]_{x=\hat{x}_{k+1|k}}, H_{\mathcal{L}_j} = \left[\frac{\partial h_{k+1,j}(x)}{\partial m_j^T}\right]_{x=\hat{x}_{k+1|k}}.$$

· Landmark measurements are assimilated in sequence. In view of the sparsity of each $H_{k+1,j}$, the computation of the residual $z_{k+1,j} - \hat{z}_{k+1|k,j}$ is sparse, as well as its covariance matrix $S_{k+1|k,j} = R_{k+1,j} + \begin{pmatrix} H_{\mathcal{R}} & H_{\mathcal{L}_j} \end{pmatrix} \begin{pmatrix} P_{\mathcal{R}\mathcal{R}k+1|k} & P_{\mathcal{R}\mathcal{L}_j k+1|k} \\ P_{\mathcal{L}_j\mathcal{R}k+1|k} & P_{\mathcal{L}_j\mathcal{L}_j k+1|k} \end{pmatrix} \begin{pmatrix} H_{\mathcal{R}}^T \\ H_{\mathcal{L}_j}^T \end{pmatrix}$, what enables its inversion in constant time. But even when assimilating a single landmark measurement $z_{k+1,j}$, the gain $K_{k+1}$ is dense so that the full map is updated, including its filtering pdf covariance.

◦ For unseen landmarks, additional stage to be inserted before SLAM Measurement update

· If $\mathcal{R}$ discovers a landmark, say $\mathcal{L}_m$, which has not yet been mapped, then the current state vector must be augmented with $m_m$. Even if, for given $r_{k+1}$, the output function $h_{k+1,m}(r_{k+1}, m_m)$ is bijective w.r.t. $m_m$, it is necessary to define a prediction $\hat{m}_{m_{k+1|k}}$ of $m_m$ in order to compute a meaningful individual Jacobian $H_{k+1,m}$. Otherwise this Jacobian may not be a valid approximation of the derivative of $h_{k+1,m}(.,.)$ over the regions where the state vector prediction pdf is high, and EKF may fail!

↪ Set for instance, with $a_{k+1,m}$ and $b_{k+1,m}$ the first and second entry of $z_{k+1,m}$

$$\hat{m}_{m_{k+1|k}} = \begin{pmatrix} \hat{r}_{x_{k+1|k}} \\ \hat{r}_{y_{k+1|k}} \end{pmatrix} + a_{k+1,m} \begin{pmatrix} \cos(\hat{\theta}_{k+1|k} + b_{k+1,m}) \\ \sin(\hat{\theta}_{k+1|k} + b_{k+1,m}) \end{pmatrix}.$$

· Generalizations and other ways to proceed do exist, including the cases when $h_{k+1,m}(r_{k+1}, m_m)$ is not bijective w.r.t. $m_m$, see for instance [Solá].

◦ Data association handling

· The easiest way to decide whether an already seen landmark in $\{\mathcal{L}_1, \ldots, \mathcal{L}_{m-1}\}$ or a new $\mathcal{L}_m$ gives rise to the observation $z_{k+1}$ is "gating".

$$\text{Compute } \delta_{k+1,j} \triangleq (z_{k+1} - \hat{z}_{k+1|k,j})^T S_{k+1|k,j}^{-1} (z_{k+1} - \hat{z}_{k+1|k,j}),\ j \in \{1, \ldots, m-1\};$$
$$\text{Define threshold } \delta_{k+1,m} \text{ as an ad hoc value};$$
$$\text{Then detect most likely } \mathcal{L}_{j^\star} \text{ by } j^\star = \arg\min_{j\in\{1,\ldots,m\}} \delta_{k+1,j}.$$