Time dependence and stability

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Contents

Introduction

- Reasons for extending the study to time dependent discs:
  - Check that the steady-state models are stable against small perturbations
  - Observations of steady discs are unlikely to give much information about the viscosity (unlike time-dependent discs)
- The main area for the study: Outbursts of dwarf novae

Time scales

- Viscous time scale (timescale on which matter diffuses through the disc under the effect of the viscous torques):

$$t_{\mathrm{visc}} \sim \frac{R^2}{\nu} \sim \frac{R}{v_{\mathrm{R}}} \tag{1}$$

- Dynamical time scale (shortest characteristic timescale of the disc):

$$t_{\phi} \sim \frac{R}{v_{\phi}} \sim \Omega_{\mathrm{K}}^{-1} \tag{2}$$

Time scales

- Deviations from hydrostatic equilibrium in the z-direction are smoothed out on a timescale:

$$t_z = \frac{H}{c_s} \sim \frac{R}{\mathcal{M} c_s} = \frac{R}{v_\phi} \sim t_\phi \tag{3}$$

- Thermal time scale (timescale for re-adjustment to thermal equilibrium, if, say, the dissipation rate is altered):

$$\begin{aligned} t_{\rm th} &= \frac{\text{heat content per unit disc area}}{\text{dissipation rate per unit disc area}} \\ &\sim \frac{\Sigma c_s^2}{D(R)} \sim \frac{R^3 c_s^2}{GM\nu} \sim \frac{c_s^2 R^2}{v_\phi^2 \nu} \sim \mathcal{M}^{-2} t_{\rm visc} \end{aligned} \tag{4}$$

Using α-parametrization and equations for thin disc, also $t_{\rm visc}$ and t_ϕ can be related:

$$t_{\rm visc} \sim \alpha^{-1} \mathcal{M}^2 t_\phi \tag{5}$$

Collecting results we have

$$t_\phi \sim t_z \sim \alpha t_{\rm th} \sim \alpha (H/R)^2 t_{\rm visc}. \tag{5.68}$$

If we assume $\alpha \lesssim 1$ there is thus a well-defined hierarchy of timescales $t_\phi \sim t_z \lesssim t_{\rm th} \ll t_{\rm visc}$. Numerically, we have, for the α-disc solutions (5.49),

$$\left.\begin{array}{l} t_\phi \sim t_z \sim \alpha t_{\rm th} \sim 100 m_1^{-1/2} R_{10}^{3/2} \text{ s}, \\ t_{\rm visc} \sim 3 \times 10^5 \alpha^{-4/5} \dot{M}_{16}^{-3/10} m_1^{1/4} R_{10}^{5/4} \text{ s}. \end{array}\right\} \tag{5.69}$$

Thus the dynamical and thermal timescales are of the order of minutes, and the viscous timescale of the order of days to weeks for typical parameters.

Implications

- Consider small perturbations: If perturbation continues to grow rather than being damped, the supposed steady solution is said to be unstable and cannot occur in reality.
- Different time scales -> Different types of instabilities
- If for example the energy balance is disturbed in the disc, any instability will grow on a timescale $t_{\rm th}$, which is much less than $t_{\rm visc}$.
- We can assume Σ (changes in $t_{\rm visc}$) being fixed during the growth of thermal instability ($t_{\rm th}$).
- Vertical structure is close to hydrostatic equilibrium, since $t_{\rm th} > t_{\phi} \sim t_{\rm z}$ for $\alpha < 1$ (disc responds rapidly).

Thermal instability

- Arise when the local (volume) cooling rate Q^- (erg $\mathrm{s}^{-1}\mathrm{cm}^{-3}$) within the disc can no longer cope with the (volume) heating rate $Q^+ \sim D/H$ due to viscous dissipation.
- If central temperature T_c is increased by small perturbation ΔT_c, and Q^+ increases faster than Q^-, a thermal instability would grow.

