§ Singular value decomposition

Let V, W be inner product spaces on $\mathbb{R}$, $\dim V, \dim W < \infty$. Let T be a linear map from V to W. Then we can define its adjoint T^*, characterized by $(Tv|w)_W = (v \mid T^*w)_V, \forall v \in V, w \in W$.

Recall:

1. $\ker T = \ker(T^*T)$, since $\ker T \subset \ker(T^*T)$ is trivial, and

   $T^*Tv = 0 \Rightarrow (v \mid T^*Tv) = 0 \Rightarrow (Tv \mid Tv) = 0 \Rightarrow Tv = 0.$
2. $\mathrm{rk}T = \mathrm{rk}T^*$, since $\dim V = \dim \operatorname{im} T + \dim \ker T$.
3. TT^*, T^*T are both self-adjoint.
4. TT^*, T^*T are both positive semi-definite, since

   $$(w \mid TT^*w) = (T^*w \mid T^*w) \geq 0$$

   and T^*T is similar.

Also recall that given $V = \mathbb{R}^n, W = \mathbb{R}^m$, std basis $e_1, .., e_n$ and std inner product, we have

$$\mathrm{End}(V) = M_{n\times n}(\mathbb{R}), \quad \mathrm{Hom}(V, W) = M_{m\times n}(\mathbb{R}), \quad T \leftrightarrow \text{matrix } A, \quad T^* \leftrightarrow A^T$$

Theorem(SVD) Set $m = \dim V, n = \dim W, T : V \to W$ linear map, then
$\exists$ ONB $v_1, ..v_n$ of $V, w_1, ..w_m$ of W and $\sigma_1 \geq .. \geq \sigma_p \geq 0$ where $p = \min(n, m)$ such that

$$Tv_i = \begin{cases} \sigma_i w_i \text{ if } 1 \leq i \leq p \\ 0 \text{ if } i > p \end{cases}$$

and σ_i is unqiuely determined by T, called the **singular values** of T.

In terms of matrices, for some $A \in M_{n\times m}(\mathbb{R}), \exists P \in M_{m\times m}(\mathbb{R}), Q \in M_{n\times n}(\mathbb{R})$, both orthogonal (i.e. $P^{-1} = P^T$), such that

$$Q^{-1}AP = \mathrm{diag}(\sigma_1, ..\sigma_p, 0, .., 0), \quad \sigma_1 \geq .. \geq \sigma_p \geq 0 : \text{unique}$$

Proof

Uniqueness: we claim that $T^*w_j = \sigma_j v_j$ satisfies the definition of T^*. This can be shown by comparing

$$\begin{aligned} (v_i \mid T^*w_j) &= (v_i \mid \sigma_j v_j) = \sigma_j \delta_{i,j} \\ (Tv_i \mid w_j) &= (\sigma_i w_i \mid w_j) = \sigma_i \delta_{i,j} \end{aligned}$$

In fact, both inner products $= \sigma_i \delta_{i,j}$, which proves our previous claim. Next we derive

$$\begin{aligned} \text{Claim} &\Rightarrow T^*Tv_i = T^*(\sigma_i w_i) = \sigma_i^2 v_i \\ &\Rightarrow \sigma_1^2, .., \sigma_p^2 \text{ are eigenvalues of } T^*T \text{ (counting mutiplicity)} \\ &\Rightarrow \text{unique!} \end{aligned}$$

Existence: we already know that T^*T is self-adjoint and positive semi-definite, therefore
$\exists$ ONB $v_1, .., v_m \in V, \lambda_1 \geq .. \geq \lambda_m \geq 0, \quad \text{s.t. } T^*Tv_i = \lambda_i v_i$.

Let $r := \mathrm{rk}(T^*T) = \mathrm{rk}(T) \leq p$, then $\lambda_1, .., \lambda_r > 0, \lambda_{r+1} = .. = 0$.

Set $\sigma_i := \sqrt{\lambda_i}$ and $w_i = \sigma_i^{-1}Tv_i$. Here w_i is indeed orthogonal:

$$(w_i \mid w_j) = \sigma_i^{-1}\sigma_j^{-1}(Tv_i \mid Tv_j) = \sigma_i^{-1}\sigma_j^{-1}(v_i \mid T^*Tv_j) = \sigma_i^{-1}\sigma_j(v_i \mid v_j) = \delta_{i,j}$$

Therefore we can extend w_i to an ONB of W, and it satisfies all requirements. ■

Remark: there are many good algorithms for calculating SVD, and it sees some applications in data science.

§ Moore-Penrose Generalized Inverse

Let V, W be inner product spaces on $\mathbb{R}$, $\dim V, \dim W < \infty$, and linear map $T : V \to W$.

Goal: seek some substitute of "T^{-1}", especially when T is not inversible.

Definition: Given $T : V \to W$. If $S : W \to V$ satisfies

- (MP1) $TST = T$,
- (MP2) $STS = S$,
- (MP3) $(TS)^* = TS$,
- (MP4) $(ST)^* = ST$,

then we say S is a **MP inverse** of T, and vice versa.

Example: T invertible $\leftrightarrow T^{-1}$ is (the) MP-inverse.

Theorem $\forall T : V \to W, \exists! S : W \to V$ that is a MP inverse of T.

Proof

Existence:

$$\forall v \in V, v = v' + v'' \text{ where } v' \in \ker T, v'' \in (\ker T)^\perp$$
$$\forall w \in W, w = w' + w'' \text{ where } w' \in \operatorname{im} T, w'' \in (\operatorname{im} T)^\perp$$

and all summands above are unique. For each $w \in W$, take any $v \in T^{-1}(w')$ and set $Sw = v''$. Under such definition, v'' only depends on $v + \ker T = T^{-1}(w')$, showing that S is well-defined. Can also check that S is linear, so we only need to show that S satisfies MP properties.

MP3:

$$\forall v \in V, STv = v'' \Rightarrow ST \text{ is orthogonal projection } V \to (\ker T)^\perp$$
$$\Rightarrow (ST)^* = ST \quad \text{(Prop 9.5.9)}$$

and MP4 is similar.

MP1: $TSTv = Tv'' = Tv$, MP2: $STSw = Sw' = Sw$.

Hence: S is a MP inverse of T!

Uniqueness: Let S, R be MP inverses of T.

$$S = STS = S(TS)^* = SS^*T^* = SS^*(TRT)^* = SS^*T^*R^*T^* = S(TS)^*(TR)^* = STSTR = STR$$
$$R = RTR = (RT)^*R = T^*R^*R = (TST)^*R^*R = T^*S^*T^*R^*R = (ST)^*(RT)^*R = STRTR = STR$$

Therfore $S = R$.■