§ Complex inner product space / Hermitian space

Definition Given V : $\mathbb{C}$-vs. A **complex / Hermitian inner product space** on V means an Hermitian form $(|)_V : V \times V \to \mathbb{C}$, positive definite. The datum $(V, (|)_V)$ is called a complex / Hermitian IPS, or simply IPS$/_{\mathbb{C}}$.

$\forall v \in V$, its **length** $\|v\| := \sqrt{(v|v)}$, so:

1. $\|v\| = 0 \iff v = 0$ since $(|)_V$ is positive definte.
2. $\|tv\| := \sqrt{(tv|tv)} = |t|\|v\|, \forall t \in \mathbb{C}$.
3. $\|v + w\|^2 = \|v\|^2 + \|w\|^2 + 2\mathrm{Re}(v|w)$. If $(v|w) = 0$, we say they are **diagonal**, written as $v \perp w$, in which case $\|v + w\|^2 = \|v\|^2 + \|w\|^2$.
4. If $\|v\| = 1$, we say v is a **unit vector**.

In general, given a subset $S \subset V$:

- S is **orthogonal** if $S \not\ni 0$, and $\forall v, w \in S, v \neq w \implies v \perp w$.
- S is **orthonormal** if S is orthogonal and $\forall v \in S, \|v\| = 1$.

Fact $S \subset V$ orthogonal $\implies S$ linearly independent.

Proof For $v = \sum_{i=1}^{m} a_i v_i, a_i \in \mathbb{C}, v_i \in S$, we can write $a_i = \frac{(v_i|v)}{(v_i|v_i)}$. Hence

$$(v_i|v) = \sum_{i=1}^{m} a_j (v_i|v_j) = a_i (v_i|v_i),$$

that is $v = 0 \iff \forall i, a_i = 0.$■

Definition Let V : IPS$/_{\mathbb{C}}$, V_0, V_1 be subspaces of V. $V_0 \perp V_1$ means $\forall v_0 \in V_0, v_1 \in V_1, v_0 \perp v_1$. Given a family of subspaces $(V_i)_{i\in I}$ of V,, if $V = \bigoplus_{i\in I} V_i$ as vector space, $i \neq j \implies V_i \perp V_j$, we say V is the **orthogonal direct sum** of $(V_i)_{i\in I}$.

Fact Given V: IPS$/_{\mathbb{C}}$, $V_0 \subset V$: subspace, $\dim V_0 < \infty$, define

$$V_0^\perp := \{v \in V \mid \forall v_0 \in V_0, (v|v_0) = 0\},$$

then $V = V_0 \oplus V_0^\perp$. Proof same as the real case with the construction

$$v = \underbrace{\sum_{i=1}^{m} (v|v_i) v_i}_{\in V_0} + \underbrace{\left(v - \sum_{i=1}^{m} (v|v_i) v_i\right)}_{\in V_0^\perp},$$

where $v_1, ..., v_m$ is an ONB of V_0. This is also the **orthogonal projection** of v to V_0.

Proposition $v \perp w \implies \|v + w\|^2 = \|v\|^2 + \|w\|^2$. Proof somewhere above.

Theorem (Cauchy-Bunyakovsky-Schwarz) In a IPS$/_{\mathbb{C}}$ V,we have

$$|(v|w)|^2 \leq (v|v)(w|w),$$

with equality $\iff v, w$ linearly dependent.

Proof Recall that in the real case we calculated the discriminant of $(tv + w|tv + w)$ and showed that $\Delta \geq 0$ is equivalent to the statement to be proved. Here we use similar tricks.

Assume $w \neq 0$. $\forall t \in \mathbb{C}, 0 \leq \|v + tw\|^2 = \|v\|^2 + |t|^2\|w\|^2 + 2\mathrm{Re}(v|tw)$. Put $t = -\frac{(w|v)}{(w|w)}$ to get

$$0 \leq \|v\|^2 - \frac{|(v|w)^2|}{(w|w)}.$$

If equality holds then $v = -tw$ for this choice of t, hence v, w are linearly dependent. Conversely, if $v = tw$ for some t, then we can easily show the equality holds. ■

Corollary (Triangle Inequality) $\|v + w\| \leq \|v\| + \|w\|$, with equality $\iff v = tw \vee w = tv$ for some $t \geq 0$.

Proof

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2 + 2\mathrm{Re}(v|w) \leq \|v\|^2 + \|w\|^2 + 2|(v|w)| \leq (\|v\| + \|w\|)^2.$$

Equality holds when $v = tw \vee w = tv, t \in \mathbb{C} \implies t \geq 0$ for 1st equality.■

Theorem (Gram-Schmidt) Let V be an IPS over $\mathbb{C}$, $v_1, ..., v_n \in V$ be linearly independent. Define deductively:

$$\begin{aligned} w_1 &= v_1, \\ w_k &= v_k - \sum_{i=1}^{k-1} \frac{(v_k|w_i)}{(w_i|w_i)} w_i, \end{aligned}$$

then $w_1, ..., w_n$ are orthogonal and $\forall k, \langle v_1, ..., v_k \rangle = \langle w_1, ..., w_k \rangle$.

