In the following discussion, V is an $\text{IPS}/_{\mathbb{C}}$ with $\dim < \infty$.

Recall $T \in \text{End}(V)$ is unitary $\iff T : (V, (|)) \simeq (V, (|))$ or $T^* = T^{-1}$.

Definition We say matrix $A \in M_{n\times n}(\mathbb{C})$ is unitary if $A^\dagger = A^{-1}$.

Thus, when $V = \mathbb{C}^n + \text{std IP}, \text{End}(V) = M_{n\times n}(\mathbb{C})$ via std basis, then unitary $T \longleftrightarrow$ unitary matrix A since adjoint $\longleftrightarrow A^\dagger$.

Let $P \in M_{n\times n}(\mathbb{C}), P = (v_1|...|v_n), v_i \in \mathbb{C}^n$.
$P^\dagger = \begin{pmatrix} v_1^\dagger \\ \vdots \\ v_n^\dagger \end{pmatrix}$, therefore the (i,j)-th entry of $P^\dagger P$ is $v_i^\dagger v_j = v_i \cdot v_j$ (std IP).

$\Longrightarrow \left(P \text{ unitary} \iff P^\dagger P = \text{id}\right)$ i.e. its column vectors are orthonormal.

Same discussions on $PP^\dagger$ gives $\left(P \text{ unitary} \iff PP^\dagger = \text{id}\right)$ i.e. its row vectors are orthonormal.

Spectral Theorem for normal operators

Recall $T \in \text{End}(V)$ is said to be normal if $TT^* = T^*T$.

Theorem The following are equivalent for $T \in \text{End}(V)$:

1. $\exists$ ONB $v_1, ..., v_n$ of V and corresponding eigenvalues $\lambda_1, ..., \lambda_n \in \mathbb{C}$ s.t. $Tv_i = \lambda_i v_i, \forall i$,
2. T is normal.

In terms of matrices, take $V = \mathbb{C}^n + \text{std IP}$, then (1) says:

$\exists P \in M_{n\times n}(\mathbb{C})$ unitary, and $\exists \lambda_1, ..., \lambda_n \in \mathbb{C}$ s.t. $P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ where $A \longleftrightarrow T$.

Proof of (1) $\Longrightarrow$ (2) Choosing an ONB of V, may assume $V = \mathbb{C}^n, T \longleftrightarrow A \in M_{n\times n}(\mathbb{C})$.

$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, A = P\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}P^{-1}$.

$T^* \longleftrightarrow A^\dagger = \left(P^{-1}\right)^\dagger \begin{pmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{pmatrix} P^\dagger = P \begin{pmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{pmatrix} P^\dagger$.

Calculation shows

$$AA^\dagger = P\begin{pmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{pmatrix}P^\dagger = A^\dagger A,$$

so A is normal by definition.

Alternatively, let $v_1, ..., v_n$ be as in (1). Can check: $T^*v_i = \overline{\lambda_i}v_i \Longrightarrow T^*T = TT^*, v_i \mapsto |\lambda_i|^2 v_i$.

Proposition Any $T \in \text{End}(V)$ decomposes uniquely into $T' + T''$ where T' is self-adjoint and T'' is anti-adjoint. Moreover, if T is normal, $T'T'' = T''T'$.

Proof (Uniqueness) If $T = T' + T'' = S' + S''$ are two such decompositions, let $T' - S' = S'' - T'' =: R \Longrightarrow -R = R^* = R. \Longrightarrow R = 0 \in \text{End}(V)$.

(Existence)

$$T' = \frac{T + T^*}{2}, T'' = \frac{T - T^*}{2}$$

is a valid decomposition. Here we prove the second statement for normal T by using $TT^* = T^*T$. $\square$

Proposition Let $V_0 \in V$: subspace be T-invariant. Then $V_0^{\perp}$ is T^*-invariant.

Proof Let $v \in V_0^{\perp}$. For all $v_0 \in V_0$, $(T^* v | v_0) = (v | T v_0) = 0$. □

Proof of (2) ⟹ (1)

1. T is self-adjoint. Claim: $\exists v_1 \in V, \|v_0\| = 1$ and $\exists \lambda_1 \in \mathbb{R}$ s.t. $Tv_1 = \lambda_1 v_1$.

   Proof Take any eigenvalue $\lambda_1 \in \mathbb{C}, v_1 \in V \setminus \{0\}, Tv_1 = \lambda_1 v_1$.

   $\overline{\lambda_1} = (Tv_1 | v_1) = (v_1 | T^* v_1) = (v_1 | T v_1) = \lambda_1 \Longrightarrow \lambda_1 \in \mathbb{R}$.

   Claim $\Longrightarrow V = \underbrace{\langle v_1 \rangle \oplus \langle v_1 \rangle^{\perp}}_{\text{both } T\text{-invariant}}, \langle v_1 \rangle = \mathbb{C} v_1$.

   Now $T|_{\langle v_1 \rangle^{\perp}}$ is still self-adjoint, and $T|_{\langle v_1 \rangle} = \lambda_1 \cdot \mathrm{id}$ is self-adjoint as well.

