Question 1

有 $P(Y \geq t) = \prod_{i=1}^{n} P(X_i \geq t) = \prod_{i=1}^{n} e^{-\lambda_i t} = \exp(-t \cdot \sum \lambda_i)$, 这说明 $Y \sim \mathrm{Exp}(\sum \lambda_i)$.

Question 2

1. $f_U(u) = \int_0^1 f_X(x) f_Y(u-x)\,\mathrm{d}x = \begin{cases} u & , \ 0 \leq u \leq 1, \\ 2-u \ , & 1 < u \leq 2. \end{cases}$

   $f_V(v) = \int_0^1 f_X(x) f_Y(x-v)\,\mathrm{d}x = \begin{cases} v+1 \ , & -1 \leq v \leq 0, \\ 1-v \ , & 0 < v \leq 1. \end{cases}$

2. $$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1 \cdot \mathrm{abs}\left(\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}\right) = \frac{1}{2}.$$

   取值范围是 $(0,0),(1,1),(2,0),(1,-1)$ 顺次连接形成的正方形内.

3. 利用几何形状直接写出 $f_U(u) = \begin{cases} u & , \ 0 \leq u \leq 1, \\ 2-u \ , & 1 < u \leq 2, \end{cases}$ $f_V(v) = \begin{cases} v+1 \ , & -1 \leq v \leq 0, \\ 1-v \ , & 0 < v \leq 1. \end{cases}$

   V 的取值范围与 U 有关, 故两者不独立.

4. • 对于 $0 \leq u \leq 1$: $f_{V|U=u}(v) = f_{U,V}(u,v)/f_U(u) = \frac{1}{2u}$.

   • 对于 $1 < u \leq 2$: $f_{V|U=u}(v) = \frac{1}{4-2u}$.

5. 先计算 $\sigma(U)$ 和 $\sigma(V)$:

$$\begin{aligned} \sigma(U) &= \sqrt{\mathrm{Var}(U)} = \sqrt{E(U^2) - E(U)^2} \\ &= \sqrt{\int_0^1 u^3\,\mathrm{d}u + \int_1^2 (2-u)u^2\,\mathrm{d}u - 1} \\ &= \sqrt{\frac{1}{4} + \left(\frac{2}{3}u^3 - \frac{1}{4}u^4\right)\Big|_1^2 - 1} \\ &= \sqrt{\frac{1}{4} + \left(\frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}\right) - 1} = \sqrt{\frac{1}{6}}. \end{aligned}$$

   V 的边际分布和 U 仅相差一个偏移, 故 $\sigma(V) = \sqrt{\frac{1}{6}}$.

   再来计算 $\mathrm{Cov}(U,V)$:

$$\mathrm{Cov}(U,V) = E(UV) - E(U)E(V),$$

   其中

$$E(UV) = \int_0^2 \int_{-u}^{u} \frac{1}{2} uv\,\mathrm{d}v\,\mathrm{d}u = \frac{1}{2}\int_0^2 u\left(\int_{-u}^{u} v\,\mathrm{d}v\right)\mathrm{d}u = 0,$$

   故 $\mathrm{Cov}(U,V) = 0$, 两变量不相关.

Question 4

1. 写出 $A = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix}$, 则 $(X,Y)^T = AG^T$, 故
$$(X,Y) \sim N(A \cdot \mathbf{0}, A \cdot 1_{n\times n} \cdot A^T) = N\left(\mathbf{0}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

2.
$$\begin{aligned} x^2 + y^2 - 2\rho xy &= (x-\rho y)^2 + (1-\rho^2)y^2 \\ &= \left(r\sqrt{1-\rho^2}\cos\theta\right)^2 + (1-\rho^2)r^2\sin^2\theta \\ &= (1-\rho^2)r^2. \end{aligned}$$

