Consulting project:
Numerical treatment of inflator holes in airbag inflation simulations

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Project short title: ESI_TechnicalAssistance2018

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1.1 Derivation

The inflator is characterized by the two user-given curves

$$T_0 = T^{total}(t) \text{ = total temperature as a function of time} \tag{1}$$

$$\rho_L u_L = \dot{m}''(t) \text{ = mass flow per unit area as a function of time} \tag{2}$$

to be valid at the location of the inflator opening orifice.

The subscript “0” is the state at rest, the subscript “L” is the state at the nozzle exit (inflator orifice) (Laval state, Ma=1.0).

Assuming polytropic change of state from “0” to “L”, we have

$$\frac{T_L}{T_0} = \frac{2}{n+1} \tag{3}$$

which is the ration of the temperatures between Ma=1 and Ma=0, and where n is the polytropic exponent. As we assume sonic flow at the orifice, we can express the sound speed by the temperature as

$$u_L = \sqrt{k \cdot R \cdot T_L} \tag{4}$$

Here, k ist he true isentropic coefficient, ie. $k = \frac{c_p}{c_p - R}$. k characterizes the sonic state, whereas n characterizes the change of state from “0” to “L”. We have $n > k$, for the time being, as we do not know further details, we work with $n = k$.

With this, we can express the density at the orifice as

$$\rho_L = \frac{\dot{m}''(t)}{u_L} = \frac{\dot{m}''(t)}{\sqrt{k \cdot R \cdot T_L}} = \frac{\dot{m}''(t)}{\sqrt{\left(\frac{2}{n+1}\right) k \cdot R \cdot T_0}} = \left(\frac{2}{n+1}\right)^{\frac{-1}{2}} \frac{1}{\sqrt{k}} \cdot \frac{\dot{m}''(t)}{\sqrt{R \cdot T_0}} \tag{5}$$

Using the assumption of perfect gases, we produce the pressure at the orifice as

$$p_L = \rho_L \cdot R \cdot T_L = \left(\frac{2}{n+1}\right)^{\frac{-1}{2}} \frac{1}{\sqrt{k}} \cdot \frac{\dot{m}''(t)}{\sqrt{R \cdot T_0}} \cdot R \cdot \frac{2}{n+1} T_0 = \left(\frac{2}{n+1}\right)^{\frac{1}{2}} \frac{1}{\sqrt{k}} \cdot \dot{m}''(t) \sqrt{R \cdot T_0} \quad \textbf{(6)}$$

With these values, we can also conclude the values at rest (“0”) for pressure

$$\begin{aligned} \frac{p_L}{p_0} &= \left(\frac{2}{n+1}\right)^{\frac{n}{n-1}} \qquad \Rightarrow \\ p_0 &= \left(\frac{2}{n+1}\right)^{\frac{-n}{n-1}} p_L \\ &= \left(\frac{2}{n+1}\right)^{\frac{-n}{n-1}} \left(\frac{2}{n+1}\right)^{\frac{1}{2}} \frac{1}{\sqrt{k}} \cdot \dot{m}''(t) \sqrt{R \cdot T_0} \\ &= \left(\frac{2}{n+1}\right)^{\frac{-(n+1)}{2(n-1)}} \frac{1}{\sqrt{k}} \cdot \dot{m}''(t) \sqrt{R \cdot T_0} \end{aligned} \quad \textbf{(7)}$$

and density:

$$\begin{aligned} \frac{\rho_L}{\rho_0} &= \left(\frac{2}{n+1}\right)^{\frac{1}{n-1}} \qquad \Rightarrow \\ \rho_0 &= \left(\frac{2}{n+1}\right)^{\frac{-1}{n-1}} \rho_L \\ &= \left(\frac{2}{n+1}\right)^{\frac{-1}{n-1}} \left(\frac{2}{n+1}\right)^{\frac{-1}{2}} \frac{1}{\sqrt{k}} \cdot \frac{\dot{m}''(t)}{\sqrt{R \cdot T_0}} \\ &= \left(\frac{2}{n+1}\right)^{\frac{-(n+1)}{2(n-1)}} \frac{1}{\sqrt{k}} \cdot \frac{\dot{m}''(t)}{\sqrt{R \cdot T_0}} \end{aligned} \quad \textbf{(8)}$$

The values “at rest” represent the state of the gas inside of the inflator before it goes into the nozzle, dependent ONLY on the measured total temperature and the measured massflow.

Finally, we can integrate the massflow over time in order to produce $M(t)$, the mass already expelled by the orifice

$$M(t) = \int \dot{m}''(t) \cdot A_{orifice} \cdot dt \quad \textbf{(9)}$$

Here, $A_{orifice}$ is the active (that means open, non-covered) part of the inflator orifice. If some part of the inflator orifice is covered by the airbag membrane, the total mass flow automatically must drop.

