Derivation of Energy Equation with turbulent kinetic

1 Classical Equations

The classical momentum and energy conservation is given by:
Momentum:

$$\frac{D(\rho\mathbf{v})}{Dt}+(\rho\mathbf{v})\cdot\nabla^T\mathbf{v}+\nabla p=\nabla^T\left(\mathbf{S}_v\right)+\rho\mathbf{g} \tag{1.1}$$

Energy

$$\frac{D(\rho E)}{Dt}+(\rho E)\cdot\nabla^T\mathbf{v}+\nabla^T(p\mathbf{v})=\nabla^T\left(\mathbf{S}_v\mathbf{v}\right)+\rho\left(\mathbf{g}^T\cdot\mathbf{v}\right) \tag{1.2}$$

Here,

$\mathbf{S}_v$ is the viscous stress tensor, and $\rho E=\rho u+\rho\frac{\mathbf{v}^2}{2}$, that means, equation (1.2) in a bit more detailed form looks as

$$\frac{D\left(\rho u+\rho\frac{\mathbf{v}^2}{2}\right)}{Dt}+\left(\rho u+\rho\frac{\mathbf{v}^2}{2}\right)\cdot\nabla^T\mathbf{v}+\nabla^T(p\mathbf{v})=\nabla^T\left(\mathbf{S}_v\mathbf{v}\right)+\rho\left(\mathbf{g}^T\cdot\mathbf{v}\right) \tag{1.3}$$

From equation (1.1), we find a formulation for the kinetic energy

$$\frac{D}{Dt}\left(\rho\frac{\mathbf{v}^2}{2}\right)+\left(\rho\frac{\mathbf{v}^2}{2}\right)\cdot\nabla^T\mathbf{v}+(\nabla p)^T\cdot\mathbf{v}=\nabla^T\mathbf{S}_v\cdot\mathbf{v}+\rho\left(\mathbf{g}^T\cdot\mathbf{v}\right) \tag{1.4}$$

and, finally, subtraction of equation (1.4) from (1.3) gives

$$\frac{D(\rho u)}{Dt}+(\rho u)\cdot\nabla^T\mathbf{v}+\nabla^T(p\mathbf{v})-\left(\nabla^T p\right)\cdot\mathbf{v}=\nabla^T\left(\mathbf{S}_v\mathbf{v}\right)-\left(\nabla^T\mathbf{S}_v\right)\cdot\mathbf{v} \tag{1.5}$$

The terms have different signification:
$(\rho u)\cdot\nabla^T\mathbf{v}$ comes from the density change due to compression or expansion

$\nabla^T (p\mathbf{v}) - (\nabla^T p) \cdot \mathbf{v}$ is the heating of the material due to compression

$\nabla^T (\mathbf{S}_v \mathbf{v}) - (\nabla^T \mathbf{S}_v) \cdot \mathbf{v}$ is the dissipation function and leads to heating due to viscous friction

2 Derive a generalized energy equation containing turbulent sources terms

Now let us have a look at the complete stress tensor

$$\mathbf{S} = \mathbf{S}_v + \mathbf{S}_t \tag{2.1}$$

where $\mathbf{S}_t$ is the turbulent addition to the stress tensor. We have to consider here only the deviatoric part of the stress tensor (hydrostatic part is the pressure p, i.e.

$$trace(\mathbf{S}_t) = trace(\mathbf{S}_v) = trace(\mathbf{S}) = 0 \tag{2.2}$$

In general, the formulation for the momentum as well as energy do not change, as they come from a global balance consideration. In fact, for momentum we have

$$\frac{D(\rho\mathbf{v})}{Dt} + (\rho\mathbf{v}) \cdot \nabla^T \mathbf{v} + \nabla p = \nabla^T (\mathbf{S}) + \rho\mathbf{g} \tag{2.3}$$

and for the energy, we have to take into account that the kinetic turbulent energy now also takes part in the global balance in the sense

$$\frac{D(\rho E + \rho k)}{Dt} + (\rho E + \rho k) \cdot \nabla^T \mathbf{v} + \nabla^T (p\mathbf{v}) = \nabla^T (\mathbf{S}\ \mathbf{v}) + \rho(\mathbf{g}^T \cdot \mathbf{v}) \tag{2.4}$$

2.1 Formulation for kinetic energy

From equation (2.3), we can derive a formulation for the mean kinetic energy (not the turbulent kinetic energy!!!!) in the following way.

