Shock waves in plasma physics

Tuomo Salmi

University of Turku

thjsal@utu.fi

November 24, 2017

Contents

Introduction

- Shock wave: A region of a small thickness over which different fluid dynamical variables change rapidly
- Fluid quantities change on lengthscales on the order of the mean free path λ and useful to regard it as a mathematical discontinuity in the fluid
- Shock driven by either Coloumb collisions or plasma instabilities

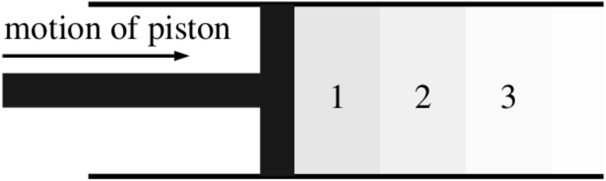
Introduction

- Situations where the bulk velocity of a gas flow makes a transition between supersonic and subsonic values
- Example: Steady adiabatic spherical, accretion
- Transition to a subsonic ‘settling’ flow, in which the gas velocity becomes very small near the stellar surface

Derivation of jump conditions

- First consider the reference frame moving with the gas velocity v, the surface of the star is pushing into the gas at a supersonic speed v_{ff}.
- Simplified geometry (see figure): A piston being pushed into a tube of gas.
- As the piston continues to accelerate into the gas, a further signal travels into the already compressed gas and acts to compress it further.
- $c_{\mathrm{ad}}^{\mathrm{s}} \propto \rho$. Hence denser zones to the left continually try to catch up the lower-density zones to the right; the compression of the left hand zones continually increases, and the density profile in front of the piston therefore steepens.

Compression of gas in a long cylinder or shock tube



The formation of a shock wave in a gas

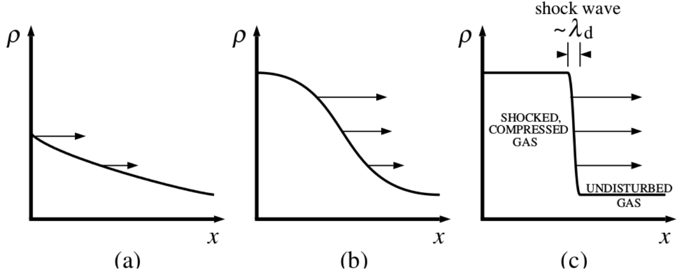


Fig. 3.7. The formation of a shock wave in a gas. (a) The denser gas in zone 1 tends to overtake the dilute gas in zone 2 since its adiabatic sound speed is greater; (b) as a result, the density gradient, and hence the overtaking tendency, increase, leading (c) to a shock wave of thickness of the order of the mean free path of the gas particles.

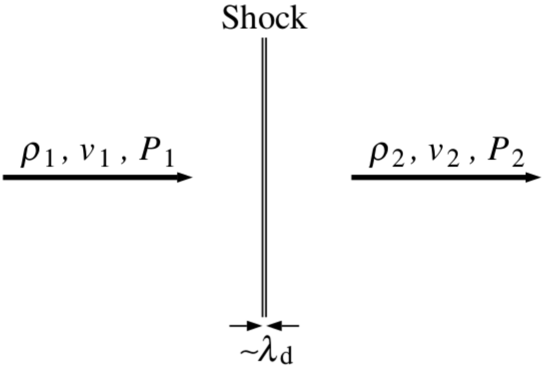
Derivation of jump conditions

- Steepening of the density profile cannot continue indefinitely: Bulk viscous forces will become important once the density scale length, $L = \rho/|\nabla\rho|$, becomes of the order of the mean free path, λ_{d}.
- Gas dissipates energy: i.e. ordered kinetic energy gets converted by viscous processes into chaotic thermal motions.
- $\frac{\text{Viscous force}}{\text{Pressure force}} \approx \frac{v}{c_{\mathrm{s}}}\frac{\lambda_{\mathrm{d}}}{L}$
- However, since the shock thickness ($\approx \lambda_d$) is much smaller than the lengthscales of gradients in the gas on each side of it, we can approximate the shock as a discontinuity in the gas flow.

