Prop Let $T : V \to W, r := \text{rk}T$. From SVD we can get ONBs $\underline{v}, \underline{w}$ and singular values σ_i. Define $S : W \to V$,

$$Sw_j = \begin{cases} \sigma_j^{-1} v_j, & 1 \le j \le r \\ 0, & j > r \end{cases}$$

Then S is an MP inverse of T.

In terms of matrices, $\exists P \in M_{m\times m}(\mathbb{R}), Q \in M_{n\times n}(\mathbb{R})$, both orthogonal, $Q^{-1}AP = \text{diag}(\sigma_i)$. Define $S \leftrightarrow$ matrix B such that $P^{-1}BQ = \text{diag}(\sigma_i^{-1})$.

Proof Use the explicit definition construction of MP inverse $Sw = v''$.

$$\begin{aligned} &\forall w \in W, w = \sum_{j=1}^{n} a_j w_j, \operatorname{im} T = \langle w_1, .., w_r \rangle \\ \implies & w' = \sum_{j=1}^{r} a_j w_j, \quad \text{Set } v := \sum_{j=1}^{r} \sigma_j^{-1} a_j v_j, Tv = w' \\ & v \in \langle v_{r+1}, .., v_m \rangle^\perp = (\ker T)^\perp \\ \implies & v = 0 + v = v' + v'' \end{aligned}$$

Therfore the definition of S agrees with the construction of MP inverse of T. ■

§ Min-Max principle

Let V be an inner product space on $\mathbb{R}$. Let B be a symmetric bilinear form $V \times V \to \mathbb{R}$. Then $\exists! S \in \text{End}(V), S^* = S$ such that $B(v_1, v_2) = (v_1 \mid Sv_2)$. In fact, we can identify V with $\mathbb{R}^n$ + std IP.
Then we observe that

$$\exists! A \in M_{n\times n}(\mathbb{R}), B(v_1, v_2) = v_1^T A v_2 = v_1 \cdot A v_2.$$

Therefore S is the A in the equation above, and S is self-adjoint since B is symmetric.

$\exists$ ONB $\underline{v}$ of $V, \lambda_1 \ge .. \ge \lambda_n$ such that $Sv_i = \lambda_i v_i$.

A simple observation:

$$\lambda_1 = \max_{\|v\|=1} B(v,v), \lambda_n = \min_{\|v\|=1} B(v,v).$$

Proof Let $v = a_1 v_1 + .. + a_n v_n$. $\|v\| = 1 \iff \sum_{i=1}^n a_i^2 = 1$.

$$\lambda_n \le \left(B(v,v) = (v \mid Sv) = \sum_{i=1}^{n} a_i^2 \lambda_i \right) \le \lambda_1 . ■$$

Remark

- $\{v \in V \mid \|v\| = 1\}$ is compact (if $V \simeq \mathbb{R}^n$) and $B(v,v)$ is continuous $\implies$ extrema exists.
- Given $S \in \text{End}(V), S = S^*$, can show $\underbrace{\max_{\|v\|=1}(v \mid Sv)}_{\text{always exists}}$ attained at v : eigenvector of S (Set $V = \mathbb{R}^n$ and use Lagrange multipiers). Can get another real proof of diagonalizable theorem.
- $\max_{\|v\|=1} B(v,v) = \max_{v\neq 0} \frac{B(v,v)}{(v|v)}$, since $B(tv,tv) = t^2 B(v,v), (tv|tv) = t^2 (v|v)$.

Theorem(Courant-Fischer) Given $S : V \to W, S = S^*, \forall 1 \le k \le n := \dim V$:

$$\lambda_k = \min_{\substack{U\subset V,\\ \dim U = n-k+1}} \left\{ \max_{\|v\|=1, v\in U} B(u,v) \right\},$$

$$\lambda_k = \max_{\substack{U\subset V,\\ \dim U = k}} \left\{ \min_{\|v\|=1, v\in U} B(v,v) \right\}.$$

Proof Replace S by $-S$: min $\leftrightarrow$ max. Suffices to prove the first equation.

Assume $U \in V, \dim U = n - k + 1$.

$$\begin{aligned}\dim U \cap \langle v_1, .., v_k \rangle &= \dim U + k - \dim U \cup \langle v_1, .., v_k \rangle \\ &\geq \dim U + k - n \\ &= 1.\end{aligned}$$

So we can take $v = \sum_{i=1}^k a_i v_i \in U, \|v\| = 1$.

$$B(v,v) = (v|Sv) = \sum_{i=1}^k \lambda_i a_i^2 \geq \lambda_k \sum_{i=1}^k a_i^2 = \lambda_k.$$

$$\max_{\|v\|=1, v\in U} B(v,v) \geq \lambda_k \Longrightarrow \inf_{\dim U = n-k+1} \left\{ \max_{\|v\|=1, v\in U} B(v,v) \geq \lambda_k \right\}.$$

To show that minimum can be attained, take $U = \langle v_k, .., v_n \rangle. \forall v = \sum_{i=1}^n a_i v_i, \|v\| = 1$,

$$B(v,v) = \sum_{i=k}^n \sigma_i a_i^2 \leq \lambda_k \sum_{i=k}^n a_i^2 = \lambda_k.$$

Therefore $\max_{\|v\|=1, v\in U} B(v,v) = \lambda_k$ for this U. ■

This is also related to SVD.