   Induction on $\dim V =: n$ to get ONB $v_2, ..., v_n$ and corresponding real eigenvalues $\lambda_2, ..., \lambda_n \in \mathbb{R}, Tv_i = \lambda_i v_i$. Merge it with Claim to extend to $n = 1$. □

2. T is anti-adjoint. Merging (1) and (2) will solve the normal case.

   $(\mathrm{i}T)^* = -\mathrm{i}T^* = \mathrm{i}T \Longrightarrow \exists$ ONB $v_1, ..., v_n \in V, \lambda_1, ..., \lambda_n \in \mathbb{R}$,

   $$(\mathrm{i}T)(v_i) = \lambda_i v_i \quad \Longrightarrow \quad Tv_i = (-\lambda_i \mathrm{i}) v_i.$$

   In fact: all $\lambda_j \in \mathrm{i}\mathbb{R}$.

3. General normal T.

   $T = T' + T'', T'T'' = T''T' \Longrightarrow$ Simultaneous Orthonormal Diagonalization.

   Let $V = V_{\mu_1} \oplus ... \oplus V_{\mu_m}$ where $\mu_1, ..., \mu_m$ are distinct eigenvalues of T', and $V_{\mu_i} \perp V_{\mu_j}$ if $i \neq j$.

   From $T'T'' = T''T'$ we know all V_{μ_i} is T''-invariant, $\because \forall v \in V_{\mu_i}, T'(T''v) = T''(T'v) = \mu_i T''v$.

   Also $\left(T''|_{V_{\mu_i}}\right)^* = -T''|_{V_{\mu_i}}$ since T'' is anti-adjoint. From (2) we know V_{μ_i} has an ONB consisting of eigenvectors of T''.

   $\Longrightarrow$ Now we have an ONB of V, both T', T'' act by scalar on any element in ONB.

   $\Longrightarrow$ So is $T = T' + T''$. □

To sum up all previous conclusions:

$$\begin{aligned} T^* = T \quad &\Longleftrightarrow \quad \forall \text{ eigenvalue } \lambda \in \mathbb{R} \\ T^* = -T \quad &\Longleftrightarrow \quad \forall \text{ eigenvalue } \lambda \in \mathrm{i}\mathbb{R} \\ T^* = T^{-1} \quad &\Longleftrightarrow \quad \forall \text{ eigenvalue } |\lambda| = 1 \end{aligned}$$

In particular, $T^* = T^{-1} \Longleftrightarrow \overline{\lambda} = \lambda^{-1} \Longleftrightarrow |\lambda| = 1$.

Many facts for IPS/$_{\mathbb{R}}$ carry over to IPS/$_{\mathbb{C}}$

Theorem Let $f : \mathbb{C}^n \to \mathbb{R}$ be an Hermitian form corresponding to $A \in M_{n \times n}(\mathbb{C}), A^{\dagger} = A$. Then

$$\begin{aligned} f \text{ is positive definite} &\Longleftrightarrow \text{all eigenvalues of } A \text{ are} > 0, \\ f \text{ is positive semi-definite} &\Longleftrightarrow \text{all eigenvalues of } A \text{ are} \geq 0. \end{aligned}$$

Proof Spectral Theorem $\Longrightarrow \exists$ unitary C s.t. $C^{\dagger} A C = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \lambda_i \in \mathbb{R}$.

$\Longrightarrow f \simeq \sum_{i=1}^{n} \lambda_i |x_i|^2$ as Hermitian forms on $\mathbb{C}$. Rescale x_i to reduce coefficients to $\{\pm 1\}$. $\square$

Let $T \in \mathrm{End}(V), T^* = T$. Then $(v_1, v_2) \mapsto (Tv_1|v_2) = (v_1|Tv_2) = \overline{(Tv_2|v_1)}$ is Hermitian. We say T is positive definite etc. if this Hermitian form is. Note that this implies that whenever we say T is positive definite, etc., **we assume T is self-adjoint**.

Let V, W be IPS$/_{\mathbb{C}}$ and $T \in \mathrm{Hom}(V, W) \Longrightarrow T^*T \in \mathrm{End}(V), TT^* \in \mathrm{End}(W)$ both positive semi-definite since $(T^*Tv|v) = (Tv|Tv) \geq 0$ and the other case is similar. Here

$$\begin{aligned} \text{Positive definite} \Longleftrightarrow \ker T &= \{0\} \quad (\text{for } T^*T), \\ \ker T^* &= \{0\} \quad (\text{for } TT^*). \end{aligned}$$

In fact, $\ker T^* = (\mathrm{im} T)^\perp$ for all $T \in \mathrm{Hom}(V, W)$,
$\because T^*w = 0 \Longleftrightarrow \forall v \in V, (v|T^*w) = 0 \Longleftrightarrow (Tv|w) = 0 \Longleftrightarrow w \in (\mathrm{im} T)^\perp$.

Definition-Proposition If $T \in \mathrm{End}(V)$ is positive definite (resp. positive semi-definite), then $\exists! S \in \mathrm{End}(V)$, positive definite (resp. positive semi-definite), and $S^2 = T$. Notation: $S = \sqrt{T}$.

Proof Same as the real case using spectral theorem. $\square$

Theorem (Polar Decomposition) Let $T \in \mathrm{End}(V)$ invertible. Then $\exists!$ pair of operators $R, U \in \mathrm{End}(V)$ s.t. $T = RU$, where R is positive definite and U is unitary.

Special case: $V = \mathbb{C}, \mathrm{End}(V) \simeq \mathbb{C}, \because$ all linear maps are equivalent to a scalar. Here
positive definite $\Longleftrightarrow \mathbb{R}_{>0}$, unitary $\Longleftrightarrow |\cdot| = 1$.

Proof Same as in the real case. In fact, $R = \sqrt{TT^*}$: positive definite, $U = R^{-1}T$. $\square$