§ Naive Set Theory

Definition Injections, bijections, cardinals.

Theorem $\forall S, T$ sets, $S \succcurlyeq T \wedge T \succcurlyeq S \Longrightarrow S \sim T$.

Propositions

- $\left(S^T\right)^W \sim S^{T \times W}$.

  Proof Let $f : T \times W \to S \in S^{T \times W}$. Define $f' : w \mapsto f(\cdot, w)$ and $f' \in \left(S^T\right)^W$. □

Propositions

- $\mathbb{R} \sim 2^{\mathbb{N}}$ using binary representation.
- $\mathbb{N} \sim \mathbb{Q}$ using $\mathbb{N} \twoheadrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Z} \times \mathbb{N}$.
- $\mathbb{R}^{\mathbb{N}} \sim \mathbb{R}$ using $\mathbb{R} \sim (0, 1), \mathbb{R} \sim \{0, 1\}^{\mathbb{N}}$ and concatenating binary digits. (We assume $S \sim T \wedge S' \sim T'$ yields $S^T \sim S'^{T'}$)

Definition Power sets, S^T.

Proposition

- $S \cup T \sim \mathbb{N} \Longrightarrow S \sim \mathbb{N} \vee T \sim \mathbb{N}$.
  - Note that $\bigcup_{i=0}^{\infty} S_i \sim \mathbb{N} \not\Rightarrow \exists i, S_i \sim \mathbb{N}$ simply by taking $S_i = \{i\}$.

Weird statements: (True in Axiom of Choice)

- Dedekind-finite S exists: $\forall T \subsetneq S, T \prec S$.
- $S \succcurlyeq T \vee T \succcurlyeq S$ (total order!)
- $\forall$ infinite $S, S \sim S \times S$.
- $S \succ T \overset{?}{\Longrightarrow} 2^S \succ 2^T, \quad 2^S \succcurlyeq 2^T \overset{?}{\Longrightarrow} S \succcurlyeq T$.