First, scalar multiplication of the equation with the mean velocity

$$\frac{D(\rho\mathbf{v})^T}{Dt} \cdot \mathbf{v} + (\rho\mathbf{v}^2) \cdot \nabla^T \mathbf{v} + (\nabla p)^T \cdot \mathbf{v} = (\nabla^T \mathbf{S})^T \cdot \mathbf{v} + \rho(\mathbf{g}^T \cdot \mathbf{v}) \tag{2.5}$$

and by adding the value of zero, we find

$$\frac{D(\rho\mathbf{v})^T}{Dt}\cdot\mathbf{v}+(\rho\mathbf{v})^T\cdot\frac{D\mathbf{v}}{Dt}-(\rho\mathbf{v})^T\cdot\frac{D\mathbf{v}}{Dt}+\left(\rho\mathbf{v}^2\right)\cdot\nabla^T\mathbf{v}+(\nabla p)^T\cdot\mathbf{v}=\left(\nabla^T\mathbf{S}\ \right)^T\cdot\mathbf{v}+\rho\left(\mathbf{g}^T\cdot\mathbf{v}\right)\quad(2.6)$$

Simplification yields

$$\frac{D\left(\rho\mathbf{v}^2\right)^T}{Dt}-(\rho\mathbf{v})^T\cdot\frac{D\mathbf{v}}{Dt}+\left(\rho\mathbf{v}^2\right)\cdot\nabla^T\mathbf{v}+(\nabla p)^T\cdot\mathbf{v}=\left(\nabla^T\mathbf{S}\ \right)^T\cdot\mathbf{v}+\rho\left(\mathbf{g}^T\cdot\mathbf{v}\right)\qquad(2.7)$$

We would like to eliminate the term $\frac{D\mathbf{v}}{Dt}$, and equation (2.3) allows an easy formulation by

$$\rho\frac{D\mathbf{v}}{Dt}+\nabla p=\nabla^T\left(\mathbf{S}\ \right)+\rho\mathbf{g}\qquad(2.8)$$

plugging (2.8) into (2.7) gives

$$\frac{D\left(\rho\mathbf{v}^2\right)^T}{Dt}+\left(\rho\mathbf{v}^2\right)\cdot\nabla^T\mathbf{v}+2(\nabla p)^T\cdot\mathbf{v}=2\left(\nabla^T\mathbf{S}\ \right)^T\cdot\mathbf{v}+2\rho\left(\mathbf{g}^T\cdot\mathbf{v}\right)\qquad(2.9)$$

which, through devision by 2, yields

$$\frac{D\left(\rho\frac{\mathbf{v}^2}{2}\right)^T}{Dt}+\left(\rho\frac{\mathbf{v}^2}{2}\right)\cdot\nabla^T\mathbf{v}+(\nabla p)^T\cdot\mathbf{v}=\left(\nabla^T\mathbf{S}\ \right)^T\cdot\mathbf{v}+\rho\left(\mathbf{g}^T\cdot\mathbf{v}\right)\qquad(2.10)$$

2.2 Generalized, assembled energy equation

The term (2.10) is now ready to be plugged into the energy equation (2.4). First, let us write (2.4) a bit more detailed in the sense

$$\frac{D\left(\rho u+\rho\frac{\mathbf{v}^2}{2}+\rho k\right)}{Dt}+\left(\rho u+\rho\frac{\mathbf{v}^2}{2}+\rho k\right)\cdot\nabla^T\mathbf{v}+\nabla^T\left(p\mathbf{v}\right)=\nabla^T\left(\mathbf{S}\ \mathbf{v}\right)+\rho\left(\mathbf{g}^T\cdot\mathbf{v}\right)\quad(2.11)$$

which shows the total energy in all of its details:

ρu = internal (thermal) energy

$\rho\frac{\mathbf{v}^2}{2}$ = kinetic energy of the mean velocity

ρk = kinetic energy of the turbulent fluctuations

2.3 Separation into turbulent energy part

In order to find out a separate formulation for turbulent energy, we have to subtract equation (2.10) from (2.11). We obtain

$$\frac{D(\rho u+\rho k)}{Dt}+(\rho u+\rho k)\cdot\nabla^T\mathbf{v}+\nabla^T(p\mathbf{v})-\nabla^T p\cdot\mathbf{v}=\nabla^T(\mathbf{S}\;\cdot\mathbf{v})-(\nabla^T\mathbf{S})\cdot\mathbf{v} \qquad (2.12)$$