1.2 Compact

To conclude, from the boundary conditions given in (1) and (2) AND under the assumption of Ma=1, we can compute the complete set of thermodynamic quantities at rest ("0") and the inflator orifice ("L"):

$$
\begin{aligned}
&T_L(t) = \frac{2}{n+1} T_0(t) \\
&u_L(t) = \sqrt{k \cdot R \cdot T_L(t)} \\
&\rho_L(t) = \frac{\dot{m}''(t)}{u_L(t)} \\
&p_L(t) = \rho_L(t) \cdot R \cdot T_L(t) \\
&p_0(t) = p_L(t) \cdot \left(\frac{2}{n+1}\right)^{\frac{-n}{n-1}} \\
&\rho_0(t) = \rho_L(t) \cdot \left(\frac{2}{n+1}\right)^{\frac{-1}{n-1}} \\
&M(t) = \int_0^t \dot{m}''(\tau) \cdot A_{orifice} \cdot d\tau
\end{aligned}
\qquad \textbf{(10)}
$$

Consequently, the thermodynamic quantities can be expressed ALSO as a function of the mass already expelled, since we have a one-to-one relation:

$$M(t) \Leftrightarrow t(M) \qquad \textbf{(11)}$$

And so we have

$$
\begin{aligned}
&T_L(M) = \frac{2}{n+1} T_0(M) \\
&u_L(M) = \sqrt{k \cdot R \cdot T_L(M)} \\
&\rho_L(M) = \frac{\dot{m}''(M)}{u_L(M)} \\
&p_L(M) = \rho_L(M) \cdot R \cdot T_L(M) \\
&p_0(M) = p_L(M) \cdot \left(\frac{2}{n+1}\right)^{\frac{-n}{n-1}} \\
&\rho_0(M) = \rho_L(M) \cdot \left(\frac{2}{n+1}\right)^{\frac{-1}{n-1}} \\
&M(M) = M \qquad \Rightarrow trivial
\end{aligned}
\qquad \textbf{(12)}
$$

The FIVE boundary conditions for the FPM inflow points can then be set to

$$
\begin{aligned}
&\mathbf{v}_L^{FPM} \cdot \mathbf{n} = u_L \\
&\mathbf{v}_L^{FPM} \cdot \mathbf{t}_1 = 0 \\
&\mathbf{v}_L^{FPM} \cdot \mathbf{t}_2 = 0 \\
&p_L^{FPM} = p_L \\
&T_L^{FPM} = T_L \quad \text{or} \quad \rho_L^{FPM} = \rho_L
\end{aligned}
\tag{13}
$$

2 Subsonic inflow

2.1 Derivation

The inflow goes over into the subsonic state, if the local pressure in front of the orifice (in front of an FPM-inflow-point) becomes bigger than the Laval pressure, i.e.

$$\tilde{p}^{FPM}(t,x) > p_L(M(t)) \tag{14}$$

The tilde means the numerical interpolation/projection of the pressure close to the boundary. A simple way of interpolation is the Shepard interpolate (least squares with error order 1):

$$\tilde{p}_i^{FPM} = \frac{\sum_{j \in N_i} W_{ij} \cdot p_j^{FPM}}{\sum_{j \in N_i} W_{ij}}, \quad W_{ij} = \exp\left(-\alpha \cdot r_{ij}\right), \quad r_{ij} = \frac{\left\|\mathbf{x}_i - \mathbf{x}_j\right\|}{\frac{1}{2}\left(h_i^2 + h_j^2\right)} \tag{15}$$

In the subsonic state, the inflow conditions change. The flow does not reach Ma=1, and the shock travels into the orifice, i.e. the measured/interpolated pressure will shortly arrive at the orifice, i.e.

$$p_B(t,x) = \tilde{p}^{FPM}(t,x) \tag{16}$$

If we assume polytropic change of state, we can compute the theoretical temperature at the orifice by

$$\frac{\tilde{p}^{FPM}}{p_0} = \frac{p_B}{p_0} = \frac{\left(T_B\right)^{\frac{n}{n-1}}}{\left(T_0\right)^{\frac{n}{n-1}}} \tag{17}$$

By using the first fundamental law of thermodynamics, we have

$$\frac{1}{2}u_B^2 = c_p \cdot \left(T_0 - T_B\right) \tag{18}$$

And furthermore, using polytropic change of state, we can compute the theoretical density at the orifice by

$$\frac{\tilde{p}^{FPM}}{p_0} = \frac{p_B}{p_0} = \frac{\left(\rho_B\right)^n}{\left(\rho_0\right)^n} \tag{19}$$

