Question 1

有 $P(Y \geq t) = \prod_{i=1}^{n} P(X_i \geq t) = \prod_{i=1}^{n} e^{-\lambda_i t} = \exp(-t \cdot \sum \lambda_i)$, 这说明 $Y \sim \text{Exp}(\sum \lambda_i)$.

Question 2

1. $f_U(u) = \int_0^1 f_X(x) f_Y(u-x) \, \mathrm{d}x = \begin{cases} u &, \ 0 \leq u \leq 1, \\ 2-u &, \ 1 < u \leq 2. \end{cases}$

   $f_V(v) = \int_0^1 f_X(x) f_Y(x-v) \, \mathrm{d}x = \begin{cases} v+1 &, \ -1 \leq v \leq 0, \\ 1-v &, \ 0 < v \leq 1. \end{cases}$

2.
$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1 \cdot \text{abs}\left(\left|\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix}\right|\right) = \frac{1}{2}.$$

   取值范围是 $(0,0), (1,1), (2,0), (1,-1)$ 顺次连接形成的正方形内.

3. 利用几何形状直接写出 $f_U(u) = \begin{cases} u &, \ 0 \leq u \leq 1, \\ 2-u &, \ 1 < u \leq 2, \end{cases} f_V(v) = \begin{cases} v+1 &, \ -1 \leq v \leq 0, \\ 1-v &, \ 0 < v \leq 1. \end{cases}$

   V 的取值范围与 U 有关, 故两者不独立.

4. • 对于 $0 \leq u \leq 1 : f_{V|U=u}(v) = f_{U,V}(u,v)/f_U(u) = \frac{1}{2u}$.

   • 对于 $1 < u \leq 2 : f_{V|U=u}(v) = \frac{1}{4-2u}$.

5. 先计算 $\sigma(U)$ 和 $\sigma(V)$:

$$\sigma(U) = \sqrt{\text{Var}(U)} = \sqrt{E(U^2) - E(U)^2}$$

$$= \sqrt{\int_0^1 u^3 \, \mathrm{d}u + \int_1^2 (2-u)u^2 \, \mathrm{d}u - 1}$$

$$= \sqrt{\frac{1}{4} + \left(\frac{2}{3}u^3 - \frac{1}{4}u^4\right)\Big|_1^2 - 1}$$

$$= \sqrt{\frac{1}{4} + \left(\frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}\right) - 1} = \sqrt{\frac{1}{6}}.$$

   V 的边际分布和 U 仅相差一个偏移, 故 $\sigma(V) = \sqrt{\frac{1}{6}}$.

   再来计算 $\text{Cov}(U,V)$:

$$\text{Cov}(U,V) = E(UV) - E(U)E(V),$$

   其中

$$E(UV) = \int_0^2 \int_{-u}^u \frac{1}{2} uv \, \mathrm{d}v \, \mathrm{d}u = \frac{1}{2} \int_0^2 u \left(\int_{-u}^u v \, \mathrm{d}v\right) \mathrm{d}u = 0,$$

   故 $\text{Cov}(U,V) = 0$, 两变量不相关.

Question 4

1. 写出 $A = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix}$, 则 $(X,Y)^T = AG^T$, 故

$$(X,Y) \sim N(A \cdot \mathbf{0}, A \cdot 1_{n \times n} \cdot A^T) = N\left(\mathbf{0}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

2.
$$x^2 + y^2 - 2\rho xy = (x - \rho y)^2 + (1 - \rho^2)y^2$$
$$= \left(r\sqrt{1-\rho^2}\cos\theta\right)^2 + (1-\rho^2)r^2\sin^2\theta$$
$$= (1-\rho^2)r^2.$$

3. 先求 $\dfrac{\partial(X,Y)}{\partial(R,\Theta)}$:

$$\frac{\partial X}{\partial R} = \sqrt{1-\rho^2}\cos\Theta + \rho\sin\Theta, \quad \frac{\partial X}{\partial \Theta} = -R\sqrt{1-\rho^2}\sin\Theta + R\rho\cos\Theta,$$

$$\frac{\partial Y}{\partial R} = \sin\Theta, \quad \frac{\partial Y}{\partial \Theta} = R\cos\Theta.$$

于是

$$|J| = \begin{vmatrix} \sqrt{1-\rho^2}\cos\Theta + \rho\sin\Theta & -R\sqrt{1-\rho^2}\sin\Theta + R\rho\cos\Theta \\ \sin\Theta & R\cos\Theta \end{vmatrix} = R\sqrt{1-\rho^2}.$$

于是

$$f_{R,\Theta}(r,\theta) = f(x,y) \cdot |J|$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right) \cdot r\sqrt{1-\rho^2}$$

$$= \frac{r}{2\pi} \exp\left(-\frac{r^2(1-\rho^2)}{2(1-\rho^2)}\right) = \frac{1}{2\pi}re^{-\frac{r^2}{2}}.$$

边际密度函数:

$$f_R(r) = re^{-\frac{r^2}{2}}, \quad f_\Theta(\theta) = \frac{1}{2\pi}.$$

R, Θ 两变量相互独立.

4. 这相当于求 $P(0 \le \Theta \le \pi/2) = 1/4$.

5. 伸缩 a, b 不改变 X, Y 的符号, 只需考虑 a, b 是单位向量或为 $\mathbf{0}$ 的情形.

先考虑 $a, b \ne 0$:
- $a \ne b$ 且 $a \ne -b$: 如上, 所求 $= 1/4$.
- $a = b$: 退化为 $P(X \ge 0)$, 由 $X \sim N(0,1)$ 知所求 $= 1/2$.
- $a = -b$: 退化为 $P(X = Y = 0)$, 所求 $= 0$.

