§ **Proposition Logic** / 谓词逻辑

A direct approximation is

$$\text{Proposition} \approx \text{Sentence} \approx \text{Statement}$$

Some sentence (in natural language) can be decomposed into smaller ones: the sentence

$$\text{If } \underbrace{\text{it rains}}, \underbrace{\text{you have to come to the classroom.}}$$

can be decomposed into the pattern If P, then Q. Here we use alphabets for

- Propositions: $P, Q, P_1, P_2, ...$
- Connectivities: $\rightarrow$ (implies), $\vee$ (or), $\wedge$ (and), $\leftrightarrow$ (iff), $\neg$ (not), $\perp$ (false).
- Auxiliary: $()$ (brackets).

Definition A sequence of alphabets above is a **proposition** iff it is
- Any atomic proposition $P_1, \perp$, etc.
- If φ, ψ are propositions, then adding $\rightarrow, \leftarrow, \leftrightarrow, \wedge, \vee$ between them generates propositions.
- If φ is a proposition, then $\neg\varphi$ is also one.

Finally, the set of all propositions $\text{PROP} \subset \Sigma^*$ is the smallest one satisfying the definition above.

Proposition Given property A, if
- $\forall$ atomic proposition P, $A(P)$ and $A(\perp)$ holds,
- $\forall\varphi, \psi \in \Sigma^*$, if $A(\varphi)$ and $A(\psi)$ holds, then $A((\varphi\square\psi))$ also holds for any connectivity $\square$.
- $\forall\varphi \in \Sigma^*$, if $A(\varphi)$ holds, then $A(\neg\varphi)$ also holds.

Then $\forall\varphi \in \text{PROP}, A(\varphi)$ holds.

To define some $F : \text{PROP} \rightarrow \Omega$, it suffices to define
- $H_{\text{atomic}} : \{P_1, P_2, ..., \perp\} \rightarrow \Omega$,
- $H_{\square} : \Omega \times \Omega \rightarrow \Omega$,
- $H_{\neg} : \Omega \rightarrow \Omega$.

If we set $\Omega = \{0, 1\}, H_{\wedge} = \min, H_{\vee} = \max, H_{\neg} : t \mapsto 1 - t$ and so on, then we call this function the **valuation** function $\nu : \text{PROP} \rightarrow \{0, 1\}$. We denote $[\varphi]_\nu := \nu(\varphi)$.

In this definition we consider the **semantics** / 语义 of the statement, whereas only the **syntax** / 语法 is considered previously.

In mathematical context we may use logical deduction

$$\varphi_1, ..., \varphi_n; \therefore \varphi.$$

Converting it to proposition logic we denote it as

$$\varphi_1, ..., \varphi_n \vDash \varphi$$

iff for any valuation ν s.t. $\nu(\varphi_1) = ... = \nu(\varphi_n) = 1$, we also have $\nu(\varphi) = 1$.

Definition If $[\varphi]_\nu = 1$ for all valuation ν i.e. for all assignment of $\nu(P)$, then we say φ is a **tautology**, denoted as $\vDash \varphi$.

Proposition If $\vDash \varphi$, then $\vDash \varphi[\psi/p]$ where $\nu'(P_1) = \nu(\psi)$ if $P_1 = \psi$ and $\nu(\varphi)$ otherwise.

§ **Natural Deduction**

Now we move on to **syntactic proofs** / 语法证明. Some deduction rules are

$$\frac{\varphi \quad \varphi \to \psi}{\psi}, \quad \frac{\varphi \quad \psi}{\varphi \land \psi}, \quad \frac{\varphi \land \psi}{\varphi}, \quad \frac{\varphi \land \psi}{\psi}, \quad \frac{[\varphi]}{\varphi \to \psi}, \quad \frac{[\neg \varphi]}{\varphi}.$$

An example of proof is proving $\vdash ((\varphi \land \psi) \to \varphi)$:

$$\frac{\frac{[\varphi \land \psi]}{\varphi}}{(\varphi \land \psi) \to \varphi}.$$

§ System K

Refer to Sequents and Trees, Section 1.2.2.

Sequents are ordered pairs $\Gamma \Rightarrow \Delta$ (or $\varphi_1, ..., \varphi_k \Rightarrow \psi_1, ..., \psi_n$ with $k, n \geq 0$). It should be interpreted as

- For $k, n > 1$, the sequent $\varphi_1, ..., \varphi_k \Rightarrow \psi_1, ..., \psi_n$ means $\varphi_1 \land ... \land \varphi_k \to \psi_1 \lor ... \lor \psi_n$ in terms of usual notations,
- $k = 0$ is denoted as $\top$, and $n = 0$ is denoted as $\bot$ as one may expect.

Given these notations, **System K** consists of the following rules:

$$(\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg)\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}$$

$$(\land \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \land \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \land)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \land \psi}$$

$$(\Rightarrow \lor)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \lor \psi} \qquad (\lor \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \lor \psi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\to)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi} \qquad (\to\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma, \psi \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}$$

Note that all rules come in symmetrical pairs.