Gas dynamics

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Notation

Eulerian description: one defines a fixed coordinate grid in space and follows how the gas quantities are changing a a given position. The Eulerian time-derivative is $\frac{\partial}{\partial \mathrm{t}}$.

Lagrangian description: one chooses a fluid element in the fluid and follows how its properties change. The Lagrangian time-derivative is $\frac{\mathrm{D}}{\mathrm{Dt}}$.

These points of view are related through:

$$\frac{\mathrm{DQ}}{\mathrm{Dt}} = \frac{\partial \mathrm{Q}}{\partial \mathrm{t}} + \mathbf{v} \cdot \nabla \mathrm{Q} \tag{1}$$

The basic equations of gas dynamisc

Continuity equation

$$\frac{\partial \rho}{\partial \mathrm{t}} + \nabla(\rho \mathbf{v}) = 0 \tag{2}$$

Euler equation

$$\rho \frac{\partial \mathbf{v}}{\partial \mathrm{t}} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \mathrm{P} + \mathbf{f} \tag{3}$$

Energy conservation

$$\frac{\partial}{\partial \mathrm{t}} \left(\frac{1}{2} \rho \mathrm{v}^2 + \rho \epsilon \right) + \nabla \cdot \left[\left(\frac{1}{2} \rho \mathrm{v}^2 + \rho \epsilon + \mathrm{P} \right) \mathbf{v} \right] = \mathbf{f} \cdot \mathbf{v} - \nabla \mathbf{q} \tag{4}$$

Contiuity equation

$$\frac{\partial \rho}{\partial \mathrm{t}} + \nabla(\rho \mathbf{v}) = 0 \tag{5}$$

ρ is the mass density (mass per unit volume), $\rho\mathbf{v}$ is the mass flux density (mass flux per unit area.)
An alternative formula is:

$$\frac{\mathrm{D}\rho}{\mathrm{Dt}} = -\rho\nabla \cdot \mathrm{v} \tag{6}$$

Equation of motion

Euler equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial \mathrm{t}} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla \mathrm{P} + \mathbf{f} \tag{7}$$

$\rho\mathbf{v}\cdot\nabla\mathbf{v}$ - advection of momentum through the fluid by velocity gradients.

$\mathbf{f}$ - force density (force per unit volume)

gravity: $\mathbf{f} = -\rho\mathbf{g} = \rho\nabla\Phi$, where Φ is gravitational potential

Poisson equation for self-gravity systems:

$$\nabla^2\Phi = 4\pi \mathrm{G}\rho \tag{8}$$

external magnetic field: $(\nabla\times\mathbf{B})\times\mathbf{B}$

viscous forces: $\rho\nu\nabla^2\mathbf{v}$, here ν is kinematic viscosity.

Energy conservation

Energy = kinetic energy $1/2\rho v^2$ + thermal energy $\rho\epsilon$
Monoatomic gas: $\epsilon = 3/2kT/(\mu m_H)$, μ is the mean molecular weight, $\mu = 1$ - neutral hydrogen, $\mu = 0.6$ - Solar composition.

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P)\,\mathbf{v}] = \mathbf{f} \cdot \mathbf{v} - \nabla \mathbf{q} \tag{9}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho\epsilon + P\right)\mathbf{v}\right] = \mathbf{f} \cdot \mathbf{v} - \nabla \mathbf{q} \tag{10}$$

here $\mathbf{q} = -K\nabla T$ is the heat flux

Steady flows

Continuity equation

$$\frac{\partial \rho}{\partial \mathrm{t}} + \nabla(\rho \mathbf{v}) = 0 \tag{11}$$

Euler equation

$$\rho \frac{\partial \mathbf{v}}{\partial \mathrm{t}} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \mathrm{P} + \mathbf{f} \tag{12}$$

Energy conservation

$$\frac{\partial}{\partial \mathrm{t}} \left(\frac{1}{2} \rho \mathrm{v}^2 + \rho \epsilon \right) + \nabla \cdot \left[\left(\frac{1}{2} \rho \mathrm{v}^2 + \rho \epsilon + \mathrm{P} \right) \mathbf{v} \right] = \mathbf{f} \cdot \mathbf{v} \tag{13}$$

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Steady flows

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$$\nabla(\rho\mathbf{v}) = 0 \tag{11}$$