3. 先求 $\dfrac{\partial(X,Y)}{\partial(R,\Theta)}$:
$$\frac{\partial X}{\partial R} = \sqrt{1-\rho^2}\cos\Theta + \rho\sin\Theta, \quad \frac{\partial X}{\partial\Theta} = -R\sqrt{1-\rho^2}\sin\Theta + R\rho\cos\Theta,$$
$$\frac{\partial Y}{\partial R} = \sin\Theta, \quad \frac{\partial Y}{\partial\Theta} = R\cos\Theta.$$
于是
$$|J| = \begin{vmatrix} \sqrt{1-\rho^2}\cos\Theta + \rho\sin\Theta & -R\sqrt{1-\rho^2}\sin\Theta + R\rho\cos\Theta \\ \sin\Theta & R\cos\Theta \end{vmatrix} = R\sqrt{1-\rho^2}.$$
于是
$$\begin{aligned} f_{R,\Theta}(r,\theta) &= f(x,y)\cdot|J| \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right)\cdot r\sqrt{1-\rho^2} \\ &= \frac{r}{2\pi}\exp\left(-\frac{r^2(1-\rho^2)}{2(1-\rho^2)}\right) = \frac{1}{2\pi}re^{-\frac{r^2}{2}}. \end{aligned}$$
边际密度函数:
$$f_R(r) = re^{-\frac{r^2}{2}}, \quad f_\Theta(\theta) = \frac{1}{2\pi}.$$
R,Θ 两变量相互独立.

4. 这相当于求 $P(0 \le \Theta \le \pi/2) = 1/4$.

5. 伸缩 a,b 不改变 X,Y 的符号, 只需考虑 a,b 是单位向量或为 $\mathbf{0}$ 的情形.

   先考虑 $a,b \ne 0$:
   - $a \ne b$ 且 $a \ne -b$: 如上, 所求 $= 1/4$.
   - $a = b$: 退化为 $P(X \ge 0)$, 由 $X \sim N(0,1)$ 知所求 $= 1/2$.
   - $a = -b$: 退化为 $P(X = Y = 0)$, 所求 $= 0$.

   再考虑 a,b 恰有一个为 0 : (不失一般性, 设 $a = 0, b \ne 0$) 退化为 $P(Y \ge 0)$, 所求 $= 1/2$.

   $a = b = 0$ 时所求 $= 1$.

Question 5

1. $X_i^2 \sim \Gamma(1/2, 1/2), Y \sim \Gamma(n/2, 1/2)$.
2. $$\begin{aligned} E\left(e^{t(Y-n)}\right) &= \int_0^{+\infty} e^{t(x-n)} \frac{(1/2)^{\frac{n}{2}}}{\Gamma(n/2)} x^{\frac{n}{2}-1} e^{-\frac{1}{2}x} \,\mathrm{d}x \\ &= \frac{1}{\Gamma\left(\frac{n}{2}\right)} e^{-tn} \left(\frac{1}{2}\right)^{\frac{n}{2}} \int_0^{+\infty} x^{\frac{n}{2}-1} e^{-\left(\frac{1}{2}-t\right)x} \,\mathrm{d}x \\ &= \frac{1}{\Gamma\left(\frac{n}{2}\right)} e^{-tn} \left(\frac{1}{2}\right)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \left(\frac{1}{2} - t\right)^{-\frac{n}{2}} \\ &= e^{-tn} \left(\frac{1}{1-2t}\right)^{\frac{n}{2}} \\ &= \exp\left(-tn - \frac{n}{2} \ln(1-2t)\right) \\ &\le \exp\left(-tn - \frac{n}{2}(-2t - 4t^2)\right) = e^{2t^2 n}. \end{aligned}$$
3. $$\begin{aligned} P(Y \ge (1+\Delta)n) &= P(Y - n \ge \Delta n) \\ &= P\left(e^{t(Y-n)} \ge e^{t\Delta n}\right) \\ &\le \exp(2t^2 n - t\Delta n). \end{aligned}$$

   取 $t = \Delta/4$ 即证.
4. $$P(Y \le (1-\Delta)n) = P\left(e^{-t(Y-n)} \ge e^{t\Delta n}\right),$$

   其中 $E\left(e^{-t(Y-n)}\right) \le \exp(2(-t)^2 n) = \exp(2t^2 n)$, 于是

   $$P(Y \le (1 - \Delta n)) \le \exp(2t^2 n - t\Delta n),$$

   取 $t = \Delta/4$ 即证.

