Question 1

1. 构造 $P(X=a)=a^{-1}, P(X=0)=1-a^{-1}$.
2. 构造 $P(X=c)=P(X=-c)=\frac{1}{2c^2}, P(X=0)=1-\frac{1}{c^2}$.

Question 2

1. $E((X-c)^2)=E(X^2)-2cE(x)+c^2=(c-E(x))^2+\mathrm{Var}(x)\geq \mathrm{Var}(X)$.
2. 在上一问中代入 $c=\frac{a+b}{2}$ 即有 $\mathrm{Var}(X)\leq E\Big(\big(X-\frac{a+b}{2}\big)^2\Big)\leq \big(\frac{b-a}{2}\big)^2$.

Question 3

1. $P(Y=k)=p(1-p)\big(p^{k-2}+(1-p)^{k-2}\big), k=2,3,...$

   其中 $P(Y=k)\geq 0, \sum_k P(Y=k)=p(1-p)\big(\frac{1}{1-p}+\frac{1}{p}\big)=1$.
2. $$\begin{aligned}E(Y)&=\sum_{k=2}^{\infty}kp(1-p)\big(p^{k-2}+(1-p)^{k-2}\big)\\&=\sum_{k=2}^{\infty}k(1-p)p^{k-1}+\sum_{k=2}^{\infty}kp(1-p)^{k-1}\\&=(1-p)\left(\sum_{k=2}^{\infty}p^k\right)'-p\left(\sum_{k=2}^{\infty}(1-p)^k\right)'\\&=(1-p)\left(\frac{1}{1-p}-p\right)'-p\left(\frac{1}{p}-(1-p)\right)'\\&=(1-p)\left(\frac{1}{(1-p)^2}-1\right)-p\left(-\frac{1}{p^2}+1\right)\\&=\frac{1}{p}+\frac{1}{1-p}-1.\end{aligned}$$
3. 利用 $\mathrm{Var}(Y)=E(Y^2)-E(Y)^2$, 计算 $E(Y^2)$ 即可.

   $$\begin{aligned}E(Y^2)&=\sum_{k=2}^{\infty}k^2p(1-p)\big(p^{k-2}+(1-p)^{k-2}\big)\\&=\sum_{k=2}^{\infty}k^2(1-p)p^{k-1}+\sum_{k=2}^{\infty}k^2p(1-p)^{k-1}.\end{aligned}$$

   其中

$$
\begin{aligned}
\sum_{k=2}^{\infty} k^2(1-p)p^{k-1} &= (1-p)\left[\sum_{k=2}^{\infty} k(k+1)p^{k-1} - \sum_{k=2}^{\infty} kp^{k-1}\right] \\
&= (1-p)\left(\sum_{k=3}^{\infty} p^k\right)'' - (1-p)\left(\sum_{k=2}^{\infty} p^k\right)' \\
&= (1-p)\left[\left(\frac{1}{1-p} - p - p^2\right)'' - \frac{1}{(1-p)^2} + 1\right] \\
&= (1-p)\left[\frac{2}{(1-p)^3} - 2 - \frac{1}{(1-p)^2} + 1\right] \\
&= \frac{1+p-(1-p)^3}{(1-p)^2}.
\end{aligned}
$$

相同的计算给出

$$
\sum_{k=2}^{\infty} k^2 p(1-p)^{k-1} = \frac{1+(1-p)-p^3}{p^2}.
$$

于是

$$
\mathrm{Var}(Y) = \left[\frac{1+p+(1-p)^3}{(1-p)^2} + \frac{1+(1-p)+p^3}{p^2}\right] - E(Y)^2.
$$

Question 4

1.
$$
E(e^{Xt}) = \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}e^{kt} = \sum_{k=0}^{\infty}\binom{n}{k}\left(pe^t\right)^k(1-p)^{n-k} = \left(pe^t+1-p\right)^n.
$$

2.
$$
\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}e^{kt} &= \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\sum_{i=0}^{\infty}k^i\frac{t^i}{i!} \\
&= \sum_{i=0}^{\infty}\frac{t^i}{i!}\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}k^i \\
&= \sum_{i=0}^{\infty}\frac{t^i}{i!}E(X^i).
\end{aligned}
$$

3. 令 $f(t) = E(e^{Xt}) = \sum_{i=0}^{\infty}\frac{t^i}{i!}E(X^i)$, 则

$$
f'(t) = \sum_{i=1}^{\infty}\frac{t^{i-1}}{(i-1)!}E(X^i) = \sum_{i=0}^{\infty}\frac{t^i}{i!}E(X^{i+1}),
$$

$$
E(X) = f'(0) = npe^t\left(pe^t+1-p\right)^{n-1}\Big|_{t=0} = np.
$$

同理可得

$$
E(X^2) = f''(0) = n(n-1)e^tp^2\left(pe^t+1-p\right)^{n-2} + npe^t\left(pe^t+1-p\right)^{n-1}\Big|_{t=0} = n(n-1)p^2+np.
$$

4. $$E(e^{Yt}) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \cdot e^{kt} = \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} e^{-\lambda} = e^{\lambda(e^t-1)}.$$

注意到第二问中证明的结论与分布列无关, 于是同理有

$$E(e^{Yt}) = \sum_{i=0}^{\infty} \frac{t^i}{i!} E(Y^i).$$

令 $g(t) = e^{\lambda(e^t-1)}$. 相似的计算给出

$$E(Y) = g'(0) = \lambda e^t e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda,$$
$$E(Y^2) = g''(0) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda + \lambda^2.$$

Question 5

1. 显然 $X_i \sim B\left(n, \frac{1}{n}\right)$, 于是 $E(X_i) = n \cdot \frac{1}{n} = 1$.
2. $$\begin{aligned} P(X_i = k) &= \frac{\binom{n}{k} \cdot (n-1)^{n-k}}{n^n} \\ &= \frac{1}{k!} \cdot \frac{(n(n-1)...(n-k+1)) \cdot (n-1)^{n-k}}{n^n} \le \frac{1}{k!}. \end{aligned}$$
3. $$\begin{aligned} P(Y \ge 4\log n) &\le nP(X_i \ge 4\log n) \\ &\le n \sum_{k=4\log n}^{n} P(X_i = k) \\ &\le n \sum_{k=4\log n}^{n} \frac{1}{k!} \\ &\le \frac{n^2}{2^{4\log n}} = \frac{1}{n^2}. \end{aligned}$$
4. $E(Y) \le 4\log n + n \cdot P(Y \ge 4\log n) = 4\log n + 1 \le 5\log n$.

Question 6

1. $$\begin{aligned} E((X - E(X) + b)^2) &= E(X^2 + E(X)^2 + b^2 - 2XE(X) - 2bE(X) + 2bX) \\ &= E(X^2) - E(X)^2 + b^2 = (\sigma(X)) + b^2. \end{aligned}$$
2. 代入 $b = \sigma(X)$ 即有

$$(\sigma(X) + t)^2 P(X \ge E(X) + t) \le E((X - E(X) + \sigma(X))^2) = 2(\sigma(X))^2,$$

即

$$P(X \ge E(X) + t) \le \frac{2(\sigma(X))^2}{(\sigma(X) + t)^2} < \frac{1}{2}.$$

3. 由上一问结论有 $m - E(X) \le \sigma(X)$. 上一问同理可得对于任意 $t > \sigma(X)$, $P(X \le E(X) - t) < \frac{1}{2}$, 于是有 $E(X) - m \le \sigma(X)$. 联立即有 $|m - E(X)| \le \sigma(X)$.