Question 1

有

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} 1_{x_i \ge \theta} \cdot \frac{\theta}{x_i^2} \\
&= 1_{x_1 \ge \theta, \ldots, x_n \ge \theta} \cdot \frac{\theta^n}{\prod x_i^2},
\end{aligned}
$$

于是 θ 的最大似然估计为 $\hat{\theta} = \min\{x_1, \ldots, x_n\}$.

对于任意 $\varepsilon > 0$, 考虑 $\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| > \varepsilon\right)$. 记 $p := P(X \le \theta + \varepsilon) > 0$, 则

$$
P\left(\left|\hat{\theta}_n - \theta\right| > \varepsilon\right) = (1-p)^n \to 0,
$$

于是 $\hat{\theta}$ 是渐进无偏估计量. 显然 $\mathbb{E}\left(\hat{\theta}\right) > \theta$, $\hat{\theta}$ 不是无偏估计量.

下面计算均方误差, 先算 $\mathbb{E}\left(\hat{\theta}\right)$ 和 $\mathbb{E}\left(\hat{\theta}^2\right)$.

$$
\begin{aligned}
F_{\hat{\theta}}(t) &= 1 - P\left(\hat{\theta} \ge t\right) = 1 - \left(\frac{\theta}{t}\right)^n, \\
f_{\hat{\theta}}(t) &= F'(t) = -n\left(\frac{\theta}{t}\right)^{n-1} \cdot \left(-\frac{\theta}{t^2}\right) = \frac{n\theta^n}{t^{n+1}}, \\
\mathbb{E}\left(\hat{\theta}\right) &= \int_{\theta}^{+\infty} t f(t) \,\mathrm{d}t = n\theta^n \int_{\theta}^{+\infty} t^{-n} \,\mathrm{d}t = \frac{n}{n-1}\theta, \\
\mathbb{E}\left(\hat{\theta}^2\right) &= \int_{\theta}^{+\infty} t^2 f(t) \,\mathrm{d}t = n\theta^n \int_{\theta}^{+\infty} t^{-n+1} \,\mathrm{d}t = \frac{n}{n-2}\theta^2.
\end{aligned}
$$

于是

$$
\begin{aligned}
\mathrm{MSE}\left(\hat{\theta}\right) &= \mathbb{E}\left(\left(\hat{\theta} - \theta\right)^2\right) = \mathbb{E}\left(\hat{\theta}^2\right) - 2\theta \cdot \mathbb{E}\left(\hat{\theta}\right) + \theta^2 \\
&= \left(1 + \frac{n}{n-2} - \frac{2n}{n-1}\right)\theta^2 \\
&= \frac{2\theta^2}{n^2 - 3n + 2}.
\end{aligned}
$$

由 $\mathrm{MSE}\left(\hat{\theta}\right) \to 0$ 知 $\hat{\theta}$ 是一致估计量.

考虑枢轴量 $G = \hat{\theta}/\theta$,

$$
F_G(t) = F_{\hat{\theta}}(\theta t) = 1 - t^{-n}, \quad t \ge 1.
$$

取 $t = \alpha^{-\frac{1}{n}}$ 即有 $P\left(\theta > t^{-1}\hat{\theta}\right) = F_G(t) = 1 - \alpha$, $\left[\alpha^{\frac{1}{n}}\hat{\theta}, \hat{\theta}\right]$ 即为置信度为 $1-\alpha$ 的置信区间.

Question 2

1. 已知 $\sum x_i \sim \Gamma(n, \lambda)$, 于是有

$$\begin{aligned}\mathbb{E}\left(\hat{\lambda}_0\right) &= n \cdot \int_0^{+\infty} \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} \cdot \frac{1}{x} \,\mathrm{d}x \\ &= n\lambda \cdot \int_0^{+\infty} \frac{\lambda^{n-1}}{\Gamma(n)} x^{n-2} e^{-\lambda x} \,\mathrm{d}x \\ &= n\lambda \cdot \frac{\Gamma(n-1)}{\Gamma(n)} \\ &= \frac{n}{n-1}\lambda.\end{aligned}$$

   于是 $\hat{\lambda}_0$ 不是无偏估计量, 是渐进无偏估计量.

2. $\hat{\lambda}_1 := \dfrac{n-1}{n}\hat{\lambda}_0$ 是无偏估计量.