Thermal instability

- The condition for thermal stability in one example:
- Assume that the disc density is so low that the vertical optical depth $\tau < 1$, and cooling rate equals thus the emissivity $4\pi j$.
- Emissivity depends on the square of the density, and has the form $4\pi j \propto \rho^2 \Lambda(T_c)$. $\Lambda(T_c)$ is a function called the cooling curve, which depends principally on how the ionization balance of the gas is maintained. It determines largely whether the instability grows or not.

$$Q^- = 4\pi j \propto \rho^2 \Lambda \sim (\Sigma/H)^2 \Lambda \sim c_s^{-2} \Lambda \sim T_c^{-1} \Lambda,$$

where we have assumed negligible radiation pressure in the last step. Similarly, we have

$$Q^+ \sim D/H \sim \nu/H \sim \alpha c_s \sim \alpha T_c^{1/2},$$

Thermal instability

- We get for the criterion:

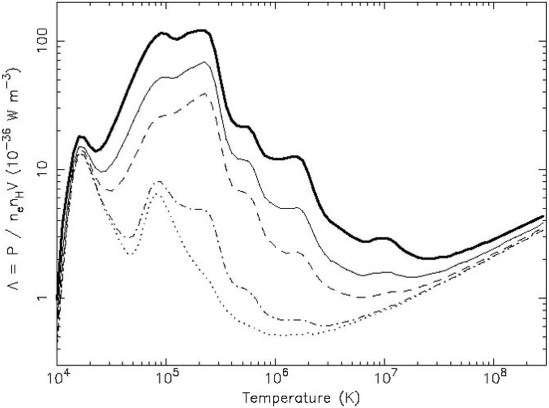
$$\frac{\mathrm{d}\ln(\Lambda/\alpha)}{\mathrm{d}\ln T_\mathrm{c}} < \frac{3}{2},$$

or, for the general case, in the alternative forms,

$$\left.\begin{array}{l} \dfrac{\mathrm{d}Q^-}{\mathrm{d}T_\mathrm{c}} < \dfrac{\mathrm{d}Q^+}{\mathrm{d}T_\mathrm{c}} \\ \dfrac{\mathrm{d}\ln Q^-}{\mathrm{d}\ln T_\mathrm{c}} < \dfrac{\mathrm{d}\ln Q^+}{\mathrm{d}\ln T_\mathrm{c}} \end{array}\right\} \Rightarrow \text{unstable.}$$

Thermal instability

- We expect it to happen for thin discs above $T_c = 10^4$ (if α does not depend on T_c), since cooling of a plasma by thermal radiation for different compositions (Sutherland, R.S., & Dopita, M.A., 1993, ApJS, 88, 253):



Thermal instability

In contrast to the optically thin case, for optically thick discs the *effective* volume cooling rate (taking account of the local re-absorption of radiant energy) is

$$Q^- = \frac{\mathrm{d}F}{\mathrm{d}z} \sim \frac{\sigma T_\mathrm{c}^4}{\kappa_\mathrm{R} \rho H^2}$$

by (5.39) and (5.37). For Kramers' opacity (5.53) this can be rewritten as

$$Q^- \sim \frac{T_\mathrm{c}^{15/2}}{\rho^2 H^2} = \frac{T_\mathrm{c}^{15/2}}{\Sigma^2} \sim T_\mathrm{c}^{15/2}$$

- Since Q^+ is same as before, the condition for thermal instability is now for optically thick case

$$\frac{d \ln \alpha}{d \ln T_c} > 7. \tag{6}$$

Viscous instability

- Consider changes in the disc structure taking place on the viscous timescale (viscous instabilities and the evolution of discs in response to changes in external conditions, such as the mass transfer rate).
- Since viscous time scale is the slowest, we may assume both thermal and hydrostatic equilibrium.
- Hence, some of equations describing steady discs apply also to time-dependent discs-