Question 6

1. $\mathrm{tr}(A^3)$ 由 $A_{ij}A_{jk}A_{ki}$ 形式的项组成, 其中
   - 至少有一个元素仅出现一次 (无妨设为 A_{ij}), 则

     $$E\left(A_{ij}A_{jk}A_{ki}\right) = E\left(A_{ij}\right)E\left(A_{jk}A_{ki}\right) = 0.$$
   - i, j, k 全相同,

     $$E\left(A_{ii}^3\right) = 0.$$

   故 $E\left(\mathrm{tr}\left(A^3\right)\right) = 0$.

   $\mathrm{tr}\left(A^4\right)$ 中期望非零的项只有 $A_{ij}A_{ji}A_{ij}A_{ji}$ 和 A_{ii}^4,

   $$E\left(\mathrm{tr}\left(A^4\right)\right) = nE\left(A_{ii}^4\right) + n(n-1)E\left(A_{ij}^2\right)E\left(A_{ji}^2\right) = 3n + n(n-1) = n^2 + 2n.$$

   其中

$$
\begin{aligned}
E(A_{ii}^4) = E(X^2) &= \int_0^{+\infty} x^2 \cdot \frac{(1/2)^{\frac{1}{2}}}{\Gamma(1/2)} x^{-\frac{1}{2}} e^{-\frac{1}{2}x} \,\mathrm{d}x \\
&= \frac{1}{\Gamma(1/2)} \cdot \frac{1}{\sqrt{2}} \int_0^{+\infty} x^{\frac{3}{2}} e^{-\frac{1}{2}x} \,\mathrm{d}x \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} (2t)^{\frac{3}{2}} e^{-t} \,\mathrm{d}t \\
&= \frac{4\Gamma(5/2)}{\sqrt{\pi}} = 3.
\end{aligned}
$$

2. $E\big(\mathrm{tr}\big(B^2\big)\big) = nE\big(B_{ii}^2\big) + n(n-1)E\big(B_{ij}^2\big) = n^2 E\big(B_{ii}^2\big) = n^2$,

   对 B^4 有贡献的项包括

   - B_{ii}^4,
   - $B_{ii}^2 B_{ij} B_{ji}, B_{ii} B_{ij} B_{ji} B_{ii}, B_{ij} B_{ji} B_{ii}^2, B_{ij} B_{jj}^2 B_{ji}, B_{ij} B_{ji} B_{ij} B_{ji}$,
   - $B_{ij} B_{ji} B_{ik} B_{ki}, B_{ij} B_{jk} B_{kj} B_{ji}$.

$$
\begin{aligned}
E\big(\mathrm{tr}\big(B^4\big)\big) &= n^2 E\big(B_{ij}^4\big) + 4n(n-1)E\big(A_{ij}^2\big)^2 + 2n(n-1)(n-2)E\big(A_{ij}^2\big)^2 \\
&= 3n^2 + 4n(n-1) + 2n(n-1)(n-2) \\
&= 3n^2 + 4n^2 - 4n + 2n^3 - 6n^2 + 4n \\
&= 2n^3 + n^2.
\end{aligned}
$$

   (利用 Wick 配对模式计算更准确)

3. 已知 B 可对角化, 若 B 有特征值 λ_i, 则 B^4 有特征值 $\lambda_i^4 \geq 0$. 于是

$$
\begin{aligned}
P\big(|\lambda_i| > 4n^{3/4}\big) &= P\big(\lambda_i^4 > 64n^3\big) \\
&\leq P\big(\mathrm{tr}\big(B^4\big) > 64n^3\big) \\
&\leq \frac{2n^3 + n^2}{64n^3} < 0.1.
\end{aligned}
$$