Question 1

1.

$$\begin{aligned}E(X) &= \sum_{x=0}^{+\infty} x \cdot P(X=x) \\ &= \sum_{x=1}^{+\infty} x \cdot (P(X>x-1) - P(X>x)) \\ &= P(X>0) + \sum_{x=1}^{+\infty} P(X>x) \cdot ((x+1)-x) \\ &= \sum_{x=0}^{+\infty} P(X>x).\end{aligned}$$

2.

$$\begin{aligned}\int_0^{+\infty} P(X>x)\,\mathrm{d}x &= \int_0^{+\infty}\left[\int_x^{+\infty} f(t)\,\mathrm{d}t\right]\mathrm{d}x \\ &= \int_0^{+\infty}\left[\int_0^{t} f(t)\,\mathrm{d}x\right]\mathrm{d}t \\ &= \int_0^{+\infty} tf(t)\,\mathrm{d}t = E(X).\end{aligned}$$

Question 2

1. • $P(Y \le t) = P\big(F^{-1}(Z) \le t\big) = P(Z \le F(t)) = F(t)$. 即 Y 和 X 的分布函数相同.
   • $P(W \le t) = P(F(X) \le t) = P\big(X \le F^{-1}(t)\big) = F\big(F^{-1}(t)\big) = t$.
2. $X \sim \exp(\lambda)$ 时 $F(x) = 1 - e^{-\lambda x}$, 所求即为 $f(t) = F^{-1}(t) = -\frac{\ln(1-t)}{\lambda}$.

Question 3

1. 显然 $E(g(X)) = \sum_{i\in[n]}(r_i - l_i)$.
2. 反证, 若不存在这样的 x, 即 $\forall x \in [0,1], g(x) < \sum_{i\in[n]}(r_i - l_i) = E(g(X))$, 则

$$E(g(X)) \le \max_{x\in[0,1]} g(x) < E(g(X)),$$

矛盾!

Question 4

1. 只需计算 $\mu = 0$ 的情形, 再加上一个平移.

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\int_0^{+\infty} ce^{-\frac{x}{b}}\,\mathrm{d}x = 2bc\int_0^{+\infty} e^{-x}\,\mathrm{d}x = 2bc = 1,$$

于是 $c = (2b)^{-1}$. 而当 $x > 0$ 时计算

$$\int_x^{+\infty} f(t)\,\mathrm{d}t = \frac{1}{2b}\int_x^{+\infty} e^{-\frac{t}{b}}\,\mathrm{d}t = \frac{1}{2}\int_{\frac{x}{b}}^{+\infty} e^{-t}\,\mathrm{d}t = \frac{1}{2}e^{-\frac{x}{b}}.$$

于是

$$F(x) = \begin{cases} \frac{1}{2}e^{\frac{\mu-x}{b}}, & x \le \mu, \\ 1-\frac{1}{2}e^{\frac{x-\mu}{b}}, & x > \mu. \end{cases}$$

2. 由对称性 $E(X)=\mu$. 计算方差时可按 $\mu=0$ 计算.

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{+\infty} x^2 f(x)\,\mathrm{d}x \\ &= \frac{1}{b}\int_0^{+\infty} x^2 e^{-\frac{x}{b}}\,\mathrm{d}x \\ &= b^2\int_0^{+\infty} x^2 e^{-x}\,\mathrm{d}x \\ &= b^2\left[-x^2e^{-x}\Big|_0^{+\infty} + \int_0^{+\infty} 2xe^{-x}\,\mathrm{d}x\right] \\ &= 2b^2\int_0^{+\infty} xe^{-x}\,\mathrm{d}x \\ &= 2b^2\left[-xe^{-x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-x}\,\mathrm{d}x\right] = 2b^2. \end{aligned}$$

于是 $\mathrm{Var}(X) = E(X^2) - E(X)^2 = 2b^2$.

