Question 1

1. $P(X \geq a) \leq \mathbb{E}(X)/a = 1/a$.
2. 不等式在 $a \in (0,1)$ 时显然成立. 设 $a > 1$.

   $P(X \geq a) = P(X - \mathbb{E}(x) \geq (a-1)\sigma(X)) \leq 1/(a-1)^2$.
3. 考虑计算 $\mathbb{E}(e^{tX})$. 下面的计算要求 $t < 1$.

$$\begin{aligned}\mathbb{E}(e^{tX}) &= \int_0^{+\infty} e^{-x} \cdot e^{tx} \,\mathrm{d}x \\ &= \int_0^{+\infty} e^{-(1-t)x} \,\mathrm{d}x \\ &= \frac{1}{t-1} e^{-(1-t)x} \bigg|_0^{+\infty} = \frac{1}{1-t} \\ &= \sum_{k=0}^{\infty} t^k.\end{aligned}$$

   于是 $\mathbb{E}(X^k) = k!, P(X \geq a) = P(X^k \geq a^k) \leq \frac{\mathbb{E}(X^k)}{a^k} = k!/a^k$.
4. 要求 $\min_t \frac{1}{1-t} e^{-ta}$. 记该函数为 $f(t)$.

$$\begin{aligned}f'(t) &= \frac{1}{(1-t)^2} e^{-ta} + \frac{1}{1-t} \cdot (-a) e^{-ta} \\ &= \frac{1}{1-t} e^{-ta} \cdot \left(\frac{1}{1-t} - a\right).\end{aligned}$$

   于是取 $t = 1 - a^{-1}$ 即得 $P(X \geq a) \leq a e^{1-a}$.
5. 所求为

$$P(X \geq a) = \int_a^{+\infty} e^{-x} \,\mathrm{d}x = e^{-a}.$$

6. 令 $K \sim \pi(ta), f(k) = \mathbb{E}(X^k)/a^k$.

$$\begin{aligned}\mathbb{E}(f(K)) &= \sum_{k=0}^{+\infty} \frac{(ta)^k}{k!} e^{-ta} \cdot \frac{k!}{a^k} \\ &= e^{-ta} \sum_{k=0}^{+\infty} t^k \\ &= \frac{e^{-ta}}{1-t}.\end{aligned}$$

   取 $t = 1 - a^{-1}$ 即有 $\mathbb{E}(f(K)) = \lambda_{\text{Chernoff}}$. 于是矩函数给出的界较 Chernoff 更紧.

Question 2

1. 对于 $\varepsilon < 1/2$, 有

$$\lim_{n\to\infty} P(|X_n - X| < \varepsilon) = \lim_{n\to\infty} \frac{1}{n} = 0,$$

   于是 $\{X_n\}$ 依概率收敛到 0. 相对的

$$P\left(\bigcup_{m=n}^{\infty}\right)P(|X_m - X| < \varepsilon) = \sum_{m=n}^{\infty}\frac{1}{m}$$

求和发散, 于是 $\{X_n\}$ 不几乎必然收敛到 0.

2. 利用 Chernoff-Hoeffding 不等式:

$$\begin{aligned}P(|Y_m - p| > \varepsilon) &= P\left(\left|\sum_{i=1}^{m} X_i - mp\right| > m\varepsilon\right)\\ &\le 2\cdot\exp\left(-\frac{2m^2\varepsilon^2}{m}\right)\\ &= 2\exp(-2m\varepsilon^2).\end{aligned}$$

于是

$$\begin{aligned}P\left(\bigcup_{m=n}^{\infty}|Y_m - p| > \varepsilon\right) &\le 2\sum_{m=n}^{+\infty}\exp(-2m\varepsilon^2)\\ &= 2\exp(-2n\varepsilon^2)\cdot\frac{1}{1-\exp(-2\varepsilon^2)} \longrightarrow 0.\end{aligned}$$

Question 3

观察到若 A 输出“合数”, 则输入为合数; 若 B 输出“质数”, 则输入为质数.

考虑如下算法: 循环执行 AB, 直至某一程序产生如上输出. 下面证明这一算法运行时间期望为 $O(T)$.