3. $\hat{\lambda}_1$ 的均方误差即为自身方差. 考虑计算 $\mathbb{E}\left(\hat{\lambda}_0^2\right)$.

$$\begin{aligned}\mathbb{E}\left(\hat{\lambda}_1^2\right) &= n^2 \cdot \int_0^{+\infty} \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} \cdot \frac{1}{x^2} \,\mathrm{d}x \\ &= \frac{n^2}{(n-1)(n-2)}\lambda^2.\end{aligned}$$

   于是

$$\begin{aligned}\mathrm{MSE}\left(\hat{\lambda}_1\right) &= \frac{(n-1)^2}{n^2}\ \mathrm{Var}\left(\hat{\lambda}_0^2\right) \\ &= \frac{(n-1)^2}{n^2}\left(\frac{n^2}{(n-1)(n-2)}\lambda^2 - \frac{n^2}{(n-1)^2}\lambda^2\right) \\ &= \frac{1}{n-2}\lambda^2.\end{aligned}$$

   $\mathrm{MSE}\left(\hat{\lambda}_1\right) \to 0$, 于是它是一致估计量.

4. 考虑 $2\lambda X_i$ 服从的分布 g: $g(x) = \frac{f\left(\frac{x}{2\lambda}\right)}{2\lambda} = \frac{1}{2}e^{-\frac{x}{2}}$, 即 $2\lambda X_i \sim \mathrm{Exp}\left(\frac{1}{2}\right), 2\lambda \sum X_i \sim \chi^2(n)$. 令 $t_0 := F^{-1}(1-\alpha)$, 则

$$P\left(0 \le \lambda \le \frac{t_0}{2n\overline{X}}\right) = P\left(0 \le 2\lambda \sum X_i \le t_0\right) = 1 - \alpha.$$

   $\left[0, \frac{F^{-1}(1-\alpha)}{2n\overline{X}}\right]$ 即为所求.

Question 3

1. 有 $F_X(x) = \int_0^x (\theta+1)t^\theta \, \mathrm{d}t = x^{\theta+1}, \quad \forall 0 < x < 1$. 于是

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x^{\theta+1} & \text{if } 0 < x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

2. 有

$$\mathbb{E}(X) = \int_0^1 (\theta+1)x^\theta \cdot x \, \mathrm{d}x = \frac{\theta+1}{\theta+2},$$

即

$$\theta = \frac{1}{1-\mathbb{E}(X)} - 2.$$

替换后即得

$$\hat{\theta}_1 = \frac{1}{1-\overline{X}} - 2.$$

3. 对概率密度函数取对数:

$$\begin{aligned} \sum_{i=1}^n \ln f(x_i) &= \sum_{i=1}^n \ln(\theta+1) + \theta \ln x_i \\ &= n \ln(\theta+1) + \theta \sum \ln x_i. \end{aligned}$$

取导数得到

$$\frac{n}{\theta+1} + \sum \ln x_i = 0,$$

即

$$\hat{\theta}_2 = -\frac{n}{\sum \ln x_i} - 1.$$

考虑 $\ln X_i$ 服从的分布: $f_{\ln X}(x) = f_X(e^x) \cdot e^x = (\theta+1)e^{(\theta+1)x}$, 于是 $-\ln X$ 服从 $\mathrm{Exp}(\theta+1)$, $-\sum \ln X_i$ 服从 $\Gamma(n, \theta+1)$. 代入计算有

$$\begin{aligned} \mathbb{E}\left(\hat{\theta}_2\right) &= -1 + \int_0^{+\infty} \frac{n}{x} \cdot \frac{(\theta+1)^n}{\Gamma(n)} x^{n-1} e^{-(\theta+1)x} \, \mathrm{d}x \\ &= -1 + n(\theta+1) \cdot \frac{\Gamma(n-1)}{\Gamma(n)} \\ &= -1 + \frac{n}{n-1}(\theta+1) \\ &= \frac{n}{n-1}\theta + \frac{1}{n-1}. \end{aligned}$$

于是 $\hat{\theta}_2$ 不是无偏估计量, 是渐进无偏估计量.