再考虑 a, b 恰有一个为 0 : (不失一般性, 设 $a = 0, b \ne 0$) 退化为 $P(Y \ge 0)$, 所求 $= 1/2$.

$a = b = 0$ 时所求 $= 1$.

Question 5

1. $X_i^2 \sim \Gamma(1/2, 1/2), Y \sim \Gamma(n/2, 1/2)$.
2.
$$
\begin{aligned}
E\big(e^{t(Y-n)}\big) &= \int_0^{+\infty} e^{t(x-n)} \frac{(1/2)^{\frac{n}{2}}}{\Gamma(n/2)} x^{\frac{n}{2}-1} e^{-\frac{1}{2}x} \, \mathrm{d}x \\
&= \frac{1}{\Gamma(\frac{n}{2})} e^{-tn} \left(\frac{1}{2}\right)^{\frac{n}{2}} \int_0^{+\infty} x^{\frac{n}{2}-1} e^{-(\frac{1}{2}-t)x} \, \mathrm{d}x \\
&= \frac{1}{\Gamma(\frac{n}{2})} e^{-tn} \left(\frac{1}{2}\right)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \left(\frac{1}{2} - t\right)^{-\frac{n}{2}} \\
&= e^{-tn} \left(\frac{1}{1-2t}\right)^{\frac{n}{2}} \\
&= \exp\left(-tn - \frac{n}{2}\ln(1-2t)\right) \\
&\leq \exp\left(-tn - \frac{n}{2}(-2t - 4t^2)\right) = e^{2t^2 n}.
\end{aligned}
$$

3.
$$
\begin{aligned}
P(Y \geq (1+\Delta)n) &= P(Y - n \geq \Delta n) \\
&= P\big(e^{t(Y-n)} \geq e^{t\Delta n}\big) \\
&\leq \exp(t\Delta n - 2t^2 n).
\end{aligned}
$$

取 $t = 1/4$ 即证.

4.
$$
P(Y \leq (1-\Delta)n) = P\big(e^{-t(Y-n)} \geq e^{-t\Delta n}\big),
$$

其中 $E\big(e^{-t(Y-n)}\big) \leq \exp(2(-t)^2 n) = \exp(2t^2 n)$, 于是
$$
P(Y \leq (1 - \Delta n)) \leq \exp(-t\Delta n - 2t^2 n),
$$

取 $t = 1/3$ 即证.

Question 6

1. $\mathrm{tr}(A^3)$ 由 $A_{ij}A_{jk}A_{ki}$ 形式的项组成, 其中
   - 至少有一个元素仅出现一次 (无妨设为 A_{ij}), 则
   $$
   E\big(A_{ij}A_{jk}A_{ki}\big) = E\big(A_{ij}\big)E\big(A_{jk}A_{ki}\big) = 0.
   $$
   - i, j, k 全相同,
   $$
   E(A_{ii}^3) = 0.
   $$

   故 $E\big(\mathrm{tr}(A^3)\big) = 0$.
   $\mathrm{tr}(A^4)$ 中期望非零的项只有 $A_{ij}A_{ji}A_{ij}A_{ji}$ 和 A_{ii}^4,
   $$
   E\big(\mathrm{tr}(A^4)\big) = nE(A_{ii}^4) + n(n-1)E(A_{ij}^2)E(A_{ji}^2) = 3n + n(n-1) = n^2 + 2n.
   $$

   其中

$$E(A_{ii}^4) = E(X^2) = \int_0^{+\infty} x^2 \cdot \frac{(1/2)^{\frac{1}{2}}}{\Gamma(1/2)} x^{-\frac{1}{2}} e^{-\frac{1}{2}x} \, \mathrm{d}x$$

$$= \frac{1}{\Gamma(1/2)} \cdot \frac{1}{\sqrt{2}} \int_0^{+\infty} x^{\frac{3}{2}} e^{-\frac{1}{2}x} \, \mathrm{d}x$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} (2t)^{\frac{3}{2}} e^{-t} \, \mathrm{d}t$$

$$= \frac{4\Gamma(5/2)}{\sqrt{\pi}} = 3.$$

2. $E\big(\mathrm{tr}(B^2)\big) = nE(B_{ii}^2) + n(n-1)E(B_{ij}^2) = n^2 E(B_{ii}^2) = n^2$,

对 B^4 有贡献的项包括

- B_{ii}^4,

- $B_{ii}^2 B_{ij} B_{ji}, B_{ii} B_{ij} B_{ji} B_{ii}, B_{ij} B_{ji} B_{ii}^2, B_{ij} B_{jj}^2 B_{ji}, B_{ij} B_{ji} B_{ij} B_{ji}$,

- $B_{ij} B_{ji} B_{ik} B_{ki}, B_{ij} B_{jk} B_{kj} B_{ji}$.

$$E(\mathrm{tr}(B^4)) = n^2 E(B_{ij}^4) + 4n(n-1)E\big(A_{ij}^2\big)^2 + 2n(n-1)(n-2)E\big(A_{ij}^2\big)^2$$

$$= 3n^2 + 4n(n-1) + 2n(n-1)(n-2)$$

$$= 3n^2 + 4n^2 - 4n + 2n^3 - 6n^2 + 4n$$

$$= 2n^3 + n^2.$$

(利用 Wick 配对模式计算更准确)

3. 已知 B 可对角化, 若 B 有特征值 λ_i, 则 B^4 有特征值 $\lambda_i^4 \geq 0$. 于是

$$P\big(|\lambda_i| > 4n^{3/4}\big) = P(\lambda_i^4 > 64n^3)$$

$$\leq P\big(\mathrm{tr}(B^4) > 64n^3\big)$$

$$\leq \frac{2n^3 + n^2}{64n^3} < 0.1.$$