Theorem(Perron) Let $A \in M_{n\times n}(\mathbb{R}), A > 0$.

1. $\rho(A) > 0, \exists v \in \mathbb{R}^n, v > 0, Av = \rho(A)v$.
2. If $\mu \in \mathbb{C}$ is an eigenvalue, $\mu \neq \rho(A)$, then $|\mu| < \rho(A)$.
3. The $\rho(A)$-eigenspace has a dimension of 1.
4. $\rho(A)$ is a single root of Char_A.

Proof (1)(2): Done.

(3) Suppose $v, v' \in \mathbb{R}, Av = \rho(A)v, Av' = \rho(A)v'$ and $v > 0$. We show that $v' \in \mathbb{R}v$ by contradiction: If not, we may choose some $\varepsilon > 0$, s.t. $v - \varepsilon v' \geq 0, \exists i$ s.t. $(v - \varepsilon v')_i > 0$. Now $v - \varepsilon v' = \rho(A)^{-1} \underbrace{A}_{>0} \underbrace{v - \varepsilon v'}_{\geq 0, \neq 0} > 0$, which contradicts the assumption that $v - \varepsilon v' > 0$.■

(4) Use our knowledge of direct sum and characteristic polynomials.

The case $n = 1$ is trivial, so we may assume that $n \geq 2$. Fix $v > 0, Av = \rho(A)v$. Since transposition doesn't change eigenvalues, $\rho(A) = \rho(A^T) \Longrightarrow \exists u \in \mathbb{R}^n, u > 0, A^T u = \rho(A)u$.

$\langle u\rangle = \mathbb{R}u$, and $\langle u\rangle^\perp$ is A-invariant since $u^T Ax = (A^T u)^T x = \rho(A)ux = 0$. From $u \neq 0$ we know that $\dim\langle u\rangle^\perp = n - 1$, and that $v \notin \langle u\rangle^\perp$ since $v, u > 0$. Hence by comparing dimensions we reach

$$\mathbb{R}^n = \langle v\rangle \oplus \langle u\rangle^\perp,$$

both summands A-invariant.

Recall that under such conditions,

$$\mathrm{Char}_A = \underbrace{\mathrm{Char}_{A|_{\langle v\rangle}}}_{X-\rho(A)} \mathrm{Char}_{A|_{\langle u\rangle^\perp}}.$$

If $(X - \rho(A))^2 \mid \mathrm{Char}_A$, then $X - \rho(A) \mid \mathrm{Char}_{A|_{\langle u\rangle^\perp}}$, which infers $\mathrm{Char}_{A_{\langle u\rangle^\perp}}(\rho(A)) = 0$. Hence we can get a $\rho(A)$-eigenvalue v' in $\langle u\rangle^\perp, v' \notin \mathbb{R}v$, contradicting (3). ■

§ Complex inner product / Hermitian inner product

Recall Real IPS = $\mathbb{R}$-vector space V + symm. bilinear form $(|) : V \times V \to \mathbb{R}$, positive definite.

Idea complex IPS = $\mathbb{C}$-vector space V + some form $(|) : V \times V \to \mathbb{C}$, positive definite.

The bilinear form we used in real IPS won't work here. To make our previous idea actually work, we introduce a new kind of "linear map":

Definition Let $T : V \to W$ be a map between $\mathbb{C}$-vector spaces. If

$$\begin{aligned} T(v_1 + v_2) &= Tv_1 + Tv_2, \\ T(zv) &= \bar{z}Tv, \qquad z \in \mathbb{C}, \end{aligned}$$

then we say T is **semi-linear**. By definition T is $\mathbb{R}$-linear.

Definition Complex IPS = $\mathbb{C}$-vector space V + map $(|) : V \times V \to \mathbb{C}$, linear in v_2, semi-linear in v_1, $\overline{(v_2|v_1)} = (v_1|v_2)$, positive definite.

In many textbooks: linear in v_1, semi-linear in v_2.

Construction Given a $\mathbb{C}$-vs. V. Consider the same set V and the same $+ : V \times V \to V$, but with

$$\begin{aligned} \odot : \mathbb{C} \times V &\to V \\ (z, v) &\mapsto \bar{z}v \end{aligned}$$

$\Longrightarrow (V, +, \odot)$ is a $\mathbb{C}$-vs. Denote it as $\overline{V}$.

Observation
1. $\overline{\overline{V}} = V$ since conjugating two times returns the original complex number.
2. Given V_1, V_2, define $V_1 \oplus V_2 = \{(v_1, v_2) | v_1 \in V_1, v_2 \in V_2\}$, then $\overline{V_1} \oplus \overline{V_2} = \overline{V_1 \oplus V_2}$.
3. $\overline{\mathbb{C}} \simeq \mathbb{C}$ by $z \mapsto \overline{z} \quad \Longrightarrow \quad \overline{\mathbb{C}^n} \simeq \mathbb{C}^n$ by $(z_i) \mapsto (\overline{z_i})$.
4. Semi-linear map $V \to W \Longrightarrow$ linear map $\overline{V} \to W$ or $V \to \overline{W}$.
   Thus $\mathrm{Hom}\left(\overline{V}, W\right) = \mathrm{Hom}\left(V, \overline{W}\right)$ as sets (but not as $\mathbb{C}$-vs. !)
5. $\overline{\mathrm{Hom}(V_1, V_2)} = \mathrm{Hom}\left(\overline{V_1}, \overline{V_2}\right)$ as $\mathbb{C}$-vs.