In order to find out the precise formulation for $\frac{D(\rho k)}{Dt}$, we have to decouple equation (2.12) furthermore. We can simply use equation (1.5) however we have to extend it by the additional heating that comes from the dissipation of turbulent kinetic energy, i.e.

$$\frac{D(\rho u)}{Dt}+(\rho u)\cdot\nabla^T\mathbf{v}+\nabla^T(p\mathbf{v})-(\nabla^T p)\cdot\mathbf{v}=\nabla^T(\mathbf{S}_v\mathbf{v})-(\nabla^T\mathbf{S}_v)\cdot\mathbf{v}+\rho\varepsilon \qquad (2.13)$$

Finally, subtracting equation (2.13) from (2.12) yields a proper formulation for the turbulent kinetic energy by

$$\frac{D(\rho k)}{Dt}+(\rho k)\cdot\nabla^T\mathbf{v}=\nabla^T(\mathbf{S}_t\cdot\mathbf{v})-(\nabla^T\mathbf{S}_t)\cdot\mathbf{v}-\rho\varepsilon \qquad (2.14)$$

This equation does not take into account effects by turbulent buoyancy, which in the case of airbag deployment do not play a role. We see clearly the production term of turbulent kinetic energy to be

$$P_k=\nabla^T(\mathbf{S}_t\cdot\mathbf{v})-(\nabla^T\mathbf{S}_t)\cdot\mathbf{v} \qquad (2.15)$$

We have now a collection of the change of all three mayor parts of energy:

- Mean kinetic energy given by the mean velocity, see equation (2.10)

$$\frac{D}{Dt}\left(\rho\frac{\mathbf{v}^2}{2}\right)+\left(\rho\frac{\mathbf{v}^2}{2}\right)\cdot\nabla^T\mathbf{v}+(\nabla p)^T\cdot\mathbf{v}=(\nabla^T\mathbf{S}\;)^T\cdot\mathbf{v}+\rho(\mathbf{g}^T\cdot\mathbf{v}) \qquad (2.16)$$

- turbulent kinetic energy, see equation (2.14)

$$\frac{D}{Dt}(\rho k)+(\rho k)\cdot\nabla^T\mathbf{v}=\nabla^T(\mathbf{S}_t\cdot\mathbf{v})-(\nabla^T\mathbf{S}_t)\cdot\mathbf{v}-\rho\varepsilon \qquad (2.17)$$

- thermal energy, see equation (2.13)

$$\frac{D}{Dt}(\rho u)+(\rho u)\cdot\nabla^T\mathbf{v}+\nabla^T(p\mathbf{v})-\left(\nabla^T p\right)\cdot\mathbf{v}=\nabla^T\left(\mathbf{S}_v\mathbf{v}\right)-\left(\nabla^T\mathbf{S}_v\right)\cdot\mathbf{v}+\rho\varepsilon \tag{2.18}$$

If we put all equations together, we find back the global energy conservation equation (2.4).

3 Detailed representation of the turbulent production term

We refer to equation (2.15):

$$P_k=\nabla^T\left(\mathbf{S}_t\cdot\mathbf{v}\right)-\left(\nabla^T\mathbf{S}_t\right)\cdot\mathbf{v} \tag{3.1}$$

The chain rule leads to

$$\begin{aligned}P_k&=\nabla^T\left(\mathbf{S}_t\cdot\mathbf{v}\right)-\left(\nabla^T\mathbf{S}_t\right)\cdot\mathbf{v}=\left(\nabla^T\mathbf{S}_t\right)\cdot\mathbf{v}+\left(\mathbf{S}_t\cdot\nabla\right)\cdot\mathbf{v}-\left(\nabla^T\mathbf{S}_t\right)\cdot\mathbf{v}\\&=\left(\mathbf{S}_t\cdot\nabla\right)\cdot\mathbf{v}\end{aligned} \tag{3.2}$$

Detailed explication of $\mathbf{S}_t$ yields

$$\mathbf{S}_t=\mu_t\begin{pmatrix}u_x+u_x-\frac{2}{3}D & u_y+v_x & u_z+w_x\\ v_x+u_y & v_y+v_y-\frac{2}{3}D & v_z+w_y\\ w_x+u_z & w_y+v_z & w_z+w_z-\frac{2}{3}D\end{pmatrix} \tag{3.3}$$

where $D=u_x+v_y+w_z$ is the divergence of the velocity, and the term $-\frac{2}{3}D$ makes the stress tensor trace free.