Proof Same as real case with due care on complex conjugation.■

Can get orthonormal $u_1, ..., u_n$ with $u_i = \frac{w_i}{\|w_i\|}$, and $w_i = v_i$ if v_i is already orthogonal.

Consequences

1. Let $v_1, ..., v_l \in V$ be orthogonal (resp. orthonormal), where V is an IPS on $\mathbb{C}$ with $\dim < \infty$ $\implies$ They can be extended to an orthogonal (resp. orthonormal) basis of V.
   $\because$ They can be first extended to a basis, then apply Gram-Schmidt to get an orthogonal (resp. orthonormal) basis.
2. $\exists$ ONB for any IPS$/_\mathbb{C}$ by setting $l = 0$ in (1).

Examples

- Standard inner product of $\mathbb{C}^n$: $x \cdot y = \sum_{i=1}^{n} \overline{x_i} y_i$ or $x^\dagger y$.
  Standard basis $e_1, ..., e_n$: $e_i = (..., 1, ...)$ with 1 placed at the i-th entry.
- Function space: Let $a < b$ in $\mathbb{R}$, $C[a, b] = \{f : [a, b] \to \mathbb{C}, \text{continuous}\}$.
  This forms a $\mathbb{C}$-vector space, and we can define a inner product $(f|g) = \int_a^b f(x)g(x) \,\mathrm{d}x$, which is positive definite and Hermitian. However $\dim = \infty$.
- Variant: limit the previous space to polynomials. Can study the result of Gram-Schmidt, etc.

Notion of isomorphisms Let V, W be IPS$/_\mathbb{C}$. If $T \in \mathrm{Hom}(V, W)$, $\|Tv\|_W = \|v\|_V, \forall v \in V$, we say T is **isometry**. If $V \underset{S}{\overset{T}{\rightleftarrows}} W$ are a pair of isometry, and $ST = \mathrm{id}_V, TS = \mathrm{id}_W$, we say they are **isomorphisms**.

Observations (same as the real case)

1. $T \in \mathrm{Hom}(V, W)$ is an isometry $\iff (Tv_1|Tv_2)_W = (v_1|v_2)_V, \forall v_1, v_2 \in V$.
   $\because \|\cdot\|^2$ determines $\mathrm{Re}(\cdot \mid \cdot)$, and by adding an i it further determines $\mathrm{Im}(\cdot \mid \cdot)$.
2. If $T \in \mathrm{Hom}(V, W)$ is an isometry, then T is an isomorphism between IPS$/_\mathbb{C}$, i.e. T^{-1} is an isometry, $\because \forall w \in W, \|T^{-1}w\|_V = \|v\|_V = \|Tv\|_W = \|w\|_W$ where $v := T^{-1}w$.
3. Suppose $\dim V =: n \in \mathbb{Z}_{\geq 1}$, then

$$\{\text{Isomorphisms of IPS}/_{\mathbb{C}} : \mathbb{C}^n + \text{std IP} \rightsquigarrow V\} \overset{1:1}{\longleftrightarrow} \{\text{Ordered ONB } v_1, ..., v_n \text{ of } V\}.$$

Since we already know

$$\{\text{Isomorphism of } \mathbb{C}\text{-vs} : \mathbb{C}^n \rightsquigarrow V\} \overset{1:1}{\longleftrightarrow} \{\text{Ordered basis } v_1, ..., v_n \text{ of } V\}.$$

- Given T as above, if T is $\simeq$ of IPS$/_{\mathbb{C}}$, then $Te_1, ..., Te_n$ is an ONB since $e_1, ..., e_n$ is an ONB.
- Given an ONB $v_1, ..., v_n \in V$, define T s.t. $Te_i = v_i$,

$$\left\|T\left(\sum_i a_i e_i\right)\right\|^2 = \left\|\sum_i a_i v_i\right\|^2 = \sum_i |a_i|^2 = \left\|\sum_i a_i e_i\right\|^2.$$

4. Since any IPS$/_{\mathbb{C}}$ with $\dim = n$ has an ONB, they are $\underset{\varphi}{\simeq} (\mathbb{C}^n, \text{std IP})$ where φ is not unique.

Definition (Unitary operator) Given V : IPS$/_{\mathbb{C}}$, a **unitary operator** on V

$$:= \text{Isomorphism } (V, (|)) \rightsquigarrow (V, (|)),$$

i.e. automorphism of IPS$/_{\mathbb{C}}$. Here the term "operator" means linear map $V \to V$.

Proposition Given V : IPS$/_{\mathbb{C}}$, $\dim < \infty$. Then T unitary $\iff T^* = T^{-1}$.
Here T^* can always be defined since $(|)_V$ is nondegenerate.

Proof Asuume T is invertible.

$$\begin{aligned} T \text{ unitary} &\iff (Tv_1|Tv_2) = (v_1|v_2), \forall v_1, v_2 \in V \\ &\iff (v_1'|Tv_2) = (T^{-1}v_1'|v_2), \forall v_1', v_2 \in V \\ &\iff T^* = T^{-1} \text{ by defintion.} \end{aligned}$$