Question 5

1.

$$\int_x^{+\infty} \frac{t}{x}e^{-\frac{t^2}{2}}\,\mathrm{d}t = \frac{1}{x}\int_x^{+\infty} te^{-\frac{t^2}{2}}\,\mathrm{d}t = -\frac{1}{x}\int_x^{+\infty}\mathrm{d}e^{-\frac{t^2}{2}} = \frac{1}{x}\cdot e^{-\frac{x^2}{2}}.$$

2.

$$\begin{aligned} g(x) &= \int_x^{+\infty} e^{-\frac{t^2}{2}}\,\mathrm{d}t - \frac{x}{x^2+1}e^{-\frac{x^2}{2}}, \\ g'(x) &= -e^{-\frac{x^2}{2}} - \frac{x^2+1-2x^2}{(x^2+1)^2}e^{-\frac{x^2}{2}} + \frac{x^2}{x^2+1}e^{-\frac{x^2}{2}} \\ &= e^{-\frac{x^2}{2}}\left(-1+\frac{x^2-1}{(x^2+1)^2}+\frac{x^2}{x^2+1}\right) > 0, \end{aligned}$$

而

$$g(0) = \int_0^{+\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x - 0 > 0.$$

明所欲证.

3. 分别应用第 1,2 小问的结论:

$$P(X\geq x)=\int_x^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,\mathrm{d}x\leq\frac{1}{\sqrt{2\pi}}\int_x^{+\infty}\frac{t}{x}e^{-\frac{t^2}{2}}\,\mathrm{d}t=\frac{1}{x\sqrt{2\pi}}e^{-\frac{x^2}{2}},$$
$$P(X\geq x)=\int_x^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,\mathrm{d}x\geq\frac{1}{\sqrt{2\pi}}\cdot\frac{x}{x^2+1}e^{-\frac{x^2}{2}}.$$

4. 令 $X\frac{:=(Y-\mu)}{\sigma}$, 有 $X\sim N(0,1)$. 于是
$$P(|Y-\mu|\leq k\sigma)=P(|X|<k)=1-2P(X\geq k),$$
代入第 3 小问结论立得.

Question 6

1.
$$\begin{aligned}E(e^{tX})&=\int_0^{+\infty}\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}e^{tx}\,\mathrm{d}x\\&=\left(\frac{\lambda}{\lambda-t}\right)^\alpha\int_0^{+\infty}\frac{((\lambda-t)x)^{\alpha-1}}{\Gamma(\alpha)}e^{-(\lambda-t)x}\,\mathrm{d}(\lambda-t)x\\&=\left(\frac{\lambda}{\lambda-t}\right)^\alpha.\end{aligned}$$
这要求 $t<\lambda$; 当 $t\geq\lambda$ 时积分发散.

2.
$$\begin{aligned}E\left(e^{tY^2}\right)&=\int_0^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}e^{tx^2}\,\mathrm{d}x\\&=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}e^{\left(t-\frac{1}{2}\right)x^2}\,\mathrm{d}x\\&=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{\frac{1}{2}-t}}\int_0^{+\infty}e^{-x^2}\,\mathrm{d}x\\&=\frac{1}{\sqrt{1-2t}}.\end{aligned}$$
这要求 $t<\frac{1}{2}$; 当 $t\geq\frac{1}{2}$ 时积分发散.

3.
$$F(x)=P(Z\leq x)=P\left(|Y|\leq\sqrt{x}\right)=2\int_0^{\sqrt{x}}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,\mathrm{d}t,$$
于是
$$\forall x\geq 0,\quad f(x)=F'(x)=\sqrt{\frac{2}{\pi}}\cdot e^{-\frac{x}{2}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{2\pi x}}\cdot e^{-\frac{x}{2}}.$$
实际上 $Z\sim\Gamma\left(\frac{1}{2},\frac{1}{2}\right)$ 即 $Z\sim\chi^2(1)$.