Recall ε-Hermitian forms $B : V \times V \to \mathbb{C}, V$: $\mathbb{C}$-vector space.

With $V = \mathbb{C}^n$, they correspond to ε-Hermitian matrices $A \in M_{n\times n}(\mathbb{C}), A^\dagger = \varepsilon A$.

In this case, left radical of B = right radical of B.

Adjoint linear maps on $\mathbb{C}$ inner product space

Definition-Proposition Let $B_1 : V_1 \times V_1' \to \mathbb{C}, B_2 : V_2 \times V_2' \to \mathbb{C}$ be sesquilinear forms with all dim $< \infty$. Assume B_1 is nondegenerate.

- $\exists$ semi-linear map $\mathrm{Hom}(V_1, V_2) \to \mathrm{Hom}(V_2', V_1'), T \mapsto T^*$ characterized by

$$B_2(Tv_1, v_2') = B_1(v_1, T^* v_2').$$

- $\exists$ semi-linear map $\mathrm{Hom}(V_1', V_2') \to \mathrm{Hom}(V_2, V_1), T \mapsto {}^*T$ characterized by

$$B_2(v_1', Tv_2) = B_1\left({}^*Tv_1', v_2\right).$$

Proof

1. Repeat the proof in the linear case.
2. Reduce to the bilinear case by passing $\overline{V_1}, \overline{V_2}$.
3. Look at matrices. Up to $\simeq$, may assume $V_i = \mathbb{C}^{m_i}, V_i' = \mathbb{C}^{m_i'}$. Here

$$B_i \longleftrightarrow A_i \in M_{m_i \times m_i'}(\mathbb{C}), \qquad A_1 \text{ : invertible.}$$

For the right adjoint:

$$B_2(Tv_1, v_2') = (Tv_1)^\dagger A_2 v_2' = v_1^\dagger T^\dagger A_2 v_2' = v_1^\dagger A_1 \underbrace{A_1^{-1} T^\dagger A_2}_{T^*} v_2'.$$

For the left adjoint:

$$B_2(v_2, Tv_1') = v_2^\dagger A_2 T v_1' = v_2^\dagger A_2 T A_1^{-1} A_1 v_1' = \left(\underbrace{(A_2 T A_1^{-1})^\dagger}_{{}^*T} v_2^\dagger\right) A_1 v_1'. \blacksquare$$

Observation

1. $(ST)^* = T^* S^*$, check by using definition.
2. If B_1, B_2 both nondegenerate, then ${}^*(T^*) = T = \left({}^*T\right)^*$.
3. If $\varepsilon = \pm 1$, both B_1, B_2 are ε-Hermitian ($V_i = V_i'$), then $T^* = {}^*T$ since $B(Tv, w) = \varepsilon \overline{B(w, Tv)} = \varepsilon \overline{B({}^*Tw, v)} = \varepsilon^2 B\left(v, {}^*Tw\right) = B\left(v, {}^*Tw\right)$.

Definition Given a nondegenerate ε-Hermitian form $B : V \times V \to \mathbb{C}, T \in \mathrm{End}(V)$.

- If $T = T^*$, we say T is **self-adjoint**,
- If $T = -T^*$, we say T is **anti/skew-adjoint**.

In term of matrices, $V \longleftrightarrow \mathbb{C}^n, B \longleftrightarrow A \in M_{n\times n}(\mathbb{C})$ s.t. $A^\dagger = \varepsilon A$, and $T \in \mathrm{End}(V) = M_{n\times n}(\mathbb{C})$, then

$$T^* = \pm T \Longleftrightarrow A^{-1} T^\dagger A = \pm T.$$

In particular if $\varepsilon = 1, A = \mathrm{id}$ (the standard complex inner product), then

$$\text{T is self-adjoint} \Longleftrightarrow T^\dagger = T,$$
$$\text{T is anti/skew-adjoint} \Longleftrightarrow T^\dagger = -T.$$

Remark Let $c \in \mathrm{i}\mathbb{R} \setminus \{0\}$, then

$$B : V \times V \to \mathbb{C} \text{ is Hermitian} \iff cB \text{ is anti-Hermitian},$$
$$T \in \mathrm{End}(V) \text{ is self-adjoint} \iff cT \text{ is anti-adjoint}.$$

Definition Let V : $\mathbb{C}$-vs., $\dim V < \infty, B : V \times V \to \mathbb{C}$, nondegenerate, ε-Hermitian. $T \in \mathrm{End}(V)$ is **normal** iff $TT^* = T^*T$.