The old equations

$$\left.\begin{array}{l}
1.\ \rho = \Sigma / H; \\
2.\ H = c_{\rm s} R^{3/2} / (GM)^{1/2}; \\
3.\ c_{\rm s}^2 = P/\rho; \\
4.\ P = \dfrac{\rho k T_{\rm c}}{\mu m_{\rm p}} + \dfrac{4\sigma}{3c} T_{\rm c}^4; \\
5.\ \dfrac{4\sigma T_{\rm c}^4}{3\tau} = \dfrac{3GM\dot{M}}{8\pi R^3} \left[1 - \left(\dfrac{R_*}{R} \right)^{1/2} \right]; \\
6.\ \tau = \Sigma \kappa_{\rm R}(\rho, T_{\rm c}) = \tau(\Sigma, \rho, T_{\rm c}); \\
7.\ \nu\Sigma = \dfrac{\dot{M}}{3\pi} \left[1 - \left(\dfrac{R_*}{R} \right)^{1/2} \right]; \\
8.\ \nu = \nu(\rho, T_{\rm c}, \Sigma, \alpha, \ldots).
\end{array}\right\}$$

The new equations

$$
\left.
\begin{array}{lrcl}
1. & \rho & = & \Sigma/H; \\
2. & H & = & c_{\mathrm{s}} R^{3/2}/(GM)^{1/2}; \\
3. & c_{\mathrm{s}}^2 & = & P/\rho; \\
4. & P & = & \dfrac{\rho k T_{\mathrm{c}}}{\mu m_{\mathrm{p}}} + \dfrac{4\sigma}{3c} T_{\mathrm{c}}^4; \\
5. & \dfrac{4\sigma}{3\tau} T_{\mathrm{c}}^4 & = & \frac{9}{8} \nu \Sigma \frac{GM}{R^3}, \\
6. & \tau & = & \Sigma \kappa_{\mathrm{R}}(\rho, T_{\mathrm{c}}) = \tau(\Sigma, \rho, T_{\mathrm{c}}); \\
7. & \dfrac{\partial \Sigma}{\partial t} & = & \frac{3}{R} \frac{\partial}{\partial R} \left\{ R^{1/2} \frac{\partial}{\partial R} (\nu \Sigma R^{1/2}) \right\}, \\
8. & \nu & = & \nu(\rho, T_{\mathrm{c}}, \Sigma, \alpha, \ldots).
\end{array}
\right\}
$$

Viscous instability

- Again 8 unknowns and 8 equations, now as functions of R, t and any parameters (e.g. α) in the viscosity prescription.
- Because eq. 7 is a diffusion equation, boundary condition must be provided.
- The equations can be simplified and the time-dependent disc problem may be reduced to one involving fewer variables:
  - All unknowns may be presented as functions of Σ and R
  - Especially the set of equations fixes ν as a function of R and Σ, making eq. 7 a nonlinear diffusion equation for $\Sigma(R, t)$.

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} = \nabla \cdot \left[D(\phi, \mathbf{r}) \, \nabla \phi(\mathbf{r}, t) \right],$$

  - Except if ν would involve an explicit t-dependence.

Viscous instability

- Solving the nonlinear equation 7 and subsequently find all the other disc variables is in general a formidable task which must be tackled numerically.
- However, some conclusions can be made by considering asxisymmetrical perturbations at each R, so that $\Sigma = \Sigma_0 + \Delta\Sigma$, where $\Sigma_0(R)$ is the steady-state distribution.
- Writing $\mu = \nu\Sigma$ we can obtain an equation governing the growth of perturbations:

$$\frac{\partial}{\partial t}(\Delta\mu) = \frac{\partial\mu}{\partial\Sigma}\frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(R^{1/2}\Delta\mu)\right].$$

Viscous instability

$$\frac{\partial}{\partial t}(\Delta\mu) = \frac{\partial\mu}{\partial\Sigma}\frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(R^{1/2}\Delta\mu)\right].$$

- Not surprisingly, $\Delta\mu$ obeys a diffusion equation, but the interesting thing is that the diffusion coefficient is proportional to $\partial\mu/\partial\Sigma$ and in principle can be either positive or negative.
- If $\partial\mu/\partial\Sigma$ is positive, perturbation decays on a viscous timescale, otherwise viscous instability.