4. 约定 $K := -\sum \ln X_i \sim \Gamma(n, \theta+1)$. 有

$$\mathbb{E}\left(\left(\hat{\theta}_2-\theta\right)^2\right)=\mathbb{E}\left(\left(\hat{\theta}_2\right)^2\right)-2\theta\mathbb{E}\left(\hat{\theta}_2\right)+\theta^2,$$

其中

$$\begin{aligned}\mathbb{E}\left(\left(\hat{\theta}_2\right)^2\right)&=\mathbb{E}\left(\frac{n^2}{K^2}\right)-2\mathbb{E}\left(\frac{n}{K}\right)+1\\&=\mathbb{E}\left(\frac{n^2}{K^2}\right)-2\frac{n(\theta+1)}{n-1}+1,\end{aligned}$$

而

$$\begin{aligned}\mathbb{E}\left(\frac{n^2}{K^2}\right)&=n^2\int_0^{+\infty}\frac{1}{k^2}\cdot\frac{(\theta+1)^n}{\Gamma(n)}k^{n-1}e^{-(\theta+1)k}\,\mathrm{d}k\\&=n^2\cdot\frac{(\theta+1)^2}{(n-2)(n-1)}.\end{aligned}$$

于是

$$\begin{aligned}\operatorname{MSE}\left(\hat{\theta}_2\right)&=\frac{n^2}{(n-2)(n-1)}(\theta+1)^2-\left(\frac{2n(\theta+1)}{n-1}\right)+1-2\theta\cdot\left(\frac{n}{n-1}\theta+\frac{1}{n-1}\right)+\theta^2\\&=\left(\frac{n^2}{(n-2)(n-1)}-\frac{2n}{n-1}+1\right)\theta^2+\left(\frac{2n^2}{(n-2)(n-1)}-\frac{2n}{n-1}-\frac{2}{n-1}\right)\theta\\&+\left(\frac{n^2}{(n-2)(n-1)}-\frac{2n}{n-1}+1\right)\\&=\frac{(\theta+1)^2(n+2)}{(n-2)(n-1)}.\end{aligned}$$

有 $\operatorname{MSE}\left(\hat{\theta}_2\right)\to 0$, 于是它是一致估计量.

Question 4

1. 有

$$\begin{aligned}\sum\ln P(X=x_i)&=\sum\ln\left(\frac{\lambda^{x_i}}{x_i!}e^{-\lambda}\right)\\&=\ln\lambda\cdot\sum x_i-n\lambda-\sum\ln(x_i!),\end{aligned}$$

求导有

$$\frac{1}{\lambda}\cdot\sum x_i-n=0,$$

于是 $\overline{X}$ 确实是 λ 最大似然估计, 于是 $\hat{p}_1$ 亦是 $e^{-\lambda}$ 最大似然估计.

记 $Y:=\sum_{i=1}^n X_i$, 则 $Y\sim\pi(n\lambda)$. 利用提示中的结论有

$$\mathbb{E}(\hat{p}_1)=\mathbb{E}\left(e^{-\frac{1}{n}Y}\right)=e^{n\lambda\left(e^{-\frac{1}{n}}-1\right)}.$$

于是 $\hat{p}_1$ 不是无偏估计量. 利用泰勒展开

$$e^{-\frac{1}{n}} - 1 = -\frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

可知 $\hat{p}_1$ 是渐进无偏估计量.

计算 $\hat{p}_1$ 的均方误差:

$$\begin{aligned}
\mathbb{E}\Big((\hat{p}_1 - p)^2\Big) &= \mathbb{E}(\hat{p}_1^2) - 2p\mathbb{E}(\hat{p}_1^2) + p^2 \\
&= e^{n\lambda\left(e^{-\frac{2}{n}}-1\right)} - 2e^{-\lambda+n\lambda\left(e^{-\frac{1}{n}}-1\right)} + e^{-2\lambda}.
\end{aligned}$$

当 $n \to \infty$ 时 $\mathrm{MSE}(\hat{p}_1) \to 0$, 于是 $\hat{p}_1$ 是一致估计量.

2. 有

$$\mathbb{E}(\hat{p}_2) = \mathbb{E}\Big(1_{X_i=0}\Big) = e^{-\lambda},$$

于是 $\hat{p}_2$ 是无偏估计量, 自然也是渐进无偏估计量. 而

$$\begin{aligned}
\mathrm{MSE}(\hat{p}_2) &= \mathrm{Var}(\hat{p}_2) \\
&= \frac{1}{n}\ \mathrm{Var}\Big(1_{X_i=0}\Big) \\
&= \frac{1}{n}e^{-\lambda}\big(1 - e^{-\lambda}\big).
\end{aligned}$$

于是 $\hat{p}_2$ 是一致估计量.