Question 1

1.
$$\begin{aligned}P(X+Y=k) &= \sum_{l=r}^{k-1} P(Y=l)\cdot P(X=k-l) \\ &= \sum_{l=r}^{k-1}\binom{l-1}{r-1}p^r(1-p)^{l-r}\cdot(1-p)^{k-l-1}p \\ &= p^{r+1}(1-p)^{k-1-r}\sum_{l=r}^{k-1}\binom{l-1}{r-1} \\ &= \binom{k-1}{r}p^{r+1}(1-p)^{k-1-r}.\end{aligned}$$

这里利用了

$$\begin{aligned}\binom{n}{m} &= \binom{n-1}{m}+\binom{n-1}{m-1} \\ &= \binom{n-2}{m}+\binom{n-2}{m-1}+\binom{n-1}{m-1} \\ &= \ldots \\ &= \sum_{k=m-1}^{n-1}\binom{k}{m-1}.\end{aligned}$$

2. $\mathrm{NB}(1,p)$ 等价于 $G(p)$, 归纳即证.

Question 2

1.
$$\begin{aligned}E(XY \mid A) &= \sum_i\sum_j x_i y_j P\big(X=x_i \wedge Y=y_j \mid A\big) \\ &= \sum_i\sum_j x_i y_j P(X=x_i \mid A)P\big(Y=y_j \mid A\big) \\ &= \left(\sum_i x_i P(X=x_i \mid A)\right)\left(\sum_j y_j P\big(Y=y_j \mid A\big)\right) \\ &= E(X \mid A)\cdot E(Y \mid A).\end{aligned}$$

2. X_i, X_j 在 $E_{i,j}$ 的条件下条件独立, 两者均服从 $X-1 \sim B(n-2, 1/2)$. 于是

$$E\big(X_i X_j \mid E_{i,j}\big) = E\big(X_i \mid E_{i,j}\big)\cdot E\big(X_j \mid E_{i,j}\big) = \frac{n^2}{4}.$$

3. 类似的有

$$E\big(X_i X_j \mid \neg E_{i,j}\big) = \frac{(n-2)^2}{4}.$$

由重期望公式有

$$
\begin{aligned}
E(X_i X_j) &= \frac{1}{2} E(X_i X_j \mid E_{i,j}) + \frac{1}{2} E(X_i X_j \mid \neg E_{i,j}) \\
&= \frac{1}{2} \cdot \frac{n^2}{4} + \frac{1}{2} \cdot \frac{(n-2)^2}{4} = \frac{(n-1)^2}{4} + \frac{1}{4}.
\end{aligned}
$$

Question 3

1. 由几何分布的无记忆性, 记事件 A 表示第一次试验未成功, 则

   $$E(X^2 \mid A) = E((X+1)^2).$$

   利用重期望公式即证. 三次方的情形是完全相同的.

   计算 $E(X^2)$:

   $$
   \begin{aligned}
   E(X^2) &= p + (1-p)\left(E(X^2) + 2 \cdot \frac{1}{p} + 1\right) \\
   \implies E(X^2) &= 1 + \frac{(p+2)(1-p)}{p^2} = \frac{2-p}{p^2}.
   \end{aligned}
   $$

   计算 $E(X^3)$:

   $$
   \begin{aligned}
   E(X^3) &= p + (1-p)\left(E(X^3) + \frac{3(2-p)}{p^2} + \frac{3}{p} + 1\right) \\
   \implies E(X^3) &= \frac{p^2 - 6p + 6}{p^3}.
   \end{aligned}
   $$

2. 令 $\overline{X} \sim G(1-p)$, 类似于上一问列出方程:

   $$
   \begin{aligned}
   E(Y) &= pE\left(\overline{X} + 1\right) + (1-p)E(X+1), \\
   E(Y^2) &= pE\left(\left(\overline{X} + 1\right)^2\right) + (1-p)E((X+1)^2).
   \end{aligned}
   $$

   计算 $E(Y)$:

   $$E(Y) = p \cdot \left(\frac{1}{1-p} + 1\right) + (1-p) \cdot \left(\frac{1}{p} + 1\right) = \frac{1}{p} + \frac{1}{1-p} - 1.$$

   计算 $E(Y^2)$:

   $$
   \begin{aligned}
   E(Y^2) &= p\left(\frac{2-(1-p)}{(1-p)^2} + \frac{2}{1-p} + 1\right) + (1-p)\left(\frac{2-p}{p^2} + \frac{2}{p} + 1\right) \\
   &= \frac{2 + (1-p) - (1-p)^3}{(1-p)^2} + \frac{2 + p - p^3}{p^2}.
   \end{aligned}
   $$

   于是 $\mathrm{Var}(Y) = E(Y^2) - E(Y)^2$.