   Proof Linear map $V_1 \to V_2$ = linear map $\overline{V_1} \to \overline{V_2}$, i.e. $\mathrm{Hom}(V_1, V_2) = \mathrm{Hom}\left(\overline{V_1}, \overline{V_2}\right)$ as sets.
   Let $T \in \mathrm{Hom}(V_1, V_2), z \in \mathbb{C}$,

$$\underbrace{(zT)}_{\mathrm{Hom}(V_1, V_2)} (v_1) = z \underbrace{T v_1}_{V_2} = \underbrace{\overline{z} \odot T v_1}_{\overline{V_2}} = \underbrace{(\overline{z} T)}_{\mathrm{Hom}\left(\overline{V_1}, \overline{V_2}\right)} (v_1)$$

   $\Longrightarrow \overline{\mathrm{Hom}(V_1, V_2)} = \mathrm{Hom}\left(\overline{V_1}, \overline{V_2}\right)$ as $\mathbb{C}$-vector spaces. ■

Recall The dual space of V is $\mathrm{Hom}(V, \mathbb{C}) =: V^\vee$. Take $W = \mathbb{C}$ to get

$$\overline{V}^\vee = \mathrm{Hom}\left(\overline{V}, \mathbb{C}\right) = \overline{\mathrm{Hom}\left(V, \overline{\mathbb{C}}\right)} \simeq \overline{\mathrm{Hom}(V, \mathbb{C})} \simeq \overline{V^\vee},$$

given by $\lambda \leftrightarrow \overline{\lambda} : w \to \overline{\lambda(w)}$.

Definition Let V, W, X be $\mathbb{C}$-vs, $B : V \times W \to X$. We say map B is **sesquilinear** if B is semi-linear in $V$and linear in W. All sesquilinear maps $V \times W \to X$ form a $\mathbb{C}$-vs. by

$$(B_1 + B_2)(v, w) = B_1(v, w) + B_2(v, w), \qquad (zB)(v, w) = zB(v, w).$$

Proof by checking directly or note that sesquilinear map $V \times W \to X$ = bilinear map $\overline{V} \times W \to X$.

Special case: $X = \mathbb{C}$ to get **sesquilinear forms**.

Definition Let $B : V \times W \to \mathbb{C}$ be sesquilinear.
- **Left radical** of $B := \{v \in V \mid B(v, \cdot) = 0\}$,
- **Right radical** of $B := \{w \in W \mid B(\cdot, w) = 0\}$.

They are subspaces of V, W respectively.

When $\dim V, \dim W < \infty$, if left radical $= \{0\} =$ right radical, then we say B is **non-degenerate**.

Fact Assume $\dim < \infty$. If non-degenerate $B : V \times W \to \mathbb{C}$, sesquilinear, then $\dim V = \dim W$.
When $\dim V = \dim W < \infty$, B non-degenerate $\Longleftrightarrow$ left radical = $\{0\}$ $\Longleftrightarrow$ right radical = $\{0\}$.

Proof Copy the arguments for bilinear case, or note that sesquilinear $B : V \times W \to \mathbb{C}$ = bilinear $\overline{V} \times W \to \mathbb{C}$, and the notion of radicals remain the same. Now everything reduces to the known case of bilinear forms.

This method also implies that: if B is non-degenerate, then $\varphi : W \simeq \overline{V}^\vee, w \mapsto B(\cdot, w)$ and $\psi : \overline{V} \simeq W^\vee$, both isomorphisms of $\mathbb{C}$-vs.

Sesquilinear forms and matrices
Take $V = \mathbb{C}^m, W = \mathbb{C}^n$ (spaces of column vectors).

Proposition

1.

$$M_{m\times n}(\mathbb{C}) \simeq \{\text{sesquilinear forms } \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}\}$$
$$A \mapsto B(v,w) = v^\dagger A w$$

where $C^\dagger = \overline{C^T}$ for some matrix C.

2.

$$B\left(\sum_i x_i e_i, \sum_j y_j e_j\right) = \sum_{i,j} a_{ij}\overline{x_i}y_j.$$

3. B non-degenerate $\iff$ A invertible.

Proof

RHS = $\left\{\text{bilinear forms } \overline{\mathbb{C}^m} \times \mathbb{C}^n \to \mathbb{C}\right\} \simeq \{\text{bilinear forms } \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}\} \simeq M_{m\times n}(\mathbb{C})$. The description of $B\left(\sum_i x_i e_i, \sum_j y_j e_j\right)$ follows in the same way by expanding

$$B\left(\sum_i x_i e_i, \sum_j y_j e_j\right) = (\overline{x_1}\ \dots\ \overline{x_m})A\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

As for the relation between non-degenerate and invertible, can direct check or reduce to the bilinear case. ■

Remark If you insist sesquilinear forms are semi-linear in w, then must take

$$B(v,w) = v^T A\overline{w} = \sum_{ij} a_{ij}x_i\overline{y_j}.$$

Definition Let V be a $\mathbb{C}$-vs., $\varepsilon \in \{\pm 1\}$, $B : V \times V \to \mathbb{C}$, sesquilinear. We say B is **ε-Hermitian** if $\overline{B(w,v)} = \varepsilon B(v,w), \forall v,w \in V$. Specifically, $\varepsilon = 1$ is called **Hermitian**, and $\varepsilon = -1$ is called **skew/anti-Hermitian**.

Note that if $V = \mathbb{C}^n$, and $B \leftrightarrow A \in M_{n\times n}(\mathbb{C})$, then

$$\underbrace{(v,w) \mapsto \overline{B(w,v)}}_{\text{corresponds to } A^\dagger} = \underbrace{\overline{w^\dagger A v}}_{\in\mathbb{C}} = \left(w^\dagger A v\right)^\dagger = v^\dagger A^\dagger w$$

So B is ε-Hermitian $\longleftrightarrow$ A is ε-Hermitian in the sense $A^\dagger = \varepsilon A$.