We find that the production rate now is

$$\begin{aligned}&P_k=\mu_t(\\&\left(u_x+u_x-\tfrac{2}{3}D\right)\cdot u_x+\left(u_y+v_x\right)\cdot u_y+\left(u_z+w_x\right)\cdot u_z+\\&\left(v_x+u_y\right)\cdot v_x+\left(v_y+v_y-\tfrac{2}{3}D\right)\cdot v_y+\left(v_z+w_y\right)\cdot v_z+\\&\left(w_x+u_z\right)\cdot w_x+\left(w_y+v_z\right)\cdot w_y+\left(w_z+w_z-\tfrac{2}{3}D\right)\cdot w_z\\&)\end{aligned} \tag{3.4}$$

simplification results in

$$
\begin{aligned}
P_k &= \mu_t \, (\\
&\left(u_x^2 + u_x^2\right) + \left(u_y^2 + v_x \cdot u_y\right) + \left(u_z^2 + w_x \cdot u_z\right) + \\
&\left(v_x^2 + u_y \cdot v_x\right) + \left(v_y^2 + v_y^2\right) + \left(v_z^2 + w_y \cdot v_z\right) + \\
&\left(w_x^2 + u_z \cdot w_x\right) + \left(w_y^2 + v_z \cdot w_y\right) + \left(w_z^2 + w_z^2\right) - \tfrac{2}{3} D^2 \\
&)
\end{aligned}
\tag{3.5}
$$

and furthermore

$$
\begin{aligned}
P_k &= \mu_t \, (\\
&\left(2u_x^2\right) + \left(u_y + v_x\right)^2 + \left(u_z + w_x\right)^2 + \\
&\left(2v_y^2\right) + \left(v_z + w_y\right)^2 + \\
&\left(2w_z^2\right) - \tfrac{2}{3} D^2 \\
&)
\end{aligned}
\tag{3.6}
$$

Another way of simplification consists in

$$
\begin{aligned}
P_k &= \mu_t \, (\\
&\left(\tfrac{4}{3} u_x^2 + \tfrac{2}{3} u_x^2\right) + \left(u_y^2 + v_x \cdot u_y\right) + \left(u_z^2 + w_x \cdot u_z\right) + \\
&\left(v_x^2 + u_y \cdot v_x\right) + \left(\tfrac{4}{3} v_y^2 + \tfrac{2}{3} v_y^2\right) + \left(v_z^2 + w_y \cdot v_z\right) + \\
&\left(w_x^2 + u_z \cdot w_x\right) + \left(w_y^2 + v_z \cdot w_y\right) + \left(\tfrac{4}{3} w_z^2 + \tfrac{2}{3} w_z^2\right) - \tfrac{2}{3} D^2 \\
&)
\end{aligned}
\tag{3.7}
$$

which yields

$$
\begin{aligned}
P_k &= \mu_t \, (\\
&\left(u_y^2 + v_x \cdot u_y\right) + \left(u_z^2 + w_x \cdot u_z\right) + \\
&\left(v_x^2 + u_y \cdot v_x\right) + \left(v_z^2 + w_y \cdot v_z\right) + \\
&\left(w_x^2 + u_z \cdot w_x\right) + \left(w_y^2 + v_z \cdot w_y\right) - \tfrac{2}{3} D^2 + \\
&\tfrac{2}{3} D^2 + \left(\tfrac{4}{3} u_x^2\right) + \left(\tfrac{4}{3} v_y^2\right) + \left(\tfrac{4}{3} w_z^2\right) - \tfrac{4}{3} u_x v_y - \tfrac{4}{3} u_x w_z - \tfrac{4}{3} v_y w_z \\
&)
\end{aligned}
\tag{3.8}
$$

which finally yields

$$
\begin{aligned}
P_k &= \mu_t \, (\\
&\left(v_x + u_y\right)^2 + \left(w_x + u_z\right)^2 + \left(v_z + w_y\right)^2 \\
&\tfrac{2}{3}\left(u_x - v_y\right)^2 + \tfrac{2}{3}\left(v_y - w_z\right)^2 + \tfrac{2}{3}\left(w_z - u_x\right)^2 \\
&)
\end{aligned}
\tag{3.9}
$$

which nicely demonstrates the positivity of the production term.