Question 4

1. 只要将所有的 Y_i 构成的元组视作同一个随机“变量” Z, 这与一般的重期望公式没有区别.
2. 令 $E_{i,j}$ 表示 i, j 是否被计入 Y. 计算 $E(Y \mid \{X_{i,j}\})$:

$$E\big(Y \mid \{X_{i,j}\}\big) = \sum_{i<j} E\big(E_{i,j} \mid \{X_{i,j}\}\big) = \frac{1}{2}\sum_{i<j} X_{i,j}.$$

对于 $i < j < k$, 显然 $E_{i,j}$ 和 $E_{j,k}$ 独立. 计算 $E\big(Y^2 \mid \{X_{i,j}\}\big)$:

$$\begin{aligned} E\big(Y^2 \mid \{X_{i,j}\}\big) &= \sum_{i<j} E\big(E_{i,j}^2 \mid \{X_{i,j}\}\big) + \sum_{\substack{i<j,\\ k<l,\\ \{i,j\}\neq\{k,l\}}} E\big(E_{i,j}E_{k,l} \mid \{X_{i,j}\}\big) \\ &= \frac{1}{2}A + \frac{1}{4}(A^2 - A) = \frac{1}{4}(A^2 + A), \end{aligned}$$

其中 $A \coloneqq \sum_{i<j} X_{i,j}$.

3. 计算 $E(Y)$:

$$E(Y) = \frac{1}{2}E(A) = \frac{n(n-1)}{8}.$$

计算 $\operatorname{Var}(Y)$:

$$\begin{aligned} \operatorname{Var}(Y) &= E(Y^2) - E(Y)^2 \\ &= \frac{1}{4}E(A^2) + \frac{1}{4}E(A) - E(Y)^2 \\ &= \frac{1}{4}E(A)^2 + \frac{1}{4}\operatorname{Var}(A) + \frac{1}{4}E(A) - \frac{1}{4}E(A)^2 \\ &= \frac{n(n-1)}{16} + \frac{1}{4}\cdot\frac{n(n-1)}{2}\cdot\frac{1}{4} \\ &= \frac{3n(n-1)}{32}. \end{aligned}$$

Question 5

1. $E(Y) = nE(X) = 0$.
2. $E(Y^2) = \sum E(X_i^2) + 2\sum_{i<j} E\big(X_iX_j\big) = n$.
3. $E\Big(\big(\sum X_i\big)^4\Big)$ 展开后有四类项:
   - X_i^4 : $\quad n \cdot 1 = n$.
   - $X_i^3X_j$: $\quad E\big(X_i^3X_j\big) = E\big(X_iX_j\big) = 0$.
   - $X_i^2X_j^2$: $\quad \binom{n}{2}\binom{4}{2} \cdot 1 = 3n(n-1)$.
   - $X_i^2X_jX_k$: $\quad E\big(X_i^2X_jX_k\big) = E\big(X_jX_k\big) = 0$.
   - $X_iX_jX_kX_l$: $\quad E\big(X_iX_jX_kX_l\big) = 0$.

   于是 $E(Y^4) = n + 3n(n-1) = 3n^2 - 2n$.
4. 要证 $P\big(Y^4 \geq k^4 \cdot n^2\big) \leq 3/k^4$. 由 Markov 不等式即证.

Question 6

1. $E(X_i) = n/m$.

2. 记事件 A_i 表示第 i 个桶中有球, $Y = A_1 + \ldots + A_m$. $P(A_i) = 1 - \left(\frac{m-1}{m}\right)^n$,

$$E(Y) = mP(A_i) = m\left(1 - \left(\frac{m-1}{m}\right)^n\right).$$

计算 $E(Y^2)$:

$$E(Y^2) = mE(A_i^2) + m(m-1)\underbrace{E(A_i A_j)}_{i \neq j}.$$

其中 $E(A_i^2) = E(A_i)$,

$$E(A_i A_j) = 1 - \left(\frac{m-2}{m}\right)^n.$$

于是

$$E(Y^2) = m\left(1 - \left(\frac{m-1}{m}\right)^n\right) + m(m-1)\left(1 - \left(\frac{m-2}{m}\right)^n\right),$$
$$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2.$$

3. 对于 $i = j$,

$$\mathrm{Cov}(X_i, X_i) = \mathrm{Var}(X_i) = n \cdot \frac{m-1}{m^2}.$$

对于 $i \neq j$,

$$\mathrm{Cov}(X_i, X_j) = E(X_i X_j) - \left(\frac{n}{m}\right)^2,$$

考虑将 X_i 写成 $\delta_{i,1} + \delta_{i,2} + \ldots + \delta_{i,n}$, 其中 $\delta_{i,k}$ 表示第 k 个球是否放到第 i 个桶里. 此时

$$\begin{aligned} E(X_i X_j) &= E\left(\left(\sum_k \delta_{i,k}\right)\left(\sum_l \delta_{j,l}\right)\right) \\ &= E\left(\sum_{k,l} \delta_{i,k}\delta_{j,l}\right). \end{aligned}$$

- 对于 $k = l$, $\delta_{i,k}$ 和 $\delta_{j,l}$ 至少有一个为 0, $E(\delta_{i,k}\delta_{j,l}) = 0$.
- 对于 $k \neq l$, $\delta_{i,k}$ 和 $\delta_{j,l}$ 独立, $E(\delta_{i,k}\delta_{j,l}) = 1/m^2$.

综上所述,

$$E(X_i X_j) = \frac{n(n-1)}{m^2}, \quad \mathrm{Cov}(X_i, X_j) = -\frac{n}{m^2}.$$