Euler equation

$$\rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla\mathrm{P} + \mathbf{f} \tag{12}$$

Energy conservation

$$\nabla\cdot\left[\left(\frac{1}{2}\rho\mathrm{v}^2 + \rho\epsilon + \mathrm{P}\right)\mathbf{v}\right] = \mathbf{f}\cdot\mathbf{v} \tag{13}$$

Steady flows

$$\nabla(\rho\mathbf{v}) = 0 \tag{14}$$

$$\rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla \mathrm{P} + \mathbf{f} \tag{15}$$

$$\nabla\cdot\left[\left(\frac{1}{2}\rho \mathrm{v}^2 + \rho\epsilon + \mathrm{P}\right)\mathbf{v}\right] = \mathbf{f}\cdot\mathbf{v} \tag{16}$$

Vector calculus: $\nabla(\mathrm{a}\cdot\mathbf{k}) = \mathrm{a}(\nabla\cdot\mathbf{k}) + \mathbf{k}(\nabla \mathrm{a})$ Using this rule and equation (14) one can obtain:

$$\rho\mathbf{v}\cdot\nabla\left(\frac{1}{2}\mathrm{v}^2 + \epsilon + \frac{\mathrm{P}}{\rho}\right) = \mathbf{f}\cdot\mathbf{v} \tag{17}$$

Then multiply equation(15) on $\mathbf{v}$:

$$\mathbf{v}\rho\cdot(\mathbf{v}\cdot\nabla)\mathbf{v} = -\mathbf{v}\cdot\nabla \mathrm{P} + \mathbf{f}\cdot\mathbf{v} \tag{18}$$

Steady flows

$$\rho \mathbf{v} \cdot \nabla \left(\frac{1}{2} \mathrm{v}^2 + \epsilon + \frac{\mathrm{P}}{\rho} \right) = \mathbf{f} \cdot \mathbf{v} \tag{19}$$

Then multiply equation(15) on $\mathbf{v}$:

$$\mathbf{v}\rho \cdot (\mathbf{v} \cdot \nabla)\mathbf{v} = -\mathbf{v} \cdot \nabla \mathrm{P} + \mathbf{f} \cdot \mathbf{v} \tag{20}$$

Here $\mathbf{v}\rho \cdot (\mathbf{v} \cdot \nabla)\mathbf{v} = \rho \mathbf{v} \cdot \nabla \left(\frac{1}{2} \mathrm{v}^2 \right)$ Then

$$\rho \mathbf{v} \cdot \nabla (\epsilon + \mathrm{P}/\rho) = \mathbf{v} \cdot \nabla \mathrm{P} \tag{21}$$

or

$$\mathbf{v} \cdot [\nabla \epsilon + \mathrm{P} \nabla (1/\rho)] = 0 \tag{22}$$

$$\mathbf{v} \cdot [\nabla \epsilon + \mathrm{P} \nabla (1/\rho)] = 0 \tag{23}$$

What is $\mathbf{v} \cdot \nabla$? Remember equation(1): $\frac{\mathrm{D}}{\mathrm{Dt}} = \frac{\partial}{\partial \mathrm{t}} + \mathbf{v} \cdot \nabla \equiv \mathrm{d}$ So,

$$\mathrm{d}\epsilon + \mathrm{Pd}(1/\rho) = 0 \tag{24}$$

where for monoatomic gas enternal energy is $\epsilon = \frac{3}{2} \frac{\mathrm{kT}}{\mu \mathrm{m_H}}$ and perfect gas law: $\mathrm{P} = \rho \mathrm{kT}/\mu \mathrm{m_H}$:

$$\rho^{-1} \mathrm{T}^{3/2} = \mathrm{constant} \tag{25}$$

or

$$\mathrm{P} \rho^{-5/3} = \mathrm{constant} \tag{26}$$

This equation describes adiabatic flows.

Adiabatic and isothermal flows

General case:

$$P\rho^{\gamma} = \text{constant} \tag{27}$$

here γ is adiabatic index, or the ratio of specific heats.
This condition is equivalent to setting the entropy of the gas constant.
Where? Very fast processes SN, thick disks
Isothermal flow with T is constant: $P\rho^{-1} = \text{constant}$ Where?
Protoplanetary disks, HII clouds. Slow processes.