由对称性无妨设输入为质数, 则 C 在 B 输出“质数”时停止. 于是

$$\mathbb{E}(T(C)) \le 2\mathbb{E}\left(G\left(\frac{1}{2}\right)\right)\cdot O(T) = O(T).$$

Question 4

1. 有

$$\begin{aligned}M_X(t) &= \left(M_{B(1,p)}(t)\right)^n = (pe^t + 1 - p)^n\\ &= \exp(n\cdot\ln(pe^t + 1 - p))\\ &\le \exp(n\cdot(pe^t - p))\\ &= \exp(\mathbb{E}(X)\cdot(e^t - 1)).\end{aligned}$$

2. 对任意 $\varepsilon > 0$,

$$\begin{aligned}P(X \ge (1+\varepsilon)\mathbb{E}(X)) &\le M_X(t)\cdot\exp(-t\cdot(1+\varepsilon)\mathbb{E}(X))\\ &\le \exp(\mathbb{E}(X)\cdot(e^t - 1 - t(1+\varepsilon))).\end{aligned}$$

对 $f(t) := e^t - 1 - t(1+\varepsilon)$ 求导:

$$f'(t) = e^t - (1+\varepsilon),$$

取 $t = \ln(1+\varepsilon)$ 即有

$$\begin{aligned}
P(X \geq (1+\varepsilon)\mathbb{E}(X)) &\leq \exp(\mathbb{E}(X) \cdot (\varepsilon - \ln(1+\varepsilon)(1+\varepsilon))) \\
&= \left(\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right)^{\mathbb{E}(X)}.
\end{aligned}$$

对任意 $0 < \varepsilon < 1$,

$$\begin{aligned}
P(X \leq (1-\varepsilon)\mathbb{E}(X)) &\leq P\big(e^{-tX} \geq e^{-t(1-\varepsilon)\mathbb{E}(X)}\big) \\
&\leq M_X(-t) \cdot \exp(t(1-\varepsilon)\mathbb{E}(X)) \\
&= \exp(\mathbb{E}(X) \cdot (e^{-t} - 1 + t(1-\varepsilon))).
\end{aligned}$$

对 $g(t) := e^{-t} - 1 + t(1-\varepsilon)$ 求导:

$$g'(t) = -e^{-t} + 1 - \varepsilon,$$

取 $t = -\ln(1-\varepsilon)$ 即有

$$\begin{aligned}
P(X \leq (1-\varepsilon)\mathbb{E}(X)) &\leq \exp(\mathbb{E}(X) \cdot (1-\varepsilon-1-\ln(1-\varepsilon)(1-\varepsilon))) \\
&= \exp(\mathbb{E}(X) \cdot (-\varepsilon - \ln(1-\varepsilon)(1-\varepsilon))) \\
&= \left(\frac{e^{-\varepsilon}}{(1-\varepsilon)^{1-\varepsilon}}\right)^{\mathbb{E}(X)}.
\end{aligned}$$

3. 由 Union Bound 只需证明 $X \sim B(n, 1/n)$ 时 $P(X \geq c_1 \ln n / \ln\ln n) \leq 1/n^2$.

$$\begin{aligned}
\ln P(X \geq c_1 \ln n / \ln\ln n) &\leq \ln\left(\frac{e^{\frac{c_1 \ln n}{\ln\ln n} - 1}}{\left(\frac{c_1 \ln n}{\ln\ln n}\right)^{\frac{c_1 \ln n}{\ln\ln n}}}\right) \\
&= \frac{c_1 \ln n}{\ln\ln n} - 1 - \left(\frac{c_1 \ln n}{\ln\ln n}\right)(\ln c_1 + \ln\ln n - \ln\ln\ln n) \\
&\leq \left[\frac{c_1 \ln n}{\ln\ln n} - \frac{c_1 \ln c_1 \ln n}{\ln\ln n}\right] - c_1 \ln n + \frac{c_1 \ln n \ln\ln\ln n}{\ln\ln n} \\
&= \ln n \cdot \left(\underbrace{\frac{c_1(1-\ln c_1)}{\ln\ln n}}_{\mathrm{A}} - c_1 + \underbrace{\frac{c_1 \ln\ln\ln n}{\ln\ln n}}_{\mathrm{B}}\right)
\end{aligned}$$

其中 $A, B \overset{n\to\infty}{\longrightarrow} 0$. 于是当 n 充分大时 $\ln P(X \geq c_1 \ln n / \ln\ln n) \leq -c_1 \ln n$, 取 $c_1 = 3$ 即证.

4. 有

$$\begin{aligned}
\mathbb{E}(Y) &\leq c_1 \ln n / \ln\ln n + n \cdot P(Y \geq c_1 \ln n / \ln\ln n) \\
&\leq c_1 \ln n / \ln\ln n + 1 \\
&\leq (c_1 + 1) \ln n / \ln\ln n.
\end{aligned}$$

Question 5

1. 考虑由 $\{y_i\}, \{-y_i\}, \{\mathbf{0}\}$ 组成的 $\{x_i\}$, 则不等式中的向量是 $\{x_i - x_j\}$ 的子集, 结论由题设成立.

