Steady Spherical Accretion

Juhani Mönkkönen

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Example

Isolated star accreting from the interstellar medium.
Super-Eddington accretion?

Consider

- Spherical symmetry $\Rightarrow$ spherical coordinates
- Steady $\Rightarrow$ no need for time, $\dot{M}$ const.
- Neglect magnetic fields, bulk momentum of star with respect to the star, ...

Continuity equation

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and further integrates into

$$r^2 \rho v = \text{const.}$$

This can be connected to the mass accretion rate:

$$\boxed{4\pi r^2 \rho(-v) = \dot{M}} \tag{3}$$

Euler equation

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \mathbf{f} \tag{4}$$

The only contribution to force density from gravity:

$$f_r = -GM\rho/r^2 \tag{5}$$

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So we can simplify (4) into

$$\boxed{v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2} = 0} \tag{6}$$

Additional relationship between variables, from the polytrope equation:

$$P = K\rho^{\gamma}, K = \text{const.} \tag{7}$$

The temperature can be solved by assuming ideal gas law:

$$T = \mu m_{\mathrm{H}} P/\rho k \tag{8}$$

Transforming the Euler equation

Use the definition of sound speed with the chain rule:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}P}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = c_s^2\frac{\mathrm{d}\rho}{\mathrm{d}r} \tag{9}$$

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To get $\frac{\mathrm{d}\rho}{\mathrm{d}r}$, we start from the continuity equation (2) with the derivation of a product

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} vr^2 + \rho \frac{\mathrm{d}}{\mathrm{d}r}\left(vr^2\right) = 0 \tag{10}$$

and can solve

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{1}{vr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(vr^2\right) \tag{11}$$

Transforming the Euler equation

Bring these to the middle term in the Euler equation (4) and get:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{c_s^2}{vr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(vr^2\right) + \frac{GM}{r^2} = 0 \tag{12}$$

which can be manipulated into:

$$\boxed{\frac{1}{2}\left(1 - \frac{c_s^2}{v^2}\right)\frac{\mathrm{d}}{\mathrm{d}r}v^2 = -\frac{GM}{r^2}\left[1 - \left(\frac{2c_s^2 r}{GM}\right)\right]} \tag{13}$$

Solutions of equation (13)

$$\frac{1}{2}\left(1-\frac{c_s^2}{v^2}\right)\frac{\mathrm{d}}{\mathrm{d}r}v^2=-\frac{GM}{r^2}\left[1-\left(\frac{2c_s^2 r}{GM}\right)\right] \tag{13}$$

For large r: $v^2 < c_s^2$ (subsonic)
Sonic point r_s: either $v^2 = c_s^2$ or $\frac{\mathrm{d}}{\mathrm{d}r}\left(v^2\right) = 0$ at $r = r_s$

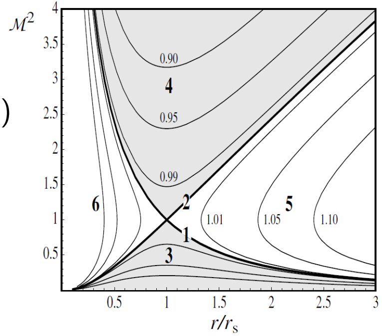
$$r_s = \frac{GM}{2c_s(r_s)^2} \sim 7.5\times 10^{13}\left(\frac{M}{M_\odot}\right)\left(\frac{10^4\mathrm{K}}{T(r_s)}\right)\mathrm{cm} \tag{14}$$

Therefore can be expected at small r: $v^2 > c_s^2$ (supersonic)

Solutions of equation (13)

Mach number vs. radius

1. Bondi accretion ($v < 0$)
2. Parker solar wind ($v > 0$)
3. sinking 'atmosphere' ($v < 0$) or stellar 'breeze' ($v > 0$)
4. everywhere supersonic
5. unphysical
6. unphysical



Integrating the Euler equation

Using the polytrope equation (7) if $\gamma \neq 1$:

$$\int \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r = \int \frac{\mathrm{d}P}{\rho} = \frac{K\gamma}{\gamma - 1}\rho^{\gamma-1} + C = \frac{c_s^2}{\gamma - 1} + C,$$

where the constant C we get from the condition at infinity, $C = -\frac{c_s(\infty)^2}{\gamma-1}$

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And we get the Bernoulli integral:

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma-1} - \frac{GM}{r} = \frac{c_s^2(\infty)}{\gamma-1} \tag{15}$$

(Note that for $v \gg c_s$, accretion happens effectively in free fall)

Sound speed and mass accretion rate

From result (15), we see that

$$c_s(r_s) = c_s(\infty)\left(\frac{2}{5-3\gamma}\right)^{1/2} \tag{16}$$

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Since $\gamma K \rho^{\gamma-1} = c_s^2$,

$$\rho(r_s) = \rho(\infty)\left(\frac{c_s(r_s)}{c_s(\infty)}\right)^{2/(\gamma-1)} \tag{17}$$

Mass accretion rate at the sonic point can be combined with the above results into

$$\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_s(\infty)^3}\left(\frac{2}{5-3\gamma}\right)^{(5-3\gamma)/2(\gamma-1)} \tag{18}$$

Velocity solvable as well

Mass accretion rate and accretion radius

Numerical values for the mass accretion rate:

$$\dot{M} \cong 1.4 \times 10^{11} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\rho(\infty)}{10^{-24}}\right) \left(\frac{c_s(\infty)}{10\text{km s}^{-1}}\right)^{-3} \text{g s}^{-1} \quad (19)$$

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Define that at the accretion radius, the gas velocity is the sound speed at infinity:

$$r \cong r_{\text{acc}} = \frac{2GM}{c_s(\infty)^2} \cong 3 \times 10^{14} \left(\frac{M}{M_\odot}\right) \left(\frac{10^4\text{K}}{T(\infty)}\right) \text{cm} \quad (20)$$

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Mass accretion with the accretion radius:

$$\dot{M} \sim \pi r_{\mathrm{acc}}^2 c_s(\infty) \rho(\infty) \quad (21)$$