T is self-adjoint and anti-adjoint $\implies T$ is normal (and many other special cases).

§ Classification of Hermitian Forms

Recall Theory of quadratic forms say over $\mathbb{R}$,

1. Quadratic forms $\sum_{i,j} a_{ij}X_iX_j, a_{ij} = a_{ji}$,
2. Symmetric matrices $A \in M_{n\times n}(\mathbb{R}), A^T = A$,
3. Symmetric bilinear forms $B : V \times V \to \mathbb{R}$,

are all the same. As for equivalent classes:

1. $/_{\text{the linear change of variables}}$, i.e., $f \sim f'$ if $f = f'(y_1, ..., y_n)$, $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = C \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$, C invertible.
2. $A \sim A'$ if $\exists C \in M_{n\times n}(\mathbb{R})$ invertible, $A = C^T A' C$.
3. $B \sim B'$ if $B(v, v) = B'(Cv, Cv)$ where $C : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$.
4. $(V, B) \sim (V', B')$ if $\exists C : V \xrightarrow{\sim} V', B(v_1, v_2) = B'(Cv_1, Cv_2)$.

Now, analogue for Hermitian forms on $\mathbb{C}$-vs. of $\dim = n$.

- Classify all $(V, B), \dim V = n, B : V \times V \to C, \varepsilon$-Hermitian up to $\simeq$.
- Classify for $V = \mathbb{C}^n$ or $A \in M_{n\times n}(\mathbb{C}), A^\dagger = A$, up to $A \sim C^\dagger AC, C$ invertible.
- Classify all maps $f : \mathbb{C}^n \to \mathbb{C}$ of the form

$$f(x_1, ..., x_n) = \sum_{i,j} a_{ij}\overline{x_i}x_j, \quad a_{ij} = \overline{a_{ji}},$$

  up to $\mathbb{C}$-linear change of variables. Note that f is $\mathbb{R}$-valued.

We now show that the matrix version of complex quadratic forms is still the equivalent of the polynomial version.

- Matrices $\implies$ Poly: take f as above with a_{ij} = the (i, j)-th entry of A.
- Poly $\implies$ Matrices:

$$\begin{aligned} f(v + w) - f(v) - f(w) &= B(v + w, v + w) - B(v, v) - B(w, w) \\ &= B(v, w) + B(w, v) \\ &= 2\mathrm{Re}B(v, w). \end{aligned}$$

  Therefore f determines the real part of B, and since $\mathrm{Im}B(v, w) = \mathrm{Re}B(\mathrm{i}v, w)$, B is completely determined and A follows B as well.

To classify them we can:

- Reduce to the quadratic forms on $\mathbb{R}$,
- Use spectral theorem for self-adjoint versions,
- Copy the arguments for quadratic forms (配方法).

  1. Diagonalization: Any $f = \sum_{i,j} a_{ij}\overline{x_i}x_j$ is equivalent with a diagonal quadratic form

$$f' = \sum_i a_i|x_i|^2.$$

  2. Rescaling: $x_i \to \sqrt{|a_i|}x_i$ when $a_i \neq 0 \implies$ reduces to $a_i \in \{0, \pm 1\}$, i.e.

$$f \simeq |x_1|^2 + \ldots + |x_p|^2 - |x_{p+1}|^2 - \ldots - |x_{p+q}|^2.$$

Proposition Given $n \in \mathbb{Z}_{\geq 1}$,

$$\{\text{Hermitian forms } f : \mathbb{C}^n \to \mathbb{C}\}/_\sim = \{(p,q) \in \mathbb{Z}^2_{\geq 0} \mid p + q \leq n\}.$$

Definition Given V : $\mathbb{C}$-vs., $B : V \times V \to \mathbb{C}$, Hermitian form. We say B is
- **Positive semi-definite** if $B(v,v) \geq 0$,
- **Positive definite** if $B(v,v) \geq 0, B(v,v) = 0 \Longleftrightarrow v = 0$.

The negative case is similar, and B is **indefinite** if none of above is true.

If $f : \mathbb{C}^n \to \mathbb{R}$ (or $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$) corresponds to (p,q), then:

$$\begin{aligned} f(\text{or } B) \text{ is positive semi-definite} &\Longleftrightarrow q = 0, \\ \text{positive definite} &\Longleftrightarrow p = n, \\ \text{indefinite} &\Longleftrightarrow p, q > 0. \end{aligned}$$

Definition $p :=$ 正惯性指数, $q :=$ 负惯性指数 of f (or B) up to $\simeq$.