Viscous instability

- $\mu = \nu\Sigma \propto \dot{M} \propto T^4(R)$.
- In case of negative $\partial\mu/\partial\Sigma$ (or $\partial\dot{M}/\partial\Sigma$ or $\partial T(R)/\partial\Sigma$):
- More material will be fed into those regions of the disc that are denser than their surroundings and material will be removed from those regions that are less dense, so that the disc will tend to break up into rings.
- To check the viscous stability of accretion discs: check the relation between T^4 and Σ at a given disc radius R.

Viscous instability

- Where gas pressure dominates (and the opacity goes as $\kappa_R \propto \rho T_c^n$):

$$T \propto \Sigma^{\frac{13-2n}{4(7-2n)}} \tag{7}$$

- Disc is unstable when $7/2 < n < 13/2$.
- This will always occur in hydrogen ionization zones, i.e. wherever T is close to the local hydrogen ionization temperature $T_\mathrm{H} \sim 6500$ K (below that $n > 13/2$ and above that Kramers' opacity with $n = -3.5$)

Viscous instability

- Where radiation pressure dominates (and with pure scattering opacity $\kappa_R =$ constant):

$$T \propto \Sigma^{-1/4}.$$

Thus again we have $\partial T/\partial\Sigma < 0$, and a viscous instability, this time known as the Lightman–Eardley instability. While the T–Σ curve will have a stable branch with positive slope at lower T, there is formally no such branch for large T. In other

Viscous instability

- What really happens when some reagion of the disc is viscously unstable?
- Disc structure evolves only on the viscous timescale, while maintaining dynamical and thermal equilibrium.
- One of these assumptions must break down, allowing the disc to evolve on a shorter timescale which then permits it to reach a new, stable, viscous equilibrium.
- Indeed thermal balance condition fails exactly when $\partial T(R)/\partial\Sigma < 0$.

Viscous+thermal instability

- The imbalance between heating and cooling:

$$\frac{\partial T}{\partial t} \propto Q^{+} - Q^{-}. \tag{5.79}$$

If $\partial T/\partial \Sigma < 0$ it is clear that a small perturbation tends to exaggerate the mismatch between Q^{+} and Q^{-}, and so causes a thermal instability.

- To understand what happens when $\partial T(R)/\partial \Sigma < 0$ we must allow for time variations of both T and Σ (for a given R):

$$\dot{T} = h(T, \Sigma) \tag{8}$$

$$\dot{\Sigma} = \epsilon k(T, \Sigma), \tag{9}$$

where $\epsilon = t_{\rm th}/t_{\rm visc}$.

- Equilibrium is found when $h = k = 0$, and this is called as a fixed point of the system in (T, Σ) phase plane.

Viscous + thermal instability

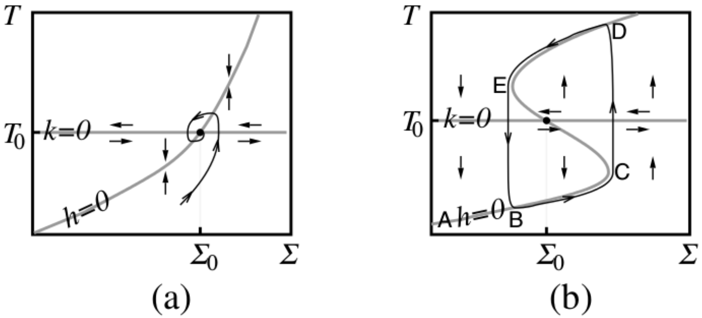


Fig. 5.12. Phase plane in the vicinity of a fixed point, indicated by a solid dot at the intersection of the two critical curves $h = 0$ and $k = 0$: (a) stable fixed point; (b) unstable fixed point and stable limit cycle.

Discussion

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The End