Acoustic waves

Here neglect transport phenomena, $\nabla \mathrm{q} = 0$. Consider a homogeneous perfect gas at rest: $\rho = \rho_0$; $\mathrm{p} = \mathrm{p}_0$; $\mathbf{v} = \mathbf{v}_0 = 0$.
Suppose that pressure is perturbed to $\mathrm{p} = \mathrm{p}_0 + \mathrm{p}_1$, $\rho = \rho_0 + \rho_1$ and $\mathbf{v} = \mathbf{v_1}$, assume that perturbations are adiabatic, so:

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\mathrm{p}}{\rho^\gamma}\right) = \frac{1}{\rho^\gamma}\frac{\mathrm{dp}_1}{\mathrm{dt}} - \gamma\frac{\mathrm{p}}{\rho^{\gamma+1}}\frac{\mathrm{d}\rho_1}{\mathrm{dt}} \tag{28}$$

from this $\mathrm{dp}_1 = \mathrm{c_s^2 d}\rho_1$ where $\mathrm{c_s^2} = \gamma\mathrm{p}/\rho$.
Assuming that the perturbations are small, $\rho_1 \ll \rho_0$ and $\mathrm{p}_1 \ll \mathrm{p}_0$, we can replace the quantities in $\mathrm{c_s^2}$ by their equilibrium values,

$$\mathrm{c_s^2} = \frac{\gamma \mathrm{p}_0}{\rho_0} \tag{29}$$

Linearizing continuity equation:

$$0 = \frac{\partial \rho_1}{\partial \mathrm{t}} + \nabla \cdot [(\rho_0 + \rho_1)\mathbf{v}_1] \simeq \frac{\partial \rho_1}{\partial \mathrm{t}} + \rho_0 \nabla \cdot \mathbf{v}_1] \tag{30}$$

then Euler equation:

$$0 = (\rho_0+\rho_1)\left[\frac{\partial \mathbf{v}_1}{\partial \mathrm{t}} + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1\right] + \nabla \mathrm{p}_1 \simeq \rho_0 \frac{\partial \mathbf{v}_1}{\partial \mathrm{t}} + \nabla \mathrm{p}_1 = \rho_0 \frac{\partial \mathbf{v}_1}{\partial \mathrm{t}} + \mathrm{c}_\mathrm{s}^2 \nabla \rho_1 \tag{31}$$

Combining both these equation one can obtain:

$$\frac{\partial^2 \rho_1}{\partial \mathrm{t}^2} - \mathrm{c}_\mathrm{s}^2 \nabla^2 \rho_1 = 0 \tag{32}$$

which is equation for acoustic waves showing that

$$\mathrm{c}_\mathrm{s}^2 = \sqrt{\frac{\gamma \mathrm{p}_0}{\rho_0}} \tag{33}$$

is the sound speed. Assuming that $\mathrm{p}_1 = (\mathrm{dp}/\mathrm{d}\rho)_0 \rho_1$ give us:

$$\mathrm{c}_\mathrm{s}^2 = \left(\frac{\mathrm{dp}}{\mathrm{d}\rho}\right)_0 \tag{34}$$

$$c_s^2 = \left(\frac{dp}{d\rho}\right)_0 \tag{35}$$

adiabatic flow:

$$c_s^{ad} = \left(\frac{5p}{3\rho}\right)^{1/2} = \left(\frac{5kT}{3\mu m_H}\right)^{1/2} \tag{36}$$

isothermal flow:

$$c_s^{iso} = \left(\frac{p}{\rho}\right)^{1/2} = \left(\frac{kT}{\mu m_H}\right)^{1/2} \tag{37}$$

c_s is the speed at which pressure disturbances travel through the gas, it limits the rapidly with which the gas can respond to pressure changes.

Subsonic and suprsonic flows

c_s is the speed at which pressure disturbances travel through the gas, it limits the rapidly with which the gas can respond to pressure changes.
Supersonic flow: $|\mathbf{v}| > c_s$ the pressure gradients has little effect on the flow.
Subsonic flow: $|\mathbf{v}| < c_s$ to a first approximation the gas behaves as if in hydrostatic equilibrium.