Derivation of jump conditions

- Aim is to find connection between the gas density, velocity and pressure (or temperature) across this idealized discontinuity by applying conservation laws.
- Choose reference frame where shock is at rest.
- Shock thickness is so small, we can regard it as locally plane.
- We regard the flow into and out of the shock as steady.
- Gass flows into the shock with density, velocity and pressure ρ_1, v_1, P_1, and emerges with ρ_2, v_2, P_2.
- Task: Integrate the laws of mass, momentum and energy conservation accross the discontinuity (shock front).

Derivation of jump conditions



Derivation of jump conditions

- From continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \implies \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho v) = 0, \tag{2}$$

integrates to

$$\rho_1 v_1 = \rho_2 v_2 = J. \tag{3}$$

- Mass flux $J = \rho v$ is a conserved quantity.

Derivation of jump conditions

- From Euler equation:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P + \mathbf{f} \implies \tag{4}$$

$$\rho v\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}P}{\mathrm{d}x} = f_x \implies \tag{5}$$

(using also eq. (2) from mass conservation):

$$\frac{\mathrm{d}}{\mathrm{d}x}(P + \rho v^2) = f_x \implies \tag{6}$$

$$(P_1 + \rho_1 v_1^2) - (P_2 + \rho_2 v_2^2) = \lim_{\mathrm{d}x\to 0}\int_{-\mathrm{d}x}^{\mathrm{d}x} f_x \mathrm{d}x = 0 \implies \tag{7}$$

Derivation of jump conditions

- From Euler equation:

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2 = I, \tag{8}$$

where the conserved quantity $I = P + \rho v^2$ is the momentum flux.

Derivation of jump conditions

- Finally from the energy equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho\epsilon\right)+\nabla\left[\left(\frac{1}{2}\rho v^2+\rho\epsilon+P\right)\mathbf{v}\right]=\mathbf{f}\cdot\mathbf{v}-\nabla\cdot\mathbf{F}_{\mathrm{rad}}-\nabla\cdot\mathbf{q} \implies \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[v\left(\frac{1}{2}\rho v^2+\rho\epsilon+P\right)\right]=f_x v \tag{10}$$

assuming that radiative losses and thermal conduction, etc., across the shock front are negligible. This is the adiabatic assumption, and the resulting jump conditions are known as the adiabatic shock conditions.

Derivation of jump conditions

- Using conservation of mass flux ($\rho v =$ constant) and $\epsilon = \frac{3}{2} P/\rho$, energy equation becomes (for monatomic gas):

$$\rho v \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2} v^2 + \frac{5}{2}\frac{P}{\rho}\right] = f_x v \tag{11}$$

After integrating over integrating over the shock front (using partial integration, again $\rho v =$ constant, and that $\int f_x v \mathrm{d}x = 0$ over the shock):

$$\frac{1}{2} v_1^2 + \frac{5}{2}\frac{P_1}{\rho_1} = \frac{1}{2} v_2^2 + \frac{5}{2}\frac{P_2}{\rho_2} \equiv E, \tag{12}$$

the conserved quantity E is the specific total energy, where account is taken of the work done by pressure in compressing the gas across the shock.

Rankine–Hugoniot conditions

- The three conservation equations are known as Rankine–Hugoniot conditions; they give three equations to determine the jumps in ρ, v and P:

$$\rho_1 v_1 = \rho_2 v_2 = J. \tag{13}$$

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2 = I, \tag{14}$$

$$\frac{1}{2} v_1^2 + \frac{5}{2} \frac{P_1}{\rho_1} = \frac{1}{2} v_2^2 + \frac{5}{2} \frac{P_2}{\rho_2} = E, \tag{15}$$

Rankine–Hugoniot relations

Using a little algebra, we can solve the jumps in ρ, *v* and *P*:

$$\frac{I}{Jv} = \frac{P}{\rho v^2} + 1.$$

Defining the Mach number $\mathcal{M}$ by

$$\mathcal{M}^2 = \frac{v^2}{(c_{\mathrm{s}}^{\mathrm{ad}})^2} = \frac{3}{5}\frac{\rho v^2}{P}, \tag{3.58}$$

we get

$$\frac{I}{Jv} = \frac{3}{5\mathcal{M}^2} + 1. \tag{3.59}$$

Rankine–Hugoniot relations

Combining equations (13),(14) and (15):

$$E = \frac{v^2}{2} + \frac{5}{2}\left(\frac{Iv}{J} - v^2\right)$$

or

$$v^2 - \frac{5}{4}\frac{I}{J}v + \frac{E}{2} = 0.$$

This is a quadratic equation for v, and therefore has two roots. Since the quadratic was derived by manipulation of the conservation laws, these roots can only be v_1, v_2. Hence, $v_1 + v_2$ is given by the sum of the roots of the quadratic:

$$v_1 + v_2 = \frac{5}{4}\frac{I}{J},$$

i.e.