Proposition Let V, W be IPSs on $\mathbb{R}$ with dim $< \infty$. Let T be a linear map $V \to W$ with singular values $\sigma_1 \geq .. \geq \sigma_i \geq ..$ and set $\sigma_i = 0$ when $i > \max\{\dim V, \dim W\}$. Then

$$\sigma_1 = \max_{v\neq 0} \frac{\|Tv\|}{\|v\|}, \qquad \sigma_{\dim V} = \min_{v\neq 0} \frac{\|Tv\|}{\|v\|}.$$

Proof Take $S = T^*T \in \mathrm{End}(V), S = S^*$ and let B be the bilinear form corresponding to S.

$$B(v_1, v_2) = (v_1|Sv_2) = (Tv_1|Tv_2), B(v,v) = \|Tv\|^2, \frac{B(v,v)}{(v|v)} = \left(\frac{\|Tv\|}{\|v\|}\right)^2.$$

Recall that $\sigma_1^2 \geq .. \geq \sigma_{\dim V}^2$ are the eigenvalues of S. From the simple observation above we proved this proposition. ■

§ Positive matrices

Let $A \in M_{m\times n}(\mathbb{R}), A \geq B$ means $\forall i, j, a_{ij} \geq b_{ij}. A > 0, A \geq 0$ is defined similarly, not to be mistaken by positive- or semi-positive definite.

Lemma $\forall A \in M_{m\times n}(\mathbb{R}), x \in \mathbb{R}^n = M_{n\times 1}(\mathbb{R})$.

- $A > 0, x \geq 0, x \neq 0 \Longrightarrow Ax > 0$,
- $A > 0, x \geq 0 \Longrightarrow Ax \geq 0$.

Proof The conclusion is obvious. ■

To study the eigenvalues and eigenvectors of positive matrices $A \in M_{m\times n}(\mathbb{R}), A > 0$, we give the following definition:

Definition For $A \in M_{m\times n}(\mathbb{R})$, its **spectral radius** $\rho(A) = \max_{\lambda\in\mathbb{C},\text{ eigenvalue}} |\lambda|$.

Theorem (Perron) Let $A \in M_{n\times n}(\mathbb{R}), A > 0$.
1. $\rho(A) > 0, \exists v \in \mathbb{R}^n, v > 0, Av = \rho(A)v$.
2. If μ : eigenvalue of A satisfies $\mu \neq \rho(A)$, then $|\mu| < \rho(A)$.
3. The $\rho(A)$-eigenspace of A has $\dim = 1$.
4. $\rho(A)$ is a single root of Char_A.

After adding more conditions, can get a more general version for $A \geq 0$ (Perron-Frobenius Theorem).

Lemma $A \in M_{n\times n}(\mathbb{R}), A > 0 \implies \exists v \in \mathbb{R}^n, v > 0, Av = \rho(A)v$.

Proof Let $S := \{x \in \mathbb{R}^n \mid \|x\| = 1, x \geq 0\}$. Consider the continuous map

$$\begin{aligned} L : S &\to \mathbb{R}_{>0} \\ x &\mapsto \min_{1\leq i\leq n, x_i\neq 0} \frac{(Ax)_i}{x_i} \end{aligned}$$

Since S is compact, $\exists v \in S$, s.t. L attains its maximum $\rho > 0$ at v. Here we claim that

$$Av = \rho v \wedge v > 0.$$

From $L(v) = \rho$ we know $Av \geq \rho v$. If $Av \neq \rho v, Av - \rho v \geq 0$, then $A(Av - \rho v) > 0$.

Therfore for some small ε,

$$A(Av - \rho v) \geq \varepsilon Av \implies A^2 v \geq (\varepsilon + \rho)Av \implies Aw \geq (\varepsilon + \rho)w$$

where $w = tAv, t \in \mathbb{R}, w \in S$. Hence we reach a contradiction $\implies Av = \rho v$.

$v = \rho^{-1} \underbrace{A}_{>0} \underbrace{v}_{\neq 0} \implies v > 0$. Also from definition of $\rho(A)$ we have $\rho \leq \rho(A)$.

To show that $\rho = \rho(A)$, let $\mu \in \mathbb{C}, w \in \mathbb{C}^n \setminus \{0\}$ s.t. $Aw = \mu w$.

For this equation we can show $\forall i, |\mu\|w_i| = |(\mu w)_i|$

$$= |\sum_{j=1}^{n} a_{ij} w_j| \leq \sum_j a_{ij} |w_j|.$$

Set $w' = (|w_1|, .., |w_n|) \in \mathbb{C}^n$, and scale it to ensure $\|w'\| = 1$.

$w' \geq 0, Aw' \geq \mu w', w' \in S \implies \rho(A) \geq \rho \geq L(w') \geq |\mu|$ for all eigenvalues $\mu \in \mathbb{C}$. Hence $\rho = \rho(A)$.■

Proof of Theorem (1) done. (2) continue the previous proof.

We already showed $\rho(A) = \max_{v\in S} L(v) \geq L(w') \geq |\mu|$, and we need to prove $|\mu| = \rho(A) \implies \mu = \rho(A)$. We reason that $|\mu| = \rho(A) \implies L(w') = \rho(A)$ from the inequality above, therefore

$$Aw' = \rho(A)w' \implies \forall i, \sum_j a_{ij} |w_j| = \rho(A) w_i = |\mu w_i| = |\sum_j a_{ij} w_j|$$

Hence all w_i are on the same ray in $\mathbb{C}$. Take some nonzero number c on this ray and $\forall i, c^{-1} w_i \in \mathbb{R}, c^{-1} w_i \geq 0$. This shows that μ is a real number. ■