Euler equation and the disks

Radial momentum equation

$$(\mathbf{v}_{\mathrm{p}} \cdot \nabla)\mathbf{v}_{\mathrm{p}} = -\frac{1}{\rho}\nabla \mathrm{P} - \nabla\Phi + \Omega^2\mathbf{R} + \text{viscous terms} \tag{38}$$

Keeping gravity + rotation

$$0 = -\nabla\Phi + \Omega^2\mathbf{R} \tag{39}$$

Gives Thin Keplerian disks

Euler equation and the disks

Radial momentum equation

$$(\mathbf{v}_\mathrm{p} \cdot \nabla)\mathbf{v}_\mathrm{p} = -\frac{1}{\rho}\nabla \mathrm{P} - \nabla\Phi + \Omega^2\mathbf{R} + \text{viscous terms} \tag{40}$$

Keeping gravity + pressure

$$0 = -\nabla\Phi - \frac{1}{\rho}\nabla \mathrm{P} \tag{41}$$

Gives Stars, stellar envelopes and atmospheres

Euler equation and the disks

Radial momentum equation

$$(\mathbf{v}_{\mathrm{p}} \cdot \nabla)\mathbf{v}_{\mathrm{p}} = -\frac{1}{\rho}\nabla \mathrm{P} - \nabla \Phi + \Omega^2 \mathbf{R} + \text{viscous terms} \tag{42}$$

Keeping advection + gravity

$$(\mathbf{v}_{\mathrm{p}} \cdot \nabla)\mathbf{v}_{\mathrm{p}} = -\nabla \Phi \tag{43}$$

Gives Gravitational collapse, dust in free-fall

Euler equation and the disks

Radial momentum equation

$$(\mathbf{v}_\mathrm{p} \cdot \nabla)\mathbf{v}_\mathrm{p} = -\frac{1}{\rho}\nabla \mathrm{P} - \nabla\Phi + \Omega^2\mathbf{R} + \text{viscous terms} \tag{44}$$

Keeping advection + gravity + rotation

$$(\mathbf{v}_\mathrm{p} \cdot \nabla)\mathbf{v}_\mathrm{p} = -\nabla\Phi + \Omega^2\mathbf{R} \tag{45}$$

Gives Slim disks

Euler equation and the disks

Radial momentum equation

$$(\mathbf{v}_\mathrm{p} \cdot \nabla)\mathbf{v}_\mathrm{p} = -\frac{1}{\rho}\nabla \mathrm{P} - \nabla \Phi + \Omega^2 \mathbf{R} + \text{viscous terms} \tag{46}$$

Keeping pressure + gravity + rotation

$$0 = -\frac{1}{\rho}\nabla \mathrm{P} - \nabla \Phi + \Omega^2 \mathbf{R} \tag{47}$$

Gives Thick disks and tori

Euler equation and the disks

Radial momentum equation

$$(\mathbf{v}_{\mathrm{p}} \cdot \nabla)\mathbf{v}_{\mathrm{p}} = -\frac{1}{\rho}\nabla \mathrm{P} - \nabla\boldsymbol{\Phi} + \Omega^2\mathbf{R} + \text{viscous terms} \tag{48}$$

Keeping advection + pressure + gravity

$$(\mathbf{v}_{\mathrm{p}} \cdot \nabla)\mathbf{v}_{\mathrm{p}} = -\frac{1}{\rho}\nabla \mathrm{P} - \nabla\boldsymbol{\Phi} \tag{49}$$

Gives Bondi-Hoyle accretion

Euler equation and the disks

Radial momentum equation

$$(\mathbf{v}_\mathrm{p} \cdot \nabla)\mathbf{v}_\mathrm{p} = -\frac{1}{\rho}\nabla \mathrm{P} - \nabla \Phi + \Omega^2 \mathbf{R} + \text{viscous terms} \tag{50}$$

Only viscous terms neglected

$$(\mathbf{v}_\mathrm{p} \cdot \nabla)\mathbf{v}_\mathrm{p} = -\frac{1}{\rho}\nabla \mathrm{P} - \nabla \Phi + \Omega^2 \mathbf{R} \tag{51}$$

Gives Sun-Keplerian ADAFs, boundary layers

What to read?

- Arnab Rai Choudhuri " The Physics of Fluids and Plasmas: An Introduction for Astrophysicists " + lecture notes by Rami Vainio
- Astrophysical Gasdynamics Lecture Notes by Garrelt Mellema
- Lecture notes "Astrophysical fluid dynamics" by Gordon Ogilvie, arxiv: 1604.03835
- Balbus, Potter "Surprises in astrophysical gasdynamics", arxiv: 1603.06489