$$1 + \frac{v_2}{v_1} = \frac{5}{4}\frac{I}{Jv_1} = \frac{5}{4}\left[\frac{3}{5\mathcal{M}_1^2} + 1\right] \tag{3.60}$$

using (3.59). Here $\mathcal{M}_1 = v_1/c_\mathrm{s}^\mathrm{ad}(1)$ is the upstream Mach number.

Rankine–Hugoniot relations in a strong shock

- In the limit of strong shock $\mathcal{M}_1 \gg 1$:

$$\frac{v_2}{v_1} = \frac{1}{4} \tag{16}$$

and

$$\frac{\rho_2}{\rho_1} = 4 \tag{17}$$

- If a disturbance tries to compress gas by a factor 4, that will give rise to a shock wave moving at infinite speed.

Rankine–Hugoniot relations in a strong shock

- Or in case of adiabatic index γ other than $\frac{5}{3}$:

$$\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{(2/\mathcal{M}_1^2) + (\gamma - 1)} \tag{18}$$

$$\frac{\rho_2}{\rho_1} \rightarrow \frac{\gamma + 1}{\gamma - 1}, \tag{19}$$

when $\mathcal{M}_1 \rightarrow \infty$.
For pressure:

$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma - 1)}{\gamma + 1} \tag{20}$$

Pressure in a strong shock

- Back to the adiabatic case:
- In the limit of strong shock $\mathcal{M}_1 \gg 1$:

$$I = \rho_1 v_1^2 \left(1 + \frac{3}{5\mathcal{M}_1^2}\right) \cong \rho_1 v_1^2 \tag{21}$$

- Thus, ahead of a strong shock the thermal pressure P_1 is negligible, compared to the ram pressure $\rho_1 v_1^2$

Pressure in a strong shock

- For the pressure behind the shock, we get

$$P_2 = \rho_1 v_1^2 - \rho_2 v_2^2 = \rho_1 v_1 (v_1 - v_2) \implies \tag{22}$$

$$P_2 = \frac{3}{4} \rho_1 v_1^2, \tag{23}$$

  since $v_2 = \frac{1}{4} v_1$.
- Behind the shock the thermal pressure is 3/4 of the ram pressure. Clearly, what has happened is that the dissipation processes (e.g. viscosity) operating in the shock front have converted most of the kinetic energy of the ordered, supersonic flow into heat – i.e. random motion of the post-shock gas particles.

The post shock gas

- The post-shock gas is subsonic, since

$$c_{\mathrm{s}}(2) = \left(\frac{5P_2}{3\rho_2}\right)^{1/2} = \left(\frac{5\rho_1 v_1^2}{16\rho_1}\right)^{1/2} = \frac{\sqrt{5}}{4} v_1 > v_2 = \frac{v_1}{4}. \quad (24)$$

- The temperature of the post-shock (with perfect gas law and v_1 as free-fall velocity):

$$T_2 = \frac{\mu m_{\mathrm{H}} P_2}{k\rho_2} = \frac{3}{16}\frac{\mu m_{\mathrm{H}}}{k} v_1^2 = \frac{3}{8}\frac{GM\mu m_{\mathrm{H}}}{kR_*} \quad (25)$$

  This is close to thermal temperature approximation:

$$T_{\mathrm{th}} = \frac{GMm_{\mathrm{p}}}{3kR_*}. \quad (26)$$

- Thus, in a strong shock the ordered kinetic energy of the supersonic pre-shock gas is converted to random thermal motions; the post-shock ion thermal velocity ($\approx c_{\mathrm{s}}(2)$) is of the order of the pre-shock bulk velocity.

Discussion

- The steepening of sound waves into shocks was inevitable, as the sound speed increased with density, and the density continually increased ahead of the pressure disturbance. In more complicated situations this need not happen, as dissipation and geometrical expansion can reverse both tendencies (e.g. every day sound waves).
- Transport processes (e.g., viscosity and heat conduction) determine the physcial thickness of the shock wave by limiting the steepening process.

The End