For the measured counter pressure $\tilde{p}^{FPM}$, there exists a unique mass flow per unit area $\dot{m}''_{crit}$:

$$\begin{aligned}
&\dot{m}''_{crit} = \\
&= \rho_B \cdot u_B \\
&= \left(\frac{p_B}{p_0}\right)^{\frac{1}{n}} \cdot \rho_0 \cdot \sqrt{2c_p \cdot \left(T_0 - T_B\right)} \\
&= \left(\frac{p_B}{p_0}\right)^{\frac{1}{n}} \cdot \rho_0 \cdot \sqrt{2c_p \cdot \left(T_0 - \left(\frac{p_B}{p_0}\right)^{\frac{n-1}{n}} \cdot T_0\right)} \\
&= \left(\frac{p_B}{p_0}\right)^{\frac{1}{n}} \sqrt{1 - \left(\frac{p_B}{p_0}\right)^{\frac{n-1}{n}}} \cdot \rho_0 \cdot \sqrt{2c_p \cdot T_0} \\
&= \left(\frac{p_B}{p_0}\right)^{\frac{1}{n}} \sqrt{1 - \left(\frac{p_B}{p_0}\right)^{\frac{n-1}{n}}} \cdot \left[\left(\frac{2}{n+1}\right)^{\frac{-(n+1)}{2(n-1)}} \frac{1}{\sqrt{k}} \cdot \frac{\dot{m}''(t)}{\sqrt{R \cdot T_0}}\right] \cdot \sqrt{2c_p \cdot T_0} \\
&= \left(\frac{p_B}{p_0}\right)^{\frac{1}{n}} \sqrt{1 - \left(\frac{p_B}{p_0}\right)^{\frac{n-1}{n}}} \cdot \left(\left(\frac{2}{n+1}\right)^{\frac{-(n+1)}{2(n-1)}} \sqrt{\frac{2c_p}{kR}}\right) \cdot \dot{m}''
\end{aligned} \tag{20}$$

Important remark: in the equation (20) above, we shall ensure

$$\frac{p_L}{p_0} = \left(\frac{2}{n+1}\right)^{\frac{n}{n-1}} \leq \left(\frac{p_B}{p_0}\right) \leq 1 \tag{21}$$

If (21) is fulfilled, then we have $0 \leq \dot{m}''_{crit} \leq \dot{m}''$

2.2 Compact

Concluding, we have the following state at the inflow, if $\tilde{p}^{FPM} > p_L(M)$

$$\begin{aligned}
p_B &= \tilde{p}^{FPM} \\
(T_B) &= \left(\frac{p_B}{p_0(M)}\right)^{\frac{n-1}{n}} \cdot \left(T_0\ (M)\right) \\
u_B &= \sqrt{2c_p \cdot \left(T_0\ (M) - T_B\right)} \\
\rho_B &= \left(\frac{p_B}{p_0(M)}\right)^{\frac{1}{n}} \cdot \rho_0(M) \\
\dot{m}''_{crit} &= \left(\frac{p_B}{p_0}\right)^{\frac{1}{n}} \sqrt{1 - \left(\frac{p_B}{p_0}\right)^{\frac{n-1}{n}}} \cdot \left(\left(\frac{2}{n+1}\right)^{\frac{-(n+1)}{2(n-1)}} \sqrt{\frac{2c_p}{kR}}\right) \cdot \dot{m}''
\end{aligned} \tag{22}$$

And the boundary conditions to be set at the inflow are

$$\begin{aligned}
\mathbf{v}_B^{FPM} \cdot \mathbf{n} &= u_B \\
\mathbf{v}_B^{FPM} \cdot \mathbf{t}_1 &= 0 \\
\mathbf{v}_B^{FPM} \cdot \mathbf{t}_2 &= 0 \\
p_B^{FPM} &= \tilde{p}^{FPM} \\
T_B^{FPM} &= T_B \quad \text{or} \quad \rho_B^{FPM} = \rho_B
\end{aligned} \tag{23}$$

3 Stagnating inflow

3.1 Classical

If the counter pressure in the neighbourhood of the orifice rises even more and becomes bigger than the theoretical stagnation pressure, i.e. if we have

$$\tilde{p}^{FPM} > p_0(M) \tag{24}$$

then, in the consequence, the flow would travel back through the orifice, which finally results in a stagnation of the velocity component in normal direction. In this case, the boundary conditions to be provided are those ones of a classical slip wall.

3.2 Alternative

Be aware, that the term $u_B = \sqrt{2c_p \cdot \left(T_0\left(M\right) - T_B\right)}$ will have a negative argument under the square root anyways, see equation (22), so one could also enhance the conditions of the subsonic inflow by re-writing (22) as

$$
\begin{aligned}
p_B &= \tilde{p}^{FPM} \\
\left(T_B\right) &= \left(\frac{p_B}{p_0\left(M\right)}\right)^{\frac{n-1}{n}} \cdot \left(T_0\left(M\right)\right) \\
u_B &= \sqrt{\max\left(2c_p \cdot \left(T_0\left(M\right) - T_B\right), 0\right)} \\
\rho_B &= \left(\frac{p_B}{p_0\left(M\right)}\right)^{\frac{1}{n}} \cdot \rho_0\left(M\right)
\end{aligned}
\tag{25}
$$

and stay with using the subsonic implementation with $u_B = 0$ as an extremal case, instead of implementing the slip-wall condition.