2. 假定 (1) 中全部命题成立, 则有

$$
\begin{aligned}
\langle \frac{1}{\sqrt{k}} A y_i, \frac{1}{\sqrt{k}} A y_j \rangle &= \frac{1}{2}\left[\left\|\frac{1}{\sqrt{k}} A(y_i + y_j)\right\|^2 - \left\|\frac{1}{\sqrt{k}} A y_i\right\|^2 - \left\|\frac{1}{\sqrt{k}} A y_j\right\|^2\right] \\
&\leq \frac{1}{2}\left[\left(1 + \frac{\varepsilon}{4}\right)\|y_i + y_j\|^2 - \left(1 - \frac{\varepsilon}{4}\right)\|y_i\|^2 - \left(1 - \frac{\varepsilon}{4}\right)\|y_j\|^2\right] \\
&= \left(1 + \frac{\varepsilon}{4}\right)\langle y_i, y_j \rangle + \frac{\varepsilon}{4}\left(\|y_i\|^2 + \|y_j\|^2\right),
\end{aligned}
$$

即

$$
\left|\langle \frac{1}{\sqrt{k}} A y_i, \frac{1}{\sqrt{k}} A y_j \rangle - \langle y_i, y_j \rangle\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon.
$$

3. 将 x_i 正则化为 $\hat{x_i} = x_i / \|x_i\|$ 即可. $x_i = \mathbf{0}$ 时显然.
4. 考虑 $\mathbb{R}^n$ 的单位正交基 $e_1, ..., e_n$, 构造如上随机矩阵 $B \in \mathrm{M}_{k \times n}, k = O(\log n / \varepsilon^2), \varepsilon$ 取 0.1. 利用上一问结论有

$$
\left|\langle \frac{1}{\sqrt{k}} B e_i, \frac{1}{\sqrt{k}} B e_j \rangle - \langle e_i, e_j \rangle\right| \leq \varepsilon.
$$

构造矩阵 $A := \frac{1}{k} B^\top B$, $A_{ij} - I_{ij}$ 即满足如上不等式, 于是 A 符合要求.

Question 6

1. 显然 Y_i 的密度函数为 $g(x) = \frac{2}{\pi(x^2+1)}, x > 0$.
2. 考虑对 Y_i 进行截断. 令 $Y_i^{(t)} := \min(Y_i, t), Z_i := Y_i - Y_i^{(t)} = \max(Y_i - t, 0)$, 则

$$
P\left(\sum Y_i > c_1 n^2\right) \leq P\left(\sum Z_i > 0\right) + P\left(\sum Y_i^{(t)} > c_1 n^2\right).
$$

- 先证 $P(\sum Z_i > 0) < 1/6$. 由 Union Bound 只需要保证 $P(Z_i > 0) < 1/6n$. 已知

$$
P(Z_i > 0) = P(Y_i > t) = \frac{2}{\pi}\left(\frac{\pi}{2} - \arctan t\right) = 1 - \frac{2}{\pi}\arctan t,
$$

洛必达法则告诉我们 $t \to +\infty$ 时 $\pi/2 - \arctan t$ 和 $1/t$ 同阶, 于是取 $t = O(n)$ 即可满足要求.

- 接下来证 $P\left(\sum Y_i^{(t)} > c_1 n^2\right) < 1/6$. 有

$$
\begin{aligned}
\mathbb{E}(Y_i) &= \int_0^t x \cdot \frac{2}{\pi(x^2+1)}\, \mathrm{d}x + \left(1 - \frac{2}{\pi}\arctan t\right) \cdot t \\
&= \frac{2}{\pi}\int_0^t \frac{x}{x^2+1}\, \mathrm{d}x + \left(1 - \frac{2}{\pi}\arctan t\right) \cdot t \\
&= \frac{1}{\pi}\ln(t^2+1) + \left(1 - \frac{2}{\pi}\arctan t\right) \cdot t.
\end{aligned}
$$

取 $t = O(n)$ 时由 Markov 不等式

$$
\begin{aligned}
P\left(\sum Y_i^{(t)} > c_1 n^2\right) &\leq \frac{\frac{1}{\pi}\ln(t^2+1) - \left(1 - \frac{2}{\pi}\arctan t\right) \cdot t}{c_1 n} \\
&\leq \frac{O(\ln n) - O\left(\frac{1}{n}\right) \cdot O(n)}{c_1 n} = O\left(\frac{\ln n}{n}\right),
\end{aligned}
$$

选取足够大的 c_1 即可保证概率不超过 1/6. 明所欲证.

3. 利用和上文类似的截断方式可知

$$\mathbb{E}\left(\sum Y_i\right) = n \cdot \mathbb{E}(Y_i) = O(n \ln t),$$

利用 Chernoff-Hoeffding 不等式即有

$$\begin{aligned} P\left(\sum Y_i^{(t)} \le c_2 n\right) &= P\left(\sum Y_i^{(t)} \le \mathbb{E}(Y_i) - O(n \ln t)\right) \\ &\le \exp\left(-O\left(\frac{n \ln^2 t}{t^2}\right)\right). \end{aligned}$$

令 $t = O(\sqrt{n})$ 并利用 $P(\sum Y_i < c_2 n) < P\left(\sum Y_i^{(t)} < c_2 n\right)$ 即证.